National Institute of Technology Karnataka, Surathkal

Identification of wave patterns in excitable media

A project as a part of NITK-SIP

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Acknowledgements

"If I have seen further, it is by standing on the shoulders of giants."

Sir Isaac Newton

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Prajwal DSouza.
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Problem Description

Targets and spirals are distinct wave patterns that have been observed in a variety of physical, chemical, and biological systems. A target pattern is produced by concentric waves travelling away from a rhythmic source, whereas a spiral wave is generated by a rotating source. Despite their different mechanisms, both wave patterns lead to indistinguishable rhythms when measured at a single point in space. [1]

Kevin Hall and Leon Glass

How to Tell a Target from a Spiral: The Two Probe Problem

(21 June 1999)

Excitable media, as the word suggests, are media in which a certain region can be excited, in other words, made to go to an excited state and after some time, will come back to the original resting state. But, even after coming back to the resting state, the region cannot be excited immediately, but only after a certain refractory period.

The existence of meta-stable excited state along with a refractory period leads to interesting travelling waves of excitation in excitable media. [2]

These travelling waves form interesting patterns. The most common patterns seen are Target and Spiral Wave patterns. The project focuses on these patterns and their identification.

The problem is with the number of probes to be placed on the medium to identify the patterns. A single probe isn't sufficient as the rhythms are indistinguishable at a given point in the medium.

We have come up with a method to identify the patterns correctly with least number of probes possible.

Proposed Solution

The solution here is an algorithm which involves finding out many features whose measurements give us certain parameters which will help us distinguish the patterns.

We solve a simpler problem first.

Identification of the direction of the wave in a given region seems to be of importance as direction of waves at various points can help us identify the pattern.

A simpler problem | How to find the direction of a plane wave?

We assume that all waves locally behave like planes waves and try to determine the direction of these waves.

Locally, both wave patterns can be approximated as plane waves as shown in figure 1.

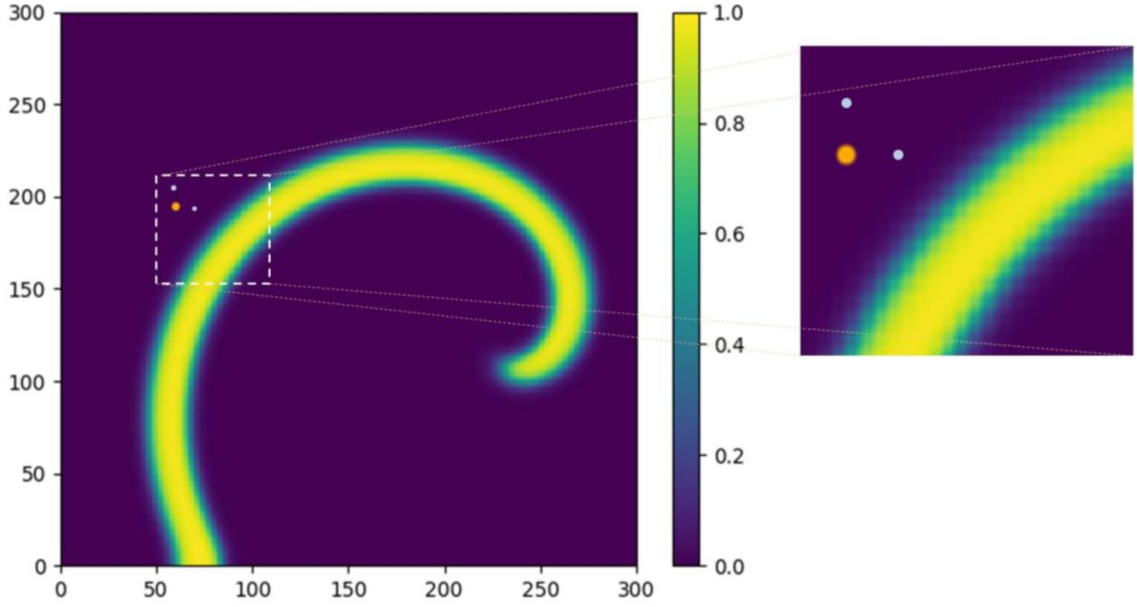

Figure 1: The arrangement of three probes in a fixed pattern in a small region. Locally, the wave is identical to a plane wave. [5]

We use a three probe method to determine the direction of the wave. Consider placing three probes closely in the excitable medium. In order to simplify calculations, we place them in a particular pattern. The pattern is shown in figure 1.

If a plane wave moves through these probes, the excitation at a given probe is shown in figure 2.

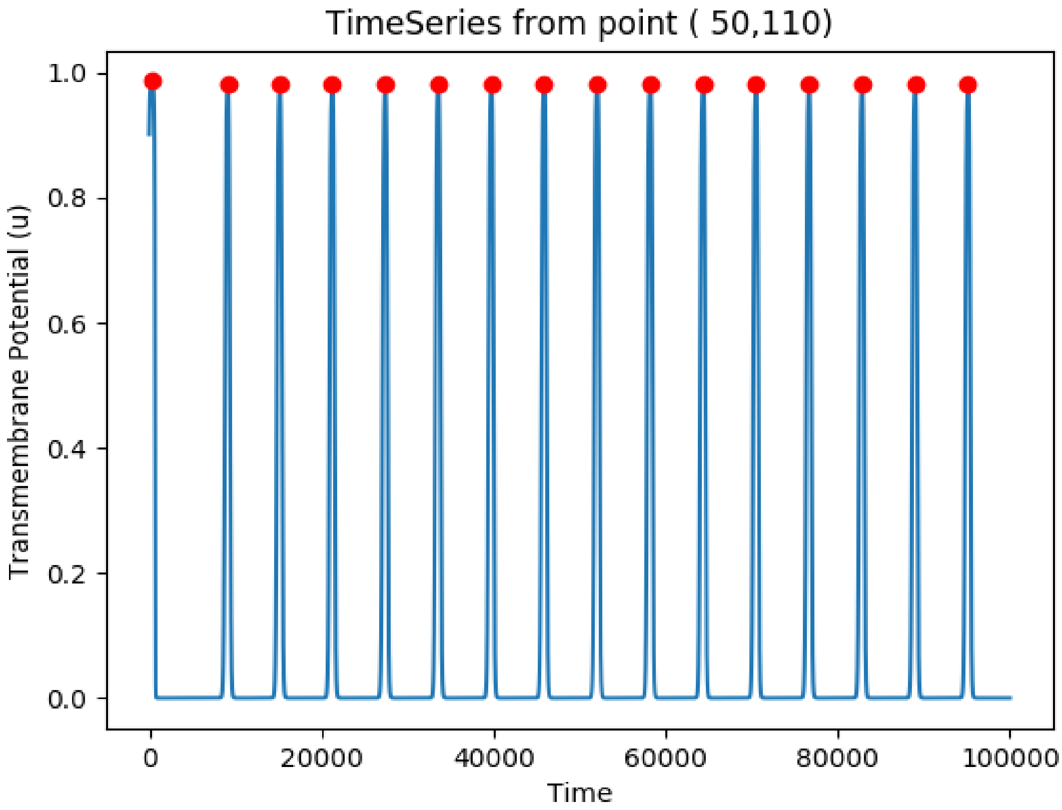

Figure 2: Time Series Data at a point (50,110) in the excitable medium. [5]

The data is periodic. It reaches a peak when the wave arrives at the probe. The time at which it arrives at the probe is called the Activation Time. We calculate the activation time, by using a peak finding algorithm. The red dots in the figure 1 represent these times.

Once we have the activation times for a given probe, we proceed only with this data, and ignore the rest. This is a key point in the analysis as we ignore most of the data.

We find the activation times for the other two probes too. Together, the activation time data (wave arrival time data) for all the three probes is given in Figure 3.

Figure 3: Activation times at three different points [5]

Each group of activation times in the figure correspond to a particular wave. We figure out the direction of that particular wave.

We model the plane wave (movement of its peaks) passing though with the equation,

$$y = mx + (c + vt) \quad \text{(eq 1)}$$

This represents a line whose intercept is changing with time t, moving with velocity v in a particular direction. Its slope is m (which isn't changing with time) and the intercept is (c + v t) which changes with time. Figure 4 describes the equation at various values of t. It can be seen that the line moves upward with increasing time t.

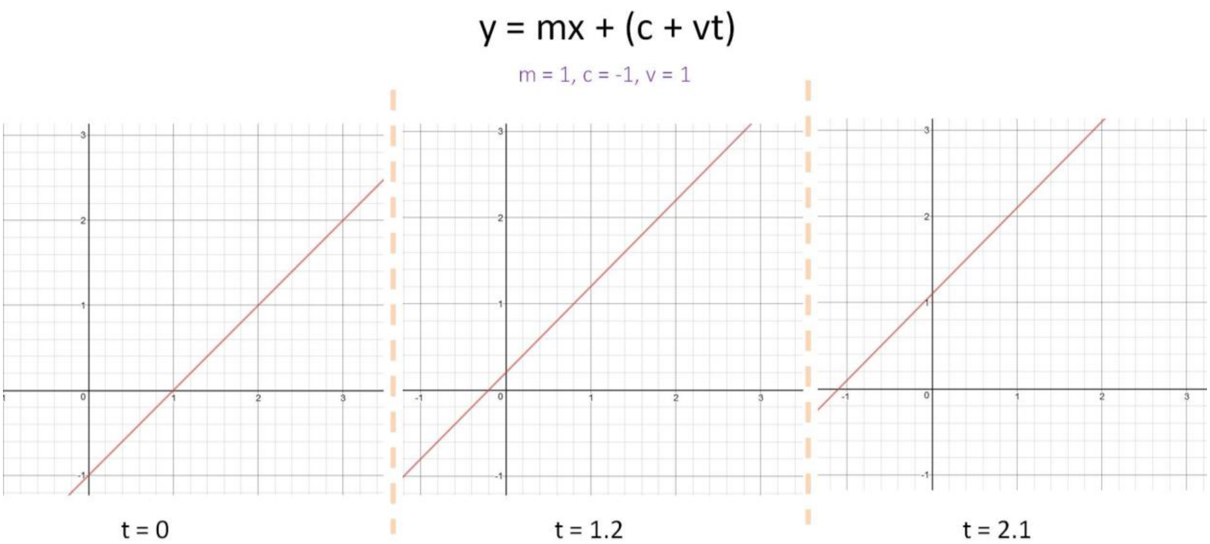

Figure 4: Graph of equation y = x + 1 + t at various values of t. [4]

