Unnati Rajpoot (section-17)

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(Linear Algebra) Question Bank - 3.

1. $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$

We know that, $AX = \lambda X$

In this question $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now, $AX = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda X$

Hence $V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ proved. and Eigen value $= 3$.

2. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$

$AX = \lambda X$

Now, $AX = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ -1-2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$

$= 3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Hence $V = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ proved and Eigen value $\lambda = 3$.

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3. $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

Characteristic Matrix $(A - \lambda I)$

$$= \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}$$

Characteristic polynomial

$$(2-\lambda)(-1-\lambda) - 4 = 0$$
$$-2 - 2\lambda + \lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - \lambda - 6 = 0$$
$$\lambda^2 - (3\lambda - 2\lambda) - 6 = 0$$
$$\lambda^2 - 3\lambda + 2\lambda - 6 = 0$$
$$\lambda(\lambda - 3) + 2(\lambda - 3) = 0$$
$$(\lambda - 3)(\lambda + 2)$$

Eigen value $\lambda = 3, -2$

Hence 3 is the Eigen value proved.

Eigenvector corrosponding to Eigen value $\lambda = 3$ is given by $(A - \lambda I)X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 2-3 & 2 \\ 2 & -1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + 2x_2 = 0$, $2x_1 - 4x_2 = 0$

let $x_2 = k$ then $-x_1 + 2k = 0$

$x_1 = 2k$

$X = \begin{bmatrix} 2k \\ k \end{bmatrix} \Rightarrow k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Eigen vector

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4. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

characteristic Matrix $(A - \lambda I)$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 0 & 2 \\ -1 & 1-\lambda & 1 \\ 2 & 0 & 1-\lambda \end{bmatrix}$$

characteristic polynomial

$$(1-\lambda)[(1-\lambda)(1-\lambda)-0]-0[-(1-\lambda)-2]+2[0-2(1-\lambda)]=0$$
$$(1-\lambda)[1+\lambda^2-2\lambda]-4(1-\lambda)=0$$
$$1+\lambda^2-2\lambda-\lambda-\lambda^3+2\lambda^2-4+4\lambda=0$$
$$-\lambda^3+3\lambda^2+\lambda-3=0$$
$$\lambda^3-3\lambda^2-\lambda+3=0$$

By hit and trial let $\lambda=1$ then, $1-3-1+3=0$

$\lambda=1$

$$(\lambda-1)\,)\,\lambda^3-3\lambda^2-\lambda+3\,(\,\lambda^2-2\lambda-3$$
$$\cancel{\lambda^3}-\lambda^2$$
$$-\quad+$$
$$-2\cancel{\lambda^2}-\lambda+3$$
$$-\cancel{2}\lambda^2+2\lambda$$
$$+\quad-$$
$$-3\cancel{\lambda}+\cancel{3}$$
$$-\cancel{3}\lambda+\cancel{3}$$
$$+\quad-$$
$$\times$$

$$(\lambda-1)(\lambda^2-2\lambda-3)=0$$
$$(\lambda-1)(\lambda^2-(3\lambda-\lambda)-3)=0$$
$$(\lambda-1)(\lambda^2-3\lambda+\lambda-3)=0$$
$$(\lambda-1)\;(\lambda(\lambda-3)+1(\lambda-3))=0$$
$$(\lambda-1)(\lambda+1)(\lambda-3)=0$$

Eigen value $\lambda = 1, -1, 3$

Hence one Eigenvalue is -1 proved

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Eigen vector corrosponding to Eigenvalue $\lambda = -1$ is given by $(A - \lambda I)X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-(-1) & 0 & 2 \\ -1 & 1-(-1) & 1 \\ 2 & 0 & 1-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + 2x_3 = 0, \quad -x_1 + 2x_2 + x_3 = 0, \quad 2x_1 + 2x_3 = 0$

Let $x_1 = k$ then, $x_3 = -k$ and $x_2 = k$

$X = k \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ Eigen vector

5. Charactersstic polynomial - the determinant of a squar matrix in which an arbitrary variable (such as λ) is substracted from each of the elements along the principal diagonal.

6. Algebraic Multiplicity - The algebraic multiplicity of an eigenvalue is the number of times it appears as a root of characterstic polynomial.

Geometric Multiplicity - The geometric multiplicity of an eigenvalue is the number of linearly independent eigenvector associated with it.

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7. $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

Characteristic matrix $(A - \lambda I)$

$$\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{bmatrix}$$

Characteristic polynomial

$$(1-\lambda)(6-\lambda) - (-6) = 0$$
$$6 - \lambda - 6\lambda + \lambda^2 + 6 = 0$$
$$\lambda^2 - 7\lambda + 12 = 0$$
$$\lambda^2 - (4\lambda + 3\lambda) + 12 = 0$$
$$\lambda^2 - 4\lambda - 3\lambda + 12 = 0$$
$$\lambda(\lambda - 4) - 3(\lambda - 4) = 0$$
$$(\lambda - 4)(\lambda - 3) = 0$$

Eigen value $\lambda = 4, 3.$

Eigen vector corrosponding to Eigen value $\lambda = 4$ is given by
$(A - \lambda I)X_1 = 0$ Where $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 1-4 & 3 \\ -2 & 6-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 3x_2 = 0$, $-2x_1 + 2x_2 = 0$

Let $x_1 = K_1$ then, $x_2 = K_1$

$X_1 = \begin{bmatrix} K_1 \\ K_1 \end{bmatrix} \Rightarrow K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Eigen Basis corrosponding to Eigenspace $K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Algebraic Multiplicity for $\lambda = 4$ is 1
Geometric Multiplicity for $\lambda = 4$ is 1.

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Eigen vector corresponding to Eigen value $\lambda=3$ is given by $(A-\lambda I)X_2$ where, $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 1-3 & 3 \\ -2 & 6-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + 3x_2 = 0$, $-2x_1 + 3x_2 = 0$

Both equation are same.

Let $x_1 = K_2$ then $x_2 = \frac{2}{3} K_2$

$X = \begin{bmatrix} K_2 \\ \frac{2}{3} K_2 \end{bmatrix} \Rightarrow K_2 \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$ Eigen Basis corresponding to Eigenspace $K_2 \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$

Algebraic Multiplicity of $\lambda=3$ is 1

Geometric Multiplicity of $\lambda=3$ is 1.

8. $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

Characteristic Matrix

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$

characteristic polynomial

$$(2-\lambda)(-\lambda) - (-1) = 0$$

$$-2\lambda + \lambda^2 + 1 = 0$$

$$(\lambda-1)^2 = 0$$

$$(\lambda-1)(\lambda-1) = 0$$

Eigen value $\lambda = 1, 1$

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Eigen vector corresponding to Eigen value $\lambda=1$ is given by $(A-\lambda I)X=0$, where $X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 2-1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1+x_2=0$ and $-x_1-x_2=0$

Both equations are same.

Let $x_1=k_1$ then $x_2=-k_1$

$X=\begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} \Rightarrow k_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Eigen Basis corresponding to Eigenspace $k_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Algebraic Multiplicity of $\lambda=1$ is 2

Geometric Multiplicity of $\lambda=1$ is 1.

9. $A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Characteristic polynomial $(A-\lambda I)$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}-\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} (1-\lambda) & 1 & 0 \\ 0 & (-2-\lambda) & 1 \\ 0 & 0 & (3-\lambda) \end{bmatrix}$$

$(1-\lambda)\left[(-2-\lambda)(3-\lambda)\right]=0$

$\cancel{\lambda=1}\ (1-\lambda)(-2-\lambda)(3-\lambda)=0$

Eigen value $\lambda=1,-2,3$.

