

# EN530.603 Applied Optimal Control

## Lecture 4: Trajectory Optimization Basics

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### 1 Variations of functions

We are interested in solving optimal control problems such as

$$\min J(x(\cdot), u(\cdot), t_f) = \int_{t_0}^{t_f} L(x(t), u(t), t) dt$$

subject to  $\dot{x} = f(x, u, t)$  and other constraints. The cost  $J$  is called a *functional*, i.e. it is a function of functions since the trajectories  $x(\cdot)$  and  $u(\cdot)$  are functions of time. It is possible to optimize a functional in a similar way we optimize a regular function. In particular, there is a functional analog to the necessary conditions for a minimum of a function  $g$  given by  $\nabla g = 0$ . This analog is

$$\text{differential of a function } \nabla g = 0 \quad \Leftrightarrow \quad \text{variation of a functional } \delta J = 0$$

Next, define the *change* in a functional, after *varying*  $x(t)$  by  $\delta x(t)$  at each  $t$ , by

$$\Delta J(x(\cdot), \delta x(\cdot)) = J(x(\cdot) + \delta x(\cdot)) - J(x(\cdot)).$$

The *variation*  $\delta J(x(\cdot), \delta x(\cdot))$  is a linear function of  $\delta x(\cdot)$  defined by the following relationship

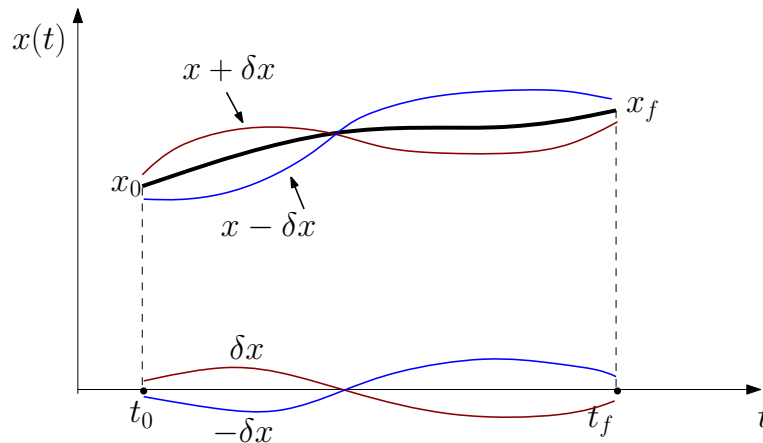
$$\Delta J(x(\cdot), \delta x(\cdot)) = \delta J(x(\cdot), \delta x(\cdot)) + o(\|\delta x\|),$$

where the *small-o* notation was used. For a positive integer  $p$  and a function  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$  we have

$$h(x) = o(\|x\|^p),$$

if  $\lim_{k \rightarrow \infty} \frac{h(x_k)}{\|x_k\|^p} = 0$  for all sequences  $x_k$  such that  $x_k \rightarrow 0$  and  $x_k \neq 0$  for all  $k$ .

In other words, roughly speaking, if  $h(x)$  denotes the the second and higher order terms in  $\Delta J$ , then  $h(x)$  has second and higher-order multiples of  $\delta x$  and thus  $h(x)/\|\delta x\|$  is at least proportional to  $\delta x$  which tends to zero as  $\delta x \rightarrow 0$ .



Similarly to standard function optimization, we can use the argument that at an optimum  $x^*(\cdot)$  we have

$$\Delta J(x^*(\cdot), \delta x(\cdot)) \geq 0$$

since any change of the cost away from optimum must be positive. Taking  $\delta x \rightarrow 0$  this implies that

$$\delta J(x^*(\cdot), \delta x(\cdot)) \geq 0.$$

But this must also hold for variations in the direction  $-\delta x(\cdot)$ , i.e.

$$\delta J(x^*(\cdot), -\delta x(\cdot)) \geq 0 \quad \Rightarrow \quad \delta J(x^*(\cdot), \delta x(\cdot)) \leq 0,$$

by the linearity of  $\delta J$ . Both conditions can only be true when

$$\delta J(x^*(\cdot), \delta x(\cdot)) = 0,$$

for any  $\delta x(\cdot)$ .

## 2 The Euler-Lagrange Equations

Consider the cost function

$$J = \int_{t_0}^{t_f} g(x(t), \dot{x}(t)) dt.$$

Its variation becomes

$$\delta J = \int_{t_0}^{t_f} [g_x(x, \dot{x}) \delta x + g_{\dot{x}}(x, \dot{x}) \delta \dot{x}] dt,$$

since the variation corresponds to taking first-order terms (i.e. derivatives) of the functions at time  $t$ . Integrate by parts (recall  $\int u \dot{v} = uv - \int \dot{u} v$ ) to get

$$\delta J = \int_{t_0}^{t_f} \left[ g_x(x, \dot{x}) \delta x - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) \delta x \right] dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f)) \delta x(t_f)$$

**Fixed boundary conditions.** If  $x(t_0)$  and  $x(t_f)$  are given then  $\delta x(0) = \delta x(t_f) = 0$  and

$$\delta J = \int_{t_0}^{t_f} \left[ g_x(x, \dot{x}) - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) \right] \delta x(t) dt$$

Since  $\delta x(t)$  are arbitrary and independent then  $\delta J = 0$  only when

$$g_x(x, \dot{x}) - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}) = 0,$$

which are called the *Euler-Lagrange equations (EL)*.

