EN.530.603 Applied Optimal Control
HW #3 Solutions

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1. Given system dynamics:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -ax_2(t) + u(t)$$
$$a > 0 \quad |u(t)| \leq 1 \quad x(t_f) = 0$$

To minimize:

$$J = \int_{t_0}^{t_f} \gamma + |u(t)| dt \quad \gamma > 0$$

(a) The Hamiltonian and adjoint equations are as follows:

$$H = \gamma + |u| + \lambda_1(x_2) + \lambda_2(-ax_2 + u)$$
$$\dot{\lambda} = -\nabla H_x$$
$$\Rightarrow \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 + a\lambda_2 \end{bmatrix}$$
$$\nabla H_u = \pm 1 + \lambda_2 \quad \text{(Does not provide u)}$$
$$H_u^* = |u| + \lambda_2 u \quad \text{(Choose u to minimize this)}$$
$$\Rightarrow u = \begin{cases} -1 & \text{if } \lambda_2 > 1 \\ 0 & \text{if } 1 > \lambda_2 > -1 \\ 1 & \text{if } -1 > \lambda_2 \end{cases}$$

(b) From the dynamics of the lagrange multipliers:

$$\lambda_1 = c_1$$
$$\lambda_2 = c_2 e^{at} + \frac{c_1}{a}$$

From the Boundary conditions, since the final time is free,

$$H(t_f) = 0 \quad \Rightarrow \gamma + |u_f| + \lambda_{2f} u_f = 0$$
$$\Rightarrow \lambda_2(t_f) = \begin{cases} -(1+\gamma) & \text{if } u_f = 1 \\ (1+\gamma) & \text{if } u_f = -1 \end{cases}$$

u_f cannot be equal to zero in the above equation. Given that $\lambda_2(t)$ is exponential, and the final value is either greater than 1 or less than -1, there can be atmost **two switches**. So the optimal control sequences are: {-1,0,1} {0,1} {1} {1,0,-1} {0,-1} {-1}

(c) The singular points where the control becomes undefined is given when $\lambda_2 = \{-1, 1\}$. Given the form of λ_2 as:

$$\lambda_2 = c_2 e^{at} + \frac{c_1}{a}$$

λ_2 is continuously increasing or decreasing unless $c_2 = 0$ in which case the boundary conditions ensure that:

$$\lambda_2 = \lambda_2(t_f) = \pm(1+\gamma)$$

which is not any of the singular values. Thus the value of λ_2 cannot stay at the singular values for any time interval. Thus there are no **singular intervals**.

(d) To determine the optimal control law we look at the curves with no switching:

i. u = -1 always. Then

$$\begin{aligned} \dot{x}_2(t) &= -ax_2 - 1 \quad x_{2f} = 0 \\ x_2(t) &= (1/a)[e^{-a(t-t_f)} - 1] \\ \dot{x}_1(t) &= x_2 \quad x_{1f} = 0 \\ x_1(t) &= (1/a)[(1/a)(1 - e^{-a(t-t_f)}) - (t - t_f))] \\ \Rightarrow x_1(t) &= \frac{1}{a^2}[-ax_2 + log(ax_2 + 1)] \quad x_2 > (-1/a) \end{aligned}$$

ii. u = 1 always. Then similar to above,

$$x_1(t) = \frac{-1}{a^2}[ax_2 + log(1 - ax_2)] \quad x_2 < (1/a)$$

iii. u = 0. Then

$$\dot{x}_2 = -ax_2 \quad \dot{x}_1 = x_2$$
$$\Rightarrow x_1 = -(x_2 - x_{20})/a + x_{10}$$

This is just a straight line with slope (-1/a).

These combine to provide the switching curves shown in Fig[1]. The general family of curves for u = {1,-1, 0} are plotted in Fig[2]. Based on the family of curves we come up with possible set of subsequences for the optimal control law presented below. Some example scenarios are presented in Figures[3][4]

$$u = \begin{cases} \{-1,0,1\} \text{ or } \{0,1\} & \text{if } x_{10} < \frac{1}{a^2}[-ax_{20} + log(1+ax_{20})] \ \ \& \ \ x_20 \geq 0 \\ \{-1,0,1\} & \text{if } x_{10} < \frac{-1}{a^2}[ax_{20} + log(1-ax_{20})] \ \ \& \ \ x_{20} \leq 0 \\ \{1,0,-1\} \text{ or } \{0,-1\} & \text{if } x_{10} > \frac{-1}{a^2}[ax_{20} + log(1-ax_{20})] \ \ \& \ \ x_{20} \leq 0 \\ \{1,0,-1\} & \text{if } x_{10} > \frac{1}{a^2}[-ax_{20} + log(1+ax_{20})] \ \ \& \ \ x_{20} \geq 0 \end{cases}$$

