Brownian motion

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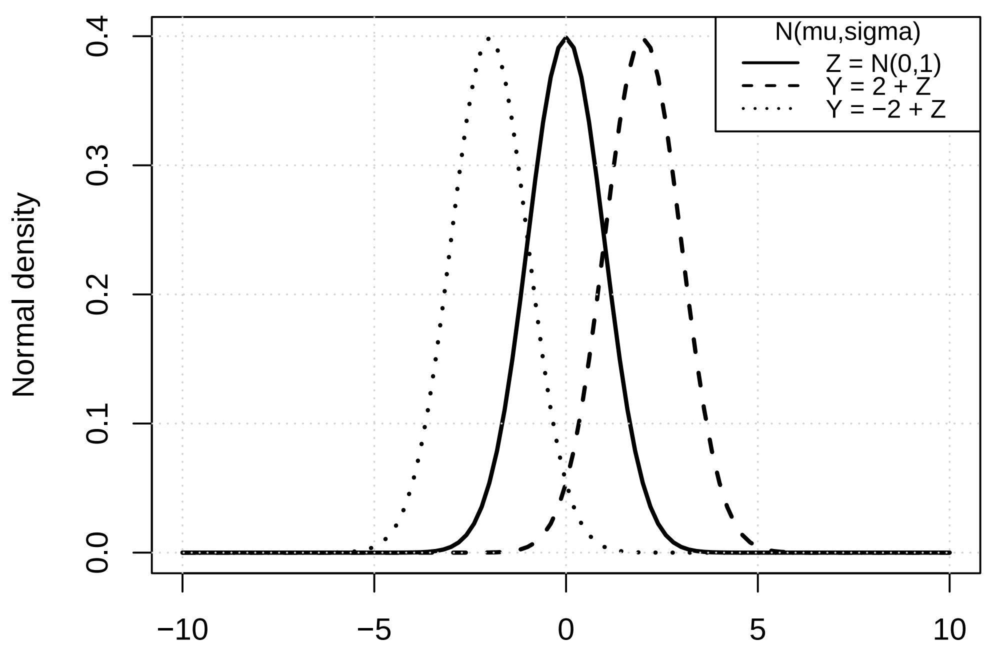
