ADA Homework 2

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Question 3 - Proof of Recurrence

Let us define some terms for sake of easiness:

$e(i, j)$ - energy obtained by merging drops from index i to index j

$d(i, j)$ - combined volume of drops from index i to index j

Motive:

Maximize $e(i, j)$ for range i to j

claim:

$e(i, j) = max(e(i,i) + e(i+1, j) + d(i,i)^2 + d(i+1, j)^2, e(i, j-1) + e(j, j) + d(i, j-1)^2 + d(j, j)^2)$

Now, energy of a single drop is 0. This gives, $e(i,i) = e(j, j) = 0$,

Therefore, $e(i, j) = max(e(i+1, j) + d(i,i)^2 + d(i+1, j)^2, e(i, j-1) + d(i, j-1)^2 + d(j, j)^2)$

Proof:

On the contrary, suppose the above claim is wrong for some $k \in (i, j)$

i.e. $e(i, j) = e(i,k) + e(k+1, j) + d(i,k)^2 + d(k+1, j)^2 > max(e(i+1, j) + d(i,i)^2 + d(i+1, j)^2, e(i, j-1) + d(i, j-1)^2 + d(j, j)^2)$

This means the following two are true

$e(i,k) + e(k+1, j) + d(i,k)^2 + d(k+1, j)^2 > e(i+1, j) + d(i,i)^2 + d(i+1, j)^2$ — ①

And

$e(i,k) + e(k+1, j) + d(i,k)^2 + d(k+1, j)^2 > e(i, j-1) + d(i, j-1)^2 + d(j, j)^2$ — ②

In ①

$e(i,k) + e(k+1, j) + d(i,k)^2 + d(k+1, j)^2 > e(i+1, j) + d(i,i)^2 + d(i+1, j)^2$

In R.H.S. breaking $e(i+1, j)$ as a combination of $e(i+1,k)$ and $e(k+1, j)$

$\implies e(i,k) + \cancel{e(k+1, j)} + d(i,k)^2 + \cancel{d(k+1, j)^2} > [e(i+1,k) + \cancel{e(k+1, j)} + d(i+1,k)^2 + \cancel{d(k+1, j)^2}] + d(i,i)^2 + d(i+1, j)^2$

$\implies e(i,k) + d(i,k)^2 > e(i+1,k) + d(i+1,k)^2 + d(i,i)^2 + d(i+1, j)^2$

In L.H.S. breaking $e(i,k)$ as a combination of $e(i,i)$ and $e(i+1,k)$

$\implies [\cancelto{0}{e(i,i)} + \cancel{e(i+1,k)} + \cancel{d(i,i)^2} + \cancel{d(i+1,k)^2}] + d(i,k)^2 > \cancel{e(i+1,k)} + \cancel{d(i+1,k)^2} + \cancel{d(i,i)^2} + d(i+1, j)^2$

$\implies d(i,k)^2 > d(i+1, j)^2$ — ③

In ②

$e(i,k)+e(k+1,j)+d(i,k)^2+d(k+1,j)^2 > e(i,j-1)+d(i,j-1)^2+d(j,j)^2$

In R.H.S. breaking $e(i,j-1)$ as a combination of $e(i,k)$ and $e(k+1,j-1)$

$\implies \cancel{e(i,k)}+e(k+1,j)+\cancel{d(i,k)^2}+d(k+1,j)^2 > [\cancel{e(i,k)}+e(k+1,j-1)+\cancel{d(i,k)^2}+d(k+1,j-1)^2]+d(i,j-1)^2+d(j,j)^2$

$\implies e(k+1,j)+d(k+1,j)^2 > e(k+1,j-1)+d(k+1,j-1)^2+d(i,j-1)^2+d(j,j)^2$

In L.H.S. breaking $e(k+1,j)$ as a combination of $e(k+1,j-1)$ and $e(j,j)$

$\implies [\cancel{e(k+1,j-1)}+\cancel{e(j,j)}^{\,0}+\cancel{d(k+1,j-1)^2}+\cancel{d(j,j)^2}]+d(k+1,j)^2 > \cancel{e(k+1,j-1)}+\cancel{d(k+1,j-1)^2}+d(i,j-1)^2+\cancel{d(j,j)^2}$

$\implies d(k+1,j)^2 > d(i,j-1)^2$ — ④

③ + ④ gives

$d(i,k)^2+d(k+1,j)^2 > d(i+1,j)^2+d(i,j-1)^2$

Using $(a+b)^2 = a^2+b^2+2*a*b$,

In R.H.S. breaking $d(i+1,j)$ for $a=d(i+1,k)$, $b=d(k+1,j)$, and $d(i,j-1)$ for $a=d(i,k)$, $b=d(k+1,j-1)$

$\implies \cancel{d(i,k)^2}+\cancel{d(k+1,j)^2} > [d(i+1,k)^2+\cancel{d(k+1,j)^2}+2*d(i+1,k)*d(k+1,j)]+[\cancel{d(i,k)^2}+d(k+1,j-1)+2*d(i,k)*d(k+1,j-1)]$

$\implies 0 > d(i+1,k)^2+2*d(i+1,k)*d(k+1,j)+d(k+1,j-1)^2+2*d(i,k)*d(k+1,j-1)$ — ⑤

Equation ⑤ can't be true as volume of all drops must be non-negative

Therefore either (or both) of the equations ① and ② must be false

This implies, $e(i,k)+e(k+1,j)+d(i,k)^2+d(k+1,j)^2 \not> max(e(i+1,j)+d(i,i)^2+d(i+1,j)^2, e(i,j-1)+d(i,j-1)^2+d(j,j)^2)$

$\implies e(i,k)+e(k+1,j)+d(i,k)^2+d(k+1,j)^2 \leq max(e(i+1,j)+d(i,i)^2+d(i+1,j)^2, e(i,j-1)+d(i,j-1)^2+d(j,j)^2)$

Hence, our claim is true.

i.e. $e(i,j) = max(e(i+1,j)+d(i,i)^2+d(i+1,j)^2, e(i,j-1)+d(i,j-1)^2+d(j,j)^2)$