1 assumptions
2 propotion test
3

Q1

Yes, it possible that
$\tilde{w}$ is +ve correlated with $\tilde{y}$
$\tilde{y}$ is +ve correlated with $\tilde{z}$.
$\tilde{w}$ can be -ve correlated with $\tilde{z}$

(a) Eg.
Consider $\tilde{a}, \tilde{b}, \tilde{c}$ to normal distr. Random variables

$$\tilde{a} = N(0,5)$$
$$\tilde{b} = N(0,2)$$
$$\tilde{c} = N(0,3).$$

Now

$$\tilde{w} = \tilde{a} + \tilde{c}$$
$$\tilde{y} = \tilde{a} + \tilde{b}$$
$$\tilde{z} = \tilde{b} - \tilde{c}$$

Because of presence of $\tilde{c}$ impacting $\tilde{w}$ & $\tilde{z}$ differently they ($\tilde{w}$ & $\tilde{z}$) will be -ve correlated.

(b) sample code & generated data attached given

Q2) Random variable : $\bar{x}$ zero mean & unit variance

$\bar{y}_i = \bar{x} + \bar{z}_i$ for $1 \le i \le n$ — ①

$\bar{z}_i$ is a zero-mean Random variable with variance σ^2

$\bar{x}, \bar{z}_1, \bar{z}_2, \bar{z}_3$ — $\bar{z}_n$ are mutually independent

$\alpha \sum_{i=1}^{n} \bar{y}_i$ is the linear estimator of x.

ⓐ What value of α minimizes the mean squared error?

given

$\hat{x} = \alpha \sum_{i=1}^{n} \bar{y}_i$
(Estimated x)

Step 1 : Finding $E[\hat{x}]$

$E[\hat{x}] = E[\alpha \sum_{i=1}^{n} \bar{y}_i]$

putting val of $\bar{y}_i$

$= E[\alpha(\sum_{i=1}^{n} (\bar{x} + \bar{z}_i))]$

By linearity of Expectation

$E[\hat{x}] = \alpha \sum_{i=1}^{n} E[\bar{x}] + \alpha \sum_{i=1}^{n} E[\bar{z}_i]$ — ②

given $E[\bar{x}] = 0$ & $E[\bar{z}_i] = 0$

Subst in ②

$E[\hat{x}] = 0$ — ③

Step 2 - finding mean squared ERROR

The ERROR Term $(\hat{x}-\bar{x})$

subst values of $\hat{x}$

$$\hat{x}-\bar{x} = \alpha \sum_{i=1}^{n} \bar{x} + \alpha \sum_{i=1}^{n} \bar{z}_i - \bar{x}$$

$$= (\alpha n-1)\bar{x} + \alpha \sum_{i=1}^{n} \bar{z}_i$$

so

$$\hat{x}-\bar{x} = (\alpha n-1)\bar{x} + \alpha \sum_{i=1}^{n} \bar{z}_i \quad — 4$$

mean squared ERROR is $E[(\hat{x}-\bar{x})^2]$ — 5

We know by variance formula.

$$VAR(\hat{x}-\bar{x}) = E[(\hat{x}-\bar{x})^2] - (E[\hat{x}-\bar{x}])^2 \quad — 4$$

Also

$E[\hat{x}-\bar{x}] = 0$ because $E[\hat{x}]=0$ $E[\bar{x}]=0$ (From Eqn-3 & Question)

so finally.

$$VAR(\hat{x}-\bar{x}) = E[(\hat{x}-\bar{x})^2] \quad — 6$$

so mean squared ERROR is equal to $VAR(\hat{x}-\bar{x})$ (using 5 & 6)

subst values from 4 in above.

$$\text{mean squared ERROR} = VAR\left((\alpha n-1)\bar{x} + \alpha \sum_{i=1}^{n} \bar{z}_i\right)$$

Since $\bar{x}$ & $\bar{z}$ are independent

$$\text{VAR}\left((\alpha n-1)\bar{x} + \alpha \sum_{i=1}^{n} \bar{z}_i\right)$$

$$= \text{VAR}((\alpha n-1)\bar{x}) + \text{VAR}\left(\alpha \sum_{i=1}^{n} \bar{z}_i\right)$$