The actual sequence can be found by solving for $\lambda(t)$ and t_f (3 unknowns) by integrating the dynamics forward for possible optimal control sequences and ensuring that the lagrange multipliers are continous at switching points and satisfy the boundary constraints ($\lambda(t_f) = \pm(1+\gamma), x(t_f) = 0$)

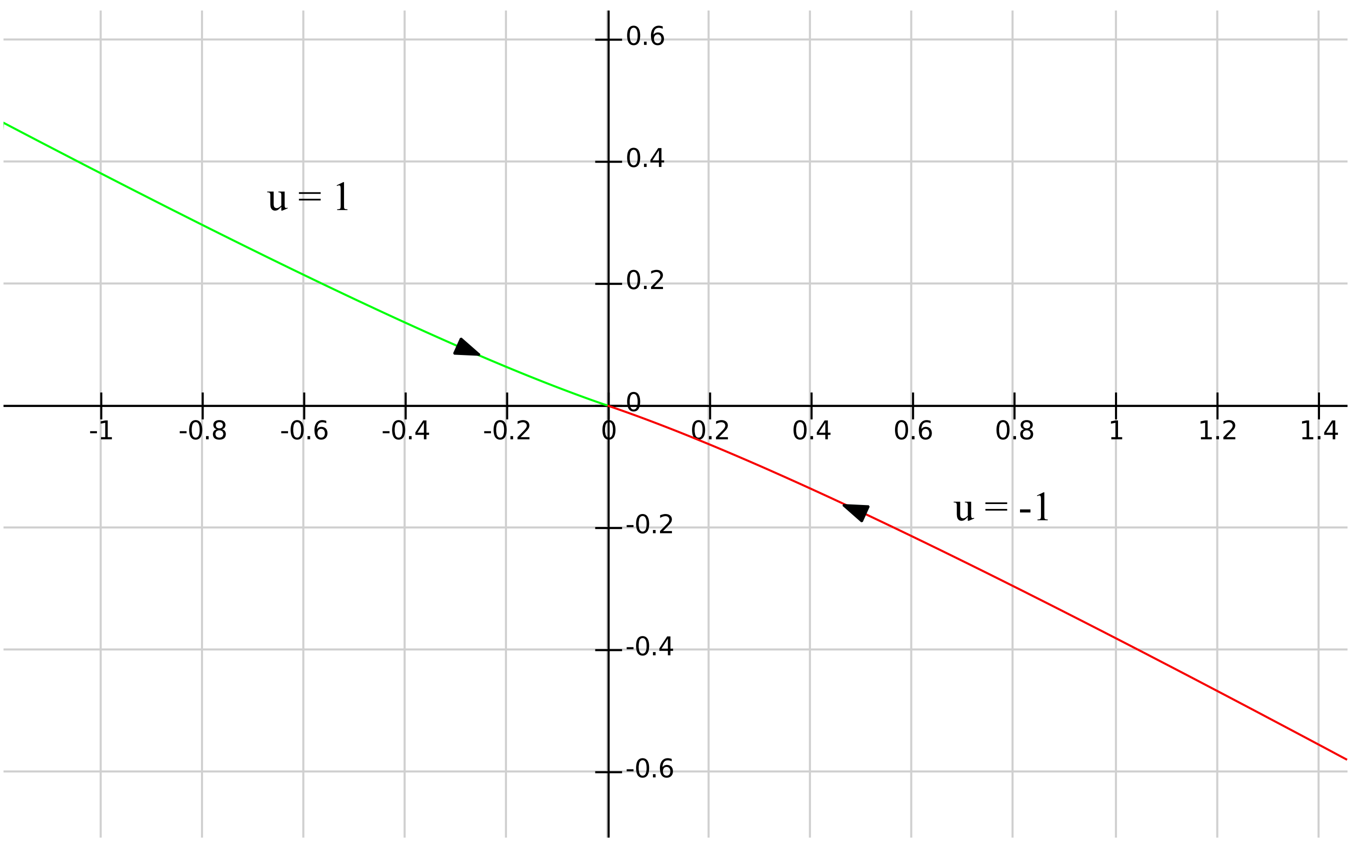

Figure 1: The switching curves $x_1(t)$ vs $x_2(t)$ passing through origin

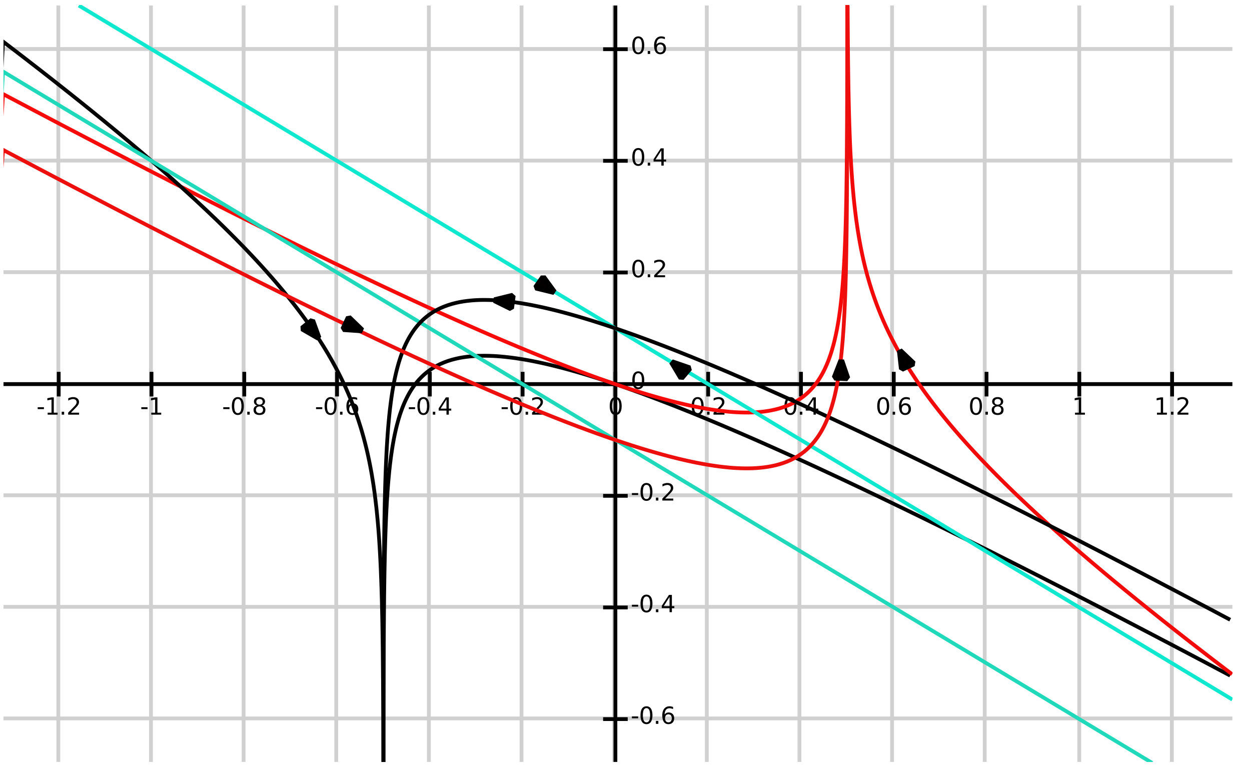

Figure 2: Plot of combined Family of curves (a) Red Lines show the family of u = -1 curves (b) Green Lines show the family of u = 0 curves (b) Black Lines show the family of u = 1 curves (a = 2 for above curves)

2. Given to minimize:

$$J = \|x(t_f)\|^2$$

subjected to the dynamics of the system:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t_f \text{ given}$$

Equating related parts of the problem to the general optimization setting, we have:

$$\Phi = x(t_f)^T x(t_f) \quad H = \lambda^T (Ax + Bu)$$

The necessary optimality conditions required are :

$$\begin{aligned} \dot{\lambda} &= -\nabla H_x = -A^T \lambda \\ \partial H_u &= \lambda^T B \\ \lambda(t_f) &= \nabla_x \Phi(t_f) = 2x(t_f) \end{aligned}$$

Since the optimal control cannot be obtained by ∂H_u we use poincare conditions to find the

