QB-2

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① Let $(F, +, \cdot)$ be a field. The element of F will be scaler, V be non empty set whose elements will be called vectors. The V is a vector space over the field F

(i) $= u + v \in V$
(ii) $u + v = v + u$
(iii) $(u+v) \cdot w = u(v+w)$
(iv) $\exists$ zero vector in V s.t $u+0 = 0+u = u$
(v) $u + (-u) = 0 \Rightarrow (-u) + u) = 0$

$(V, +)$ if satisfy all property known as albelion group
- rational, real, integer, complex

(vi) $cu \in V$
(vii) $c(u+v)$
(viii) $(c+d)u = (cd)u$
(ix) $c(du) = (cd)u$
(x) $1u = u \; (1 \in F)$

$\Rightarrow M(\mathbb{R}) = \left\{ \begin{bmatrix} 1 & a \\ b & c \end{bmatrix} \; n, b, c \in \mathbb{R} \right\}$

e.g $\Rightarrow \mathbb{R}^2(\mathbb{R})$
$u, (a_1, a_2) \quad v, (b_1, b_2) \quad w(c_1, c_2)$
$u + v = (a_1, a_2) + (b_1, b_2)$
$(a_1 + b_1, a_2 + b_2)$
$\because a_1, b_1 \in \mathbb{R}$
$\Rightarrow a_1 + b_1 \in \mathbb{R}$
$a_2 + b_2 \in \mathbb{R}$
$\therefore (a_1 + b_1, a_2 + b_2) = \mathbb{R}$

② $V = \mathbb{R}$ Addition $(a, b) + (c, d) = (a+c, b+d)$
multiplication: $K(a, b) = (K \{a, b)$?

$\Rightarrow$ Addition $\Rightarrow$ $u + v = (a, b) + (c, d)$
$(a+c, b+d)$
$\therefore a + c \in \mathbb{R}$
$b + d \in \mathbb{R}$
$\therefore (a + c + b + d) = \mathbb{R}$

(19) $c\sin x + d\cos x = \sin 2x$ and try to determine c and d so that this eq is true. Since they are function the eq must be true for all values of x. setting $x=0$, we have

$$c\sin 0 + d\cos 0 = \sin 0 \quad c(0) + d(1) = 0$$

$$d = 0$$

$$\sin x = \frac{\pi}{2}$$

$$c\sin(\pi/2) + d\cos(\pi/2) = \sin 2(\pi/2)$$

$$c(1) + d(0) = c = 0$$

$\Rightarrow \sin 2x = 0(\sin x) + 0(\cos x) = 0$

since $2x$ is not the zero function, so $\sin 2x$ is not in span.

20) $r(x) = 1 - 4x + 6x^2$ $\quad p(x) = 1 - x + x^2$ $\quad q(x) = 2 + x - 3x^2$

$$c_1(px) + c_2(qx) = r(x)$$

$$c_1(1 - x + x^2) + c_2(2 + x - 3x^2) = 1 - 4x + 6x^2$$

$$c_1 - c_1x + c_1x^2 + 2c_2 + c_2x - c_2 3x^2 = 1 - 4x + 6x^2$$

$$c_1 + 2c_2 = 1$$

$$(c_2 - c_1)x = -4x$$

$$c_2 - c_1 = -4$$

$-1 + 4 = c_1 \Rightarrow \boxed{c_1 = 3}$ $\quad \boxed{c_2 + 4 = c_1}$

$c_2 + 4 + 2c_2 = 1$

$3c_2 = -3$ $\quad \boxed{c_2 = -1}$

27) $B = \{1+x, x+x^2, 1+x^2\}$ (b)

$$c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = a + bx + cx^2$$

$$(c_1 + c_3) + (c_2 + c_1)x + (c_2 + c_3)x^2 = a + bx + cx^2$$

$c_1 + c_3 = a$

$c_2 + c_1 = b$

$c_2 + c_3 = c$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

rank = 2

so c_1, c_2, c_3 has non zero values so B spans $\mathbb{R}$

(26) ⇒ A subset B of a vector space V is a basis of for V if (6)

1. B space V and
2. B is linearly independent.

Dimen → no of vectors in the basis of vector space

(28) (6)

$C_1(1+x) + C_2(x+x^2) + C_3(1+x^3) = 1-4x+6x^2$

$C_1 + C_3 = 1$, $(C_1+C_2) = -4$, $C_2 + C_3 = 6$

$$\begin{bmatrix} 1 & 0 & 6 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -4 \\ 0 & 1 & 1 & 6 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 6 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -4 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 2 & 11 \end{array}\right]$$ unique solⁿ

$C_3 = \frac{11}{2}$, $C_2 = \frac{5}{2} + \frac{11}{2}$, $C_1 = -4 - \frac{1}{2}$

$$\begin{bmatrix} -9/2 \\ 1/2 \\ 11/2 \end{bmatrix}$$ Ans

(29) (6)

