CHAPTER 11

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Chapter Objectives

- Introduction
- Classification of second order equations
- Finite-difference approximations
- Elliptic equations to partial derivatives
- Solution of Laplace equation
- Solution of Poisson's equation
- Solution of elliptic equations by relaxation
- Parabolic equations method
- Solution of one-dimensional heat equation
- Solution of two-dimensional heat equation
- Hyperbolic equations
- Solution of wave equation

11.1 Introduction

Partial differential equations arise in the study of many branches of applied mathematics, e.g., in fluid dynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics, and electro-magnetic theory. Only a few of these equations can be solved by analytical methods which are also complicated by requiring use of advanced mathematical techniques. In most of the cases, it is easier to develop approximate solutions by numerical methods. Of all the numerical methods available for the solution of partial differential equations, the method of finite differences is most commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equation is changed to a system of linear equations which are solved by iterative procedures. This process is slow but produces good results in many boundary value problems. An added advantage of this method is that the computation can be carried by electronic computers. To accelerate the solution, sometimes the method of relaxation proves quite effective.

Besides discussing the finite difference method, we shall briefly describe the relaxation method also in this chapter.

11.2 Classification of Second Order Equations

The general linear partial differential equation of the second order in two independent variables is of the form

$$A(x,y)\frac{\partial^2 u}{dx^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+\left(x,y,u\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \qquad (1)$$

Such a partial differential equation is said to be

(*i*) **elliptic** *if* $B^2 - 4AC < 0$, (*ii*) **parabolic** *if* $B^2 - 4AC = 0$, and (*iii*) **hyperbolic** *if* $B^2 - 4AC > 0$.

NOTE **Obs.** *A partial equation is classified according to the region in which it is desired to be solved. For instance, the partial differential equation* $f_{xx} + f_{yy} = 0$ *is elliptic if* $y > 0$, *parabolic if* $y = 0$, *and hyperbolic if* $y < 0$.

EXAMPLE 11.1

Classify the following equations:

(i) $\dfrac{\partial^2 u}{\partial x^2}+4\dfrac{\partial^2 u}{\partial x\partial y}+4\dfrac{\partial^2 u}{\partial y^2}-\dfrac{\partial u}{\partial x}+2\dfrac{\partial u}{\partial y}=0$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2}+\left(1-y^2\right)\dfrac{\partial^2 u}{\partial y^2}=0,-\infty<x<\infty,-<y<1$

(iii) $\left(1+x^2\right)\dfrac{\partial^2 u}{\partial x^2}+\left(5+2x^2\right)\dfrac{\partial^2 u}{\partial x \partial t}+\left(4+x^2\right)\dfrac{\partial^2 u}{\partial t^2}=0.$

Solution:

(i) Comparing this equation with (1) above, we find that t^2

$$A = 1, B = 4, C = 4$$

$$\therefore \quad B^2 - 4AC = (4)^2 - 4 \times 1 \times 4 = 0$$

So the equation is parabolic.

(ii) Here $A = x^2$, $B = 0$, $C = 1 - y^2$

$$B^2 - 4AC = 0 - 4x^2 (1 - y^2) = 4x^2(y^2 - 1)$$

For all x between $-\infty$ and ∞, x^2 is positive

For all y between -1 and 1, $y^2 < 1$

$$B^2 - 4AC < 0$$

Hence the equation is elliptic

(iii) Here $A = 1 + x^2$, $B = 5 + 2x^2$, $C = 4 + x^2$

$$\therefore \quad B^2 - 4AC = (5 + 2x^2)^2 - 4(1 + x^2)(4 + x2) = 9 \text{ } i.e. > 0$$

So the equation is hyperbolic

Exercises 11.1

1. What is the classification of the equation $f_{xx} + 2f_{xy} + f_{yy} = 0$.

2. Determine whether the following equation is elliptic or hyperbolic?

$(x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0.$

3. Classify the equation

(i) $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} + 2u_x - 3u = 0.$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2}+y^2\dfrac{\partial^2 u}{\partial y^2}=x\dfrac{\partial u}{\partial x}-y\dfrac{\partial u}{\partial y}$

(iii) $3\dfrac{\partial^2 u}{\partial x^2}+4\dfrac{\partial^2 u}{\partial x \partial y}+6\dfrac{\partial^2 u}{\partial y^2}-2\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}-u=0$

4. In which parts of the (x, y) plane is the following equation elliptic?

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x \partial y}+\left(x^2+4y^2\right)\frac{\partial^2 u}{\partial y^2}=2\sin\left(xy\right).$$

11.3 Finite Difference Approximations to Partial Derivatives

Consider a rectangular region R in the x, y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in Figure 11.1. The points of intersection of the dividing lines are called *mesh points, nodal points, or grid points*

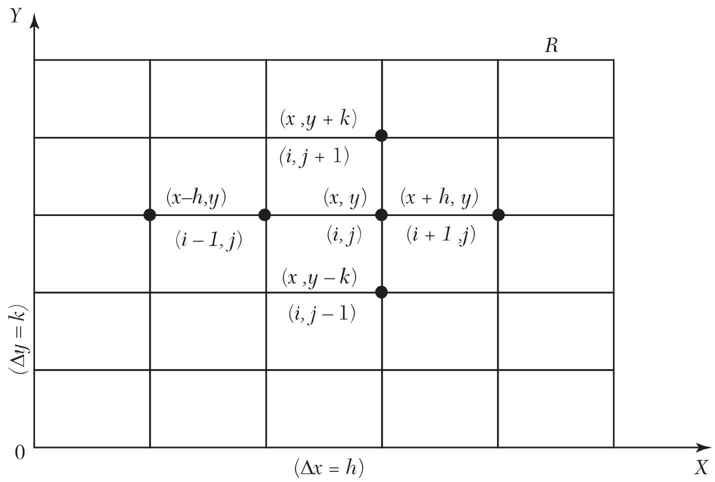

FIGURE11.1

Then we have the finite difference approximations for the partial derivatives in x-direction (Section 10.17):

$$\frac{\partial u}{\partial x} = \frac{u(x+h,y)-u(x,y)}{h} + O(h) = \frac{u(x,y)-u(x-h,y)}{h} + O(h)$$

$$= \frac{u(x+h,y)-u(x-h,y)}{2h} + O(h^{2)}$$

And $$\frac{\partial^2 u}{\partial x^2} = \frac{(x-h,y)-2u(x,y)+u(x+h,y)}{h^2} + O(h^2)$$

Writing $u(x, y) = u(ih, jk)$ as simply $u_{i,j}$, the above approximations become

$$ux = \frac{u_{i+1}, j - U_{i,j}}{h} + (h) \tag{1}$$

$$= \frac{u_{i,j} - u_{i-1}, j}{h} + O(h) \tag{2}$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \tag{3}$$

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \tag{4}$$

Similarly we have the approximations for the derivatives w.r.t. y:

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \tag{5}$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \tag{6}$$

$$= \frac{u_{i,j} + 1 - u_{i,j} - 1}{2k} + O\left(k^2\right) \tag{7}$$

and $$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \tag{8}$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations (1) to (8), we obtain the finite-difference analogues of the given equation.

11.4 Elliptic Equations

The Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and the Poisson's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ are Example s of elliptic partial differential equations.

The Laplace equation arises in steady-state flow and potential problems. Poisson's equation arises in fluid mechanics, electricity and magnetism and torsion problems.

The solution of these equations is a function $u(x, y)$ which is satisfied at every point of a region R subject to certain boundary conditions specified on the closed curve C (Figure 11.2).

In general, problems concerning steady viscous flow, equilibrium stresses in elastic structures etc., lead to elliptic type of equations.

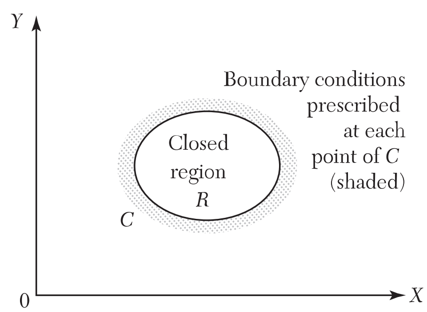

FIGURE 11.2

11.5 Solution of Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

Consider a rectangular region R for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side h, as shown in Figure 11.3 (assuming that an exact sub-division of R is possible). Replacing the derivatives in (1) by their difference approximations, we have

$$\frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] + \frac{1}{h^2}\left[u_{i,j-1} - 2_{ui,j} + u_{i,j+1}\right] = 0$$

or $$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{ij+1} + u_{i,j-1}\right] \tag{2}$$

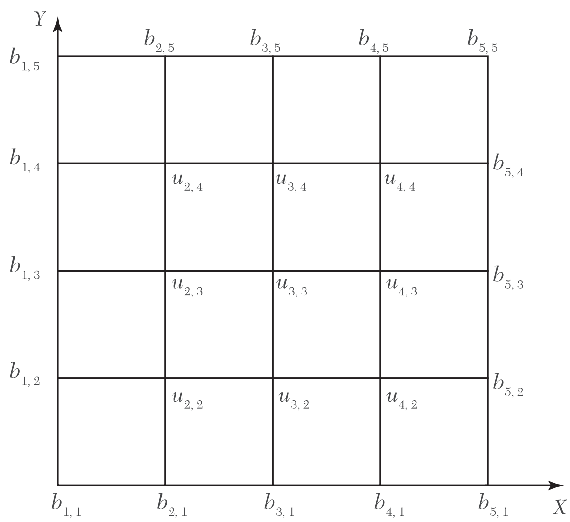

FIGURE 11.3

This shows that the value of u at any interior mesh point is the average of its values at four neighboring points to the left, right, above and below. (2) is called the **standard 5-point formula** which is exhibited in Figure 11.4.

Sometimes a formula similar to (2) is used which is given by

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \tag{3}$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighboring diagonal mesh points. (3) is called the **diagonal five-point**

formula which is represented in Figure 11.5. Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

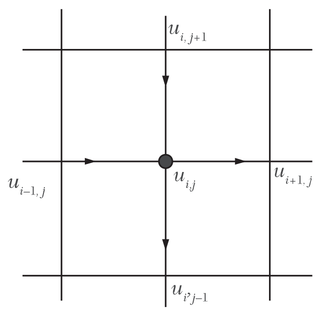

FIGURE 11.4

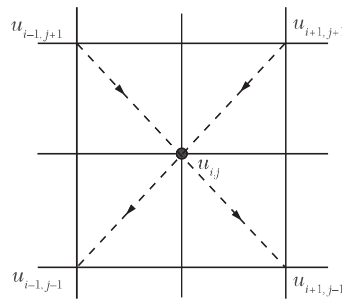

FIGURE 11.5

Now to find the initial values of u at the interior mesh points, we first use the diagonal five-point formula (3) and compute $u_{3,3}$, $u_{2,4}$, $u_{4,4}$, $u_{4,2}$ and $u_{2,2}$, in this order. Thus we get,

$$u_{3,3} = \frac{1}{4}\ (b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1});$$

$$u_{2,4} = \frac{1}{4}\ (b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3})$$

$$u_{4,4} = \frac{1}{4}\ (b_{3,5} + b_{5,3} + b_{3,5} + u_{3,3});\ u_{4,2}$$

$$= \frac{1}{4}\ (u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3})$$

$$u_{2,2} = \frac{1}{4}(b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1})$$

The values at the remaining interior points, *i.e.*, $u_{2,3}$, $u_{3,4,}$ $u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we obtain

$$u_{2,3} = \frac{1}{4}(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2}),\ u_{3,4}$$

$$= \frac{1}{4}(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3})$$

$$u_{4,3} = \frac{1}{4}(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2}),\ u_{3,2}$$

$$= \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1})$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by either of the following iterative methods. In each case, the method is repeated until the difference between two consecutive iterates becomes negligible.