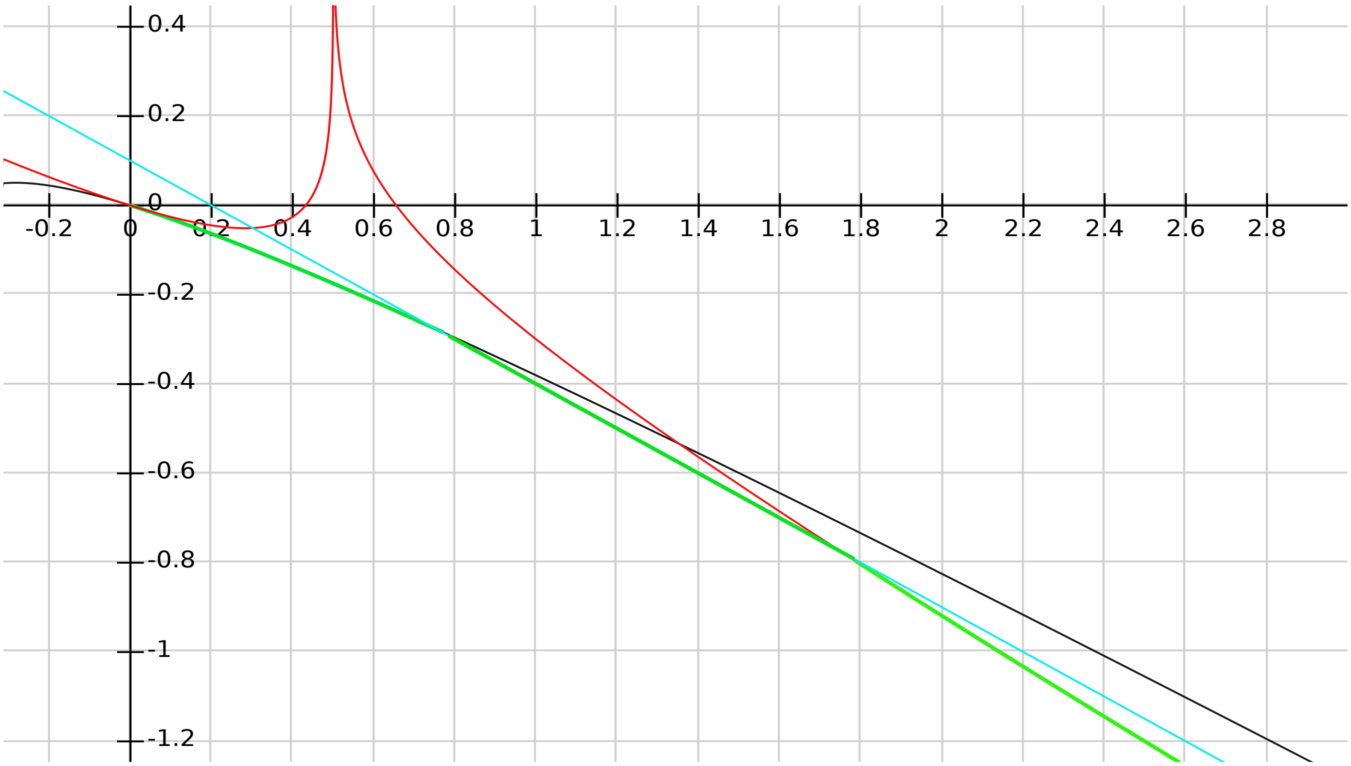

Figure 3: Example Scenario 1 optimal control sequence(-1,0,1)

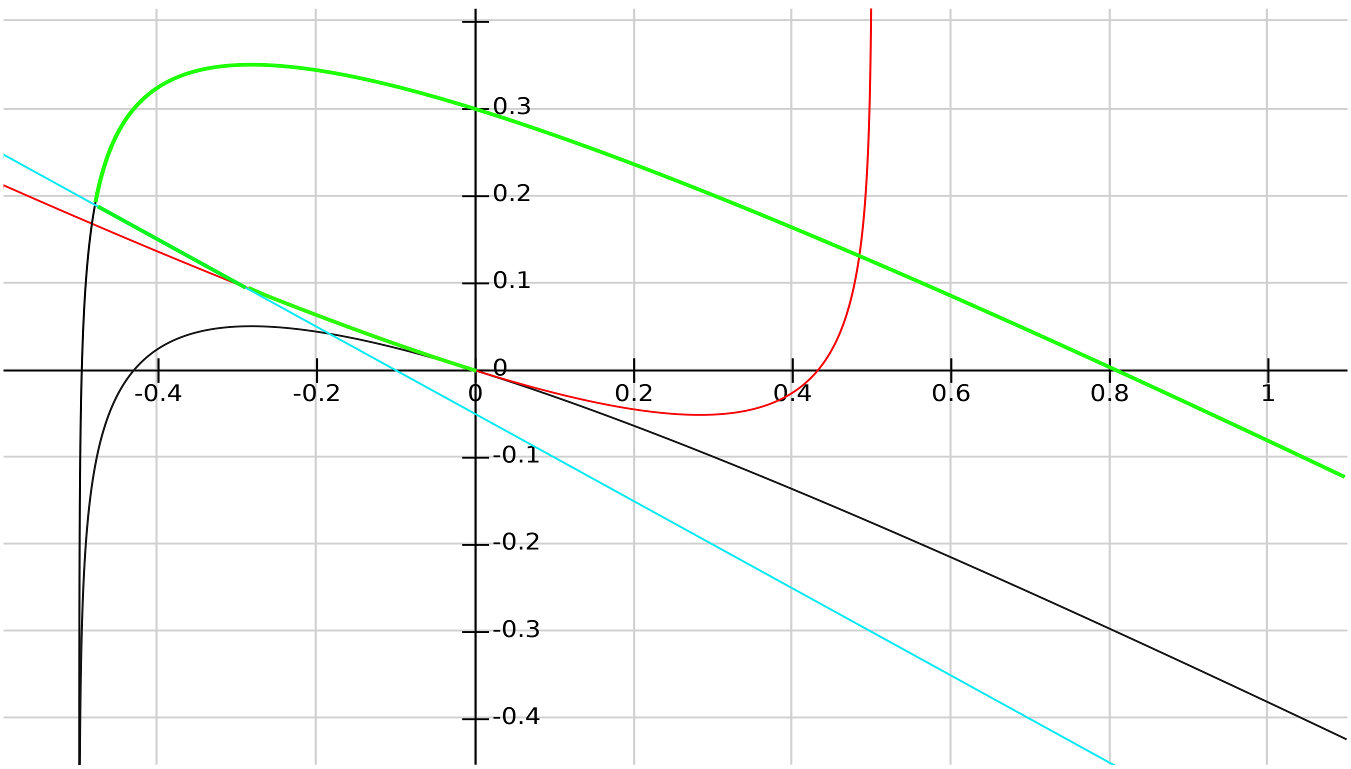

Figure 4: Example Scenario 2 optimal control sequence(1,0,-1)

