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Applied Optimal Control HW #5

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1) 2/4

$\dot{\theta} = \omega - \beta - \eta_v$

$\dot{\beta} = \eta_u$

$\eta_v \sim N(0, \sigma_v^2)$

$\eta_u \sim N(0, \sigma_u^2)$

$z_k = \theta_k + v_k \qquad v_k \sim N(0, \sigma_N^2)$

$$\dot{x}(t) = F(t)\underbrace{x(t)}_{\text{state }(\theta)} + G(t)\underbrace{u(t)}_{\text{control }(\omega)} + L(t)\underbrace{w(t)}_{\text{noise}}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\beta} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}_{F(t)} \begin{bmatrix} \theta \\ \beta \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{G(t)} u(t) + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{L(t)} w(t)$$

Δt = constant time sampling interval

$$Q_c(t) = \begin{bmatrix} \sigma_v^2 & \\ & \sigma_u^2 \end{bmatrix}$$

$$\Phi_k = e^{\Delta t F} = e^{\begin{bmatrix} 0 & -\Delta t \\ 0 & 0 \end{bmatrix}}$$

$$e^{\overbrace{\begin{bmatrix} 0 & -\Delta t \\ 0 & 0 \end{bmatrix}}^{A}} = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\Delta t \\ 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdots$$

$$\boxed{= \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix} = \Phi_k}$$

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1 continued

$$\Gamma_k = \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)\, G(\tau)\, d\tau$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$$

$$\Gamma_k = \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$$

$$\boxed{\Gamma_k = \begin{bmatrix} \Delta t \\ 0 \end{bmatrix}}$$

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$$Q_k \approx L'(t_k)\; Q_c'(t_k)\; L^T(t_k)\, \Delta t$$

$$= \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \sigma_v^2 & \\ & \sigma_u^2 \end{bmatrix} \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}^T \Delta t =$$

??

1b -> Wrong state propagation Line 59 in ur code.
Check soln (3/4)

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2.) $x_k = f(x_{k-1}, u_{k-1}) + w_{k-1}$

$z_k = h_k(x_k) + v_k$

$w_k \sim N(0, Q_k) \quad v_k \sim N(0, R_k)$

$x_{k|k-1} \sim N(\hat{x}_{k|k-1}, P_{k|k-1})$

$$J(x) = \frac{1}{2}(x - \hat{x}_{k|k-1})^T P_{k|k-1}^{-1}(x - \hat{x}_{k|k-1}) + \frac{1}{2}[z_k - h(x)]^T R_k^{-1}[z_k - h(x)]$$

Drop all $k|k-1$ subscripts for conciseness

$F = [A + BCD]^{-1}$ → all arbitrary matrices

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$F = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$

$$F^{-1}F = I - B[(DA^{-1}B + C^{-1})^{-1} - C + CDA^{-1}B(DA^{-1}B + C^{-1})^{-1}]DA^{-1}$$

$$(DA^{-1}B + C^{-1})^{-1} = C - CDA^{-1}B(DA^{-1}B + C^{-1})^{-1} \quad \Big| (DA^{-1}B + C^{-1}) \times$$

Wrong solution, Check solution for correct answer

$$I = C(DA^{-1}B + C^{-1}) - CDA^{-1}B$$

$F = P_{k+1}$
$A = P_k^{-1}$
$B = H_{k+1}^T$
$C = R_{k+1}$
$D = H_{k+1}$

$$P_{k+1} = P_k - P_k H_{k+1}^T (H_{k+1} P_k H_{k+1}^T + W_{k+1}^{-1})^{-1} H_{k+1} P_k$$

$$K_{k+1} = P_k H_k^T (H_{k+1} P_k H_{k+1}^T + W_{k+1}^{-1})^{-1} \times [W_{k+1} + H_{k+1} P_k H_{k+1}^T - H_{k+1} - H_{k+1} P_k H_{k+1}^T] W_{k+1}$$

2 continued

Change indices: $k+1 \rightarrow k$, $k \rightarrow k|k-1$

Cross Product goes away because two components $(x \times y) = x$ - for these two matrices

$$\boxed{K_k = P_{k|k-1} H_k^T \left(H_k P_{k|k-1} H_k^T + R_k\right)^{-1}}$$

$$dx = K_k \, dz$$

$$d\hat{x} = x - \hat{x}$$

$$dz = z - h(\hat{x})$$

$$x - \hat{x} = K_k\left(z_k - h(\hat{x})\right)$$

$$\boxed{x = \hat{x} + K_k\left(z_k - h(\hat{x})\right)}$$

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3b) We can reconstruct the full state of the system using only the bearing sensors. However, the state estimation is not very accurate and the range sensors help significantly reduce the error in the estimation and without them the system is very susceptible to the noise of the bearing sensors.