Suppose this wave arrives at three points, (x_1, y_1), (x_2, y_2) and (x_3, y_3), at times t_1, t_2, and t_3 respectively, then we can solve for the three constants in equation 1.

$$y_1 = mx_1 + (c + vt_1)$$

$$y_2 = mx_2 + (c + vt_2)$$

$$y_3 = mx_3 + (c + vt_3) \quad \text{(eq 2)}$$

For simplification, if the points are (0, 0), (h, 0) and (0, h), then,

$$m = \frac{t_1 - t_2}{t_3 - t_1} \quad \text{(eq 3)}$$

Due to time translation symmetry, we can set $t_1 = 0$, then,

$$m = \frac{-t_2}{t_3} \quad \text{(eq 4)}$$

The vector in the direction perpendicular to the slope m is given by,

$$\overrightarrow{k} = \begin{bmatrix} -t_3 \\ t_2 \end{bmatrix}$$ (eq 5)

Normalizing this vector gives the unit vector in the direction of propagation of the plane wave.

$$\widehat{k} = \frac{\overrightarrow{k}}{\left\|\overrightarrow{k}\right\|}$$ (eq 6)

So, locally, by using three probes we can find the direction of the wave. We call these three probes a probe cluster, or simply, a cluster.

So, we now have a unit vector that describes the direction of the moving wave at a given point.

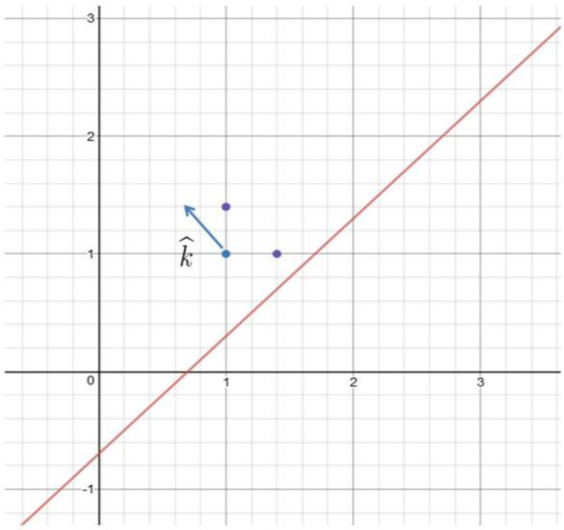

Figure 5: Probe placements and the computed unit direction vector. [4]

We can now determine the direction of the wave at a given point. Finding the directions at many other points should give us insight into its pattern.

A more difficult problem | How to find the centre of a target wave?

Once we know the direction of the wave in a given region, we can do the same with a cluster in a different region.

Let us assume that the wave is a target wave. Since we now know the direction of the wave in two regions and for a target these waves go outward, originating from a single point, retracing the directions from two regions we can find the centre of the target wave, as in figure 6.

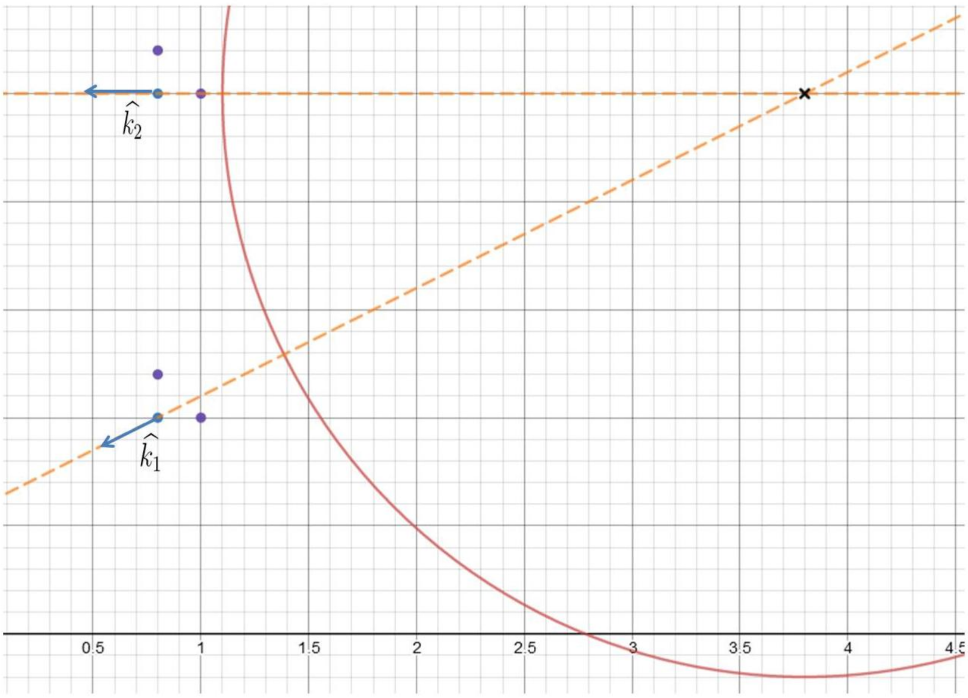

Figure 6: Finding the centre by knowing the directions of the wave at two regions. [4]

Let the directions of the wave at two regions/points be

$$\widehat{k_1} = \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix} \quad \widehat{k_2} = \begin{bmatrix} k_{21} \\ k_{22} \end{bmatrix} \qquad \text{(eq 7)}$$

Then, we can find the intersection of the directions in the following way.

We construct a line in the direction of the direction vector using the vector representation of the line.

For direction vector k_1

$$\overrightarrow{L_1} = r_1\widehat{k_1} + \overrightarrow{p_1} \qquad \text{(eq 8)}$$

where r_1 is a scalar, which varies.
p_1 is a vector representing any point on the line. We take this point to be the one where the cluster is placed. This convention is going to be of use later.

Similarly for the other direction k_2

$$\overrightarrow{L_2} = r_2\widehat{k_2} + \overrightarrow{p_2} \qquad \text{(eq 9)}$$

We have the point of intersection when,

$$\overrightarrow{L_1} = \overrightarrow{L_2}$$

$$r_1\widehat{k_1} + \overrightarrow{p_1} = r_2\widehat{k_2} + \overrightarrow{p_2} \qquad \text{(eq 10)}$$

For simplicity, we choose the position of the 1st cluster to be (0, 0) and 2nd cluster to be (0, d). Two clusters are separated by a distance d in the y-direction.

Thus we have,

$$\begin{bmatrix} k_{11} & -k_{21} \\ k_{12} & -k_{22} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(eq 11)}$$

Solving the matrix equation gives, r_1, from which the point of intersection can be found.

$$\overrightarrow{L}_{center} = r_1\widehat{k_1} + p_1 \qquad \text{(eq 12)}$$

Thus components of **L** $_{centre}$ are the co-ordinates of the intersection.

Here, the two clusters were used to find the centre. Hence we call them “centre finding clusters”.

A Target wave has the same centre, irrespective of which region we calculate it from. Any deviation to this, should indicate, non-Target behaviour.

So, once we know the centre of the target wave, how to we prove that it is a target wave?

Confirming Cluster

We have assumed that the wave a Target. Now, if we consider a third region, a third point in fact, and place a cluster there to get the direction of the wave. If we construct a line at the point in the direction of the direction vector k_3, the line must also pass though the same centre as it also originated from the same point. So if this criteria, is satisfied, we call the wave, a target. The 3rd cluster helps us to confirm that the position of the centre is the same. Hence, we call it the confirming Cluster.

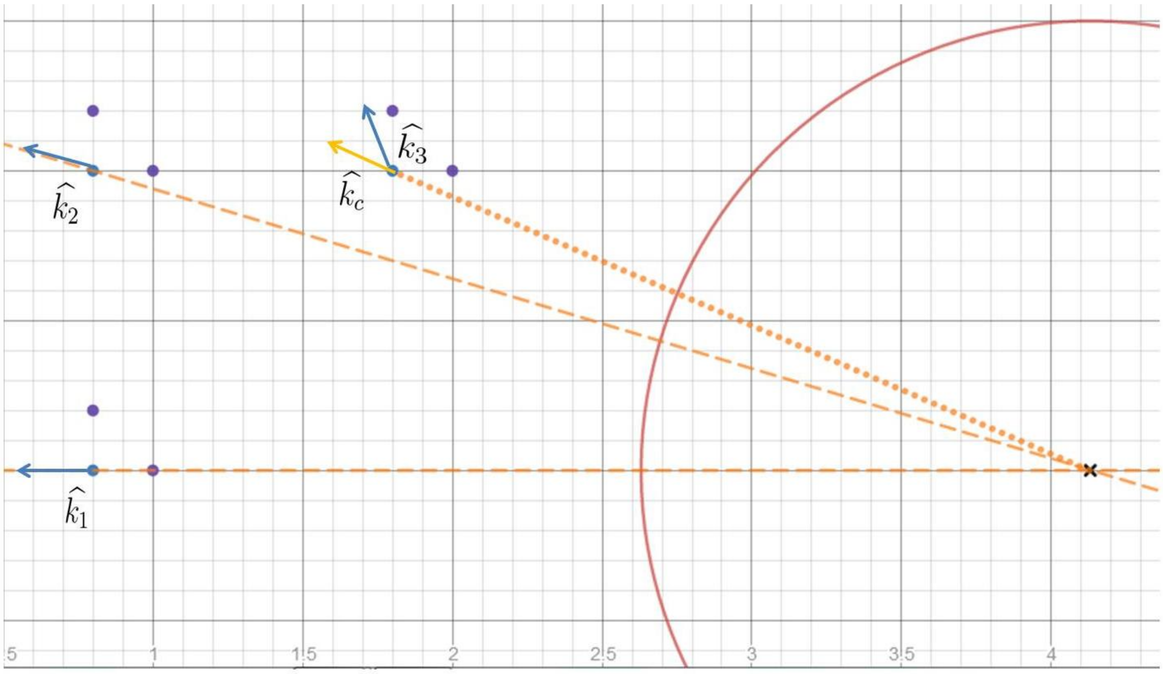

Figure 7: The blue unit vectors labelled k_1, k_2 and k_3 are measured direction vectors in their respective regions. k_1 and k_2 are "centre finding clusters" and k_3 is the unit vector corresponding to the confirming cluster. k_c is the expected unit vector if the wave is indeed a target. [4]