**Free boundary conditions.** When  $x(t_f)$  is not fixed, in addition to the EL equation, the following must hold

$$g_{\dot{x}}(x(t_f), \dot{x}(t_f)) = 0.$$

## 2.1 Example: shortest path curve

Consider a one-dimensional problem with state  $x(t)$  and the cost function

$$J = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}^2} dt,$$

which in fact corresponds to the length of a curve in the  $(x, t)$  plane. The goal is to compute the shortest length curve between two given points  $(x_0, t_0)$  and  $(x_f, t_f)$ .

Applying the EL equations we get

$$\frac{d}{dt} \frac{\dot{x}}{(1 + \dot{x}^2)^{1/2}} = 0 \quad \Leftrightarrow \quad \frac{\ddot{x}}{(1 + \dot{x}^2)^{3/2}} = 0,$$

which is satisfied when  $\ddot{x} = 0$  or when

$$x(t) = c_1 t + c_0,$$

i.e. when  $x(t)$  is a straight line. Let  $t_0 = 0$  and  $t_f = 1$ . It is easy to see that  $c_0 = x_0$  and  $c_1 = x_f - x_0$ .

## 2.2 Particle in 3-D

Let  $g(x, \dot{x}, t) = \frac{1}{2}m\|\dot{x}\|^2 - V(x)$ , where  $m$  denotes the mass and  $V$  denotes the potential energy of a particle with position  $x \in \mathbb{R}^3$ . The function  $g$  is actually the Lagrangian of the particle and the EL equations lead to

$$m\ddot{x} = -\nabla_x V,$$

which is simply Newton's law. This is one of the simplest examples that illustrates that Lagrangian mechanics can be considered as a special case of optimal control.

## 3 Free final-time

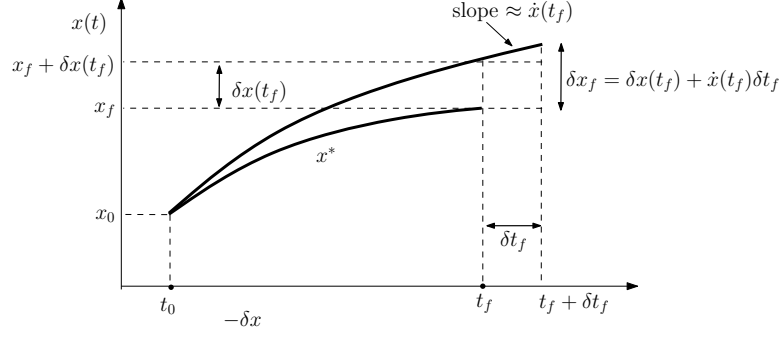
When the final time  $t_f$  is allowed to vary, the variation of  $J$  is expressed as

$$\begin{aligned} \delta J = & \int_{t_0}^{t_f} \left[ g_x(x^*, \dot{x}^*, t) \delta x - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) \delta x \right] dt \\ & + g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t) \delta x(t_f) + g(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f, \end{aligned}$$

where we employ the spatially-optimized trajectory  $x^*(t)$  instead of just  $x(t)$  to signify that after finding the optimal path without considering  $\delta t_f$ , then variations of  $t_f$  infinitesimally contribute to the cost by the term

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) \delta t_f$$

which is simply the cost at the last points multiplied by the time variations: think of it as a first-order approximation to the integral  $\int_{t_f}^{t_f + \delta t_f} g(x^*, \dot{x}^*, t) dt$  which is what must be added to the cost when varying time.



Define the total space-time variation  $\delta x_f$  by

$$\delta x_f = \delta x(t_f) + \dot{x}(t_f)\delta t_f,$$

i.e. this variation combines the effects of varying the final state by keeping time  $t_f$  fixed, and then by varying  $t_f$  (See figure above).

We have

$$\begin{aligned} \delta J = & \int_{t_0}^{t_f} \left[ g_x(x^*, \dot{x}^*, t) \delta x - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) \delta x \right] dt \\ & + \left( g_{\dot{x}}(x^*, \dot{x}^*, t_f) \delta x_f + [g(x^*, \dot{x}^*, t_f) - g_{\dot{x}}(x^*, \dot{x}^*, t_f) \dot{x}] \delta t_f \right)_{t=t_f}. \end{aligned}$$

**Unrelated  $t_f$  and  $x(t_f)$ .** We next consider the case when  $t_f$  and  $x(t_f)$  are unrelated in which we have

$$g_{\dot{x}}(t_f) = 0, \quad [g(t_f) - g_{\dot{x}}(t_f) \dot{x}] \delta t_f = 0 \quad \Rightarrow \quad g(t_f) = 0$$

**Function  $\Theta(t_f)$ .** We next consider the case when the boundary constraint is given by

$$x(t_f) = \Theta(t_f)$$

Variations  $\delta t_f$  and  $\delta x_f$  are related by

$$\delta x_f = \frac{d\Theta}{dt} \delta t_f$$

and hence the necessary conditions become

$$g_{\dot{x}}(t_f) \left[ \frac{d\Theta}{dt} - \dot{x}^* \right]_{t=t_f} + g(t_f) = 0,$$

which are called *transversality conditions*.

