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1. Given,

$$Y = |Z|$$

$$Z \sim N(0,1)$$

$$\mu = 0, \sigma^2 = 1$$

a) $f(Y) = ?$

$$f(y) = f(|Z|)$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$f(y) = f_Z(y) + f_Z(-y) \quad y \geq 0$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \quad y \geq 0$$

$$= \frac{2}{\sqrt{2\pi}} e^{-y^2/2} \quad y \geq 0$$

b) $$E(Y) = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$= \int_0^{\infty} y \cdot \frac{2}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

$$y^2 = t$$

$$2y\,dy = dt$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-t/2}}{\sqrt{2\pi}} \, dt \, .$$

$$= \frac{2}{\sqrt{2\pi}}$$

c) $Var(Y) = E(Y^2) - (E(Y))^2$

$$E[y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy$$

$$= \int_0^{\infty} z^2 f_Y(y)\,dy$$

$E[y^2] = E[|z|^2] = E[z^2]$

So, $E[y^2] = 1$

$$Var(Y) = 1 - \left(\frac{2}{\sqrt{2\pi}}\right)^2$$

$$= 1 - \frac{2}{\pi}$$

d) In general

$$E(Y^k) = \int_0^\infty y^k \cdot f_Y(y)\,dy$$

$$= \int_0^\infty y^k \cdot \frac{2}{\sqrt{2\pi}} e^{-y^2/2}\,dy$$

$$\frac{y^2}{2} = t \Rightarrow y = \sqrt{2t}$$

$$y\,dy = dt$$

$$= \int_0^\infty y^{k-1} \cdot y\,dy \cdot \frac{2}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= \int_0^\infty (\sqrt{2t})^{k-1} \cdot dt \cdot \frac{\cancel{2}^{\sqrt{2}}}{\cancel{\sqrt{2}}\sqrt{\pi}} e^{-t}$$

$$= \frac{(\sqrt{2})^k}{\sqrt{\pi}} \int_0^\infty t^{\frac{k-1}{2}} e^{-t}\,dt$$

$$= \frac{(\sqrt{2})^k}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right)$$

$$E(Y^3) = \frac{2\sqrt{2}}{\sqrt{\pi}} \Gamma(2) \Rightarrow \frac{2\sqrt{2}}{\sqrt{\pi}}$$