EN.530.603 Applied Optimal Control
HW #2 Solutions

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1. To find the curve x^* which minimizes J(x) passing throught 0 and 4.

$$J(x) = \int_0^1 \left[\frac{1}{2}\dot{x}^2(t) + 3x(t)\dot{x}(t) + 2x^2(t) + 3x(t)\right] dt \tag{1}$$

From Euler-Lagrange equations:

$$J(x) = \int_0^1 g(x, \dot{x}, t)dt$$

Necessary Conditions

$$g_x - \frac{d}{dt}g_{\dot{x}} = (3\dot{x} + 4x + 3) - \frac{d}{dt}(\dot{x} + 3x) = 0$$
$$\Rightarrow \ddot{x} = 4x + 3$$

Homogenous solution:

$$\ddot{x} = 4x \Rightarrow x(t) = \lambda_1 e^{2t} + \lambda_2 e^{-2t} \quad \lambda_1, \lambda_2 \in \mathbf{R}$$

Non Homogenous solution:

$$x(t) = -3/4 \Rightarrow \ddot{x} = 0 = 4(-3/4) + 3$$

Combined Solution:

$$x(t) = \lambda_1 e^{2t} + \lambda_2 e^{-2t} - \frac{3}{4}$$

Boundary Conditions:

$$x(1) = 4 \quad \& \quad x(0) = 0$$
$$\Rightarrow \lambda_1 + \lambda_2 = 3/4$$
$$\Rightarrow \lambda_1 e^2 + \lambda_2 e^{-2} = 19/4$$
$$\Rightarrow \lambda_1 = \frac{-3e^{-2} + 19}{4(e^2 - e^{-2})} = 0.6408 \quad \lambda_2 = \frac{3e^2 - 19}{4(e^2 - e^{-2})} = 0.1091$$

Thus the final solution for extremal trajectory x(t) is given as:

$$x(t) = 0.6408e^{2t} + 0.1091e^{-2t} - 0.75$$

2. To find extremals for

$$J(x) = \int_0^{t_f} [\dot{x_1}^2(t) + \dot{x_2}^2(t) + 3x_1(t)x_2(t)]dt$$
$$g(x,\dot{x}) = [\dot{x_1}^2(t) + \dot{x_2}^2(t) + 3x_1(t)x_2(t)]$$

Using necessary conditions on the trajectory, the Euler-Lagrange equations give the extremal trajectory as:

$$\begin{aligned}
&g_x - \frac{d}{dt}g_{\dot{x}} = [3x_2 \quad 3x_1] - \frac{d}{dt}[2\dot{x_1} \quad 2\dot{x_2}] = 0 \\
&\quad \Rightarrow 2\ddot{x}_1 = 3x_2 \quad \& \quad 2\ddot{x}_2 = 3x_1 \\
&\quad \Rightarrow x_1^{(4)} = \frac{3}{2}\ddot{x}_2 = \frac{9}{4}x_1 \\
&\quad \Rightarrow x_1(t) = \lambda_1 sinh(\omega t) + \lambda_2 sin(\omega t) + \lambda_3 cosh(\omega t) + \lambda_4 cos(\omega t) \quad \omega = \sqrt{3/2}; \\
&\quad \Rightarrow x_2(t) = \frac{2}{3}\ddot{x}_1 = \lambda_1 sinh(\omega t) - \lambda_2 sin(\omega t) + \lambda_3 sinh(\omega t) - \lambda_4 sin(\omega t) \;\; \lambda_{1,2,3,4} \in \mathbb{R}
\end{aligned}$$

Given $x_1(0) = 0$ $x_2(0) = 0$ implies that:

$$\lambda_3 + \lambda_4 = 0 \;\; \& \;\; \lambda_3 - \lambda_4 = 0$$

which gives $\lambda_3 = \lambda_4 = 0$.

$$\Rightarrow x_1(t) = \lambda_1 sinh(\omega t) + \lambda_2 sin(\omega t) \tag{2}$$
$$\Rightarrow x_2(t) = \frac{2}{3}\ddot{x}_1 = \lambda_1 sinh(\omega t) - \lambda_2 sin(\omega t) \tag{3}$$