To measure the extent to which this criteria, is satisfied, we draw a line through the calculated centre and the 3rd cluster point. We calculate the direction vector k_c which indicates the direction of this line. This is the expected direction vector if the sample is a target.

$$\overrightarrow{k_c} = \overrightarrow{L}_{center} - \overrightarrow{p_3} \qquad \text{(eq 13)}$$

and the expected unit vector,

$$\widehat{k_c} = \frac{\overrightarrow{k_c}}{\left\|\overrightarrow{k_c}\right\|}$$

(eq 14)

For a target, since k_c and k_3 are unit vectors which point in the same direction, the dot product between them, should be 1 or close to 1. This is how we claim that the wave is a target. Figure 5 depicts this.
Since the Dot product can be between -1 and 1, we use feature scaling and represent this value between 0 and 1.

$$T = \frac{(\widehat{k_c} \cdot \widehat{k_3}) + 1}{2}$$

(eq 15)

T is called the Target value. This measures the deviation of the behaviour from the Target. For a Target, the value is close to 1.
There is more to this though. The directions of the direction vectors k_1 and k_2 also matter. They must be pointing away from the centre. This can be assured by the solution to the matrix equation. If the solution r_1 and r_2 are less than zero, then the direction vectors are pointing away from the centre. This is precisely because of the convention we chose at eq *.

So, if a wave pattern is Target, then

1) r_1 and r_2 must be less than zero.
2) Target value must be close to 1.

The confirming cluster is actually calculating the centre of the target wave with respect to the direction of the wave at its point. The Target value is an indirect measure of the shift in the position of the centre. If it's a target wave, the centre should not change.

Target Value Threshold

If target value is above the Target Value Threshold, it will be classified as a target.

We can now measure the target nature of a wave pattern. But, how confident are we?

The Confidence Parameter

If we are very far from the centre of a spiral or a target wave, the behaviour is similar and appears to be a wave front moving radially outward. Hence the algorithm would predict the wave pattern to be a target. We need a measure of how far we are from the centre. Turns out, the dot product between k_1 and k_2 is a good measure. If we are very far from the centre, then k_1 and k_2 are almost in the same direction. Hence, the dot product will be nearly equal to 1. Now, if we take, a quantity, which is one minus the dot product and apply feature scale to make it lie between 0 and 1, we have

$$C = \frac{1 - (\widehat{k_1} \cdot \widehat{k_2})}{2} \qquad \text{(eq 16)}$$

This is called the confidence parameter. It's based on the dot product between the wave directions at the "centre finding clusters". If its value is close to 1, then dot product of k_1 and k_2 is equal to -1, implying that the direction vectors are in the opposite directions. If the distance between the clusters is small, then we can say that we are very close to the centre. In fact, the clusters are on opposite sides with respect to the centre.

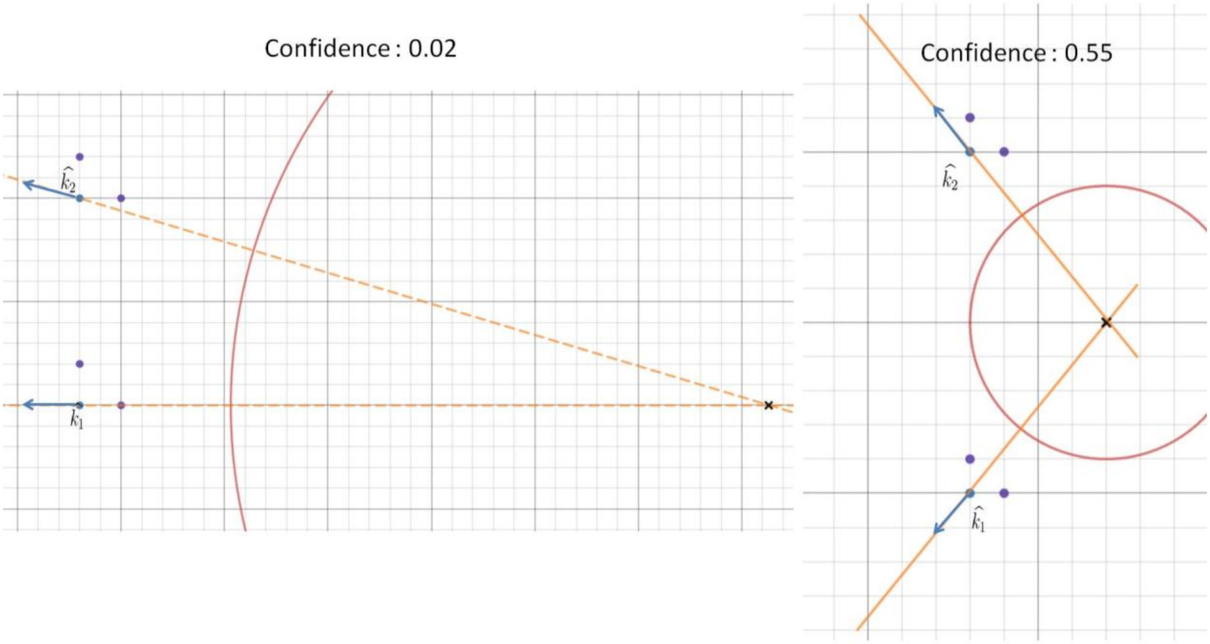

Figure 8: The blue unit vectors labelled k_1 and k_2 are measured direction vectors in their respective regions. The confidence value is negative of the dot product between k_1 and k_2 scaled to be in between 0 and 1. We can clearly see that confidence value is higher for the situation on the right than on the left. [4]

So, what is the process to determine if the wave pattern is a Target?

The Procedure

We assume that the wave pattern is a Target.

Step 1: Place two probe clusters in the medium at different points. These will be centre finding clusters. Determine the direction of the waves at those points. Find the confidence value and the centre.

Step 2:

a. If the confidence value is sufficiently high, place the confirming probe cluster compute the target value. If the target value is above the target threshold value, it is classified as a Target wave as assumed initially.
b. If the confidence value is low, since we now have better idea of where the centre could be located (as the sample is assumed to be a target), place the third cluster close to the predicted centre. Calculate the target value. Although this isn't quite reliable in real experimental conditions where there can be noise.

   To improve accuracy, add a fourth cluster close to the computed centre. We now have improvement in confidence values as there are two clusters very close to the centre. The target value is calculated again considering the fourth probe. We can always go on adding more clusters to improve accuracy as long as we have very high confidence value.

Also, if there is a change in position of the centre, it indicates that the wave pattern is not a target, as target waves have the same centre.

The whole process requires about 3 - 5 clusters, which equates to about 9 - 15 probes.

We try to increase the confidence value throughout the algorithm till it reaches a certain threshold.

Experiments and Results

Testing the algorithm on samples generated by simulation.

Experiment 1: 3 clusters, 9 probes, fixed pattern.

For simplicity, we just used three clusters here. The first two clusters were used to find the centre and Third cluster was used as the confirming cluster to measure the target value. Here we did not bother about the confidence value.

A spiral wave and a target wave were generated using the Barkley Model with parameters a = 0.53, b = 0.05 and ε = 0.02[3]. The dimension of the medium was 300 × 300. We placed the clusters in a particular pattern. The distance between the probes within the cluster was taken to be h = 10.The distance between the clusters was given by the parameter d.

For simplicity, the first, second and third clusters were placed at (x_1, y_1), (x_1, y_1+d) and (x_1+d, y_1+d) respectively as shown in figure 9.

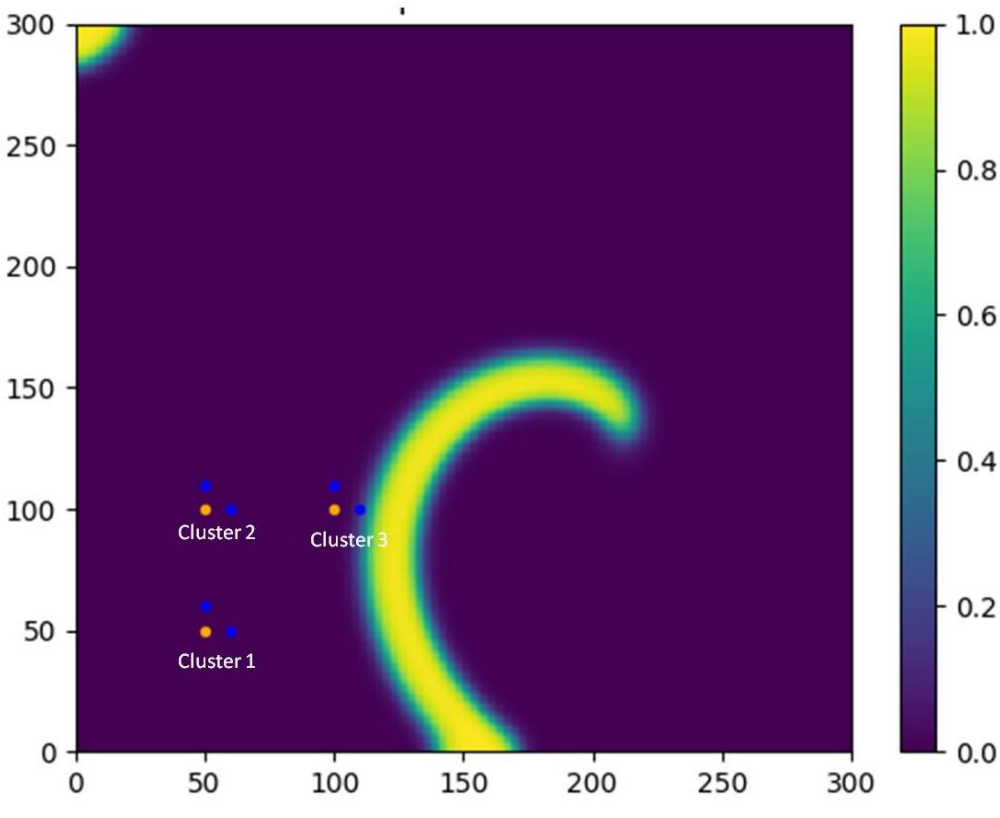

Figure 9: The sample in the above example is a spiral. The clusters are spaced in the format (x_1, y_1), (x_1, y_1+d) and (x_1+d, y_1+d), [5]

Spiral Sample

For the spiral, we placed the 1st cluster at (50, 50). The distance between the clusters was d = 50. The Target Value Threshold was 0.95. If target value was above 0.95, it will be classified as a target.

