EN530.603 Applied Optimal Control
Lecture 5: Continuous Optimal Control Basics

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1 Continuous Systems with Terminal Constraints

Consider the cost

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt,$$

subject to q constraints

$$\psi(x(t_f), t_f) = 0$$

and the dynamics

$$\dot{x}(t) = f(x(t), u(t), t), \qquad t_0 \text{ and } x(t_0) \text{ are given.}$$

It will be useful to employ the shorthand notation $f(t) \equiv f(x(t), u(t), t)$, or $\phi(t) \equiv \phi(x(t), t)$, etc... Sometimes, f (or any other function) could also be without arguments, i.e. $f \equiv f(x(t), u(t), t)$.

To obtain the necessary conditions, form the augmented cost

$$J_a = \phi(t_f) + \nu^T\psi(t_f) + \int_{t_0}^{t_f} \left\{L(x, u, t) + \lambda^T(t)[f(x, u, t) - \dot{x}]\right\} dt.$$

Let $\Phi = \phi + \nu^T\psi$ and define the Hamiltonian H by

$$H(x, u, \lambda, t) = L(x, u, t) + \lambda^T(t)f(x, u, t).$$

Taking variations with respect to all variables including final time t_f we obtain

$$\delta J_a = \Big((\partial_t\Phi + L)\delta t_f + \partial_x\Phi \cdot \delta x_f\Big)_{t=t_f} + \int_{t_0}^{t_f} \left(\partial_x H \cdot \delta x + \partial_u H \cdot \delta u - \lambda^T \delta\dot{x}\right) dt.$$

Integrating by parts and using the relationship

$$\delta x_f = \delta x(t_f) + \dot{x}\delta t_f,$$

we obtain

$$\delta J_a = \Big(\left[\partial_t\Phi + L + \lambda^T\dot{x}\right]\delta t_f + \left[\partial_x\Phi - \lambda^T\right]\delta x_f\Big)_{t=t_f} + \int_{t_0}^{t_f} \Big[\Big(\partial_x H + \dot{\lambda}^T\Big)\delta x + \partial_u H \cdot \delta u\Big] dt$$

Since all variations are arbitrary and independent the necessary conditions become

$$\dot{\lambda}^T = -\partial_x H = -\lambda^T\partial_x f - \partial_x L, \tag{1}$$

$$\lambda(t_f)^T = \partial_x\Phi\big|_{t=t_f} = \left(\partial_x\phi + \nu^T\partial_x\psi\right)_{t=t_f}, \tag{2}$$

$$\partial_u H = \lambda^T\partial_u f + \partial_u L = 0, \tag{3}$$

$$\left(\partial_t\Phi + L + \lambda^T\dot{x}\right)_{t=t_f} = \left(\frac{d\Phi}{dt} + L\right)_{t=t_f} = 0, \tag{4}$$

where

$$\frac{d\Phi}{dt} = \partial_t \Phi + \partial_x \Phi \cdot \dot{x}.$$

After substituting the expression for $\lambda(t_f)$ the necessary conditions are summarized according to:

$$\begin{aligned}
&\dot{x} = f(x, u, t) && (5)\\
&\dot{\lambda} = -\nabla_x H = -\nabla_x f \cdot \lambda - \nabla_x L, && (6)\\
&\nabla_u H = \nabla_u f \cdot \lambda + \nabla_u L = 0, && (7)\\
&\lambda(t_f) = \nabla_x \phi(x(t_f), t_f) + \nabla_x \psi(x(t_f), t_f) \cdot \nu, && (8)\\
&\Big(\partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f\Big)_{t=t_f} = 0, && (9)
\end{aligned}$$

When the final time t_f is fixed the last relationship (9) can be dropped.