(i) **Jacobi's method.** Denoting the nth iterative value of $u_{i,j}$, by $u^{(n)}_{i,j}$, the iterative formula to solve (2) is

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n)}_{i,j+1} + u^{(n)}_{i,j-1}] \tag{4}$$

It gives improved values of $u_{i,j}$ at the interior mesh points and is called the *point Jacobi's formula*.

(ii) **Gauss-Seidal method.** In this method, the iteration formula is

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j+1} + u^{(n)}_{i,j-1}]$$

It utilizes the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows.

NOTE **Obs.** *The Gauss-Seidal method is simple and can be adapted to computer calculations. Its convergence being slow, the working is somewhat lengthy. It can however, be shown that the Gauss-Seidal scheme converges twice as fast as Jacobi's scheme.*

The accuracy of calculations depends on the mesh-size, *i.e.*, smaller the h, the better the accuracy. But if h is too small, it may increase rounding-off errors and also increases the labor of computation.

EXAMPLE 11.2

Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in Figure 11.6.

Solution:

Let $u_1, u_2, \cdots, u_9$ be the values of u at the interior mesh-points. Since the boundary values of u are symmetrical about AB,

$\therefore\ u_7 = u_1, u_8 = u_2, u_9 = u_3.$

Also the values of u being symmetrical about CD. $u3 = u_1$, $u_6 = u_4$, $u_9 = u_7$.

Thus it is sufficient to find the values u_1, u_2, u_4, and u_5.

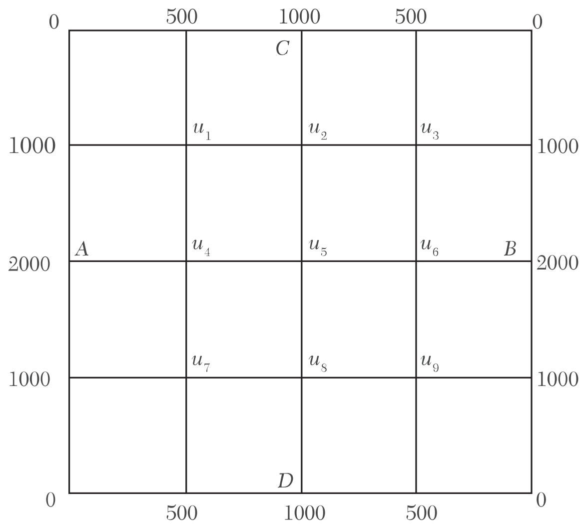

FIGURE 11.6

Now we find their initial values in the following order:

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \qquad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125 \qquad \text{(Diag.formula)}$$

$$u_2 = \frac{1}{4}\ (1125 + 1125 + 1000 + 1500) \approx 1188 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}\ 14\ (2000 + 1500 + 1125 + 1125) \approx 1438 \qquad \text{(Std. formula)}$$

Now we carry out the iteration process using the standard formulae:

$$u_1^{(n+1)} = \frac{1}{4}\left[1000 + u_2^{(n)} + 500 + u_4^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + u_1^{(n)} + 1000 + u_5^{(n)}\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[2000 + u_5^{(n)} + u_1^{(n+1)} + u_1^{(n)}\right]$$

$$u_5^{(n+1)} = \frac{1}{4}\left[u_4^{(n+1)} + u_4^{(n)} + u_2^{(n+1)} + u_2^{(n)}\right]$$

First iteration: (put $n = 0$ in the above results)

$$u_1^{(1)} = \frac{1}{4}(1000 + 1188 + 500 + 1438) \approx 1032$$

$$u_2^{(1)} = \frac{1}{4}(1032 + 1125 + 1000 + 1500) = 1164$$

$$^{(1)} \qquad -(2000 + 1500 + 1032 + 1125) = 1414$$

$$u_5^{(1)} = \frac{1}{4}(1414 + 1438 + 1164 + 1188) = 1301$$

Second iteration: (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(1000 + 1164 + 500 + 1414) = 1020$$

$$u_2^{(2)} = \frac{1}{4}(1020 + 1032 + 1000 + 1301) = 1088$$

$$u_4^{(2)} = \frac{1}{4}(2000 + 1301 + 1020 + 1032) = 1338$$

$$u_5^{(2)} = \frac{1}{4}(1338 + 1414 + 1088 + 1164) = 1251$$

Third iteration:

$$u_1^{(3)} = \frac{1}{4}(1000 + 1088 + 500 + 1338) = 982$$

$$u_2^{(3)} = \frac{1}{4}(982 + 1020 + 1000 + 1251) = 1063$$

$$u_4^{(3)} = \frac{1}{4}(2000 + 1251 + 982 + 1020) = 1313$$

$$u_5^{(3)} = \frac{1}{4}(1313 + 1338 + 1063 + 1088) = 1201$$

Fourth iteration:

$$u_1^{(4)} = \frac{1}{4}(1000 + 1063 + 500 + 1313) = 969$$

$$u_2^{(4)} = \frac{1}{4}(969 + 982 + 1000 + 1201) = 1038$$

$$u_4^{(4)} = \frac{1}{4}(2000 + 1201 + 969 + 982) = 1288$$

$$u_5^{(4)} = \frac{1}{4}(1288 + 1313 + 1038 + 1063) = 1176$$

Fifth iteration:

$$u_1^{(5)} = \frac{1}{4}(1000 + 1038 + 500 + 1288) = 957$$

$$u_2^{(5)} = \frac{1}{4}(957 + 969 + 1000 + 1176) \approx 1026$$

$$u_4^{(5)} = \frac{1}{4}(2000 + 1176 + 957 + 969) \approx 1276$$

$$u_5^{(5)} = \frac{1}{4}(1276 + 1288 + 1026 + 1038) = 1157$$

Similarly,

$$u_1^{(6)} = 951, u_2^{(6)} = 1016, u_4^{(6)} = 1266, u_5^{(6)} = 1146$$

$$u_1^{(7)} = 946, u_2^{(7)} = 1011, u_4^{(7)} = 1260, u_5^{(7)} = 1138$$

$$u_1^{(8)} = 943, u_2^{(8)} = 1007, u_4^{(8)} = 1257, u_5^{(8)} = 1134$$

$$u_1^{(9)} = 941, u_2^{(9)} = 1005, u_4^{(9)} = 1255, u_5^{(9)} = 1131$$

$$u_1^{(10} = 940, u_2^{(10)} = 1003, u_4^{(10)} = 1253, u_5^{(10} = 1129$$

$$u_1^{(11)} = 939, u_2^{(11)} = 1002, u_4^{(11)} = 1252, u_5^{(11)} = 1128$$

$$u_1^{(12)} \approx 939, u_2^{(12)} \approx 1001, u_4^{(12)} \approx 1251, u_5^{(12)} = 1126$$

There is a negligible difference between the values obtained in the eleventh and twelfth iterations.

Hence $u_1 = 939$, $u_2 = 1001$, $u_4 = 1251$ and $u_5 = 1126$.

EXAMPLE 11.3

Given the values of $u(x, y)$ on the boundary of the square in the Figure 11.7, evaluate the function $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure by

(*a*) *Jacobi's method* (*b*) *Gauss-Seidal method*

Solution:

To get the initial values of u_1, u_2, u_3, u_4, we assume that $u_4 = 0$. Then

$$u_1 = \frac{1}{4}\ (1000 + 0 + 1000 + 2000) = 1000 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}\ (1000 + 500 + 1000 + 0) = 625 \qquad \text{(Std. formula)}$$

$$u_3 = \frac{1}{4}\ (2000 + 0 + 1000 + 500) = 875 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \qquad \text{(Std. formula)}$$

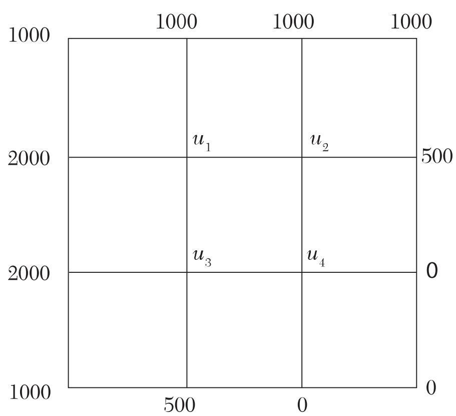

FIGURE 11.7

(a) We carry out the successive iterations, using Jacobi's formulae:

$$u_1^{(n+1)} = \frac{1}{4}\left[2000 + u_2^{(n)} + 1000 + u_3^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n)} + 500 + 1000 + u_4^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[2000 + u_4^{(n)} + u_1^{(n)} + 500\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[u_3^{(n)} + 0 + u_2^{(n)} + 0\right]$$

First iteration: (put $n = 0$ in the above results)

$$u_1^{(1)} = \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125$$

$$u_2^{(1)} = \frac{1}{4}(1000 + 500 + 1000 + 375) \approx 719$$

$$u_3^{(1)} = \frac{1}{4}(2000 + 375 + 1000 + 500) \approx 969$$

$$u_4^{(1)} = \frac{1}{4}(875 + 0 + 625 + 0) \approx 375$$

Second iteration: (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(2000 + 719 + 1000 + 969) = 1172$$

$$u_3^{(2)} = \frac{1}{4}(1125 + 500 + 1000 + 375) = 750$$

$$u_3^{(2)} = \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000$$

$$u_4^{(2)} = \frac{1}{4}(969 + 0 + 719 + 0) = 422$$

Similarly,
$$u_1^{(3)} \approx 1188, u_2^{(3)} \approx 774, u_3^{(3)} \approx 1024, u_4^{(3)} \approx 438$$
$$u_1^{(4)} \approx 1200, u_2^{(4)} \approx 782, u_3^{(4)} \approx 1032, u_4^{(4)} \approx 450$$
$$u_1^{(5)} \approx 1204, u_2^{(5)} \approx 788, u_3^{(5)} \approx 1038, u_4^{(5)} \approx 454$$
$$u_1^{(6)} \approx 1206.5, u_2^{(6)} \approx 790, u_3^{(6)} \approx 1040, u_4^{(6)} \approx 456.5$$
$$u_1^{(7)} \approx 1208, u_2^{(7)} \approx 791, u_3^{(7)} \approx 1041, u_4^{(7)} \approx 458$$

and
$$u_1^{(8)} \approx 1208, u_2^{(8)} \approx 791.5, u_3^{(8)} \approx 1041.5, u_4^{(8)} \approx 458 \; .$$

There is no significant difference between the seventh and eighth iteration values.

