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Assignment : 3 Differential Calculus

1) a) $\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y} = \frac{(-1)\,8}{-1+2} = -8$

b) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2} = L$

From path $x=0 \Rightarrow L=0$

From path $y=x^3$:

$\lim_{(x,y)\to(0,0)} \frac{x^3 \cdot x^3}{2x^6} = \frac{1}{2}$

$\Rightarrow$ Limit DNE

c) $\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{y}{x}\right)$

$$\left.\begin{array}{l}\text{From path } y=x \;:\; L=\pi/4 \\ \text{From path } y=0 \;:\; L=0\end{array}\right\}\text{DNE}$$

d) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

$0 \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \frac{\sqrt{x^2+y^2}}{2}$ $\Rightarrow$ By squeeze theorem limit $=0$

e) $\lim_{(x,y)\to(0,0)} \frac{\sin^2(x+y)}{|x|+|y|}$

$0 \le \left|\frac{\sin^2(x+y)}{|x|+|y|}\right| \le \frac{\sin^2|x+y|}{|x+y|}$

$\lim_{(x,y)\to(0,0)} \sin|x+y| \cdot \frac{\sin|x+y|}{|x+y|} = 0 \cdot 1 = 0$

f) $f(x,y) = \begin{cases} 1 & \text{if } x+y \geq 2 \\ -1 & \text{if } x+y < 2 \end{cases}$ find $\lim_{(x,y)\to(1,1)} f(x,y)$

if both $x \to 1^+$ and $y \to 1^+ \Rightarrow x+y \to 2^+$

hence we use $\lim_{(x,y)\to(1,1)} f(x,y) = 1$

if both $x \to 1^-$ and $y \to 1^- \Rightarrow x+y \to 2^-$

$\Rightarrow \lim_{(x,y)\to(1,1)} f(x,y) = -1$

g) $\lim_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^2+y^2}$

$0 \leq \left| \dfrac{2x^2y}{x^2+y^2} \right| \leq |x|$ $\quad (x^2+y^2 \geq 2|x|\cdot|y|$

$\Rightarrow$ limt $= 0$

② a) $f(x,y) = \dfrac{x+y}{x-y}$, $(x,y) \in \mathbb{R}^2$

$\lim_{x\to 0} \left[\lim_{y\to 0} \dfrac{x+y}{x-y} \right] = \lim_{x\to 0} 1 = 1$

$\lim_{y\to 0} \left[\lim_{x\to 0} \dfrac{x+y}{x-y} \right] = \lim_{y\to 0} (-1) = -1$

As iterated limit aren't equal $\Rightarrow$

$\lim_{(x,y)\to(0,0)} f(x,y) \to$ DNE

b) $f(x,y) = \begin{cases} 0 & ; y = 0 \\ x \cdot \sin \frac{1}{y} & ; y \neq 0 \end{cases}$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} x \cdot \sin \dfrac{1}{y} = 0$

as, $0 \leq \left| x \sin \dfrac{1}{y} \right| \leq |x|$

$$\lim_{x \to 0}\left[\lim_{y \to 0} x \cdot \sin\frac{1}{y}\right] \to \text{DNE} \quad \text{as } -1 \leq \sin\frac{1}{y} \leq 1$$

$$\lim_{y \to 0}\left[\lim_{x \to 0} x \cdot \sin\frac{1}{y}\right] = \lim_{y \to 0} 0 = 0$$

c) $$f(x,y) = \frac{x^2 y^2}{x^2 y^2 + (x-y)^2}$$

$$\lim_{x \to 0}\left[\lim_{y \to 0} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2}\right] = \lim_{x \to 0} 0 = 0$$

$$\lim_{y \to 0}\left[\lim_{x \to 0} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2}\right] = \lim_{y \to 0} 0 = 0$$

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} = L$$

$$\left.\begin{array}{lll} \text{for path } x = y & : & L = 1 \\ \text{for path } x = 0 & : & L = 0 \end{array}\right\} \text{DNE}$$

③ $$f(x,y) = \begin{cases} 0 & x : \text{rational} \\ 1 & x : \text{irrational} \end{cases}$$

$$\lim_{(x,y) \to (a,b)} f(x,y) = 0$$ if we approach (a,b) through rational points i.e $(x,y) \in \mathbb{R}$

$$= 1$$ if we approach (a,b) through irrational points i.e $(x,y) \in \mathbb{Q}^c$

hence limit is dependent on path so it DNE

④ $$f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & (x,y) \neq (0,0) \\ \dfrac{1}{2} & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} = \frac{1}{2}$$