Eigen vector corresponding to Eigen value $\lambda=1$ is given by $(A-\lambda I)X_1=0$ $\quad X_1=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-1 & 1 & 0 \\ 0 & -2-1 & 1 \\ 0 & 0 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$, $-3x_2 + x_3 = 0$, $x_3 = 0$

Let $x_1 = k_1$

$X_1 = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ Basis corresponding to Eigen Space $k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Algebraic Multiplicity of $\lambda = 1$ is 1
Geometric Multiplicity of $\lambda = 1$ is 1

Eigen vector corresponding to Eigen value $\lambda = -2$ is given by $(A - \lambda I)X_2 = 0$, where $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-(-2) & 1 & 0 \\ 0 & -2(-2) & 1 \\ 0 & 0 & 3-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 + x_2 = 0$, $x_3 = 0$, $5x_3 = 0$

Let $x_2 = k_2$ then $x_1 = -\frac{k_2}{3}$

$X_2 = \begin{bmatrix} -k_2/3 \\ k_2 \\ 0 \end{bmatrix} \Rightarrow k_2 \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$ Eigen Basic corresponding to Eigenspace $k_2 \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$

Algebraic Multiplicity of $\lambda = -2$ is 1
~~Algebraic~~ Geometric Multiplicity of $\lambda = -2$ is 1

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Eigen vector corresponding to Eigen value $\lambda = 3$ is given by $(A - \lambda I) X_3 = 0$ where $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-3 & 1 & 0 \\ 0 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + x_2 = 0$, $-5x_2 + x_3 = 0$

Let $x_2 = k_3$ then, $x_1 = \frac{k_3}{3}$ and $x_3 = 5k_2$

$$X_3 = \begin{bmatrix} k_3/3 \\ k_3 \\ 5k_3 \end{bmatrix} \Rightarrow k_3 \begin{bmatrix} 1/3 \\ 1 \\ 5 \end{bmatrix}$$

Eigen Basis corresponding to Eigenspace $k_3 \begin{bmatrix} 1/3 \\ 1 \\ 5 \end{bmatrix}$ is $\begin{bmatrix} 1/3 \\ 1 \\ 5 \end{bmatrix}$

Algebraic Multiplicity of $\lambda = 3$ is 1
Geometric Multiplicity of $\lambda = 3$ is 1.

10. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Characteristic Matrix $(A - \lambda I)$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

characteristic polynomial

$$(1-\lambda)\left[(1-\lambda)(-\lambda) - 1\right] + 1\left[0 - (1-\lambda)\right] = 0$$
$$(1-\lambda)\left[-\lambda + \lambda^2 - 1\right] - 1 + \lambda = 0$$
$$-\cancel{\lambda} + \lambda^2 - 1 + \lambda^2 - \lambda^3 + \cancel{\lambda} - 1 + \lambda = 0$$
$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$
$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

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By hit and trial let $\lambda = 1$, $-1+2+1-2=0$

$$(\lambda-1)\overline{)\lambda^3-2\lambda^2-\lambda+2}(\lambda^2-\lambda-2$$
$$\cancel{\lambda^3}-\lambda^2$$
$$-\lambda^2-\lambda+2$$
$$-\cancel{\lambda^2}+\lambda$$
$$-2\lambda+2$$
$$-2\lambda+2$$
$$\times$$

$$(\lambda-1)(\lambda^2-\lambda-2)=0$$
$$(\lambda-1)(\lambda^2-(2\lambda-1\lambda)-2)=0$$
$$(\lambda-1)(\lambda^2-2\lambda+\lambda-2)=0$$
$$(\lambda-1(\lambda(\lambda-2)+1(\lambda-2)=0$$
$$(\lambda-1)(\lambda+1)(\lambda-2)=0$$

Eigen value $\lambda = 1, -1, 2$

Eigen vector corrosponding to Eigen value $\lambda = 1$ is given by
$(A-\lambda I)X_1 = 0$ where $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-1 & 0 & 1 \\ 0 & 1-1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = 0$, $x_3 = 0$, $x_1 + x_2 - x_3 = 0$

let $x_1 = k_1$ then, $x_2 = -k_1$

$$X_1 = \begin{bmatrix} k_1 \\ -k_1 \\ 0 \end{bmatrix} \Rightarrow k_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Eigen Basis corrosponding to Eigen space $k_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Algebraic multiplicity of $\lambda = 1$ is 1
Geometric multiplicity of $\lambda = 1$ is 1

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Eigen vector corresponding to Eigen value -1 is given by $(A-\lambda I)X_2$ where $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-(-1) & 0 & 1 \\ 0 & 1-(-1) & 1 \\ 1 & 1 & -(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$2x_1 + x_3 = 0$, $2x_2 + x_3 = 0$ and $x_1 + x_2 + x_3 = 0$.

Let $x_3 = K_2$ then, $x_1 = -\frac{K_2}{2}$ and $x_2 = -\frac{K_2}{2}$

$$X_1 = \begin{bmatrix} -\frac{K_2}{2} \\ -K_2/2 \\ K_2 \end{bmatrix} \Rightarrow K_2 \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Eigen Basis corresponding to Eigen space $K_2 \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

Algebraic Multiplicity of $\lambda = -1$ is 1
Geometric Multiplicity of $\lambda = -1$ is 1

Eigen vector corresponding to Eigen value $\lambda = 2$ is given by $(A-\lambda I)X_3$, where $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-2 & 0 & 1 \\ 0 & 1-2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_3 = 0$, $-x_2 + x_3 = 0$, $x_1 + x_2 - 2x_3 = 0$

Let $x_3 = K_3$, then, $x_1 = K_3$ and $x_2 = K_3$

$$X_3 = \begin{bmatrix} K_3 \\ K_3 \\ K_3 \end{bmatrix} \Rightarrow K_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigen Basis corrosponding to Eigenspace $K_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Algebraic Multiplicity of $\lambda = 2$ is 1
Geometric Multiplicity of $\lambda = 2$ is 1.

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11. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Characteristic Matrix $(A-\lambda I)$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}.$$

Characteristic polynomial

$$(1-\lambda)[(-1-\lambda)(1-\lambda)-1]-2[-(1-\lambda)-0]=0$$
$$(1-\lambda)[-1+\cancel{\lambda}-\cancel{\lambda}+\lambda^2-1]+2-2\lambda=0$$
$$(1-\lambda)[\lambda^2-2]+2-2\lambda=0$$
$$\lambda^2-\cancel{2}-\lambda^3+\cancel{2\lambda}+\cancel{2}-\cancel{2\lambda}=0$$
$$\lambda^2-\lambda^3=0$$
$$\lambda^3-\lambda^2=0$$
$$\lambda^2(\lambda-1)=0$$
$\lambda^2=0 \quad \Rightarrow \lambda=0,0$

$\lambda-1=0 \quad \Rightarrow \lambda=1$

Eigen value $=0,0,1$

Eigenvector corrosponding to Eigenvalue $\lambda=0$ is given by $(A-\lambda I)X_1=0$ where $X_1=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-0 & 2 & 0 \\ -1 & -1-0 & 1 \\ 0 & 1 & 1-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1+2x_2=0$, $-x_1-x_2+x_3=0$ $\quad x_2+x_3=0$

Let $x_2=k_1$ then $x_1=-2k_1$ $\quad x_3=-k_1$

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$X_1 = \begin{bmatrix} -2k_1 \\ k_1 \\ -k_1 \end{bmatrix} \Rightarrow k_1 \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ Eigen Basis corresponding to Eigen-space $X_1 \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

Algebraic Multiplicity of $\lambda = 0$ is 2
Geometric Multiplicity of $\lambda = 0$ is 1

Eigen vector corresponding to Eigen value $\lambda = 1$ is given by $(A - \lambda I) X_2 = 0$ $\quad X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-1 & 2 & 0 \\ -1 & -1-1 & 1 \\ 0 & 1 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_2 = 0 \qquad -x_1 - 2x_2 + x_3 = 0 \qquad , x_2 = 0$

Let $x_1 = k_2$ $\qquad$ then, $-k_1 - 2 \times 0 + x_3 = 0$

$x_3 = k_2$

$X_2 = \begin{bmatrix} k_2 \\ 0 \\ k_2 \end{bmatrix} \Rightarrow k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ Eigen Basis corresponding to Eigenspace $k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Algebraic Multiplicity of $\lambda = 1$ is 1
Geometric Multiplicity of $\lambda = 1$ is 1.

