Linear Algebra

Q.B-3

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1) $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$

$\left| \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0 \Rightarrow \left| \begin{bmatrix} -\lambda & 3 \\ 3 & -\lambda \end{bmatrix} \right| = 0$

$\begin{vmatrix} -\lambda & 3 \\ 3 & -\lambda \end{vmatrix} = 0 \xrightarrow{\quad} |\lambda^2 - 9| = 0$
$- (i)$

$\lambda = \pm 3$

Eigen values of A are 3 & -3.

For $\lambda = 3$ (also same process to calculate value of V for $\lambda = -3$)

$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -3V_1 + 3V_2 \\ 3V_1 - 3V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-3V_1 + 3V_2 = 0 \Rightarrow V_1 = V_2$

$3V_1 - 3V_2 = 0 \quad V_1 = V_2$

$\therefore V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Hence, V is an eigenvector of A

2) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & -2 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)[(1-\lambda)^2 - 0] = 0$

$(3-\lambda)(1-\lambda)(1-\lambda) = 0$

$\lambda = 3, 1, 1$

For $\lambda = 1$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2V_1 = 0$$
$$-2V_3 = 0$$
$$V_1 = 0$$

$$V = \begin{bmatrix} 0 \\ V_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$

$$(A - \lambda I)V = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2V_2 - 2V_3 = 0 \Rightarrow V_2 = -V_3$$
$$V_1 - 2V_3 = 0 \Rightarrow V_1 = 2V_3$$

$$\therefore V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 2V_3 \\ -V_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Thus $(2, -1, 1)$ is an eigen vector corresponding to $\lambda = 3$

3) $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$ ⇒ $(2-\lambda)(-1-\lambda) - 4 = 0$

$\Rightarrow -2 - 2\lambda + \lambda + \lambda^2 - 4 = 0$

$\lambda^2 - \lambda - 6 = 0$

$\lambda^2 - 3\lambda + 2\lambda - 6 = 0$

$\lambda(\lambda - 3) + 2(\lambda - 3) = 0$

$(\lambda + 2)(\lambda - 3) = 0$

$\lambda = -2, 3$

Eigenvectors using $\lambda = 3$

$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-V_1 + 2V_2 = 0$

$2V_1 - 4V_2 = 0$

$V_1 = 2V_2$

$V_1 = 2V_2$

$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ 2V_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

4) $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

$|A - I\lambda| = 0$

$\begin{vmatrix} 1-\lambda & 0 & 2 \\ -1 & 1-\lambda & 1 \\ 2 & 0 & 1-\lambda \end{vmatrix} = 0$ $\Rightarrow (1-\lambda)(1-\lambda)(1-\lambda) + 2(2\lambda - 2)$

$(1-\lambda)[(1-\lambda)(1-\lambda) - 4] = 0$

$(1-\lambda)(\lambda^2 - 2\lambda - 3) = 0$

$(1-\lambda)(\lambda+1)(\lambda-3) = 0$

$\lambda = 1, -1, 3$

Eigen vectors using $\lambda = -1$

$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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$$2V_1 + 2V_3 = 0 \quad \text{--- (i)}$$

$$-V_1 + 2V_2 + V_3 = 0 \quad \text{--- (ii)}$$

$$2V_1 + 2V_3 = 0 \quad \text{--- (iii)}$$

$$V_1 = V_3$$

from (ii) $\Rightarrow -V_1 + 2V_2 + V_3 = 0$

$$\cancel{-V_1} + 2V_2 + \cancel{V_1} = 0$$

$$V_2 = 0$$

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

5) A characteristics polynomial of a square matrix is defined as a polynomial that contains the eigenvalues as roots and is invariant under matrix similarity.

6) The algebraic multiplicity of an eigenvalue e is the power to which $(\lambda - e)$ divides the characteristics polynomial. The geometric multiplicity of an eigenvalue is the number of linearly independent eigenvectors associated with it.

7) $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

$$|A - I\lambda| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = 0 \quad \Rightarrow (1-\lambda)(6-\lambda) + 6 = 0$$

$$\cancel{\Rightarrow 6 - \lambda - 6\lambda + \lambda^2 = 0}$$

$$\cancel{\Rightarrow \lambda^2 - 7\lambda + 6 = 0}$$

$$\cancel{\lambda^2 - 6\lambda - \lambda + 6 = 0}$$

$$\Rightarrow 6 - \lambda - 6\lambda + 6 + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 12 = 0$$

$$\lambda^2 - 4\lambda - 3\lambda + 12 = 0$$

$$(\lambda - 4)(\lambda - 3) = 0$$

$$\lambda = 4 \;\&\; 3$$

(a) characteristics polynomial is $(\lambda-4)(\lambda-3)=0$

(b) $\therefore$ eigen value of A is 4 and 3

(c) Basic

for $\lambda=4$, $\begin{bmatrix}-3 & 3\\ -2 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=0 \Rightarrow -3v_1+3v_2=0 \Rightarrow v_1=v_2$ — (i)