control on the boundaries:

$$u^* = argminH^*(u) = \lambda^T Bu$$
$$\Rightarrow u^* = \begin{cases} -1 & \lambda^T B > 0 \\ 1 & \lambda^T B < 0 \\ \text{singular} & \lambda^T B = 0 \end{cases}$$

This shows that the control for optimization is bang bang i.e the control either stays at +1 or -1 if $\lambda^T B \neq 0$.

For the double integrator the dynamics of the system are given as follows:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The control is then given as :

$$\Rightarrow u^* = \begin{cases} -1 & \lambda_2 > 0 \\ 1 & \lambda_2 < 0 \\ \text{singular} & \lambda_2 = 0 \end{cases}$$

Now to find the general trajectories for this case, we look at the system and multiplier dynamics.

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \\ \lambda(t_f) &= 2x(t_f) \\ \Rightarrow \lambda_1 &= \text{ const} \\ \Rightarrow \lambda_2 &= -\lambda_1 t + c_1 \end{aligned}$$

The linearity of λ_2 implies that we can either switch once or not switch at all. But when λ_2 goes to zero we cannot decide on the control since it is singular. Thus the possible optimal control sequences are $\{1\}, \{-1\}, \{1, -1\}, \{-1, 1\}$.

We explain the optimal control law based on one example scenario and then provide the general optimal control law after that. For example consider our starting condition to be $x_0 < 0$(Third quadrant). Then the possible control sequences are $\{1\}$ or $\{1, -1\}$. The other sequences take you away from origin and are clearly not optimal. In both the control sequences, we start with control u = 1. To choose the right sequence, we use the boundary conditions $\lambda(t_f) = x(t_f)$. If t_f is less than the minimum time required to cross to fourth quadrant, then $x_2 f < 0$. This implies that $\lambda_2 f < 0$ and control $u_f = 1$. Thus the optimal sequence in this case is $u = 1$ throughout. In the case when t_f is sufficient to cross over to fourth quadrant, $x_2 f > 0$ and

$\lambda_2 f > 0$. This implies $u_f = -1$ and the optimal control sequence is $\{1, -1\}$. In this way similar analysis can be done for other quadrants.

If the control is always +1 then initial position should lie on:

$$x_1(t_0) = x_2(t_0)^2 \quad x_2(t_0) < 0$$
$$\text{OR } x_1(t_0) < 0, \;\; x_2(t_0) < 0 \quad \text{(Third Quadrant with small } t_f\text{)}$$

else if the control is always -1 then initial position should lie on:

$$x_1(t0) = -x_2(t_0)^2 \quad x_2(t_0) > 0$$
$$\text{OR } x_1(t_0) > 0, \;\; x_2(t_0) > 0 \quad \text{(First Quadrant with small } t_f\text{)}$$

else if the control sequence is {-1,1} then the initial position should lie in:

$$x_1(t_0) > x_2(t_0)^2 \;\; x_2(t_0) < 0$$
$$\text{OR } x_1(t_0) > 0, \;\; x_2(t_0) > 0 \quad \text{(First Quadrant with large } t_f\text{)}$$

else if the control sequence is {1,-1} then the initial position should lie in:

$$x_1(t_0) < -x_2(t_0)^2 \;\; x_2(t_0) > 0$$
$$\text{OR } x_1(t_0) < 0, \;\; x_2(t_0) < 0 \quad \text{(Third Quadrant with large } t_f\text{)}$$

(Here large t_f is large enough to cross from one quadrant to another). It should be noted that, when t_f is long enough to reach origin, we reach the least possible cost (0). So for every initial starting position, there is a t_f^* above which the control law does not change.

