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Problem 1.

1. a) $\dot{\theta} = \omega - \beta - \eta_v$ $\quad$ $\eta_v, \eta_u$ uncorrelated Gaussian with $\sigma_v^2, \sigma_u^2$

$\dot{\beta} = \eta_u$

$z_k = \theta_k + v_k$ $\quad$ $v_k$ time-uncorrelated with $\sigma_n^2$

a) Determine $\Phi_k, \Gamma_k, Q_k$ assuming a constant sampling interval $\Delta t$ during which there is a constant input $\omega_k$

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$$\theta_k = \Phi_k \theta_{k-1} + \Gamma_k \omega_{k-1}$$

$$P_k = \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + Q_{k-1}$$

In state-space form, $\underbrace{\begin{bmatrix} \dot{\theta} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \beta \end{bmatrix}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_v \\ \eta_u \end{bmatrix}$

soln of $\dot{x} = Ax$ is $\Phi x_0$

$\Phi_k = \begin{bmatrix} 1 & e^{-t} \\ 0 & 1 \end{bmatrix}$, but for sampling interval $\Delta t$ during which control is constant:

$$\Phi_k = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix}$$ 1

$$\Gamma_k = \int_{t_{k-1}}^{t_k} \Phi_k G(\tau)\, d\tau = \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau = \begin{bmatrix} \Delta t \\ 0 \end{bmatrix}$$ 1

$$\begin{bmatrix} \theta_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_{k-1} \\ \beta_{k-1} \end{bmatrix} + \begin{bmatrix} \Delta t \\ 0 \end{bmatrix} \omega_{k-1}$$

$$Q_k = \int_{t_{k-1}}^{t_k} \Phi L Q_c' L^T \Phi^T d\tau = q_c \int_{t_{k-1}}^{t_k} \begin{bmatrix} -1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -\Delta t & 1 \end{bmatrix} d\tau = q_c \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 + \Delta t^2 & -\Delta t \\ -\Delta t & -1 \end{bmatrix} d\tau$$

$$Q_k = \begin{bmatrix} \sigma_v & 0 \\ 0 & \sigma_u \end{bmatrix} \begin{bmatrix} \Delta t + \frac{\Delta t^3}{3} & -\frac{\Delta t^2}{2} \\ -\frac{\Delta t^2}{2} & -\Delta t \end{bmatrix}$$

Check soln 1

1. b) True state propagation is wrong in the code. Check solution

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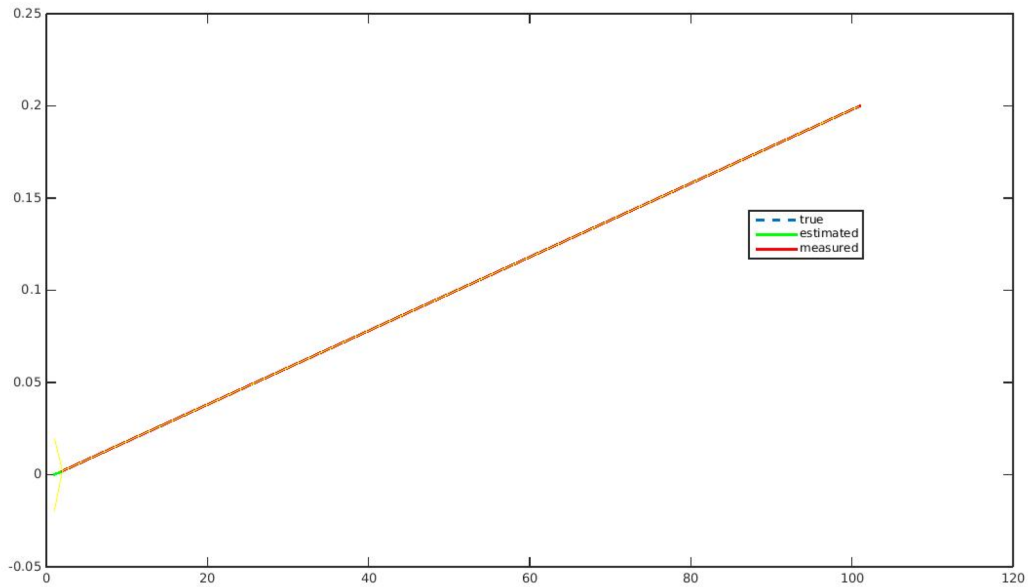


