

Optimization

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2018/06/11

Background

Problems in finance and economics are often concerned with the behavior of agents who are considered to be utility maximizers. Utility functions are thought to be monotonic in their variables—more of a utility enhancing variable is better than less—and are thought to obey diminishing returns as the variables scale.¹ Additionally, real-life constraints ensure that variables are bounded. Hence optimization of functions under constraints forms an important discipline in the study of such subjects.

After linear optimization, quadratic optimization problems are the simplest since they can be captured in *quadratic forms* and can be represented via symmetric matrices with special properties. A distinguishing feature of quadratic optimization is the presence of first and second order conditions that characterize the nature of the extreme point.

Quadratic Optimization

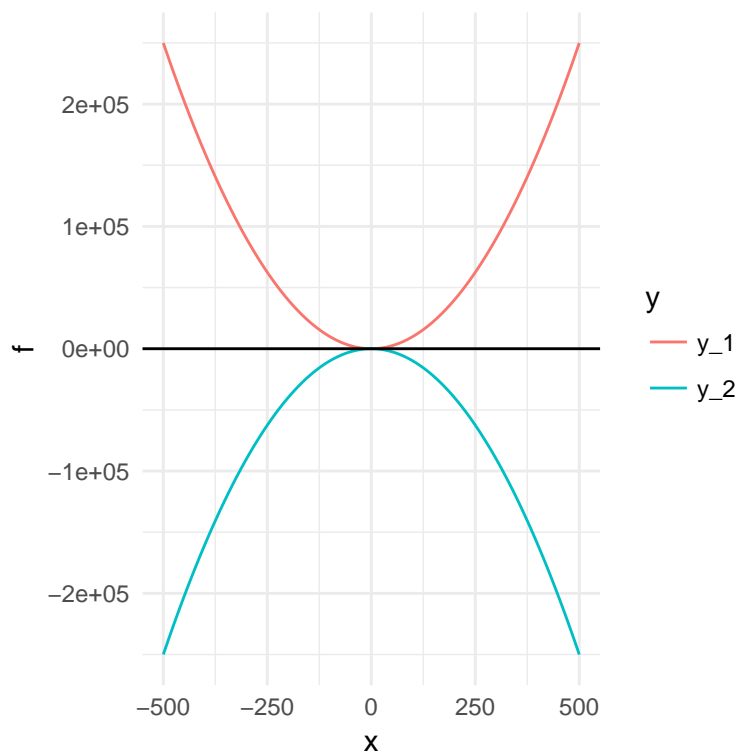
In one dimension, $x \in \mathbb{R}$ the simplest quadratic objective functions can be $f(x) = \{x^2, -x^2\}$ with the first attaining a global minimum and the second a global maximum at $x = 0$.

```
x <- -500:500
y_1 <- x^2
y_2 <- -x^2
data_1 <- cbind(x, y_1, y_2) %>%
  dplyr::as_tibble() %>% #wide format
```

¹This utility function can assume a variety of forms. For example, if the ‘agent’ is a firm, its utility is its profit function; if its a government, it could be some (aggregate) social welfare function etc.

```
tidyr::gather(.,
  y_1:y_2,
  key = 'y',
  value = 'f') #long format

ggplot(data_l, aes(x, f, color = y)) +
  geom_line() +
  geom_hline(yintercept = 0) +
  theme_minimal()
```



Quadratic Forms

For the case when $x = (x_1, x_2) \in \mathbb{R}^2$, a general quadratic form is the following:

$$Q(x) = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2$$

which may be represented as a matrix:

$$Q(x) = [x_1 x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Similarly, when $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, the form becomes

$$Q(x) = [x_1 x_2 x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and in general, when $x = (x_1, \dots, x_n) \in \mathbb{R}^n$:

$$Q(x) = [x_1 \dots x_n] \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{1n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

It's easy to see that $Q(0) = 0$. If $a > 0, ax^2 > 0$ and we can call the quadratic form “positive definite”. Similarly, if $a < 0, ax^2 < 0$ and we call the form “negative definite”. This carries over when $x \in \mathbb{R}^2$, since $Q(x_1, x_2) = x_1^2 + x_2^2 > 0$ (positive definite) and $Q(x_1, x_2) = -x_1^2 - x_2^2 > 0$ (negative definite). Additionally for the case when $Q(x_1, x_2) = x_1^2 - x_2^2 > 0$, the quadratic form is *indefinite*.

In general, symmetric matrices are called positive (semi)definite, negative (semi)definite etc. according to the definiteness of the expression $Q(x) = x^\top Ax$.²

Unconstrained Optimization

While it's clear what a maximum is in one dimension, what should be its definition in two dimensions and beyond? The central idea remains the same: x^* is the maximum if there is no other x in the domain of $f(x)$ such that $f(x) \geq f(x^*)$. Note that this definition is general and independent of the dimension of the underlying space from where we pick x . The only difference in dimensions one and beyond is the structure of the domain is; and while in one dimension it must take the form $x_L \leq x \leq x_H$, one needs such conditions for each component when $x \in \mathbb{R}^n : x_{L1} \leq x_1 \leq x_{H1}, \dots, x_{Ln} \leq x_n \leq x_{Hn}$. With this interpretation in mind, we are ready to formulate the necessary and sufficient conditions for the optimal of an unconstrained optimization program.

²Hence if $\forall x \neq 0 \in \mathbb{R}^n, x^\top Ax \geq 0$ the form is positive semidefinite and so on.