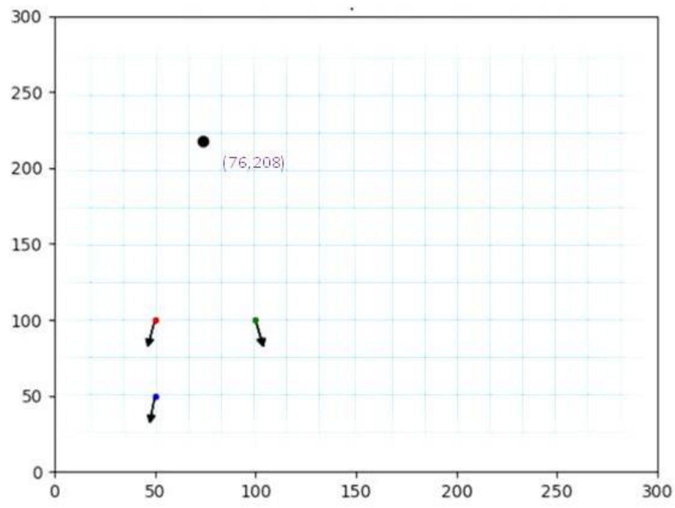

Figure 10: The direction of the wave at the three clusters. First cluster is at (50, 50). Probes are not displayed here. The expected direction vector k_c isn't visible in this case as it is almost in the same direction as k_3. [5]

The generated sample had 15 wave passes. Average Target Value was 0.999854. But, the confidence value was 0.001036. Predicted centre if the sample is a target, was (76.540442, 208.113270). Even though this would be identified as a target, the confidence is very low.

We tried another location, but this time we moved the clusters close to the expected centre. The first cluster was placed at (60,195) with the same distance parameter, d = 50.

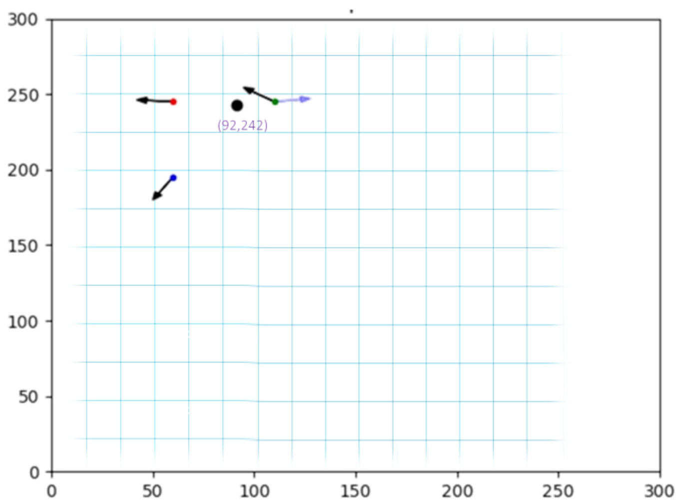

Figure 11: The direction of the wave at the three clusters for one of the wave passes. First cluster is at (60, 195). Probes are not displayed here. [5]

The generated sample again had 15 wave passes. Average Target Value was 0.111470 and the confidence value was 0.250012. Predicted centre if the sample is a target was (92.390373, 242.573360).

The centre had changed. The Target Value was as low as 0.111470. Hence we can conclude that this indeed was a Spiral wave.

In order to see the results for various locations, the cluster placement was done at many other points. We get wave directions at different locations.

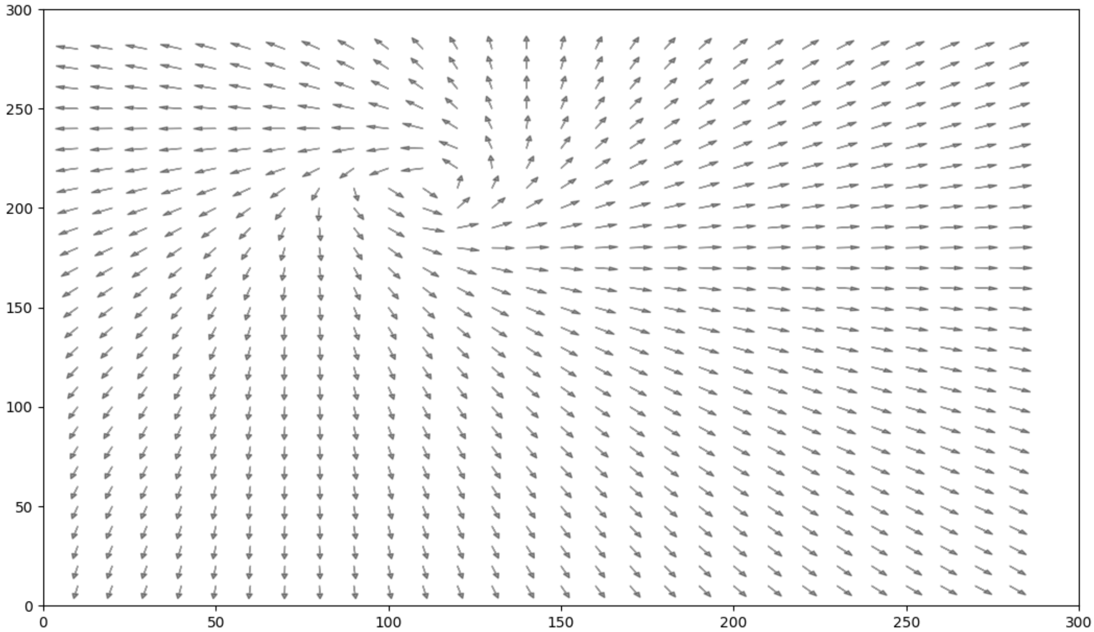

Figure 12: The direction of the wave at many points in the medium. A cluster was placed at each of those points. [5]

We placed the clusters in the same pattern (fixed distance parameter d, between the clusters) at various points in the medium.

The region enclosed by the clusters used for a given analysis is called an Analysis Region.

If it was identified as a Target, we colour the region of the probes in blue, and if it wasn't, we colour the region red. The resulting plot is shown in Figure 13.

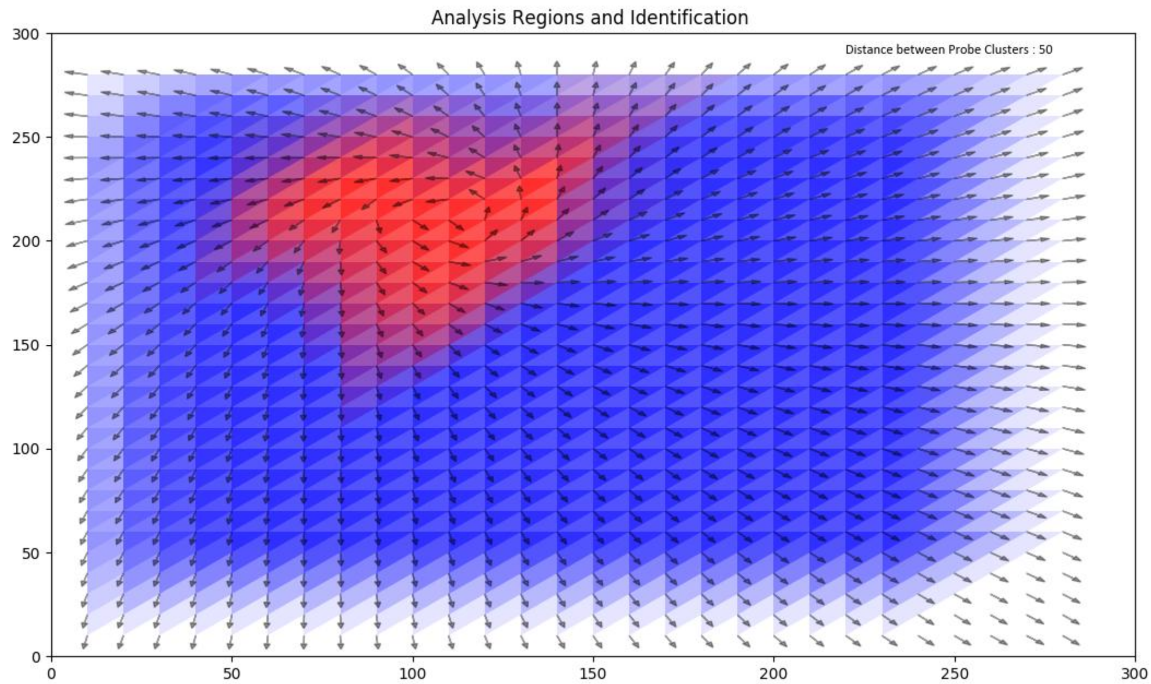

Figure 13: Clusters are kept at specific regions. If the value is target value is above 0.95, they are identified as Target and whole region between the clusters called the 'Analysis Region' is shaded blue. Else, it is shaded red. Distance parameter d = 50. [5]

It can be seen that, it is identified correctly as a spiral at some distance from the centre of the spiral. For the analysis, the distance parameter was d = 50.