(a) Given the boundary conditions we evaluate the unknowns λ_1 and λ_2 :

$$\begin{aligned}
&\text{If } \; t_f = 1 \text{ and } x_2(t_f) = 1 \\
&\text{then } \; \lambda_1 sinh(\omega) - \lambda_2 sin(\omega) = 1 \quad \text{ From Eqn[2]} \\
&\text{If } \; x_1(t_f) \text{ is free} \\
&\text{then } \; g_{\dot{x}}(1)|_{t_f} = \left[2\dot{x_1}(t_f) \quad 2\dot{x_2}(t_f)\right](1) = 0 \\
&\Rightarrow \;\; 2\dot{x_1}(t)|_{t=1} = 0 \\
&\Rightarrow \;\; \lambda_1 cosh(\omega) + \lambda_2 cos(\omega) = 0 \\
&\Rightarrow \;\; \begin{bmatrix} sinh(\omega) & -sin(\omega) \\ cosh(\omega) & cos(\omega) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} cos(\omega) \\ -cosh(\omega) \end{bmatrix} \\
&\text{where } \; \alpha = cos(\omega)sinh(\omega) + cosh(\omega)sin(\omega)
\end{aligned}$$

This solves the optimization problem for the given boundary conditions completely as

$$x_1(t) = \frac{1}{\alpha}(cos(\omega)sinh(\omega t) - cosh(\omega)sin(\omega t))$$
$$x_2(t) = \frac{1}{\alpha}(cos(\omega)sinh(\omega t) + cosh(\omega)sin(\omega t))$$
$$\omega = \sqrt{3/2} \;\; \& \;\; \alpha = cos(\omega)sinh(\omega) + cosh(\omega)sin(\omega)$$

(b) Given t_f is free but the final position should lie on the surface:

$$\psi(x(t), t) = x_1(t) + 3x_2(t) + 5t - 15 = 0$$

With this boundary condition, the optimization problem is modified as

$$J'(x) = \nu(\psi(x(t_f), t_f)) + \int_0^{t_f} g(x, \dot{x}, t)dt$$
$$g(x, \dot{x}, t) = \dot{x_1}^2(t) + \dot{x_2}^2(t) + 3x_1(t)x_2(t)$$

which provides the necessary boundary conditions as:

$$g_{\dot{x}}(t_f) + \nu(\psi_x(t_f)) = 0$$
$$g(t_f) - g_{\dot{x}}(t_f)\dot{x}(t_f) + \nu(\psi_t(t_f)) = 0$$
$$\psi(x(t_f), t_f) = 0$$

We need to solve for the unknowns $t_f, \lambda_1, \lambda_2, \nu$ with the conditions as described below.

$$g_{\dot{x}}(t_f) + \nu(\psi_x(t_f)) = 0$$
$$\Rightarrow 2\begin{bmatrix}\dot{x}_1(t_f) & \dot{x}_2(t_f)\end{bmatrix} + \nu\begin{bmatrix}1 & 3\end{bmatrix} = 0$$
$$\Rightarrow \dot{x}(t_f) = \frac{-\nu}{2}\theta \text{ where } \theta = [1 \;\; 3]^T$$
$$\Rightarrow \dot{x}(t_f) = \omega\begin{bmatrix}cosh(\omega t_f) & cos(\omega t_f)\\ cosh(\omega t_f) & -cos(\omega t_f)\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = -\frac{\nu}{2}\begin{bmatrix}1\\ 3\end{bmatrix} \quad \text{From Eqns(2, 3)}$$
$$\Rightarrow \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \frac{\nu}{2\omega}\begin{bmatrix}-2sech(\omega t_f)\\ sec(\omega t_f)\end{bmatrix}$$
$$\Rightarrow x(t_f) = \begin{bmatrix}sinh(\omega t_f) & sin(\omega t_f)\\ sinh(\omega t_f) & -sin(\omega t_f)\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \frac{\nu}{2\omega}\begin{bmatrix}-2tanh(\omega t_f) + tan(\omega t_f)\\ -2tanh(\omega t_f) - tan(\omega t_f)\end{bmatrix}$$

Now using the above conditions, we solve for ν in terms of t_f using the second boundary condition:

$$g(t_f) - g_{\dot{x}}(t_f)\dot{x}(t_f) + \nu(\psi_x(t_f)) = 0$$
$$\Rightarrow \dot{x}(t_f)^T\dot{x}(t_f) + 3x_1(t_f)x_2(t_f) - 2\dot{x}(t_f)^T\dot{x}(t_f) + 5\nu = 0$$
$$\Rightarrow -\frac{5}{2}\nu^2 + 3x_{1f}x_{2f} + 5\nu = 0$$
$$\Rightarrow -\frac{5}{2}\nu^2 + 5\nu + \frac{\nu^2}{2}(4tanh^2(\omega t_f) - tan^2(\omega t_f)) = 0$$
$$\nu\left[4tanh^2(\omega t_f) - tan^2(\omega t_f) - 5\right] = -10 \quad \text{Since } \nu \neq 0$$