Figure 1. Plot for dt = 1 sec and n = 100

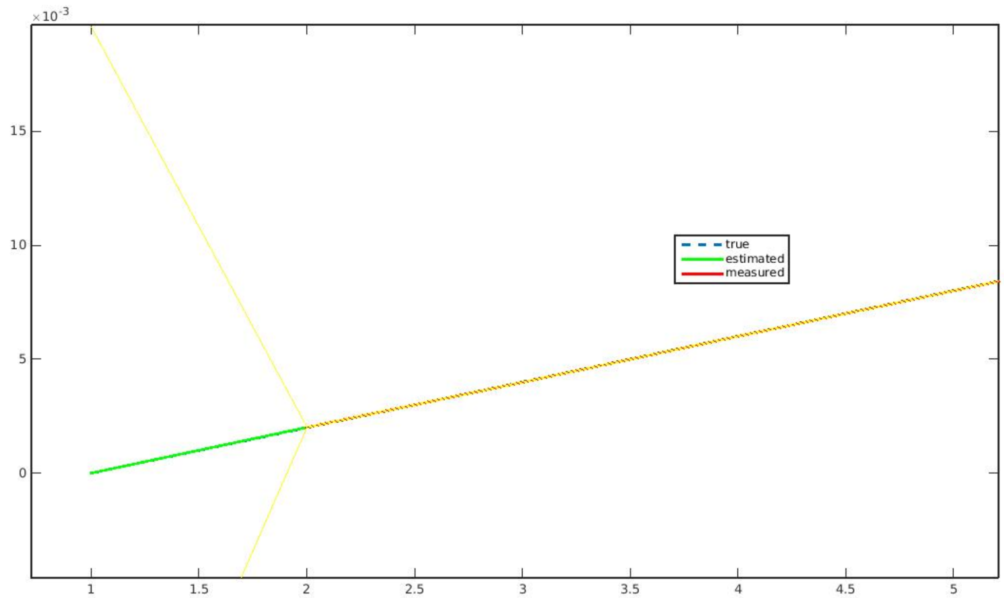


Figure 2. Zoom in on beginning of plot

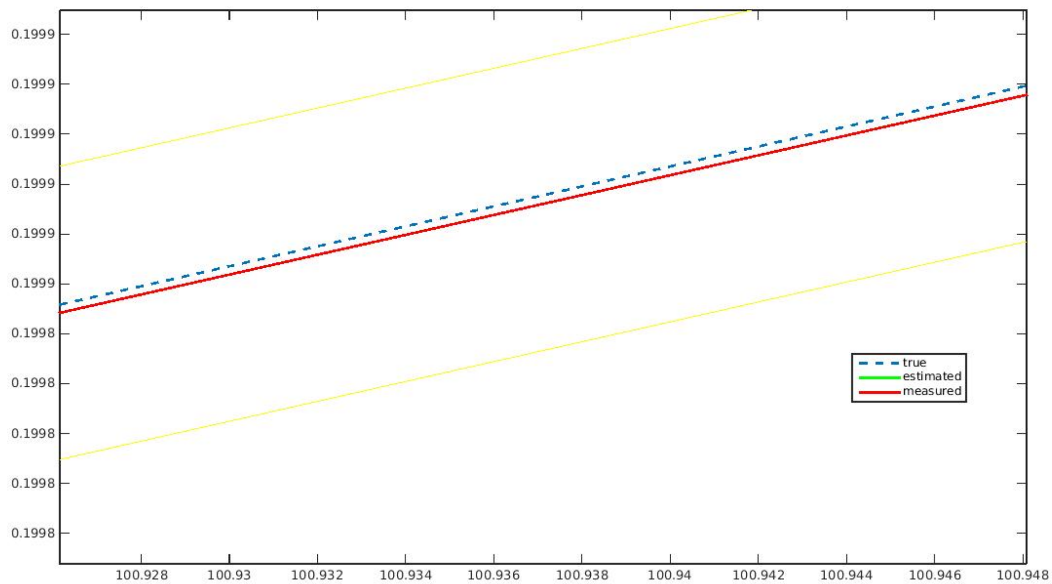


Figure 3. Zoom in on end of plot

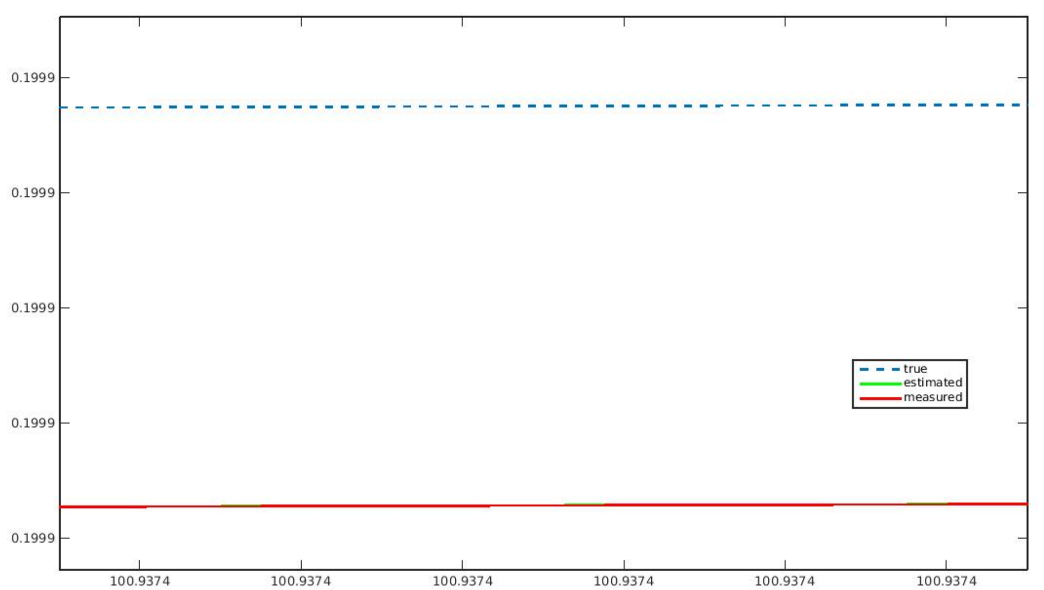


Figure 4. Zoom in to show true states, estimated states, measured states

Problem 2.

2. Consider a nonlinear discrete-time model with additive noise given by

$$x_k = f(x_{k-1}, u_{k-1}) + w_{k-1}$$
$$z_k = h_k(x_k) + v_k$$

Show that the minimizer of $J(x) = \frac{1}{2}(x-\hat{x})^T P^{-1}(x-\hat{x}) + \frac{1}{2}(z-h(x))^T R^{-1}(z-h(x))$

Corresponds to $x = \hat{x} + K(z - h(\hat{x}))$

$$K = PH^T(HPH^T+R)^{-1}$$

To minimize $J$, $\nabla J = 0$, $dx = x - \hat{x}$, $dz = z - h(\hat{x}_0)$

$$\nabla J = P^{-1}dx - H^T R^{-1}(dz - H dx) = 0$$
$$(P^{-1} + H^T R^{-1} H)dx = H^T R^{-1} dz$$
$$dx = (P^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1} dz$$

by matrix inversion lemma

$$dx = [P - PH^T(HPH^T+R)^{-1}HP]H^T R^{-1} dz$$
$$dx = [I - \underbrace{PH^T(HPH^T+R)^{-1}}_{=K}H]PH^T R^{-1} dz$$
$$dx = \underbrace{(I - KH)P}_{P_k}H^T R^{-1} dz$$
