and $$k_3 = hf\left(x_0 + h, y_0 + k'\right)$$

Finally compute, $k = \frac{1}{6}\left(k_1 + 4k_2 + k_3\right)$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, and k_3).

EXAMPLE 10.14

Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here we have $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2(1) = 0.200$$

$$k_2 = hf\left(x_0 + \frac{1}{2}hy_0 + \frac{1}{2}k_1\right) = 0.2f(0.1, 1.1) \quad 0.240$$

$$k' = hf\left(x_0 + h, y_0 + k_1\right) = 0.2f(0.2, 1.2) = 0.280$$

and $$k_3 = hf\left(x_0 + h, y_0 + k'\right) = 0.2f(0.1, 1.28) = 0.296$$

$$\therefore \quad k = \frac{1}{6}\left(\mathrm{k}_1 + 4\mathrm{k}_2 + \mathrm{k}_3\right) = \frac{1}{6}(0.200 + 0.960 + 0.296) = 0.2426$$

Hence the required approximate value of y is 1.2426.

10.7 Runge-Kutta Method*

The Taylor's series method of solving differential equations numerically is restricted by the labor involved in finding the higher order derivatives. However, there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution up to the term in h^r where r differs from method to method and is called the *order of that method*.

First order R-K method. We have seen that Euler's method (Section 10.4) gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy'_0 \qquad [\because y' = f(x, y)]$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \cdots$$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in h.

Hence, *Euler's method is the Runge-Kutta method of the first order*.

Second order R-K method. The modified Euler's method gives

$$y_1 = y + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1)\right] \tag{1}$$

Substituting $y_1 = y_0 + hf(x_0, y_0)$ on the right-hand side of (1), we obtain

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h), y_0 + hf_0\right] \quad \text{where} \quad f_0 = (x_0, y_0) \tag{2}$$

Expanding L.H.S. by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \tag{3}$$

Expanding $f(x_0 + h, y_0 + hf_0)$ by Taylor's series for a function of two variables, (2) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f_0 = (x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)^{**}\right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0\right\} + O(h^3)\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + O(h^3) \qquad \left[\because \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]$$

$$= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + O(h^3) \tag{4}$$

Comparing (3) and (4), it follows that the modified Euler's method agrees with the Taylor's series solution upto the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

$^{**}O(h^2)$ means "terms containing second and higher powers of h" and is read as *order of* h^2.

∴ *The second order Runge-Kutta formula is*

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

Where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k)$

(*iii*) *Third order R-K method.* Similarly, it can be seen that Runge's method (Section 10.6) agrees with the Taylor's series solution upto the term in h^3.

As such, *Runge's method is the Runge-Kutta method of the third order.*

∴ *The third order Runge-Kutta formula is*

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where, $k_1 = hf(x_0, y_0), k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

And $\quad k_3 = hf\left(x_0 + h, y_0 + k'\right)$, where $k' = k_3 = hf(x_0 + h, y_0 + k_1)$.

(*iv*) *Fourth order R-K method.* This method is most commonly used and is often referred to as the *Runge-Kutta method* only.

Working rule *for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from*

$$\frac{dy}{dx} = f(x, y), y(x_0)$$

is as follows*:*

Calculate successively $k_1 = hf(x_0, y_0)$,

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

and $$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

Finally compute $$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3, and k_4).

NOTE **Obs.** *One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.*

EXAMPLE 10.15

Apply the Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here $\quad x_0 = 0,\ y_0 = 1,\ h = 0.2,\ f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $\quad k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$$

$$= \frac{1}{6} \times (1.4568) = 0.2428$$

Hence the required approximate value of y is 1.2428.

EXAMPLE 10.16

Using the Runge-Kutta method of fourth order, solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.

Solution:

We have $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$

Hence $x_0 = 0,\ y_0 = 1,\ h = 0.2$

$$k_1 = hf(x_0, y_0) = 0.2 f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] = 0.19599$$

Hence $y(0.2) = y_0 + k = 1.196$.

To find $y(0.4)$:

Here $x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2$.

$$k_1 = hf(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752$.

EXAMPLE 10.17

Apply the Runge-Kutta method to find the approximate value of y for $x = 0.2$, in steps of 0.1, if $dy/dx = x + y^2$, $y = 1$ where $x = 0$.