$[p(x)] = 2 - 3x + 5x^2$ $\{1, x, x^2\}$

$p(x) = (2)1 + (-3)x + (5)x^2$ is already a linear combination so co-ordinate vector are

$$[p(x)]_B = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$ Ans

(30) (6) $W_3 = \{(x_1, x_2, x_3, x_4, x_5) : 3x_1 + x_2 + x_3 = 0\}$

$x_2 = 3x_1 + x_2$

$(x_1 + 3x_1 + x_2 + x_3 + x_4 + x_5)$

⇒ $x_1(1,3,0,0,0) + x_3(0,1,1,0,0)$ ~~$+ x_4(0) + x_5()$~~

$x_4(0,0,0,1,0) + x_5(0,0,0,0,1)$ Ans

Dim = 4

(29) $F(x, y) = \{x, 2x-y, 3x+4y)$

$Ker(T) = \{x, y \mid T(x, y) = 0\}$

$= T(x, y) = 0$

$x, 2x-y, 3x+4y = (0, 0, 0)$

$x = 0 \quad y = 0$

$(0, 0)$

$Ker(T) = \{0, 0\}$

$Nullity(T) = 0$

$Rank = dim(\mathbb{R}^2) - Nullity$

$= 2$ Ans

$Range = \{(x, 2x-y, 3x+4y\}$

$\Rightarrow \{x(1, 2, 3) + y(0, -1, 4)\}$

$RS = \{(1, 2, 3)\ (0, -1, 4)\}$ Ans

(31) $V = \{(x, y, z) \in \mathbb{R}^3 ;\ det(A) = 0\}$: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ x & y & z \end{bmatrix}$

$\Rightarrow 1(2z-3y) - 1(2z-3x) + 1(2y-2x) = 0$

$\Rightarrow \cancel{2z} - 3y - \cancel{2z} + 3x + 2y - 2x = 0$

$x - y = 0 \Rightarrow x = y$

$V = \{(x, x, z)\} = \{x(1, 1, 0), z(0, 0, 1)\}$

$Dim = 2$ Ans

(32) $= V = \{(x, y, z, w) \in \mathbb{R}^4 ;\ x+y-z = 0,\ y+z+w = 0,\ 2x+y-3z-w = 0\}$

$\Rightarrow z = x+y$

$w = -x-2y$

$V = \{(x, y, x+y, -x-2y)\}$

$\{x(1, 0, 1, -1) ;\ y(0, 1, 1, -2)\}$

~~Dim~~

basis of $V = \{(1, 0, 1, -1)\ (0, 1, 1, -2)\}$

(27) $a(1+x, x+x^2, 1+x^2) + b(1+x, x+x^2, 1+x^2)$

$\cancel{\Rightarrow a + ax + ax + ax^2 + a + ax^2 + b + x}$

$a+b + ax+bx + ax + bx + ax^2 + bx^2 + a+b + ax^2 + bx$

$= 2a + 2b + 2ax + 2bx + 2ax^2 + 2bx^2$

$\Rightarrow x^2(2a+2b) + x(2a+2b) + (2a+2b) \in p_2(x)$

(2) $(\alpha_1 . \alpha_2)(a,b) = ((\alpha_1 + \alpha_2) a, b)$

$= (\alpha_1 a, \alpha_2 a, b)$ — (i)

$(\alpha_1 + \alpha_2)(a,b) = [\alpha_1(a,b) + \alpha_2(a,b)]$

$= [(a\alpha_1 + b\alpha_1) + (a\alpha_2 + b\alpha_2)]$

$= [(a\alpha_1 + a\alpha_2) + (b\alpha_1 + b\alpha_2)]$ — (ii)

$\because$ (i) & (ii) are not same

$\therefore$ V is not a vector space

(3) $a = x^n + x^{n-1} + x^{n-2} + --- + 1$

$b = y^n + y^{n-1} + y^{n-2} + --- + 1$

$a+b (x^n + y^n) + (x^{n-1} + y^{n-1}) + (x^{n-2} + y^{n-2}) + --- + 2$

$\Rightarrow C.a = Cx^n + Cx^{n-1} + Cx^{n-2} + --- + C$

$\Rightarrow$ P is a vector space

(5) $A = (x, y)$ $\quad B = (a, b)$

$\therefore A + B = (x + a, y + b) \in R^2$

$C.A = (Cx. Cy) \in R^2$ $\quad C \in R^3$

$\Rightarrow R^2 \longrightarrow R^3$ is a subspace

(6) $\alpha = (a_1, b_1, 0)$ $\quad c \in R$

$\beta = (a_2, b_2, 0)$

$\alpha + \beta = (a_1 + a_2, b_1 + b_2, 0) \in \omega$

$c\alpha = (ca_1, cb_1, 0) \in \omega$

$\therefore \omega$ is a subspace of V

(0) $\Rightarrow \alpha = \begin{bmatrix} a_1 & a_1 + 1 \\ 0 & b_1 \end{bmatrix}$ $\quad c \in R$

