Graph Theory

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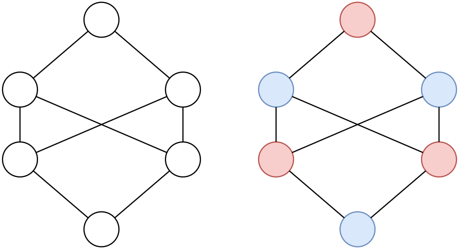
4 Coloring

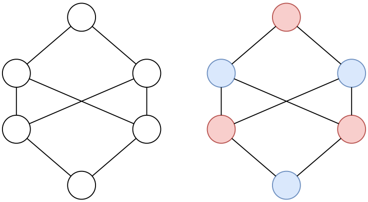
- Lower and upper bounds
- Colouring maps and planar graphs
- Colouring vertices
- List colouring
- Applications in network traffic and shunting

- In the *graph coloring problem*, one has to assign a color to each node ...
- ... so that every two nodes that are adjacent have a different color.
- A coloring is a mapping $f : V \rightarrow \{1, ..., n\}$ with

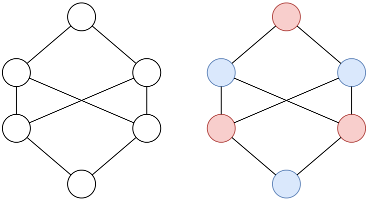
$$f(u) \neq f(v) \; \forall (u, v) \in E$$

- We want to use a minimum number of colors.
- The minimal number of colors needed to color a graph is called *chromatic number* and denoted by $\chi(G)$.





- This problem has many applications and has been studied extensively.
- Related problems are, for example, the *graph partition* or the *graph covering* problem.
- In the context of the graph coloring problem, in every feasible coloring of G, all nodes sharing the same color imply a stable set in G.
- Thus we have a partition of G into stable sets.
- But it is also possible to use a set covering formulation, where the set of vertices has to be covered by a minimum number of stable sets.
- Many optimization problems on graphs can be formulated as set covering problems.



- T. R. Jensen, B. Toft: Graph Coloring Problems (Wiley, 1995)
- We will come back to this topic when discussing Perfect Graphs.

Example 4.1

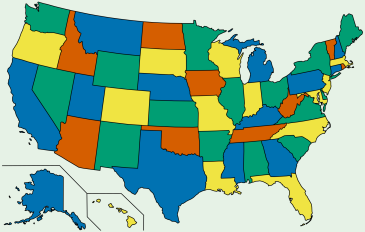


Figure 3: https://commons.wikimedia.org/wiki/File:Map_of_United_States_accessible_colors_shown.s

- How many colors are required to color the regions of any map so that no two adjacent regions have the same color?

Example 4.2

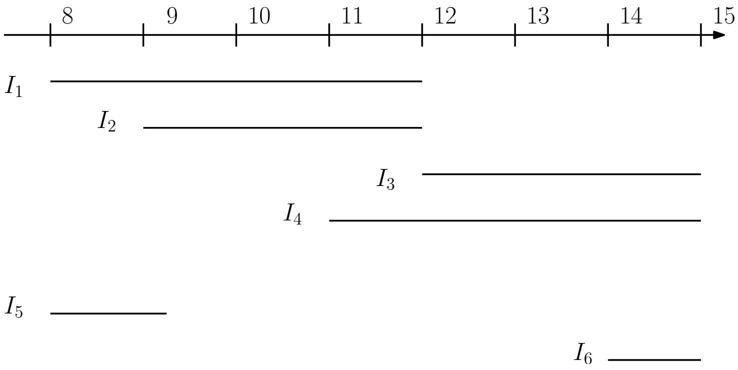
- Let S be a set of students and V be a set of lectures.
- For $v \in V$, let $h(v) \subseteq S$ be the set of students who want to take an exam on v.
- Goal: We want to organize a minimal number of exam dates.
- Two exams may overlap only if no examinee wants to participate in both of them.
- We define the graph:

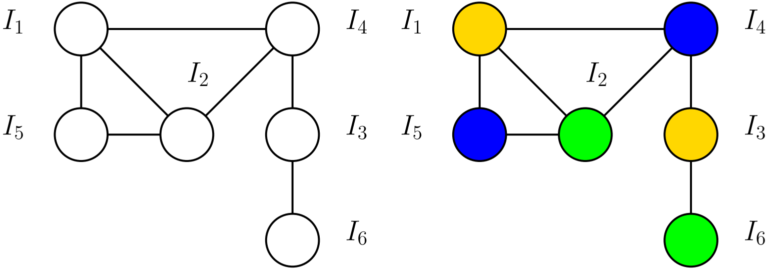
$$G = (V, E) \text{ with} (u, v) \in E \Leftrightarrow h(u) \cap h(v) \neq \emptyset$$

- The minimum number of test dates is equal to $\chi(G)$.