Now we use the third condition which requires the final point to be on the surface to find the final time t_f which solves all the other variables in terms of t_f:

$$\begin{aligned}
\theta^T x(t_f) + 5t_f - 15 = 0 \\
\Rightarrow \frac{\nu}{2\omega}\theta^T \begin{bmatrix} -2tanh(\omega t_f) + tan(\omega t_f) \\ -2tanh(\omega t_f) - tan(\omega t_f) \end{bmatrix} + 5t_f - 15 = 0 \\
\Rightarrow \frac{\nu}{2\omega}(-8tanh(\omega t_f) - 2tan(\omega t_f)) + 5t_f - 15 = 0
\end{aligned}$$

The above equation is solved in MATLAB to obtain the value of t_f. The function has multiple solutions $\{3.6665,\ 4.3404,\ 5.9142,\ 7.2788,\ 8.1885\}$. The family of extremals is given by:

$$\begin{aligned}
t_f &= \{3.6665, 4.3404, 5.9142, 7.2788, 8.1885\} \\
\omega &= \sqrt{3/2} \\
\nu &= \frac{-10}{4tanh^2(\omega t_f) - tan^2(\omega t_f) - 5} \\
\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} &= \frac{\nu}{2\omega} \begin{bmatrix} -2sech(\omega t_f) \\ sec(\omega t_f) \end{bmatrix} \\
x_1(t) &= \lambda_1 sinh(\omega t) + \lambda_2 sin(\omega t) \\
x_2(t) &= \lambda_1 sinh(\omega t) - \lambda_2 sin(\omega t)
\end{aligned}$$

We pick $t_f = 3.6665$ to verify the constraints. The initial boundary conditions are obvious since $sinh(0) = sin(0) = 0 :\ \ x_1(0) = x_2(0) = 0$. For the final time, we have to ensure the trajectory lies on the surface $\psi(x(t_f), t_f) = 0$

$$\begin{aligned}
x_1(t_f = 3.665) &= -0.0089sinh(3.665\omega) - 0.8982sin(3.665\omega) = 0.4810 \\
x_2(t_f = 3.665) &= -0.0089sinh(3.665\omega) + 0.8982sin(3.665\omega) = -1.2714 \\
\psi(x(t_f), t_f) &= [1\ \ 3] \begin{bmatrix} 0.4810 \\ -1.2714 \end{bmatrix} + 5(3.6488) - 15 = -3.333 - 15 + 18.325 = 0.00082 \sim 0
\end{aligned}$$

Similar checks can be done for other solutions also.

3. Given a function $\eta(t) \in \mathbf{C}^2$ (twice continous) defined on $\{t_0, t_f\}$ such that $\eta(t_0) = \eta(t_f) = 0$. We need to derive the Euler-Lagrange necessary conditions for fixed boundary conditions using the following evaluation form:

$$
\begin{aligned}
&J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt \\
&\left.\frac{dJ(x^* + \epsilon\eta)}{d\epsilon}\right|_{\epsilon=0} = 0 \\
&\Rightarrow \int_{t_0}^{t_f} \left.\frac{dg(x^* + \epsilon\eta, \dot{x}^* + \epsilon\dot{\eta})}{d\epsilon}\right|_{\epsilon=0} dt = 0 \\
&\Rightarrow \frac{dg(x^* + \epsilon\eta, \dot{x}^* + \epsilon\dot{\eta})}{d\epsilon} = \frac{\partial g}{\partial(x^* + \epsilon\eta)} \frac{d(x^* + \epsilon\eta)}{d\epsilon} + \frac{\partial g}{\partial(\dot{x}^* + \epsilon\dot{\eta})} \frac{d(\dot{x} + \epsilon\dot{\eta})}{d\epsilon} \\
&\Rightarrow \left.\frac{dg(x^* + \epsilon\eta, \dot{x}^* + \epsilon\dot{\eta})}{d\epsilon}\right|_{\epsilon=0} = \frac{\partial g}{\partial x^*}\eta + \frac{\partial g}{\partial \dot{x}^*}\dot{\eta} \\
&\Rightarrow \int_{t_0}^{t_f} \left(\frac{\partial g}{\partial x^*}\eta + \frac{\partial g}{\partial \dot{x}^*}\dot{\eta}\right) dt = 0 \\
&\Rightarrow \int_{t_0}^{t_f} \frac{\partial g}{\partial x^*}\eta(t)dt + \left.\frac{\partial g}{\partial \dot{x}^*}\eta(t)\right|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}^*}\right)\eta(t)dt = 0 \\
&\Rightarrow \int_{t_0}^{t_f} \left[\frac{\partial g}{\partial x^*} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}^*}\right)\right]\eta(t)dt = 0 \quad \text{Since } \eta(t_0) = \eta(t_f) = 0
\end{aligned}
$$