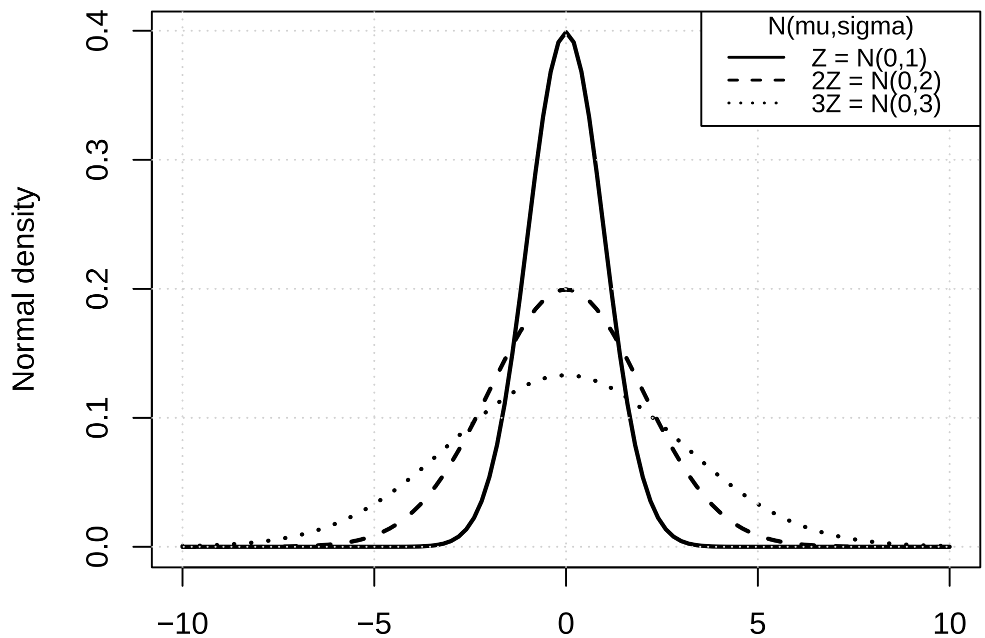
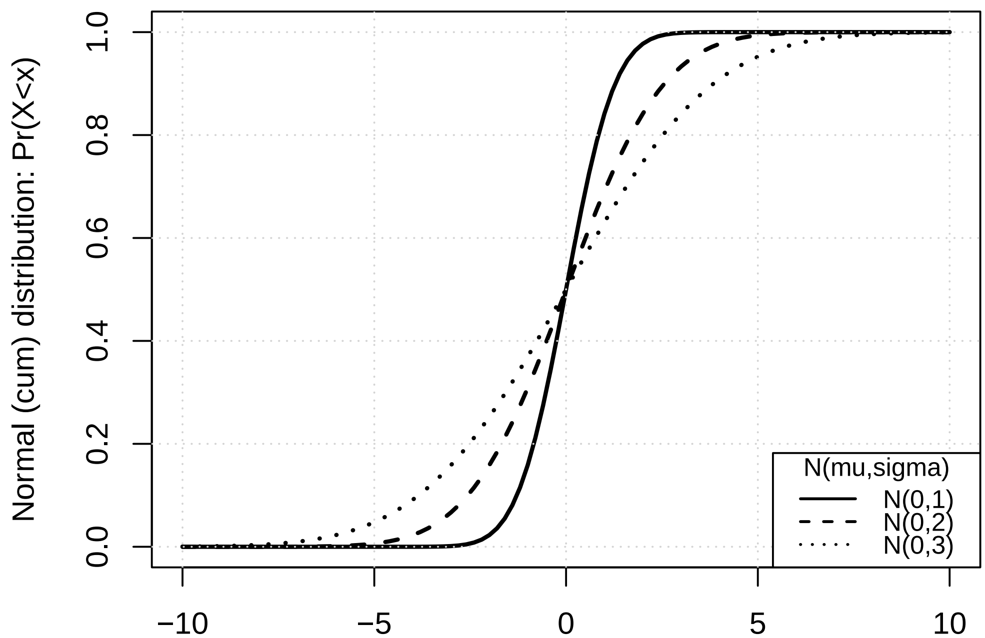
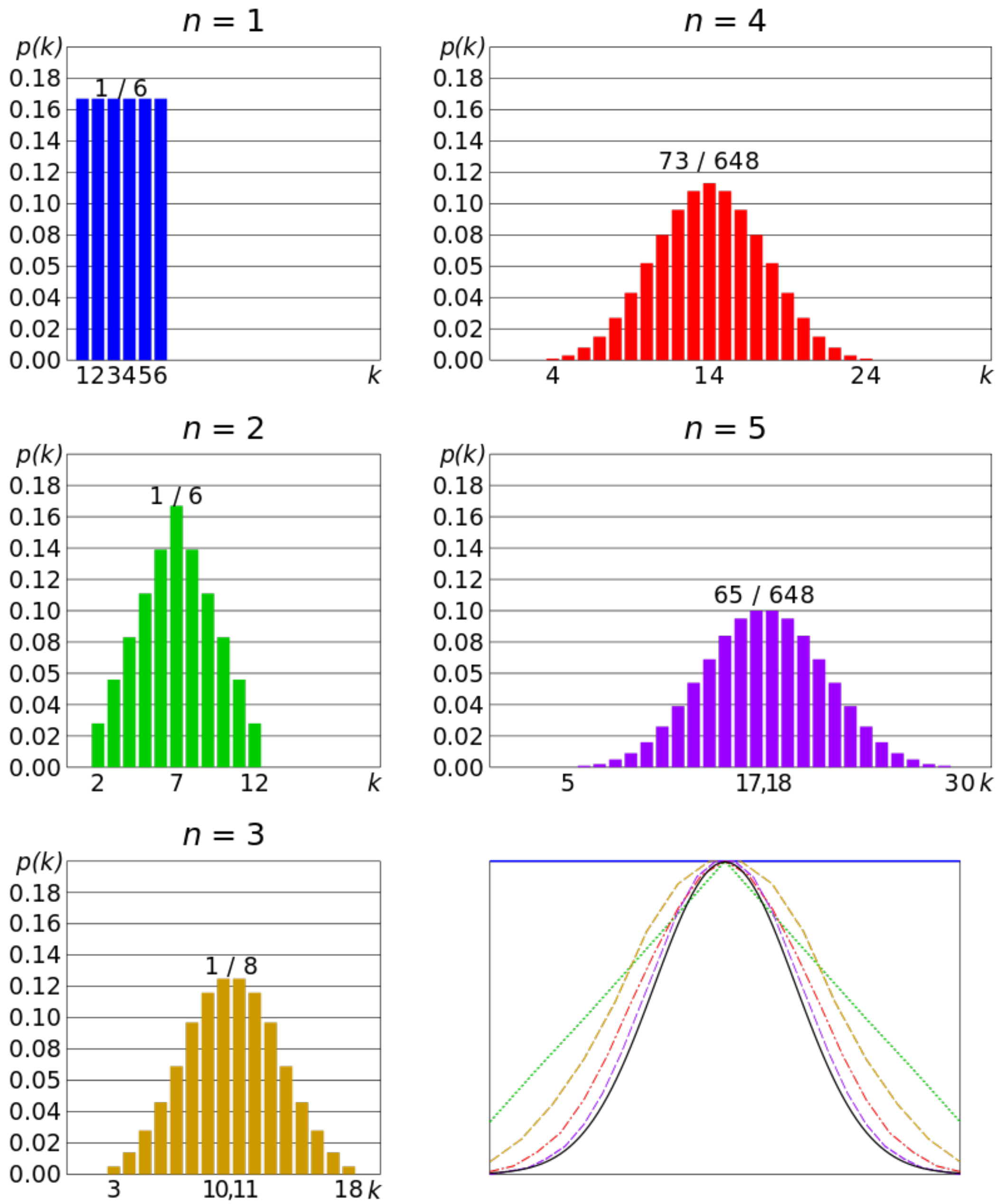
Some background




