Interpolation

Suppose we are given the following values of $y = f(x)$ for a set of values of x:-

x: x_0 x_1 x_2 ---- x_n

y: y_0 y_1 y_2 ---- y_n

Then, the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation and the process of finding the value of the function outside the given range is called extrapolation.

A polynomial $P_n(x)$ of degree $\leq n$ which fits the given data exactly,

i.e. $P_n(x_i) = f(x_i)$, $i = 0, 1, 2, --- n$.

is called interpolating polynomial.

Interpolation with evenly spaced points

(1) Newton's Forward Interpolation Formula

Let the f^n $y = f(x)$ take the values $y_0, y_1, --- y_n$ corresponding to the values $x_0, x_1, ---, x_n$ of x. Let these values of x be equi-spaced such that

$x_i = x_0 + ih$, $i = 0, 1, ----$

Assuming $y(x)$ to be a polynomial of n^{th} degree in x such that $y(x_0) = y_0,\ y(x_1) = y_1, \ldots, y(x_n) = y_n$.

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\,h^3}(x - x_0)(x - x_1)(x - x_2) + \ldots$$

$= y_0 + p\,\Delta y_0$

Where $\Delta y_0, \Delta y_1, \ldots \Delta y_{n-1}$ are first forward difference, defined as

$$\Delta y_r = y_{r+1} - y_r$$

Second forward difference, is defined as

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r .$$

Similarly, $\Delta^3 y_r = \Delta^2 y_{r+1} - \Delta^2 y_r$

$$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^p y_r .$$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	y_3			
x_4	y_4			
⋮	⋮	⋮		

Q. Find a polynomial which takes the following values:-

x	0	1	2	3
y	1	2	1	10

Hence, find $f(4)$.

Soln:- $x_0 = 0$, $h = 1$

Using Newton's forward Inter^n formula,

$$y(x) = y_0 + \Delta y_0 (x-0) + \frac{\Delta^2 y_0}{2!}(x-0)(x-1) + \frac{\Delta^3 y_0}{3!}(x-0)(x-1)(x-2) \quad \text{— (1)}$$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$n-2$

Putting these values in (1), we get

$$y(x) = 1 + 1\cdot x + \frac{(-2)}{2}\cdot x(x-1) + \frac{12}{6}\cdot x(x-1)(x-2)$$

$$= 1 + x - x^2 + x + 2x(x^2-3x+2)$$

$$= 1 + 2x - x^2 + 2x^3 - 6x^2 + 4x$$

$$y(x) = 2x^3 - 7x^2 + 6x + 1$$

$$y(4) = 2\cdot 64 - 7(16) + 24 + 1 = \underline{\underline{41.}}$$

Obs. 2. *The first two terms of this formula give the linear interpolation while the first three terms give a parabolic interpolation and so on.*

7.3. NEWTON'S BACKWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values x_0, $x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \qquad [\because E^{-1} = 1 - \nabla$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\ldots\right] y_n$$

[using Binomial theorem

i.e.

$$\mathbf{y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots} \qquad \ldots(1)$$

It is called *Newton's backward interpolation formula* as (1) contains y_n and backward differences of y_n.

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead* (to the right) *of* y_n.

Example 7.1. *The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface :*

x = height :	*100*	*150*	*200*	*250*	*300*	*350*	*400*
y = distance :	*10.63*	*13.03*	*15.04*	*16.81*	*18.42*	*19.90*	*21.27*

Find the values of y when

(i) x = 160 ft. *(V.T.U., B.E., 2013)*

(ii) x = 410.

Sol. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	**13.03**		– 0.39		
		2.01		0.15	
200	15.04		**– 0.24**		– 0.07
		1.77		**0.08**	
250	16.81		– 0.16		**– 0.05**
		1.61		0.03	
300	18.42		– 0.13		– 0.01
		1.48		**0.02**	
350	19.90		**– 0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 150$, then $y_0 = 13.03$, $\Delta y_0 = 2.01$, $\Delta^2 y_0 = -0.24$, $\Delta^3 = 0.08$, $\Delta^4 y_0 = -0.05$

Since $x = 160$ and $h = 50$, $\quad \therefore \quad p = \frac{x - x_0}{h} = \frac{10}{50} = 0.2$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots$$

$\nabla y_p = y_{p+1} - y_p$

$\Delta y_0 \cdot y_0$

$$y_{160} = 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168 = 13.46 \text{ nautical miles}$$

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $\quad x_n = 400,\ p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

Using the line of backward difference

$$y_n = 21.27,\ \nabla y_n = 1.37,\ \nabla^2 y_n = -0.11,\ \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400}$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$$= 21.53 \text{ nautical miles.}$$

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Example 7.2. *From the following table, estimate the number of students who obtained marks between 40 and 45 :*

Marks :	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
No. of students :	*31*	*42*	*51*	*35*	*31*

(M.T.U., B. Tech., 2013)

Sol. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of students (y_x) :	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**– 25**	
60	124		**– 16**		**37**
		35		**12**	
70	159		**– 4**		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45. Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \qquad [\because \; h = 10$$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\,\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{3!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$= 47.87, \text{ on simplification.}$$

The number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

Example 7.3. *Find the cubic polynomial which takes the following values :*

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate f(4). *(Anna, B. Tech., 2011)*

Sol. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		– 2	
		– 1		12
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \dfrac{x-0}{h} = x$ [$\because$ $h = 1$

$\therefore$ Using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1,$$

which is the required polynomial.

To compute $f(4)$, we take $x_n = 3$, $x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ [$\because$ $h = 1$

Obs. Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 = 41$$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

The above example shows that *if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.*

Example 7.4. *Using Newton's backward difference formula, construct an interpolating polynomial of degree 3 for the data : $f(-0.75) = -0.0718125$, $f(-0.5) = -0.02475$, $f(-0.25) = 0.3349375$, $f(0) = 1.10100$. Hence find $f(-1/3)$.*

Sol. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 0.75	– 0.0718125			
		0.0470625		
– 0.5	– 0.02475		0.312625	
		0.3596875		**0.09375**
– 0.25	0.3349375		**0.400375**	
		0.7660625		
0	**1.10100**			

We use Newton's backward difference formula

$$y(x) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_3$$

taking $$x_3 = 0,\ p = \frac{x-0}{h} = \frac{x}{0.25} = 4x, \qquad [\because\ h = 0.25]$$

$$\therefore \quad y(x) = 1.10100 + 4x\,(0.7660625) + \frac{4x\,(4x+1)}{2}(0.400375)$$

$$+ \frac{4x\,(4x+1)\,(4x+2)}{6}(0.09375)$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x$$

$$= x^3 + 4.001x^2 + 4.002x + 1.101.$$

Put $$x = -\frac{1}{3}, \text{ so that}$$

$$y\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 4.001\left(-\frac{1}{3}\right)^2 + 4.002\left(-\frac{1}{3}\right) + 1.101$$

$$= 0.1745$$

Example 7.5. *In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :*

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

(Anna, B.E., 2007)

Sol. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3$, $x = 1$, $h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$