Hamiltonian conservation. Note that whenever the Hamiltonian does not depend on time (that is when f and L do not depend on time)

$$\partial_t H(x, u, \lambda, t) = 0$$

then H is a conserved quantity along optimal trajectories $x^*(t), u^*(t), \lambda^*(t)$, i.e. we have that

$$\begin{aligned}
\dot{H}(x, u, \lambda, t) &= \partial_x H \cdot \dot{x} + \partial_u H \cdot \dot{u} + \partial_\lambda H \cdot \dot{\lambda} + \partial_t H && (10)\\
&= -\dot{\lambda}^T f(x, u, t) + 0 \cdot \dot{u} + f(x, u, t)^T \dot{\lambda} + 0 = 0 && (11)
\end{aligned}$$

Therefore, in this case we have $H(t) = \text{const}$ for all $t \in [t_0, t_f]$. Furthermore, in the special case when $\partial_t \phi = 0$ and $\partial_t \psi = 0$ the last condition (9) reduces to $H(t) = 0$.

Minimum-time problems. For minimum-time problems we have $\phi = 0$ and $L = 1$ so that condition (9) reduces to

$$\Big(\nu^T [\partial_t \psi + \nabla_x \psi^T \cdot f] + 1\Big)_{t=t_f} = 0,$$

which can be used along with the constraint $\psi(x(t_f), t_f) = 0$ to determine the multipliers ν and final time t_f.

Solution Methods

We are faced with solving the differential equations for $t = [t_0, t_f)$:

$$\text{Euler-Lagrange (EL) :} \qquad \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} f(x, u, t) \\ -\nabla_x H \end{pmatrix} \tag{12}$$

where $u(t)$ is computed by minimizing H which corresponds to the condition

$$\text{Control optimization :} \qquad \nabla_u H = 0,$$

which we assume can be solved and that $u(t)$ is then expressed as a function of $x(t)$ and $\lambda(t)$, subject to the boundary constraints

$$\text{Transversality Conditions (TC):} \qquad \begin{aligned} &\psi(x(t_f), t_f) = 0 \\ &\lambda(t_f) = \nabla_x \phi(x(t_f), t_f) + \nabla_x \psi(x(t_f), t_f) \cdot \nu, \\ &\left(\partial_t \phi + \nu^T \partial_t \psi + L + \lambda^T f\right)_{t=t_f} = 0, \end{aligned} \tag{13}$$

The following solution methods are applicable based on whether EL can be integrated in closed-form and whether TC can be solved in closed form:

- general: two-point boundary value problem (BVP) works with any EL and TC, the conditions are satisfied using a numerical "collocation" procedure
- EL integrable: pick $\lambda(0)$ integrate from t_0 to t_f and solve TC as an implicit equality for the unknown $(\lambda(0), \nu)$. When final time t_f is free then solve for $(\lambda(0), \nu, t_f)$.
- EL integrable and TC solvable: closed-form solution.

Example 1. Minimum Control Effort Landing Consider a second order system with state $x = (p, v) \in \mathbb{R}^4$ where $p \in \mathbb{R}^2$ is the position and $v \in \mathbb{R}^2$ is the velocity. The system has a *double integrator dynamics* given by

$$\begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix},$$

where $u \in \mathbb{R}^2$ is the acceleration control input. The system starts with known initial state $x_0 = (p_0, v_0)$ and must "land" with a prescribed velocity v_f somewhere on a unit circle centered at the origin, i.e. the final configuration must satisfy $\psi(x(t_f)) = 0$, where

$$\psi(x) = p^T p - 1.$$

The objective function is the control effort given by

$$L(x, u) = \frac{1}{2}\|u\|^2$$

We start with the Hamiltonian, and the multipliers $\lambda = (\lambda_p, \lambda_v)$

$$H = \frac{1}{2}u^T u + \lambda_p^T v + \lambda_v^T u,$$

We have

$$\begin{aligned} &\dot{\lambda} = -\nabla_x H \quad \Rightarrow \quad \dot{\lambda}_p = 0, \quad \dot{\lambda}_v = -\lambda_p \\ &\nabla_u H = 0 \quad \Rightarrow \quad u = -\lambda_v, \end{aligned}$$

from which we get

$$\ddot{u} = -\ddot{\lambda}_v = \dot{\lambda}_p = 0,$$

which means that the path $p(t)$ is a cubic spline that can be written according to

$$p(t_0 + t) = c_3 t^3 + c_2 t^2 + v_0 t + p_0, \tag{14}$$

while the velocity is

$$v(t_0 + t) = 3c_3 t^2 + 2c_2 t + v_0. \tag{15}$$

Now from

$$\lambda_p(t_f) = \nabla_p \psi(x(t_f))\nu = 2p(t_f)\nu.$$

Note that above since the velocity is not present in the terminal constraint ψ, then there is no additional condition on $\lambda_v(t_f)$.