The switching curves which go to zero are shown in Fig[5] and the optimal trajectories for some of the sequences discussed above are shown in Fig[6]

3. Given a robotic arm with two degrees of freedom (θ_1, θ_2). The state of the arm is given as:

$$x = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

The forward kinematics of the RR manipulator is given as:

$$p_t = \begin{pmatrix} cos(\theta_1)l_1 + cos(\theta_1 + \theta_2)l_2 \\ sin(\theta_1)l_1 + sin(\theta_1 + \theta_2)l_2 \end{pmatrix}$$

The state inequality for obstacle avoidance is given as:

$$\|p_t - p_0\|^2 \geq r^2$$
$$c(x(t), t) = r^2 - (p_t - p_0)^T (p_t - p_0) \leq 0$$

Since the above state inequality does not directly say anything about the control, we differentiate c(x(t),t) until controls show up. Thus the new inequality are obtained as:

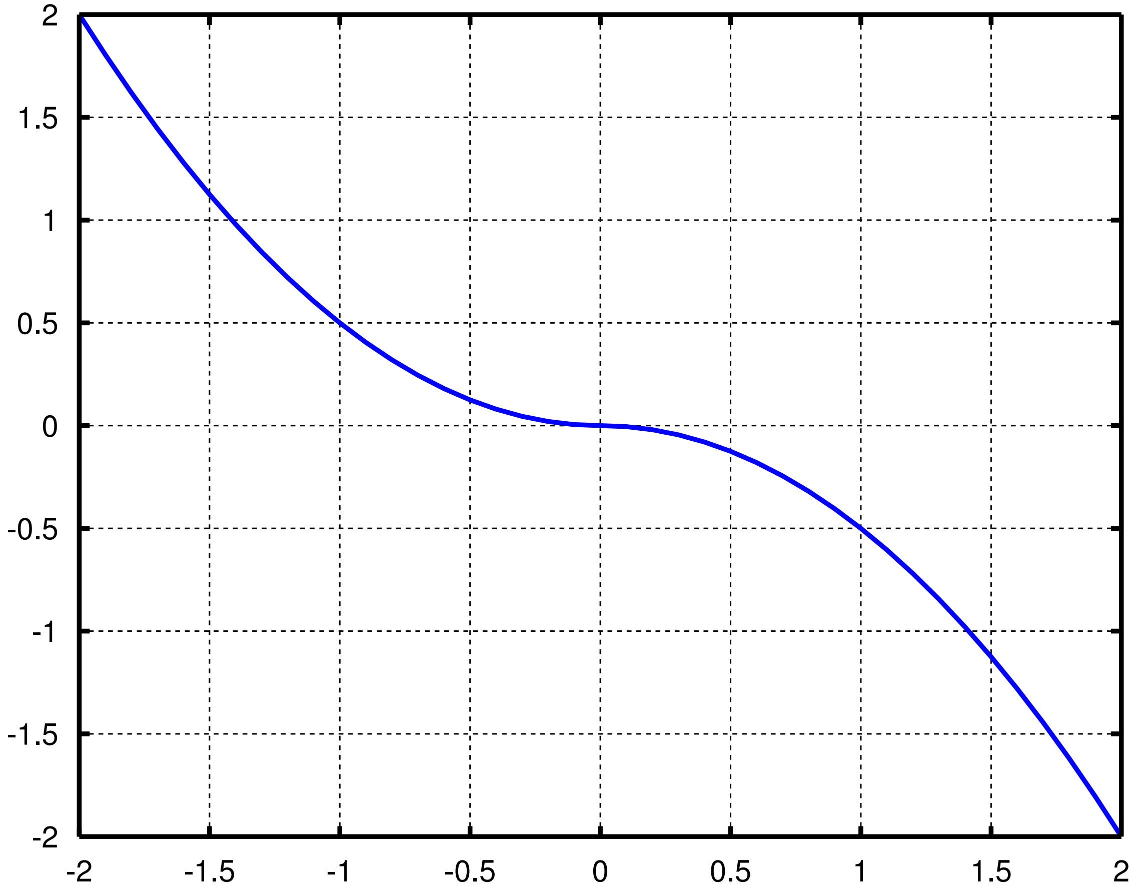

Figure 5: The switching curves $x_1(t)$ vs $x_2(t)$ passing through origin

$$\dot{c} = 2(p_0 - p_t)^T \dot{p}_t = 2(p_0 - p_t)^T (\partial_x p_t \dot{x})$$

$$\text{Where } \partial_x p_t = \begin{pmatrix} -sin(\theta_1)l_1 - sin(\theta_1 + \theta_2)l_2 & -sin(\theta_1 + \theta_2)l_2 & 0 & 0 \\ cos(\theta_1)l_1 + cos(\theta_1 + \theta_2)l_2 & cos(\theta_1 + \theta_2)l_2 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \dot{c} = 2(p_0 - p_t)^T \begin{pmatrix} -sin(\theta_1)l_1\dot{\theta}_1 - sin(\theta_1 + \theta_2)l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ cos(\theta_1)l_1\dot{\theta}_1 + cos(\theta_1 + \theta_2)l_2(\dot{\theta}_1 + \dot{\theta}_2) \end{pmatrix}$$