Now,

as $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = \frac{1}{2} \Rightarrow$ Continuous

$$f_x(0,0): \quad \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin^{-1} h}{\tan^{-1} 2h} - \frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{2\sin^{-1} h - \tan^{-1} 2h}{2h\tan^{-1} 2h}$$

$$= \lim_{x\to 0} \frac{2\left(\cancel{x} + \frac{x^3}{6}\right) - \left(\cancel{2x} - \frac{8x^3}{3}\right)}{2x\left(2x - \frac{8x^3}{3}\right)}$$

$$= \lim_{x\to 0} \frac{3x^3}{2x \cdot x\left(2 - \frac{8x^2}{3}\right)} = 0$$

Similarly $f_y(0,0) = 0$.

b) $\lim_{(x,y)\to(0,0)} x\sin\frac{1}{x} + y\sin\frac{1}{y} = 0 = f(0,0)$

$\rightarrow$ Continuous

~~As of $\frac{1}{x}, \frac{1}{y}$ in sine function~~ $\Rightarrow f_x(0,0) \rightarrow 0$

$F_y(0,0) \rightarrow 0$

c) $f(x,y) = \begin{cases} xy\log(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

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$$\lim_{(x,y)\to(0,0)} xy\log(x^2+y^2) \qquad \text{let} \quad x = r\cos\theta, \quad y = r\sin\theta$$

$$\lim_{r\to 0} r^2\, \text{s.c} \log r^2 = \lim_{r\to 0} \text{s.c} \frac{\log r^2}{1/r^2}$$

$$= \lim_{r\to 0} \text{s.c} \frac{\frac{1}{r^2}\cdot 2r}{-\frac{2}{r^3}} = 0 = f(0,0)$$

$\rightarrow$ Continuous.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly,

$$f_y(0,0) = 0$$

d) $$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & x \neq y \\ 0 & x = y \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y} = \lim_{r\to 0} \frac{r^3(\cos^3\theta + \sin^3\theta)}{r(\cos\theta - \sin\theta)} \rightarrow \text{DNE}$$

as $1/(\cos\theta - \sin\theta)$ is unbounded

5) $$f(x,y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_x(0,y) : \lim_{h\to 0} \frac{f(h+0,y) - f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{y\dfrac{h^2-y^2}{h^2+y^2} + \dfrac{y\cdot y^2}{y^2}}{h}$$

$$= \lim y \frac{2h^2}{(h^2+y^2)\cdot h} = 0$$

$$f_y(x,0) : \lim_{h \to 0} \frac{f(x, h+0) - f(x,0)}{h}$$

$$= \lim_{h \to 0} \frac{h\left(\frac{x^2 - h^2}{x^2 + h^2}\right) - 0}{h} = 1$$

$$f_x(0,0) : \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$f_y(0,0) : \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h(-h^2)}{h^2} - 0}{h} = -1$$

6) $$f(x,y) = \begin{cases} -xy & |y| \geq |x| \\ xy & |y| < |x| \end{cases}$$

$$f_{yx}(0,0) : \lim_{k \to 0} \frac{f_y(0+k, 0) - f_y(0,0)}{k}$$

$$f_y(k,0) : \lim_{h \to 0} \frac{f(k, 0+h) - f(k,0)}{h}$$

$$= \lim_{h \to 0} \frac{k \cdot h - 0}{h} = k$$

$$f_y(0,0) : \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h} = 0$$

$h \to 0$

$h \to 0$

xy

$|h| \leq |$

$h \to 0$

$h \to 0$

$|K| \leq |h|$

xy

8

$f_{xy} \Rightarrow \lim_{k \to 0} \dfrac{k}{k} = 1$

$f_{xy}(0,0)$: $\lim_{k \to 0} \dfrac{f_x(0, 0+k) - f_x(0,0)}{k}$

$f_x(0,k)$: $\lim_{h \to 0} \dfrac{f(0+h, k) - f(0,0)}{h} = -k$

$f_x(0,0)$: $\lim_{h \to 0} \dfrac{f(0+h, 0) - f(0,0)}{h} = 0$

$f_{xy} \Rightarrow \lim_{k \to 0} \dfrac{-k}{k} = -1$

$\Rightarrow f_{xy}(0,0) = 1 \neq f_{yx}(0,0) = -1$

7) $\lim_{(x,y) \to (0,0)} |x| + |y| = f(0,0) = 0$

no matter what side we reach $(0,0)$ all ~~and~~ possible combinations of addition or substraction of two zero results zero.