Now considering that $\lambda_p(t_f) = \dot{u}(t_f) = 6c_3$ the above is equivalent to

$$6c_3 = 2p(t_f)\nu.$$

Finally, assuming t_f is given we can solve for ν, c_2, c_3 (5 unknowns) the implicit equations (5 equations):

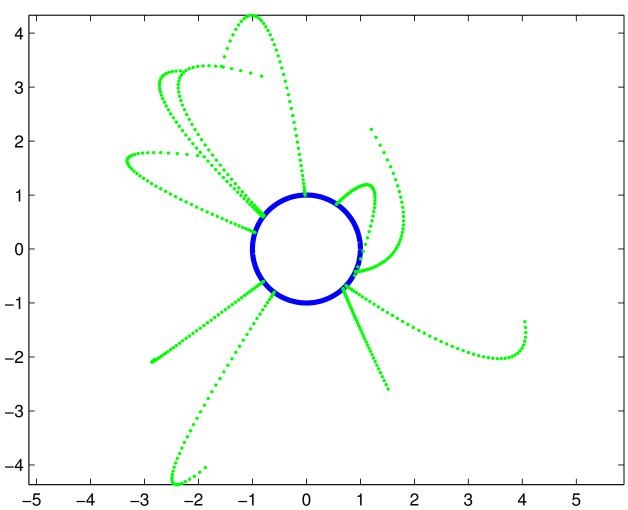
$$6c_3 - 2p(t_f)\nu = 0, \tag{16}$$

$$p(t_f)^T p(t_f) - 1 = 0, \tag{17}$$

$$v(t_f) - v_f = 0, \tag{18}$$

where $p(t_f)$ and $v(t_f)$ are given by (14) and (15). Note that it is necessary that $\nu \neq 0$ to ensure that the constraint is satisfied.

Examples of the resulting trajectories from randomly initialized states are given. In all examples we have $v_f = (0, 0)$



Example 2. Example: Zermelo's problem (Bryson §2.7) Consider a ship with dynamics

$$\dot{x} = V\cos\theta + u(x, y) \tag{19}$$

$$\dot{y} = V\sin\theta + v(x, y), \tag{20}$$

where (x, y) is the position, V is a constant velocity, θ is the heading angle input and u and v denote velocity due to currents. The goal is to travel between points A and B in minimum time.

The Hamiltonian is

$$H = \lambda_x(V\cos\theta + u) + \lambda_y(V\sin\theta + v) + 1.$$

The Euler-Lagrange equations are

$$\dot{\lambda}_x = -\partial_x H = -\lambda_x \partial_x u - \lambda_y \partial_x v \tag{21}$$

$$\dot{\lambda}_y = -\partial_y H = -\lambda_x \partial_x y - \lambda_y \partial_y v \tag{22}$$

$$0 = \partial_\theta H = V(-\lambda_x \sin\theta + \lambda_y \cos\theta) \Rightarrow \tan\theta = \frac{\lambda_y}{\lambda_x} \tag{23}$$

Since this is a minimum-time problem we have $H = 0$ and from (23) that

$$\lambda_x = \frac{-\cos\theta}{V + u\cos\theta + v\sin\theta}, \quad \lambda_y = \frac{-\sin\theta}{V + u\cos\theta + v\sin\theta}$$

This leads to

$$\dot{\theta} = \sin^2\theta \partial_x v + \sin\theta\cos\theta(\partial_x u - \partial_y v) - \cos^2\theta \partial_y u$$

Now, in order to reach B one has to select the start angle θ_A and the final time t_f.