Hence $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$.

(b) We carry out the successive iterations, using Gauss-Seidal formulae

$$u_1^{(n+1)} = \frac{1}{4}\left[2000 + u_2^{(n)} + 1000 + u_3^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} = 500 + 1000 + u_4^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[2000 + u_4^{(n)} + u_1^{(n+1)} = 500\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0\right]$$

First iteration:(put $n = 0$ in the above results)

$$u_1^{(1)} = \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125$$

$$u_2^{(1)} = \frac{1}{4}(1125 + 500 + 1000 + 375) = 750$$

$$u_3^{(1)} = \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000$$

$$u_4^{(1)} = \frac{1}{4}(1000 + 0 + 750 + 0) \approx 438$$

Second iteration: (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(2000 + 750 + 1000 + 1000) \approx 1188$$

$$u_2^{(2)} = \frac{1}{4}(1188 + 500 + 1000 + 438) \approx 782$$

$$u_3^{(2)} = \frac{1}{4}(2000 + 438 + 1188 + 500) \approx 1032$$

$$u_4^{(2)} = \frac{1}{4}(1032 + 0 + 782 + 0) \approx 454$$

Similarly $u_1^{(3)} \approx 1204, u_2^{(3)} \approx 789, u_3^{(3)} \approx 1040, u_4^{(3)} \approx 458$

$u_1^{(4)} \approx 1207, u_2^{(4)} \approx 791, u_3^{(4)} \approx 1041, u_4^{(4)} = 458$

and $u_1^{(5)} = 1208, u_2^{(5)} \approx 791.5, u_3^{(5)} \approx 1041.5, u_4^{(5)} \approx 458.25$

Thus there is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$.

EXAMPLE 11.4

Solve the Laplace equation $u_{xx} + u_{yy} = 0$ given that

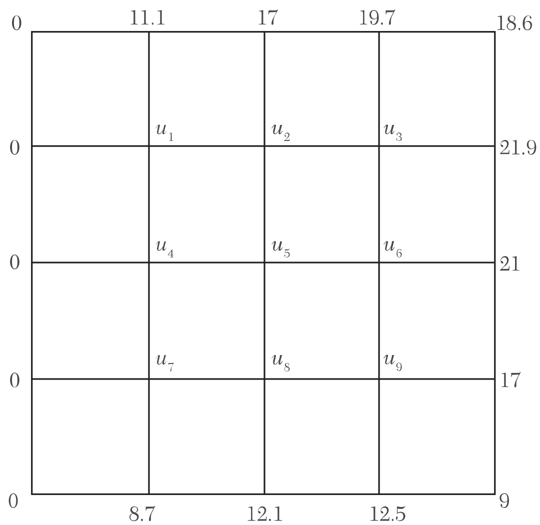

FIGURE 11.8

Solution:

We first find the initial values in the following order:

$$u_5 = \frac{1}{4}(0 + 17 + 21 + 12.1) = 12.5 \qquad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 12.5 + 0 + 17) = 7.4 \qquad \text{(Diag. formula)}$$

$$u_3 = \frac{1}{4}(12.5 + 18.6 + 17 + 21) = 17.28 \qquad \text{(Diag. formula)}$$

$$u_7 = \frac{1}{4}(12.5 + 0 + 0 + 12.1) = 6.15 \qquad \text{(Diag. formula)}$$

$$u_9 = \frac{1}{4}(12.5 + 9 + 21 + 12.1) = 13.65 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(17 + 12.5 + 7.4 + 17.3) = 13.55 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(7.4 + 6.2 + 0 + 12.5) = 6.52 \qquad \text{(Std. formula)}$$

$$u_6 = \frac{1}{4}(17.3 + 13.7 + 12.5 + 21) = 16.12 \qquad \text{(Std. formula)}$$

$$u_8 = \frac{1}{4}(12.5 + 12.1 + 6.2 + 13.7) = 11.12 \qquad \text{(Std. formula)}$$

Now we carry out the iteration process using the standard formula:

$$u_1^{(n+1)} = \frac{1}{4}[0 + 11.1 + u_4^{(n)} + u_2^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + 17 + u_5^{(n)} + u_3^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + 19.7 + u_6^{(n)} + 219]$$

$$u_4^{(n+1)} = \frac{1}{4}[0 + u_1^{(n+1)} + u_7^{(n)} + u_5^{(n)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_2^{(n+1)} + u_8^{(n)} + u_6^{(n)}]$$

$$u_6^{(n+1)} = \frac{1}{4}[u_5^{(n+1)} + u_3^{(n+1)} + u_9^{(n)} + 21]$$

$$u_7^{(n+1)} = \frac{1}{4}[0 + u_4^{(n+1)} + 8.7 + u_8^{(n)}]$$

$$u_8^{(n+1)} = \frac{1}{4}[u_7^{(n+1)} + u_5^{(n+1)} + 12.1 + u_9^{(n)}]$$

$$u_9^{(n+1)} = \frac{1}{4}[u_8^{(n+1)} + u_6^{(n+1)} + 12.8 + 17]$$

First iteration: (put $n = 0$, in the above results)

$$u_1^{(1)} = \frac{1}{4}(0 + 11.1 + u_4^{(0)} + u_2^{(0)})$$

$$= \frac{1}{4}(0 + 11.1 + 6.52 + 13.55) = 7.79$$

$$u_2^{(1)} = \frac{1}{4}(7.79 + 17 + 12.5 + 17.28) = 13.64$$

$$u_3^{(1)} = \frac{1}{4}(13.64 + 19.7 + 16.12 + 21.9) = 12.84$$

$$u_4^{(1)} = \frac{1}{4}(0 + 7.79 + 6.15 + 12.5) = 6.61$$

$$u_5^{(1)} = \frac{1}{4}(6.61 + 13.64 + 11.12 + 16.12) = 11.88$$

$$u_6^{(1)} = \frac{1}{4}(11.88 + 17.84 + 13.65 + 21) = 16.09$$

$$u_7^{(1)} = \frac{1}{4}(0 + 6.61 + 8.7 + 11.12) = 6.61$$

$$u_8^{(1)} = \frac{1}{4}(6.61 + 11.88 + 12.1 + 13.65) = 11.06$$

$$u_9^{(1)} = \frac{1}{4}(11.06 + 16.09 + 12.8 + 17) = 12.238$$

Second iteration: (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(0 + 11.1 + 6.61 + 13.64) = 7.84$$

$$u_2^{(2)} = \frac{1}{4}(7.84 + 17 + 11.88 + 17.84) = 16.64$$

$$u_3^{(2)} = \frac{1}{4}(13.64 + 19.7 + 16.09 + 21.9) = 17.83$$

$$u_4^{(2)} = \frac{1}{4}(0 + 7.84 + 6.61 + 11.88) = 6.58$$

$$u_5^{(2)} = \frac{1}{4}(6.58 + 13.64 + 11.06 + 16.09) = 11.84$$

$$u_6^{(2)} = \frac{1}{4}(11.84 + 17.83 + 14.24 + 21) = 16.23$$

$$u_7^{(2)} = \frac{1}{4}(0 + 6.58 + 8.7 + 11.06) = 6.58$$

$$u_8^{(2)} = \frac{1}{4}(6.58 + 11.84 + 12.1 + 14.24) = 11.19$$

$$u_9^{(2)} = \frac{1}{4}(11.19 + 16.23 + 12.8 + 17) = 14.30$$

Third iteration: (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4}(0 + 11.1 + 6.58 + 13.64) = 7.83$$

$$u_2^{(3)} = \frac{1}{4}(7.83 + 17 + 11.84 + 17.83) = 13.637$$

$$u_3^{(3)} = \frac{1}{4}(13.63 + 19.7 + 16.23 + 21.9) = 17.86$$

$$u_4^{(3)} = \frac{1}{4}(0 + 7.83 + 6.58 + 11.84) = 6.56$$

$$u_5^{(3)} = \frac{1}{4}(6.56 + 13.63 + 11.19 + 16.23) = 11.90$$

$$u_6^{(3)} = \frac{1}{4}(11.90 + 17.86 + 14.30 + 21) = 16.27$$

$$u_7^{(3)} = \frac{1}{4}(0 + 6.56 + 8.7 + 11.19) = 6.61$$

$$u_8^{(3)} = \frac{1}{4}(6.61 + 11.90 + 12.1 + 14.30) = 11.23$$

$$u_9^{(3)} = \frac{1}{4}(11.23 + 16.27 + 12.8 + 17) = 14.32$$

Similarly

$$u_1^{(4)} = 7.82, u_2^{(4)} = 13.65, u_3^{(4)} = 17.88, u_4^{(4)} = 6.58, u_5^{(4)} = 11.94,$$
$$u_6^{(4)} = 16.28, u_7^{(4)} = 6.63, u_8^{(4)} = 11.25, u_9^{(4)} = 14.33$$
$$u_1^{(5)} = 7.83, u_2^{(5)} = 13.66, u_3^{(5)} = 17.89, u_4^{(5)} = 6.50, u_5^{(5)} = 11.95,$$
$$u_6^{(5)} = 16.29, u_7^{(5)} = 6.64, u_8^{(5)} = 11.25, u_9^{(5)} = 14.34$$

There is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 7.83$, $u_2 = 13.66$, $u_3 = 17.89$, $u_4 = 6.6$, $u_5 = 11.95$, $u_6 = 16.29$, $u_7 = 6.64$, $u_8 = 11.25$, $u_9 = 14.34$.

11.6 Solution of Poisson's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{1}$$

Its method of solution is similar to that of the Laplace equation. Here the standard five-point formula for (1) takes the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) \tag{2}$$

By applying (2) at each interior mesh point, we arrive at linear equations in the nodal values $u_{i,j}$. These equations can be solved by the Gauss-Seidal method.

NOTE **Obs.** *The error in replacing u_{xx} by the finite difference approximation is of the order $O(h^2)$. Since $k = h$, the error in replacing u_{yy} by the difference approximation is also of the order $O(h^2)$. Hence the error in solving Laplace and Poisson's equations by finite difference method is of the order $O(h^2)$.*

EXAMPLE 11.5

Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = 1/3$.

Solution:

Here $h = 1/3$.

