ARMA GARCH Processes

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The Autocorrelation Function (ACF)

The correlation between random variables X_1, X_2 is a measure of their linear dependence and is defined as:

$$\rho_{12} := \frac{\text{cov}(X_1, X_2)}{\sqrt{\text{var}(X_1)\text{var}(X_2)}} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

It lies between -1 and 1 and for normal random variables $\rho_{12} = 0$ implies that the variables are independent.

If we have a sample $\{x_{1,t}, x_{2,t}\}_{t=1}^T$ the correlation can be consistently estimated by computing sample correlation:

$$\hat{\rho}_{12} = \frac{\hat{\sigma}_{12}}{\hat{\sigma}_1 \hat{\sigma}_2}$$

For a time series r_t which is weakly stationary, the lag-l autocorrelation function is the correlation between r_t and r_{t-l}:

$$\rho_l = \frac{\sigma_{t,t-l}}{\sigma_t \sigma_{t-l}} = \frac{\sigma_{t,t-l}}{\sigma_t^2} = \frac{\gamma_l}{\gamma_0}$$

This follows from weak stationarity: $\sigma_t^2 = \sigma_{t-l}^2 = \gamma_0$ and $\text{cov}(r_t, r_{t-l}) = \gamma_l$.

We claim that there is no autocorrelation if $\rho_l = 0 \; \forall l > 0$.

To estimate the autocorrelation function of lag (say) 1, we use its sample counterpart:

$$\hat{\rho}_1 = \frac{\sum_{t=2}^T (r_t - \bar{r})(r_{t-1} - \bar{r})}{\sum_{t=1}^T (r_t - \bar{r})^2}$$

In general for lag l we consistently estimate it as:

$$\hat{\rho}_l = \frac{\sum_{t=l+1}^T (r_t - \bar{r})(r_{t-1} - \bar{r})}{\sum_{t=1}^T (r_t - \bar{r})^2}$$

The statistic $\hat{\rho}_1, \hat{\rho}_2, \ldots$ is the *sample autocorrelation function* of r_t and is key to capturing the linear dependence nature of the time series in question.

Autoregressive (AR) Processes

Perhaps last period's returns may have some significant impact on the value of the returns this period. If so, its lag-1 autocorrelation may be useful for predicting the current period's value:

$$r_t = \phi_0 + \phi_1 r_{t-1} + u_t$$

where u_t is weakly stationary with mean 0 and variance σ_u^2. This is simply equivalent to a regression where r_{t-1} is the explanatory or independent variable.

It's straightforward to check the conditional mean and variance of such a process:

$$\mathbb{E}(r_t|r_{t-1}) = \phi_0 + \phi_1 r_{t-1}$$
$$\text{var}(r_t|r_{t-1}) = \sigma_u^2$$

And more generally there could be defined autoregressive processes of order p ($AR(p)$):

$$r_t = \phi_0 + \phi_1 r_{t-1} + \ldots + \phi_p r_{t-p} + u_t$$

AR(1) processes

Is the AR(1) process $r_t = \phi_0 + \phi_1 r_{t-1} + u_t$ weakly stationary? This will imply that its unconditional mean and variance must be fixed in time and lag-l covariance must depend only on the lag length l.

$$\mathbb{E}(r_t) = \phi_0 + \phi_1 \mathbb{E}(r_{t-1}) + \mathbb{E}(u_t)$$
$$\mathbb{E}(r_t) = \phi_0 + \phi_1 \mu$$

$$\mu = \frac{\phi_0}{1-\phi_1}$$

This clearly implies that for the mean of an AR(1) process to exist, $\phi_1 \neq 1$ and $\phi_0 = \mu \cdot (1-\phi_1) = \mu - \mu\phi_1$.

Hence a weakly stationary AR(1) process is:

$$r_t = \mu - \mu\phi_1 + \phi_1 r_{t-1} + u_t$$

$$r_t - \mu = (r_{t-1} - \mu)\phi_1 + u_t$$

$$r_t - \mu = ((r_{t-2} - \mu)\phi_2 + u_{t-1})\phi_1 + u_t$$

$$\vdots$$

$$r_t - \mu = u_t + \phi_1 u_{t-1} + \phi_1^2 u_{t-2} + \dots$$

$$r_t = \mu + \sum_{i=0}^{\infty} \phi_1^i \cdot u_{t-i}$$

Additionally,

$$\text{var}(r_t) = \phi_1^2 \text{var}(r_{t-1}) + \sigma_u^2$$

Since for weakly stationary AR(1) processes $\text{var}(r_t) = \text{var}(r_{t-1}) = \gamma_0$ we have

$$\gamma_0 = \frac{\sigma_u^2}{1-\phi_1^2}$$

Weak stationarity immediately implies that $\phi_1 \in (-1, 1)$.