$$dx = P_k H^T R^{-1} dz$$

$$K = P_k H^T R^{-1} = (I - KH)P_{k|k-1}H^T R^{-1}$$
$$= [P - PH^T(HPH^T + R^{-1})HP]H^T R^{-1}$$

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Check
Soln

$$dx = K dz$$
$$x - \hat{x} = K(z - h(\hat{x}))$$
$$x = \hat{x} + K(z - h(\hat{x}))$$

Problem 3.

3. a)

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3.a)

$$\begin{pmatrix} x_k \\ y_k \\ \theta_k \\ r_k \end{pmatrix} = \begin{pmatrix} x_{k-1} + \Delta t \cos(\theta_{k-1}) r_{k-1} \Omega_{k-1} \\ y_{k-1} + \Delta t \sin(\theta_{k-1}) r_{k-1} \Omega_{k-1} \\ \theta_{k-1} + \Delta t \omega_{k-1} \\ r_{k-1} \end{pmatrix} + w_{k-1}$$

$$F_k = \partial_x f(\hat{x}_{k|k}, u_k) = \begin{bmatrix} 1 & 0 & -\Delta t \sin(\theta) r\Omega & \Delta t \cos(\theta)\Omega \\ 0 & 1 & \Delta t \cos(\theta) r\Omega & \Delta t \sin(\theta)\Omega \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_k = \partial_x h(\hat{x}_{k|k-1}) = \begin{bmatrix} \frac{y'-y}{\|p'-p\|^2} & \frac{-x'-x}{\|p'-p\|^2} & -1 & 0 \\ \frac{x-x'}{\|p'-p\|} & \frac{y-y'}{\|p'-p\|} & 0 & 0 \end{bmatrix}$$