The position of the centre calculated from each analysis region was plotted. The resulting plot

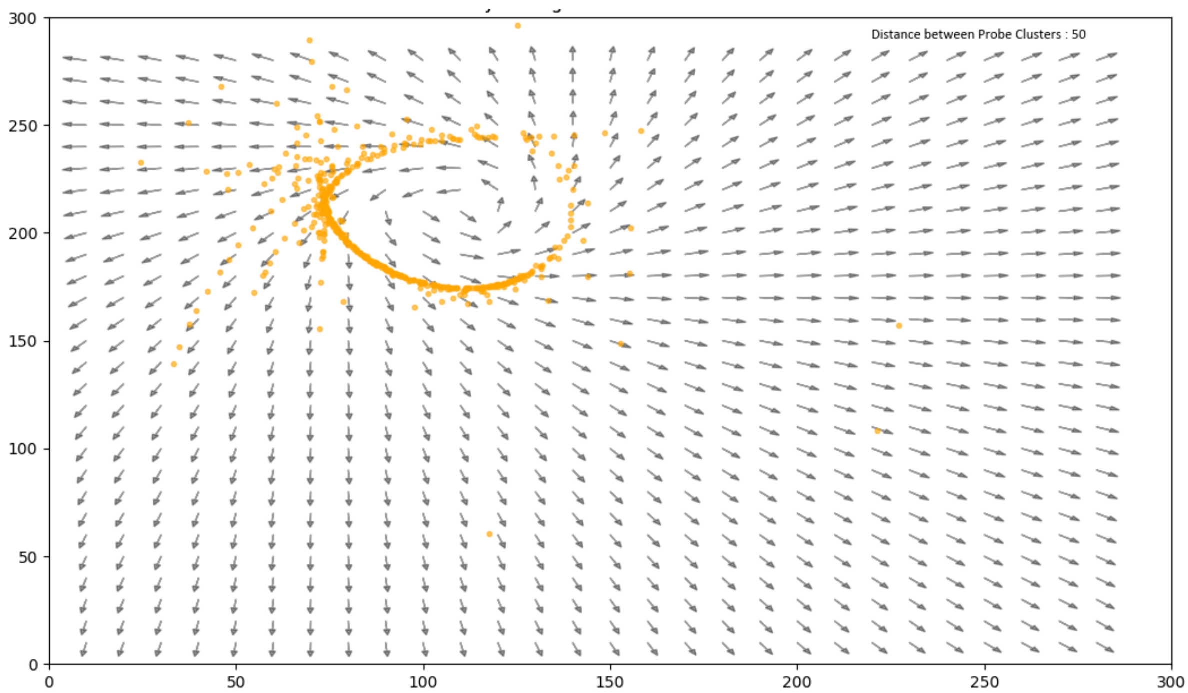

Figure 14: Each dot is a centre computed from different analysis regions. Distance parameter d = 50. [5]

The identifications for distance parameter d = 150 were

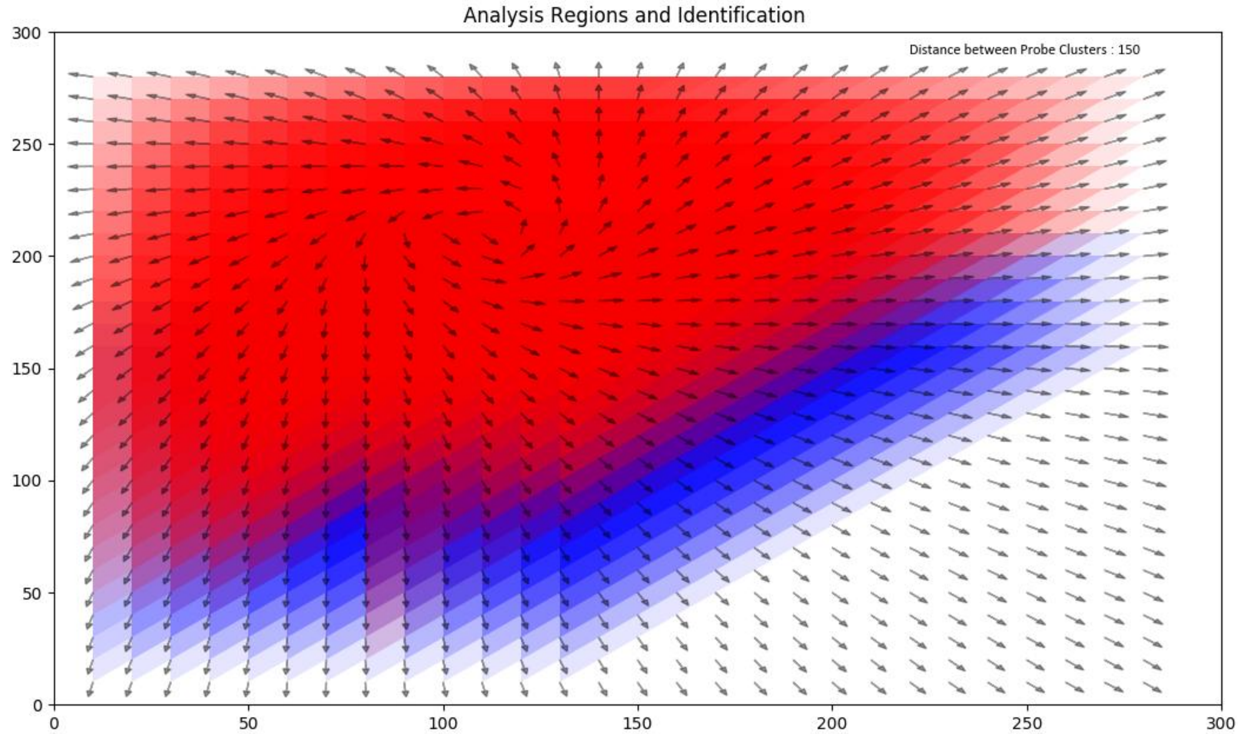

Figure 15: Clusters are kept at specific regions. If the value is target value is above 0.95, they are identified as Target and whole region between the clusters called the 'Analysis Region' is shaded blue. Else, it is shaded red. Distance parameter d = 150. [5]

Target Sample

Clusters were placed in the same pattern. The first cluster was at (50, 50).

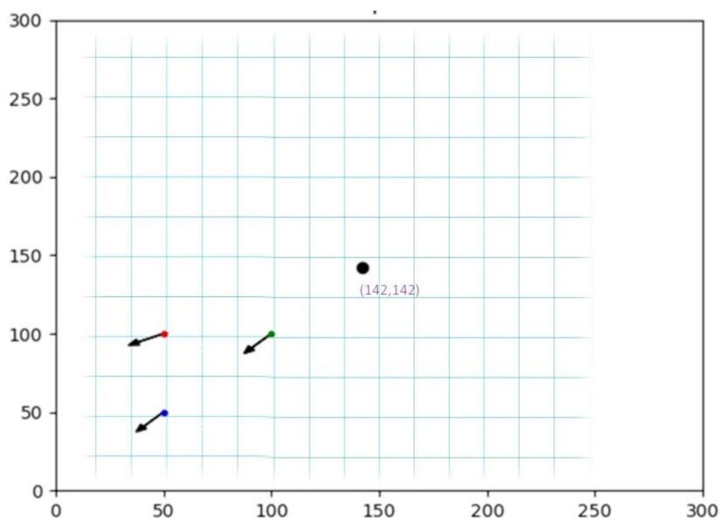

Figure 16: The direction of the wave at the three clusters. First cluster is at (50, 50). Probes are not displayed here. The expected direction vector k_c isn't visible in this case as it is almost in the same direction as k_3. [5]

For the Target, Its centre was actually at (150, 150). The Target Value Threshold was 0.95. The generated sample had 15 wave passes. Average Target Value was 1. But, the confidence value was 0.031351. Predicted centre if the sample is a target was (142.198582, 142.198582). Even though this would be identified as a target, the confidence is low.

We tried another location. We moved the clusters close to the expected centre. The first cluster was placed at (147,125) with the same distance parameter, d = 50.

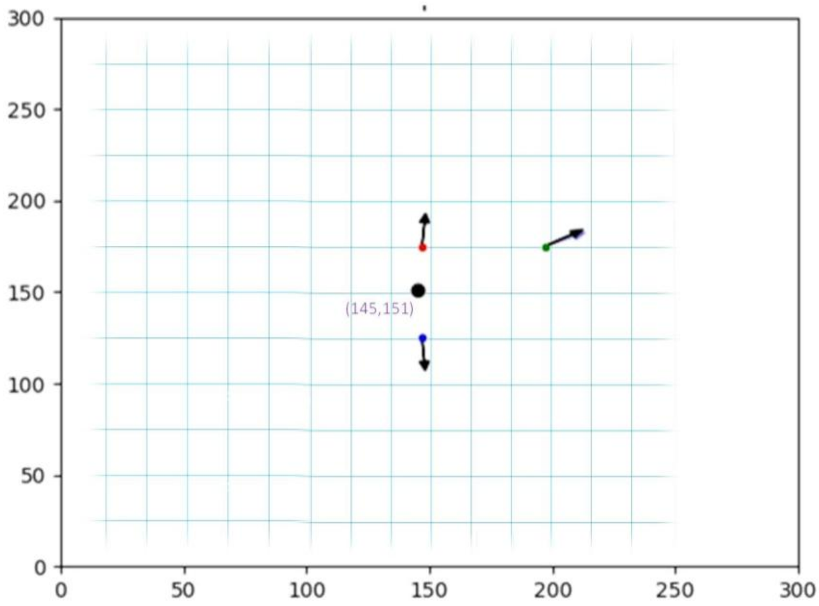

Figure 17: The direction of the wave at the three clusters. First cluster is at (147, 125). Probes are not displayed here. The expected direction vector k_c isn't visible in this case as it is almost in the same direction as k_3. [5]

The generated sample had 15 wave passes. Average Target Value was 0.997562 and the confidence value was 0.994134. Predicted centre if the sample is a target was (145.083599, 151.128766).