The standard five-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4_{ui,j} = h^2 f(ih, jh)$$
$$= h^2[-81(ih\,.\,jh)] = h^4(-81)\,ij = -ij \qquad (i)$$

For u_1 $(i = 1, j = 2)$, (i) gives $0 + u_2 + u_3 + 100 - 4u_1 = -2$

i.e., $\quad -4u_1 + u_2 + u_3 = -102 \qquad (ii)$

For u_2 $(i = 2, j = 2)$, (i) gives $u_1 + 100 + u_4 + 100 - 4u_2 = -4$

i.e., $\quad u_1 - 4u_2 + u_4 = -204 \qquad (iii)$

For u_3 $(i = 1, j = 1)$, (i) gives $0 + u_4 + 0 + u_1 - 4u_3 = -1$

i.e., $\quad u_1 - 4u_3 + u_4 = -1 \qquad (iv)$

For u_4 $(i = 2, j = 1)$ gives $u_3 + 100 + u_2 - 4u_4 = -2$

i.e., $\quad u_2 + u_3 - 4u_4 = -102 \qquad (v)$

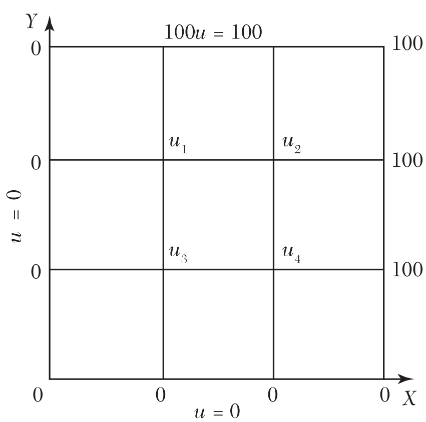

FIGURE 11.9

Subtracting (v) from (ii), $-4u_1 + 4u_4 = 0$, *i.e.,* $u_1 = u_4$

Then (iii) becomes $2u_1 - 4u_2 = -204 \qquad (vi)$

and (iv) becomes $2u_1 - 4u_3 = -1 \qquad (vii)$

Now (4) × (*ii*) + (*vi*) gives $-14u_1 + 4u_3 = -612$ (*viii*)

(*vii*) + (*viii*) gives $-12u_1 = -613$

Thus $u_1 = 613/12 = 51.0833 = u_4$.

From (*vi*), $u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$

From (*vii*), $u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916$

EXAMPLE 11.6

Solve the equation $\nabla^2 u - 10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length = 1.

Solution:

Here $h = 1$.

∴ The standard five-point formula for the given equation is

$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4_{ui,j} = -10(i^2 + j^2 + 10)$ (*i*)

For u_1 $(i = 1, j = 2)$, (*i*) gives $0 + u_2 + 0 + u_3 - 4u_1 = -10(1 + 4 + 10)$

i.e., $u_1 = \frac{1}{4}(u_2 + u_3 + 150)$ (*ii*)

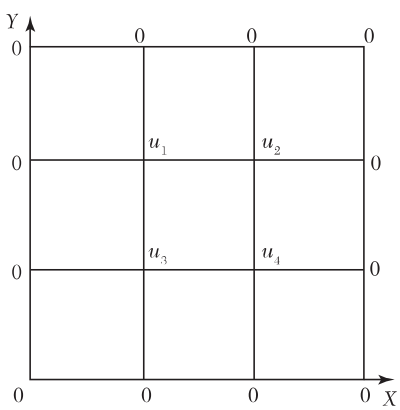

FIGURE 11.10

For u_2 $(i = 2, j = 2)$, (*i*) gives $u_2 = \frac{1}{4}(u_1 + u_4 + 180)$ (*iii*)

For $u_3 (i = 1, j = 1)$, we have $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$ (*iv*)

For u_4 $(i = 2, j = 1)$, we have $\quad u_4 = \frac{1}{4}(u_2 + u_3 + 150) \qquad (v)$

Equations (ii) and (v) show that $u_4 = u_1$. Thus the above equations reduce to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150),\ u_2 = \frac{1}{4}(u_2 + 90),\ u_3 = \frac{1}{4}(u_1 + 60)$$

Now let us solve these equations by the Gauss-Seidal iteration method.

First iteration: Starting from the approximations $u_2 = 0$, $u_3 = 0$, we obtain $u_1^{(1)} = 37.5$

Then $\quad u_2^{(1)} = \frac{1}{2}(37.5 + 90) \approx 64$

$u_2^{(1)} = \frac{1}{2}(37.5 + 60) \approx 49$

Second iteration: $\quad u_1^{(2)} = \frac{1}{4}(64 + 49 + 150) \approx 66, u_2^{(2)} = \frac{1}{2}(66 + 90) = 78$

$u_3^{(2)} = \frac{1}{2}(66 + 60) = 63$

Third iteration: $\quad u_1^{(3)} = \frac{1}{4}(78 + 63 + 150) \approx 73, u_2^{(3)} = \frac{1}{2}(73 + 90) \approx 82,$

$u_3^{(3)} = (73 + 60) \approx 67$

Fourth iteration: $\quad u_1^{(4)} = (82 + 67 + 150) \approx 75, u_2^{(4)} = (75 + 90) = 82.5,$

$u_3^{(4)} = (75 + 60) = 67.5$

Fifth iteration: $\quad u_1^{(5)} = (82.5 + 67.5 + 150) = 75, u_2^{(5)} = (75 + 90) = 82.5,$

$u_3^{(5)} = (75 + 90) = 67.5$

Since these values are the same as those of fourth iteration, we have $u_1 = 75$, $u_2 = 82.5, u_3 = 67.5$ and $u_4 = 75$.

Exercises 11.2

1. Solve the equation $u_{xx} + u_{yy} = 0$ for the square mesh with the boundary values as shown in Figure 11.11.

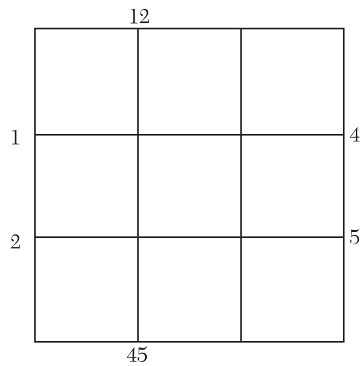

FIGURE 11.11

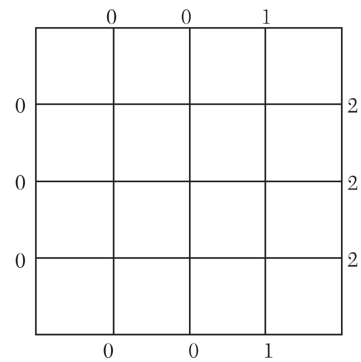

FIGURE 11.12

2. Solve $u_{xx} + u_{yy} = 0$ over the square mesh of side four units satisfying the following boundary conditions: $u\ (0, y) = 0$ for $0 \le y \le 4$, $u\ (4, y) = 12 + y$ for $0 \le y \le 4$; $u(x, 0) = 3x$ for $0 \le x \le 4$,$u\ (x, 4) = x^2$ for $0 \le x \le 4$.

3. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Figure 11.12. Iterate until the maximum difference between successive values at any point is less than 0.005.

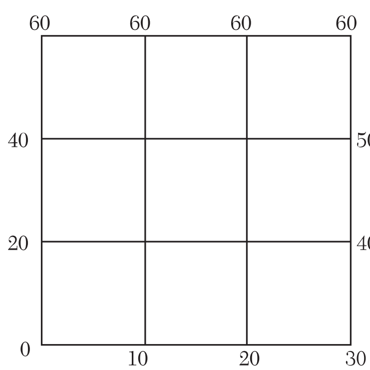

FIGURE 11.13

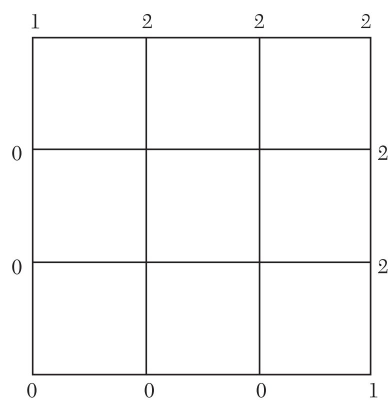

FIGURE 11.14

4. Using central-difference approximation solve $\nabla^2 u = 0$ at the nodal points of the square grid of Figure 11.13 using the boundary values indicated.

5. Solve $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Figure 11.14. Iterate till the mesh values are correct to two decimal places.

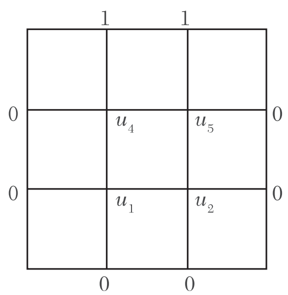

FIGURE 11.15

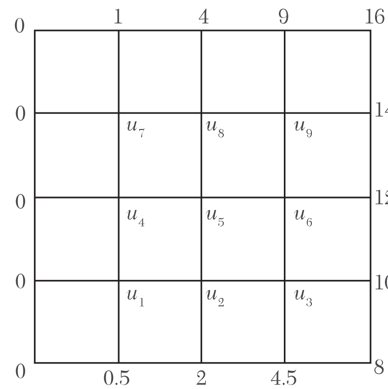

FIGURE 11.16

6. Solve the Laplace's equation $u_{xx} + u_{yy} = 0$ in the domain of Figure 11.15 by (*a*) Jacobi's method, (*b*) Gauss-Seidal method.

7. Solve the Laplace's equation $\nabla^2 u = 0$ in the domain of the Figure 11.16.

8. Solve the Poisson's equation $\nabla^2 u = 8x^2y^2$ for the square mesh of Figure 11.17 with $u(x, y) = 0$ on the boundary and mesh length = 1.

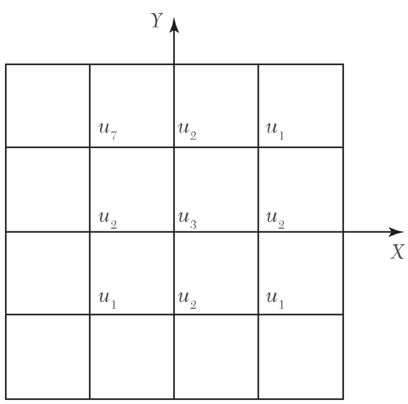

FIGURE 11.17

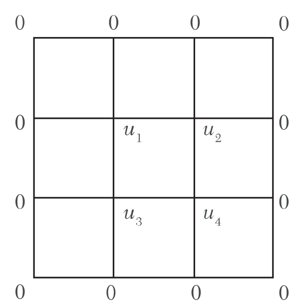

FIGURE 11.18

11.7 Solution of Elliptic Equations by Relaxation Method

If the equations for all the mesh points are written using (2) of Section 11.6, we get a system of equations which can be solved by any method. For this purpose, the method of relaxation is particularly well-suited. Here we shall describe this method in relation to elliptic equations.

Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We take a square region and divide it into a square net of mesh size h. Let the value of u at A be u_0 and its values at the four adjacent points be u_1, u_2, u_3, u_4 (Figure 11.19). Then

$$\frac{\partial^2 u}{\partial x^2} - \approx \frac{u_1 + u_3 - 2u_0}{h^2} \text{ and } \frac{\partial^2 u}{\partial y^2} \approx \frac{u_2 + u_4 - 2u_0}{h^2}$$

If (1) is satisfied at A, then

$$\frac{u_1 - u_3 - 2u_0}{h^2} + \frac{u_2 + u_4 - 2u_0}{h^2} \approx 0$$

or $$u_1 + u_2 + u_3 + u_4 - 4u_0 \approx 0$$

If r_0 be the residual (discrepancy) at the mesh point A,

then $$r_0 = u_1 + u_2 + u_3 + u_4 - 4u_0 \tag{2}$$

Similarly the residual at the point B, is given by

$$r_1 = u_0 + u_5 + u_6 + u_7 - 4u_1 \text{ and so on} \tag{3}$$

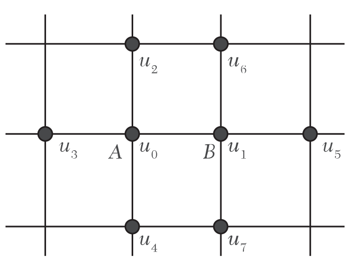

FIGURE 11.19

The main aim of the relaxation process is to reduce all the residuals to zero by making them as small as possible step by step. We, therefore, try to adjust the value of u at an internal mesh point so as to make the residual thereat zero. But when the value of u is changing at a mesh point, the values of the residuals at the neighboring interior points will also be changed. If u_0 is given an increment 1, then

(i) (2) shows that r_0 is changed by -4.

