Wiener Processes

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Wiener Processes

For discrete time financial time series, the building blocks are the "innovations" or the error terms that are assumed to be emanating from some distribution with presumably constant mean and variance. In other words, the "increments" ϵ_t to the financial time series are iid with fixed mean and variances. This idea can be used to characterize increments dW_t to a stochastic process W_t in continous time—the Wiener Process—that is the continuous counterpart to a discrete financial time series.

A continuous stochastic process W_t is a Wiener Process if its increments: $W_{t+\delta t} - W_t = \delta W_t$ are:

1. *Normal Increments*: $W_{\delta t} \sim \mathcal{N}(0, \delta t)$
2. *Independent Increments*: $W_{\delta t}$ is independent of $W_{s<t}$

From 1. it follows that we can consider $\delta W_t = \epsilon\sqrt{\delta t}$, where $\epsilon \sim \mathcal{N}(0, 1)$.[1]

If we consider $W_T - W_0$ as a sum of small increments $\delta t = T/n$ or $T - 0 = n\delta t$,

$$W_T - W_0 = W_{n\delta t} - W_0 = \sum_{i=1}^{n} \delta w_i = \sum_{i=1}^{n} \epsilon_i \sqrt{\delta t}$$

And since ϵ_i are iid,

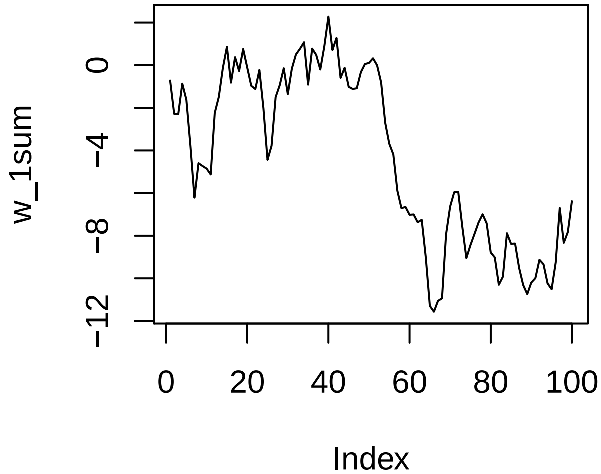
$$\mathbb{E}(W_T - W_0) = 0$$
$$\text{var}(W_T - W_0) = \sum_{i=1}^{n} \delta t = n\delta t = T$$

[1] More formally, a Wiener process is a real valued, (almost surely) continuous stochastic process with independent and stationary increments.

Hence each segment from time 0 to T is distributed as $W_T - W_0 \sim \mathcal{N}(0, T)$. More generally, $W_t - W_s \sim \mathcal{N}(\mu(t-s), \sigma^2(t-s))$.

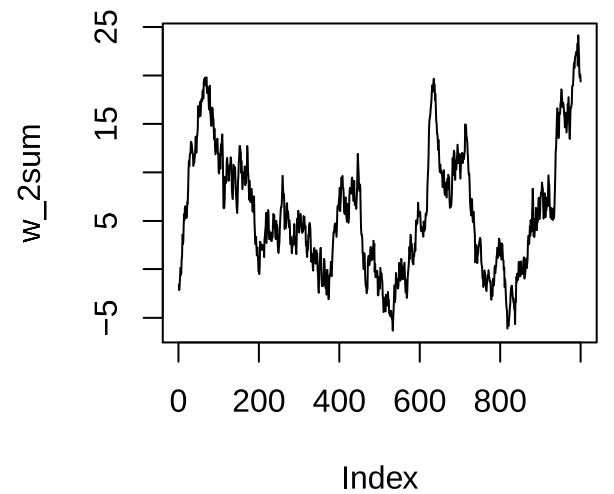
```
# T = 100
T_1 = 100
w_1 <- rnorm(T_1)
w_1sum <- cumsum(w_1)

plot(w_1sum,
     type = "l"
     )
```



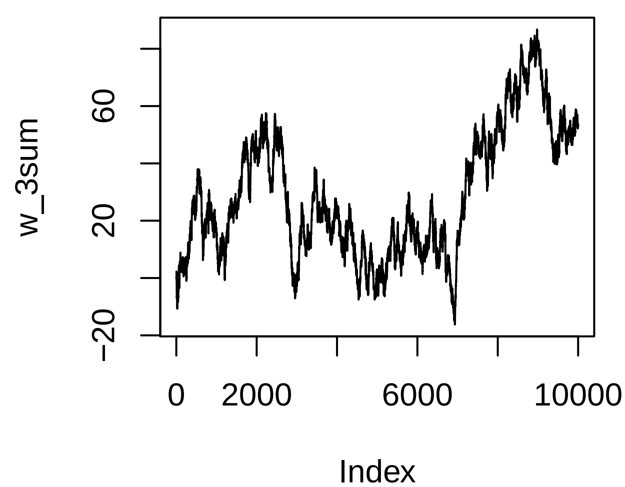

```
# T = 1000
T_2 = 1000
w_2 <- rnorm(T_2)
w_2sum <- cumsum(w_2)

plot(w_2sum,
     type = "l"
     )
```




```r
# T = 10000
T_3 = 10000
w_3 <- rnorm(T_3)
w_3sum <- cumsum(w_3)

plot(w_3sum,
     type = "l"
     )
```



While the paths of Wiener Processes are continuous (almost surely) they are also non-differentiable (almost surely). As a result, we cannot use our classical ideas of Riemannian integration anymore.