The centre hadn't changed much. The Target Value was close to 1. Hence we can conclude that this indeed is a Target wave.

Clusters were placed at different regions and the wave directions were calculated at those points. Resulting plot is

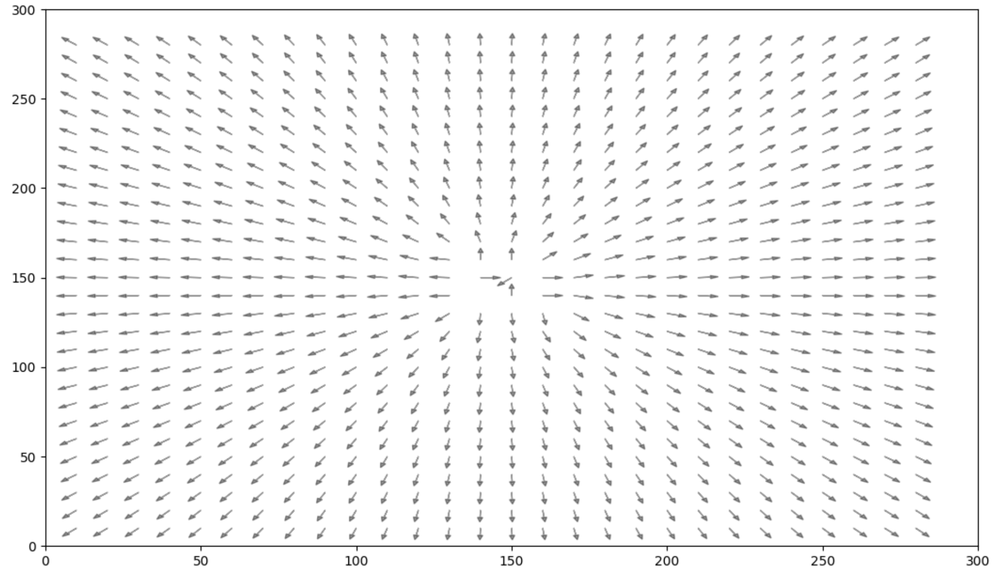

Figure 18: The direction of the wave at many points in the medium. A cluster was placed at each of those points. [5]

The cluster placement was done in the same pattern (fixed distance parameter d, between the clusters) at various points in the medium. If it was identified as a Target, we coloured the region of the probes in blue, and if it wasn't, we coloured the region red. The resulting plot is shown in Figure 19.

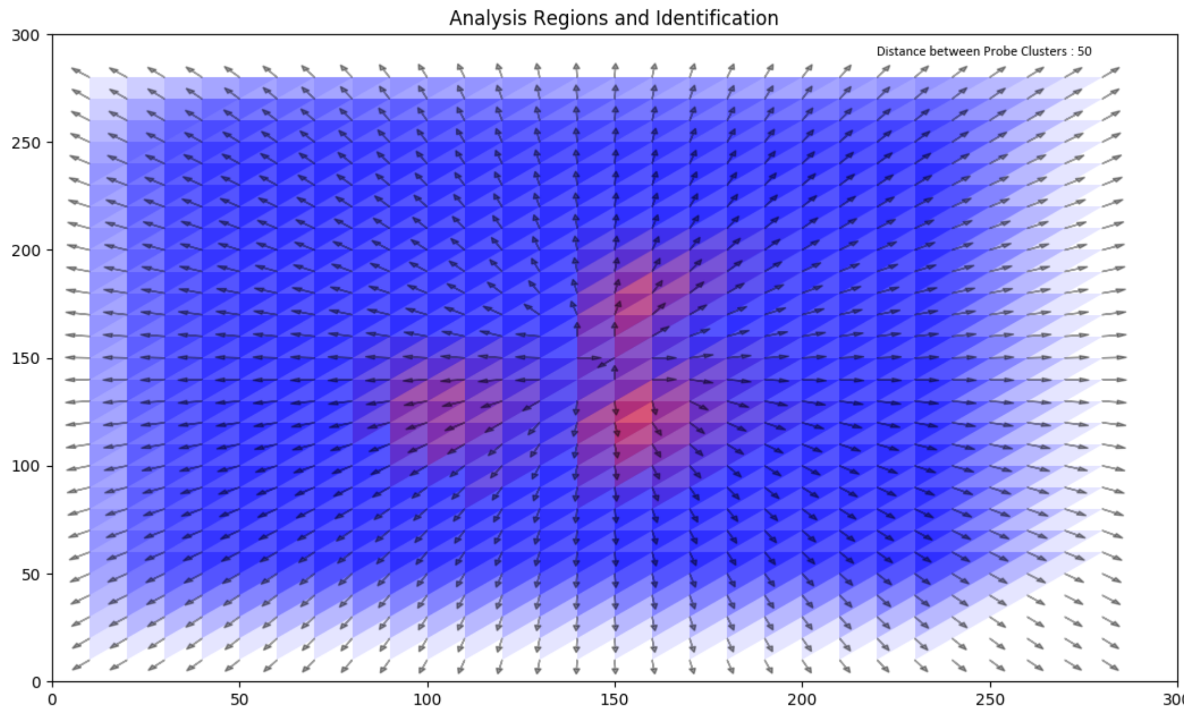

Figure 19: Clusters are kept at specific regions. If the value is target value is above 0.95, they are identified as Target and whole region between the clusters called the 'Analysis Region' is shaded blue. Else, it is shaded red. Distance parameter d = 150. [5]

It can be seen that, it was identified correctly as a target at all distances from the centre. But, at the centre there was some confusion. This is because it is hard to calculate the direction of the wave at the centre. In the analysis, the distance parameter was d = 50.

The position of the centre calculated from each analysis region was plotted. The resulting plot

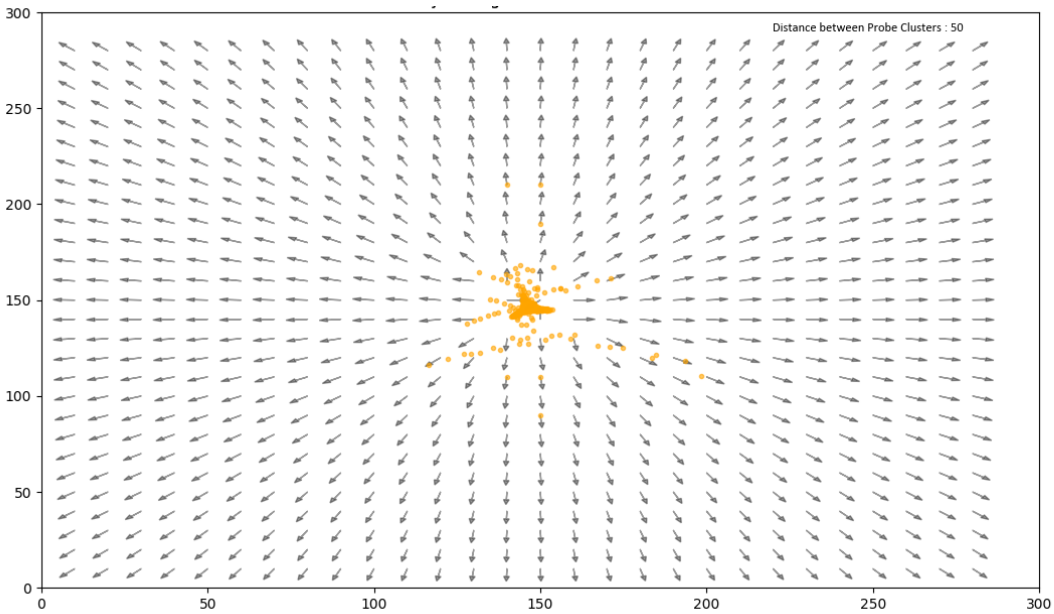

Figure 20: Each dot is a centre computed from different analysis regions. Distance parameter d = 50. [5]

The identifications for distance parameter d = 150, we get

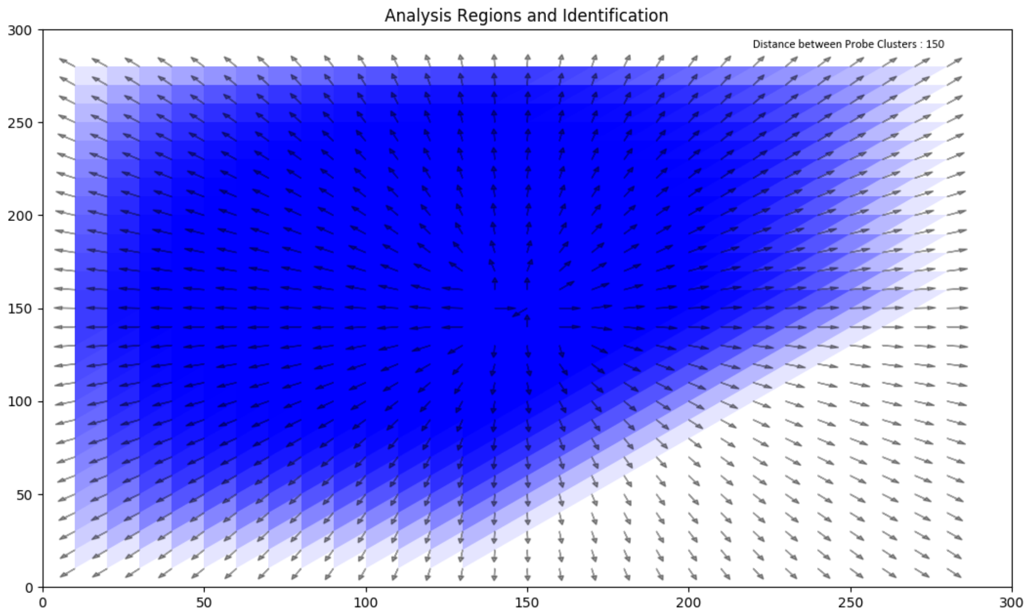

Figure 21: Clusters are kept at specific regions. If the value is target value is above 0.95, they are identified as Target and whole region between the clusters called the 'Analysis Region' is shaded blue. Else, it is shaded red. Distance parameter d = 150. [5]

Experiment 2: 3 clusters, 9 probes, random arrangement with certain constraints.

