EN530.603 Applied Optimal Control
Lecture 7: Constrained Optimal Control

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We will consider optimal control problems subject to constraints of the form

$$c(x, u, t) \leq 0.$$

Such constraints occur in many practical applications. Control constraints on u arise due to e.g. maximum current available in a motor, maximum thrust in a rocket, maximum power produced by an engine, maximum purchasing power in a financial transaction, etc... State constraints on x generally describe forbidden regions in state space that are unsafe, or simply invalid for the system, such as obstacles in the environment.

For instance, consider a the dynamics of a simple car with state (x, y, θ, v, ϕ) given by

$$\dot{x} = \cos\theta v, \tag{1}$$
$$\dot{y} = \sin\theta v, \tag{2}$$
$$\dot{\theta} = \frac{\tan\phi}{l} v, \tag{3}$$
$$\dot{v} = u_1, \tag{4}$$
$$\dot{\phi} = u_2, \tag{5}$$

where ϕ is the steering angle and the inputs are the forward acceleration and steering angle rate. We have various constraints such as

$a_{min} \leq u_1 \leq a_{max}$,	maximum engine power in forward and reverse
$\lvert u_2 \rvert \leq u_2^{max}$	maximum steering rate,
$v_{min} \leq v \leq v_{max}$,	maximum speed in forward and reverse
$\lvert\phi\rvert \leq \phi_{max}$,	maximum steering angle (e.g. $\phi_{max} = \pi/6$)
$\lVert(x, y) - (x_o, y_o)\rVert \geq r_o$,	stay at least r_o meters away from obstacle at (x_o, y_o).

1 Pontryagin's Minimum Principle

Recall that at a minimum we have

$$J(u) - J(u^*) = \delta J \geq 0$$

and

$$\delta J(u^*, \delta u) = \delta J(u^*, \delta u) + o(\lVert\delta u\rVert)$$

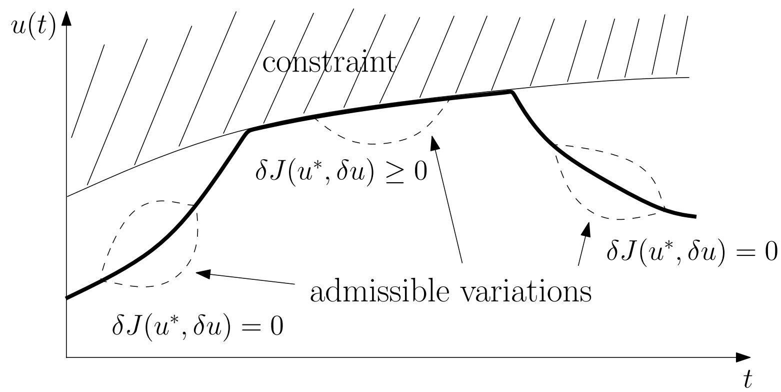
The necessary conditions without constraints were that for arbitrary variations δu

$$\delta J(u^*, \delta u) \geq 0, \text{ and } \delta J(u^*, -\delta u) \geq 0$$

which holds only when the necessary condition

$$\delta J(u^*, \delta u) = 0$$

holds. But when u^* is at the boundary of the admissible controls, then if the control $u + \delta u$ is admissible, then $u - \delta u$ is *not* admissible.



Hence, at a boundary the necessary condition is

$$J(u^*, \delta u) \geq 0,$$

for *admissible variations* δu, i.e. such that $u + \delta u$ does not violate the constraint.

With this definition, let's look at the actual variations for the augmented cost functional

$$J_a = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} [L(x, u, t) + \lambda^T (f(x, u, t) - \dot{x})] dt$$

We have, (recall $H = L + \lambda^T f$) that

$$\begin{aligned} \delta J_a(u^*, \delta u) = & \left\{ [\varphi_x - \lambda^T] \delta x_f + [H + \phi_t] \delta t_f \right\} \Big|_{t=t_f} \\ & + \int_{t_0}^{t_f} \left\{ [\dot{\lambda}^T + \partial_x H] \delta x(t) + [\partial_u H] \delta u(t) + \delta \lambda(t)^T [f - \dot{x}] \right\} dt \end{aligned}$$

Now assume that the state dynamics is satisfied, and $\lambda(t)$ is selected to satisfy the conditions due to variations $\delta x(t)$, and it the boundary conditions (for variations δx_f and δt_f) are satisfied.