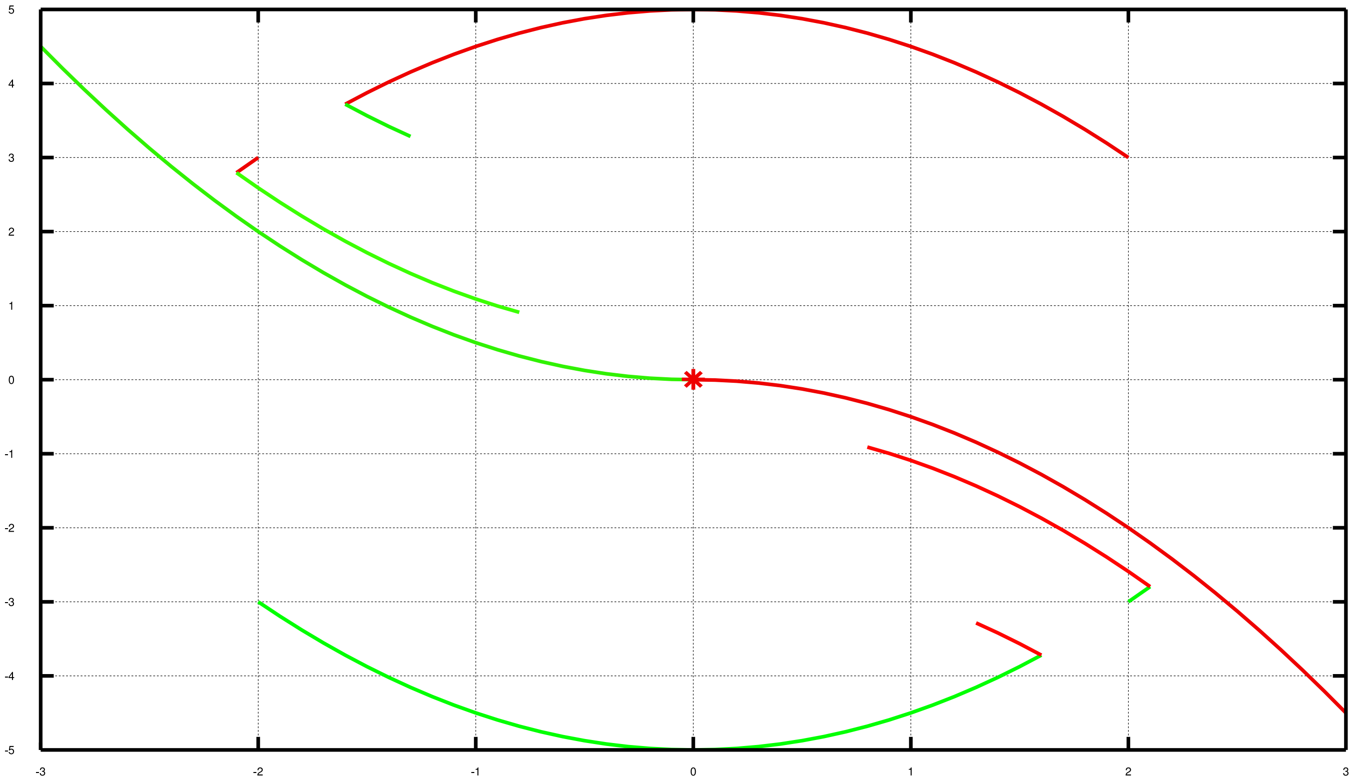

Figure 6: The optimal trajectories $x_1(t)$ vs $x_2(t)$ depicting various control sequences plotted by simulating the dynamics in MATLAB.(a) The red curves represent u = -1 and green represent u = +1 (b) Once the paths reach origin, they do not move any further even if the final time is increased. The control remains zeros thereafter.

Since $\dot{c}$ does not have information about u1 and u2 we differentiate again to get

$$\ddot{c} = 2(p_0 - p_t)^T(\partial_x(\partial_x p_t \dot{x})\dot{x}) + 2(\partial_x p_t \dot{x})^T(\partial_x p_t \dot{x})$$

$$\partial_x(\partial_x p_t) = \begin{pmatrix} -l_1 cos(\theta_1)\dot{\theta}_1 - l_2 cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) & \cdots & \\ -l_2 cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) & -l_1 sin(\theta_1) - l_2 sin(\theta_1+\theta_2) & -l_2 sin(\theta_1+\theta_2) \\ -l_1 sin(\theta_1)\dot{\theta}_1 - l_2 sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) & \cdots & \\ -l_2 sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) & l_1 cos(\theta_1) + l_2 cos(\theta_1+\theta_2) & l_2 cos(\theta_1+\theta_2) \end{pmatrix}$$

$$\partial_x(\partial_x p_t)\dot{x} = \begin{pmatrix} -l_1 c_1 \dot{\theta}_1^2 - l_2 c_{12}(\dot{\theta}_1+\dot{\theta}_2)^2 + u_1(-l_1 s_1 - l_2 s_{12}) + u_2(-l_2 s_{12}) \\ -l_1 s_1 \dot{\theta}_1^2 - l_2 s_{12}(\dot{\theta}_1+\dot{\theta}_2)^2 + u_1(l_1 c_1 + l_2 c_{12}) + u_2(l_2 c_{12}) \end{pmatrix}$$

$$\partial_x p_t \dot{x} = \begin{pmatrix} -sin(\theta_1) l_1 \dot{\theta}_1 - sin(\theta_1+\theta_2) l_2(\dot{\theta}_1+\dot{\theta}_2) \\ cos(\theta_1) l_1 \dot{\theta}_1 + cos(\theta_1+\theta_2) l_2(\dot{\theta}_1+\dot{\theta}_2) \end{pmatrix}$$

