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CLASS 2

Solution of $AX = B$

where A is $m \times n$ matrix
X is $n \times 1$ matrix
B is $m \times 1$ matrix

$A \rightarrow$ coefficient matrix
$(A/B) \rightarrow$ augmented matrix
$n \rightarrow$ number of unknowns.

rank$(A) = r$
rank$(A|B) = r_1$

Theorem: (A) if rank (A) = rank $(A|B)$
then system of equation is consistent (solutions exist)

(i) if rank(A) = rank $(A|B) = n$
then unique solution exists

(ii) if rank(A) = rank $(A/B) \neq n$
then infinitely many solutions exist.

and, $n - r$ variables can be selected arbitrarily.

Example:

(1)
$-x + y + 2z = 2$
$3x - y + z = 6$
$-x + 3y + 4z = 4$

Find the solution using Gauss elimination method or row reduction method.

$$(A|b) = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{array}\right]$$

$R_2 \rightarrow R_2 + 3R_1$
$R_3 \rightarrow R_3 - R_1$

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$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

$\hookrightarrow$ Echelon form ; rank(A) = 3

rank (A|B) = 3

n = 3, ∴ unique solⁿ exists.

3rd row : $-5z = -10$; $z = 2$

2nd row : $2y + 7z = 12$

$2y = -2$

$y = (-1)$

1st row : $-x + y + 2z = 2$

$-x - 1 + 4 = 2$

$x = 1$

② $3x + 2y + z = 3$

$2x + y + z = 0$

$6x + 2y + 4z = 6$

$$(A/b) = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$\approx$

$R_2 \to R_2 - \frac{2}{3}R_1$

$R_3 \to R_3 - 2R_1$

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$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$\approx$

$R_3 \to R_3 - 6R_2$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

$0x + 0y + 0z = 12$; $0 = 12$

no solution exist

(i) rank $(A) = 2$

(ii) rank $(B) = 3$

(iii) $n = 3$

$r \neq r_1$

(3) $8x - 4y - 2z = -6$

$16x + 2y + z = 3$

Solve it using row reduction method of gauss elimination.

$$(A|B) = \left[\begin{array}{ccc|c} 8 & -4 & -2 & -6 \\ 16 & 2 & 1 & 3 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$= \left[\begin{array}{ccc|c} 8 & -4 & -2 & -6 \\ 0 & 10 & 5 & 15 \end{array}\right]$$

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$0x + 10y + 5z = 15$

$2y + z = 3$

From eqn (ii),

$x = 0$

$4y + 2z = 3$, from (i)

↓

infinitely many solns

$\because$ Rank(A) = 2

Rank (B) = 2

$n = 3$

(i) = (ii) $\neq$ (iii)

$n - r = 1$.

1 variable can be selected arbitrarily.

(4) $-2x + y + z = a$

$x - 2y + z = b$

$x + y - 2z = c$

Find value of unknown constant (s) such that

1. no solution
2. unique solution
3. infinitely many solution

$$[A \mid B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right]$$

$R_2 \to R_2 + \frac{1}{2}R_1$

$R_3 \to R_3 + \frac{1}{2}R_1$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 0 & -1.5 & 1.5 & b + a/2 \\ 0 & 1.5 & -1.5 & c + a/2 \end{array}\right]$$

$\approx$

$R_3 \to R_3 + R_2$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 0 & -1.5 & 1.5 & b + a/2 \\ 0 & 0 & 0 & b + c + a \end{array}\right]$$

(i) no solution

$\text{Rank}(A) \neq \text{Ran}(A/b)$

$a + b + c \neq 0$

(ii) unique sol^n

Condition: $\text{Rank}(A) = \text{Rank}(B) = 3$

$\downarrow$

$= 2$, not possible

(iii) infinitely many

$\text{Rank}(A) = \text{Rank}(B) \neq 3$

$\downarrow$

2

$\Rightarrow a + b + c = 0$

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Linear Combination of n vectors

$\alpha_1 v_1 + \alpha_2 v_2 \dots + \alpha_n v_n$

$\alpha_1 \dots \alpha_n$ are scalars

$v_1 \dots v_n$ are vectors

This is a trivial linear combination if all the scalars are 0.
otherwise, it's called non trivial linear combination

Ex: $0v_1 + 0v_2 \dots + 0v_n \rightarrow$ trivial L.C

$0v_1 + 1v_2 \dots + 0v_n \rightarrow$ non trivial L.C

Linearly dependent vectors (L.D)

A set, $S = \{v_1, v_2 \dots v_n\}$ is called linearly dependent vectors if there exists a non-trivial linear combination of $v_1, v_2 \dots v_n$ that equals the zero vector.

Example: $S = \{(1,0,1), (1,1,0), (-1,0,-1)\}$ is L.D

$\alpha(1,0,1) + \beta(1,1,0) + \gamma(-1,0,-1) = \vec{0} = (0,0,0)$

$(\alpha+\beta+\gamma, \beta, \alpha-\gamma) = (0,0,0)$

$\beta = 0$ | $\beta = 0$
$\alpha = \gamma$ $\Rightarrow$ $\alpha = \gamma$
$\alpha + \beta - \gamma = 0$ $\downarrow$

infinite sol^n.

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Linearly Independent (L.I)

A set, $S = \{v_1, v_2 \ldots\ldots v_n\}$ is called linearly independent if no non trivial L.C of $v_1, v_2 \ldots\ldots v_n$ equals 0, i.e. if

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots\ldots \alpha_n v_n = \vec{0}$$

then $\alpha_1 = 0$

$\alpha_2 = 0$

$\vdots$

$\alpha_n = 0$

Eg: $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ show L.I

$$\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = \vec{0}$$

$$(\alpha, \beta, \gamma) = (0,0,0)$$

* Let there be n - vectors which are kept as rows of a matrix A, if rank(A) = n, these n - vectors are L.I

if rank(A) < n, then these are L.D.

Q. $S = \{(1,2,-2)\ (-1,3,0)\ (0,-2,1)\}$

Find if they L.I or LD

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

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$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

$\approx$

$R_3 \to R_3 + \frac{2}{5} R_2$

rank(A) = 3

n = 3.

=> they are L.I.

$\approx$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & 0 & 1/5 \end{bmatrix}$$

Q. $S = \{ (1, 0, 1), (1, 1, 0) \; (-1, 0, -1) \}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank(A) = 2

n = 3

rank(A) < 3

Linearly dependent.