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PS1 Report

Problem 1 Proofs

1. **Associativity**

$$
\begin{aligned}
(f*g)*h &= f*(g*h) && \text{Definition of Associativity}\\
&= ((g*h)*f)(t) && \text{rearrange and expand}\\
&= \int_0^t ((g*h)(s)f(t-s)\,ds && \text{substitute definition}\\
&= \int_0^t \left(\int_0^s g(v)h(s-v)\,dv\right) f(t-s)\,ds && \text{substitute definition}\\
&= \int_0^t g(v)\left(\int_0^{t-v} h(s-v)f(t-s)\,ds\right)\,dv && \text{swap w/ Fubini's}\\
&= \int_0^t g(v)\left(\int_0^{t-v} h(u)f(t-v-u)\,du\right)\,dv && \text{simplify, let u = s-v}\\
&= \int_0^t g(v)(h*f)(t-v)\,dv && \text{definition}\\
&= g*(h*f)
\end{aligned}
$$

2. **Distributivity**

$$
\begin{aligned}
f*(g+h) &= f*g+f*h && \text{Definition of Distributivity}\\
&= (f*g)(t)+(f*h)(t) && \text{expand}\\
&= \int_{-\infty}^{\infty} f(s)g(t-s)\,ds + \int_{-\infty}^{\infty} f(s)h(t-s)\,ds && \text{substitute definition}\\
&= \int_{-\infty}^{\infty} f(s)(g(t-s)+h(t-s))\,ds && \text{combine integrals}\\
&= \int_{-\infty}^{\infty} f(s)((g+h)(t-s))\,ds && \text{substitute definition}\\
&= (f*(g+h))(t)
\end{aligned}
$$

3. **Differentiation Rule**

$$
\begin{aligned}
(f * g)' &= f' * g = f * g' \\
F(f * g) &= F(f * g) \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} f(s)g(t-s)\,ds && \text{expand} \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} g(t-s)\,ds && \text{Leibniz rule} \\
&= \int_{-\infty}^{\infty} f(s)g'(t-s)\,ds && \text{simplify} \\
&= f * g' && \text{substitute definition}
\end{aligned}
$$

4. **Convolution Theorem**

$$
\begin{aligned}
F(f * g) &= F(f)F(g) = \hat{f}\hat{g} \\
&= F^{-1}(\hat{f}\hat{g})(x) && \text{inverse Fourier transform} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{i\omega x}\,d\omega && \text{substitute definition of} \\
& && \text{inverse Fourier Transform} \\
&= \int_{-\infty}^{\infty} g(y)\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega(x-y)}\,d\omega\,dy && \text{Fubini's Theorem} \\
&= \int_{-\infty}^{\infty} g(y)f(x-y)\,dy && \text{Substitute definition of} \\
& && \text{inverse Fourier Transform} \\
& && \text{for } f \text{ at } (x-y) \\
&= (g * f)(x) && \text{definition}
\end{aligned}
$$

Problem 2

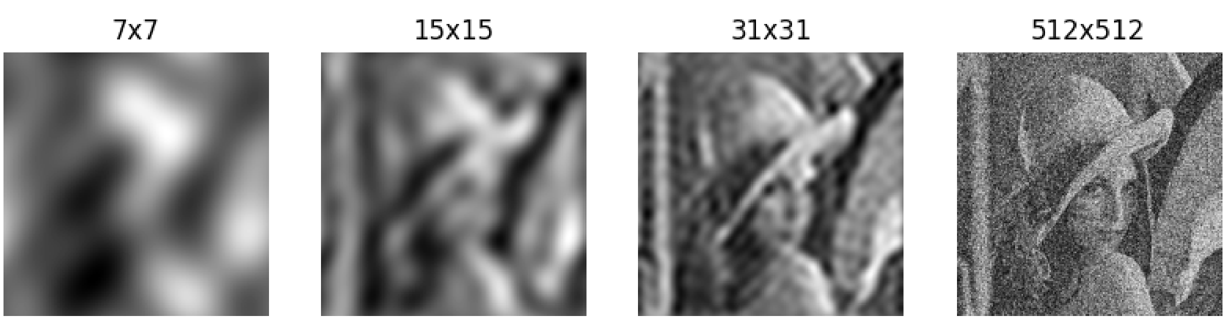

Figure 1: Lenas w/ different Filter sizes!