Then we have

$$\delta J_a(u^*, \delta u) = \int_{t_0}^{t_f} \partial_u H(x^*(t), u^*(t), \lambda^*(t), t) \cdot \delta u(t) dt$$

This can be approximate by

$$\delta J_a(u^*, \delta u) = \int_{t_0}^{t_f} [H(x^*(t), u^*(t) + \delta u(t), \lambda^*(t), t) - H(x^*(t), u^*(t), \lambda^*(t), t)] \, dt + o(\|\delta u\|)$$

Taking $\delta u \to 0$ since $\delta J(u^*, \delta u) \geq 0$ we must have

$$\int_{t_0}^{t_f} [H(x^*(t), u^*(t) + \delta u(t), \lambda^*(t), t) - H(x^*(t), u^*(t), \lambda^*(t), t)]\, dt \geq 0,$$

for all admissible $\delta u(t)$ which can only happen if

$$H(x^*(t), u^*(t) + \delta u(t), \lambda^*(t), t) \geq H(x^*(t), u^*(t), \lambda^*(t), t) \tag{6}$$

for all admissible $\delta u(t)$ and for all $t \in [t_0, t_f]$. The relationship (6) is known as the *Pontryagin's minimum principle*. It states that an optimal control must minimize the Hamiltonian.

Here, u^* must be the global absolute minimum that minimizes H. Pontryagin's principle works for any bounds on u which is not the case for the condition $\partial_u H = 0$. Yet, it is still only a necessary condition.

The sufficient conditions for a *local minimum* require that

$$\partial_u^2 H(x^*(t), u^*(t), \lambda^*(t), t)$$

is positive definite.

Example 1. Consider

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_2 + u \end{aligned}$$

with cost

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2}[x_1^2 + u^2]dt$$

with t_f given and final state $x(t_f)$ free. The Hamiltonian is

$$H = \frac{1}{2}x_1^2 + \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2(-x_2 + u)$$

The necessary conditions are

$$\dot{\lambda}_1 = -x_1 \tag{7}$$

$$\dot{\lambda}_2 = -\lambda_1 + \lambda_2 \tag{8}$$

and, if the controls are unconstrained we have

$$\partial_u H = u + \lambda_2 = 0$$

Since $\partial_u^2 H = 1 > 0$ then $u^* = -\lambda_2$ is in fact the minimum of H. The boundary condition is $\lambda(t_f) = 0$.

Now if the control was bounded by

$$-1 \leq u(t) \leq 1, \qquad \text{for all } t \in [t_0, t_f]$$

then we must select u to minimize H subject to the bounds on u.

The terms in H depending on u are

$$\frac{1}{2}u^2 + \lambda_2 u.$$

When the optimal control is not saturated we have $u^* = -\lambda_2$ which will occur when $|\lambda| \leq 1$. When $|\lambda_2| > 1$ the minimizing control must be

$$\begin{aligned} u &= -1, \text{ for } \lambda_2 > 1 \\ u &= 1, \text{ for } \lambda_2 < 1 \end{aligned}$$

In summary we have

$$\begin{aligned} &u^*(t) = -1, && \text{for } \lambda_2(t) > 1 \\ &u^*(t) = -\lambda_2, && \text{for } -1 \leq \lambda(t) \leq 1 \\ &u^*(t) = 1, && \text{for } \lambda_2(t) < 1 \end{aligned}$$

Note that, in genera, it will *not* be the case that bounds are handled by simply computing the unbounded solution and then saturating it as above.

