Preliminary Single-Phase Variable-Area Results

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- The variable-area Euler equations are [1]:

$$\frac{\partial\left(\mathcal{A}\mathbf{U}\right)}{\partial t}+\frac{\partial\left(\mathcal{A}\mathbf{F}\right)}{\partial x}=\begin{bmatrix}0\\ p\frac{\partial\mathcal{A}}{\partial x}\\ 0\end{bmatrix} \tag{1}$$

where $\mathbf{U}$ and $\mathbf{F}$ are given by

$$\mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho E\end{bmatrix}\qquad \mathbf{F}=\begin{bmatrix}\rho u\\ \rho u^2+p\\ u\left(\rho E+p\right)\end{bmatrix} \tag{2}$$

- Define the change of variables $\mathbf{V} \equiv \mathcal{A}\mathbf{U} = [V_0, V_1, V_2]^T$, and let

$$\tilde{\mathbf{F}} \equiv \mathcal{A}\mathbf{F} = \begin{bmatrix} V_1 \\ \frac{V_1^2}{V_0} + p\mathcal{A} \\ \frac{V_1}{V_0}\left(V_2 + p\mathcal{A}\right) \end{bmatrix} \tag{3}$$

- Then the variable-area equations can be written as

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \tilde{\mathbf{F}}}{\partial x} = \begin{bmatrix} 0 \\ p\frac{\partial \mathcal{A}}{\partial x} \\ 0 \end{bmatrix} \tag{4}$$

- The weak statement, which proceeds by dotting (4) with a test function $\mathbf{W}$ and integrating over the domain Ω, is: find $\mathbf{V}$ such that

$$\int_{\Omega} \left(\frac{\partial \mathbf{V}}{\partial t} \cdot \mathbf{W} - \tilde{\mathbf{F}} \cdot \frac{\partial \mathbf{W}}{\partial x} - \mathbf{S} \cdot \mathbf{W} \right) \mathrm{d}x + \int_{\Gamma} \left(\tilde{\mathbf{F}} \cdot \mathbf{W} \right) \hat{n}_x \, \mathrm{d}s = 0 \tag{5}$$

holds for all admissible $\mathbf{W}$, where

$$\mathbf{S} \equiv \left[0, p \frac{\partial \mathcal{A}}{\partial x}, 0 \right]^T \tag{6}$$

- The analytical area function $\mathcal{A}(x)$ should not be used directly.
- Instead, $\mathcal{A}$ and $\frac{\partial \mathcal{A}}{\partial x}$ should be replaced by

$$\mathcal{A}^h(x) \equiv \sum_i \mathcal{A}(x_i)\phi_i(x) \tag{7}$$

$$\frac{\partial \mathcal{A}^h}{\partial x} = \sum_i \mathcal{A}(x_i)\frac{\partial \phi_i}{\partial x} \tag{8}$$

respectively.

- In "quasi-linear" form, we write the flux term as:

$$\begin{aligned} \frac{\partial \tilde{\mathbf{F}}}{\partial x} &= \frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{V}} \frac{\partial \mathbf{V}}{\partial x} \\ &\equiv \tilde{\mathbf{A}} \frac{\partial \mathbf{V}}{\partial x} \end{aligned} \tag{9}$$

  where $\tilde{\mathbf{A}}$ is the "flux Jacobian" matrix.
- How are $\tilde{\mathbf{A}}$ and $\mathbf{A}$ from the constant-area equations related?

- The change of variables $\mathbf{V} \equiv \mathcal{A}\mathbf{U}$ implies

$$\frac{\partial U_i}{\partial V_j} = \frac{1}{\mathcal{A}}\delta_{ij} \tag{10}$$

- Since $p = p(\mathbf{U}(\mathbf{V}))$, the chain rule yields

$$\begin{aligned} \frac{\partial p}{\partial V_0} &= \frac{\partial p}{\partial U_i}\frac{\partial U_i}{\partial V_0} \\ &= \frac{\partial p}{\partial U_0}\frac{1}{\mathcal{A}} \end{aligned} \tag{11}$$

and similarly for derivatives with respect to V_1 and V_2.