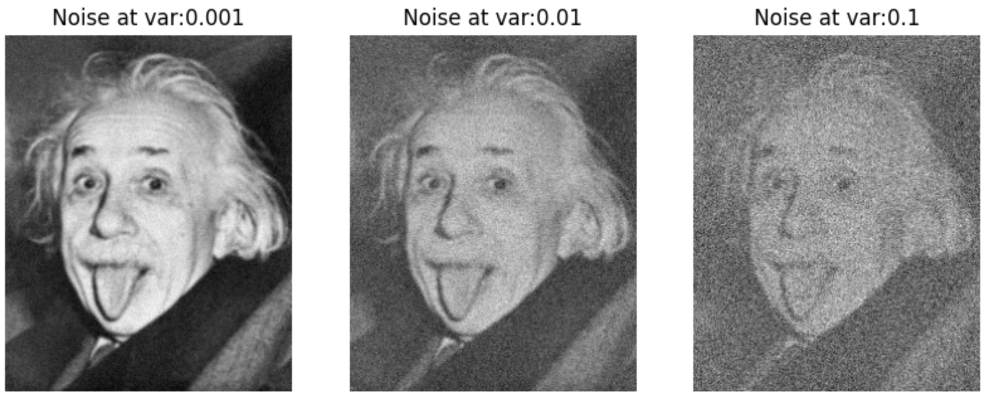

Figure 2: Einsteins with Gaussian Noise added at different Variances!

Problem 3

The objective function that we are trying to minimize is:

$$min(E(u)) = \lambda \|f - u\|^2_{L^2} + \int_{\Omega} \|\nabla u_i\| \tag{1}$$

where λ is the regularization parameter, u is the denoised image, f is the observed noisy image, Ω is the domain of the image, and $|\nabla u|$ is the total variation of u.

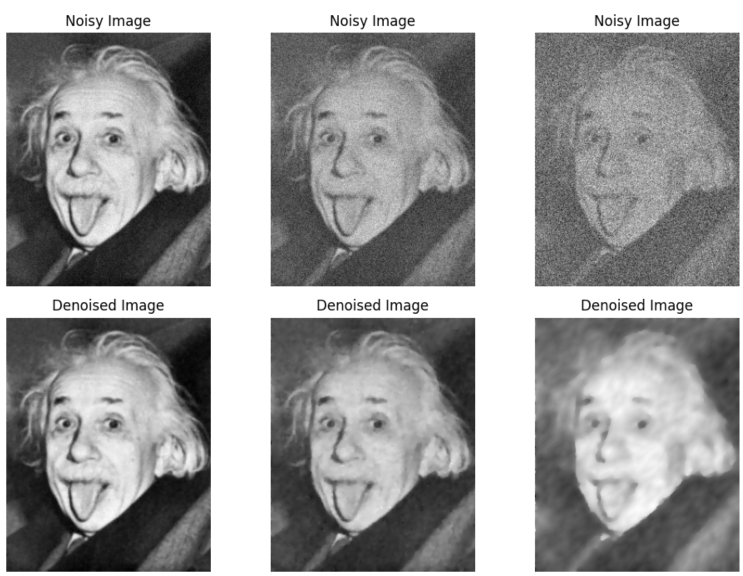

Figure 3: Noisy Einsteins with added Noise on top and denoised Einsteins after running Total Variation Denoising Algorithm. The hyperparameters used were $\lambda = 0.1$, learning rate = 0.01, tolerance = $1\text{x}10^{-8}$.

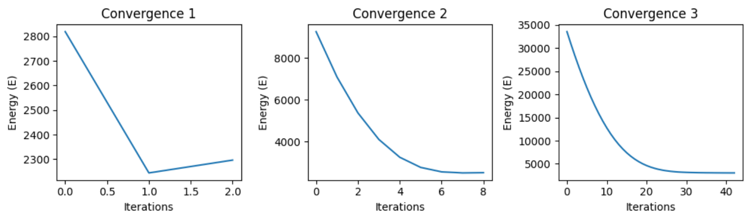

Figure 4: Graphs of calculated energy vs number of iterations for the respective Einstein images above.