Since $\eta(t)$ is arbitrary function except for the endpoints, the above function will be zero only if the pre-multiplier to η is zero over the trajectory between t_0 and t_f. This gives the Euler-Lagrange equations as:

$$\Rightarrow \frac{\partial g}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) = 0 \quad \forall x(t) = x^*(t) \;\; t \in \{t_0, t_f\}$$

4. Given optimization problem :

$$
\begin{aligned}
&\dot{x} = ax - bu \quad x(t_0) \text{ given} \\
&J = \frac{1}{2}c\,[x(t_f)]^2 + \frac{1}{2}\int_{t_0}^{t_f} [u(t)]^2\, dt
\end{aligned}
$$

Equating the relevant parts of this problem to the general optimization problem we get the following:

$$
\begin{aligned}
&J_a = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} \left(H(x(t), u(t), \lambda, t), -\lambda^T\dot{x}\right) dt \quad \text{(General optimization problem)} \\
&\phi(x(t_f), t_f) = \frac{1}{2}c\,[x(t_f)]^2 \\
&H(x(t), u(t), \lambda, t) = \frac{1}{2}[u(t)]^2 + \lambda(ax - bu)
\end{aligned}
$$

The necessary conditions for optimal control is :

$$\begin{aligned} H_u &= u - \lambda b = 0 \\ \Rightarrow u &= \lambda b \\ \dot{\lambda} &= -\partial_x H = -a\lambda \\ \Rightarrow \lambda &= me^{-at} \\ \Rightarrow u &= mbe^{-at} \end{aligned}$$

With this input we solve the dynamics of the system as follows:

$$\begin{aligned} \dot{x} &= ax - b(mbe^{-at}) \\ \Rightarrow x(t) &= e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} b\left(-me^{-a\tau}b\right) d\tau \\ \Rightarrow x(t) &= e^{a(t-t_0)}x(t_0) - \frac{mb^2e^{at}}{2a}\left[-e^{-2at} + e^{-2at_0}\right] \end{aligned}$$

Now we apply the boundary condition for λ to get the value of m :

$$\begin{aligned} \lambda(t_f) &= \partial_x \phi(x,t)|_{t=t_f} \\ \Rightarrow \lambda(t_f) &= cx(t_f) = c\left[e^{a(t_f-t_0)}x(t_0) - \frac{mb^2e^{at_f}}{2a}\left(e^{-2at_0} - e^{-2at_f}\right)\right] \\ \Rightarrow me^{-at_f} &= ce^{a(t_f-t_0)}x(t_0) - m\{\frac{cb^2e^{at_f}}{2a}\left(e^{-2at_0} - e^{-2at_f}\right)\} \\ \Rightarrow m &= \frac{ce^{a(t_f-t_0)}x(t_0)}{e^{-at_f} + \frac{cb^2e^{at_f}}{2a}\left(e^{-2at_0} - e^{-2at_f}\right)} \end{aligned}$$

Which solves the problem by providing explicitly the control u(t) as:

$$u(t) = mbe^{-at} = \frac{bce^{a(t_f-t_0)}x(t_0)}{1 + \frac{cb^2}{2a}\left(e^{2a(t_f-t_0)} - 1\right)} e^{a(t_f-t)}$$

This can be show after a lot of intermediate steps to be equivalent to:

$$u(t) = \frac{bce^{2a(t_f-t)}}{1 - \frac{b^2c}{2a}\left(1 - e^{2a(t_f-t)}\right)} x(t)$$