Like the previous ones, we just used three clusters here. The first two clusters were used to find the centre and the third cluster was used as the confirming cluster. Here, there was no fixed pattern. But, there was a constraint on how far the clusters are from one another. The clusters were confined to be at a distance from one another. This distance was between a certain upper threshold and above a certain minimum, a lower threshold.

The clusters were placed within 60 units from one another, but above 40 units. This gave us an approximate equilateral arrangement.
The Target value threshold was 0.95.
The Target value was computed in each case and the region between the clusters, (also called Analysis region) was shaded blue if it was identified as a Target and shaded red if it was not.

To get an idea of the shape of the analysis regions, we only have 10 analysis regions here. The resulting plot was,

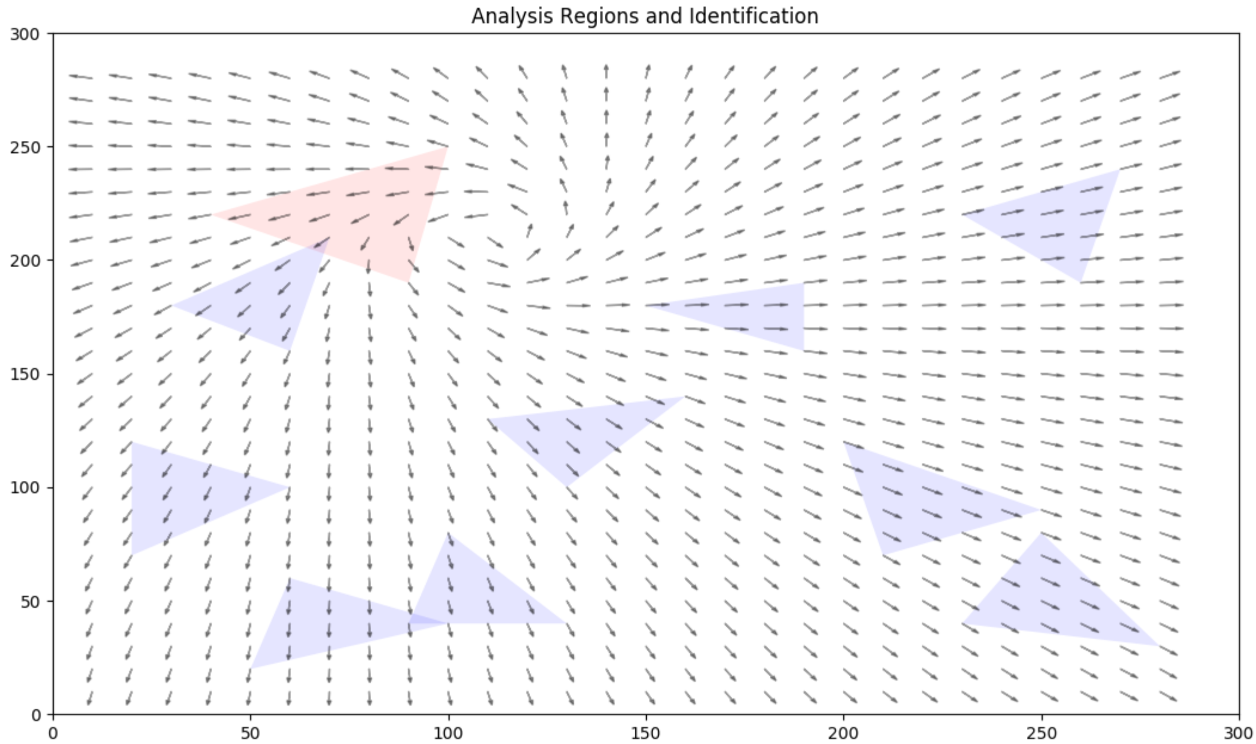

Figure 22: Analysis Regions of roughly equilateral shape. Shaded blue if identified as target and red if not. Target Value threshold was 0.95. There are only 10 analysis regions here. [5]

If we plot 1000 Analysis Regions which are randomly picked, but are of an approximate equilateral shape, then we get

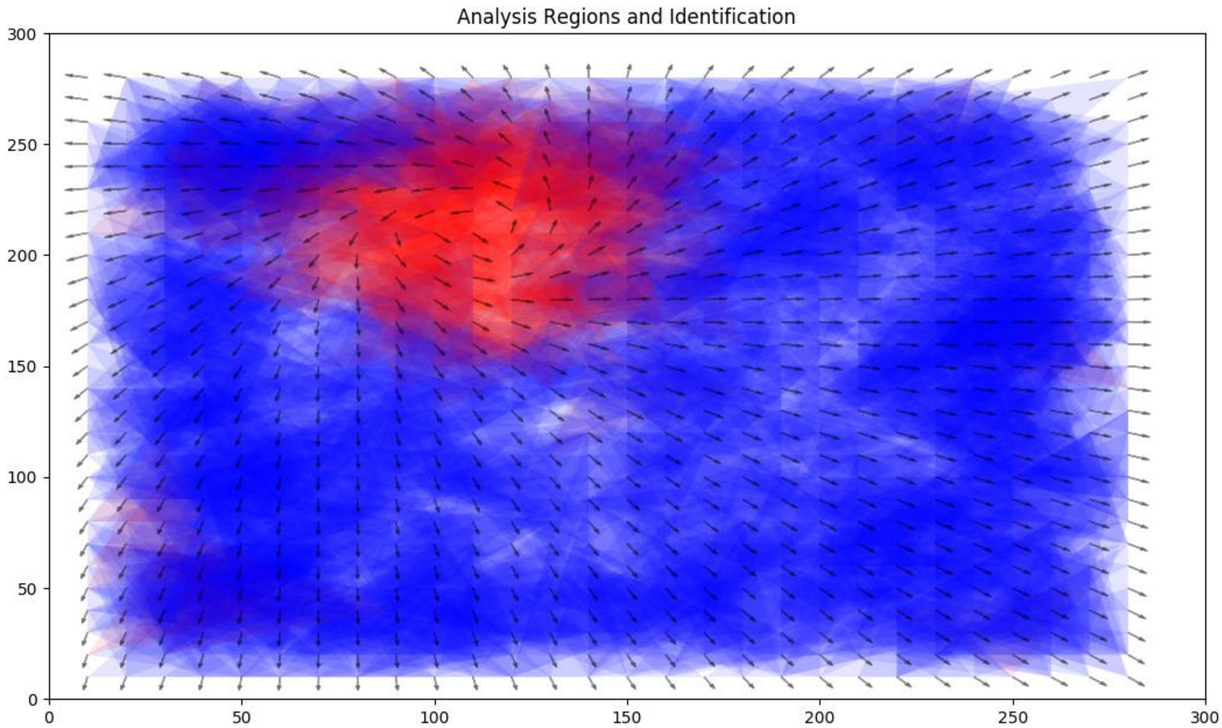

Figure 23: Analysis Regions of roughly equilateral shape. Shaded blue if identified as target and red if not. Target Value threshold was 0.95. There are 1000 analysis regions here. [5]

This was a spiral. For a target sample, we got

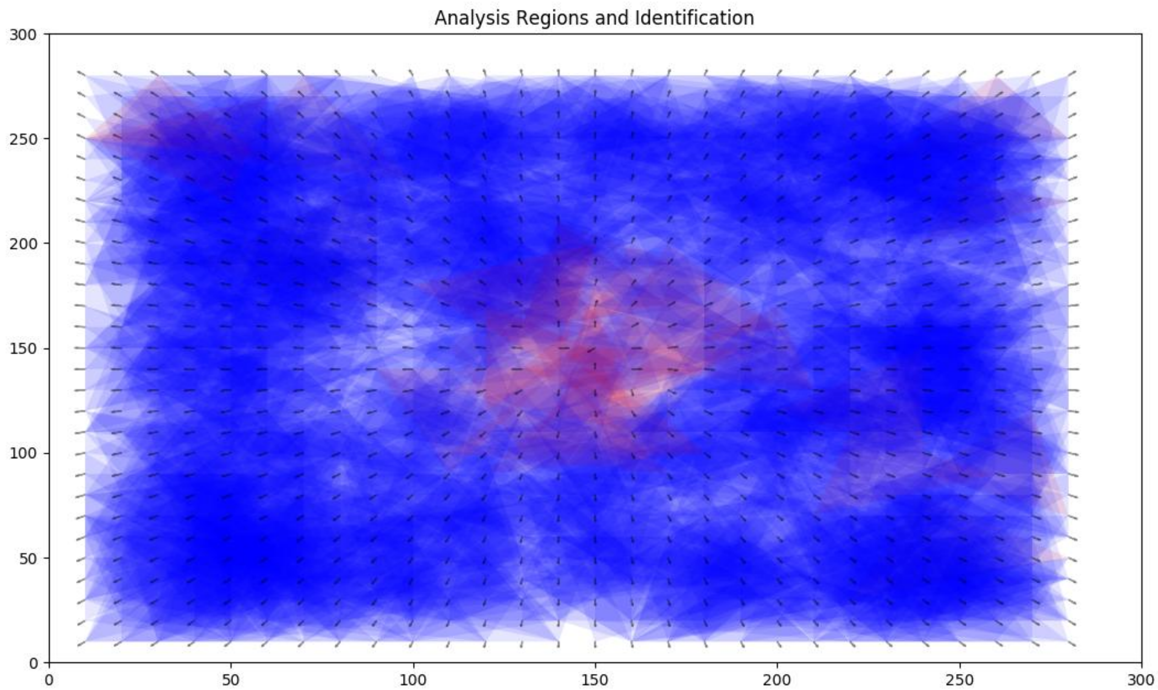

Figure 24: Analysis Regions of roughly equilateral shape. Shaded blue if identified as target and red if not. Target Value threshold was 0.95. There are 1000 analysis regions here. [5]

Earlier, the clusters were placed within 60 units from one another, but above 40 units. Now they were placed within 120 units, but above 100 units.