$v=\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}v_1\\ v_1\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$

for $\lambda=3$

$\begin{bmatrix}-2 & 3\\ -2 & 3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=0 \Rightarrow -2v_1+3v_2=0 \Rightarrow v_1=\frac{3}{2}v_2$ — (ii)

$v=\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}\frac{3}{2}v_2\\ v_2\end{bmatrix}=\begin{bmatrix}3/2\\ 1\end{bmatrix}=\begin{bmatrix}3\\ 2\end{bmatrix}$

$\therefore$ Basics $=(1,1)$ & $(3,2)$

(d) A.M of $\lambda=4$ is 1

A.M of $\lambda=3$ is 1

G.M

For $\lambda=4$

$[A-4I]=\begin{bmatrix}-3 & 3\\ -2 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}-1 & 1\\ -2 & 2\end{bmatrix}$ $(R_1 \to R_1-R_2)$

$\Rightarrow \begin{bmatrix}0 & 0\\ -2 & 2\end{bmatrix}$ $(R_1 \to 2R_1-R_2)$

$\rho(A-4I)=1$

nulity $=1$

$\therefore$ G.M $=n(A-4I)=n-\rho(A-4I)=2-1=1$

A.M $=$ G.M $=1$

For $\lambda=3$

$[A-3I]=\begin{bmatrix}-2 & 3\\ -2 & 3\end{bmatrix}(R_1 \to R_1-R_2)=\begin{bmatrix}0 & 0\\ -2 & 3\end{bmatrix}$ $\therefore \rho(A-3I)=1$

nulity $=1$

G.M $=n(A-3I)=n-\rho(A-3I)=2-1=1$

A.M $=$ G.M $=1$

8) $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

$|A - I\lambda| = 0$

$\begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(2-\lambda) + 1 = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1, 1$

a) characteristics polynomial is $(\lambda - 1) = 0$

b) eigen value ~~of~~ is 1

~~c) $[A - \lambda I][=[0$~~

c) $[A - \lambda I][V] = [0] \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow V_1 + V_2 = 0$

$\Rightarrow V_1 = -V_2$

$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$ Basis $= (1, -1)$

d) A.M of $\lambda = 1$ is 2

For $\lambda = 1$, G.M

$(A - I) \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} (R_1 \to R_1 + R_2) \Rightarrow \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$ rank = 1

$\rho(A - I) = 2$

$G.M = n(A - I) = n - \rho(A - I) = 2 - 1 = 1$ Ans

9) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda)(3-\lambda) = 0$

$\lambda = 1, -2 \;\&\; 3$

a) characteristics polynomial is $(1-\lambda)(-2-\lambda)(3-\lambda) = 0$

b) eigen value is 1

(c) Basics $\rightarrow$ Similar way as we calculate in

Q-no- 7 & 8.

(d) A.M of $\lambda = 1$ is 1
A.M of $\lambda = -2$ is 1
A.M of $\lambda = 3$ is 1

* G.M
For $\lambda = 1$

$$[A - I] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{array}{l} \text{rank} = \rho(A-I) = 3 \\ G.M = n - \rho(A-I) = 3-3=0 \end{array}$$

For $\lambda = -2$
~~[A+2I]~~

$$[A + 2I] \Rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow \begin{array}{l} \text{rank} = \rho(A-I) = 2 \\ G.M = n - \rho(A+2I) = 3-2 = 1 \end{array}$$

For $\lambda = 3$

$$[A - 3I] = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} \text{rank} = \rho(A-I) = 2 \\ G.M = n - \rho(A-3I) = 3-2 = 1 \end{array}$$

* ~~So~~ 10, 11 & 12 questions are same as 7, 8 & 9 solve by yourselves.