2 General Constraints

Now consider general constraints $c(x, u, t) \leq 0$. The Hamiltonian is defined by

$$H = L + \lambda^T f + \mu^T c, \quad \text{where } \begin{cases} \mu = 0 \text{ if } c < 0 \\ \mu \geq 0 \text{ if } c = 0 \end{cases}$$

The adjoint equations are

$$\dot{\lambda} = \begin{cases} -\nabla_x L - \nabla_x f^T \lambda, & c < 0 \\ -\nabla_x L - \nabla_x f^T \lambda - \nabla_x c^T \mu, & c = 0 \end{cases}$$

The control is found by setting $\nabla_u H = 0$ which corresponds to

$$0 = \begin{cases} \nabla_u L + \nabla_u f^T \lambda, & c < 0 \\ \nabla_u L + \nabla_u f^T \lambda + \nabla_u c^T \mu, & c = 0. \end{cases}$$

Note that when the constraints are not active the conditions reduce to the standard unconstrained case. In some cases the condition above cannot be used directly to compute the optimal u^* and one must apply the more general Pontryagin's principle. The main problem in dealing with such constraints is to determine when the constraint becomes active.

To reiterate, the difficulty is that there are many problems in which H cannot be directly minimized by setting $\partial_u H = 0$ and assuming that H is locally convex. In such cases, the control is found by the minimum principle.

Note that control-independent constraints in the form

$$c(x(t), t) \leq 0$$

are more difficult to handle and require differentiation until the control appears explicitly. For instance,

$$c(x(t), t) \leq 0 \quad \Rightarrow \quad \dot{c} = c_x \cdot \dot{x} = c_x \cdot f(x, u, t) \leq 0,$$

and if u appears, then the new constraint is the combined $(c, \dot{c}) \leq 0$. If not this can be repeated q times until u shows up explicitly and $(c, \dot{c}, \ldots, c^{(q-1)}) \leq 0$ is enforced as the constraint.

3 Minimum-time problems

Consider the linear system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0,$$

with a single control input u constrained by

$$|u(t)| - 1 \leq 0$$

which is required to reach to origin $x(t_f) = 0$ in minimum time defined by the cost

$$J = \int_{t_0}^{t_f} (1) dt.$$

There are two ways to proceed. Since the constrained is a simple bound on the control we can use the standard Hamiltonian

$$H = 1 + \lambda^T (Ax + Bu)$$

and apply the Pontryagin approach in §1. Alternatively we can use the Hamiltonian

$$H = 1 + \lambda^T (Ax + Bu) + \mu(|u| - 1).$$

and apply the approach in §2. Both will be equivalent.

Applying the former approach with $H = 1 + \lambda^T (Ax + Bu)$, the adjoint equation is

$$\dot{\lambda} = -\nabla_x H = -A^T \lambda$$

subject to

$$\lambda(t_f) = 0, \qquad H(t_f) = 0.$$

The first and second derivatives of H are

$$\nabla_u H = B^T \lambda, \qquad \nabla_u^2 H = 0,$$

Since u does not appear in $\nabla_u H$ these equations cannot be used to find the optimal u, and furthermore H is not convex since $\nabla_u^2 H = 0$.

According to the minimum principle though, H is minimized with respect to u when we have

$$\begin{aligned} u &= +1, \text{ if } \lambda^T B < 0, \\ u &= -1, \text{ if } \lambda^T B > 0, \end{aligned}$$

where $\lambda^T B$ is called the *switching function*.

It turns out that $\nabla_u H$ is zero only at the *switching points* i.e. when the control reverses sign. This is an example of *bang-bang* control or "maximum-effort" since the control is always at its max or min.

In addition, to ensure smoothness in the trajectory during transition between constraints it is necessary to enforce the constraints

$$\begin{aligned} \lambda({t_1}^-) &= \lambda({t_1}^+), \\ H({t_1}^-) &= H({t_1}^+), \end{aligned}$$

where t_1 is the time of transition, t_1^- is the time infinitesimally before t_1 and t_1^+ infinitesimally after t_1. These conditions are called *Weierstrass-Erdmann conditions* and are related to well-definedness of the action integral used in the variational formulation.

In other words, λ, H, and ∂H_u must be continuous despite that u is discontinuous. Note that even when the control is discontinuous the state remains continuous since only $\dot{x}$ (the rate-of-change of state) "jumps" while

$$x(t_1^-) = x(t_1^+).$$

Example 2. Consider computing the optimal trajectory reaching the origin in minimum time t_f of the double integrator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned}$$

starting at an arbitrary state $x(t_0) = x_0$.