(ii) (3) shows that r_1 is changed by 1.

i.e., if the value of the function is increased by 1 *at a mesh point* (shown by a double ring), *then the residual at that point is decreased by* 4 *while the residuals at the adjacent interior points* (shown by a single ring), *get increased each by* 1. This relaxation pattern is shown in Figure 11.20.

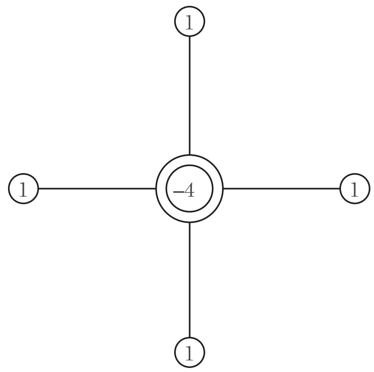

FIGURE 11.20

Working procedure *to solve an equation by the relaxation method*:

I. *Write down by trial, the initial values of u at the interior mesh points by diagonal averaging or cross-averaging.*

II. *Calculate the residuals at each of these points by* (2) *above*. If we apply this formula at a point near the boundary, one or more end points get chopped off since there are no residuals at the boundary.

III. *Write the residuals at a mesh-point on the right of this point and the value of u on its left.*

IV. *Obtain the solution by reducing the residuals to zero, one by one, by giving suitable increments to u and using Figure* 11.20. *At each step, we reduce the numerically largest residual to zero and record the increment of u on the left* (below the earlier value thereat) *and the modified residual on the right* (below the earlier residual).

V. *When a round of relaxation is completed, the value of u and its increments are added at each point. Using these values, calculate all the residuals afresh. If some of there calculated residuals are large, liquidate these again.*

VI. *Stop the relaxation process, when the current values of the residuals are quite small. The solution will be the current value of u at each of the nodes.*

NOTE **Obs.** *Relaxation method combines simplicity with the speed of convergence. Its only drawback is its unsuitability for computer calculations.*

EXAMPLE 11.7

Solve by relaxation method, the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ inside the square bounded by the lines $x = 0$, $x = 4$, $y = 0$, $y = 4$, given that $u = x^2y^2$ on the boundary.

Solution:

Taking $h = 1$, we find u on the boundary from $u = x^2y^2$. The initial values of u at the nine mesh points are estimated to be 24, 56, 104; 16, 32, 56; 8, 16, 24 as shown on the left of the points in Figure 11.21.

$\therefore$ Residual at A, *i.e.*, $r_A = 0 + 56 + 16 + 16 - 4 \times 24 = -8$

Similarly $r_B = 0$, $r_C = -16$, $r_D = 0$, $r_E = 16$, $r_F = 0$, $r_G = 0$, $r_H = 0$, $r_I = -8$.

(*i*) The numerically largest residual is 16 at E. To liquidate it, we increase u by 4 so that the residual becomes zero and the residuals at neighboring nodes get increased by 4.

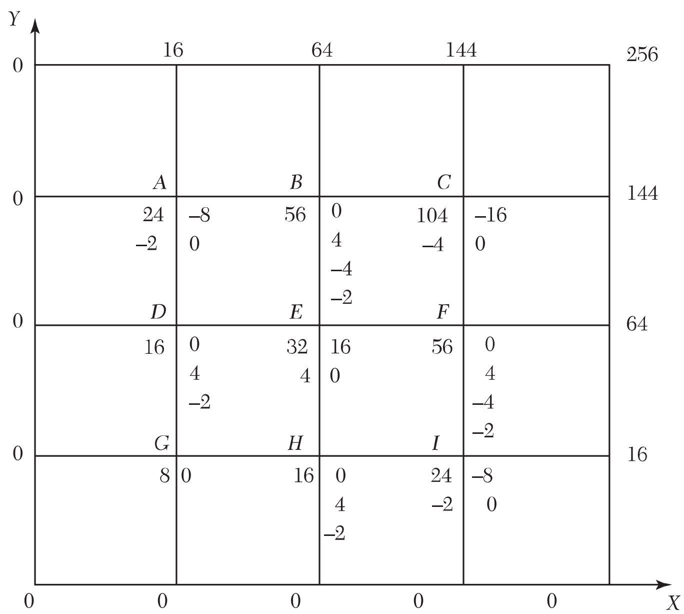

FIGURE 11.21

(*ii*) Next, the numerically largest residual is – 16 at C. To reduce it to zero, we increase u by – 4 so that the residuals at the adjacent nodes are increased by – 4.

(*iii*) Now, the numerically largest residual is – 8 at A. To liquidate it, we increase u by– 2 so that the residuals at the adjacent nodes are increased by – 2.

(*iv*) Finally, the largest residual is – 8 at I. To liquidate it, we increase u by – 2 so that the residuals at the adjacent points are increased by – 2.

(*v*) The numerically largest current residual being 2, we stop the relaxation process. Hence the final values of u are:

$$u_A = 22, \quad u_B = 56, \quad u_C = 100,$$
$$u_D = 16, \quad u_E = 36, \quad u_F = 56,$$
$$u_G = 8, \quad u_H = 16, \quad u_I = 22.$$

EXAMPLE 11.8

Solve by relaxation method Example 11.3.

Solution:

(*i*) The initial values of u at A, B, C, and D are estimated to be 1000, 625, 875, and 375 [Figure 11.22 (*i*)].

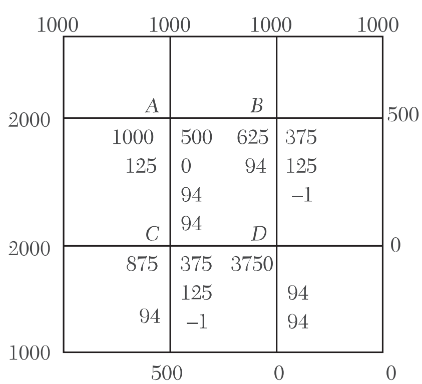

FIGURE 11.22 (I)

$\therefore \quad r_A = 500, r_B = 375, r_C = 375, r_D = 0$

To liquidate r_A, increase u by 125

To liquidate r_B, increase u by 94

To liquidate r_C, increase u by 94

(*ii*) Modified values of u are 1125, 719, 969, 375 [Figure (*ii*)]

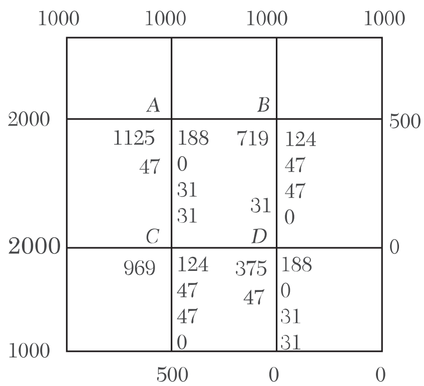

FIGURE 11.22 (II)

$\therefore \qquad r_A = 188,\ r_B = 124,\ r_C = 124,\ r_D = 188.$

To liquidate r_A, r_D, r_B, r_C increase u by 47, 47, 31, 31 in turn.

(*iii*) Revised values of u are 1172, 750, 1000, 422 [Figure (*iii*)]

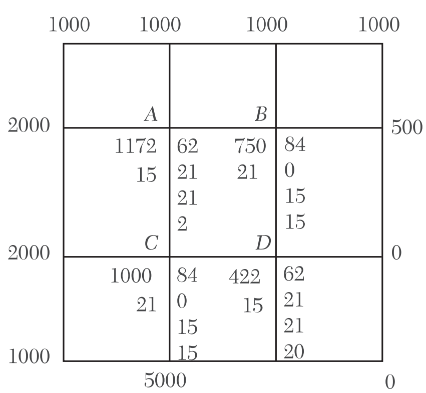

FIGURE 11.22 (III)

$\therefore \qquad r_A = 62,\ r_B = 84,\ r_C = 84,\ r_D = 62$

To liquidate r_B, r_C, r_A, r_D increase u by 21, 21, 15, 15, respectively.

(*iv*) Improved values of u are 1187, 771, 1021, 437 [Figure (*iv*)]

$\therefore \qquad r_A = 44,\ r_B = 40,\ r_C = 40,\ r_D = 44.$

To liquidate r_A, r_D, r_B, r_C increase u by 11, 11, 10, 10, respectively

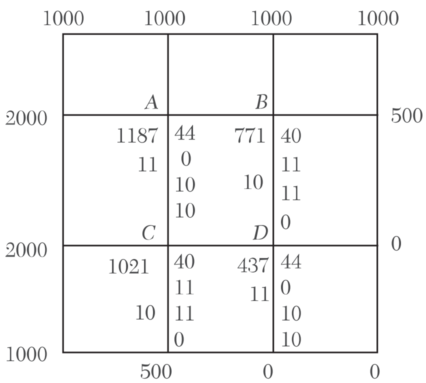

FIGURE 11.22 (IV)

(v) Modified values of u are 1198, 781, 1031, 448 [Figure (v)]

$\therefore \qquad r_A = 20, r_B = 22, r_C = 22, r_D = 20.$

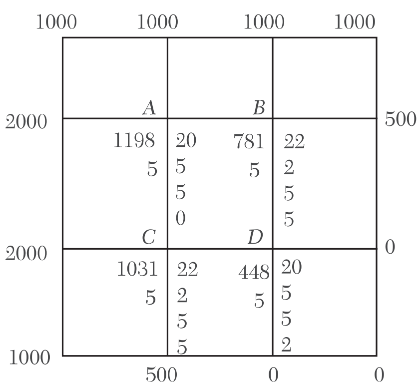

FIGURE 11.22 (V)

To liquidate r_B, r_C, r_A, r_D increase u by 5, 5, 5, 5, respectively.

(vi) Revised values of u are 1203, 786, 1036, 453 [Figure (vi)]

$\therefore \qquad r_A = 10, r_B = 12, r_C = 12, r_D = 10$

To liquidate r_B, r_C, r_A, r_D increase u by 3, 3, 2, 2, respectively.

