Active Sensing – Derivation of HTF

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1 System

State equation of a 1-D system with state variable $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where x_1 is the 1-D position , $x_2 = \dot{x_1}$ is the 1-D velocity and control is the linear 1D acceleration is,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -b/M \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u \tag{1}$$

Where b is the linear damping term and m is the mass of the system. The output equation is given by,

$$\begin{aligned} y &= \frac{d}{dt} s(x_1) \\ &= g(x_1) x_2 \end{aligned} \tag{2}$$

where $g(x_1) = s'(x_1)$

2 Linearizing to give LTP system

Linearizing the system around $x_1^* =$Cos(ωt) , $x_2^* = \dot{x_1^*} = \omega$Sin$(\omega t)$ gives,

$$\begin{aligned} \dot{\delta x} &= \begin{bmatrix} 0 & 1 \\ 0 & -b/M \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} \delta u \\ &= A\delta x + B\delta u \end{aligned} \tag{3}$$

where $\delta x = x - x^*$ and $\delta u = u - u^*$.

Now linearizing the output gives,

$$\begin{aligned}\delta y &= \begin{bmatrix} g'(x_1^*)x_2^* & g(x_1^*) \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \\ &= \begin{bmatrix} g'\left(Cos(\omega t)\right)\omega Sin(\omega t) & g\left(Cos(\omega t)\right) \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \\ &= C(t)\delta x \end{aligned} \tag{4}$$

3 HTF of the periodic function–Harmonic Balance

From HTF theory,

$$y(t) = \sum_k \left(h_k(t) * u(t)\right) e^{jk\omega_o t} \tag{5}$$

where ω_o is the fundamental frequency of the system .

Now for the system as described in above sections, $H_n(s) = \mathcal{L}h_n$ is given by

$$\mathcal{H}_{n,m}(s) = C_{n-m}\left(s_m I - A\right)^{-1} B \tag{6}$$

Where C_k's are the Fourier coefficients of $C(t)$, s_m=$s + jm\omega_o$. In our case $\omega_o = \omega$

Substituting for A,B in the above equation gives,

$$\begin{aligned}\mathcal{H}_{n,m}(s) &= C_{n-m}\left(s_m I - A\right)^{-1} B \\ &= C_{n-m} \begin{bmatrix} s + jm\omega & -1 \\ 0 & s + jm\omega + b/M \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/M \end{bmatrix} \\ &= C_{n-m} \frac{1}{(s + jm\omega + b/M)(s + jm\omega)} \begin{bmatrix} s + jm\omega + b/M & 1 \\ 0 & s + jm\omega \end{bmatrix} \begin{bmatrix} 0 \\ 1/M \end{bmatrix} \\ &= \frac{1}{M}\frac{1}{(s + jm\omega + b/M)(s + jm\omega)} C_{n-m} \begin{bmatrix} 1 \\ s + jm\omega \end{bmatrix} \end{aligned} \tag{7}$$

Now,

$$C(t) = \begin{bmatrix} g'\left(Cos(\omega t)\right)\omega Sin(\omega t) & g\left(Cos(\omega t)\right) \end{bmatrix} \tag{8}$$

For $g(x) = d_1 * x + e_1$, the fourier co-effs of C(t) are

$$\left\{ -\omega \left(\left\{ \begin{array}{cc} -\frac{\text{id1}}{2} & n = 1 \\ \frac{\text{id1}}{2} & n = -1 \end{array} \right. \right), \left\{ \begin{array}{cc} \frac{\text{d1}}{2} & n = -1, n = 1 \\ \text{e1} & n = 0 \end{array} \right. \right\}$$

$H_0 = \mathcal{H}_{0,0}$

$H_1 = \mathcal{H}_{1,0}$

$H_m 1 = \mathcal{H}_{-1,0}$

4 HTF of the periodic function–IRF

$$y(t) = \int_0^t h(t, \tau) u(\tau) d\tau \tag{9}$$

For LTP system,

$$\begin{aligned} h(t+T, \tau+T) &= h(t, \tau) \\ \implies h(t+T, t-r+T) &= h(t, t-r) \end{aligned} \tag{10}$$

Therefore, $h(t, t-r)$ is periodic in T. Now,

$$\begin{aligned} h(t, t-r) &= \sum h_k(r) e^{jk\omega_o t} \\ \implies h(t, \tau) &= \sum h_k(t-\tau) e^{jk\omega_o t} \end{aligned} \tag{11}$$

$$\begin{aligned} y(t) &= \int_0^t h(t, \tau) u(\tau) d\tau \\ &= \int_0^t \sum h_k(t-\tau) e^{jk\omega_o t} u(\tau) d\tau \\ &= \int_0^t \sum h_k(t-\tau) e^{jk\omega_o (t-\tau)} u(\tau) e^{jk\omega_o \tau} d\tau \\ &= \sum \int_0^t h_k(t-\tau) e^{jk\omega_o (t-\tau)} u(\tau) e^{jk\omega_o \tau} d\tau \\ &= \sum_k \left(h_k(t) * u(t) \right) e^{jk\omega_o t} \end{aligned} \tag{12}$$

Where h_k's are the fourier co-effs of $h(t, t-r)$.

For our case,

$$
\begin{aligned}
h(t, t-r) &= C(t)\Phi(t, t-r)B \\
&= C(t) \begin{bmatrix} 1 & (1-e^{-br/m})\frac{m}{b} \\ 0 & e^{-br/m} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\
&= C(t) \begin{bmatrix} (1-e^{-br/m})\frac{1}{b} \\ \frac{1}{m}e^{-br/m} \end{bmatrix} \\
&= C(t)\alpha(r)
\end{aligned} \tag{13}
$$

Now, $h_k(r) = C_k.\alpha(r)$ Where C_k fourier co-effs of C(t)

5 Transfer function of plant

Just considering h_0, h_1, h_{-1} the output of the linearized system is,

$$y = h_0 * u + (h_1 * u)e^{j\omega_o t} + (h_{-1} * u)e^{-j\omega_o t} \tag{14}$$

Now, modulating with $Cos\omega t$ gives

$$yCos\omega t = (h_0 * u)Cos\omega t + Re[(h_1 * u)](1 + Cos2\omega t) - Im[(h_1 * u)](Sin2\omega t) \tag{15}$$

Now low pass filtering it gives the plant transfer function as

$$y_p = Re(h_1) * u \tag{16}$$

For $g(x) = d_1 * x + e_1$ this transfer function is, $\frac{d_1}{2(b+ms)}$