We have

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

the necessary conditions become

$$\begin{aligned} u &= +1, \quad \text{if } \lambda_2 < 0, \\ u &= -1, \quad \text{if } \lambda_2 > 0, \end{aligned}$$

The adjoint equations require that

$$\begin{aligned} \dot{\lambda}_1 &= 0, \\ \dot{\lambda}_2 &= -\lambda_1 \end{aligned} \quad \Rightarrow \left\{ \begin{aligned} \lambda_1(t) &= c_1, \\ \lambda_2(t) &= -c_1 t + c_2 \end{aligned} \right. ,$$

where c_1 and c_2 are constants of integration Since λ_2 is a linear function of time, then it will change sign at most once. Therefore we have the following situations

$$u^*(t) = \begin{cases} +1, & \text{for all } t \in [t_0, t^*], \text{ or} \\ -1, & \text{for all } t \in [t_0, t^*], \text{ or} \\ +1, & \text{for all } t \in [t_0, t_1] \text{ and } -1, \text{ for all } t \in [t_1, t^*], \text{ or} \\ -1, & \text{for all } t \in [t_0, t_1] \text{ and } +1, \text{ for all } t \in [t_1, t^*]. \end{cases}$$

Thus, since $u = \pm 1$ in any case, we have

$$\begin{aligned} x_1 &= \pm \frac{1}{2} t^2 + c_3 t + c_4 \\ x_2 &= \pm t + c_3, \end{aligned}$$

where c_3 and c_4 are constants of integration.

This is equivalent to the condition (by removing time)

$$x_1(t) = \frac{1}{2} x_2^2(t) + c_5, \qquad \text{for } u = +1 \qquad (\text{parabola } A(c_5)) \tag{9}$$

and

$$x_1(t) = -\frac{1}{2}x_2^2(t) + c_6, \qquad \text{for } u = -1 \qquad (\text{parabola } B(c_6)) \tag{10}$$

where c_5 and c_6 are constants determined by the initial conditions.

Now if $u^*(t) = +1$ for all t then we must have $c_5 = 0$ and the motion is completely determined by the parabola $A(0)$ defined in (9). Now if $u^*(t) = -1$ for all t then we must have $c_6 = 0$ the motion is completely determined by the parabola $B(0)$ defined in (10).

If the system is guided by $u^*(t) = +1$ for $t \in [t_0, t_1]$ and then by $u^*(t) = -1$ for $t \in [t_1, t_f]$ then it must be that the second arc lies on $B(0)$. The first arc is one of the parabolas $A(c_6)$. Note that only curves with $c_5 < 0$ and that start underneath $B(0)$ will actually intersect $B(0)$.

With the same reasoning, if the system starts with $u^*(t) = -1$ for $t \in [t_0, t_1]$ and then by $u^*(t) = +1$ for $t \in [t_1, t_f]$ then it must be that the second arc lies on $A(0)$. The first arc is one of the parabolas $B(c_5)$. Note that only curves with $c_6 < 0$ and that start above $A(0)$ will actually intersect $A(0)$.

We see that the curves $A(0)$ and $B(0)$ act as switching curves. In particular, combining the conditions above results in the curve defined by

$$x_1(t) = -\frac{1}{2}x_2(t)|x_2(t)|$$

as the combined *switching curve* (we can also denote it by $AB(0)$). To state the actual control law we define the *switching function*

$$s(x(t)) \equiv x_1(t) + \frac{1}{2}x_2(t)|x_2(t)|$$

and the control law is summarized according to

$$u^*(t) = \begin{cases} -1, & \text{for } x(t) \text{ such that } s(x(t)) > 0, \\ +1, & \text{for } x(t) \text{ such that } s(x(t)) < 0, \\ -1, & \text{for } x(t) \text{ such that } s(x(t)) = 0 \text{ and } x_2 > 0, \\ +1, & \text{for } x(t) \text{ such that } s(x(t)) = 0 \text{ and } x_2 < 0, \\ 0, & \text{for } x(t) = 0. \end{cases}$$