Solution:

Given $f(x, y) = x + y^2$.

Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x0 = 0$, $y0 = 1$, $h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.1f(0,1) = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1f(0.05, 1.1) = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1f(0.05, 1.1152) = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1168) = 0.1347$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165$

Step II. $x_1 = x_0 + h = 0.1$, $y_1 = 1.1165$, $h = 0.1$

$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1165) = 0.1347$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1f(0.15, 1.1838) = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1f(0.15, 1.194) = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1f(0.2, 1.1576) = 0.1823$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$

EXAMPLE 10.18

Using the Runge-Kutta method of fourth order, solve for y at $x = 1.2$, 1.4

From $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ given $x_0 = 1$, $y_0 = 0$

Solution:

We have $f(x, y) = \dfrac{2xy + e^x}{x^2 + xe^x}$

To find $y(1.2)$:

Here $x_0=1$, $y_0=0$, $h=0.2$

$$k_1 = hf(x_0, y_0) = 0.2\frac{0+e'}{1+e'} = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left\{\frac{2(1+0.1)(0+0.073)e^{1+0.1}}{(1+0.1)^2 + (1+0.1)e^{1+0.1}}\right\}$$

$$= 0.1402$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2\left\{\frac{2(1+0.1)(0+0.07)e^{1.1}}{(1+0.1)^2 + (1+0.1)e^{1.1}}\right\}$$

$$= 0.1399$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = 0.2\left\{\frac{2(1.2)(0.1399)e^{1.2}}{(1.2)^2 + (1.2)e^{1.2}}\right\}$$

$$= 0.1348$$

and $$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \frac{1}{6}[0.1462 + 0.2804 + 0.2798 + 0.1348]$$

$$= 0.1402$$

Hence $y(1.2) = y0 + k = 0 + 0.1402 = 0.1402.$

To find y (1.4):

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$

$$k_1 = hf(x_1, y_1) = 0.2f(1.2, 0) = 0.1348$$

$$k_2 = hf\left(x_1 + h/2, y_1 + k_1/2\right) = 0.2f(1.3, 0.2076) = 0.1303$$

$$k_3 = hf\left(x_1 + h/2, y_1 + k_2/2\right) = 0.2f(1.3, 0.2053) = 0.1301$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(1.3, 0.2703) = 0.1260$$

$$\therefore \quad k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}[0.1348 + 0.2606 + 0.2602 + 0.1260]$$

$$= 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705.$

Exercises 10.3

1. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.

2. Using the Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + y^2$, $y(0) = 1$ taking $h = 0.1$.

3. Using the Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from $10\dfrac{dy}{dx} = x^2 + y^2 \ \ y(0) = 1$, taking $h = 0.1$.

4. Use the Runge Kutta method to find y when $x = 1.2$ in steps of 0.1, given that $dy/dx = x^2 + y^2$ and $y(1) = 1.5$.

5. Given $dy/dx = x^3 + y$, $y(0) = 2$. Compute $y(0.2)$, $y(0.4)$, and $y(0.6)$ by the Runge-Kutta method of fourth order.

6. Find $y(0.1)$ and $y(0.2)$ using the Runge-Kutta fourth order formula, given that $y' = x^2 - y$ and $y(0) = 1$.

7. Using fourth order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3$. $dy/dx\ (4x/y - xy)$. Calculate y for $x = 0.1$ and 0.2.

8. Find by the Runge-Kutta method an approximate value of y for $x = 0.6$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x+y)}$

9. Using the Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, , $y(0) = 1$. Take $h = 0.2$.

10. Using fourth order Runge-Kutta method, integrate $y' = -2x^3 + 12x^2 - 20x + 8.5$, using a step size of 0.5 and initial condition of $y = 1$ at $x = 0$.

11. Using the fourth order Runge-Kutta method, find y at $x = 0.1$ given that $dy/dx = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$.

12. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ at $x = 0$, find y for $x = 0.1$, 0.2, 0.3, 0.4, and 0.5.