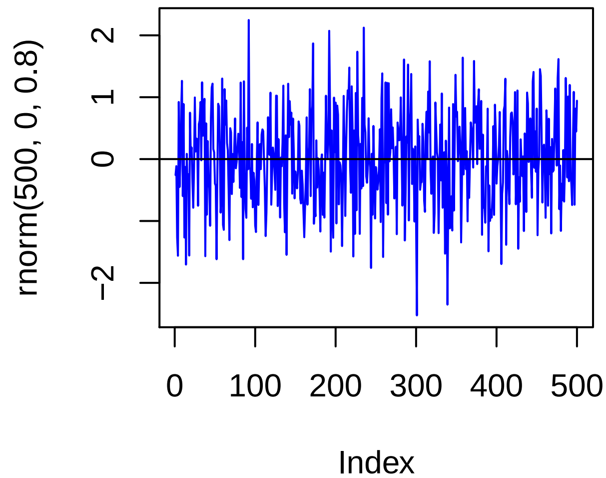
Hence taken together, for an AR(1) process to be weakly stationary it is necessary and sufficient that $\phi_1 \in (-1, 1)$;[1] and the canonical AR(1) series can be written as:

$$r_t = (1-\phi_1)\mu + \phi_1 r_{t-1} + u_t$$

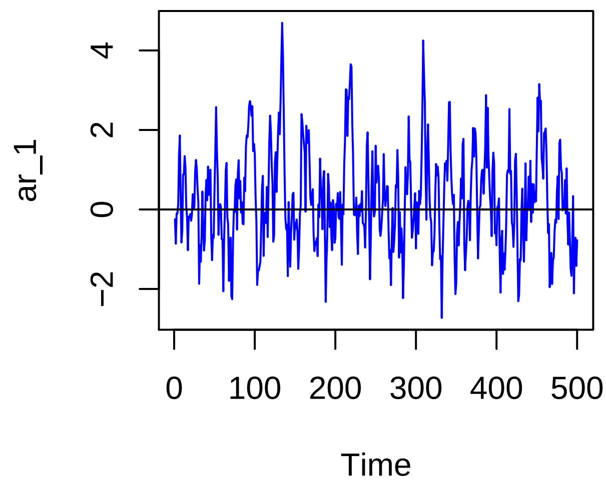
We can plot some hypothetical autoregressive processes by simulation via the function `arima.sim()` include in the `stats` package that loads by default.

[1] For a general AR(p) process, the corresponding condition is: $|\phi_1| + |\phi_2| + \ldots + |\phi_p| < 1$.

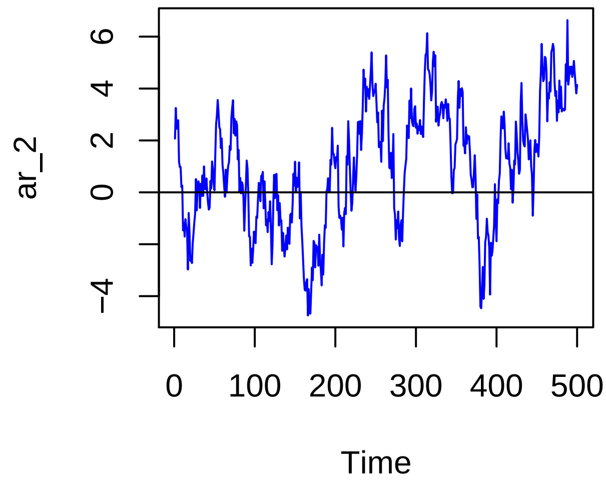
```r
# AR(0)
plot(rnorm(500, 0, 0.8), type = "l", col = "blue")
abline(h = 0)
```