12. $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

Characteristic Matrix $(A - \lambda I)$

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 0 & 2 \\ 3 & -1-\lambda & 3 \\ 2 & 0 & 1-\lambda \end{bmatrix}$$

Characteristic ~~Matri~~ polynomial

$$(1-\lambda)[(-1-\lambda)(1-\lambda)] + 2[0-2(-1-\lambda)] = 0$$

$$(1-\lambda)[(1+\lambda)(\lambda-1)] + 4 + 4\lambda = 0$$

$$(1-\lambda)(\lambda^2-1) + 4 + 4\lambda = 0$$

$$\lambda^2 - 1 - \lambda^3 + \lambda + 4 + 4\lambda = 0$$

$$-\lambda^3 + \lambda^2 + 5\lambda + 3 = 0$$

$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

By hit and trial let $\lambda = -1$ then,

$$\Rightarrow -1 - 1 + 5 - 3 = 0$$

$$-5 + 5 = 0$$

$$(\lambda+1) \overline{) \cancel{\lambda^3} - \lambda^2 - 5\lambda - 3} (\lambda^2 - 2\lambda - 3$$

$$\cancel{\lambda^3} + \lambda^2$$

$$- \quad -$$

$$-2\cancel{\lambda^2} - 5\lambda - 3$$

$$-\cancel{2\lambda^2} - 2\lambda$$

$$+ \quad +$$

$$-3\cancel{\lambda} - \cancel{3}$$

$$-\cancel{3\lambda} - \cancel{3}$$

$$+ \quad +$$

$$\times$$

$$(\lambda+1)(\lambda^2 - 2\lambda - 3) = 0$$

$$(\lambda+1)(\lambda^2 - (3\lambda - \lambda) - 3] = 0$$

$$(\lambda+1)[\lambda^2 - 3\lambda + \lambda - 3] = 0$$

$$(\lambda+1)[\lambda(\lambda-3) + 1(\lambda-3)] = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-3) = 0$$

Eigen value $= -1, -1, 3$.

Eigen Vector corresponding to Eigenvalue $\lambda = 3$ is given by
$(A - \lambda I) X_1 = 0$

$$\begin{bmatrix} 1-3 & 0 & 2 \\ 3 & -1-3 & 3 \\ 2 & 0 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 & 2 \\ 3 & -4 & 3 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + 2x_3 = 0, \quad 3x_1 - 4x_2 + 3x_3 = 0, \quad 2x_1 - 2x_3 = 0$

Let $x_1 = k_1$ then $x_3 = k_1$

$$3k_1 - 4x_2 + 3k_1 = 0$$

$$x_2 = \frac{3}{2} k_1$$

$$X_1 = \begin{bmatrix} k_1 \\ \frac{3}{2} k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix}$$

Eigen basis corresponding to Eigenspace $k_1 \begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix}$

Algebraic Multiplicity of $\lambda = 3$ is 1
Geometric Multiplicity of $\lambda = 3$ is 1

Eigen vector corresponding to Eigen value $\lambda = -1$ is given by $(A - \lambda I) X_2 = 0$ where, $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1-(-1) & 0 & 2 \\ 3 & -1-(-1) & 3 \\ 2 & 0 & 1-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 3 & 0 & 3 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + 2x_3 = 0, \quad 3x_1 + 3x_2 = 0, \quad 2x_1 + 2x_3 = 0$

All equation are equivalent.

Let $x_1 = k_2$ and $x_3 = -k_2$ $\quad x_2 = 0$

$$X_2 = \begin{bmatrix} k_2 \\ 0 \\ -k_2 \end{bmatrix} \Rightarrow k_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Eigen Basis corresponding Eigenspace $k_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Algebraic Multiplicity = 2
Geometric " = 1

13. Product of eigenvalues is determinant of given matrix.

If Eigen values are Non zero ⇒ Determinat is non-Zero.

Hence, matrix is invertible or non-singular.

17. $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Characteristic Matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix}$$

Characteristic polynomial

$$(1-\lambda)(-1-\lambda) - 1 = 0$$
$$-(1-\lambda)(1+\lambda) - 1 = 0$$
$$-(1-\lambda^2) - 1 = 0$$
$$\lambda^2 - 1 - 1 = 0$$
$$\lambda^2 = +2$$
$$\lambda = \pm\sqrt{2}$$

Eigen value $\lambda_1 = +\sqrt{2}$, $\lambda_2 = -\sqrt{2}$

$$(\lambda_1)^{19} = (\sqrt{2})^{19} = [(2)^{1/2}]^{19} \Rightarrow 2^9 \times \sqrt{2} = 512\sqrt{2}$$

$$(\lambda_2)^{19} = (-\sqrt{2})^{19} = [-(2)^{1/2}]^{19} \Rightarrow (-2)^9 \times \sqrt{2} = -512\sqrt{2}$$

Eigen values of the matrix A^{19} is $+512\sqrt{2}$ and $-512\sqrt{2}$

18. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$

Characteristic Matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 7 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

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Characteristic polynomial.

$(1-\lambda)[(4-\lambda)(3-\lambda)-0]-2[0]+3[0]=0$
$(1-\lambda)(4-\lambda)(3-\lambda)=0$

Eigen value $\lambda = 1, 4, 3$.

11. $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$

Eigen value $= 4, 8$.
Trace of matrix = Sum of Eigen value
$= 4+8$
$= 12$

We know that sum of elements on the principal diagonal of a matrix is equal to the trace of matrix

$2+y = 12$
$y = 12-2 = 10$

Also, we know that product of Eigen values is equal to the determinant of matrix, so,

$4\times 8 = 2y - 3x$ (put value of $y=10$)
$32 = 2\times 10 - 3x$
$3x = 20-32$
$3x = -12$
$x = -4$

$\boxed{x=-4 \text{ and } y=10}$

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20. Eigen value = $-1, -2$

Let Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

characteristic Matrix is $= \begin{bmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{bmatrix}$

Eigen vector for $\lambda = -1$

$$\begin{bmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (a+1) & b \\ c & (d+1) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(a+1) - b = 0$ $\qquad$ $c - (d+1) = 0$

$a - b = -1 \quad —(1)$ $\qquad$ $c - d = 1 \quad —(2)$

For $\lambda = -2$, Eigen vector is

$$\begin{bmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} (a+2) & b \\ c & (d+2) \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$a+2 - 2b = 0$ $\qquad$ $c - 2(d+2) = 0$

