System of Linear Equations : (Linear System)

A linear system of m equations in n unknowns $x_1, \ldots, x_n$ is a set of equations of the form

$$\text{(1)} \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

The system is called linear because each variable x_j appears in the first power only, just as in the equations of a straight line. $a_{11}, a_{12}, a_{21}, \ldots, a_{mn}$ are given numbers, called the coefficients of the system. $b_1, \ldots, b_m$ on the right are also given numbers.

If all the b_j are zero, then linear system is called Homogeneous system.

If at least one b_j is not zero, then linear system is called Nonhomogeneous system.

A solution of (1) is a set of numbers $x_1, \ldots, x_n$ that satisfies all the m equations and written as column vector.

If the system (1) is homogeneous, it always has at least the trivial solution $x_1 = 0, \ldots, x_n = 0$.

Matrix form of Linear System : Form the definition of matrix multiplication, the m equations of (1) may be written as a single vector equation

$$AX = b$$

where the coefficient matrix $A = [a_{jk}]$ is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m \times n} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} ; \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

are column vectors.

The matrix $[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ is called the

Augmented matrix of the system ①.

The linear system ① is called consistent if ① has at least one solution. It is inconsistent if system ① has no solution at all.

Fundamental Theorem for linear Systems:

I. If rank of A and rank of $[A|b]$ are equal, then system is consistent.

ⓐ If rank A = rank $[A|b] = n$ (no. of variables)

then unique solution exists.

ⓑ If rank A = rank$[A|b] < n$ (no. of variables)

then infinitely many solutions exist in terms of $(n-r)$ arbitrary constants (Parameters), where r is rank of A.

II. If rank of A and rank of $[A|b]$ are not equal then the system is inconsistent and has no solution at all.

Solutions of Linear System by Gauss Elimination Method:

Consider the augmented matrix $[A|b]$. Apply elementary row operations on $[A|b]$ such that A reduces to Echelon form.

Find rank of A and rank of $[A|b]$ by Echelon form.

By Fundamental Thm for linear system,

if system has unique soln then take back substitution, to find unknowns $x_1, \ldots, x_n$.

if system has infinite soln then write some unknowns into $n-r$ unknowns.

Ex. Determine whether the given system of linear equations is consistent. If consistent, find solution.

$$2x_2 + 3x_3 = 8$$
$$2x_1 + 3x_2 + x_3 = 5$$
$$x_1 - x_2 - 2x_3 = -5$$

Soln:- The augmented matrix is $[A|b] = \left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right]$

$R_1 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right]$ $R_2 \to R_2 - 2R_1$ $\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right]$

$R_2 \to \frac{1}{5} R_2$ $\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}\right]$ $R_3 \to R_3 - 2R_2$ $\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Echelon form

So,
rank $A = 3 =$ rank $[A|b]$
and no. of unknowns are 3.

$\therefore$ By Fundamental thm, This system has unique solution.

So, by back substitution, (first write echelon form into system)

$$x_1 - x_2 - 2x_3 = -5 \quad \text{—— (i)}$$
$$x_2 + x_3 = 3 \quad \text{—— (ii)}$$
$$x_3 = 2 \quad \text{—— (iii)}$$

from (ii) & (iii), $x_2 = 3 - x_3 = 3 - 2 = 1$

from (i) & $x_1 = -5 + x_2 + 2x_3 = -5 + 1 + 2 \times 2 = 0$

$\therefore$ The solution is $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Ex. Solve $x_1+x_2-x_3=0$, $2x_1-x_2+x_3=3$, $4x_1+2x_2-2x_3=2$

Soln:- The augmented matrix is $[A|b]=\left[\begin{array}{ccc|c}1&1&-1&0\\2&-1&1&3\\4&2&-2&2\end{array}\right]$

Apply row operations to convert into Echelon form

$R_2\to R_2-2R_1$, $R_3\to R_3-4R_1$ $\left[\begin{array}{ccc|c}1&1&-1&0\\0&-3&3&3\\0&-2&2&2\end{array}\right]$ $R_2\to(-\frac{1}{3})R_2$ $\left[\begin{array}{ccc|c}1&1&-1&0\\0&1&-1&-1\\0&-2&2&2\end{array}\right]$

$R_3\to R_3+2R_2$ $\left[\begin{array}{ccc|c}1&1&-1&0\\0&1&-1&-1\\0&0&0&0\end{array}\right]$

$\therefore$ rank $A=2$ & rank $[A|b]=2$

Both are equal but less than no. of unknowns 3.

$\therefore$ By Funda. Thm., this system has infinite solns.

Now, write reduced augmented form into unknowns form -

$$x_1+x_2-x_3=0\ ;\ x_2-x_3=-1$$

$\therefore$ $x_2=x_3-1$ & $x_1=-x_2+x_3=-x_3+1+x_3=1$

Thus, the solutions are $\begin{bmatrix}1\\x_3-1\\x_3\end{bmatrix}$ or $\begin{bmatrix}1\\k-1\\k\end{bmatrix}$ where k is arbitrary constants.

Ex. Solve $x_1-x_2+2x_3=3$; $x_1+2x_2-x_3=-3$; $2x_2-2x_3=1$

The augmented matrix is $[A|b]=\left[\begin{array}{ccc|c}1&-1&2&3\\1&2&-1&-3\\0&2&-2&1\end{array}\right]$

R_2-R_1 $\left[\begin{array}{ccc|c}1&-1&2&3\\0&3&-3&-6\\0&2&-2&1\end{array}\right]$ $R_2\to\frac{1}{3}R_2$ $\left[\begin{array}{ccc|c}1&-1&2&3\\0&1&-1&-2\\0&2&-2&1\end{array}\right]$

$R_3\to R_3-2R_2$ $\left[\begin{array}{ccc|c}1&-1&2&3\\0&1&-1&-2\\0&0&0&5\end{array}\right]$

Rank $A=2$ & Rank $[A|b]=3$. Both are unequal. So system is inconsistent.

Solution of linear system by Cramer's rule :

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

If the determinant of coefficient matrix A is non-zero, then

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

where D is the determinant of A and

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{31} \\ a_{21} & b_2 & a_{32} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{21} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{vmatrix}$$

[Note:- D_i is the determinant obtained from D by replacing the i^{th} column in D by constant column vector b.]

(52) Ex. Solve $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$.

First find determinant of the coefficient matrix

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$
$$= 1(1+3) + 1(2+3) - 1(2-1)$$
$$= 4 + 5 + 1 = 10 \neq 0$$

$$D_1 = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 4(1+3) + 2(3-$$
$$= 16 + 4 = 20.$$

$$D_2 = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} - (3)\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = -2(4-2) + 3(2-4)$$
$$= -4 - 6 = -10$$

$$D_3 = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 2 & +1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 4(2-1) + 2(1+2)$$
$$= 4 + 6 = 10$$

$\therefore x_1 = D_1/D = 2, \; x_2 = D_2/D = -1, \; x_3 = 1$