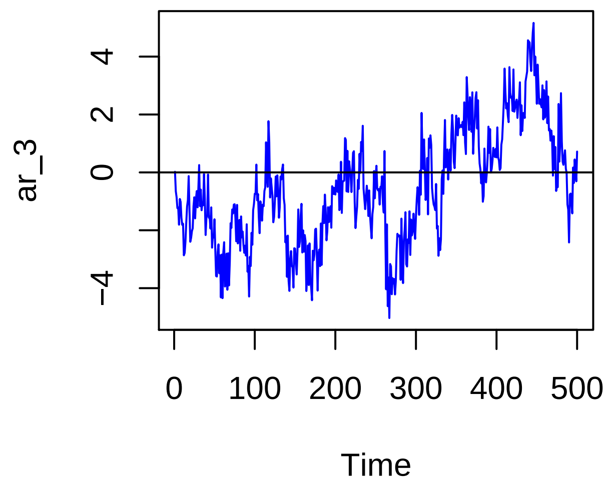

```r
# AR(1)
ar_1 <- arima.sim(n = 500, list(ar = c(0.8)), sd = 0.8)
plot(ar_1, col = "blue")
abline(h = 0)
```




```r
# AR(2)
ar_2 <- arima.sim(n = 500, list(ar = c(0.8, 0.15)), sd = 0.8)
plot(ar_2, col = "blue")
abline(h = 0)
```




```r
# AR(3)
ar_3 <- arima.sim(n = 500, list(ar = c(0.5, 0.3, 0.15)), sd = 0.8)
plot(ar_3, col = "blue")
abline(h = 0)
```



Autocorrelation Function for AR(1) processes

We can easily check that for positive lags $l > 0$, the lagged covariance follows:

$$\gamma_l = \phi_1 \gamma_{l-1}$$

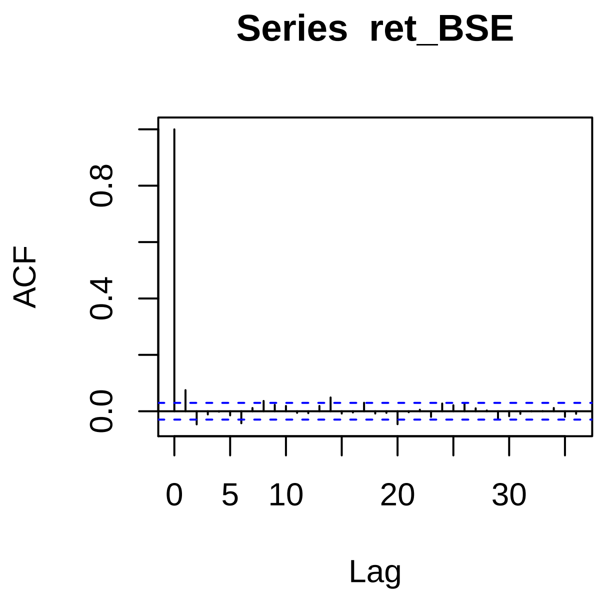
Hence it follows that for the autocorrelation function $\rho_l = \phi_1 \rho_{l-1}$; and becasue

$\rho_0 = 1$, $\rho_l = \phi_1^l$. This implies that the autocorrelation function of an AR(1) series decays exponentially with rate ϕ_1 and starting value 1. If $\phi_1 < 0$ the series alternates between positive and negative terms.

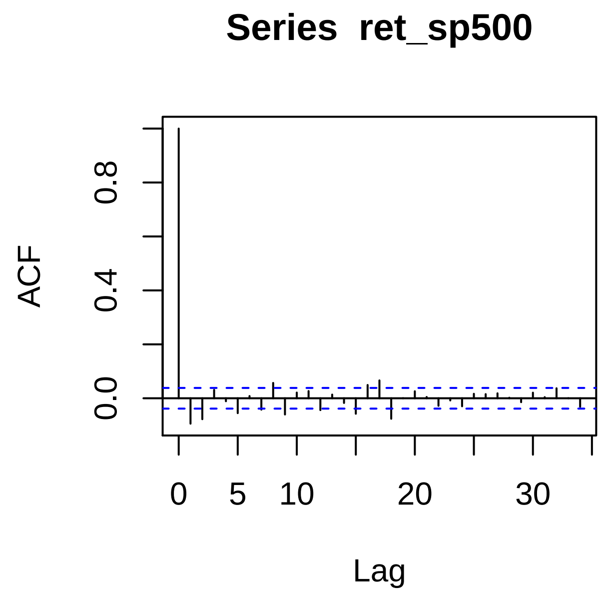
Illustration

For example let's compute the sample autocorrelation function (ACF) for the financial market indices.