$a - 2b = -2 \quad —(3)$ $\qquad$ $c - 2d = 4 \quad —(4)$

From (i) and 3

$$\begin{array}{l} \cancel{a} - b = -1 \\ \cancel{a} - 2b = -2 \\ - \quad + \quad\quad + \\ \hline b = 1 \end{array}$$

putting $b = 1$ in eq^n 1

$a - b = -1$

$a - 1 = -1$

$a = -1 + 1$

$a = 0$

On Substracting (2) and (4) we get

$$\cancel{c} - d = 1$$
$$-\cancel{c} \mp 2d = -4$$
$$d = -3$$

putting $d = -3$ in eq^n (2)

$$c - d = 1$$
$$c - (-3) = 1$$
$$c + 4 = 1$$
$$c = -4 + 1 = -3$$
$$c = -3$$

Matrix $A = \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix}$

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22. $A = \begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$

Charatesstic Matrix

$$\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} (3-\lambda) & 5 & 2 \\ 5 & (12-\lambda) & 7 \\ 2 & 7 & (5-\lambda) \end{bmatrix}$$

Charatesstic polynomial

$$(3-\lambda)[(12-\lambda)(5-\lambda)-49]-5[5(5-\lambda)-14]+2[35-2(12-\lambda)]=0$$

$$(3-\lambda)[(60-12\lambda-5\lambda+\lambda^2-49)]-5[25-5\lambda-14]+2[35-24+2\lambda]=0$$

$$(3-\lambda)[\lambda^2-17\lambda+11]-5[11-5\lambda]+2[11+2\lambda]=0$$

$$3\lambda^2-51\lambda+33-\lambda^3+17\lambda^2-11\lambda-55+25\lambda+22+4\lambda=0$$

$$-\lambda^3+20\lambda^2-36\lambda=0$$

$$\lambda^3-20\lambda^2+36\lambda=0 \quad \text{—(i)}$$

By hit and trial $\lambda = 0$

On Dividing Eq^n (i) by $(\lambda-0)$ we get

$$\lambda^2-20\lambda+36=0$$

$$\lambda^2-(18\lambda+2\lambda)+36=0$$

$$\lambda^2-18\lambda-2\lambda+36=0$$

$$\lambda(\lambda-18)-2(\lambda-18)=0$$

$$(\lambda-2)(\lambda-18)=0$$

Eigen value $\lambda = 0, 2, 18$.

23.
$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

Characteristic matrix

$$\begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 & -1 \\ -6 & (-11-\lambda) & 6 \\ -6 & -11 & (5-\lambda) \end{bmatrix}$$

Characteristic polynomial.

$$\Rightarrow -\lambda[(-11-\lambda)(5-\lambda)-(-66)]-1[-6(5-\lambda)-(-36)]-1[66+6(-11-\lambda)]=0$$

$$\Rightarrow -\lambda[-55+11\lambda-5\lambda+\lambda^2+66]-1[-30+6\lambda+36]-1[\cancel{66}-\cancel{66}-6\lambda]=0$$

$$\Rightarrow -\lambda[\lambda^2+6\lambda+11]-[6\lambda+6]+6\lambda=0$$

$$-\lambda^3-6\lambda^2-11\lambda-\cancel{6\lambda}-6+\cancel{6\lambda}=0$$

$$-\lambda^3-6\lambda^2-11\lambda-6=0$$

$$\lambda^3+6\lambda^2+11\lambda+6=0 \quad -(1)$$

By hit and trial Let $\lambda=-1$ then,

$$=(-1)^3+6(-1)^2+11(-1)+6$$

$$=-1+6-11+6$$

$$=0$$

Divide Eqn 1 by $(\lambda+1)$

$$(\lambda+1)\overline{)\lambda^3+6\lambda^2+11\lambda+6}\,(\lambda^2+5\lambda+6$$

$$\cancel{\lambda^3}+\lambda^2$$

$$5\cancel{\lambda^2}+11\lambda+6$$

$$\cancel{5\lambda^2}+5\lambda$$

$$6\cancel{\lambda}+\cancel{6}$$

$$\cancel{6\lambda}+\cancel{6}$$

$$\times$$

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$(\lambda+1)[\lambda^2+5\lambda+6]=0$

$(\lambda+1)[\lambda^2+(3\lambda+2\lambda)+6]=0$

$(\lambda+1)[\lambda^2+3\lambda+2\lambda+6]=0$

$(\lambda+1)[\lambda(\lambda+3)+2(\lambda+3)]=0$

$(\lambda+1)(\lambda+2)(\lambda+3)=0$

Eigen value $\lambda = -1, -2, -3$

maximum Eigen value $= -1$
Minimum Eigen value $= -3$

Ratio of maximum and minimum Eigen value is

$= \frac{(-1)}{(-3)} = \frac{1}{3}$ Ans

24. Let A and B are square matrix of same order. The matrix A is said to be similar to the matrix B if there exists an invertible matrix P such that

$A = P^{-1}BP$ or $PA = BP$ — (1)

Multiplying both side of eq^n 1 by P^{-1} we get

$PAP^{-1} = B$

Therefore, A is similar to B iff B is similar to A.
The matrix P is called similarity matrix.
The transformation in eqn (1) is called a similarity transformation.

25. $A = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix}$

If A and B are similar then it should have equal characterstic polynomial

Characterstic polynomial of Matrix A

$$\begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ -4 & 6-\lambda \end{bmatrix}$$

$$(2-\lambda)(6-\lambda) - (-4) = 0$$
$$12 - 2\lambda - 6\lambda + \lambda^2 + 4 = 0$$
$$\lambda^2 - 8\lambda + 16 = 0$$

Characterstic polynomial of Matrix B.

$$\begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 \\ -5 & 7-\lambda \end{bmatrix}$$

$$(3-\lambda)(7-\lambda) - 5 = 0$$
$$21 - 3\lambda - 7\lambda + \lambda^2 - 5 = 0$$
$$\lambda^2 - 10\lambda + 16 = 0$$

characterstic polynomial of A is does not equal to characterstic polynomial B, so the Matrix A and B are not similar.

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26. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Characteristic polynomial of A

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 2 & 0 \\ 0 & (1-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{bmatrix}$$

$$(1-\lambda)[(1-\lambda)(1-\lambda)-1]-2[0]+0=0$$

$$\cancel{(1-\lambda)[1-\lambda^2-1]}$$

$$(1-\lambda)[\lambda^2+\cancel{1}-2\lambda-\cancel{1}]=0$$

$$(1-\lambda)(\lambda^2-2\lambda)=0$$

$$\lambda^2-2\lambda-\lambda^3+2\lambda^2=0$$

$$-\lambda^3+3\lambda^2-2\lambda=0$$

$$\lambda^3-3\lambda^2+2\lambda=0$$

Characteristic polynomial of B

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix}$$

$$(2-\lambda)[(1-\lambda)(1-\lambda)-0]+1[0-2(1-\lambda)]=0$$

$$(2-\lambda)(\lambda^2+1-2\lambda)-2+2\lambda=0$$

$$2\lambda^2+\cancel{2}-4\lambda-\lambda^3-\lambda+2\lambda^2-\cancel{2}+2\lambda=0$$

$$-\lambda^3+4\lambda^2-3\lambda=0$$

$$\lambda^3-4\lambda^2+3\lambda=0$$

Characteristic polynomial of A $\neq$ characteristic polynomial of B
So, Matrix A and B are not Similar.