4 Minimum Control Effort Problems

Consider a nonlinear system with affine controls defined by

$$\dot{x} = a(x(t), t) + B(x(t), t)u(t),$$

where B is an $n \times m$ matrix. The cost function is

$$J = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} |u_i(t)| \right] dt$$

subject to

$$-1 \le u_i(t) \le 1, \qquad i = 1, \ldots, m.$$

The Hamiltonian is

$$H(x,u,\lambda,t) = \sum_{i=1}^{m} |u_i| + \lambda^T a + \lambda^T B u$$

and the minimum principle requires that

$$\sum_{i=1}^{m} |u_i^*| + \lambda^{*T} B u^* \leq \sum_{i=1}^{m} |u_i| + \lambda^{*T} B u$$

for all admissible $u(t)$. If B is expressed as

$$B = [b_1 \mid b_2 \mid \cdots \mid b_n]$$

If we assume that the components of u are independent of one another we have

$$|u_i^*| + \lambda^{*T} b_i u_i^* \leq |u_i| + \lambda^{*T} b_i u_i$$

Note that

$$|u_i| + \lambda^{*T} b_i u_i = [1 + \lambda^{*T} b_i] u_i, \text{ for } u_i \geq 0, \tag{11}$$

and

$$|u_i| + \lambda^{*T} b_i u_i = [-1 + \lambda^{*T} b_i] u_i, \text{ for } u_i \leq 0, \tag{12}$$

If $\lambda^{*T} b_i > 1$ then the minimum of (11) is 0 for $u \geq 0$ while the minimum of (12) is attained at $u_i = -1$.

If $\lambda^{*T} b_i = 1$ then the minimum of (11) is attained at $u_i = 0$ while the minimum of (12) is attained for any $u_i \leq 0$.

If $0 \leq \lambda^{*T} b_i < 1$ the minimum of (11) and (12) is attained at $u_i = 0$.

In summary, the optimal control is

$$u_i^* = \begin{cases} 1, & \text{for } \lambda^{*T} b_i < -1 \\ 0, & \text{for } -1 < \lambda^{*T} b_i < -1 \\ -1, & \text{for } 1 < \lambda^{*T} b_i \\ \text{undetermined nonnegative value if } \lambda^{*T} b_i = -1 \\ \text{undetermined nonpositive value if } \lambda^{*T} b_i = 1 \end{cases}$$

5 Singular Controls.

The example above illustrates the possibility of *singular controls*, i.e. controls which cannot be directly determined by neither the optimality conditions (away from constraints) nor the minimum principle (at the control boundary).

In the unconstrained case this generally occurs when $\nabla_u H = 0$ cannot be solved for u which is caused by the conditions

$$\nabla_u^2 H = 0,$$

i.e. when the Hamiltonian is not convex. To illustrate the situation consider the LQR setting in which $\nabla_u^2 H = R$, and consider the case $R = 0$. (The same reasoning will apply for any singular R)

The first order condition is

$$H_u = Ru + B^T\lambda = B^T\lambda = 0,$$

which does not provide information about u. The solution is to consider higher-order derivative of H_u until u appears explicitly:

$$\frac{d}{dt}H_u = B^T\dot{\lambda} = -B^T(Qx + A^T\lambda) = 0,$$

and one more time

$$\frac{d^2}{dt^2}H_u = -B^T[Q(Ax + Bu) + A^T(Qx + A^T\lambda)] = 0,$$

which now provides enough information to obtain the singular control u.

6 Corner Conditions.

More generally, path constraints that become active at a particular time t cause a discontinuity in the controls, which in turn corresponds to a change in the slope of the state trajectory $x(\cdot)$. Such points $x(t)$ are called *corners*. For problems with control bounds the corner conditions are

$$\begin{aligned}
&\lambda(t^-) = \lambda(t^+),\\
&H(t^-) = H(t^+),\\
&H_u(t^-) = H_u(t^+),
\end{aligned}$$

where t^- and t^+ denote the time immediately before and after the constraint becomes active. For problems with state inequality constraints we have

$$\begin{aligned}
&\lambda(t^-) = \lambda(t^+) + \nabla c_x^T\mu\\
&H(t^-) = H(t^+) - \partial_t c^T\lambda,\\
&H_u(t^-) = H_u(t^+).
\end{aligned}$$