# Interpolation with Unequal Intervals

## Lagrange's Interpolation formula

If $y = f(x)$ takes the values $y_0, y_1, ---, y_n$ corresponding to $x_0, x_1, ---, x_n$, then ~~if~~ since there are $(n+1)$ points, so we can represent $f(x)$ by a polynomial in $x$ of degree $n$. Let this polynomial be of the form:-

$$y = f(x) = a_0(x-x_1)(x-x_2)---(x-x_n) + a_1(x-x_0)(x-x_2)---(x-x_n) + ---+ a_n(x-x_0)(x-x_1)---(x-x_{n-1}). \quad (1)$$

Putting $x = x_0$, $y = y_0$ in (1), we get

$$y_0 = f(x_0) = a_0(x_0-x_1)(x_0-x_2)--(x_0-x_n)$$

$$\Rightarrow \boxed{a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)--(x_0-x_n)}}$$

Putting $x = x_1$ & $y = y_1$ in (2), we get

$$y_1 = f(x_1) = a_1(x_1-x_0)(x_1-x_2)--(x_1-x_n)$$

$$\Rightarrow \boxed{a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)--(x_1-x_n)}}$$

Proceeding the same way, we find $a_1, a_2, --a_n$.

Substituting the values of $a_0, a_1, \ldots a_n$ in (1),

$$f(x) = \frac{(x_0 - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} y_0 +$$

$$\frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} \cdot y_1 +$$

$$\cdots + \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} \cdot y_n$$

**Example 7.17.** *Given the values*

| | | | | | |
|---|---|---|---|---|---|
| *x* : | 5 | 7 | 11 | 13 | 17 |
| *f(x)* : | 150 | 392 | 1452 | 2366 | 5202, |

*(V.T.U., B. Tech., 2006)*

*evaluate f(9), using Lagrange's formula*

**Sol.** (*i*) Here $x_0 = 5, \quad x_1 = 7, \quad x_2 = 11, \quad x_3 = 13, \quad x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810$$

**Example 7.18.** *Find the polynomial f (x) by using Lagrange's formula and hence find f (3) for*

| *x* : | 0 | 1 | 2 | 5 |
|---|---|---|---|---|
| *f (x)* : | 2 | 3 | 12 | 147 |

*(Anna, B. Tech., 2012)*

**Sol.** Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147.$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $f(x) = x^3 + x^2 - x + 2$

$\therefore$ $f(3) = 27 + 9 - 3 + 2 = 35.$

**Example 7.19.** *A curve passes through the points (0, 18), (1, 10), (3, – 18) and (6, 90). Find the slope of the curve at x = 2.*

*(J.N.T.U., B.Tech., 2009)*

**Sol.** Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$

Since the values of $x$ are unequally spaced, we use the Lagrange's formula :

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10)$$

$$+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

*i.e.*, $$y = 2x^3 - 10x^2 + 18$$

Thus the slope of the curve at $x = 2 = \left(\frac{dy}{dx}\right)_{x=2}$

$$\therefore \quad = (6x^2 - 20x)_{x=2} = -16.$$