$f_x(0,0)$: $\lim_{h \to 0} \dfrac{f(0+h, 0) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{|h|}{h} \to$ DNE

Partial derivates DNE hence non-Differentiable at $(x,y) = (0,0)$

8) $f(x,y) = \begin{cases} \dfrac{(x+y)\{\sqrt{x^2+y^2} + xy\}}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

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$$\Delta z = dz + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = f_{x(x_0,y_0)}\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$(\Delta x+\Delta y)\frac{\sqrt{\Delta x^2+\Delta y^2}+\Delta x\cdot\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} = \Delta x + \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

as $f_x(0,0) : \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = 1$

$$\Rightarrow (\Delta x+\Delta y)\frac{\Delta x\cdot\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} = \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$\frac{\Delta x^2}{\sqrt{\frac{\Delta x^2}{\Delta y^2}+1}}\cdot\frac{\Delta y}{|\Delta y|} + \frac{\Delta y^2}{\sqrt{\frac{\Delta y^2}{\Delta x^2}+1}}\cdot\frac{\Delta x}{|\Delta x|} = \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$\Rightarrow \epsilon_1 = \frac{\pm\Delta x}{\sqrt{\frac{\Delta x^2}{\Delta y^2}+1}} \quad \text{and} \quad \epsilon_2 = \pm\frac{\Delta y}{\sqrt{\frac{\Delta y^2}{\Delta x^2}+1}}$$

~~Also~~ $\epsilon_1 \to 0$ as $\Delta x \to 0$, $\epsilon_2 \to 0$ as $\Delta y \to 0$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x^2\,\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} \le \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x(\Delta x^2+\Delta y^2)}{2\sqrt{\Delta x^2+\Delta y^2}} = 0$$

similarly $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

9)

$$f(x,y) = \begin{cases} x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2} & xy \neq 0 \\ x^3 \sin\frac{1}{x^2} & x\neq 0,\ y=0 \\ y^3 \sin\frac{1}{y^2} & x=0,\ y\neq 0 \\ 0 & x=y=0 \end{cases}$$

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$$f_x(0,0) : \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h^3 \sin\frac{1}{h^2}}{h} = 0$$

$$f_y(0,0) : \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = 0$$

$$\Delta z = dz + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$f(0+\Delta x, 0+\Delta y) - f(0,0) = f_x(0,0)\Delta x + f_y(0,0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$\Rightarrow \Delta x^3 \sin\frac{1}{\Delta x^2} + \Delta y^3 \sin\frac{1}{\Delta y^2} = \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

On comparing both sides we have,

$$\epsilon_1 = \Delta x^2 \sin\frac{1}{\Delta x^2} \quad \text{and} \quad \epsilon_2 = \Delta y^2 \sin\frac{1}{\Delta y^2}$$

Now,

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \epsilon_1 = 0 = \lim_{(\Delta x,\Delta y)\to(0,0)} \epsilon_2$$

hence $f(x,y) \rightarrow$ differentiable at $(0,0)$.

Continuity of $f_x(x_0,y_0)$ at $(x_0,y_0) = (0,0)$:

$$f_x(x_0,y_0) : \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

$$\lim_{(x_0,y_0)\to(0,0)} \left[\lim_{h\to 0} \frac{(x_0+h)^3 \cdot \sin\frac{1}{(x_0+h)^2} + y_0^3 \sin\frac{1}{y_0^2} - x_0^3 \sin\frac{1}{x_0^2} - y_0^3 \sin\frac{1}{y_0^2}}{h} \right]$$

$$\lim_{(x_0,y_0)\to(0,0)} \left[\lim_{h\to 0} \frac{(x_0+h)^3 \cdot \sin\frac{1}{(x_0+h)^2} - x_0^3 \sin\frac{1}{x_0^2}}{h} \right]$$

Employing ~~chain~~ L.H rule

$$\lim_{(x_0,y_0)\to(0,0)} \left[\lim_{h\to 0} 3(h+x_0)^2 \sin\frac{1}{(h+x_0)^2} - 2\cos\frac{1}{(h+x_0)^2} \right]$$

$$\lim_{(x_0,y_0)\to(0,0)} 3x_0^2 \cdot \sin\frac{1}{x_0^2} - 2\cos\frac{1}{x_0^2} \Rightarrow \text{DNE}$$

Similarly $f_y(x,y)$ not continuous at $(x,y) = (0,0)$

10) $f(x,y) = \begin{cases} (x^2+y^2)\cos\dfrac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$f_x(0,0)$: $\lim_{h\to 0} \dfrac{f(h+0,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{h^2 \cos\frac{1}{|h|}}{h} = 0$$