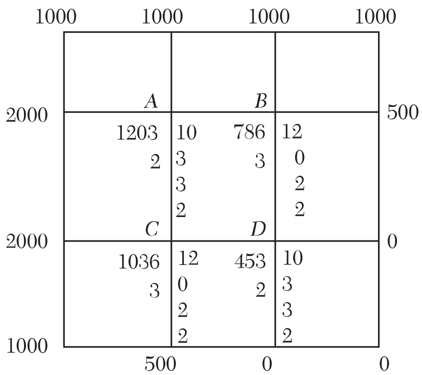

FIGURE 11.22 (VI)

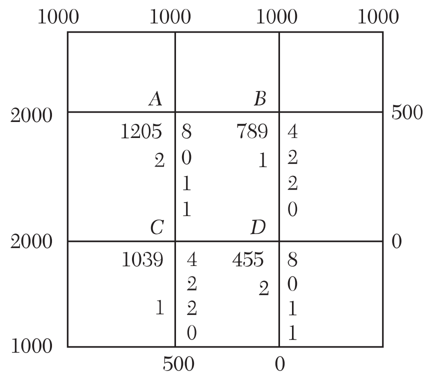

FIGURE 11.22 (VII)

(*vii*) Improved values of u are 1205, 789, 1039, 455 [Figure (*vii*)]

$\therefore \qquad r_A = 8, r_B = 4, r_C = 4, r_D = 8.$

To liquidate r_A, r_D, r_B, r_C increase u by 2, 2, 1, 1.

(*viii*) Finally the current residuals being 1, 0, 0, 1, we stop the relaxation process.

Hence the values of u at A, B, C, D are 1207, 790, 1040, 457.

Exercises 11.3

1. Given that $u(x, y)$ satisfies the equation $\nabla^2 u = 0$ and the boundary conditions are $u(0, y)= 0$, $u(4, y) = 8 + 2y$, $u(x, 0) = \frac{1}{2}x^2$, $u(x, 4) = x^2$, find the values $u(i, j)$, $i = 1, 2, 3; j = 1, 2, 3$by the relaxation method.
2. Apply the relaxation method to solve the equation $\nabla^2 u = -400$, when the region of u is the square bounded by $x = 0$, $y = 0$, $x = 4$, and $y = 4$ and u is zero on the boundary of the square.
3. Solve by relaxation method, the equation $\nabla^2 u = 0$ in the square region with square meshes(Figure 11.23) starting with the initial values $u_1 = u_2 = u_3 = u_4 = 1$.

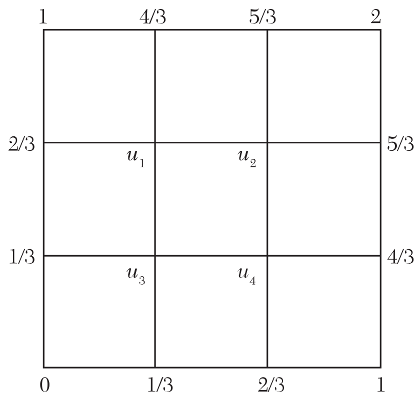

FIGURE 11.23

11.8 Parabolic Equations

The one-dimensional heat conduction equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is a well-known Example of parabolic partial differential equations. The solution of this equation is a temperature function u(x, t) which is defined for values of x from 0 to l and for values of time t from 0 to ∞.The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions (Figure 11.24).

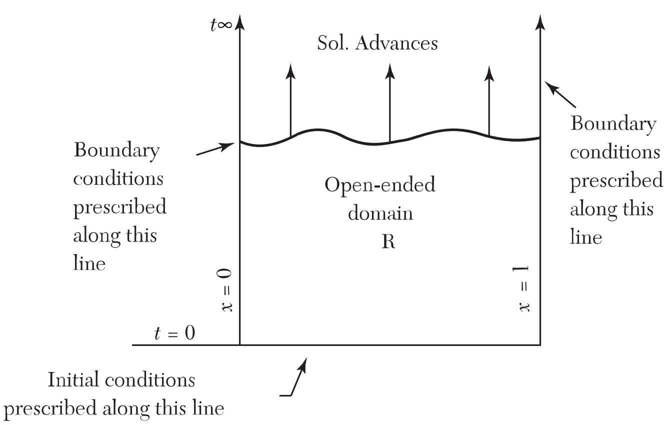

FIGURE 11.24

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

11.9 Solution of One Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{i}$$

where $c^2 = k/s\rho$ is the diffusivity of the substance (cm²/sec.)

Schmidt method. Consider a rectangular mesh in the x-t plane with spacing h along x direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \text{[by (5) Section 11.3.}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad \text{[by (4) Section 11.3.}$$

Substituting these in (1), we obtain $u_{i,j+1} - u_{i,j} = \dfrac{kc^2}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$

or $$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha u_{i+1,j} \tag{2}$$

where $\alpha = kc^2/h^2$ is the mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j + 1)$th mesh point in terms of the known function values at the points x_{i-1}, x_i, and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels $j + 1$ and j and is therefore, called a *two-level formula*. In schematic form (2) is shown in Figure 11.25.

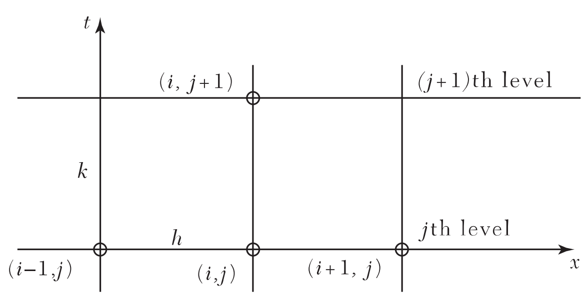

FIGURE 11.25

Hence (2) is called the *Schmidt explicit formula* which is valid only for $0 < \alpha \leq 12$.

NOTE

Obs. *In particular when α = 1/2, (2) reduces to*

$$u_{i,j+1} = 1/2, (u_{i-1,j} + u_{i+1,j}) \tag{3}$$

which shows that the value of u at x_i at time t_{j+1} is the mean of the u-values at x_{i-1} and x_{i+1} at time t_j. This relation, known as Bendre-Schmidt recurrence relation, gives the values of u at the internal mesh points with the help of boundary conditions.

Crank-Nicolson method. We have seen that the Schmidt scheme is computationally simple and for convergent results $\alpha \le 12$ *i.e.,* $k \le h^2/2c^2$. To obtain more accurate results, h should be small *i.e.* k is necessarily very small. This makes the computations exceptionally lengthy as more time levels would be required to cover the region. A method that does not restrict α and also reduces the volume of calculations was proposed by Crank and Nicolson in 1947.

According to this method, $\partial^2u/\partial x^2$ is replaced by the average of its central-difference approximations on the jth and $(j + 1)$th time rows. Thus (1) is reduced to

$$\frac{u_{i,j+1} - u_{i,j}}{h} = c^2 \frac{1}{2}\left\{\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right\} + \left\{\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right\}$$

or $$-\alpha u_{i-1,j+1} + (2 + 2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j} + (2 - 2\alpha)u_{i,j} + \alpha u_{i+1,j} \tag{4}$$

where $\alpha = kc^2/h^2$.

Clearly the left side of (4) contains three unknown values of u at the $(j + 1)$th level while all the three values on the right are known values at the jth level. Thus (4) is a two *level implicit relation* and is known as *Crank-Nicolson formula*. It is convergent for all finite values of α. Its computational model is given in Figure 11.26.

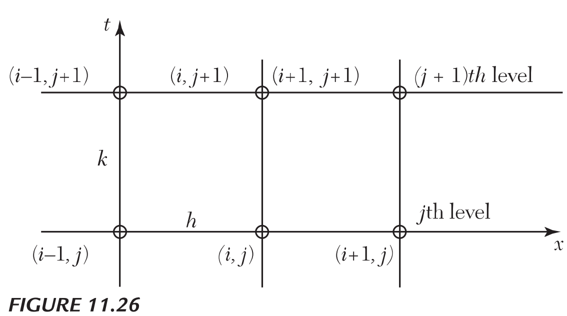

FIGURE 11.26

If there are n internal mesh points on each row, then the relation (4) gives n simultaneous equations for the n unknown values in terms of the known boundary values. These equations can be solved to obtain the values at these mesh points. Similarly, the values at the internal mesh points on all rows can be found. A method such as this in which the calculation of an unknown mesh value necessitates the solution of a set of simultaneous equations, is known as an *implicit scheme*.

Iterative methods of solution *for an implicit scheme*.

From (4), we have

$$(1+\alpha)\, u_{i,j+1} = -\ \alpha(u_{i-1,j+1+ui+1,j+1}) + u_{i,j} + \frac{1}{2}\ \alpha(u_{i-1,j} - 2_{ui,j} + u_{i+1,j}) \tag{5}$$

Here only $u_{i,\,j+1}$, $u_{i-1,\,j+1}$ and $u_{i+1,\,j\,+1}$ are unknown while all others are known since these were already computed in the *j*th step.

Writing $$b_i = u_{i,j} + \frac{\alpha}{2}\left(u_{i-1,j} -\ 2u_{i,j} + u_{i+1,j}\right)$$

and dropping *j*'s (5) becomes $u_i = \dfrac{\alpha}{2(1+\alpha)}\left(u_{i-1} + u_{i+1}\right) + \dfrac{b}{1+\alpha}$

This gives the iteration formula

$$u_i^{(n+1)} = \frac{\alpha}{2(1+\alpha)}\left\{u_{i-1}^{(n)} + u_{i+1}^{(n)}\right\} + \frac{b_i}{1+\alpha} \tag{6}$$

which expresses the $(n + 1)$th iterates in terms of the nth iterates only. This is known as the *Jacobi's iteration formula*.

As the latest value of u_{i-1} *i.e.*, $u_{i-1}^{(n+1)}$ is already available, the convergence of the iteration formula (6) can be improved by replacing $u_{i-1}^{(n)}$ by $u_{i-1}^{(n+1)}$. Accordingly (6) may be written as

$$u_i^{(n+1)} = \frac{\alpha}{2(1+\alpha)}\left\{u_{i-1}^{(n+1)} + u_{i+1}^{(n)}\right\} + \frac{b_i}{1+\alpha} \tag{7}$$

which is known as the *Gauss-Seidal iteration formula*.

NOTE **Obs.** *Gauss-Seidal iteration scheme is valid for all finite values of* α *and converges twice as fast as Jacobi's scheme.*

Du Fort and Frankel method. If we replace the derivatives in (1) by the central difference approximations

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \qquad \text{[From (7) Section 11.3]}$$

and $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j-1+} u_{i+1,j}}{h^2}$$ [From (4) Section 11.3]

we obtain $$u_{i,j+1} - u_{i,j-1} = \frac{2kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

i.e., $$u_{i,j+1} = u_{i,j-1} + 2\alpha\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] \tag{8}$$

where $\alpha = kc^2/h^2$. This difference equation is called the *Richardson scheme* which is a *three-level method*.