10.8 Predictor-Corrector Methods

If x_{i-1} and x_i are two consecutive mesh points, we have $x_i = x_{i-1} + h$. In Euler's method (Section 10.4), we have

$$y_i = y_{i-1} + hf(x_0 + \overline{i-1}h, y_{i-1}); \quad i = 1,2,3\cdots \tag{1}$$

The modified Euler's method (Section 10.5), gives

$$y_i = y_{i-1} + \frac{h}{2}\left[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)\right]$$

The value of y_i is first estimated by using (1), then this value is inserted on the right side of (2), giving a better approximation of y_i. This value of y_i is again substituted in (2) to find a still better approximation of y_i. This step is repeated until two consecutive values of y_i agree. *This technique of refining an initially crude estimate of y_i by means of a more accurate formula is known as* **predictor-corrector method.** The equation (1) is therefore called the *predictor* while (2) serves as a *corrector* of y_i.

In the methods so far described to solve a differential equation over an interval, only the value of y at the beginning of the interval was required. In the predictor-corrector methods, four prior values are needed for finding the value of y at x_i. Though slightly complex, these methods have the advantage of giving an estimate of error from successive approximations to y_i.

We now describe two such methods, namely: Milne's method and Adams-Bashforth method.

10.9 Milne's Method

Given $dy/dx = f(x, y)$ and $y = y_0$, $x = x_0$; to find an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as follows:

The value $y_0 = y(x_0)$ being given, we compute

$$y_1 = y(x_0 + h),\ y_2 = y(x_0 + 2h),\ y_3 = y(x_0 + 3h),$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0), f_1 = f(x_0 + h, y_1), f_2 = f(x_0 + 2h, y_2), f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x,y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + \cdots$$

In the relation

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x,y)dx$$

$$y_4 = y_0 + \int_{x_0}^{x_0+4h}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots\right)dx$$

[Put $x = x_0 + nh,\ dx = hdn$]

$$= y_0 + \int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots\right)dn$$

$$= y_0 + h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \ldots\right)$$

Neglecting fourth and higher order differences and expressing $\Delta f_0, \Delta^2 f_0$ and $\Delta^3 f_0$ and in terms of the function values, we get

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

which is called a *predictor*.

Having found y_4, we obtain a first approximation to

$$f_4 = f(x_0 + 4h, y_4)$$

Then a better value of y_4 is found by Simpson's rule as

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

which is called a *corrector*.

Then an improved value of f_4 is computed and again the corrector is applied to find a still better value of y_4. We repeat this step until y_4 remains unchanged. Once y_4 and f_4 are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the *predictor* as

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of y_5 is obtained from the *corrector* as

$$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$$

We repeat this step until y_5 becomes stationary and, then proceed to calculate y_6 as before.

This is *Milne's predictor-corrector method*. To insure greater accuracy, we must first improve the accuracy of the starting values and then subdivide the intervals.

EXAMPLE 10.19

Apply Milne's method, to find a solution of the differential equation $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary condition $y = 0$ at $x = 0$.

Solution:

Using Picard's method, we have

$$y = y(0) + \int_0^x f(x, y)dx, \text{ where } f(x, y) = x - y^2$$

To get the first approximation, we put $y = 0$ in $f(x, y)$,

Giving $y_1 = 0 + \int_0^x x dx = \frac{x^2}{2}$

To find the second approximation, we put

Giving $y_2 = \int_0^x \left(x - \frac{x^4}{4} \right) dx = \frac{x^2}{2} - \frac{x^5}{20}$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20} \right)^2 \right] dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \qquad (i)$$

Now let us determine the starting values of the Milne's method from (i), by choosing $h = 0.2$.

$x_0 = 0.0, \quad y_0 = 0.0000, \quad f_0 = 0.0000$

$x_1 = 0.2, \quad y_1 = 0.020, \quad f_1 = 0.1996$

$x_2 = 0.4, \quad y_2 = 0.0795 \quad f_2 = 0.3937$

$x_3 = 0.5, \quad y_3 = 0.1762, \quad f_3 = 0.5689$

Using the predictor, $y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$x = 0.8 \quad y_4^{(p)} = 0.3049, \quad f_4 = 0.7070$

and the corrector, $y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$, yields

$$y_4^{(c)} = 0.3046 \qquad f_4 = 0.7072 \qquad (ii)$$

Again using the *corrector*,

$$y_4^{(c)} = 0.3046, \text{, which is the same as in (ii)}$$

Now using the predictor,

$$y_4^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

$x = 0.1$, $\quad y_5^{(p)} = 0.4554 \qquad f_5 = 0.7926$

and the corrector $y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$ gives

$$y_5^{(c)} = 0.4555 \qquad f_5 = 0.7925$$

Again using the corrector,

$$y_5^{(c)} = 0.4555, \text{ a value which is the same as before.}$$

Hence $y(1) = 0.4555$.