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21. $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

Characteristic polynomial

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix}$$

$$(5-\lambda)(5-\lambda) - 4 = 0$$
$$25 + \lambda^2 - 10\lambda - 4 = 0$$
$$\lambda^2 - 10\lambda + 21 = 0$$
$$\lambda^2 - (7\lambda + 3\lambda) + 21 = 0$$
$$\lambda^2 - 7\lambda - 3\lambda + 21 = 0$$
$$\lambda(\lambda - 7) - 3(\lambda - 7) = 0$$
$$(\lambda - 3)(\lambda - 7) = 0$$

Eigen value $\lambda = 3, 7$

Matrix A is diagonalizable because it has distinct Eigen values

Eigen vector for $\lambda = 3$.

$$\begin{bmatrix} 5-3 & 2 \\ 2 & 5-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + 2x_2 = 0$, $2x_1 + 2x_2 = 0$

Both the equation are equivalent of each other

Let $x_1 = k_1$ then $x_2 = -k_1$

$$X_1 = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} \Rightarrow k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Algebraic Multiplicity of $\lambda = 3$ is 1

Geometric " " " is 1

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Eigen vector for $\lambda = 7$

$$\begin{bmatrix} 5-7 & 2 \\ 2 & (5-7) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + 2x_2 = 0$, $2x_1 - 2x_2 = 0$

Both the equation are equivalent to each others

let $x_1 = k_2$ then $x_2 = k_2$

$$X_2 = \begin{bmatrix} k_2 \\ k_2 \end{bmatrix} \Rightarrow k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Algebraic Multiplicity of $\lambda = 7$ is 1

Geometric " " " is 1.

The matrix P is $= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}$

because we assume that the entries of diagonal matrix is Eigen value ~~determinant~~ of P and then satisfy the condition.

We know that

$$P^{-1}AP = D$$

Multiply Both side with P we get

$$PP^{-1}AP = PD$$

$$AP = PD$$

$$AP = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5+2 & 5-2 \\ 2+5 & 2-5 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 7 & -3 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 7 & -3 \end{bmatrix}$$

Since $AP = PD$ then,

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}$$

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28. $\begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$

characteristic polynomial

$\begin{bmatrix} (-3-\lambda) & 4 \\ -1 & (1-\lambda) \end{bmatrix} \Rightarrow (-3-\lambda)(1-\lambda)+4=0$

$-3+3\lambda-\lambda+\lambda^2+4=0$

$\lambda^2+2\lambda+1=0$

$\lambda^2+(1\lambda+1\lambda)+1=0$

$\lambda^2+\lambda+\lambda+1=0$

$\lambda(\lambda+1)+1(\lambda+1)=0$

$(\lambda+1)(\lambda+1)$.

Eigen value $\lambda = -1, -1$

Since Eigen value are not distint so the matrix is not diagonalizable

29. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Characteristic polynomial $(A-\lambda I)=0$

$\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$

$(3-\lambda)[(3-\lambda)(3-\lambda)-0]-1[0]+0=0$

$(3-\lambda)(3-\lambda)(3-\lambda)=0$

Eigen value $\lambda = 3, 3, 3$

Since Eigen value are not distint so the matrix is not diagonalizable.

30. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

characterstic polynomial, $(A - \lambda I) = 0$

$$\begin{bmatrix} (1-\lambda) & 0 & 1 \\ 0 & (1-\lambda) & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

$$(1-\lambda)\left[-\lambda(1-\lambda) - 1\right] - 0 + 1\left[0 - (1-\lambda)\right] = 0$$

$$(1-\lambda)\left[-\lambda + \lambda^2 - 1\right] - 1 + \lambda = 0$$

$$\cancel{-\lambda} + \lambda^2 - 1 + \lambda^2 - \lambda^3 + \cancel{\lambda} - 1 + \lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Let $\lambda = 1$ by hit and trial

$$= (1)^3 - 2(1)^2 - (1) + 2$$

$$= 1 - 2 - 1 + 2$$

$$= 0$$

On Dividing Eqn by $(\lambda - 1)$ we get

$$(\lambda - 1)\ \lambda^3 - 2\lambda^2 - \lambda + 2\ (\lambda^2 - \lambda - 2$$

$$\cancel{\lambda^3} - \lambda^2$$

$$-\cancel{\lambda^2} - \lambda + 2$$

$$-\cancel{\lambda^2} + \lambda$$

$$-\cancel{2\lambda} + 2$$

$$-\cancel{2\lambda} + \cancel{2}$$

$$\times$$

$(\lambda-1)(\lambda^2-\lambda-2)=0$

$(\lambda-1)[\lambda^2-(2\lambda-\lambda)-2]=0$

$(\lambda-1)[\lambda^2-2\lambda+\lambda-2]=0$

$(\lambda-1)[\lambda(\lambda-2)+1(\lambda-2)]=0$

$(\lambda-1)(\lambda+1)(\lambda-2)=0$

Eigen value $\lambda = 1, -1, -2$

Eigen value are distinct so, the matrix is diagonalizable.

Eigen vector for $\lambda = 1$

$$\begin{bmatrix} (1-1) & 0 & 1 \\ 0 & 1-1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = 0, \quad x_3 = 0, \quad x_1 + x_2 - x_3 = 0$

Let $x_1 = k_1$ then $\quad k_1 + x_2 - 0 = 0$

$x_2 = -k_1$

$$X_1 = \begin{bmatrix} k_1 \\ -k_1 \\ 0 \end{bmatrix} \Rightarrow k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Algebraic Multiplicity of $\lambda = 1$ is 1

Geometric " " " is 1

Eigen vector for $\lambda = -1$

$$\begin{bmatrix} 1-(-1) & 0 & 1 \\ 0 & 1-(-1) & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_3 = 0, \quad 2x_2 + x_3 = 0, \quad x_1 + x_2 + x_3 = 0$

Let $x_3 = k_2$, then $2x_1 + k_2 = 0$ and $2x_2 + k_2 = 0$

$x_1 = -\frac{k_2}{2} \qquad x_2 = -\frac{k_2}{2}$

$$X_2 = \begin{bmatrix} -k_2/2 \\ -k_2/2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = k_2' \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Algebraic Multiplicity of $\lambda = -1$ is 1

Geometric " " " is 1

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Eigen vector for $\lambda=2$

$$\begin{bmatrix} 1-2 & 0 & 1 \\ 0 & 1-2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_3 = 0$, $-x_2 + x_3 = 0$, $x_1 + x_2 - 2x_3 = 0$

Let $x_3 = K_3$, then $x_1 = K_3$ and $x_2 = K_3$

$$X_3 = \begin{bmatrix} K_3 \\ K_3 \\ K_3 \end{bmatrix} \Rightarrow K_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Algebraic Multiplicity of $\lambda = 2$ is 1
Geometric " " " is 1

The Matrix P is $\begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

because we assume that the entries of diagonal matrix is Eigen value and then we have to satisfy the condition

We know

$$P^{-1}AP = D$$
$$PP^{-1}AP = PD$$
$$AP = PD$$

$$AP = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1+2 & 1+1 \\ -1 & -1+2 & 1+1 \\ 1-1 & -1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

since $AP = PD$ then,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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31. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

Characteristic polynomial

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & 1 \\ 3 & 0 & 1-\lambda \end{bmatrix}$$

$$(1-\lambda)\left[(2-\lambda) * (1-\lambda) - 0\right] - 0 + 0 = 0$$
$$(1-\lambda)(2-\lambda(1-\lambda) = 0$$

Eigen value $\lambda = 1, 2, 1$

Two Eigen value are same and one is distinct so, if we get Algebraic Multiplicity equal to Geometric Multiplicity for each Eigen value then only Matrix is diagonalizable otherwise not.