If we replace $u_{i,j}$ by the mean of the values $u_{i,j-1}$ and $u_{i,j+1}$

i.e., $u_{i,j} = (u_{i,j\text{-}1} + \frac{1}{2}\ u_{i,j+1})$in (8), then we get

$$u_{i,j+1} = u_{i,j-1} + 2\alpha[u_{i-1,j} - (u_{i,j-1} + u_{i,j+1}) + u_{i+1,j}]$$

On simplification, it can be written as

$$u_{i,j+1} = \frac{1-2\alpha}{1+2\alpha}u_{i,j-1} + \frac{2\alpha}{1+2\alpha}\left\{u_{i-1,j} + u_{i+1,j}\right\} \tag{9}$$

This difference scheme is called *Du Fort-Frankel method* which is a *three level explicit method*. Its computational model is given in Figure 11.27

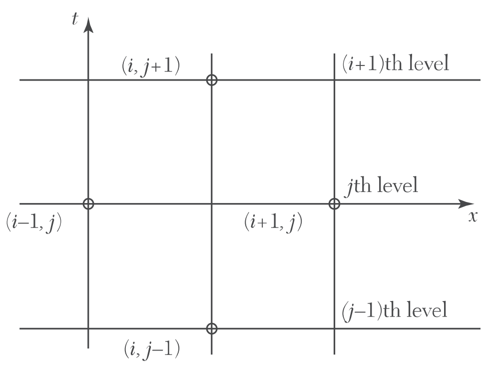

FIGURE 11.27

EXAMPLE 11.9

Solve $\frac{\partial u}{\partial u} = \frac{\partial^2 u}{\partial^2 x^2}$ in $0 < x < 5$, $t \geq 0$ given that $u(x, 0) = 20$, $u(0, t) = 0$, $u(5, t) = 100$. Compute u for the time-step with $h = 1$ by the Crank-Nicholson method.

Solution:

Here $c^2 = 1$ and $h = 1$.

Taking α (*i.e.*, c^2k/h) = 1, we get $k = 1$.

Also we have

J \ I	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Then Crank-Nicholson formula becomes

$$4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}$$

$\therefore$ $4u_1 = 0 + 20 + 0 + u_2$ *i.e.*, $4u_1 - u_2 = 20$ (1)

$4u_2 = 20 + 20 + u_1 + u_3$ *i.e.*, $u_1 - 4u_2 + u_3 = -40$ (2)

$4u_3 = 20 + 20 + u_2 + u_4$ *i.e.*, $u_2 - 4u_3 + u_4 = -40$ (3)

$4u_4 = 20 + 100 + u_3 + 100$ *i.e.*, $u_3 - 4u_4 = -220$ (4)

Now (1) – 4(2) gives $15u_2 - 4u_3 = 180$ (5)

4(3) + (4) gives $4u_2 - 15u_3 = -380$ (6)

Then 15(5) – 4(6) gives $209\, u_2 = 4220$ *i.e.*, $u_2 = 20.2$

From (5), we get $4u_3 = 15 \times 20.2 - 180$ *i.e.*, $u_3 = 30.75$

From (1), $4u_1 = 20 + 20.2$ *i.e.*, $u1 = 10.05$

From (4), $4u_4 = 220 + 30.75$ *i.e.*, $u4 = 62.69$

Thus the required values are 10.05, 20.2, 30.75 and 62.68.

EXAMPLE 11.10

Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t)= u(1, t) = 0$ and $u(x, 0) = \sin px$, $0 \le x \le 1$ using the Schmidt method (Take $h = 0.2$ and $\alpha = 1/2$).

Solution:

Since $h = 0.2$ and $\alpha = ½$

$\therefore$ $\alpha = \dfrac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = 1/2$, we use the Bendre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}(i_{i-1,j} + u_{i+1,j}) \tag{i}$$

We have $u(0, 0) = 0$, $u(0.2, 0) = \sin \pi/5 = 0.5875$

$u(0.4, 0) = \sin 2\pi/5 = 0.9511$, $u(0.6, 0) = \sin 3\pi/5 = 0.9511$

$u(0.8, 0) = \sin 4\pi/5 = 0.5875$, $u(1, 0) = \sin \pi = 0$

The values of u at the mesh points can be obtained by using the recurrence relation (*i*) as shown in the table below:

$x \rightarrow$		0	0.2	0.4	0.6	0.8	1.0
t ↓ 0	j \ i	0	1	2	3	4	5
	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

EXAMPLE 11.11

Find the values of $u(x, t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - (1/2)\, x^2$ at the points $x = i{:}i = 0, 1, 2, \cdots, 7$ and $t = 1/8\ j{:}\ j = 0, 1, 2, \cdots, 5$

Solution:

Here $c^2 = 4$, $h = 1$ and $k = 1/8$. Then $\alpha = c^2k/h^2 = 1/2$.

∴ We have Bendre-Schmidt's recurrence relation

$$u_{i,j+1} = 1/2\,(u_{i-1,j} + u=) \tag{i}$$

Now since $\quad u(0, t) = 0 = u(8, t)$

∴ $u_{0,i} = 0$ and $u_{8,j} = 0$ for all values of j, *i.e.*, the entries in the first and last column are zero.

Since $u(x, 0) = 4x - (1/2)\,x^2$

$\therefore$ $u_{i,\,0} = 4i - (1/2)\,i^2$

$= 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$ for $i = 0, 1, 2, 3, 4, 5, 6, 7$

at $t = 0$

These are the entries of the first row.

Putting $j = 0$ in (i), we have $u_{i,\,1} = (1/2)\,(u_{i-1,\,0} + u_{i+1,\,0})$

Taking $i = 1, 2, \cdots, 7$ successively, we get

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0+6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(3.5+7.5) = 5.5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{3}(6+8) = 7$$

$u_{4,\,1} = 7.5$, $u_{5,\,1} = 7$, $u_{6,\,1} = 5.5$, $u_{7,\,1} = 3$.

These are the entries in the second row.

Putting $j = 1$ in (i), the entries of the third row are given by

$$u_{1,2} = \frac{1}{2}\left(u_{i-1,1} + u_{i+1,1}\right)$$

Similarly putting $j = 2, 3, 4$ successively in (i), the entries of the fourth, fifth, and sixth rows are obtained.

Hence the values of $u_{i,\,j}$ are as given in the following table:

$j \backslash i$	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	2	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

EXAMPLE 11.12

Solve the equation $\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x, 0) = \sin \pi x$, $0 \le x \le 1; u(0, t) = u(1, t) = 0$, using (*a*) Schmidt method, (*b*) Crank-Nicolson method, (*c*) Du Fort-Frankel method. Carryout computations for two levels, taking $h = 1/3$, $k = 1/36$.

Solution:

Here $c^2 = 1$, $h = 1/3$, $k = 1/36$ so that $\alpha = kc^2/h^2 = 1/4$.

Also $u_{1,0} = \sin \pi/3 = \sqrt{3}/2$, $u_{2,0} = \sin 2\pi/3 = \sqrt{3}/2$ and all boundary values are zero as shown in Figure 11.28.

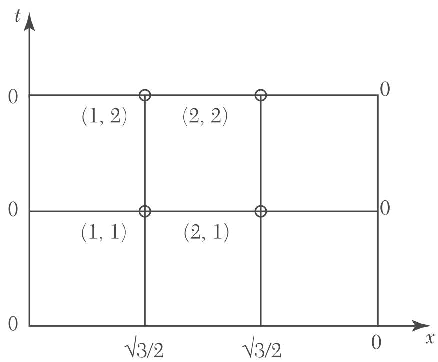

FIGURE 11.28

(*a*) *Schmidt's formula* [(2) of Section 11.9]

$$u_{i, j+1} = \alpha\, u_{i-1, j} + (1 - 2\alpha)\, u_{i, j} + \alpha\, u_{i+1, j}$$

becomes $$u_{i, j+1} = \frac{1}{4}[u_{i-1, j} + 2u_{i, j} + u_{i+1, j}]$$

For $i = 1, 2; j = 0$:

$$u_{1,1} = \frac{1}{4}\ [u_{0,0} + 2u_{1,0} + u_{2,0}] = \frac{1}{4}\ (0 + 2 \times \sqrt{3}/2 + 3/2) = 0.65$$

$$u_{2,1} = \frac{1}{2}\ [u_{1,0} + 2u_{2,0} + u_{3,0}] = \frac{1}{4}\ (\sqrt{3}/2 + 2 \times \sqrt{3}/2 + 0) = 0.65$$

For $i = 1, 2; j = 1$:

$$u_{1,2} = \frac{1}{4}\ (u_{0,1} + 2u_{1,1} + u_{2,1}) = 0.49$$

$$u_{2,2} = \frac{1}{4}\ (u_{1,1} + 2u_{2,1} + u_{3,1}) = 0.49$$

(*b*) *Crank-Nicolson formula* [(4) of Section 11.9] becomes

$$-\frac{1}{4}u_{i-1,j+1}+\frac{5}{2}u_{i,j+1}-\frac{1}{4}u_{i+1,j+1}=\frac{1}{4}u_{i-1,j}+\frac{3}{2}u_{i,j}+\frac{1}{4}u_{i+1,j}$$

For $i = 1, 2; j = 0$:

$$-u_{0,1}+10u_{1,1}-u_{2,1}=u_{0,0}+6u_{1,0}+u_{2,0}$$

i.e., $$10u_{1,1}-u_{2,1}=7\sqrt{3}/2$$

$$-u_{1,1}+10u_{2,1}-u_{3,1}=u_{1,0}+6u_{2,0}+u_{3,0}$$

i.e., $$-u_{1,1}+10u_{2,1}=7\sqrt{3}/2$$

Solving these equations, we find

$$u_{1,1}=u_{2,1}=0.67$$

For $i = 1, 2; j = 1$:

$$-u_{0,2}+10u_{1,2}-u_{2,2}=u_{0,1}+6u_{1,1}+u_{2,1}$$

i.e., $$10u_{1,2}-u_{2,2}=4.69$$

$$-u_{1,2}+10u_{2,2}-u_{3,2}=u_{1,1}+6u_{2,1}+u_{3,1}$$

i.e, $$-u_{1,2}+10_{u2,2}=4.69$$

Solving these equations, we get $u_{1,2} = u_{2,2} = 0.52$.

(*c*) *Du Fort-Frankel formula* [(8) of Section 11.9] becomes $u_{i,j+1}=\frac{1}{3}(u_{i,j-1}+u_{i-1,j}+u_{i+1,j})$

To start the calculations, we need $u_{1,1}$ and $u_{2,1}$.

We may take $u_{1,1} = u_{2,1} = 0.65$ from Schmidt method.

For $i = 1, 2; j = 1$:

$$u_{1,2}=\frac{1}{3}(u_{1,0}+u_{0,1}+u_{2,1})=\frac{1}{3}(\sqrt{3}/2+0+0.65)=0.5$$

$$u_{2,2}=\frac{1}{3}(u_{2,0}+u_{1,1}+u_{3,1})=\frac{1}{3}(\sqrt{3}/2+0.65+0)=0.5.$$

11.10 Solution of Two Dimensional Heat Equation

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

The methods employed for the solution of one dimensional heat equation can be readily extended to the solution of (1).

Consider a square region $0 \le x \le y \le a$ and assume that u is known at all points within and on the boundary of this square.

If h is the step-size then a mesh point $(x, y, t) = (ih, jh, nl)$ may be denoted as simply (i, j, n).