- Since $(p\mathcal{A})$ always appears as a product in $\tilde{\mathbf{F}}$, it's easy to show that $\tilde{\mathbf{A}} = \mathbf{A}$! Thus, we can write

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{V}}{\partial x} = \begin{bmatrix} 0 \\ p\frac{\partial \mathcal{A}}{\partial x} \\ 0 \end{bmatrix} \tag{12}$$

- Where

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 & 0 \\ p_{,0} - u^2 & p_{,1} + 2u & p_{,2} \\ u\left(p_{,0} - H\right) & up_{,1} + H & u\left(1 + p_{,2}\right) \end{bmatrix} \tag{13}$$

$$p_{,i} \equiv \frac{\partial p}{\partial U_i}, i = 0, 1, 2 \tag{14}$$

- This implies that the SUPG formulation carries over (almost directly) from the constant-area case.

- The eigenvectors and eigenvalues are the same, so no new analysis needed.

- Of course, the strong form of the residual is slightly different. . .

- Unfortunately, the quasi-linear form does not seem to pass a basic sanity check!
- The equations *must* satisfy the trivial steady state solution: $\rho, \rho E, p = const$, $u = 0$, $\frac{\partial \mathcal{A}}{\partial x} \neq 0$.
- In conservative form, we get

$$\frac{\partial \tilde{\mathbf{F}}}{\partial x} = \begin{bmatrix} 0 \\ \frac{\partial (p\mathcal{A})}{\partial x} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ p\frac{\partial \mathcal{A}}{\partial x} \\ 0 \end{bmatrix} \tag{15}$$

which, once combined with **S** on the right-hand side, clearly satisfies the governing equations.

- In quasi-linear form, we obtain

$$\mathbf{A}\frac{\partial \mathbf{V}}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ p_{,0} & p_{,1} & p_{,2} \\ 0 & H & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial(\rho\mathcal{A})}{\partial x} \\ 0 \\ \frac{\partial(\rho E\mathcal{A})}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ p_{,0}\frac{\partial(\rho\mathcal{A})}{\partial x} + p_{,2}\frac{\partial(\rho E\mathcal{A})}{\partial x} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ p_{,0}\rho + p_{,2}\rho E \\ 0 \end{bmatrix} \frac{\partial \mathcal{A}}{\partial x} \tag{16}$$

- Hence, the quasi-linear form satisfies the trivial solution if and only if

$$p_{,0}\rho + p_{,2}\rho e = p \tag{17}$$

- We note that (17) holds for an ideal gas, $p = (\gamma - 1)\rho e$, since

$$\begin{aligned} p_{,0} &= \frac{(\gamma-1)}{2}u^2 \overset{(u=0)}{=} 0 \\ p_{,2} &= \gamma - 1 \qquad \overset{(u=0)}{=} \gamma - 1 \end{aligned}$$

- But *not* for a stiffened gas, $p = (\gamma - 1)\rho(e - q) - \gamma p_\infty$, since

$$p_{,0} = (\gamma - 1)\left(\frac{1}{2}u^2 - q\right) \overset{(u=0)}{=} (\gamma - 1)(-q)$$
$$p_{,2} = \gamma - 1 \qquad \overset{(u=0)}{=} \gamma - 1$$

so

$$\begin{aligned} p_{,0}\rho + p_{,2}\rho E &= (\gamma - 1)(-q)\,\rho + (\gamma - 1)\rho e \\ &= (\gamma - 1)\rho\,(e - q) \\ &\neq p \end{aligned} \tag{18}$$

- The condition does not seem to hold in general for arbitrary EOS. . .

E. F. Toro, *Riemann solvers and numerical methods for fluid dynamics: A practical introduction, 2nd ed.*
Berlin: Springer-Verlag, 1999.