- Exams are held at one school. The math exam takes place from 8-12, the German exam from 9-12, the English exam from 12-15 and the geography exam from 11-15. Further, a room is needed for a conference from 8-9 and for a adult education course from 14-15. How many rooms does the school need to provide?
- We construct the associated interval graph. The intervals are

$$
\begin{aligned}
I_1 &= [8, 12] \\
I_2 &= [9, 12] \\
I_3 &= [12, 15] \\
I_4 &= [11, 15] \\
I_5 &= [8, 9] \\
I_6 &= [14, 15]
\end{aligned}
$$


8
9
10
11
12
13
14
15
I_1
I_2
I_3
I_4
I_5
I_6


I_1
I_4
I_2
I_5
I_3
I_6
I_1
I_4
I_2
I_5
I_3
I_6

- We assume simple graphs, because a graph with loops is not colorable and multiple edges do not influence the colorability.

Observations:

- Given a graph G with k nodes:
  - $\chi(G) = k \Leftrightarrow G = K_k$
  - $\chi(G) = 1 \Leftrightarrow G = \overline{K_k}$
- $\chi(G) \leq 2 \Leftrightarrow G$ is bipartite
- $\chi(G) \geq 3 \Leftrightarrow G$ contains a path with odd length
- $\chi(C_n) = 3$, if $n \geq 3$ and odd
- $\chi(H) \leq \chi(G)$ for every subgraph $H \subseteq G$

- Computing $\chi(G)$ is $\mathcal{NP}$-complete.
- ‘Given a fixed $k \geq 3$, can we color a graph with exactly k colors?’ is $\mathcal{NP}$-complete.

Lemma 4.3

$$\chi(G) \geq \omega(G)$$

Lemma 4.4

Given a graph G with m edges:

$$\chi(G) \leq \sqrt{2m + \frac{1}{4}} + \frac{1}{2}$$

Proof.

- Let f be a coloring with $k = \chi(G)$ colors.
- Between two colors at least one edge exists. Thus there are $\binom{k}{2}$ possible permutations of pairs of colours, and since any edge cannot connect more than 2 colours, that means that there must be $\binom{k}{2}$ edges of the graph.
- Thus $m \geq \binom{k}{2} = \frac{1}{2}k(k-1)$.
- $\Leftrightarrow 2m \geq k^2 - k$
- $\Leftrightarrow 2m + \frac{1}{4} \geq k^2 - k + \frac{1}{4}$
- $\Leftrightarrow 2m + \frac{1}{4} \geq (k + \frac{1}{2})^2$

Algorithm 5 Heuristic for Node-Coloring

1: number all nodes $v_1, ..., v_n$
2: color v_1 with color 1
3: **for** $i = 2$ to n **do**
4: color v_i with the smallest possible color not in $N(v_i)$

Lemma 4.5

Given a graph G $\chi(G) \leq \Delta(G) + 1$.

Lemma 4.6

Given a graph G $\chi(G) \leq \Delta(G) + 1$.

Proof.

- Algorithm 5 provides a feasible coloring.
- In G any node has not more than $\Delta(G)$ neighbors.
- Thus, we need not more than $\Delta(G) + 1$ colors.

$\square$

Lemma 4.7 (Szekeres-Wilf, 1968)

$$\chi(G) \leq 1 + \max\{\delta(H), H \text{ (spanned) subgraphs of } G\}$$

Lemma 4.8 (Brooks, 1941)

Let G be a connected graph with $n \geq 2$ vertices that is neither complete nor an odd circle. Then $\chi(G) \leq \Delta(G)$ holds.