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17) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 1 = 0$

$-1-\cancel{\lambda}+\cancel{\lambda}+\lambda^2-1=0$

$\lambda^2 = 2,\ \lambda = \pm\sqrt{2}$

$\therefore$ The eigen values of A^{19} are $(\sqrt{19})^{19}$ & $(-\sqrt{2})^{19}$

$\left.\begin{matrix}(\sqrt{2})^{19} = 512\sqrt{2} \\ (-\sqrt{2})^{19} = -512\sqrt{2}\end{matrix}\right]$ Ans

18) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$

$|A-\lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 7 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda)(3-\lambda) = 0$

$\lambda = 1, 4$ & 3

$\therefore$ Hence, eigen values are $1, 4$ & 3. Ans

19) $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$

$|A-\lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 3 \\ x & y-\lambda \end{vmatrix} = 0$

$(2-\lambda)(y-\lambda) - 3x = 0$ — (i)

given eigen values, $\lambda = 4$ & 8, we get from eq (i)

$(i) \xrightarrow{4}$ $-2(y-4)-3x=0$

$-3x-2y+8=0$ — (ii)

$(i) \xrightarrow{8}$ $-6(y-8)-3x=0$

$-3x-6y+48=0$ — (iii)

Solving eq (ii) & (iii), we get

~~x = 10~~ $x = -4$, & $y = 10$ Ans

22) $A = \begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$ $(R_1 \to R_1 + R_3)$

$$\cancel{|A - \lambda I| = 0}$$

$$\cancel{\begin{vmatrix} 3-\lambda & 5 & 2 \\ 5 & 12-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{vmatrix} = 0}$$

$$\begin{bmatrix} 5 & 12 & 7 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$

Two rows are same, so $|A| = 0$
From the properties of eigen values, product of eigen values is equal to determinant.
Hence, product of eigen values is zero. That implies one of the eigen values is zero

Hence, minimum value of eigen value is zero

24) **Similar matrix** – Similar matrix represents the same linear map under two (possibly) different bases with P being the change of basis matrix a transformation $A \longleftrightarrow P^{-1}AP$ is called similar matrix transformation

25) $A = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 \\ -4 & 6-\lambda \end{vmatrix} = 0$

$(2-\lambda)(6-\lambda) + 4 = 0$

$12 - 2\lambda - 6\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0$

$\lambda = 4 \,\&\, 4$

$|B - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & -1 \\ -5 & 7-\lambda \end{vmatrix} = 0$

$(3-\lambda)(7-\lambda) - 5 = 0$

$21 - 3\lambda - 7\lambda + \lambda^2 - 5 = 0$

$\lambda^2 - 10\lambda + 16 = 0$

$(\lambda - 2)(\lambda - 8) = 0$

$\lambda = 2 \,\&\, 8$

$\therefore$ eigen values of $[A] \neq [B]$

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We know that if eigenvectors of two matrix is not equal than matrix are not similar.

26) Similar as 25

27) $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)^2 - 4 = 0 \Rightarrow (5-\lambda)^2 = 4$

$\Rightarrow 5 - \lambda = \pm 2$

$\Rightarrow \lambda = \cancel{9, 1}\ 7 \,\&\, 3$

A has different eigen value hence matrix A is diagonisable.

$D = P^{-1}AP$

Let $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$

For $\lambda = 3$

$[A - 3I][V] = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \Rightarrow 2V_1 + 2V_2 = 0 \Rightarrow V_2 = -V_1$

$V(\text{at } \lambda = 3) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 7$

$[A - 7I][V] = 0$

$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2V_1 + 2V_2 = 0 \Rightarrow V_1 = V_2$

$V(\text{at } \lambda = 7) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore P = [V_{\lambda=3} \quad V_{\lambda=7}]$

$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$AP = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -3 & 7 \end{bmatrix}$

$PD = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -3 & 7 \end{bmatrix}$

$\therefore AP = PD \Rightarrow P^{-1}AP = D$

proved

28) $A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (-3-\lambda)(1-\lambda) + 4 = 0$

$-3 + 3\lambda - \lambda + \lambda^2 + 4 = 0$

$\Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)(\lambda+1) = 0, \lambda = -1, -1$

eigen value are not distinct Hence, matrix A is not diagonisable.

29) Same as Q. no-27

~~30) Same as Q. no-27~~

31) Same as Q. no-27

32) $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2 - 1) - 2(\lambda - 1) + 1(-1+\lambda) = 0$

$\lambda^2 \cancel{-1} - \lambda^3 + \cancel{\lambda} - \cancel{2\lambda} + \cancel{2} \cancel{-1} + \cancel{\lambda}$

$\lambda^2 - \lambda^3 = 0$

$\lambda^2(1-\lambda)^2 = 0$

$\lambda = 0, 0 \ \& \ 1$

→ eigen value are not distinct. Hence, matrix A is not diagonisable.

30) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

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$(1-\lambda)[-\lambda(1-\lambda)-1] + 1(\lambda-1) = 0$

$(1-\lambda)(-\lambda+\lambda^2-2) = 0$

$(1-\lambda)(\lambda^2-\lambda-2) = 0$

$(1-\lambda)(\lambda^2-2\lambda+\lambda-2) = 0$

$(1-\lambda)(\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda = 1, -1 \& 2$

Since eigen values are different.,
Hence, matrix A is diagonisable. , $D = P^{-1}AP$

Let $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

For $\lambda_1 = 1$

$[A-I][V] = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & \cancel{0} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$V_3 = 0$, $V_1 + V_2 = 0$, $V_1 = -V_2$

$\therefore$

$V_{\lambda=1} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda = -1$

$[A+I][V] = 0 \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$

$2V_1 + V_3 = 0$
$V_3 = -2V_1$
$2V_2 + V_3 = 0 \Rightarrow V_3 = -2V_2$
$V_1 + V_2 + V_3 = 0$, $V_2 = -V_1 + 2V_1$
$= V_1$

$V_{\lambda=-1} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_1 \\ -2V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 2$

$[A-2I][V] = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$

$-V_1 + V_3 = 0 \Rightarrow V_1 = V_3$
$-V_2 = V_3 = 0 \Rightarrow V_2 = V_3$
$V_1 = V_2 = V_3$

$V_{\lambda=2} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$d :: P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$AP = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$\therefore AP = PD \Rightarrow D = P^{-1}AP$ proved

33) $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}^9$

$\cancel{|A - \lambda I| = 0}$

$\cancel{\begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = 0 \rightarrow (-4-\lambda)(5-\lambda) + 18 = 0}$

$\cancel{-20 + 4\lambda - 5\lambda + \lambda^2 + 18 = 0}$

$\cancel{\lambda^2 + \lambda + 2 = 0}$

$$A^2 = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} -14 & 30 \\ -15 & 31 \end{bmatrix}$$

$$A^8 = A^4 \cdot A^4 = \begin{bmatrix} -14 & 30 \\ -15 & 31 \end{bmatrix} \begin{bmatrix} -14 & 30 \\ -15 & 31 \end{bmatrix} = \begin{bmatrix} -254 & -510 \\ -255 & 511 \end{bmatrix}$$

$$A^9 = A^8 A = \begin{bmatrix} -254 & -510 \\ -255 & 511 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2546 & -4074 \\ 513 & 1025 \end{bmatrix}$$

34) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

~~Given~~ To find $\Rightarrow [A]^{2018}$

$[A]^{2018} = [A^2]^{1009} = [I]^{1009} = I$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

35) (i) $A = \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix}$

$|A - k\lambda| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 0 & k-\lambda \end{vmatrix} = 0 \quad (1-\lambda)(k-\lambda) = 0$

$\lambda = 1$ & $\lambda = k$

Hence, value of λ is any real number except 1

$\lambda = R - [1]$

41) A norm is a function from a real or complex vector space to the non-negative real number that behaves in certain ways like the distance from the origin.

42) Orthogonally in R^n means a vectors $(V_1, V_2, V_3, \dots, V_k\}$ in R_n is called an orthogonal set if all the vectors in the set are pairwise orthogonal.

43) $V_1 = (2,1,-1)$, $V_2 = (0,1,1)$, $V_3 = (1,-1,1)$

$\langle V_1 \cdot V_2 \rangle = 0 + 1 - 1 = 0$

$\langle V_2 \cdot V_3 \rangle = 0 - 1 + 1 = 0$

$\langle V_1 \cdot V_3 \rangle = 2 - 1 - 1 = 0$

Hence, $\{V_1, V_2, V_3\}$ is an orthogonal set in R_3.

44) Orthogonal ~~sets:~~ basic :- An orthogonal basis for a subspace W of R^n is a basis of W that is an orthogonal set

$\{(0,1),(1,0)\}$ is an orthogonal basis of $\mathbb{R}^2$.

46) Orthonormal sets :- A set of vectors in R^2 is an orthonormal set if it is an orthogonal set of unit vectors.

47) $q_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ & $q_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$\langle q_1 \cdot q_2 \rangle = \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$

~~$\|q_1\| = \langle q_1 q_1 \rangle = \sqrt{\left(\frac{1}{\sqrt3}\right)^2}$~~

$\|q_1\| = q_1 \cdot q_1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$

$\|q_2\| = q_2 \cdot q_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{6}{6} = 1$

Now $\langle q_1 \cdot q_2 \rangle = \frac{1}{\sqrt3} + \frac{1}{\sqrt6} + \left(\frac{-1}{\sqrt3}\right)\left(\frac{2}{\sqrt6}\right) + \frac{1}{\sqrt6} \cdot \frac{1}{\sqrt3} = \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} \cdot \frac{1}{\sqrt{18}}$

$\therefore \{q_1 \cdot q_2\}$ is an orthogonal set in $\mathbb{R}^3$.