EXAMPLE 10.20

Using Milne's method find $y(4.5)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$; $y(4.4) = 1.0187$.

Solution:

We have $y' = (2 - y^2)/5x = f(x)$ [say]

Then the starting values of the Milne's method are

$x_0 = 0$, $\quad y_0 = 1$, $\quad f_0 = \frac{2-12}{5\times 4} = 0.05$

$x_1 = 4.1$, $\quad y_1 = 1.0049$, $\quad f_1 = 0.0485$

$x_2 = 4.2$, $\quad y_2 = 1.0097$, $\quad f_2 = 0.0467$

$x_3 = 4.3$, $\quad y_3 = 1.0143$, $\quad f_3 = 0.0452$

$x_4 = 4.4$, $\quad y_4 = 1.0187$, $\quad f_4 = 0.0437$

Since y_5 is required, we use the predictor

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4') \qquad (h = 0.1)$$

$$x = 4.5,\ y_5^{(p)} = 1.0049 + \frac{4(0.1)}{3}(2\times 2.0467 - 0.0452 + 2\times 0.0437) = 1.023$$

$$f_5 = \frac{2 - y_5^2}{5x_5} = \frac{2-(1.023)^2}{5 \times 4.5} = 0.0424$$

Now using the corrector $y_5^{(c)} = y_3 + \frac{h}{3}\left(f_3 + 4f_4 + f_5\right)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3}(0.0452 + 4 \times 0.0437 + 0.0424) = 1.023$$

Hence $y(4.5) = 1.023$

EXAMPLE 10.21

Given $y' = x(x^2 + y^2)\, e^{-x}$, $y(0) = 1$, find y at $x = 0.1$, 0.2, and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.

Solution:

Given $y(0) = 1$ and $h = 0.1$

We have $\quad y'(x) = x(x^2 + y^2)e^{-x} \qquad y'(0) = 0$

$$\therefore \quad y''(x) = \left[(x^3 + xy^2)(e^{-x}) + (3x^2 + y^2 + x(2y)y')\right]e^{-x}$$

$$= e^{-x}\left[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy'\right]; \qquad y''(0) = 1$$

$$y'''(x) = e^{-x}\left[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy'\right]$$

$$y'''(0) = 2$$

Substituting these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \cdots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \cdots$$

$$= 1 + 0.005 - 0.0003 = 1.0047, \textit{ i.e.}, 1.005$$

Now taking $\quad x = 0.1$, $y(0.1) = 1.005$, $h = 0.1$

$$y'(0.1) = 0.092, y''(0.1) = 0.849, y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \cdots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{6}(-1247) + \cdots$$

$$= 1.018$$

Now taking $x = 0.2, y(0.2) = 1.018, h = 0.1$

$$y'(0.2) = 0.176, y''(0.2) = 0.77, y'''(0.2) = 0.819$$

Substituting these values in the Taylor's series

$$y(0.2) = y(0.2) + \frac{0.1}{1!}y''(0.2) + \frac{(0.1)^2}{2!}y''(0.2) + \frac{(0.1)^3}{3!}y'''(0.2) + \cdots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001$$

$$= 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$$\begin{array}{lll} x_0 = 0.0, & y_0 = 1 & f_0 = y_0 = 0 \\ x_1 = 0.1, & y_1 = 1.005 & f_1 = 0.092 \\ x_2 = 0.2, & y_2 = 1.018 & f_2 = 0.176 \\ x_3 = 0.3, & y_3 = 1.04 & f_3 = 0.26 \end{array}$$

Using the predictor, $y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}[2(0.092) - 0.176 + 2(0.26)]$$

$$= 1.09.$$

$x = 0.4 \quad y_4^{(p)} = 1.09, \qquad f_4 = y'(0.4) = 0.362$

Using the corrector, $y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$, yields

$$y_4^{(c)} = 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $y(0.4) = 1.071$

EXAMPLE 10.22

Using the Runge-Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at $x = 0.4$ using Milne's method.