Lemma 4.9 (Reed, 1998)

Let G be a graph with n nodes:

$$\chi(G) \leq \left\lfloor \frac{n + \omega(G)}{2} \right\rfloor$$

Definition 4.10

Graphs that can be drawn in such a way that no two edges meet in a point other than a common end are called *plane graphs*; abstract graphs that can be drawn in this way are called *planar*.
This means a graph is called planar, if it is isomorphic to a plane graph

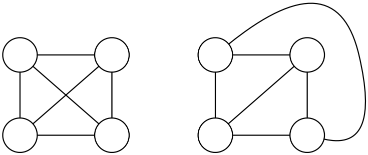


Figure 4: The K_4 is a planar graph.

- The plane graph which is isomorphic to a given planar graph G is said to be embedded in the plane.
- A plane graph isomorphic to G is called its drawing.

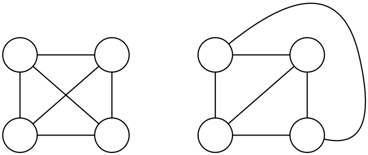


Figure 5: The K_4 is a planar graph.

- The faces of a plane drawing of a planar graph G are the regions of the plane that are separated from each other by the edges of G.

Theorem 4.11 (Euler)

Let G be a plane drawing of a connected planar graph. Let n, m, and f denote the number of vertices, edges, and faces. Then

$$n - m + f = 2 \tag{13}$$

Proof.

The proposition will be proved by induction on m.

- If $m = 0$, then G has one vertex, no edges, and one face:

$$n = 1,\ m = 0,\ f = 1$$

- So Equation 13 holds for the base case.
- Suppose Equation 13 is true for any graph with at most $m - 1$ edges.
- If G is a tree, then $m = n - 1$, and $f = 1$; so Equation 13 holds.
- If G is not a tree, let e be an edge of some cycle of G.
- Then G with the edge e removed is a connected plane graph with n vertices, $m - 1$ edges, and $f - 1$ faces. So

$$n - (m - 1) + (f - 1) = 2$$

  by the induction hypothesis.
- Hence $n - m + f = 2$.

□

Theorem 4.12 (Four color theorem)

Each planar graph can be colored with four colors.

- Despite numerous attempts, it took more than a hundred years until Appel and Haken found a first proof in 1976.
- This is controversial, since it consists to a large extent of a case discrimination of more than 1000 cases, which was accomplished with computer support.
- In 1997 the approach was simplified, but still consists of more than 600 individual cases.

Definition 4.13

A *map graph* is an undirected graph which represents the intersections of a finite set of regions in the euclidean plane.

- All map graphs are planar graphs.

Corollary 4.14

Let G be a plane drawing of a connected planar graph. Let n and m denote the number of vertices and edges and let $m \geq 3$. Then

$$m \leq 3n - 6$$

Proof.

Exercise. □

Corollary 4.15

Any planar graph contains a vertex of degree at most 5.

Proof.

- Assume every vertex has degree 6 or more.
- Then $6n \leq 2m$ since every vertex is the end of at least six edges.
- So $3n \leq m$.
- But then $3n \leq 3n - 6$ by corollary 4.14, a contradiction for all n.
- Therefore there must be at least a single vertex with degree at most 5.

□

Theorem 4.16

All planar graphs are 6-colorable.

Proof.

- Let G be a planar graph with $n \geq 6$ vertices.
- According to Corollary 4.15, G contains a vertex v of degree at most five.
- Remove v and all the edges connected to it.
- The graph left has $n - 1$ vertices, and is therefore 6-colorable.
- Return the vertex v and all the edges that were previously present (at most five of them) to the graph and color v a different color from vertices connected to it.
- Thus there exists a coloring of G with at most six colors.

□

Theorem 4.17

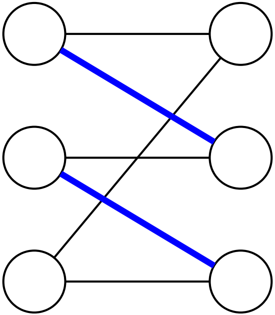
Each planar graph can be colored with fife colors.

Proof.

Exercise. □

Definition 4.18

A set M of edges without a common terminal node in a graph $G = (V, E)$ is called a *matching*.
M is a matching of $U \subseteq V$ if every vertex in U is incident with an edge in M. The vertices in U are then called matched (by M); vertices not incident with any edge of M are unmatched.