Similarly $f_y(0,0) = 0$

$$\Delta z = dz + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$f(0+\Delta x, 0+\Delta y) - f(0,0) = f_x(0,0)\Delta x + f_y(0,0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$(\Delta x^2 + \Delta y^2)\cos\frac{1}{\sqrt{\Delta x^2+\Delta y^2}} = \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$\Rightarrow \epsilon_1 = \Delta x \cdot \cos\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}} \ ; \ \epsilon_2 = \Delta y \cdot \cos\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \Delta x \cos\frac{1}{\sqrt{\Delta x^2+\Delta y^2}} = 0 = \lim_{(\Delta x,\Delta y)\to(0,0)} \Delta y \cdot \cos\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}$$

Continuity of $f_x(x_0, y_0)$ at $(x_0, y_0) = (0,0)$:

$$f_x(x_0, y_0) : \lim_{h \to 0} \frac{f(h+x_0, y_0) - f(x_0, y_0)}{h}$$

$$\lim_{(x_0, y_0) \to (0,0)} \left[\lim_{h \to 0} \frac{[(h+x_0)^2 + y_0^2] \cos \frac{1}{\sqrt{(h+x_0)^2 + y_0}}}{h} - (x_0^2 + y_0^2) \cos \frac{1}{\sqrt{x_0^2 + y_0^2}} \right]$$

$$\lim_{(x_0, y_0) \to (0,0)} \left[\lim_{h \to 0} \frac{(h+x_0)^2 \cdot \cos \frac{1}{\sqrt{(h+x_0)^2 + y_0^2}}}{h} - x_0^2 \cos \frac{1}{\sqrt{x_0^2 + y_0^2}} \right]$$

LH ~~Chain~~ Rule :

$$\lim_{(x_0, y_0) \to (0,0)} \left[\lim_{h \to 0} \frac{(h+x_0)^3 \sin \frac{1}{\sqrt{(h+x_0)^2 + y_0^2}}}{((h+x)^2 + y^2)^{3/2}} + 2(h+x_0) \cos \frac{1}{\sqrt{(h+x_0)^2 + y^2}} \right]$$

$$\lim_{(x_0, y_0) \to (0,0)} \frac{x_0^3 \sin \frac{1}{\sqrt{x_0^2 + y_0^2}}}{(x^2 + y^2)^{3/2}} + 2x_0 \cos \frac{1}{\sqrt{x_0^2 + y_0^2}}$$

from path $y_0 = 0$

$$\lim_{x_0 \to 0} \frac{\cancel{x_0^3} \sin \frac{1}{|x_0|}}{\cancel{x_0^3}} + 2x_0 \cdot \cos \frac{1}{|x_0|} \rightarrow \text{DNE}$$

similarly for $f_y(x_0, y_0)$ at $(x_0, y_0) = (0,0)$

11) $$f(x,y) = \begin{cases} |xy|^p & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

Δy^2

$$f_x(0,0) : \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) : \lim_{h \to 0} \frac{f(0, h+0) - f(0,0)}{h} = 0$$

$$\Delta z = dz + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$f(0+\Delta x, 0+\Delta y) - f(0,0) = 0\,\Delta x + 0\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$|\Delta x \Delta y|^p = \epsilon_1 \Delta x + \epsilon_2 \Delta y \rightarrow (1)$$

$$\epsilon_1 = |\Delta y|^p \cdot \frac{|\Delta x|^p}{\Delta x} \text{ and } \epsilon_2 = |\Delta y|^p \times 0 = 0$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|\Delta x \Delta y|^p}{\Delta x} = 0 \quad \text{for it to differentiable}$$

$$0 \leq \left| \frac{|\Delta x \Delta y|^p}{\Delta x} \right| = |\Delta x|^{p-1} \cdot |\Delta y|^p$$

$$\Rightarrow p > 1$$

From (1) other combination would be

$$\epsilon_1 = \epsilon_2 = \frac{|\Delta x \cdot \Delta y|^p}{\Delta x + \Delta y}$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|\Delta x \cdot \Delta y|^p}{\Delta x + \Delta y} \text{ should be 0 for differentiability}$$

$$0 \leq \frac{|\Delta x \cdot \Delta y|^p}{\Delta x + \Delta y} \leq \frac{1}{2} \cdot \frac{1}{(\Delta x \cdot \Delta y)^{p-\frac{1}{2}}} \quad \because \left(\Delta x + \Delta y \geq 2\sqrt{\Delta x \Delta y}\right)$$

$$= \frac{1}{2} |\Delta x \cdot \Delta y|^{-(p-\frac{1}{2})}$$

$$\Rightarrow p > \frac{1}{2}$$

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Continuity :

$$\lim_{(x,y)\to 0} f(x,y) = f(0,0) = 0$$

$$\lim_{(x,y)\to 0} |xy|^p = 0 \implies p > 0$$