EN530.603 Applied Optimal Control
Homework #5

November 12, 2014

Due: November 26, 2013 (hand-in in Hackerman 117 or email scanned copy)

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1. **Linear Estimation (attitude filtering):** Consider the single-axis attitude estimation problem using angle measurements and angular rate information from a gyroscope. The gyro rate, denoted by ω typically drifts over time and must be compensated by subtracting the random drift term called the *bias* and denoted by β. The state of the system that must be estimated is (θ, β) where θ is the angle. The dynamics is

$$\dot{\theta} = \omega - \beta - \eta_v, \tag{1}$$

$$\dot{\beta} = \eta_u, \tag{2}$$

where η_v and η_u are uncorrelated Gaussian noise processes with variances σ_v^2 and σ_u^2, respectively. Note that here ω is regarded as the known control input to the system. The angle measurements z at each time t_k are defined by

$$z_k = \theta_k + v_k,$$

where v_k is a Gaussian time-uncorrelated process with variance given by $R = \sigma_n^2$.

   (a) Determine the discrete-time dynamics of system, i.e. the matrices Φ_k and Γ_k, and Q_k assuming a constant sampling interval Δt during which there is a constant input ω_k

   (b) Implement a Kalman filter using the following setup: assume that the true angular rate of the system is $\dot{\theta} = 0.02$ rad/sec and $\Delta t = 1$ sec. The noise is $\sigma_n = 1.5 \times 10^{-5}$, $\sigma_u = 3 \times 10^{-9}$, and $\sigma_v = 3 \times 10^{-6}$. The initial angle is $\theta_0 = 0$, the initial bias is $\beta_0 = 1.7 \times 10^{-7}$ and the initial covariance is $P_0 = \text{diag}([1 \times 10^{-4},\ 1 \times 10^{-12}])$. Measurements must be generated by evolving the true angle and adding appropriate noise terms. Show the resulting true angle, measured angle, and the estimated angle as well as 95% confidence intervals. You are free to use/modify `int_test.m` which implements a double-integrator system and can serve as a template for your code.

   Note: the simulated control input ω must be consistent with the true anglular velocity. This is accomplished by first generating a bias path $\beta_0, \beta_1, \ldots, \beta_N$ using the discrete version of $\dot{\beta} = \eta_u$ (which will involve sampling η_u). This discrete bias path $\beta_{0:N}$ will be regarded as the *true bias*. The control input is then generated according to

$$\omega_k = 0.02 + \beta_k + \eta_v,$$

   by sampling η_v, i.e. by setting $\eta_v = \sigma_v \texttt{randn}(1)$.

2. Consider the nonlinear discrete-time model with additive noise given by

$$x_k = f(x_{k-1}, u_{k-1}) + w_{k-1}, \tag{3}$$
$$z_k = h_k(x_k) + v_k. \tag{4}$$

where the noise terms $w_k \sim \mathcal{N}(0, Q_k)$ and $v_k \sim \mathcal{N}(0, R_k)$ satisfy the standard assumptions of being uncorrelated in time. Assume that the distribution at time k before measurement processing is known and denoted by

$$x_{k|k-1} \sim \mathcal{N}(\hat{x}_{k|k-1}, P_{k|k-1})$$

Following the least-squares approach show that the minimizer of the cost function

$$J(x) = \frac{1}{2}(x - \hat{x}_{k|k-1})^T P_{k|k-1}^{-1}(x - \hat{x}_{k|k-1}) + \frac{1}{2}[z_k - h(x)]^T R_k^{-1}[z_k - h(x)],$$

after linearizing the function $h(x)$ around $\hat{x}_{k|k-1}$ corresponds to the EKF correction:

Correction:

$$x = \hat{x}_{k|k-1} + K_k[z_k - h(\hat{x}_{k|k-1})],$$
$$K_k = P_{k|k-1}H_k^T(H_k P_{k|k-1} H_k^T + R_k)^{-1}$$

Note: the subscript $_{k|k-1}$ can be dropped for conciseness in derivation.

3. **Nonlinear Estimation (simple car model with beacon measurements):** Consider a wheeled robot with state (x, y, θ) where $p = (x, y)$ is the position and θ is its orientation. The robot is controlled using forward and angular velocities denoted by the inputs $u_k = (v_k, \omega_k)$. The dynamics is given by

$$\begin{pmatrix} x_k \\ y_k \\ \theta_k \end{pmatrix} = \begin{pmatrix} x_{k-1} + \Delta t \cos(\theta_{k-1}) v_{k-1} \\ y_{k-1} + \Delta t \sin(\theta_{k-1}) v_{k-1} \\ \theta_{k-1} + \Delta t \omega_{k-1} \end{pmatrix} + w_{k-1}, \tag{5}$$