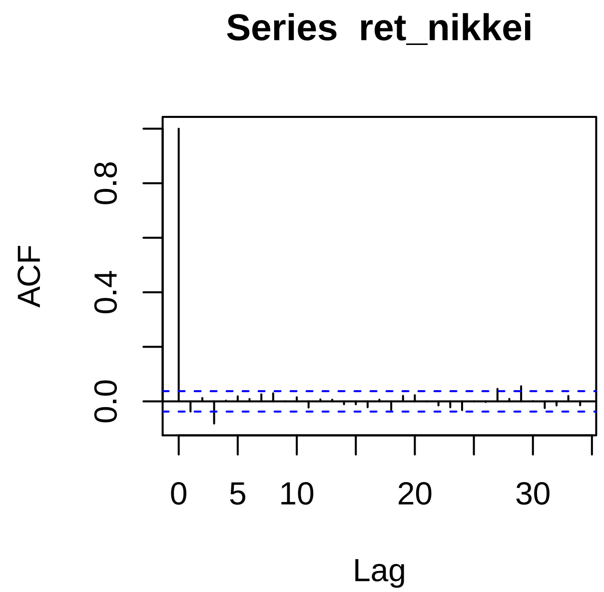
```
acf(ret_BSE, na.action = na.pass)
```




```
acf(ret_sp500, na.action = na.pass)
```



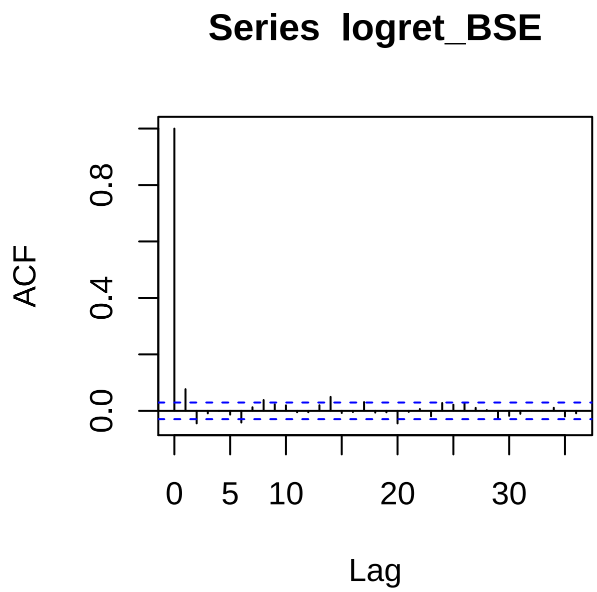

```
acf(ret_nikkei, na.action = na.pass)
```



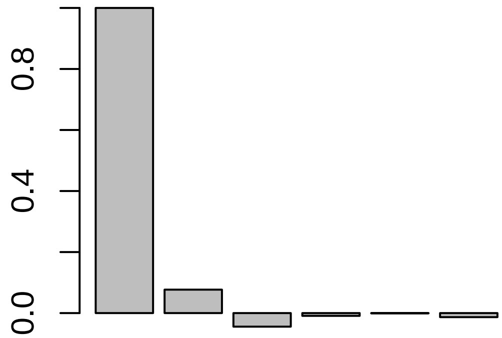
What about log-returns?

```
logret_BSE <- func_pr_to_logret(index_bse$Close)
logret_SP <- func_pr_to_logret(ind_sp500$SP500)
logret_Nikkei <- func_pr_to_logret(ind_nikkei$NIKKEI225)
```