Solution:

We have $\quad f(x, y) = xy + y^2.$

To find $y(0.1)$:

Here $x_0 = 0, y_0 = 1, h = 0.1.$

$\therefore \qquad k_1 = hf(x_0, y_0) = (0.1) \times f(0,1) = 0.1000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) \times f(0.05, 1.05) = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) \times f(0.05, 1.0577) = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) \times f(0.1, 1.1172) = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$

To find $y(0.2)$:

Here $x_1 = 0.1$, $y_1 = 1.1169$, h=0.1

$$k_1 = hf(x_1, y_1) = (0.1) \times f(0.1, 1.1169) = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) \times f(0.15, 1.1848) = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) \times f(0.15, 1.1959) = 0.1609$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) \times f(0.2, 1.2778) = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773.$

To find $y(0.3)$:

Here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$.

$$k_1 = hf(x_2, y_2) = (0.1) \times f(0.2, 1.2773) = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1) \times f(0.25, 1.3716) = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\, f(0.25, 1.3885) = 0.2275$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\, f(0.3, 1.5048) = 0.2716$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.2267$$

$$y(0.3) = y_3 = y_2 + k = 1.504$$

Now the starting values for the Milne's method are:

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the *predictor*

$$y_4^{(p)} = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$x_4 = 0.4 \quad y_4^{(p)} = 1.8344 \quad f_4 = 4.0988$$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right)$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\left[1.8869 + 4(2.7132) + 4.098\right]$$

$$=1.8397 \qquad f_4 = 4.1159.$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\left[1.8869 + 4(2.7132) + 4.1159\right]$$

$$= 1.8391 \qquad f_4 = 4.1182 \tag{i}$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\left[1.8869 + 4(2.7132) + 4.1182\right]$$

$$= 1.8392 \text{ which is same as (i)}$$

Hence $y(0.4) = 1.8392$.

Exercises 10.4

1. Given $\frac{dy}{dx} = x^3 + y,\ y(0) = 2$. The values of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are gotten by the R.K. method of the order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$

2. Given $2\,dy/dx = (1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne's predictor corrector method.

3. Solve that initial value problem

$$\frac{dy}{dx} = 1 + xy^2, y(0) = 1$$

for $x = 0.4$ by using Milne's method, when it is given that

x:	0.1	0.2	0.3
y:	1.105	1.223	1.355

4. From the data given below, find y at $x = 1.4$, using Milne's predictor-corrector formula: $dy/dx = x^2 + y/2$:

$x = 1$	1.1	1.2	1.3
$y = 2$	2.2156	2.4549	2.7514

5. Using Taylor's series method, solve $\dfrac{dy}{dx} = xy + x^2$, $y(0) = 1$; at $x = 0.1$, 0.2, 0.3. Continue the solution at $x = 0.4$ by Milne's predictor-corrector method.

6. If $y = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.01$, $y(0.2) = 2.04$, and $y = 2.09$, find $y(0.4)$ using Milne's predictor-corrector method.

7. Using the Runge-Kutta method, calculate y (0.1), $y(0.2)$, and $y(0.3)$ given that $\dfrac{dy}{dx} = \dfrac{2xy}{1+x^2} = 1.$ $y(0) = 0$. Taking these values as starting values, find $y(0.4)$ by Milne's method.

10.10 Adams-Bashforth Method

Given $\dfrac{dy}{dx} = f(x,y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h), y_{-2} = y(x_0 - 2h), y_{-3} = y(x_0 - 3h)$$

by Taylor's series or Euler's method or the Runge-Kutta method.

Next we calculate

$$f_{-1} = f(x_0 - h, y_{-1}), f_{-2} = f(x_0 - 2h, y_{-2}), f_{-3} = f(x_0 - 3h, y_{-3})$$

Then to find y_1, we substitute Newton's backward interpolation formula

$$f(x,y) = f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2}{6}\nabla^3 f_0 + \cdots$$