# Milne's method:

Given IVP $y' = f(x, y)$, $y(x_0) = y_0$ ①

to find the approximate value of the solution curve $y(x)$ for $x = x_0 + nh$ by Milne's method, we proceed as follows:

If the values of $y_0, y_1, y_2, y_3$ given at $x_0, x_1, x_2, x_3$ then it is fine otherwise, we compute

$$y_1 = y(x_0 + h), \quad y_2 = y(x_0 + 2h), \quad y_3 = y(x_0 + 3h),$$

by Taylor series method or Euler method or R-K method.

Next we calculate

$$f_0 = f(x_0, y_0), \quad f_1 = f(x_1, y_1), \quad f_2 = f(x_2, y_2), \quad f_3 = f(x_3, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \cdots \quad ①$$

$$p = \frac{x - x_0}{h}$$

in the relation

$$\int_{x_0}^{x_4} y' = \int_{x_0}^{x_4} f(x, y)\,dx$$

$$\Rightarrow y_4 - y_0 = \int_{x_0}^{x_0+4h} \left[f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \cdots\right] dx$$

$$= \int_0^4 \left[f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \cdots\right] h\,dp$$

$$= h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \cdots\right).$$

Neglecting fourth and higher order differences [as part of T.E.] and ~~expressing~~ taking

$$\Delta f_0 = f_1 - f_0$$

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0$$

$$\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$$

we get

$$\boxed{y_4^{(P)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)} \quad (2)$$

which is called a predictor.

Having found $y_4$, we obtain a first approximation to

$$f_4 = f(x_0 + 4h, y_4)$$

Then a better value of $y_4$ is found by

Simpson's $\frac{1}{3}$ rule as

$$\int_{x_2}^{x_4} y' \, dx = \int_{x_2}^{x_4} f(x, y) \, dx$$

$$y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

which is called a corrector.

Then an improved value of $f_4$ is computed and again the corrector is applied to find a still better value of $y_4$.