EN530.603 Applied Optimal Control
Midterm #1
October 16, 2013

1. Find the optimum of the function

$$J(x_1, x_2) = x_1^2 + 4x_2^2 + x_1x_2 + x_2,$$

subject to

$$f(x_1, x_2) = x_1 - 3x_2 - 10 \geq 0.$$

Solution: Let

$$H(x, \lambda) = x_1^2 + 4x_2^2 + x_1x_2 + x_2 - \lambda(x_1 - 3x_2 - 10),$$

where $\lambda \geq 0$ when $f(x) = 0$ and $\lambda = 0$ when $f(x) < 0$. The necessary condition is

$$\nabla_x H = \begin{bmatrix} 2x_1 + x_2 - \lambda \\ 8x_2 + x_1 + 1 + 3\lambda \end{bmatrix} = 0,$$

along with $f(x) \leq 0$. We have that

$$\lambda = 2x_1 + x_2,$$

and hence

$$-7x_1 - 11x_2 = 1$$

Setting $\lambda = 0$ results in $x_1 = \frac{1}{15}$, $x_2 = -\frac{2}{15}$ which violates the constraint. Thus, we need to solve

$$-7x_1 - 11x_2 = 1, \text{ and } \; x_1 - 3x_2 - 10 = 0,$$

which results in

$$x_1 = \frac{107}{32}, \quad x_2 = -\frac{71}{32}.$$

Note: since the constraint is linear we could have also solved for one of the variables directly without resorting to Lagrange multipliers.

2. Consider a linear system with state $x \in \mathbb{R}^n$, controls $u \in \mathbb{R}^m$, dynamics

$$\dot{x} = Ax + Bu$$

and cost function

$$J = \frac{1}{2}x(t_f)^T P_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[x(t)^T Q x(t) + 2x(t)^T N u(t) + u(t)^T R u(t)\right] dt,$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite, $R \in \mathbb{R}^{m\times m}$ is symmetric positive definite, and $N \in \mathbb{R}^{n\times m}$. The final state $x(t_f)$ is free and the final time t_f is fixed.

Find the optimal controller in a feedback form, i.e. in the form $u(t) = -K(t)x(t)$.

In the process, show that there is a matrix $P(t)$ such that choosing the multipliers according to $\lambda(t) = P(t)x(t)$ satisfies the optimality conditions. Derive the *generalized* Riccati ODE for $P(t)$ and specify any required boundary conditions. The matrix $P(t)$ can then be used to determine $K(t)$.

Solution:

Let

$$H = \frac{1}{2}\left[x(t)^T Q x(t) + 2x(t)^T N u(t) + u(t)^T R u(t)\right] + \lambda^T(Ax + Bu).$$

The first-order necessary conditions require that

$$\dot{\lambda} = -\nabla_x H = -Qx - Nu - A^T\lambda, \tag{1}$$

and

$$0 = \nabla_u H = N^T x + Ru + B^T\lambda \quad \Rightarrow \quad u = -R^{-1}(N^T x + B^T\lambda). \tag{2}$$

Assume that

$$\lambda(t) = P(t)x(t).$$

From (2) we have

$$u = -R^{-1}(N^T x + B^T P x) = -R^{-1}(N^T + B^T P)x =: -Kx, \tag{3}$$

where $K = R^{-1}(N^T + B^T P)$ while (1) can be equivalently expressed as

$$\dot{\lambda} = -Qx + NKx - A^T P x = (-Q + NK - A^T P)x, \tag{4}$$

Differentiating $\lambda = Px$ we have

$$\dot{\lambda} = \dot{P}x + P\dot{x},$$

which, after substituting (4) and (3) becomes

$$(-Q + NK - A^T P)x = \dot{P}x + P(A - BK)x,$$

which will hold for all x if we choose $\dot{P}(t)$ so that

$$-\dot{P} = PA + A^T P + Q - (PB + N)K$$

with boundary condition

$$P(t_f) = P_f,$$

in order to satisfy the tranversality condition $\lambda(t_f) = \nabla_x(\frac{1}{2}x(t_f)^T P_f x(t_f))$.

3. Consider the triple integrator system with state $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and dynamics

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u,$$

where u is the control input constrained by

$$|u| \leq 1.$$

Describe the procedure for computing the minimum time control law taking the system from a starting state $x(t_0) = x_0$ to a final fixed state $x(t_f) = 0$. The cost function is $J = \int_{t_0}^{t_f} 1 dt$. In the process, determine whether there are any singular intervals and find the number of switching times that are expected to occur along the optimal solution.

Note: the solution might involve finding the roots of static nonlinear equations. You are only required to specify what these equations are and not to solve them analytically.

Solution: Define

$$H = 1 + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u.$$

The necessary conditions result in

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1, \quad \dot{\lambda}_3 = -\lambda_2,$$

which are equivalent to

$$\lambda_1 = c_1, \quad \lambda_2 = -c_1 t + c_2, \quad \lambda_3 = \frac{c_1}{2} t^2 - c_2 t + c_3.$$

Since $H_u = 0$ cannot be used directly to solve for u we employ the maximum principle to find u which minimizes $\lambda_3 u$, i.e.

$$\begin{aligned} \lambda_3 < 0 \quad &\Rightarrow \quad u = 1 \\ \lambda_3 = 0 \quad &\Rightarrow \quad u \text{ undetermined} \\ \lambda_3 > 0 \quad &\Rightarrow \quad u = -1. \end{aligned}$$

Since λ_3 is quadratic it can be zero maximum two times. Furthermore, we have that

$$0 = H(t_f) = 1 + \lambda_3(t_f) u(t_f),$$

at the required final state. Since $u(t_f) = \pm 1$ then $\lambda_3(t_f) = \mp 1$ which means that singular intervals do not exist.

Thus possible control sequences are $u = \{-1, 1, -1\}$ and $u = \{1, -1, 1\}$ at some switching times t_1 and t_2. We can find the unknown times t_1, t_2, t_f by integrating the dynamics using each sequence and solving the three nonlinear equations $x(t_f) = 0$.