- *Colouring vertices* is a mapping $f : E \to \{1, ..., n\}$ with

$$f(e_1) \neq f(e_2) \text{ if } e_1, e_2 \text{ adjacent}$$

- An edge coloring thus decomposes the edge set into disjoint matchings.
- The chromatic index $\chi'(G)$ is the smallest number of colors with which G can be edge colored.

- With $\alpha'(G)$ we denote the size of a maximal matching in G.

Lemma 4.19

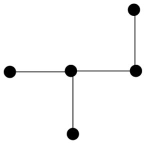
Given a graph G with $M \geq 1$ edges:

$$\chi'(G) \geq \frac{m}{\alpha'(G)}$$

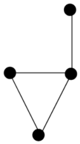
Proof.

Every color is a matching. $\square$

- Given a graph $G = (V, E)$
- we define $L(G) = (E, F)$ and
- $(e_1, e_2) \in F$ if they have a common node.
- $L(G)$ is called *line graph* of G.



G



$L(G)$

Observations:

- $\chi'(G) = \chi(L(G))$.

- Computing $\chi'(G)$ is $\mathcal{NP}$-complete.

Theorem 4.20 (König, 1916)

Given a bipartite graph $G = (V, E)$ then $\chi'(G) = \Delta(G)$.

Theorem 4.21 (Vizing, 1964)

Given a simple graph $G = (V, E)$ then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

- For every $v \in V$ exist a list of colors $L(v)$.
- We seek a coloring that assign a color of this list to every node.
- We call G k-list-colorable if
  - If $|L(v)| = k \ \forall v \in V$
  - then G list-colorable.
- This means: If for every note k colors are available, then a list-coloring exists.
- The list-chromatic number $\chi_L(G)$ is the smallest $k \in \mathbb{N}$ so that G is k-list-colorable.

- Let G be k-list-colorable.
- Choose for every $v \in V$ the list $L(v) = \{1, ..., k\}$.
- Then G is k-colorable which means

$$\chi(G) \leq \chi_L(G)$$

Example 4.22

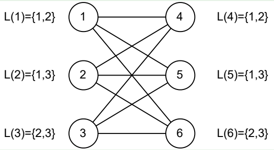


Figure 6: G is bipartite, but not 2-list-colorable.

Lemma 4.23

Let $t = \binom{2k-1}{k}$ *then* $K_{t,t}$ *is not* t*-list-colorable.*

Proof.

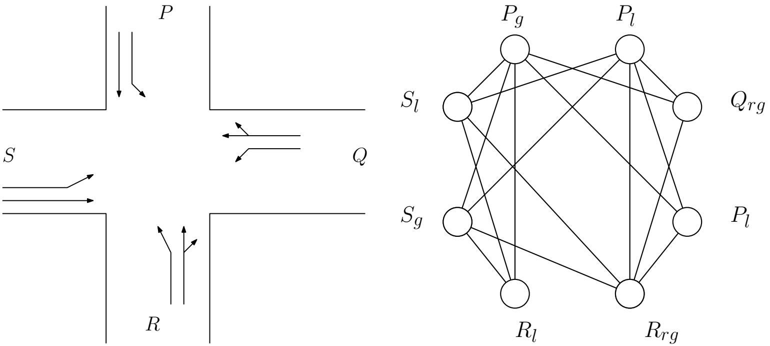
Exercise. $\square$

Theorem 4.24 (Vizing, 1976; Erdös et. al., 1979)

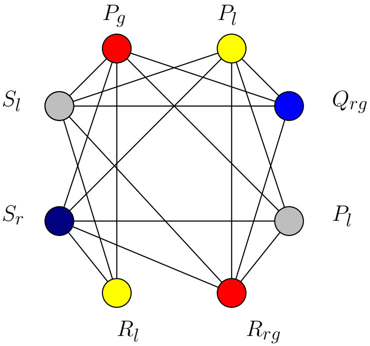
Let G be a connected graph with $n \geq 2$ vertices that is neither complete nor an odd circle. Then $\chi_L(G) \leq \Delta(G)$.

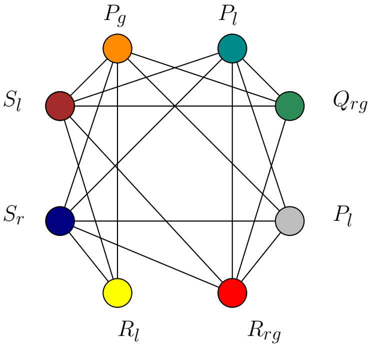
Theorem 4.25 (Voigt, 1993; Thomassen, 1993)

Each planar graph is 5-list colorable.