(Note: This problem can also be easily solved using riccati equation and arrive at the second solution. The steps for the second method are outlined as below:) The Riccati equation for the above problem is given as:

$$\dot{P} = -2aP + P^2b^2$$

Substituting $b^2P = \frac{-\dot{q}}{q}$, we get:

$$\ddot{q} + 2a\dot{q} = 0$$

Using boundary conditions $q(t_f) = 1, \dot{q}(t_f) = -cb^2$, we get :

$$q(t) = \left(1 - \frac{cb^2}{2a}\right) + \frac{cb^2}{2a} e^{2a(t_f - t)}$$
$$P(t) = \frac{ce^{2a(t_f-t)}}{1 - \frac{cb^2}{2a}\left(1 - e^{2a(t_f-t)}\right)}$$

Then the control can be specified as:

$$u(t) = bP(t)x(t) = \frac{bce^{2a(t_f-t)}}{1 - \frac{b^2c}{2a}\left(1 - e^{2a(t_f-t)}\right)} x(t)$$

The solution matches with the above solution and shows the linear feedback form for the control in LQR type of problems.

5. The matlab code for the problem is given below:

```
function [ ] = hw2( )
%HW3 Summary of this function goes here
%   Detailed explanation goes here
% In the given problem:  xdot = Ax + Bu
%  A = [0, 1; 3, -1] B = [0;1]
%  Q = diag(1, 1/2) (not unique actually , R = 1/2
% ( Q is positive semidefinite, R symm pos def
% Optimality for the cost function ensures that,
% lambdadot = -Qx - A'lambda;
% u = -Rinv*B'*lambda
% lamdba = P*x
% Pdot = -A'*P - PA + PBRinv*B'*P - Q and P(t_f) = 0
%this gives control law u

%constants for the problem: A,B,Q,R,tf
close all;
A = [0 1; 3 -1];
B = [0;1];
Q = [1 0;0 0.5];
R = 0.5;
tf = 20;
Pf = zeros(2,2);
Δt = 0.1;
%Solve riccati equation for tf = 20, Pf = 0
[t,P] = ode45(@(t,P)Pdot(t,P,A,B,Q,R),tf:-Δt:0,Pf);
m1 = figure;
for i = 1:4
    subplot(2,2,i),  plot(t,P(:,i));
    ylabel(strcat('P_',int2str(i),'(t)'));
    xlabel('time(t)');
end
%plot2svg('riccati.svg',m1);
%exportfig('pic1.pdf',...
   % 'width',3.7,...
 %   'color','rgb');
```

```
%Solve for control input and state x(t)
N = length(t);
X = zeros(2,N);
U = zeros(N,1);
X(:,N) = [-5; 5];%X(0)
for count = N:-1:2
    [xdotcount,U(count)] = xdot(t(count),X(:,count),A,B,P(count,:),R);
    X(:,count-1) = X(:,count) + ∆t*xdotcount;
end
m2 = figure;
for i = 1:2
    subplot(2,2,i), plot(t,X(i,:));
    ylabel(strcat('X_',int2str(i),'(t)'));
    xlabel('time(t)');
end
subplot(2,1,2), plot(t,U);
ylabel('U(t)');
xlabel('time(t)');
%exportfig('pic2.pdf',...
 %   'width',3.7,...
%    'color','rgb');
%plot2svg('XandU.svg',m2);
end

function out1 = Pdot(t,P,A,B,Q,R)
P1 = reshape(P,2,2);
out = -(A')*P1 - P1*A + P1*B*(B')*P1*(1/R) - Q; %R is scalar
out1 = out(:);
end

function [out1, u] = xdot(t,x,A,B,P,R)
Pt = reshape(P',2,2);
u = -(1/R)*(B')*Pt*x;
out1 = A*x + B*u;
end
```

The results of the above code are shown below:

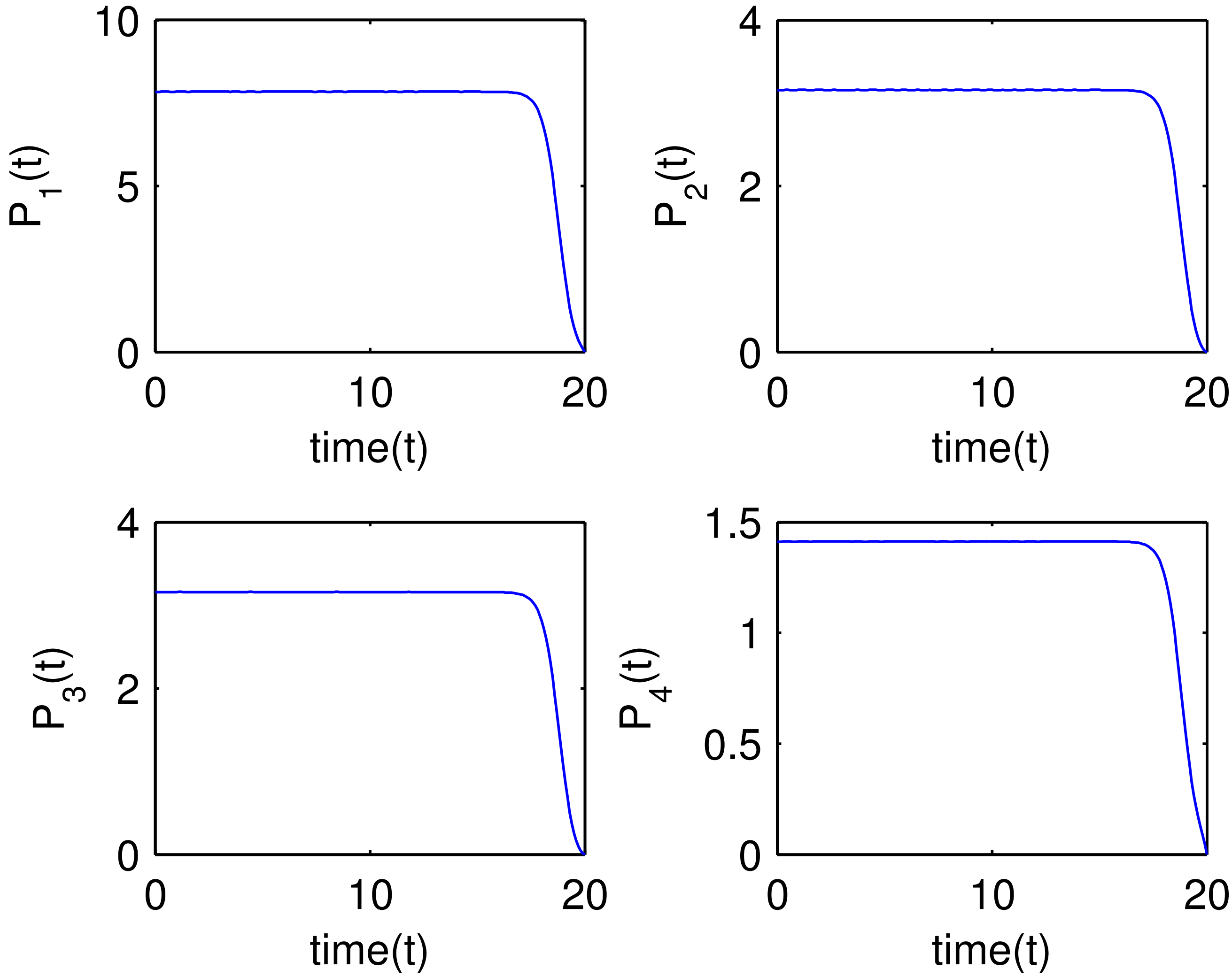

Figure 1: The solution to riccati equation P(t) with Pf = 0

6. Riccati equations are first order differential equations with a special structure i.e equations have maximum differential order 1 and maximum polynomial order of the variable as 2. Riccati equations can be written in the form[Source **Wiki**]:

$$y'(x) = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x)$$

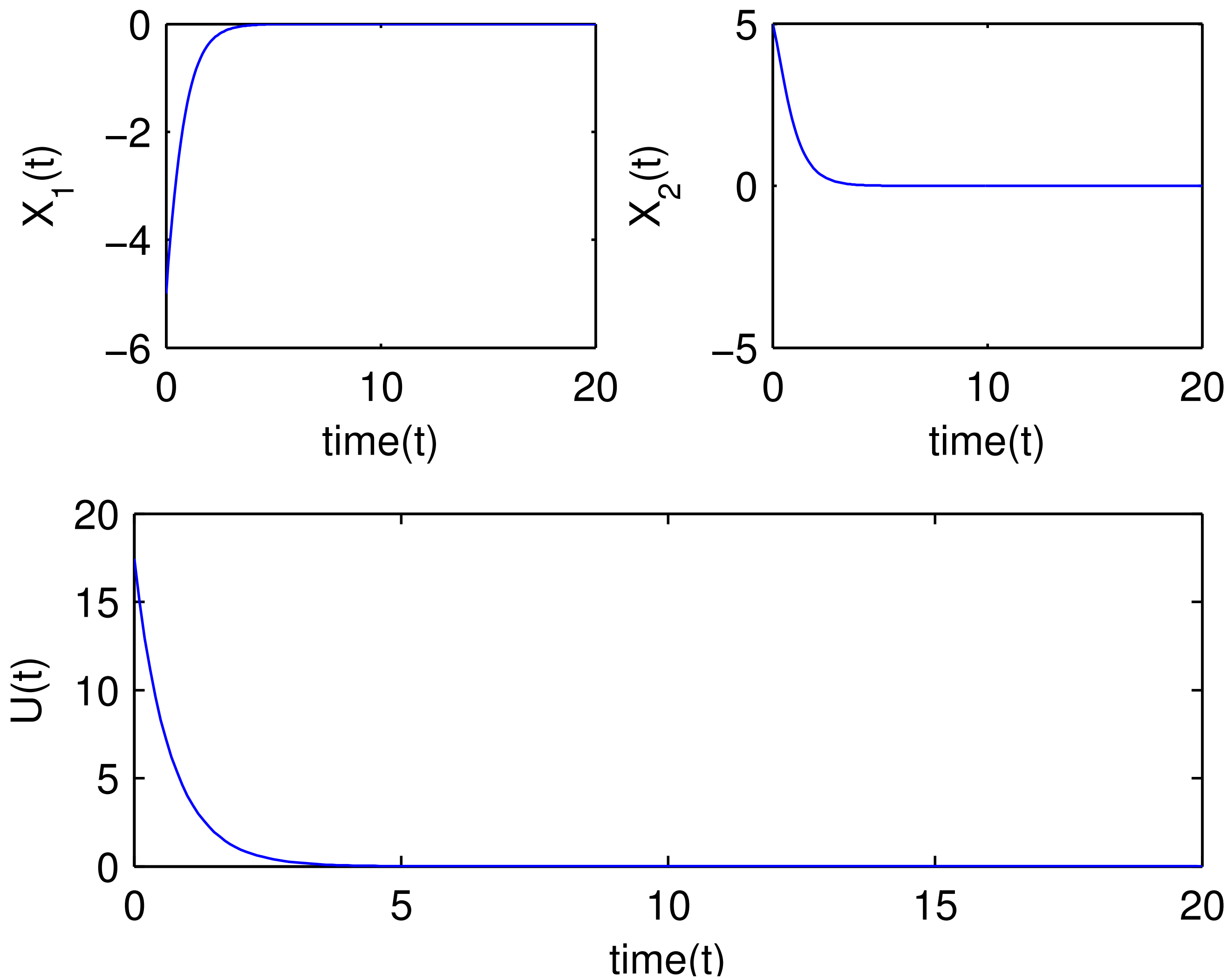

Figure 2: The state X(t) and the control U(t)

This equation can be reduced to second order linear differential equation [Source **Wiki**]:

$$\text{Substituting: } y(x)q_2(x) = \frac{-u'}{u}$$

$$\Rightarrow y(x){q_2}'(x) + y'(x)q_2(x) = \frac{-u''}{u} + \left(\frac{u'}{u}\right)^2$$

$$\Rightarrow y(x){q_2}'(x) + \{q_0(x) + q_1(x)y(x) + q_2(x)y^2(x)\}q_2(x) = \frac{-u''}{u} + \left(\frac{u'}{u}\right)^2$$

$$\Rightarrow q_2(x)q_0(x) + y(x)\{{q_2}'(x) + q_2(x)q_1(x)\} + q_2(x)^2y^2(x) = \frac{-u''}{u} + \{q_2(x)y(x)\}^2$$

$$\Rightarrow u'' = -q_2(x)q_0(x)u - uy(q_2q_1 + {q_2}')$$

$$\text{Using : } R(x) = q_1(x) + \frac{q_2'(x)}{q_2(x)} \quad S(x) = q_2(x)q_0(x) \quad u' = -u(x)y(x)q_2(x)$$

$$u'' - Ru' + Su = 0$$

The second order equation can be solved to compute the solutions for y(x)

[THIS is just for your information; I don't expect everyone to write this]

Acknowledgements

I hereby declare that I have not discussed this homework with anyone. The solutions written here are my own work and from lecture notes and sample code provided by the professor. Any external references are mentioned in the text.

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