Eigen value for $\lambda = 1$

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 2 & 2-1 & 1 \\ 3 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 + x_3 = 0$, $3x_1 = 0$

Let $x_2 = k_1$ then $x_3 = -k_1$

$$X_1 = \begin{bmatrix} 0 \\ k_1 \\ -k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Algebraic Multiplicity of $\lambda = 1$ is 2
Geometric Multiplicity of $\lambda = 1$ is 1

Since,

Algebraic Multiplicity $\neq$ Geometric multiplicity

then, the Matrix is not diagonalizable

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32. $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Characteristic polynomial

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 2 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

$\Rightarrow (1-\lambda)[\lambda^2 - 1] - 2[\lambda - 1] + 1[-1+\lambda) = 0$

$\lambda^2 - 1 - \lambda^3 + \cancel{\lambda} - 2\lambda + 2 - 1 + \cancel{\lambda} = 0$

$-\lambda^3 + \lambda^2 = 0$

$-\lambda^2(\lambda - 1) = 0$

$\lambda^2 = 0$

$\lambda = 0, 0$ and $(\lambda - 1) = 0$

$\lambda = 1$

Eigen value $\lambda = 0, 0, 1$

Eigen vector corresponding to Eigen value $\lambda = 0$

$$\begin{bmatrix} 1-0 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 + x_3 = 0$, $-x_1 + x_3 = 0$, $x_1 + x_2 = 0$

Let $x_1 = k_1$ then $x_3 = k_1$ and $x_2 = -k_1$

$X = k_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ Algebraic Multiplicity of $\lambda = 0$ is 2

Geometric Multiplicity of $\lambda = 0$ is 1

Since Algebraic multiplicity $\neq$ Geometric multiplicity then, the matrix is not diagonalizable.

33 Compute $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}^9$

Characteristic polynomial

$$\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{bmatrix}$$

$$(-4-\lambda)(5-\lambda) + 18 = 0$$
$$-20 + 4\lambda - 5\lambda + \lambda^2 + 18 = 0$$
$$\lambda^2 - \lambda - 2 = 0$$
$$\lambda^2 - (2\lambda - \lambda) - 2 = 0$$
$$\lambda^2 - 2\lambda + \lambda - 2 = 0$$
$$\lambda(\lambda - 2) + 1(\lambda - 2) = 0$$
$$(\lambda + 1)(\lambda - 2) = 0$$

Eigen value $\lambda = -1, 2$.

Eigen vector for $\lambda = -1$

$$\begin{bmatrix} -4+1 & 6 \\ -3 & 5+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 6x_2 = 0$, $-3x_1 + 6x_2 = 0$

Both Eqn are equivalent

Let $x_1 = k_1$ then, $-3k_1 + 6x_2 = 0$

$6x_2 = 3k_1$, $x_2 = \frac{k_1}{2}$

$X_1 = k_1 \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ or $X_1 = k_1' \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Eigen vector for $\lambda = 2$.

$$\begin{bmatrix} -4-2 & 6 \\ -3 & 5-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$-6x_1 + 6x_2 = 0$, $-3x_1 + 3x_2 = 0$

Both eq^n are equivalent, let $x_1 = k_2$ then $x_2 = k_2$

$X_2 = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$\det(P) = 2 - 1 = 1$

$\text{adj}P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$P^{-1} = \dfrac{\text{adj}P}{\det(P)} \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

We know,

$D = P^{-1} A P$

or $A = P D P^{-1}$

$A^m = P D^m P^{-1}$

$A^9 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^9 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 512 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2 & 512 \\ -1 & 512 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2 - 512 & 2 + 1028 \\ -1 - 512 & 1 + 1028 \end{bmatrix}$

$A^9 = \begin{bmatrix} -514 & 1030 \\ -513 & 1029 \end{bmatrix}$

Q. Compute $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{2018}$

Characteristic polynomial is given by $(A - \lambda I) = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \Rightarrow (1-\lambda)\left[(-1-\lambda)(1-\lambda)-0\right] - 0 + 0 = 0$$

$$(1-\lambda)(-1-\lambda)(1-\lambda) = 0$$

Eigen Value $\lambda = 1, -1, 1.$

Eigen vector for $\lambda = 1$

$$\begin{bmatrix} 1-1 & 1 & 1 \\ 0 & -1-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 + x_3 = 0$, $-2x_2 = 0$

$x_2 = 0$, $x_3 = 0$ let $x_1 = K_1$

$$X_1 = K_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector for $\lambda = -1$

$$\begin{bmatrix} 1-(-1) & 1 & 1 \\ 0 & -1-(-1) & 0 \\ 0 & 0 & 1-(-1) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 + x_3 = 0$

Let $x_2 = K_2$, $x_3 = K_3$

$$x_1 = \frac{-x_2 - x_3}{2} = \frac{-K_2 - K_3}{2}$$

$$X = \begin{bmatrix} \frac{-K_2 - K_3}{2} \\ K_2 \\ K_3 \end{bmatrix} = K_1 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

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Eigen vectos fos λ-1 is

$$x_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow X_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow X_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow Det(P) = 1(4-0)+(0)+0$$
$$= 4$$

$$adj\,P = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We know that $P^{-1} = \dfrac{Adj\,P}{Det\,P}$

$$P^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that

$$D = P^{-1}AP \quad \text{OR} \quad A = PDP^{-1}$$
$$A^m = PD^mP^{-1}$$

$$A^{2018} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2018} \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{2018} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{2018} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$A^{2018} = \frac{1}{2}\begin{bmatrix} 2 & 1-1 & 1-1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{2018} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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38. Let $V(\mathbb{F} = \mathbb{R}$ or $\mathbb{C})$ be a vector space over a field F then is said to be an Inner product space. If following properties hold for all vector u, v and w in V and all space c.

(i) $\langle u, v\rangle = \langle v, u\rangle$
(ii) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$
(iii) $\langle cu, v\rangle = c\langle u, v\rangle$
(iv) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$

A vector space with an inner product is called an inner product space.

39. $u = (a, b)$, $v = (c, d)$

Given, $\langle u, v\rangle = 2ac + 3bd = 2ca + 3db = \langle v, u\rangle$

(i) Let $w = (w_1, w_2)$
So, we have $v + w = \{(c + w_1), (d + w_2)\}$

$$\langle u, v+w\rangle = 2a(c+w_1) + 3b(d+w_2)$$
$$= 2ac + 2aw_1 + 3bd + 3bw_2$$
$$= (2ac + 3bd) + (2aw_1 + 3bw_2)$$
$$= \langle u, v\rangle + \langle u, w\rangle$$

First property is satisfied

(ii) For any scalar x

$$\langle xu, v\rangle = 2xac + 3xbd$$
$$= x(2ac + 3bd)$$
$$= x\langle u, v\rangle$$

Second property is also satisfied

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(iii) $\langle u, u\rangle = 2a^2 + 3b^2 \geq 0$

if $u = 0 \Rightarrow a = 0, b = 0$

$\langle u, u\rangle = 0^2 + 0^2 = 0$

Third property is also satisfied.

Since all properties are satisfied so, it defines an inner product.

40. $u = (a, b)$ and $v = (c, d)$

also given,

$$\langle u, v\rangle = (ac - bd)$$

$$\langle u, u\rangle = a \times a - b \times b$$

$$= a^2 - b^2 \geq 0$$

This will never satisfy the property

For ex- $a = 1, b = 3$

$$\langle u, u\rangle = a^2 - b^2$$

$$= 1 - 9$$

$$= -8 \geq 0$$

Hence $\langle u, v\rangle$ is not inner product.

41. The Euclidean distance of a vector from the origin is a norm, called the Euclidean Norm. It is also defined as the square root of the inner product of a vector with itself.

$$\text{Norm } \|V\| = \sqrt{\langle V, V\rangle}$$

42. An orthogonal set of vectors in a inner product space V is a set $\{V_1, V_2 \dots V_k\}$ of vector from V such that $\langle V_i, V_j\rangle = 0$ where $V_i \neq V_j$.