For the Spiral, the resulting plot is,

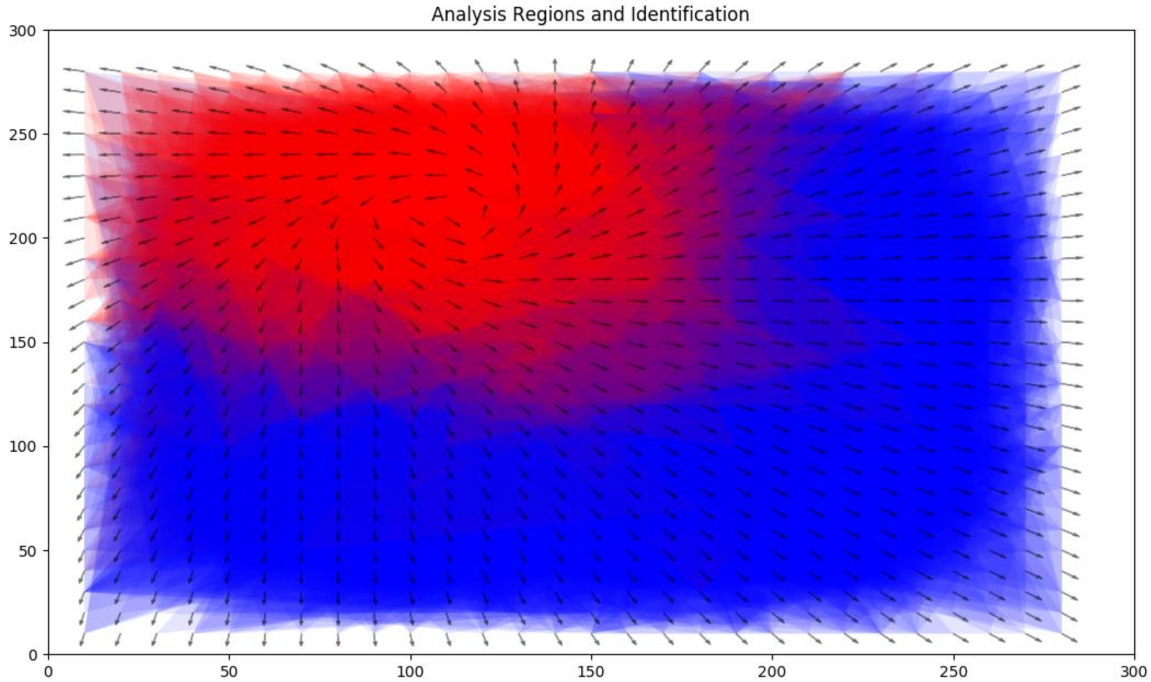

Figure 25: Analysis Regions of roughly equilateral shape. Shaded blue if identified as target and red if not. Target Value threshold was 0.95. There are 1000 analysis regions here. [5]

For a Target, we got

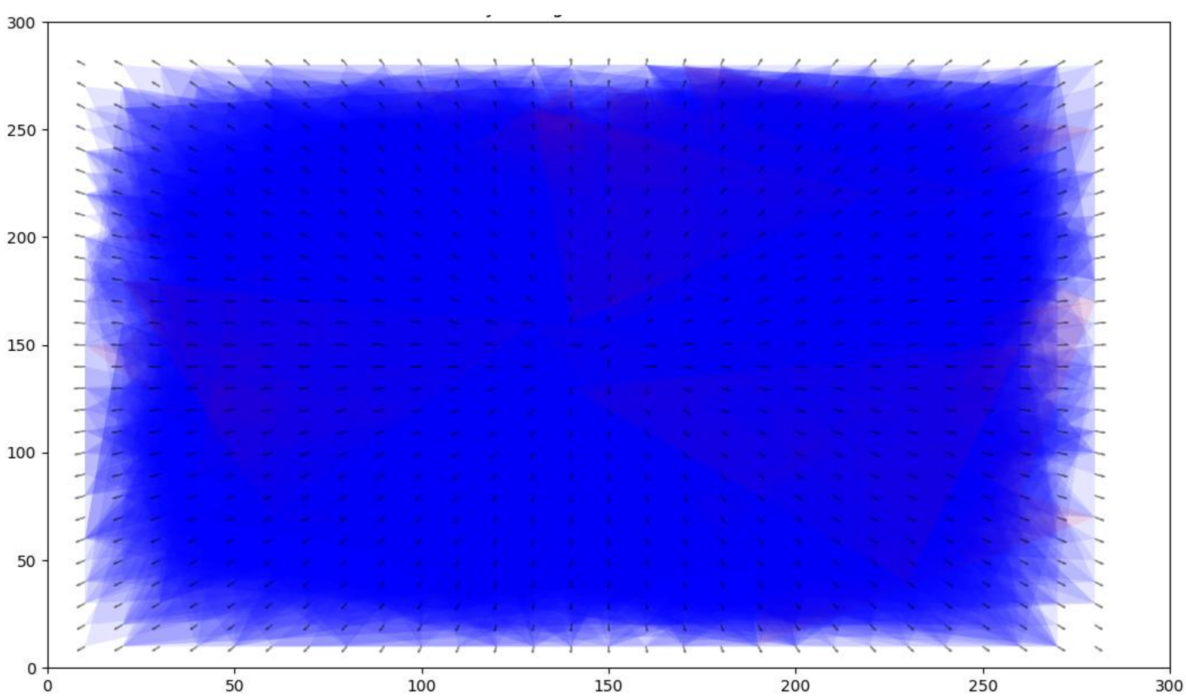

Figure 26: Analysis Regions of roughly equilateral shape. Shaded blue if identified as target and red if not. Target Value threshold was 0.95. There are 1000 analysis regions here. [5]

We need to define two terms, Effective Distance and Effective Region.
Effective Distance is the minimum distance the probes must be placed from the centre of the wave pattern so as to be identified correctly, because beyond a certain distance for the centre, the wave pattern appears to be target.
Effective Region is the region whose points are within the Effective Distance from the centre of the wave pattern.

If we compare the plots for smaller analysis regions and larger ones, it's clear that the effective region goes on widening with larger analysis regions. So, in order to get better identification, the clusters must be placed a larger distances and hence have larger analysis regions.

Conclusions

The Algorithm does work, but is effective only within a certain region.

We have tried the simpler version of the algorithm with only three clusters being placed. And in this version, we have realized that the algorithm can identify the pattern effectively within a certain region.

But, there are parameters which we can change. We must note that we used Target Threshold Value of 0.95. This can be increased. It increases the effective region. Another factor that can increase this region is the average distance between the clusters. As we increase this distance, the effective region grows.

Of the two factors, we can say that increasing Target Value threshold beyond a certain value doesn't improve the analysis region since it value cannot be increased beyond 1. But, the distance between the clusters could be increased for greater distances. But, will the effective region grow indefinitely? It appears so, but, this cannot be concluded. It's only a conjecture.

Another important realization one has in these experiments is that, "If you are looking at a region of the medium and cannot distinguish if it's a spiral or a target, it's most likely that even a machine cannot." Even with data at all the points of a giving region, if you cannot distinguish the pattern, the machine most likely won't which implies, you can never come up with an algorithm which will.

This brings us to another point. We used 3 clusters, hence 9 probes. What is the minimum number of probes? This is more of a problem of an information limit. What is the minimum information necessary to distinguish the patterns? For the algorithm we have developed, it turns out to be 9 probes unless we are very far from the centre of the wave pattern. We can increase number of the probes for accuracy.

We need to also note that we just used the arrival times/activation times to understand the pattern, ignoring all the rhythmic data that arrives at the probe. We may have just ignored some useful information. But, for now, capturing activation points at 9 locations has done well for us.

Future Development Scope and Prospects

Many parameters were discussed, not all variations of them were studied. Also, the algorithm has general applications beyond excitable media.

As we saw earlier, we have defined parameters such as Target Value Threshold, confidence, distance between clusters. This brings into question, what are most optimal parameter values to get better identification? The answer is that it depends on the situation presented, the type of medium etc. This can be studied for better understanding.

Classification is an important problem in machine learning. It's where machine models find the best set of parameters learning from certain known classifications and then classify samples presented before it. Machine learning can be used to automate the identification algorithm by choosing parameters such as Target Value Threshold etc.
Machine Learning can be used to reduce errors in experimental setups. Since there can be noise in the data acquired from a real excitable medium such as the heart tissue.

We did make a conjecture that the Effective region for the algorithm increases in size with increasing size of the analysis region. But, this may not be true, and this needs verification.

Although it appears to have applications only to excitable media, the algorithm has general applications and can identify wave patterns in any media. For example, it's hard to identify if the waves emitted by a pulsar in a distant galaxy are radial or spiral.[1] Placing clusters/probe/detectors in particular ways could help us with this problem. But, placing probes/detectors in distant galaxies is harder.

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[4] Graph Plots for figures 4 – 8 were done using Desmos Online Graphing Calculator.
(https://www.desmos.com/calculator)

[5] Rest of the figures were plotted using Python's module Matplotlib.
(https://matplotlib.org/)

[6] Coding Language: Python and its modules numpy and scipy.
(https://www.python.org/)