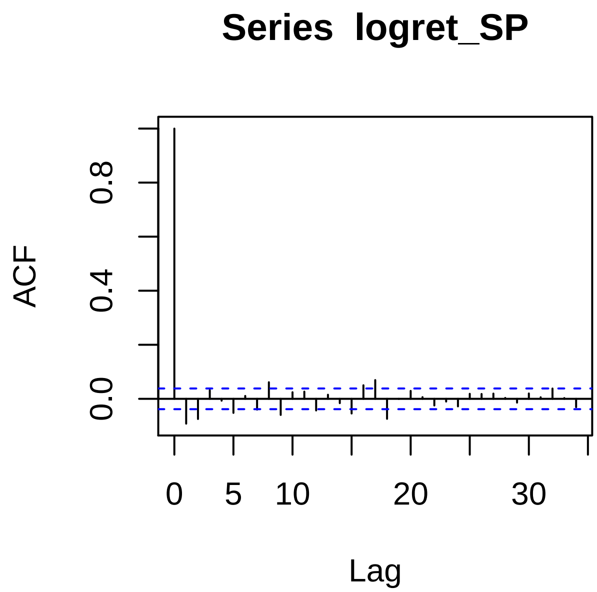
```
ACF_BSE <- acf(logret_BSE, na.action = na.pass)
```



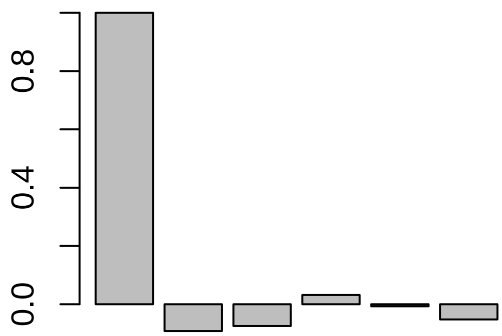

```
barplot(head(ACF_BSE$acf))
```



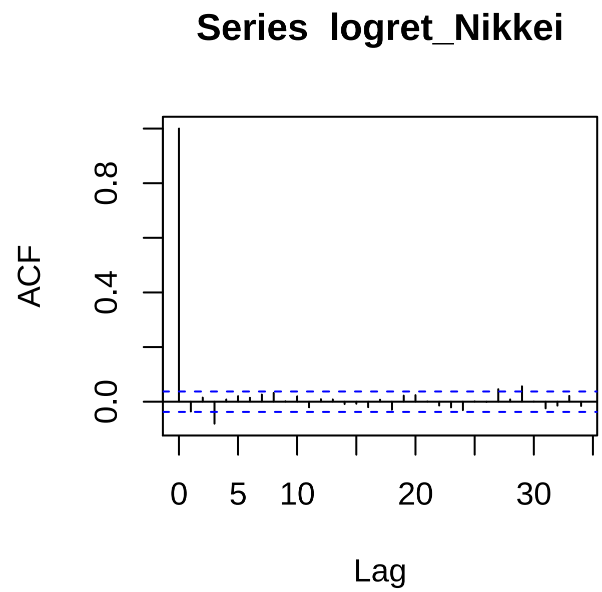

```
ACF_SP <- acf(logret_SP, na.action = na.pass)
```



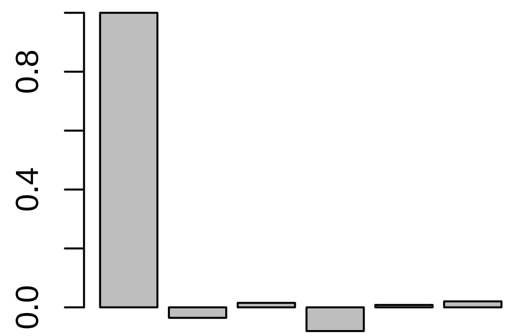

```
barplot(head(ACF_SP$acf))
```




```
ACF_Nikkei <- acf(logret_Nikkei, na.action = na.pass)
```




```r
barplot(head(ACF_Nikkei$acf))
```



Partial Autocorrelation Functions (PACF)

Is there a way to know how many lags to include for an autoregressive return series? This issue is solved via the usage of *partial autocorrelation functions* as shown below.

Consider the following sequences of AR processes:

$$r_t = \phi_{01} + \phi_{11} r_{t-1} + u_{1t}$$

$$r_t = \phi_{02} + \phi_{12} r_{t-1} + \phi_{22} r_{t-2} + u_{2t}$$

$$r_t = \phi_{03} + \phi_{13} r_{t-1} + \phi_{23} r_{t-2} + \phi_{33} r_{t-3} + u_{3t}$$

