Preliminary Single-Phase Variable-Area Results

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June 7, 2013

- The variable-area Euler equations are [1]:

$$\frac{\partial\left(\mathcal{A}\mathbf{U}\right)}{\partial t}+\frac{\partial\left(\mathcal{A}\mathbf{F}\right)}{\partial x}=\begin{bmatrix}0\\ p\frac{\partial\mathcal{A}}{\partial x}\\ 0\end{bmatrix} \tag{1}$$

where **U** and **F** are given by

$$\mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho E\end{bmatrix}\qquad \mathbf{F}=\begin{bmatrix}\rho u\\ \rho u^2+p\\ u\left(\rho E+p\right)\end{bmatrix} \tag{2}$$

- Define the change of variables $\mathbf{V} \equiv \mathcal{A}\mathbf{U} = [V_0, V_1, V_2]^T$, and let

$$\tilde{\mathbf{F}} \equiv \mathcal{A}\mathbf{F} = \begin{bmatrix} V_1 \\ \frac{V_1^2}{V_0} + p\mathcal{A} \\ \frac{V_1}{V_0}\left(V_2 + p\mathcal{A}\right) \end{bmatrix} \tag{3}$$

- Then the variable-area equations can be written as

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \tilde{\mathbf{F}}}{\partial x} = \begin{bmatrix} 0 \\ p\frac{\partial \mathcal{A}}{\partial x} \\ 0 \end{bmatrix} \tag{4}$$

- The weak statement, which proceeds by dotting (4) with a test function $\mathbf{W}$ and integrating over the domain Ω, is: find $\mathbf{V}$ such that

$$\int_{\Omega}\left(\frac{\partial \mathbf{V}}{\partial t}\cdot\mathbf{W}-\tilde{\mathbf{F}}\cdot\frac{\partial \mathbf{W}}{\partial x}-\mathbf{S}\cdot\mathbf{W}\right)\mathrm{d}x+\int_{\Gamma}\left(\tilde{\mathbf{F}}\cdot\mathbf{W}\right)\hat{n}_x\,\mathrm{d}s=0 \tag{5}$$

holds for all admissible $\mathbf{W}$, where

$$\mathbf{S}\equiv\left[0,p\frac{\partial\mathcal{A}}{\partial x},0\right]^T \tag{6}$$

- The analytical area function $\mathcal{A}(x)$ should not be used directly.
- Instead, $\mathcal{A}$ and $\frac{\partial \mathcal{A}}{\partial x}$ should be replaced by

$$\mathcal{A}^h(x) \equiv \sum_i \mathcal{A}(x_i)\phi_i(x) \tag{7}$$

$$\frac{\partial \mathcal{A}^h}{\partial x} = \sum_i \mathcal{A}(x_i)\frac{\partial \phi_i}{\partial x} \tag{8}$$

respectively.

- To motivate the quasi-linear form, consider the identities

$$\mathbf{A}\frac{\partial \mathbf{V}}{\partial x} = \mathbf{A}\left(\mathcal{A}\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathcal{A}}{\partial x}\mathbf{U}\right)$$

$$= \mathcal{A}\frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathcal{A}}{\partial x}\mathbf{A}\mathbf{U} \quad (9)$$

$$\frac{\partial \tilde{\mathbf{F}}}{\partial x} = \mathcal{A}\frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathcal{A}}{\partial x}\mathbf{F} \quad (10)$$

  where $\mathbf{A}$ is the constant-area flux Jacobian, $\mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = \frac{\partial \mathbf{F}}{\partial x}$.
- Rearranging (9) and substituting into (10), we get

$$\frac{\partial \tilde{\mathbf{F}}}{\partial x} = \mathbf{A}\frac{\partial \mathbf{V}}{\partial x} + (\mathbf{F} - \mathbf{A}\mathbf{U})\frac{\partial \mathcal{A}}{\partial x} \quad (11)$$

- Thus, the quasi-linear form of the governing equations reads

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{V}}{\partial x} + \left(\mathbf{F} - \mathbf{AU}\right)\frac{\partial \mathcal{A}}{\partial x} - \mathbf{S} = \mathbf{0} \tag{12}$$

- We note that in the special case where $\mathbf{F}$ is a "homogeneous function of degree 1," we have $\mathbf{F} = \mathbf{AU}$, and the term in red vanishes.
- $\mathbf{F}$ is HFOD1 for the ideal gas equation of state, but not in general.
- The term in red also vanishes when there is no change in area.

- For a general EOS, we have

$$\mathbf{F} - \mathbf{A}\mathbf{U} = \begin{bmatrix} 0 \\ \hat{p} \\ u\hat{p} \end{bmatrix} \tag{13}$$

where

$$\begin{aligned} \hat{p} &\equiv p - \left(\frac{\partial p}{\partial U_0}, \frac{\partial p}{\partial U_1}, \frac{\partial p}{\partial U_2} \right) \cdot \mathbf{U} \\ &= p - p_{,0}\rho - p_{,1}\rho u - p_{,2}\rho E \end{aligned} \tag{14}$$

- For an ideal gas, where

$$
\begin{aligned}
p &= (\gamma - 1)\rho e \\
p_{,0} &= \frac{(\gamma - 1)}{2} u^2 \\
p_{,1} &= (\gamma - 1)\,(-u) \\
p_{,2} &= \gamma - 1
\end{aligned}
$$

it is easy to check that $\hat{p} = 0$.

- This implies that the SUPG formulation carries over (almost directly) from the constant-area case.
- The eigenvectors and eigenvalues are the same, so no new analysis needed.
- Of course, the strong form of the residual is slightly different. . .

E. F. Toro, *Riemann solvers and numerical methods for fluid dynamics: A practical introduction, 2nd ed.*
Berlin: Springer-Verlag, 1999.