= Using variance properties

$$(\alpha n-1)^2 \,\text{VAR}(\bar{x}) + \alpha^2 \sum_{i=1}^{n} \text{VAR}(\bar{z}_i) \quad - \quad (7)$$

Given

$\text{VAR}(\bar{x}) = 1$ & $\text{VAR}(\bar{z}) = \sigma^2$

Subst these values in equation 7

$$(\alpha n-1)^2 \times 1 + \alpha^2 n \sigma^2$$

So

Mean squared ERROR $= (\alpha n-1)^2 \times 1 + \alpha^2 n\sigma^2$

Step-3 : Minimizing the Mean squared ERROR funcⁿ

$$\frac{d\, (\alpha n-1)^2 \times 1 + \alpha^2 n\sigma^2}{d\alpha} = 0$$

$$2(\alpha n-1)n + 2\alpha n\sigma^2 = 0$$

$$2n(\alpha n-1) + 2\alpha n\sigma^2 = 0$$

$$n(\alpha n-1) + \alpha n\sigma^2 = 0$$

$$n\alpha n + \alpha n\sigma^2 = n$$

$$\alpha n(n+\sigma^2) = n$$

$n+\sigma^2$

So value of α that minimizes the mean squared ERROR

$$\boxed{\alpha = \frac{1}{n+\sigma^2}}$$

(b) For $\sigma^2 = 0$, that means no noise $\bar{z}_i$

$\alpha = 1/n$ Estimation $= \frac{1}{n} \times \sum_{i=1}^{n} \bar{y}_i$

Since there is no noise now

$\bar{y}_i = \bar{x}$

Estimator $= \frac{1}{n} \times n\bar{x} = \bar{x}$ (perfect capture of $\bar{x}$)

For $\sigma^2 = \infty$

$\alpha \Rightarrow 0$ so Estimator $\to 0$

Basically we are capturing excessive noise.
hence Estimator $\to 0$

(c) For $\alpha = \frac{1}{n+\sigma^2}$, the mean squared ERROR is minimised

The mean-squared ERROR is $E[(\hat{x}-\bar{x})^2]$

$E[(\hat{x}-\bar{x})^2] = \text{VAR}((\hat{x}-\bar{x})^2)$ [proven in part-A]

$\text{VAR}[(\hat{x}-\bar{x})^2] = (\alpha n-1)^2 + \alpha^2 n\sigma^2$ (proven in part-A) — (1)

Subst. $\alpha = \frac{1}{n+\sigma^2}$ in (1)

MSE is

$$\left(\frac{n}{n+\sigma^2} - xn - 1\right)^2 + \left(\frac{1}{n+\sigma^2}\right)^2 n\sigma^2$$

$$= \frac{-\sigma^2}{n+\sigma^2} + \frac{n\sigma^2}{(n+\sigma^2)^2}$$

$$= \frac{\sigma^4 + n\sigma^2}{(n+\sigma^2)^2}$$

$$= \frac{\sigma^2(\sigma^2+n)}{(n+\sigma^2)^2}$$

MSE (mean squared ERROR) is :-

$$\boxed{\frac{\sigma^2(\sigma^2+n)}{(n+\sigma^2)^2}}$$

Scaling with n

With we have a large amount of n (measurements) as compared to variance of noise then - MSE $\propto 1/n$

given in the Question :

Signal of interest is Random Variable $\tilde{a}$ < mean: μ, variance: σ^2

The signal is observed as

$$\tilde{y} = \tilde{w}\tilde{a}$$

$$P_{\tilde{w}=-1} = \frac{1}{2} \qquad P_{\tilde{w}=0} = \frac{1}{2}$$

$\tilde{w}$ & $\tilde{a}$ are independent.