Special Case. For the special case when

$$u = -V(y/h), \quad v = 0$$

consider starting at (x_0, y_0) with the goal to reach the origin $(0, 0)$. We have

$$\dot{\lambda}_x = 0 \;\Rightarrow\; \lambda_x = \text{const}$$

and therefore

$$\frac{-\cos\theta}{V - V(y/h)\cos\theta} = \frac{-\cos\theta_f}{V} = -\text{const} \quad \Rightarrow \quad \cos\theta = \frac{\cos\theta_f}{1 + (y/h)\cos\theta_f}$$

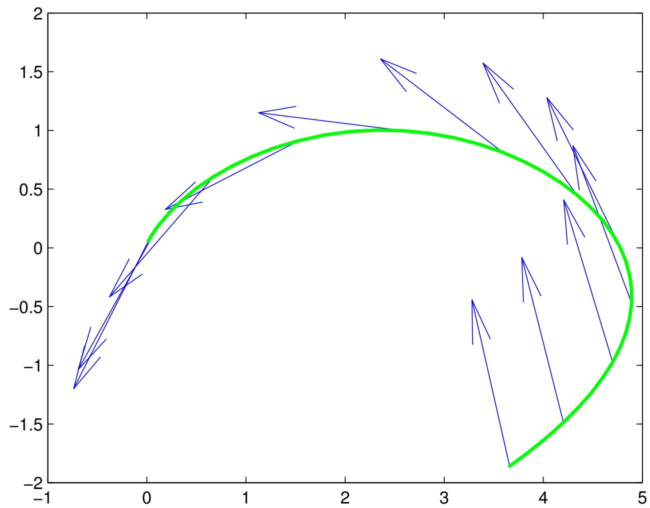
In the above, it turned out that it is convenient to work in terms of θ_f rather than t_f. The solution can be obtained analytically as

$$x = \frac{h}{2}\left[\sec\theta_f(\tan\theta_f - \tan\theta) - \tan\theta(\sec\theta_f - \sec\theta) + \log\frac{\tan\theta_f + \sec\theta_f}{\tan\theta + \sec\theta}\right], \tag{24}$$

$$y = h(\sec\theta - \sec\theta_f), \tag{25}$$

from which one can compute the initial angle θ and final angles θ_f to achieve given final position (x, y).

The computed path with initial conditions given by $x_0 = 3.66$ and $y_0 = -1.86$ with $h = 1$, $V = .3$ are given below



Example 3. Minimum Control Effort Landing with Optimal Time Consider the minimum control effort landing §1 with free final time t_f and a cost function given by

$$L(x, u) = bt_f + \frac{1}{2}\|u\|^2,$$

for some constant $b > 0$ which controls the balance between penalizing total time and total control effort.

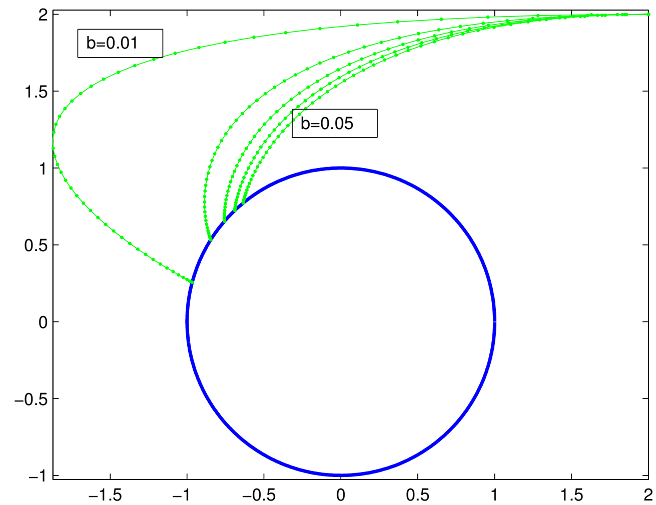
We need to add the third transversality condition from (13)

$$\partial_t \phi(t_f) + H(t_f) = 0,$$

which in our case is

$$b - \frac{1}{2}\|u(t_f)\|^2 + \dot{u}(t_f)^T v(t_f) = 0$$

This can be solved along with the other five conditions to obtain the unknowns c_2, c_3, ν, t_f. Plots of computed trajectories with varying b are given below.



References