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dynamics: ① $\dot{\theta} = w - \beta - \eta_v$

② $\dot{\beta} = \eta_u$

angle measurement: $z_k = \theta_k + v_k$

θ: angle

w: gyro rate (control input)

β: random drift term (bias)

η_v: Gaussian noise with σ_v^2 variance

η_u: Gaussian noise with σ_u^2 variance

z: angle measurement

v: Gaussian time-uncorrelated process with σ_n^2 variance

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) Determine the discrete-time dynamic of system: Φ_k, Γ_k, Q_k

(Assume constant Δt and constant input w_k)

State: $x = (\theta, \beta)$, control: $u = (w)$, noise: $w = \begin{bmatrix} \eta_v \\ \eta_u \end{bmatrix}$

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) + L(t)w(t)$$

$$F = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_c(t) = \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}$$

Because we have a sampling rate of Δt during which the control is constant,

$$\Phi_k = e^{\Delta t F} \approx I + \Delta t F(t_{k-1}) = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix}$$

$$\Gamma_{k-1} = \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) G(\tau)\, d\tau = \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix} G(\tau)\, d\tau = \begin{bmatrix} \Delta t & -\frac{\Delta t^2}{2} \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Delta t \\ 0 \end{bmatrix}$$

$$Q_{k-1} = \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) L(\tau) Q_c' L(\tau)^T \Phi^T(t_k, \tau)\, d\tau$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 & -(\tau - t_{k-1}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & -(\tau - t_{k-1}) \\ 0 & 1 \end{bmatrix}^T d\tau$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} -1 & -\tau + t_{k-1} \\ 0 & 1 \end{bmatrix} Q_c \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(\tau - t_{k-1}) & 1 \end{bmatrix} d\tau$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 & -(\tau - t_{k-1}) \\ 0 & 1 \end{bmatrix} Q_c \begin{bmatrix} 1 & 0 \\ -(\tau - t_{k-1}) & 1 \end{bmatrix} d\tau$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} 1 & -(\tau - t_{k-1}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_v & 0 \\ 0 & \sigma_u \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(\tau - t_{k-1}) & 1 \end{bmatrix} d\tau$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} \sigma_v & -(\tau - t_{k-1})\sigma_u \\ 0 & \sigma_u \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(\tau - t_{k-1}) & 1 \end{bmatrix} d\tau$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} \sigma_v + (\tau - t_{k-1})^2 \sigma_u & -(\tau - t_{k-1})\sigma_u \end{bmatrix} \quad \begin{bmatrix} \sigma_v^2 \Delta t + \frac{\Delta t^3}{3}\sigma_u^2 & -\frac{\Delta t^2}{2}\sigma_u^2 \end{bmatrix}$$

2. Nonlinear discrete-time

$$\begin{cases} x_k = f(x_{k-1}, u_{k-1}) + w_{k-1} \\ z_k = h_k(x_k) + v_k. \end{cases} \qquad (h_k(x_k) \text{ is a non-linear function})$$

$w_k \sim N(0, Q_k)$, $v_k \sim N(0, R_k)$. $\Rightarrow$ uncorrelated in time

$x_{k|k-1} \sim N(\hat{x}_{k|k-1}, P_{k|k-1})$

least-square approach

$e_k = z_k - h_k(x_k)\hat{x}_k$ where $\hat{x}_k$ is the optimal estimate.

↳ estimate error

Since we want to minimize the error, the cost function is

$$J = \frac{1}{2} e_z^T e_z = \frac{1}{2}(z_k - h_k(x_k)\hat{x}_k)^T (z_k - h_k(x_k)\hat{x}_k)$$

To incoporate that the bigger standard deviation will have the bigger estimated error, the cost function can be formulated as follow:

$$J(\hat{x}) = \frac{1}{2}(\hat{x} - \hat{x}_0)^T P_0^{-1}(\hat{x} - \hat{x}_0) + \frac{1}{2}[z - h(\hat{x})]^T R^{-1}[z - h(\hat{x})]$$

in discrete model:

$$J(x) = \frac{1}{2}(x - \hat{x}_{k|k-1})^T P_{k|k-1}^{-1}(x - \hat{x}_{k|k-1}) + \frac{1}{2}[z_k - h(x)]^T R_k^{-1}[z_k - h(x)]$$

first-order expansion of nonlinear

$z \approx h(\hat{x}_0) + \partial h(x_{k|k-1})(x - x_{k|k-1}) + v$ (Taylor's approximation first order)