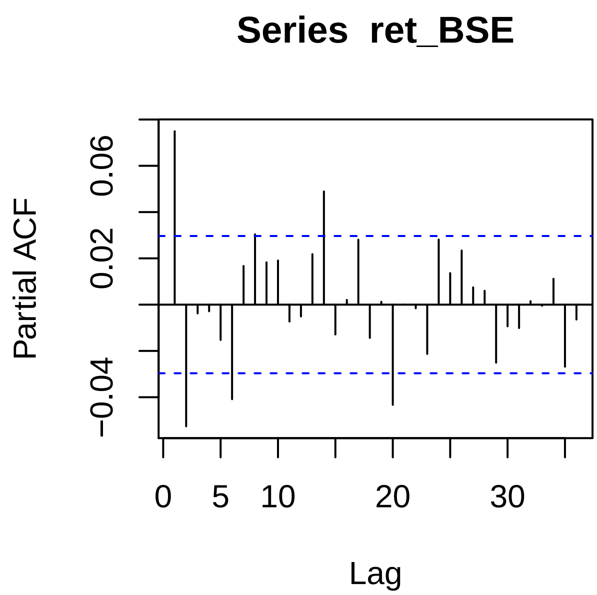
$$r_t = \phi_{04} + \phi_{14} r_{t-1} + \phi_{24} r_{t-2} + \phi_{34} r_{t-3} + \phi_{44} r_{t-4} + u_{4t}$$
$$\vdots$$

These models are merely multiple regressions and can be estimated via the standard least squares method.

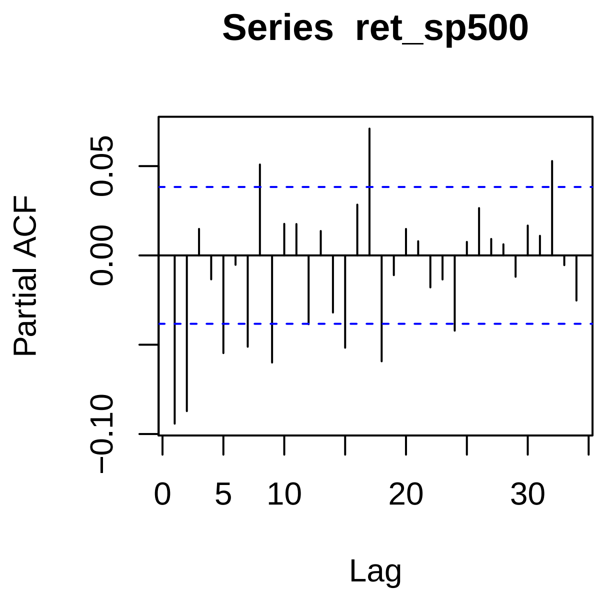
In these models, $\hat{\phi}_{11}$ is called the lag 1 sample PACF of r_t, $\hat{\phi}_{22}$ of the second equation is the lag 2 sample PACF of r_t and so on. By construction, the lag 2 $\hat{\phi}_{22}$ is the marginal contribution of r_{t-2} in explaining r_t over the AR(1) model and so on. Hence if the underlying model is say AR(p) then all sample PACFs $\hat{\phi}_{11}, \ldots, \hat{\phi}_{pp}$ must be different from 0 but all sample PACFs from then on: $\hat{\phi}_{p+1,p+1}, \ldots = 0$. This property can be used to find the order p.

Armed with this knowledge, let's compute the PACFs for the three financial market indices:

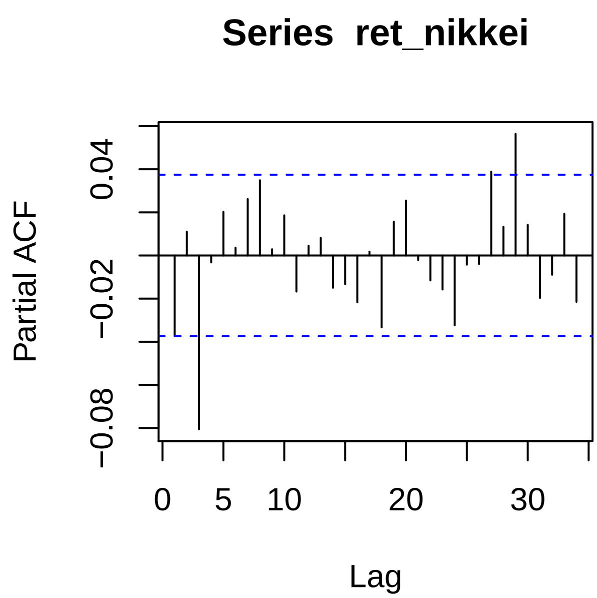
```r
pacf(ret_BSE, na.action = na.pass)
```




```r
pacf(ret_sp500, na.action = na.pass)
```




```r
pacf(ret_nikkei, na.action = na.pass)
```