An orthogonal set of vectors is then an orthogonal set of unit vector.

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43. $v_1 = (2,1,-1)$, $v_2 = (0,1,1)$, $v_3 = (1,-1,1)$

We know that set $\{v_1, v_2 \dots v_k\}$ is orthogonal if $v_i v_j = 0$ where $v_i \neq v_j$

$v_1 v_2 = (2,1,-1)(0,1,1) = 0+1-1 = 0$
$v_2 v_3 = (0,1,1)(1,-1,1) = 0-1+1 = 0$
$v_3 v_1 = (1,-1,1)(2,1,-1) = 2-1-1 = 0$
Hence $V = \{v_1, v_2, v_3\}$ is orthogonal set.

44. An orthogonal basis for a subspace W of V is just a basis for W that is an orthogonal set.

45.
$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \right.$$

we have $x - y + 2z = 0$
let $z = k_1$, $y = k_2$ then $x = k_2 - 2k_1$

$$X = \begin{bmatrix} k_2 - 2k_1 \\ k_2 \\ k_1 \end{bmatrix}$$

Firstly let $k_1 = 0$ then secondly let $k_2 = 0$ then we have

$$X = k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Now we have, $v_1 = (1,1,0)$ and $v_2 = (-2,0,1)$

$w_1 = v_1 = (1,1,0)$

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$$W_2 = V_2 - \frac{\langle V_2, W_1 \rangle}{\|w_1\|^2} \cdot w_1$$

$$= (-2,0,1) - \frac{\{(-2,0,1)(1,1,0)\}}{(1^2+1^2+0^2)} \cdot (1,1,0)$$

$$= (-2,0,1) - \frac{(-\cancel{2})(1,1,0)}{\cancel{2}}$$

$$= (-2,0,1) + (1,1;0)$$

$$= (-1,1,1)$$

$W = \{(1,1,0), (-1,1,1)\}$ orthogonal bases.

46. A set of vector is orthonormal if it is an orthogonal set having the property that every vector is a unit vector (a vector of magnitude 1).

47. $q_1 = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $q_2 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$$q_1 \cdot q_2 = \left(\frac{+1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$= \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}}$$

$$= \frac{1}{\sqrt{18}} (\cancel{1} - \cancel{2} + \cancel{1}) = 0$$

Hence $\{q_1, q_2\}$ is orthogonal set.

Further.

$$|q_1| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{-1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{3}{3}} = 1$$

$$q_2 : \sqrt{\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{6}{6}} = 1$$

(q_1, q_2) are unit vector, Hence, $\{q_1, q_2\}$ is orthonormal set.

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48. An orthonormal basis for a subspace W of V is a basis for W that is an orthonormal set.

49. The orthogonal matrix is defined as a matrix for which its inverse is its transpose and its relation to the inner product is developed.

50.
$$A = \begin{bmatrix} 1/3 & 1/2 & 1/5 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.3 & -0.5 & 0.2 \\ -0.3 & 0 & 0.4 \end{bmatrix}$$

$AA^T = I$.

$$= \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.3 & -0.5 & 0.2 \\ -0.3 & 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.3 & 0.3 & -0.3 \\ 0.5 & -0.5 & 0 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9+2.5+0.4 & 0.9-2.5+1 & -0.9+0.8 \\ 0.9-2.5+0.4 & 0.9+2.5+1 & -0.9+0.8 \\ -0.9+0.8 & -0.9+2 & 0.9+1.6 \end{bmatrix}$$

$$= \begin{bmatrix} 3.8 & -0.6 & -0.1 \\ -1.2 & 4.4 & -0.1 \\ -0.1 & 1.1 & 2.5 \end{bmatrix} \neq I$$

A is not orthogonal.

51.

$$A = \begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

A is a orthogonal matrix if $AA^T = I$

$$\begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} [\cos^2\theta\sin^2\theta + \cos^2\theta + \sin^4\theta] & [\cos^3\theta\sin\theta - \cos\theta\sin\theta + \cos\theta\sin^3\theta] & [\cos\theta\cancel{\sin^2\theta} - \cancel{\sin^2\theta}\cos\theta] \\ [\cos^3\theta\sin\theta - \sin\theta\cos\theta + \cos\theta\sin^3\theta] & [\cos^4\theta + \sin^2\theta + \sin^2\theta\cos^2\theta] & [\sin\theta\cancel{\cos^2\theta} - \cancel{\cos^2\theta}\sin\theta] \\ [\cos\theta\sin^2\cancel{\theta} + 0 - \cos\theta\cancel{\sin^2\theta}] & [\cos^2\cancel{\theta}\sin\theta - \cos^2\cancel{\theta}\sin\theta] & [\sin^2\theta + \cos^2\theta] \end{bmatrix}$$

$$\begin{bmatrix} [\sin^2\theta(\cos^2\theta + \sin^2\theta) + \cos^2\theta] & [\cos\theta\sin\theta(\cos^2\theta + \sin^2\theta) - \cos\theta\sin\theta] & 0 \\ [\cos\theta\sin\theta(\cos^2\theta + \sin^2\theta) - \sin\theta\cos\theta] & [\cos^2\theta(\cos^2\theta + \sin^2\theta) + \sin^2\theta] & 0 \\ [\cos\theta\sin\theta(\cos^2\theta + \sin^2\theta) - \sin\theta\cos\theta & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \sin^2\theta + \cos^2\theta & \cos\theta\cancel{\sin\theta} - \cos\cancel{\theta}\sin\theta & 0 \\ \cos\theta\cancel{\sin\theta} - \sin\cancel{\theta\cos\theta} & \cos^2\theta + \sin^2\theta & 0 \\ \cos\theta\cancel{\sin\theta} - \sin\cancel{\theta\cos\theta} & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence A is orthogonal and $A^{-1} = A^T$

$$A^{-1} = \begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}$$

52. Orthographic projection is a means of representing 3-D objects in 2-D.

53. $W = \text{span}(x_1, x_2)$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$x_1 = (1,1,0)$ and $x_2 = (-2,0,1)$

$W_1 = x_1 = (1,1,0)$

$$W_2 = x_2 - \frac{\langle x_2, w_1 \rangle}{\|w_1\|^2} \cdot w_1$$

$$= (-2,0,1) - \frac{\{(-2,0,1)(1,1,0)\}}{(1^2 + 1^2 + 0)} \cdot (1,1,0)$$

$$= (-2,0,1) - \frac{(-2)}{2}(1,1,0)$$

$$= (-2,0,1) + (1,1,0)$$

$$= (-1,1,1)$$

$w = \{(1,1,0), (-1,1,1)\}$ is orthogonal basis

54. Any finite dimensional inner product space has orthognal basis $\beta = \{V_1, V_2, V_3 \ldots\ldots V_n\}$

$\dim V(F) = n$

Then there exists

$\beta' = \{\hat{W}_1, \hat{\omega}_2, \hat{\omega}_3 \ldots\ldots \hat{\omega}_n\}$ such that β' is orthonormal basis.

where $\omega_1 = v_1$

$$\omega_2 = V_2 - \frac{\langle V_2, W_1 \rangle}{\|\omega_1\|^2} \omega_1$$

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$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2}\cdot w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2}\cdot w_2$$

$$w_n = v_n - \frac{\langle v_n, w_1\rangle}{\|w_1\|^2}\cdot w_1 - \frac{\langle v_n, w_2\rangle}{\|w_2\|^2}\cdot w_2 - \frac{\langle v_n, w_3\rangle}{\|w_3\|^2}\cdot w_3 - \cdots$$

$$\cdots \frac{\langle v_n, w_n\rangle}{\|w_n\|^2}\cdot w_n$$

$$B' = \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_2}{\|w_3\|} \cdots \frac{|w_n|}{\|w_n\|} \right\}$$

55. $W = \mathrm{span}(x_1, x_2, x_3)$.