where $w_{k-1} \sim \mathcal{N}(0, Q_{k-1})$ is the process noise. The robot observes beacons in the environment at fixed positions $p' = (x', y') \in \mathbb{R}^2$ through measurements $z = (\phi, r)$, where ϕ is the relative angle between the robot and the beacon and r is the range to the beacon. The sensor model is

$$z_k = h([x_k, y_k, \theta_k]) + v_k,$$

where

$$h([x, y, \theta]) = \begin{pmatrix} \arctan \frac{y'-y}{x'-x} - \theta \\ \|p' - p\| \end{pmatrix}$$

for a single beacon (and can be augmented accordingly for more beacons). Here $v_k \sim \mathcal{N}(0, R_k)$ is the sensor noise. The EKF can be implemented using the Jacobians

$$F = \begin{bmatrix} 1 & 0 & -\Delta t v \sin\theta \\ 0 & 1 & \Delta t v \cos\theta \\ 0 & 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} \frac{y'-y}{\|p'-p\|^2} & -\frac{x'-x}{\|p'-p\|^2} & -1 \\ \frac{x-x'}{\|p'-p\|} & \frac{y-y'}{\|p'-p\|} & 0 \end{bmatrix}.$$

The Jacobian H can be extended accordingly for multiple beacons. An EKF implementation of this model is provided in file `uni_ekf_test.m`

(a) The goal is to extend this model to the case when the robot wheel radius is not perfectly known. This situtation occurs in practice e.g. when the vehicle is under different loads or speeds, or simply when the radius is not known a priori. The velocity v is expressed as

$$v(t) = r(t)\Omega(t),$$

where $r(t)$ is the wheel radius in meters and $\Omega(t)$ is the commanded wheel speed in rad/s. The changing parameter $r(t)$ can be typically modeled as

$$\dot{r} = \eta_r,$$

where η_r is a Gaussian process. In discrete-time, the complete model is now

$$\begin{pmatrix} x_k \\ y_k \\ \theta_k \\ r_k \end{pmatrix} = \begin{pmatrix} x_{k-1} + \Delta t \cos(\theta_{k-1}) r_{k-1}\Omega_{k-1} \\ y_{k-1} + \Delta t \sin(\theta_{k-1}) r_{k-1}\Omega_{k-1} \\ \theta_{k-1} + \Delta t \omega_{k-1} \\ r_{k-1} \end{pmatrix} + w_{k-1}, \tag{6}$$

where the inputs now are $u = (\Omega, \omega)$ and w_{k-1} is now a four-dimensional noise term.

Your task is to implement an EKF for this model. Assume that the true wheel radius is $r = 1$ and that the true initial state is $(0, 0, 0, 1)$ while the covariance is $P_0 = \text{diag}([.01, .01, .01, .04])$. The initial state is a Gaussian perturbation of the the true initial state (as shown in provided template). The dynamics noise covariance is assumed to be

$$Q_k = \Delta t^2 \text{diag}([.01, .01, .01, .0001]),$$

i.e. you do not have to derive it. You do need to derive the Jacobian F of the new system. Extend `uni_test.m` to account for this scenario. Modify the error plots to display all four errors and demonstrate that the estimated $\hat{r}_k$ stabilizes around its true value.

(b) Consider the case when the sensor only provides bearing measurements, i.e. when

$$h([x, y, \theta]) = \left(\arctan \frac{y' - y}{x' - x} - \theta \right)$$

Implement this sensor model in a file called `uni_ekf_test2.m` and test the filter performance compared to the one in a). Based on these experiments, is a bearing-only sensor enough to reconstruct the full state?

4. *Optional: for extra credit.* **Static shape estimation.** Consider the estimation of a static shape defined as a noisy quadratic function z over the planar coordinates $(p, q) \in \mathbb{R}^2$ and parametrized using coefficients $x = (x_1, \ldots, x_6)$:

$$z = x_1 p^2 + x_2 q^2 + x_3 pq + x_4 p + x_5 q + x_6 + v \equiv Hx + v,$$

where $v \sim \mathcal{N}(0, r)$ is noise with variance r. You are provided a matlab file `shape_fit.m` which implements static *batch* estimation by combining $k = 8$ measurements $z_1, \ldots, z_k$ into one multidimensional measurement $z = (z_1, \ldots, z_k)$ and applying the linear weighted least-squares approach. Your task is to extend this formulation by:

- considering a given prior on the parameters $x \sim \mathcal{N}(m_0, P_0)$ where e.g. $m = (1.2, 1.3, 1, 1, 1, 1)$ and $P_0 = diag(16, 16, 16, 16, 16, 16)$, and
- implementing iterative weighted least-squares estimation by performing 4 iterations with 2 measurements each (you can use the same 8 measurements as in the provided code)
- compare and discuss the resulting quality of fit compared to batch estimation; change the prior and comment on its effect

Note: email your all of your code to marin@jhu.edu with a subject line starting with: **EN530.603.F2014.HW5** in addition attach a printout of the code and plots to your homework solutions.