N(mu,sigma)
Z = N(0,1)
2Z = N(0,2)
3Z = N(0,3)
Normal density
0.0
0.1
0.2
0.3
0.4
−10
−5
0
5
10


Normal (cum) distribution: Pr(X<x)
0.0
0.2
0.4
0.6
0.8
1.0
−10
−5
0
5
10
N(mu,sigma)
N(0,1)
N(0,2)
N(0,3)



From the central limit theorem, if random variables $X_1, \ldots, X_n$ are iid,

$$\frac{X_1 + \ldots + X_n}{n} = \bar{X} \rightarrow \mathcal{N}(\mu, \frac{\sigma^2}{n}) \tag{1}$$

$$\because \mathbb{E}\left(\frac{X_1 + \ldots + X_n}{n}\right) = \frac{\mathbb{E}(X_1) + \ldots + \mathbb{E}(X_n)}{n} = \frac{n\mu}{n} = \mu \tag{2}$$

$$\text{and } var\left(\frac{X_1 + \ldots + X_n}{n}\right) = \frac{1}{n^2}var(X_1) + \ldots + \frac{1}{n^2}var(X_n) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \tag{3}$$

$$\text{Consequently } X_1 + \ldots + X_n \rightarrow \mathcal{N}(n\mu, n\sigma^2) \equiv sd(\sum_{i=1}^{n} X_i) = \sqrt{n}\sigma \tag{4}$$

Brownian motion

A Brownian motion or Wiener process is a stochastic process for random variables X_t, and has two fundamental, defining properties:

1. Normal increments: Changes in the process Δz over an interval Δt are normally distributed:
$$\Delta z = \epsilon\sqrt{\Delta t}$$
2. Independent increments: Changes in the process Δz for two non-overlapping intervals of time are independent

Implication: For an interval $T = n\Delta t$,

$$z(T) - z(0) = \epsilon_1\sqrt{\Delta t} + \ldots + \epsilon_n\sqrt{\Delta t}$$

$$\mathbb{E}[z(T) - z(0)] = 0 + \ldots + 0 = 0$$

$$var[z(T) - z(0)] = n\Delta t = T \Rightarrow \sigma(z(T) - z(0)) = \sqrt{T}$$

In general we can observe a drift in the process μ which contains information about the conditional mean. Hence the general Brownian motion can assume the form:

$$\Delta X = a\Delta t + b\Delta z \tag{5}$$
$$\Delta X = a\Delta t + b\epsilon\sqrt{\Delta t} \tag{6}$$