$\beta = \begin{bmatrix} a_2 & a_2 + 1 \\ 0 & b_2 \end{bmatrix}$

$\therefore \alpha + \beta = \begin{bmatrix} a_1 + a_2 & a_1 + a_2 + 2 \\ 0 & b_1 + b_2 \end{bmatrix} \in \omega$

$c\alpha = \begin{bmatrix} ca_1 & ca_1 + c \\ 0 & cb_1 \end{bmatrix}$ $\quad \therefore \omega$ is a subspac of A

(14) $a(1,-3,2) + b(2,-4,-1) + c(1,-5,7) \neq (2,-5,3)$

$\Rightarrow a + 2b + c = 2$ $\quad 5b - 5c = 1$

$\Rightarrow -3a - 4b - 5c = -5$ $\quad 2b - 2c = 1$

$\Rightarrow 2a - b + 7c = 3$

$\Rightarrow$ V isn't a linear combination

(15) $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 7R_1$ $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 10 & -6 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow V_3$ is lin. dependent on V_1 & V_2

(18) Let $c \in R$

$cx^2 + cx + c \in P_2(x)$ $\therefore 1, x, x^2$ spans $P_2(x)$

(20) $\begin{bmatrix} 6 & -4 & 1 \\ 1 & -1 & 1 \\ -3 & 1 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_2$, $R_2 \to R_2 - 6R_1$, $R_3 \to R_3 + 3R_1$ $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -5 \\ 0 & -2 & 5 \end{bmatrix}$ $R_3 \to R_3 + R_2$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore p(x), q(x)$ spans

(21) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ $R_3 \to R_3 - R_1$, $R_3 \to R_3 + R_2$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $\therefore$ set is LD L.I

(23) $\alpha \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \neq \gamma \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

$\alpha + \beta = 2\gamma$ $\quad \therefore \alpha + \beta = \gamma$

$\alpha - \beta = 0$

$\beta = \gamma$ $\quad$ not ~~LD~~ L.I

$\alpha = \gamma$

(34) $P(x+y) = P((a+b)+(c,d))$

$P((a+c)+(b+d))$

$\Rightarrow (a+c, 2a+2c-b-d, 3a+3c, 4b+4d)$

$T(x)+T(y) \Rightarrow T(a,b) + T(c,d)$

$\Rightarrow (a, 2a-b, 3a+4b) + (c, 2c-d, 3c+4d)$

$\Rightarrow (a+c, 2a+2c-b-d, 3a+3c+4b+4d)$

$\Rightarrow$

$T(\alpha x) \Rightarrow T(\alpha a, \alpha b)$

$\Rightarrow \alpha a, 2\alpha a - \alpha b, 3\alpha a + 4\alpha b$

$\Rightarrow \alpha(a, 2a-b, 3a+4b)$

$\Rightarrow \alpha T(a,b)$

$= \alpha T(x)$

(36) $\Rightarrow Ker(T) = \{(x,y,z) \mid h(x,y,z) = 2x+z + 3x-z, x+y+z)\}$

$\Rightarrow 2x+z=0 \quad 3x-z=0 \quad x+y+z=0$

$\Rightarrow x=0, y=0 \quad z=0$

$Ker(T) = \{(0,0,0)\}$

null space $= 1$

(37) $Ker(T) \Rightarrow x=0, y=0, z=0$

$Ker(T) = \{(0,0,0)\}$

null space $= 1$

$Range(T) = \{x(1,0,0), y(0,4,0), z(0,0,-1)\}$

$\{(1,0,0)\ (0,4,0)\ (0,0,-1)\}$

Rank $= 3$ Ans

(39) $Ker(T) = x=0 \quad \Rightarrow x=0 \quad y=0$

$x+y=0$

$y=0$

$Ker(T) = \{(0,0)\}$

Nullity $= 0$ Ans

(40) Ker (T) $\Rightarrow x + z = 0$

$2x + y + 3z = 0 \qquad x = z,\ y = -z,\ z = z,\ \omega = 0$

$2y + 2z = 0$

$\omega = 0$

$\therefore$ Kernal (T) = $\{(-z, -z, z, 0)\}$

null space = $\{(-z, -z, z, 0\}$

Range (T) = $\{x + z, 2x + y + 3z + 2y + 2z \cancel{+\omega}\}$

$\Rightarrow \{x(1, 2, 0, 0), y(0, 1, 2, 0), z(1, 3, 2, 0)$

$\omega(0, 0, 0, 0)\}$

Rank $\Rightarrow$ 3 Ans