EN530.603 Applied Optimal Control
Homework #1

September 10, 2014

Due: September 24, 2014 (before class)

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1. Find the stationary points (i.e. that satisfy $\nabla L = 0$) of the following and determine whether they are maxima, minima, or saddle points:

   (a) $L(x) = (1 - x_1)^2 + 200(x_2 - x_1^2)^2$

   (b) $L(u) = (u - 1)(u + 2)(u - 3)$

   (c) $L(u) = (u_1^2 + 3u_1 - 4)(u_2^2 - u_2 + 3)$

2. Find the stationary points of the following and determine whether they are maxima, minima, or saddle points:

   (a)

$$\text{minimize } L(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \tag{1}$$

$$\text{subject to } f(x) = x_1 + x_2 + x_3 = 0 \tag{2}$$

   (b)

$$\text{minimize } L(u) = (u_1^2 + 3u_1 - 4)(u_2^2 - u_2 + 3) \tag{3}$$

$$\text{subject to } f(u) = u_1 - 2u_2 = 0 \tag{4}$$

3. (a) Consider the optimization of a quadratic cost subject to linear constraints, i.e. minimize

$$L(x, u) = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u,$$

   subject to

$$f(x, u) = Ax + Bu + c = 0,$$

   where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; $Q \geq 0$ (positive semidefinite matrix) and $R > 0$ (positive definite matrix); $A \in \mathbb{R}^{n \times n}$ and $B^{n \times m}$ and $c \in \mathbb{R}^n$.

   - Derive the necessary and sufficient conditions for an optimal solution using the Lagrangian multiplier approach. Be careful which matrices you are allowed to invert.
   - Assume that A is full rank and compute the actual optimal solution.

(b) Consider the optimization of a quadratic cost subject to linear constraints, i.e. minimize

$$L(y) = \frac{1}{2} y^T M y$$

subject to

$$f(y) = Ay + c = 0,$$

where $y \in \mathbb{R}^n$, $M > 0$ is positive definite, $A \in \mathbb{R}^{m \times n}$ for $m < n$ is full rank, and $c \in \mathbb{R}^m$. Compute the optimal solution y^* and show that it is a global minimum.

4. Implementation: you are free to use parts of the code provided at the course homepage.

   (a) Write a MATLAB function which implements gradient descent to optimize the cost-function in problem 1a given by

   $$L(x) = (1 - x_1)^2 + 200(x_2 - x_1^2)^2.$$

   You can use either a constant or a variable stepsize. What is the effect of the step-size choice? Use a starting point at $x = (0, 0)$.

   (b) Find analytically the optimum of the following problem:

   $$\text{minimize } L(x,u) = x^2 + 20u^2 \tag{5}$$
   $$\text{subject to } f(x,u) = x - 2u + 3 \leq 0 \tag{6}$$

   and use MATLAB fmincon function to verify your solution.

   Note: email your code to marin@jhu.edu with subject line starting with: EN530.603.F2014.HW1; in addition attach a printout of the code to your homework solutions.

5. Consider the minimization of $f(x)$ for $x \in \mathbb{R}^n$. Newton's method is derived by finding the direction $d^k \in \mathbb{R}^n$ which minimizes the local quadratic approximation f^k of f at x^k defined by

   $$f^k(d) = f(x^k) + \nabla f(x^k)^T d + \frac{1}{2} d^T \nabla^2 f(x^k) d.$$

   In contrast, the search direction d^k in a *trust-region* Newton method is derived by solving the *constrained* optimization

   $$\text{minimize} \quad f^k(d) \quad \text{subject to} \quad \|d\| \leq \gamma^k,$$

   for a given $\gamma^k > 0$ called the trust-region radius. Using the Lagrangian multiplier approach prove that this optimization is equivalent to solving

   $$(\nabla^2 f(x^k) + \delta^k I) d^k = -\nabla f(x^k),$$

   where $\delta^k \geq 0$. How do you interpret the value δ^k. Can you propose a reasonable choice for δ^k considering the properties of $\nabla^2 f(x^k)$.

6. Read one (or both) of the following two historical perspectives on optimal control and write a one-paragraph summary of the paper you choose:

   (a) Arthur Bryson "Optimal Control– 1950 to 1985", IEEE Control Systems, 1996

   (b) Sussman and Willems, "300 Years of Optimal Control: From the Brachystochrone to the Maximum Principle", IEEE Control Systems, 1997