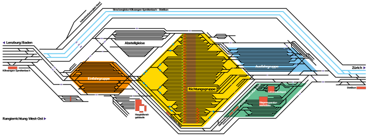


- We consider the intersection and select a node for each turning lane. Now we consider which lanes should ideally not travel at the same time if they are not to obstruct each other. We connect these with an edge.





- If we now want to construct an ideal traffic light circuit, i.e. we want the traffic to flow as smoothly as possible with as few traffic light phases as possible, then we must therefore find the maximum number of nodes that are not connected to each other in each case.



- Rearranging cars of an incoming train in a hump yard is a widely discussed topic.
- Sorting of Rolling Stock Problems can be described in several scenarios and with several constrains.
- We focus on the train marshalling problem where the incoming cars of a train are distributed to a certain number of sorting tracks. When pulled out again to build the outgoing train, cars sharing the same destination should appear consecutively. The goal is to minimize the number of sorting tracks.

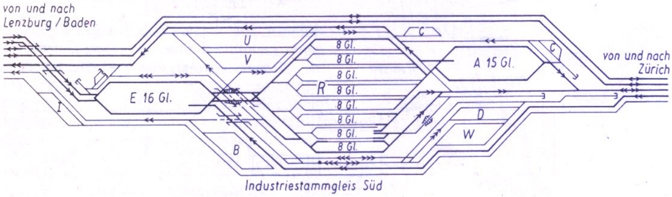
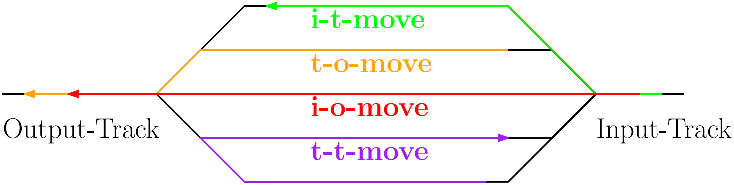


Bild 12.19 Gleisplanschema Rbf Zürich-Limmattal (SBB)
E Einfahrgruppe, A Ausfahrgruppe, R Richtungsgruppe, U und V Umspann- und Leerwagengruppe, W Wagenreparatur, I Übergabegruppen für die Industrie, C Lokwarte- und Dienstbegleitwagengleise, D Lokdepot, B Gleise für Baudienste



Sorting of rolling Stock Problems (SRSP, SRS)

Track Topology					
Design					
stacks	queues	stacks/queues	sido	diso	dido
Length					
unbounded	b-bounded				
Sorting Mode					
Shunting					
no-shunting	h-hump-shunting				
Timing					
sequential	concurrent	time-window			
Splitting					
s-split	split	chain-split			
Structure of Output Sequence					
free	ordered				
g-blocks	g-pattern				

Definition 4.26 (Sorting of rolling Stock Problems, Hansmann, 2010)

Sorting of rolling Stock Problems can be defined as triples $\nu := \alpha|\beta|\gamma$ with

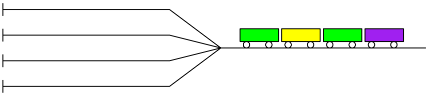
$$\alpha \in \{\{st, qu, sq, sd, ds, dd, \cdot\} \times \{ub, b-bd, \cdot\}\}$$

$$\beta \in \{\{nsh, h-hsh, \cdot\} \times \{se, co, tw, \cdot\} \times \{s-sp, sp, csp, \cdot\}\}$$

$$\gamma \in \{\{fr, or\cdot\} \times \{g-bl, g-pa, \cdot\}\}$$

Definition 4.27 (Train Marshalling Problem)

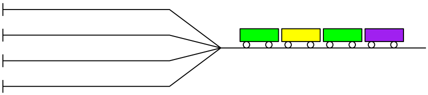
Train Marshalling Problem (TMP) is defined as triples $st, ub|nsh, se, csp|fr, g-bl$.



A *train* σ of length n is accordingly a sequence of cars

$$\sigma = (\sigma_1, ..., \sigma_n)$$

with $\sigma_i \in \{1, ..., d\}$ for $i \in \{1, ...n\}$.