~~48) Orthonormal basis :- An orthonormal basis for a of W that is an orthogonal set.~~

48) Orthonormal basis :- An orthonormal basis for a subsbace W of $\mathbb{R}^n$ is a basis of W that is an ~~orthogon~~ orthonormal set.

49) Orthogonal matrix :- An $n \times n$ matrix Q whose column form an orthonormal set is called an orthogonal matrix.

50) $A = \begin{bmatrix} 1/3 & 1/2 & 1/5 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{bmatrix}$, $\Rightarrow A^T = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/2 & -1/2 & 0 \\ 1/5 & 1/5 & 2/5 \end{bmatrix}$

For orthogonal matrix, $AA^T = 0$

$\begin{bmatrix} 1/3 & 1/2 & 1/5 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{bmatrix}\begin{bmatrix} 1/3 & \frac{1}{3} & -1/3 \\ 1/2 & -\frac{1}{2} & 0 \\ 1/5 & \frac{1}{5} & \frac{2}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} \left(\frac{1}{9}+\frac{1}{4}+\frac{1}{25}\right), & \left(\frac{1}{9}-\frac{1}{4}+\frac{1}{25}\right), & \left(\frac{1}{9}+0+\frac{2}{25}\right) \\ \left(\frac{1}{9}-\frac{1}{4}+\frac{1}{25}\right), & \left(\frac{1}{9}-\frac{1}{4}+\frac{1}{25}\right), & \left(\frac{1}{9}+0+\frac{2}{25}\right) \\ \left(\frac{1}{9}+0+\frac{2}{25}\right), & \left(\frac{1}{9}+0+\frac{2}{25}\right), & \left(\frac{1}{9}+\frac{4}{25}\right) \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \frac{1}{9} & \left(\frac{1}{9}-\frac{1}{2}\right) & \frac{1}{9} \\ \frac{1}{2} & 0 & 0 \\ \left(-\frac{1}{4}+\frac{1}{25}\right) & \left(\frac{1}{4}+\frac{1}{25}\right) & \frac{2}{25} \end{bmatrix}$ $R_1 \to (R_1 + R_2 - R_3)$
$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$= I$ $\therefore A$ is orthogonal matrix $\therefore A^{-1} = A^T$

51) $A = \begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}$

For orthogonal matrix, $AA^T = I$

$\Rightarrow \begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \cos^2\theta\sin^2\theta + \cos^2\theta + \sin^4\theta & \cos^3\theta\sin\theta - \sin\theta\cos\theta + \cos\theta\sin^3\theta & \cos\theta\sin^3\theta - \cos\theta\sin^2\theta \\ \cos^3\theta\sin\theta - \sin\theta\cos\theta + \cos\theta\sin^3\theta & \cos^4\theta + \sin^2\theta + \cos^2\theta\sin^2\theta & \cos^2\theta\sin\theta - \cos^2\theta \\ \cos\theta\sin^2\theta - \cos\theta\sin^2\theta & \cos^2\theta\sin\theta - \cos^2\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= I$

$\therefore$ A is orthogonal matrix $\therefore A^{-1} = A^T$

52) Orthogonal projection :- The projection of a vector on a plane is its orthogonal projection on that plane.

62) $A = \begin{bmatrix} 1 & 4 \\ b & a \end{bmatrix}$ eigen values are $= -1$ & 7

$|A - \lambda I|$

$\begin{vmatrix} 1-\lambda & 4 \\ b & a-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(a-\lambda) - 4b = 0$ — (i)

when $\lambda = -1 \Rightarrow 2(a+1) - 4b = 0$ — (ii)

when $\lambda = 7 \Rightarrow -6(a-7) - 4b = 0$ — (iii)

Solving eq^n (ii) & (iii)

$2a + 2 - 4\cancel{b} = 0$

$\mp 6a \pm 42 = 4\cancel{b} = 0$

$8a - 40 = 0 \Rightarrow a = 5, \; b = 3$

$\therefore A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$

61) Determinant of a matrix is the product of its eigen values.
Determinant of inverse of matrix is reciprocal to the inverse of value of determinant.

Eigenvalue of matrix A are 1, 2 & 4.
Determinant of matrix $A = 1 \times 2 \times 4 = 8$

Determinant of inverse of $A = \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{8} = 0.125$

$\det(A^{-1}) = \det(A^{-1})^T = 0.125$