The above inequality has u_1 and u_2. Hence the collection $(c \;\; \dot{c} \;\; \ddot{c}) \leq 0$ gives the state control inequality conditions which must be satisfied on the boundary. Thus the constraints are:

$$c(x(t),t) = r^2 - (p_t - p_0)^T(p_t - p_0) \leq 0$$
$$\dot{c}(x(t),t) = 2(p_0 - p_t)^T \begin{pmatrix} -sin(\theta_1)l_1\dot{\theta}_1 - sin(\theta_1+\theta_2)l_2(\dot{\theta}_1+\dot{\theta}_2) \\ cos(\theta_1)l_1\dot{\theta}_1 + cos(\theta_1+\theta_2)l_2(\dot{\theta}_1+\dot{\theta}_2) \end{pmatrix} \leq 0$$
$$(p_0 - p_t)^T \begin{bmatrix} -l_1 sin(\theta_1) - l_2 sin(\theta_1+\theta_2) & -l_2 sin(\theta_1+\theta_2) \\ l_1 cos(\theta_1) + l_2 cos(\theta_1+\theta_2) & l_2 cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \leq -(\partial_x p_t \dot{x})^T(\partial_x p_t \dot{x}) - (p_0 - p_t)^T M(x)$$

4. Given rocket dynamics as :

$$\dot{x}_1(t) = \frac{cu(t)}{x_2(t)} - \frac{D}{x_2(t)}$$
$$\dot{x}_2(t) = -u(t)$$

To maximize the range of rocket, the optimization problem is set as:

$$\text{maximize } J = \int_{t_0}^{t_f} x_1(t)dt \quad x(t_0) \text{ and } x(t_f) \text{ are given}$$

and terminal time is free. Equating to the general optimal control setting we get:

$$H = x_1(t) + \lambda_1\left(\frac{cu(t)}{x_2(t)} - \frac{D}{x_2(t)}\right) + \lambda_2(-u(t))$$

Since terminal time is free and there are no terminal constraints, $H(t_f) = 0$ and since H is not explicitly dependent on time (t), H = const = 0 throught the interval. The adjoint equations for the control problem are given as:

$$\dot{\lambda}_1 = -1$$
$$\dot{\lambda}_2 = -\lambda_1\left(\frac{cu - D}{{x_2}^2}\right)$$

To investigate the possibility of singular control intervals, we try to find the necessary optimal conditions:

$$H_u = \frac{c\lambda_1}{x_2} - \lambda_2 = 0$$

Since H_u does not provide the optimal control directly, we check if we are on the boundaries.

$$\text{maximize } H^*(u) = \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) u$$

The boundary condition on u is given as $0 \leq u(t) \leq u_m$. Hence to maximize Hamiltonian, the Optimal control is given as:

$$u = \begin{cases} u_m & \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) > 0 \\ 0 & \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) < 0 \\ \text{singular} & \left(\frac{c\lambda_1}{x_2} - \lambda_2\right) = 0 \end{cases}$$

To evaluate the possiblity of singular intervals, we want to evaluate the singular condition over a period of time. During the singular interval,

$$\begin{aligned} \dot{H}_u &= \frac{-c\lambda_1}{{x_2}^2}(-u) + \frac{c(-1)}{x_2} - (-\lambda_1)\left(\frac{cu - D}{{x_2}^2}\right) = 0 \\ \Rightarrow \dot{H}_u &= \frac{-c}{x_2} + \frac{\lambda_1 D}{{x_2}^2} = 0 \\ \Rightarrow \lambda_1 &= \frac{cx_2}{D} \\ \Rightarrow \ddot{H}_u &= \frac{c}{{x_2}^2}(-u) - \frac{2\lambda_1 D}{{x_2}^3}(-u) - \frac{D}{{x_2}^2} = 0 \\ \Rightarrow u^* &= \frac{Dx_2}{2\lambda_1 D - cx_2} = \frac{D}{c} \end{aligned}$$

So if the fuel burn rate is constant at $\frac{D}{c}$, and $\lambda_1 = c\frac{x_2}{D}$, and $\lambda_2 = \frac{c^2}{D}$ for some period of time, then that interval is a singular interval. But if $\frac{D}{c}$ is greater than u_m, then we do not have a singular interval.

Acknowledgements

I hereby declare that I have not discussed this homework with anyone. The solutions written here are my own work and from lecture notes and sample code provided by the professor. Any external references are mentioned in the text.

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