$x_1 = (1, -1, -1, 1)$, $x_2 = (2, 1, 0, 1)$, $x_3 = (2, 2, 1, 2)$

$w_1 = x_1 = (1, -1, -1, 1)$

$$w_2 = x_2 - \frac{\langle x_2, w_1\rangle}{\|w_1\|^2}\cdot w_1$$

$$= (2,1,0,1) - \frac{[(2,1,0,1)(1,-1,-1,1)]}{((1)^2 + (-1)^2 + (-1)^2 + 1^2)}\cdot(1,-1,-1,1)$$

$$= (2,1,0,1) - \frac{[2 - \cancel{1} + \cancel{1}]}{4}(1,-1,-1,1)$$

$$= (2,1,0,1) - \left(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}\right)$$

$$= \left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2}(3, 3, 1, 1)$$

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$$w_3 = x_3 - \frac{\langle x_3, w_1\rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle x_3, w_2\rangle}{\|w_2\|^2} \cdot w_2$$

$$= (2,2,1,2) - \frac{[(2,2,1,2)(1,-1,-1,1)]}{1^2+(-1)^2+(-1)^2+1^2} \cdot (1,-1,-1,1)$$

$$- \frac{[(2,2,1,2)(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})]}{(\frac{3}{2})^2+(\frac{3}{2})^2+(\frac{1}{2})^2+(\frac{1}{2})^2} \cdot \left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}\right)$$

$$= (2,2,1,2) - \frac{[2-2-1+2]}{4}(1,-1,-1,1) - \frac{(3+3+\frac{1}{2}+1)}{\frac{20 5}{4}}\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}\right)$$

$$= (2,2,1,2) - \left(\frac{1}{4},\frac{-1}{4},\frac{-1}{4},\frac{1}{4}\right) - \frac{\cancel{15}\,3}{2\times \cancel{5}}\left[\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]$$

$$= \left(\frac{7}{4},\frac{9}{4},\frac{5}{4},\frac{7}{4}\right) - \left(\frac{9}{4},\frac{9}{4},\frac{3}{4},\frac{3}{4}\right)$$

$$= \left(\frac{-2}{4}, 0, \frac{2}{4}, \frac{4}{4}\right)$$

$$w_3 = \left(-\frac{1}{2}, 0, \frac{1}{2}, 1\right)$$

56.

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Let $x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

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By Gram-Schmidt process)

$w_1 = x_1 = (1,2,3)$

$$w_2 = x_2 - \frac{\langle x_2, w_1\rangle}{\|w_1\|} \cdot w_1$$

$$= (0,1,0) - \frac{[(0,1,0)(1,2,3)]}{(1^2+2^2+3^2)}(1,2,3)$$

$$= (0,1,0) - \frac{[\cancel{2})}{\cancel{14}\ 7}(1,2,3)$$

$$= (0,1,0) - \left[\frac{1}{7}, \frac{2}{7}, \frac{3}{7}\right)$$

$$= \left(\frac{-1}{7}, \frac{5}{7}, \frac{-3}{7}\right)$$

$$= \frac{1}{7}(-1, 5, -3)$$

$$w_3 = x_3 - \frac{\langle x_3, w_1\rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle x_3, w_2\rangle}{\|w_2\|^2} \cdot w_2$$

$$\Rightarrow (0,0,1) - \frac{[(0,0,1)(1,2,3)] \cdot (1,2,3)]}{14}$$

$$- \frac{\left[(0,0,1)\left(\frac{-1}{7}, \frac{5}{7}, \frac{-3}{7}\right)\right]}{\frac{1+25+9}{49}} \times \frac{1}{7}(-1,5,-3)$$

$$\Rightarrow (0,0,1) - \frac{3}{14}(1,2,3) + \frac{3}{\cancel{7}} \times \frac{\cancel{49}\,7}{\cancel{35}\,5} \times \frac{1}{7}(-1,5,-3)$$

$$\Rightarrow (0,0,1) - \left(\frac{3}{14}, \frac{6}{14}, \frac{9}{14}\right) + \left(\frac{-3}{35}, \frac{15}{35}, \frac{-9}{35}\right)$$

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$$w_3 = \left(\frac{-3}{14}, \frac{-6}{14}, \frac{5}{14}\right) + \left(\frac{-3}{35}, \frac{15}{35}, \frac{-9}{35}\right)$$

$$\cancel{w_3 = \frac{(-105-42) + (-210+210)}{490}}$$

$$w_3 = \frac{(-105-42)}{490}, \frac{(-210+210)}{490}, \frac{(175-126)}{490}$$

$$w_3 = \frac{-147}{490}, 0, \frac{49}{490}$$

$$w_3 = \frac{-3}{10}, 0, \frac{1}{10}$$

57. $\lambda^3 - 4\lambda^2 + a\lambda + 30$

one of $\lambda = 2$. and factor is $(\lambda - 2)$

$(2)^3 - 4(2)^2 + 2a + 30 = 0$

$8 - 16 + 2a + 30 = 0$

$2a = -22$

$a = -11$

$\lambda^3 - 4\lambda^2 - 11\lambda + 30$

$(\lambda - 2)\ )\ \lambda^3 - 4\lambda^2 - 11\lambda + 30\ (\ \lambda^2 - 2\lambda - 15$

$\lambda^3 - 2\lambda^2$

$-\quad +$

$-2\lambda^2 - 11\lambda + 30$

$-2\lambda^2 + 4\lambda$

$+\quad -$

$-15\lambda + 30$

$-15\lambda + 30$

$+\quad -$

$\times$

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$$(\lambda-2)(\lambda^2-2\lambda-15)=0$$

$$(\lambda-2)(\lambda^2-(5\lambda-3\lambda)-15)=0$$

$$(\lambda-2)[\lambda^2-5\lambda+3\lambda-15)=0$$

$$(\lambda-2)[\lambda(\lambda-5)+3(\lambda-5)]=0$$

$$(\lambda-2)(\lambda-5)(\lambda+3)=0$$

Eigen value $\lambda = 2, 5, -3$

largest Eigen value is 5.

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60. Eigenvalues of A are : 1, 2, and 4

We know, A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots \frac{1}{\lambda_n}$

Eigen values of $A^{-1} = 1, \frac{1}{2}$ and $\frac{1}{4}$.

Any square matrix and its trasponse A' have the same Eigen value.

Eigen value of $(A^{-1})^T = 1, \frac{1}{2}$ and $\frac{1}{4}$.

We know that, Producet of Eigen value = Determinat

$1 \times \frac{1}{2} \times \frac{1}{4}$ = Determinant

$\boxed{\frac{1}{8} = \text{Determinat}}$

62. $A = \begin{bmatrix} 1 & 4 \\ b & a \end{bmatrix}$ Eigen value = -1, 7.

Trace of matrix = Sum of Eigen value

$= -1 + 7$

$= 6.$

We know that sum of element on principal diagonal of a matrix is equal to the trace of matrix

$1 + a = 6$

$a = 6 - 1$

$a = 5$

Also, we know that product of Eigen values is equal to the determinant of matrix, so,

$-1 \times 7 = a - 4b$

$-7 = 5 - 4b$

$4b = 7 + 5$

$b = \frac{\cancel{12}\,3}{\cancel{4}}$

$b = 3$

$a = 5$ and $b = 3.$