Wiener Process with Drift

If the drift rate for a Wiener Process X_t is μ and the variance change rate is σ^2, we can generalize such a process as:

$$dX_t = \mu dt + \sigma dW_t$$

where W_t is a standard Wiener Process. A discretized version takes the following form:

$$X_t - X_0 = \mu t + \sigma\epsilon\sqrt{t}$$

which yields $\mathbb{E}(X_t - X_0) = \mu t$ and $\text{var}(X_t - X_0) = \sigma^2 t$. More generally, the drift and volatility could depend on X_t in which case we get the classic Ito Process:

$$X_t - X_0 = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

Ito's Lemma

Consider some function $G(x_1, x_2)$. From calculus we know that:

$$\delta G = \frac{\partial G}{\partial x_1}\cdot\delta x_1 + \frac{\partial G}{\partial x_2}\cdot\delta x_2 + \frac{1}{2!}\frac{\partial^2 G}{\partial x_1^2}\cdot(\delta x_1)^2 + \frac{1}{2!}\frac{\partial^2 G}{\partial x_2^2}\cdot(\delta x_2)^2 + \frac{\partial^2 G}{\partial x_1 \partial x_2}\cdot(\delta x_1)(\delta x_2) + \dots$$

As $\delta x_1, \delta x_2 \to 0$

$$dG = \frac{\partial G}{\partial x_1}dx_1 + \frac{\partial G}{\partial x_2}dx_2$$

Now let us suppose that G depends on X_t and t where X_t is some Ito Process.

$$\delta G = \frac{\partial G}{\partial x}\cdot\delta x + \frac{\partial G}{\partial t}\cdot\delta t + \frac{1}{2!}\frac{\partial^2 G}{\partial x^2}\cdot(\delta x)^2 + \frac{1}{2!}\frac{\partial^2 G}{\partial t^2}\cdot(\delta t)^2 + \frac{\partial^2 G}{\partial x \partial t}\cdot(\delta x)(\delta t) + \dots$$

A discretized Ito Process is:

$$\delta x = \mu \delta t + \sigma \epsilon \sqrt{\delta t}$$

This may be used to compute $(\delta x)^2$.

$$(\delta x)^2 = \mu^2 (\delta t)^2 + \sigma^2 \epsilon^2 \delta t + 2\mu\sigma\epsilon(\delta t)^{3/2} = \sigma^2 \epsilon^2 \delta t + H(\delta t)$$

where $H(\cdot)$ denote higher order terms in δt.

It's clearly seen here that the $(\delta x)^2$ term depends on order δt. We also notice that

$$\mathbb{E}(\sigma^2 \epsilon^2 \delta t) = \sigma^2 \delta t$$

$$\text{var}(\sigma^2 \epsilon^2 \delta t) = \mathbb{E}(\sigma^4 \epsilon^4 (\delta t)^2) - [\mathbb{E}(\sigma^2 \epsilon^2 \delta t)]^2 = 2\sigma^4 (\delta t)^2$$

where we used $\mathbb{E}(\epsilon^4) = 3$ for standard normal random variables. Hence we can claim that:

$$\lim_{\delta t \to 0} \sigma^2 \epsilon^2 \delta t = \sigma^2 \delta t$$

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$$(\delta x)^2 = \mu^2 (\delta t)^2 + \sigma^2 \epsilon^2 \delta t + 2\mu\sigma\epsilon(\delta t)^{3/2} = \sigma^2 \epsilon^2 \delta t + H(\delta t)$$

we can observe that as $\delta t \to 0$

$$(\delta x)^2 \to \sigma^2 \delta t$$

This gives us an expansion of an Ito Process which is slightly different from the results of classical calculus:

$$\delta G = \frac{\partial G}{\partial x} \cdot \delta x + \frac{\partial G}{\partial t} \cdot \delta t + \frac{1}{2!} \frac{\partial^2 G}{\partial x^2} \cdot (\delta x)^2 + \frac{1}{2!} \frac{\partial^2 G}{\partial t^2} \cdot (\delta t)^2 + \frac{\partial^2 G}{\partial x \partial t} \cdot (\delta x)(\delta t) + \dots$$

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial G}{\partial x^2}\sigma^2 dt$$

If we consider $X_t = \mu dt + \sigma dW_t$, we get the classic Ito's Lemma

$$dG = (\frac{\partial G}{\partial x}\mu + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2)dt + \frac{\partial G}{\partial x}\sigma dW_t$$

This may be stated more simply as:

$$dG(X_t, t) = G'(X_t, t)dX_t + \frac{1}{2}G''(X_t)\sigma_t^2 dt$$

Application: $G(W_t) = W_t^2$

Consider a squared standard Wiener Process: $G(X_t) = W_t^2$. What is its equation?