$\Rightarrow dz = H dx + v$ where $H = \partial h(x_{k|k-1})$, $dz = z - h(x)$, $dx = x_k - x_{k|k-1}$

$$\begin{aligned}
J(dx) &= \frac{1}{2} dx^T P_0^{-1} dx + \frac{1}{2}[dz - H dx]^T R^{-1}[dz - H dx] \\
&= \frac{1}{2} dx^T P_0^{-1} dx + \frac{1}{2}[dz^T - dx^T H^T] R^{-1}[dz - H dx] \\
&= \frac{1}{2} dx^T P_0^{-1} dx + \frac{1}{2}[dz^T R^{-1} - dx^T H^T R^{-1}][dz - H dx] \\
&= \frac{1}{2} dx^T P_0^{-1} dx + \frac{1}{2}[dz^T R^{-1} dz - dz^T R^{-1} H dx - dx^T H^T R^{-1} dz + dx^T H^T R^{-1} H dx] \\
&= \frac{1}{2} dx^T P_0^{-1} dx + \frac{1}{2} dz^T R^{-1} dz - \frac{1}{2} dz^T R^{-1} H dx - \frac{1}{2} dx^T H^T R^{-1} dz + \frac{1}{2} dx^T H^T R^{-1} H dx \\
&= \frac{1}{2} dx^T P_0^{-1} dx + \frac{1}{2}[H dx + v]^T R^{-1}[H dx + v] - \frac{1}{2}[H dx + v]^T R^{-1} H dx - \frac{1}{2} dx^T H^T R^{-1}[H dx + v] \\
&\quad + \frac{1}{2} dx^T H^T R^{-1} H dx \\
&= \frac{1}{2} dx^T P_0^{-1} dx + \frac{1}{2}[dx^T H^T R^{-1} + v^T R^{-1}][H dx + v] - \frac{1}{2}[dx^T H^T R^{-1} H dx + v^T R^{-1} H dx] \\
&\quad - \frac{1}{2} dx^T H^T R^{-1} H dx - \frac{1}{2} dx^T H^T R^{-1} v + \frac{1}{2} dx^T H^T R^{-1} H dx \\
&= \frac{1}{2} dx^T P_0^{-1} dx + \frac{1}{2} dx^T H^T R^{-1} H dx + \frac{1}{2} dx^T H^T R^{-1} v + \frac{1}{2} v^T R^{-1} H dx + \frac{1}{2} v^T R^{-1} v - \frac{1}{2} dx^T H^T R^{-1} H dx \\
&\quad - \frac{1}{2} v^T R^{-1} H dx - \frac{1}{2} dx^T H^T R^{-1} H dx - \frac{1}{2} dx^T H^T R^{-1} v + \frac{1}{2} dx^T H^T R^{-1} H dx
\end{aligned}$$

$$\begin{aligned}
\nabla J(dx) &= P_0^{-1} dx - H^T R^{-1} H dx + \frac{1}{2} H^T R^{-1} v + \frac{1}{2} v^T R^{-1} H - H^T R^{-1} H dx - \frac{1}{2} v^T R^{-1} H - \frac{1}{2} v^T R^{-1} H - H^T R^{-1} H dx \\
&\quad - \frac{1}{2} H^T R^{-1} v + H^T R^{-1} H dx \\
&= P_0^{-1} dx - 2 H^T R^{-1} H dx - \frac{1}{2} v^T R^{-1} H = 0 \quad (\text{necessary condition})
\end{aligned}$$

$(P_0^{-1} - 2H^T R^{-1} H) dx = \frac{1}{2} v^T R^{-1} H \quad \Rightarrow \quad dx = \frac{1}{2}(P_0^{-1} - 2H^T R^{-1} H)^{-1} v^T R^{-1} H$

$\nabla J(dx) = \frac{1}{2} dz^T P_0^{-1} dx + \frac{1}{2} dz^T R^{-1} dz - \frac{1}{2} dz^T R^{-1} H dx - \frac{1}{2} dx^T H^T R^{-1} dz + \frac{1}{2} dx^T H^T R^{-1} H dx$

$\nabla J(dx) = P_0^{-1} dx - \frac{1}{2} H^T R^{-1} dz - \frac{1}{2} H^T R^{-1} dz + H^T R^{-1} H dx$

$= [P_0^{-1} + H^T R^{-1} H] dx - H^T R^{-1} dz$

$= 0$ (necessary condition)

$[P_0^{-1} + H^T R^{-1} H] dx = H^T R^{-1} dz$

$dx = [P_0^{-1} + H^T R^{-1} H]^{-1} H^T R^{-1} dz$

$x - \hat{x}_{k|k-1} = [P_{k|k-1}^{-1} + H_k^T R_k^{-1} H_k]^{-1} H_k^T R_k^{-1} [z_k - h(\hat{x}_{k|k-1})]$

$$\begin{cases} x = \hat{x}_{k|k-1} + K_k [z_k - h(\hat{x}_{k|k-1})] \\ K_k = [P_{k|k-1}^{-1} + H_k^T R_k^{-1} H_k]^{-1} H_k^T R_k^{-1} \end{cases}$$

3/4

Show the equivalence between this and the form given in homework

1b) Wrong true state update provided 3/4

3b) Is bearing sensor enough to reconstruct the state.
The results for 3b are not correct. Check 2/4
3a) 4/4