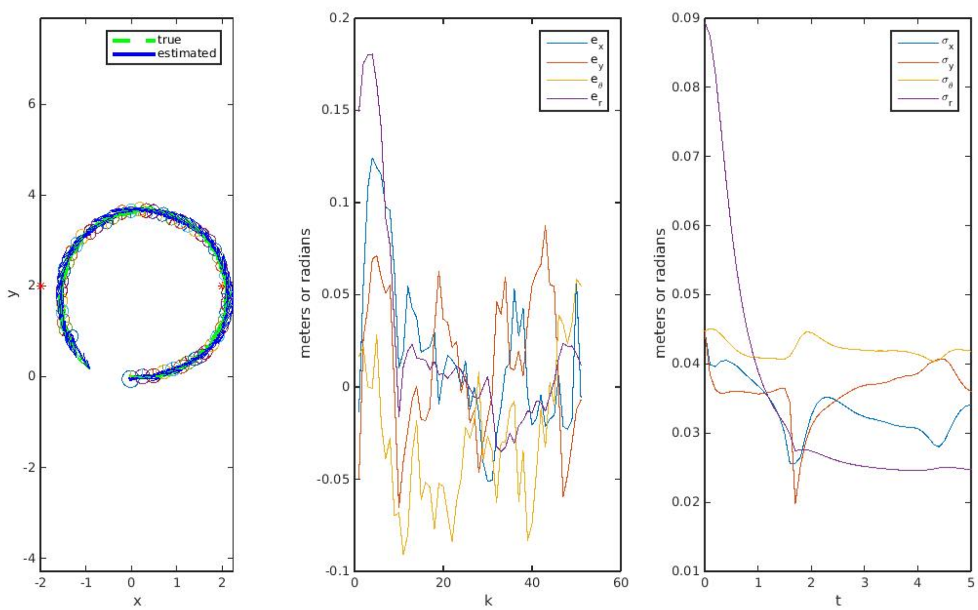


Figure 5. The error in r is shown to stabilize about zero

3. b) With only one bearing sensor, you cannot reconstruct the complete state. Check solution for more info. With two or more bearing sensors, depending on the placement you can reconstruct the state information.

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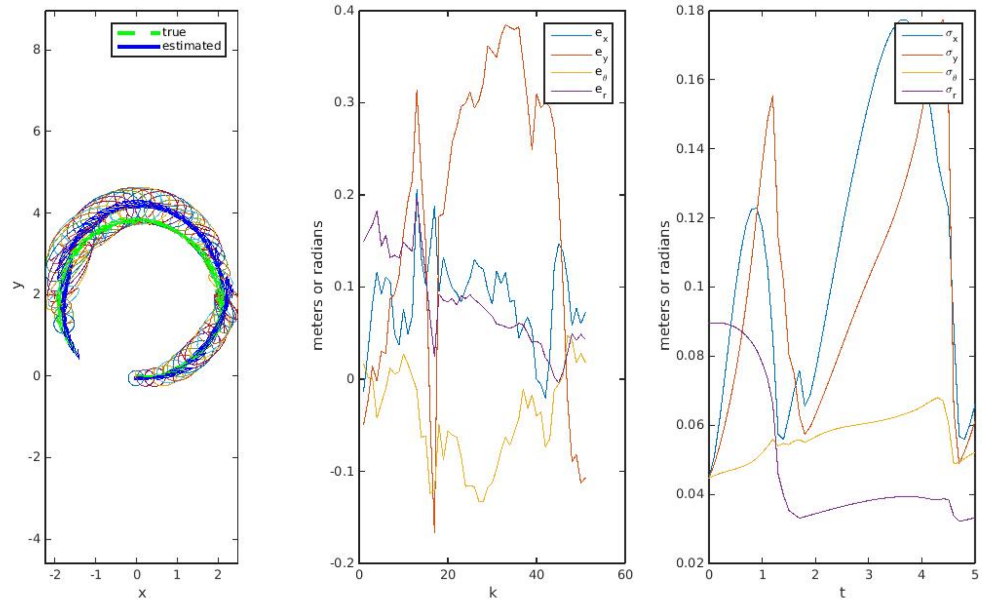


Figure 6. In the absence of range measurements, the error and standard deviation do not look as good.