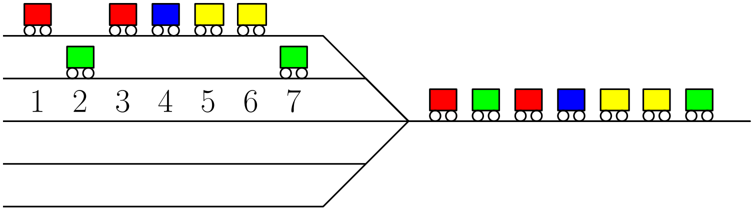


A *train* σ of length n is accordingly a sequence of cars

$$\sigma = (\sigma_1, ..., \sigma_n)$$

with $\sigma_i \in \{1, ..., d\}$ for $i \in \{1, ...n\}$.

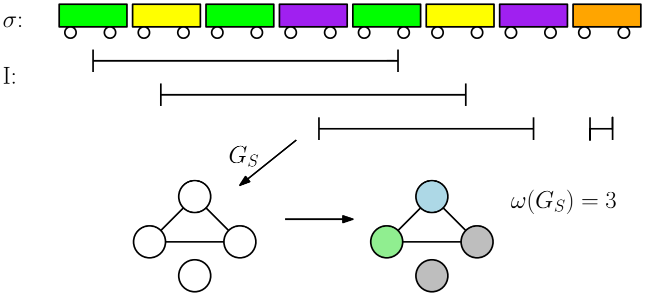
1. All cars with the same destination should be sorted into one block.
2. Only one shunting movement is allowed per track.
3. So the number of directional tracks $K(\sigma)$ is minimized.



$$\sigma = (1, 2, 1, 3, 4, 4, 2)$$

Train Marshalling Problem

Beygang [p. 50 ff. 2] (Greedy): A graph coloring of the interval graph G_S corresponds to a (not necessarily minimal) solution of the TMP with $K(S) \leq \omega(G)$.

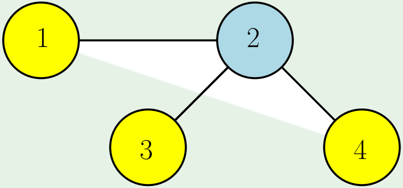


Example 4.28

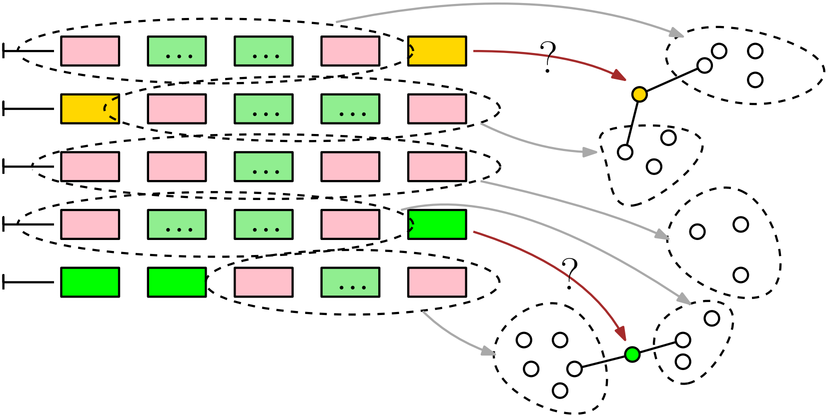
Let $\sigma = 1, 2, 1, 3, 4, 4, 2$, so the train has length $n = 7$ with 4 destinations.

$$\begin{array}{ll} \text{Track 1}: & \sigma_1\,\sigma_3\,\sigma_4\,\sigma_5\,\sigma_6 \\ \text{Track 2}: & \sigma_2\,\sigma_7 \end{array}$$

The intervals of each target $1, 2, 3, 4$ correspond to $I_1 = [1, 3]$, $I_2 = [2, 8]$, $I_3 = [4, 4]$, $I_4 = [5, 6]$. Thus, the set of nodes of the corresponding interval graph consists of nodes $V = 1, 2, 3, 4$ and edges $(1, 2)$, $(2, 3)$, $(2, 4)$.



Train Marshalling Problem



Summary

- Lower and upper bounds
- Colouring maps and planar graphs
- Planar graphs are 4-colorable
- Colouring vertices
- List colouring
- Applications: Planning problems with competing resources, e.g. in network traffic and shunting