We know that

$$dG = \frac{\partial G}{\partial W}\sigma dW + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial W^2}dt + \frac{\partial G}{\partial W}\mu dt$$

Hence the (stochastic differential) equation for this process is:

$$dW_t^2 = dt + 2W_t dW_t$$

What about a general process $G = W_t^n$?

$$G'(X_t, t) = nW_t^{n-1}$$

$$G''(X_t, t) = n(n-1)W_t^{n-2}$$

Hence

$$dW_t^n = \frac{1}{2}n(n-1)W_t^{n-2}dt + nW_t^{n-1}dW_t$$

Application: Log Price Process

A special application is the evolution of log prices. Suppose P_t be the price at time t. It is often assumed that the price follows the following Ito Process—the Geometric Brownian Motion—with constant drift and diffusion:

$$dP_t = \mu P_t dt + \sigma P_t dW_t$$

For the log price process: $G = \ln(P_t)$

$$d(\ln P_t) = (\frac{\partial \ln P_t}{\partial t} + \frac{1}{2}\frac{\partial^2 \ln P_t}{\partial P_t^2})dt + \frac{\partial \ln P_t}{\partial P_t} dW_t$$

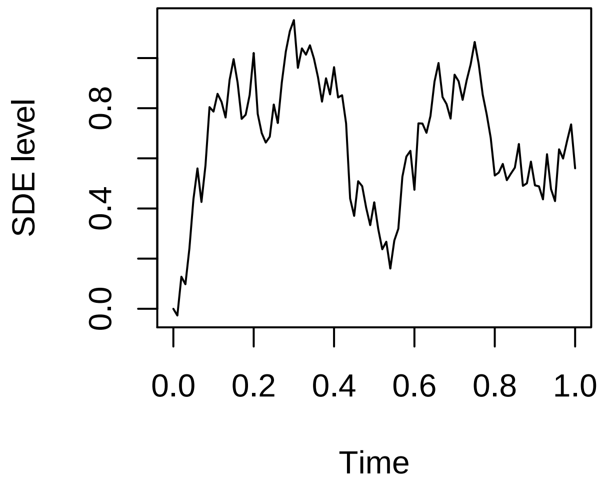
$$d(\ln P_t) = (\mu - \sigma^2/2)dt + \sigma dW_t$$

The package `sde`

We use the package `sde` for analyzing stochastic differential equations

Standard Wiener Process

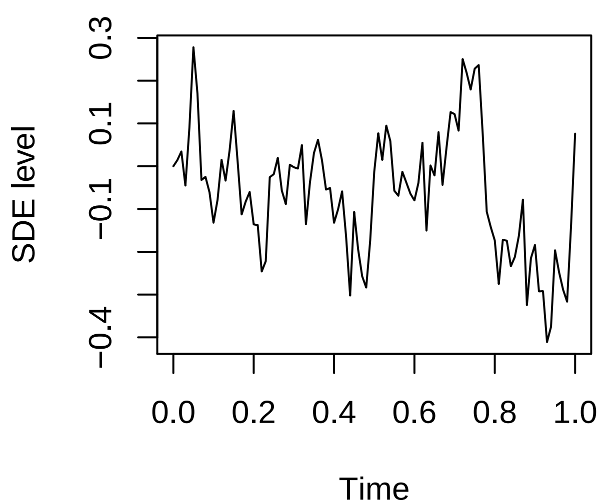
```
# Standard Wiener Process
plot(sde::sde.sim(X0 = 0, t0 = 0,
                  drift = expression(1), #note expression
                  sigma = expression(1)
                  ),
     ylab = "SDE level"
     )
```



A more complex motion as shown below can also be simulated.

$$dX_t = (-11x + 6x^2 - x^3)dt + dW_t$$

```
plot(sde::sde.sim(X0 = 0,
                  t0 = 0,
                  drift = expression(-11*x + 6*x^2 -x^3),
                  sigma = expression(1)
                  ),
     ylab = "SDE level"
     )
```



References

Jondeau, Eric, Ser-Huang Poon, and Michael Rockinger. 2007. *Financial Modeling Under Non-Gaussian Distributions*. Springer Finance.

Tsay, Ruey S. 2010. *Analysis of Financial Time Series*. Third Edition. John Wiley; Sons.