## 4 Differential Constraints

Consider the optimization of

$$J = \int_{t_0}^{t_f} g(x, \dot{x}, t) dt$$

where  $x \in \mathbb{R}^{n+m}$  subject to

$$f(x(t), \dot{x}(t), t) = 0,$$

where  $f = (f_1, \dots, f_n)$  are  $n$  constraints, and  $x(t_f)$  and  $t_f$  are fixed. To obtain the necessary conditions, we define the Lagrangian multipliers  $\lambda : [t_0, t_f] \rightarrow \mathbb{R}^n$  and the augmented cost

$$J_a = \int_{t_0}^{t_f} \{g(x, \dot{x}, t) + \lambda^T f(x, \dot{x}, t)\} dt$$

Taking variations

$$J_a = \int_{t_0}^{t_f} \{[g_x + \lambda^T f_x] \delta x(t) + [g_{\dot{x}} + \lambda^T f_{\dot{x}}] \delta \dot{x} + \delta \lambda^T f\} dt$$

Integrating by parts we get

$$J_a = \int_{t_0}^{t_f} \left\{ [g_x + \lambda^T f_x - \frac{d}{dt} [g_{\dot{x}} + \lambda^T f_{\dot{x}}] \delta x + \delta \lambda^T f \right\} dt.$$

If we define the augmented cost  $g_a$  by

$$g_a = g + \lambda^T f$$

the Euler-Lagrange equations become

$$\frac{\partial g_a}{\partial x} - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{x}} = 0$$

which along with the constraint

$$f(x, \dot{x}, t) = 0$$

constitute the necessary conditions.

## 5 General Boundary Constraints

Let  $x \in \mathbb{R}^n$  and consider the optimization of

$$J(x(\cdot), t_f) = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) dt$$

subject to the free final time  $t_f$  and general boundary conditions

$$\psi(x(t_f), t_f) = 0,$$

where  $\psi$  is a vector of  $m$  functions. To obtain the necessary conditions define the augmented cost

$$J_a = \varphi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) dt,$$

where  $\nu \in \mathbb{R}^m$ . Let

$$w(x(t_f), \nu, t_f) = \varphi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f)$$

Taking variations

$$\delta J_a = w_x(t_f)\delta x_f + w_{t_f}(t_f)\delta t_f + \delta \nu^T \psi(t_f) + \int_{t_0}^{t_f} [g_x \cdot \delta x(t) + g_{\dot{x}} \cdot \delta \dot{x}(t)] dt + g(t_f)\delta t_f,$$

where the last term is due to cost accrued from final time variations. Using integration by parts as well as the total variation definition  $\delta x_f = \delta x(t_f) + \dot{x}(t_f)\delta t_f$  we have

$$\begin{aligned} \int_{t_0}^{t_f} g_{\dot{x}} \cdot \delta \dot{x}(t) dt &= g_{\dot{x}}(t_f)\delta x(t_f) - \int_{t_0}^{t_f} \frac{d}{dt} g_{\dot{x}} \cdot \delta x(t) \\ &= g_{\dot{x}}(t_f)(\delta x_f - \dot{x}\delta t_f) - \int_{t_0}^{t_f} \frac{d}{dt} g_{\dot{x}} \cdot \delta x(t) \end{aligned}$$

which results in

$$\delta J_a = [w_x(t_f) + g_{\dot{x}}]\delta x_{t_f} + [w_{t_f}(t_f) + g(t_f) - g_{\dot{x}}(t_f)\dot{x}(t_f)]\delta t_f + \delta \nu^T \psi(t_f) + \int_{t_0}^{t_f} [g_x - \frac{d}{dt} g_{\dot{x}}]\delta x(t) dt,$$

The necessary conditions require that  $\delta J_a = 0$  for arbitrary  $\delta x(t), \delta \nu$  which is only possible if the following necessary conditions hold:

$$\nabla_x w(x(t_f), \nu, t_f) + \nabla_{\dot{x}} g(x(t_f), \dot{x}(t_f), t_f) = 0, \quad (1)$$

$$\frac{\partial}{\partial t_f} w(x(t_f), \nu, t_f) + g(x(t_f), \dot{x}(t_f), t_f) - \nabla_{\dot{x}} g(x(t_f), \dot{x}(t_f), t_f)^T \dot{x}(t_f) = 0, \quad (2)$$

$$\psi(x(t_f), t_f) = 0 \quad (3)$$

$$\nabla_x g(x(t), \dot{x}(t), t) - \frac{d}{dt} \nabla_{\dot{x}} g(x(t), \dot{x}(t), t) = 0, \quad t \in (t_0, t_f). \quad (4)$$