This leads to

$$\mathbb{E}(\Delta X) = \mathbb{E}(a\Delta t) + \mathbb{E}(b\epsilon\sqrt{\Delta t}) = a\Delta t + b\sqrt{\Delta t}\,\underbrace{\mathbb{E}(\epsilon)}_{=0} = a\Delta t$$

$$var(\Delta X) = var(a\Delta t) + var(b\epsilon\sqrt{\Delta t}) = 0 + (b^2\Delta t)\,\underbrace{var(\epsilon)}_{=1} = b^2\Delta t$$

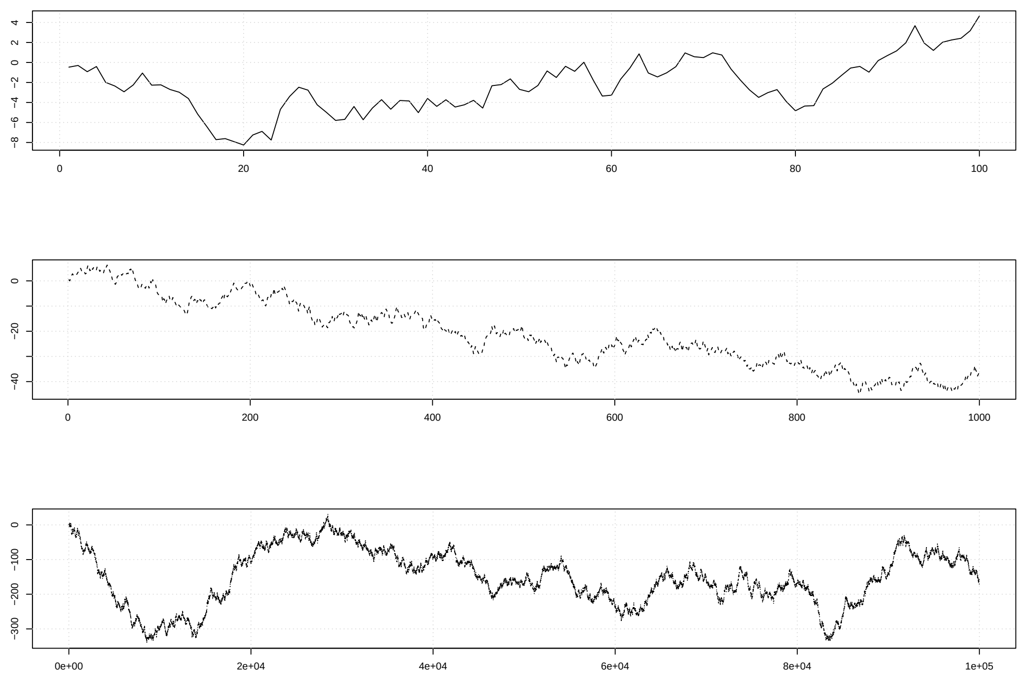
$$\text{or equivalently, standard deviation } \sigma(\Delta X) = \sqrt{var(X)} = b\sqrt{\Delta t}$$

$$\text{Thus } \Delta X \sim \mathcal{N}(a\Delta t, b\sqrt{\Delta t})$$

Brownian motion as $\Delta t \to 0$

As the interval $\Delta t \to 0$, we get the differential version of the process:

$$dX = adt + bdz$$



Stock price process

The relative, or percentage change in stock prices evolve as a generalized Wiener process. This is referred to as 'geometric Brownian motion'.

$$\frac{\Delta S}{S} \sim \mathcal{N}(\mu \Delta t, \sigma \sqrt{\Delta t}) \tag{7}$$

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t} \tag{8}$$

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{9}$$

$$dS = \mu S dt + \sigma S dz \tag{10}$$

References

Hull, John. 2014. *Options, Futures and Other Derivatives*. Ninth Edition. Pearson.

Jondeau, Eric, Ser-Huang Poon, and Michael Rockinger. 2007. *Financial Modeling Under Non-Gaussian Distributions*. Springer Finance.

Tsay, Ruey S. 2010. *Analysis of Financial Time Series*. Third Edition. John Wiley; Sons.