ⓐ $E[\tilde{w}] = 0$ because of being 1 & -1 with equal prob.

$$E[\tilde{y}] = E[\tilde{w}\tilde{a}] = E[\tilde{w}]E[\tilde{a}] = 0 \times \mu = 0$$

(because $E[\tilde{w}] = 0$)

$$E[\tilde{y}^2] = E[\tilde{w}^2\tilde{a}^2] = E[\tilde{w}^2]E[\tilde{a}^2] \quad — ①$$

$$E[\tilde{w}^2] = (-1)^2 \times \frac{1}{2} + (1)^2 \times \frac{1}{2} = 1.$$

$$E[\tilde{a}^2] = \text{var}(\tilde{a}) + (E[\tilde{a}])^2 = \sigma^2 + \mu^2$$

Subst. values in ①

$$E[\tilde{y}^2] = 1 \times (\sigma^2 + \mu^2) = \sigma^2 + \mu^2$$

Variance of $\tilde{y}$ is

$$\text{var}(\tilde{y}) = E[\tilde{y}^2] - (E[\tilde{y}])^2$$

$$\text{var}(\tilde{y}) = \sigma^2 + \mu^2 - 0$$

$$\text{var}(\tilde{y}) = \sigma^2 + \mu^2 \quad — ②$$

Q3-a-(2)

$$\text{cov}(\tilde{a},\tilde{y}) = E[\tilde{a}\tilde{y}] - E[\tilde{a}]E[\tilde{y}] \quad — (3)$$

$E[\tilde{a}] = \mu$.
$E[\tilde{y}] = 0$

$$E[\tilde{a}\tilde{y}] = E[\tilde{a}\tilde{w}\tilde{a}] = E[\tilde{w}][\tilde{a}^2] = 0 \times (\sigma^2+\mu^2) = 0$$

subst in (3)

$$\text{cov}(\tilde{a},\tilde{y}) = 0 - \mu \times 0 = 0 \quad — (4)$$

The linear MMSE for $\tilde{a}$ given $\tilde{y}$

$$\tilde{a} = \frac{\sigma_{\tilde{a}} \times \rho \times (\tilde{y} - E[\tilde{y}])}{\sigma_{\tilde{y}}} + E[\tilde{a}]$$

splitting corr. coeff.

$$\tilde{a} = \sigma_{\tilde{a}} \times \frac{\text{cov}(\tilde{a},\tilde{y})}{\sigma_{\tilde{y}} \times \sigma_{\tilde{y}} \times \sigma_{\tilde{a}}} \times (\tilde{y} - E[\tilde{y}]) + E[\tilde{a}]$$

$$\tilde{a} = \frac{\text{cov}(\tilde{a},\tilde{y})}{\text{var}(\tilde{y})} \times (\tilde{y} - E[\tilde{y}]) + E[\tilde{a}]$$

Subst. values

Linear MMSE

$$\boxed{\tilde{a} = \frac{0}{\sigma^2+\mu^2}(y-0) + \mu = \mu.}$$

Q3-b

(b) MSE = $E[(\breve{a}-\breve{a}_{MMSE})^2]$

Best linear estimator in terms of mean squared error
is μ so (derived in part-a).

$E[(\breve{a}-\mu)^2$

and this is equal to variance $= \sigma^2$

(c)

We know:

$\bar{y}$ is measuring signal $\breve{a}$

$\bar{y} = \breve{w}\breve{a}$

where catch depending on $\breve{w}$ value.

$\bar{y} = -a$ or $\bar{y} = \breve{a}$

if we take estimator as $|\bar{y}|$ i.e absolute value
of $\bar{y}$ then we will eliminate the noise $\breve{w}$
& the always correctly capture $\breve{a}$ with
MSE of 0 which better (less) than the MSE of linear
estimator μ.

The equation of least squares regression line

$$y = a + bx \quad — ①$$

a = y intercept
b = slope

We know :-
The Regression coefficient is given by :-

$$b = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \quad — ②$$

The intercept is given by

$$a = \bar{y} - b\bar{x} \quad — ③$$

To show the least squares regression line passes through $(\bar{x}, \bar{y})$

Substitute $x = \bar{x}$ in eq^n - ①

$$\bar{y} = a + b\bar{x}$$

Substitute a from eq^n - ③

$$\bar{y} = (\bar{y} - b\bar{x}) + b\bar{x}$$

$$\bar{y} = \bar{y}$$

Hence proved, when $x = \bar{x}$, $y = \bar{y}$ proving least square regression line passes through $(\bar{x}, \bar{y})$.