Information Criteria

Apart from PACF, another way to find the number of lags is the use of likelihood based information criteria. Here we look at the two most famous ones: the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC).

$$\text{AIC}(l) = -\frac{2}{T} \cdot \ln(\text{likelihood}) + \frac{2}{T} \cdot (\#\text{parameters})$$

For a Gaussian AR(l), $AIC = \ln(\hat{\sigma}^2_{u,MLE}) + 2\frac{l}{T}$. The first term measures the goodness of fit of the model while the second penalizes the usage of parameters.

The Bayesian Information Criterion (BIC) uses a different penalty function. For a Gaussian AR(l) is takes the following form:

$$BIC(l) = \ln(\hat{\sigma}^2_{u,MLE}) + \ln(T) \cdot \frac{l}{T}$$

Moving Average (MA) Processes

Consider an infinitely long autoregressive process:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{r-2} + \ldots + u_t$$

If this series is to be weakly stationary, the coefficients ϕ_j must decay sufficiently fast. One way to ensure this is to assume that $\phi_j = -\theta^j$ for some $\theta \in (0, 1)$.

$$r_t = \phi_0 - \theta_1 r_{t-1} - \theta_1^2 r_{t-2} - \ldots + u_t$$

$$r_t + \sum_{j=1}^{\infty} \theta_1^j r_{t-j} = \phi_0 + u_t$$

The same form can be written for r_{t-2}:

$$r_{t-1} + \sum_{j=2}^{\infty} \theta_1^j r_{t-j} = \phi_0 + u_{t-1}$$

Solving for the above two equations, we get:

$$r_t = \phi_0(1 - \theta_1) + u_t - \theta_1 u_{t-1}$$

This indicates the AR model is a weighted average of shocks u_t, u_{t-1} and a constant. This is a *moving average* form of order 1 or MA(1). It's straightforward to check that unlike AR processes, MA processes are *always* stationary. (Can you see why?)

The general form for the MA(q) process is:

$$r_t = c_0 + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2} - \ldots - \theta_q u_{t-q}$$

Autocorrelation Function

Consider the MA(1) model with the constant term 0:

$$r_t = u_t - \theta_1 u_{t-1}$$

$$r_{t-l} r_t = u_t r_{t-l} - \theta_1 u_{t-1} r_{t-l}$$

$$\mathbb{E}(r_{t-l} r_t) = 0 - \theta_1 \mathbb{E}(u_{t-1} r_{t-l})$$

From this we see that:

$$\gamma_1 = -\theta_1 \sigma_u^2$$

$$\gamma_{l>1} = 0$$

Also, $\mathrm{var}(r_t) = \gamma_0 = (1 + \theta_1^2)\sigma_u^2$ and this implies that for $\rho_l = \frac{\gamma_l}{\gamma_0}$ becomes:

$$\rho_0 = 1$$

$$\rho_1 = -\frac{\theta_1}{1 + \theta_1^2}$$

$$\rho_{l>1} = 0$$

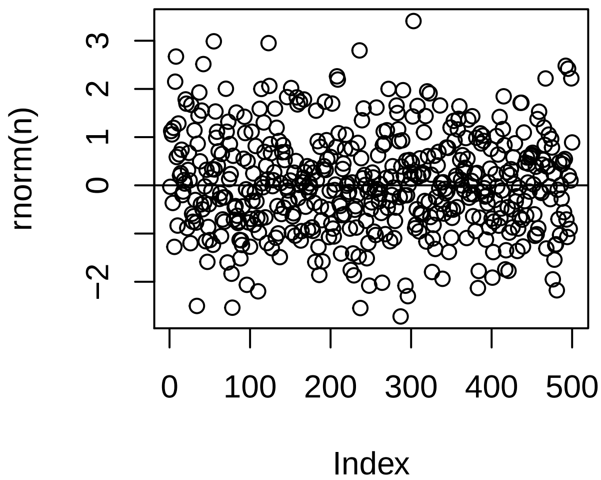
Hence for MA(1) processes, while the first lag autocorrelation is nonzero, all further lags produce zero autocorrelation. This property can be exploited to locate the order of the MA process. In general, for an MA(q) process, the ACF cuts off at lag q. Since the MA(q) process only relies on its past $q - 1$ realization, it's often called a 'finite memory' process.