Replacing the derivatives in (1) by their finite difference approximations, we get

$$\frac{u_{li,j,n+1} - u_{i,j,n}}{l} = \frac{c^2}{h^2}\{(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}) + (u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n})\}$$

i.e., $$u_{i,j,n+1} = u_{i,j,n} + \alpha(u_{i-1,j,n} + u_{i+1,j,n} + u{+}{+}\,u_{i,j-1,n} - 4u_{i,j,n}) \qquad (2)$$

where $\alpha = lc^2/h^2$. This equation needs the five points available on the nth plane (Figure 11.29).

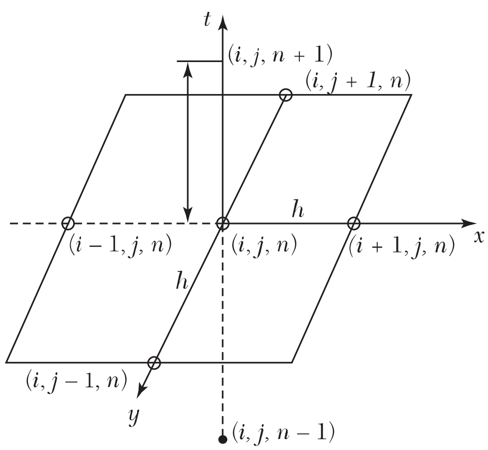

FIGURE 11.29

The computation process consists of point-by-point evaluation in the $(n + 1)$th plane using the points on the nth plane. It is followed by plane-by-plane evaluation. This method is known as *ADE* (*Alternating Direction Explicit*) *method*.

EXAMPLE 11.13

Solve the equation $\dfrac{\partial u}{\partial u} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ subject to the initial conditions $u(x,y,0) = \sin 2\pi x \sin 2\pi y$, $0 \le x, y \le 1$, and the conditions $u(x, y, t) = 0$,

$t > 0$ on the boundaries, using ADE method with $h = 1/3$ and $\alpha = 1/8$. (Calculate the results for one time level).

Solution:

The equation (2) above becomes

$$u_{i,j,n+1} = u_{i,j,n} + \frac{1}{8}\,(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j+1,n} + u_{i,j-1,n} - 4u_{i,j,n})$$

i.e., $$u_{i,j,n+1} = \frac{1}{2}\,u_{i,j,n} + \frac{1}{8}\,(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j+1,n} + u_{i,j-1,n}) \tag{1}$$

The mesh points and the computational model are given in Figure 11.30.

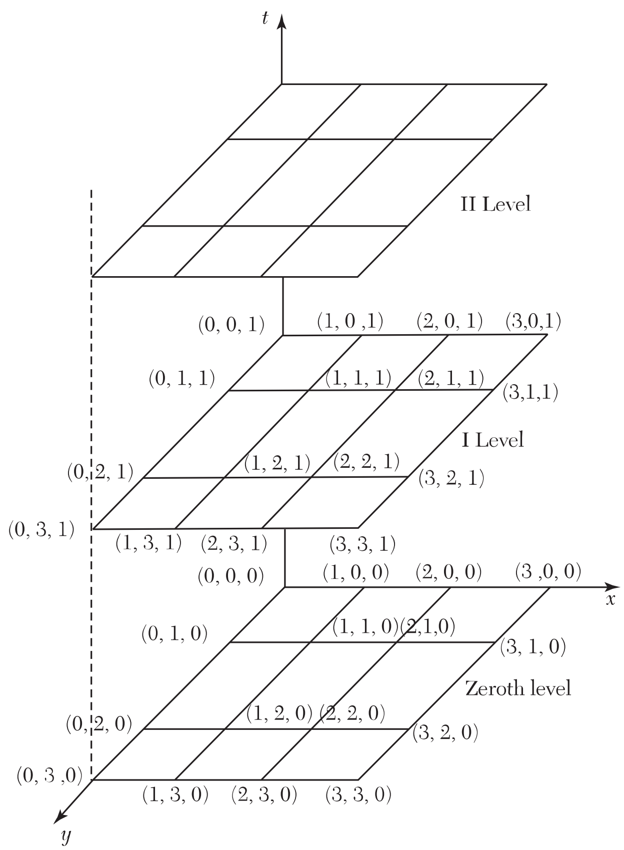

FIGURE 11.30

At the zero level ($n = 0$), the initial and boundary conditions are

$$u_{i,j,0} = \sin\frac{2\pi i}{3}\sin\frac{2\pi i}{3}$$

and $\quad u_i, 0, 0 = u_0, j, 0 = u_3, j, 0 = u_i, 3, 0 = 0; i, j = 0, 1, 2, 3.$

Now we calculate the mesh values at the first level:

For $n = 0$, (1) gives

$$u_{i,j,1} = \frac{1}{2}\ u_{i,j,0} + \frac{1}{8}\ (u_{i-1,j,0} + u_{i+1,j,0} + u_{i,j+1,0} + u_{i,j-1,0}) \tag{2}$$

(i) Put $i = j = 1$ in (2):

$$u_{1,1,1} = \frac{1}{2}\ u_{1,1,0} + \frac{1}{8}\ (u_{0,1,0} + u_{2,1,0} + u_{1,2,0} + u_{1,0,0})$$

$$= \frac{1}{2}\left(\sin\frac{2\pi}{3}\right)^2 + \frac{1}{8}\left(0 + \sin\frac{4\pi}{3}\sin\frac{2\pi}{3} + \sin\frac{2\pi}{3}\sin\frac{4\pi}{3} + 0\right)$$

$$= \frac{3}{8} + \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}\times\right) = \frac{3}{16}$$

(ii) Put $i = 2, j = 1$ in (2)

$$u_{2,1,1} = \frac{1}{2}u_{2,1,0} + \frac{1}{8}(u_{1,1,0} + u_{3,1,0} + u_{2,2,0} + u_{2,0,0})$$

$$= \frac{1}{2}\sin\frac{4\pi}{3}\sin\frac{2\pi}{3} + \frac{1}{8}\left\{\left(\sin\frac{2\pi}{3}\right)^2\right\} + 0 + \left(\sin\frac{4\pi}{3}\right)^2 + 0$$

$$= -\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{8}\left\{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2\right\} = -\frac{3}{16}$$

(iii) Put $i = 1, j = 2$ in (2):

$$u_{1,2,1} = \frac{1}{2}\ u_{1,2,0} + \frac{1}{8}\ (u_{0,2,0} + u_{2,2,0} + u_{1,1,0})$$

$$= \frac{1}{2}\sin\frac{2\pi}{3}\sin\frac{4\pi}{3} + \frac{1}{8}\left\{0 + \left(\sin\frac{4\pi}{3}\right)^2 + 0 + \left(\sin\frac{2\pi}{3}\right)^2\right\}$$

$$= -\frac{3}{8} + \frac{1}{8}\left(\frac{3}{4} + \frac{3}{4}\right) = -\frac{3}{16}$$

(*iv*) Put $i = 2, j = 2$ in (2):

$$u_{2,2,1} = \frac{1}{2}\ u_{2,2,0} + \frac{1}{8}\ (u_{1,2,0} + u_{3,2,0} + u_{2,3,0} + u_{2,1,0})$$

$$= \frac{1}{2}\left(\sin\frac{4\pi}{3}\right)^2 + \frac{1}{8}\left(\sin\frac{2\pi}{3}\sin\frac{4\pi}{3} + 0 + 0 + \sin\frac{4\pi}{3}\sin\frac{2\pi}{3}\right)$$

$$= \frac{3}{8} + \frac{1}{8}\left(-\frac{3}{4} - \frac{3}{4}\right) = -\frac{3}{16}$$

Similarly the mesh values at the second and higher levels can be calculated.

Exercises 11.4

1. Find the solution of the parabolic equation $u_{xx} = 2u_t$ when $u(0, t) = u(4, t) = 0$ and $u(x, 0) = x(4 - x)$, taking $h = 1$. Find the values up to $t = 5$.

2. Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with the conditions $u(0, t) = 0$, $u(x, 0) = x(1 - x)$, and $u(1, t) = 0$. Assume $h = 0.1$. Tabulate u for $t = k$, $2k$ and $3k$ choosing an appropriate value of k.

3. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial f} = 0$; $f(0, t) = f(5, t) = 0$, $f(x, 0) = x^2(25 - x^2)$; find the values of f for $x = ih$ ($i = 0, 1, ..., 5$) and $t = jk$ ($j = 0, 1, ..., 6$) with $h = 1$ and $k = 1/2$, using the explicit method.

4. Given $\partial u/\partial t = \partial^2 u/\partial t^2$, $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = x/3(16 - x^2)$. Obtain $u_{i,j}$ for10 = 1, 2, 3, 4 and $j = 1, 2$ using Crank-Nicholson's method.

5. Solve the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(0, t) = u(1, t) = 0$ and

$$u(x,0) = \begin{cases} 2x \text{ for } 0 \le x \le 1/2 \\ 2(1-x) \text{ for } 1/2 \le x \le 1 \end{cases}$$

Take $h = 1/4$ and k according to the Bandre-Schmidt equation.

6. Solve the two dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ satisfying the initial condition: $u(x, y, 0) = \sin \pi x \sin \pi y$, $0 \le x, y \le 1$ and the boundary conditions: $u = 0$ at $x = 0$ and $x = 1$for $t > 0$. Obtain the solution up to two time levels with $h = 1/3$ and $\alpha = 18$.

11.11 Hyperbolic Equations

The wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is the simplest Example of hyperbolic partial differential equations. Its solution is the displacement function $u(x, t)$ defined for values of x from 0 to l and for t from 0 to ∞, satisfying the initial and boundary conditions. The solution, as for parabolic equations, advances in an open-ended region (Figure 11.24). In the case of hyperbolic equations however, we have two initial conditions and two boundary conditions.

Such equations arise from convective type of problems in vibrations, wave mechanics, and gas dynamics.

11.12 Solution of Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

subject to the initial conditions: $u = f(x)$, $\partial u/\partial t = g(x)$, $0 \le x \le 1$ at $t = 0$ (2)

and the boundary conditions: $u(0, t) = \phi(t)$, $u(1, t) = \psi(t)$ (3)

Consider a rectangular mesh in the x-t plane spacing h along x direction and k along time direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i+1,j}}{h^2} \text{ and } \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}$$

Replacing the derivatives in (1) by their above approximations, we obtain

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{c^2k^2}{h^2}\ (u_{i-1,j} - 2u_{i,j} + u_{i,j+1})$$

or $$u_{i,j+1} = 2(1 - \alpha^2c^2)\, u_{i,j} + \alpha^2c^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \tag{4}$$

where $\alpha = k/h$.

Now replacing the derivative in (2) by its central difference approximation, we get

$$\frac{ui,j+1 - ui,j-1}{2k} = \frac{\partial u}{\partial t} = g(x)$$

or $$u_{i,j+1} = u_{i,j-1} + 2kg(x) \text{ at } t = 0$$

i.e., $$u_{i,1} = u_{i,-1} + 2kg(x) \text{ for } j = 0 \tag{5}$$