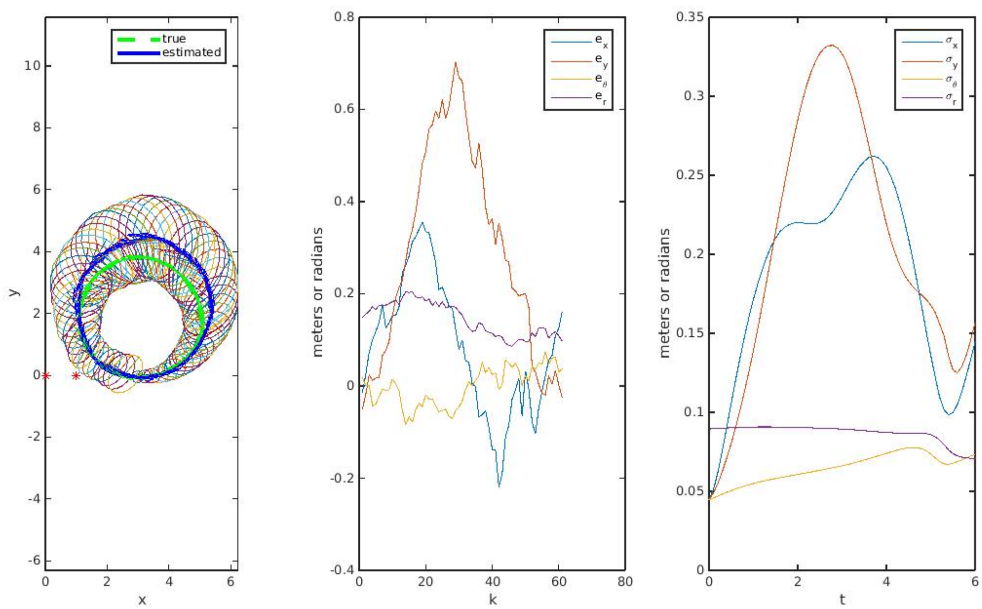


Figure 7. The full-state can still be reconstructed, however, even with poor placement of the beacons.

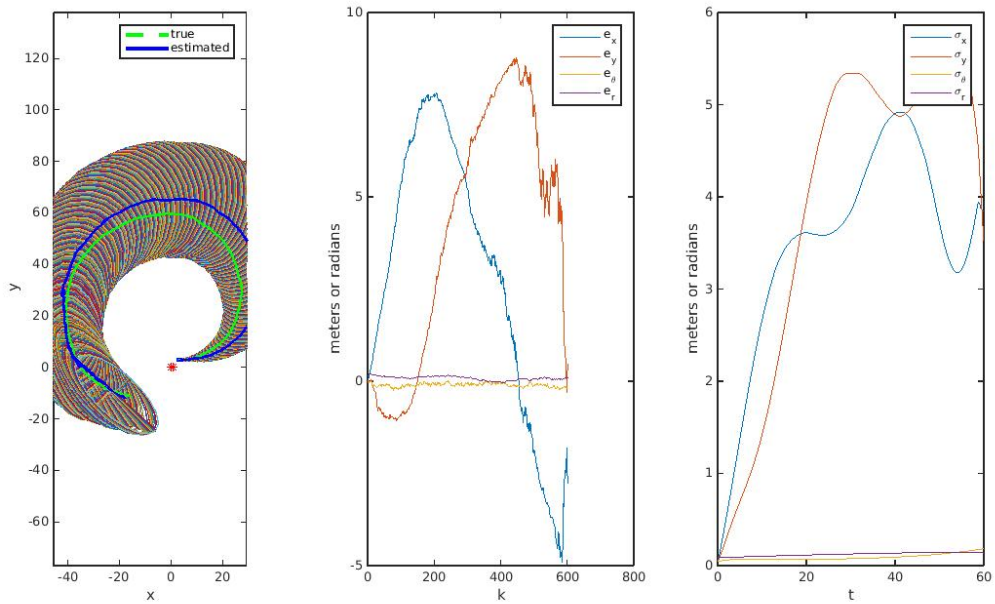


Figure 8. With a longer run time, and beacons very close together, the error is high, but the full state can be reconstructed.