Illustrations

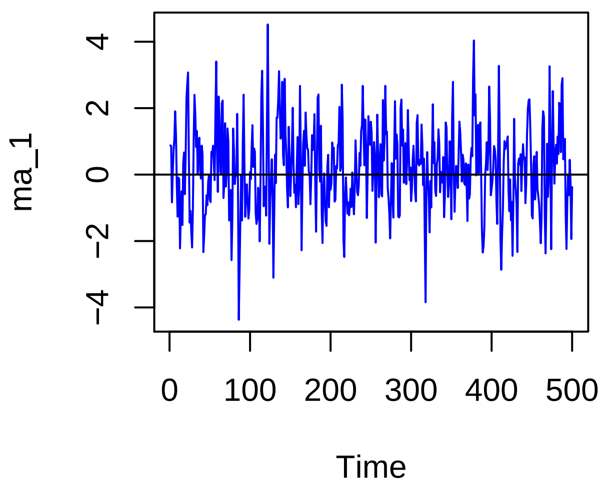
We simulate some moving average processes below:

```r
n <- 500

# MA(0)
plot(rnorm(n))
abline(h = 0)
```

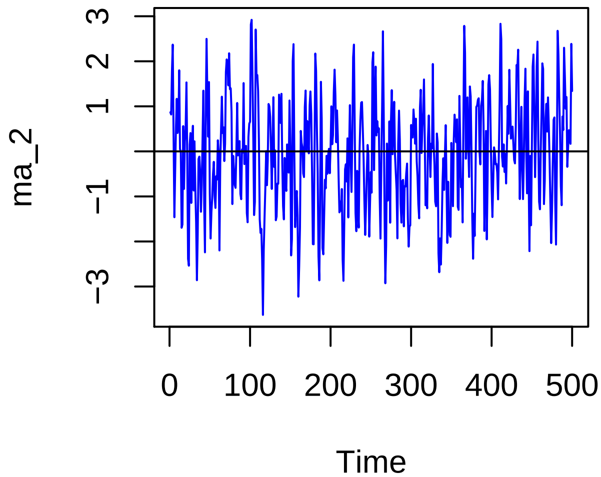



```r
# MA(1)
ma_1 <- arima.sim(n = n, list(ma = c(0.8)), innov=rnorm(n))
plot(ma_1, col = "blue")
abline(h = 0)
```

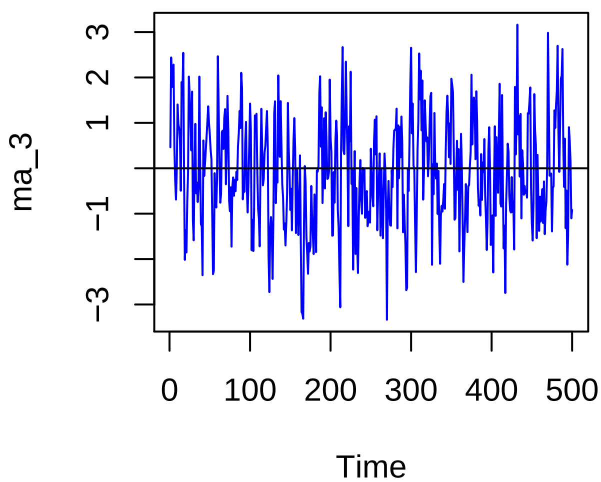



```r
# MA(2)
ma_2 <- arima.sim(n = n, list(ma = c(0.8, 0.15)), innov=rnorm(n))
```

```r
plot(ma_2, col = "blue")
abline(h = 0)
```




```r
# MA(3)
ma_3 <- arima.sim(n = n, list(ma = c(0.5, 0.3, 0.15)), innov=rnorm(n))
plot(ma_3, col = "blue")
abline(h = 0)
```



References

Jondeau, Eric, Ser-Huang Poon, and Michael Rockinger. 2007. *Financial Modeling Under Non-Gaussian Distributions*. Springer Finance.

Tsay, Ruey S. 2010. *Analysis of Financial Time Series*. Third Edition. John Wiley; Sons.