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28/11/20

Eigen Values and Eigen Vectors

Def^n: Let $A_{n\times n}$ be square matrix of order n. The values of λ, for which the eq^n $AX = \lambda X$ has non trivial solutions ($X \neq 0$), are called the eigen values of A and the non zero vectors X are called the corresponding eigen vectors of A.

Then, $(A - \lambda I)X = 0$ has non trivials if

$$|A - \lambda I| = 0$$

Characteristic eq^n of A is $|A - \lambda I| = 0$.

Eigen values (λ) are the roots of this eq^n.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}_{3\times 3}$

Find Eigen Values and Eigen vectors.

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

characteristic eq^n.

$$(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$\Rightarrow \lambda = 1, 2, 3$

Eigen Values $\rightarrow$ 1, 2, 3

Eigen Vectors $\rightarrow$

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① $\lambda = 1$

$(A - I)X = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i $\to$ $0 = 0$

ii $x_2 + x_3 = 0$

$x_1 + x_3 = 0$

$$X = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = -x_3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad x_3 \neq 0.$$

Let, $x_3 = -1$

$X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is the eigen vector.

② $\lambda = 2$

$(A - 2I)X = 0$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x_1 = 0$

$x_3 = 0$

$2x_1 + x_3 = 0$

$x_1 = 0$, $x_3 = 0$

x_2 is arbitrary

$= x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $x_2 \neq 0$.

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* Any non zero scalar multiple of an eigen vector is also an eigen vector.

Result :

① Corresponding to n distinct Eigen values of matrix $A_{n \times n}$, there are n L.I eigen vectors.

② It may or may not be possible to get m L.I eigen vectors of A corresponding to the E.V repeating m-times.

Example

①

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \qquad \lambda = 1, 2, 2$$

A has only 2-LI eigen vectors.

② Some properties : A is n×n matrix.

① A and A' have same eigen values.

To prove : $|A' - \lambda I|$

Proof : $|A - \lambda I| = 0$, λ is eigen value of A.

$$(A - \lambda I)^T = A^T - \lambda I^T$$
$$= A^T - \lambda I \quad — ①$$

Now, $|A - \lambda I| = |A - \lambda I|^T$, from ①.
$$= |A^T - \lambda I|$$

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Now, if $|A - \lambda I| = 0$, $\Rightarrow |A^T - \lambda I| = 0$.

x —— x

(3) Eigen values of a Δ^r matrix are the principal diagonal elements of that matrix.

Proof : $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$

$|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$

(4) The eigen values of a diagonal matrix A are the main diagonal elements of A.

$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$

$|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$

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(5) Let $A^2 = A$

A is idempotent.

To prove : $\lambda = 0$ or 1

Proof : $AX = \lambda X$ — (1)

$A^2X = A(\lambda X)$

$A^2 X = \lambda (A x)$

$= \lambda(\lambda X)$

$A^2 X = \lambda^2 X$ — (11)

(1) & (11)

$\lambda X = \lambda^2 X$

$\lambda(1 - \lambda) X = 0$

$\Rightarrow \lambda = 0, 1$

(6) The sum of all the eigen values is equal to Trace(A)

Trace (A) = sum of diagonal elements.

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_n \ldots\ldots \lambda_n = \text{Trace}(A)$

$= a_{11} + a_{22} + \ldots\ldots a_{nn}.$

(7) $\lambda_1 \times \lambda_2 \times \lambda_3 \ldots\ldots \lambda_n = |A|$

Proof of This *

(8) If $\lambda \neq 0$ is an Eigen value of A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

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Proof : $AX = \lambda X$

$A^{-1}(AX) = A^{-1}(\lambda X)$

$X = \lambda(A^{-1}X)$

$A^{-1}X = \frac{1}{\lambda}X$

$\Rightarrow \frac{1}{\lambda}$ is eigen value of A^{-1}.

with same Eigen vector.

(9) If λ is Eigen Value of A,
then λ^k is eigen value of A^k

Proof : $AX = \lambda X$

$A^2X = \lambda(AX)$

$A^2X = \lambda(\lambda X)$

$A^2X = \lambda^2 X$

$A^3X = A(A^2X) = A(\lambda^2 X) = \lambda^2(AX) = \lambda^3 X$

⋮ so ~~on~~ and so forth.

$A^kX = \lambda^k X$

$\Rightarrow \lambda^k$ is eigen value of A^k, where k is positive integer.

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Suppose Matrix A exists as,

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} \cdots\cdots & a_{2n} \\ \vdots & \vdots & \\ a_{n1} & a_{n2} \cdots\cdots & a_{nn} \end{bmatrix}$$

Let λ be the Eigen values, then

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & a_{22}-\lambda \cdots\cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} \cdots\cdots & & a_{nn}-\lambda \end{vmatrix} = 0$$

Expanding the determinant, we obtain a polynomial of degree n in λ, which is of the form,

$$P(\lambda) = |A - \lambda I| = (-1)^n \left[\lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} \cdots\cdots + (-1)^n c_n \right] = 0$$

or

$$P(\lambda) = \lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} \cdots\cdots + (-1)^n c_n = 0 . \quad — (1)$$

where $c_1, c_2, c_3 \cdots\cdots c_n$ can be expressed in terms of the elements a_{ij} of the matrix. A.

Let $r_1, r_2 \cdots r_n$ be the roots of an n-order polynomial

$$P(\lambda) = (r_1 - \lambda)(r_2 - \lambda) \cdots\cdots (r_n - \lambda)$$

$$P(\lambda) = \prod_{i=1}^{n} r_i + \cdots\cdots + \sum_{i=1}^{n} r_i (-\lambda)^{n-1} + (-\lambda)^n \quad \text{—(ii)}$$

Since the eigenvalues are roots of a matrix polynomial, we can match (i) & (ii)

(i)
$$\sum_{i=1}^{n} \lambda_i = \frac{-a_1}{a_0} = C_1 = \text{Trace}(A)$$

(ii)
$$\prod_{i=1}^{n} \lambda_i = (-1)^n \frac{a_n}{a_0} = \text{Det}(A)$$