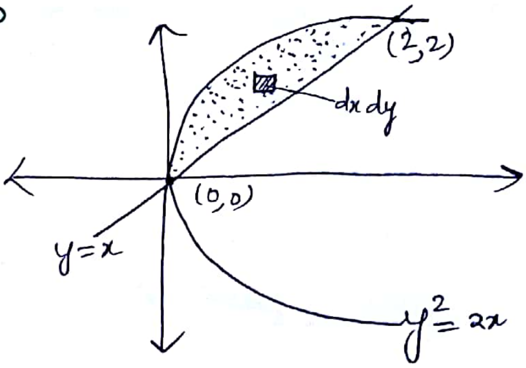
Assignment - 6

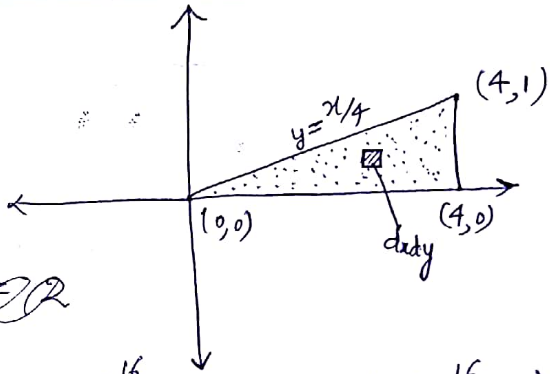
(1). $y^2 = 2x \ , \ y = x$



$$A = \int_{x=0}^{2}\int_{y=x}^{\sqrt{2x}} dx\,dy = \int_0^2 (\sqrt{2x} - x)\,dx$$

$$= \left[\frac{2\sqrt{2}}{3} x^{3/2} - \frac{x^2}{2}\right]_0^2 = \left[\frac{2\sqrt{2}}{3}\cdot 2^{3/2} - \frac{2^2}{2}\right] = \left[\frac{8}{3} - \frac{4}{2}\right] = \frac{2}{3}$$

(2) (a) $\int_0^1 \int_{4y}^{4} e^{x^2}\,dx\,dy$



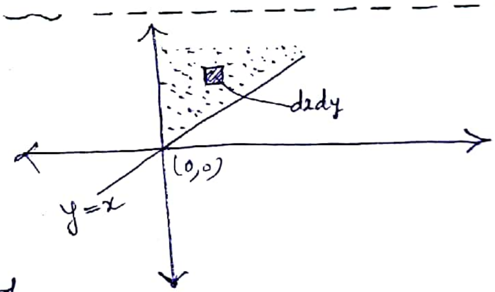
$y=0, \quad x = 4y$
$y=1, \quad x = 4.$

$x^2 = t$

$$A = \int_{x=0}^{4}\int_{y=0}^{x/4} e^{x^2}\,dx\,dy$$

$$= \int_0^4 \frac{x}{4} e^{x^2}\,dx = \frac{1}{8}\int_0^4 2x\, e^{x^2}\,dx = \frac{1}{8}\int_0^{16} e^{t}\,dt = \frac{1}{8}(e^{16} - 1)$$

(b) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

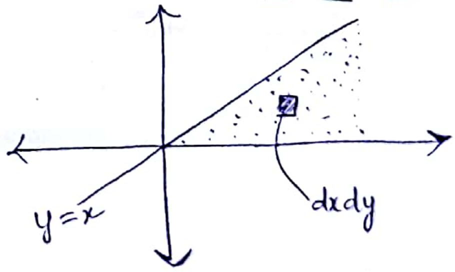


$x = 0 \quad y = x$
$x = \infty \quad y = \infty$

$$A = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y}[y - 0]\,dy$$

$$= \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty = \left[-0 + 1\right] = 1$$

(c). $\int_0^\infty \int_0^x x e^{-x^2/y}\,dy\,dx$



$x = 0, \infty \quad y = 0, x$

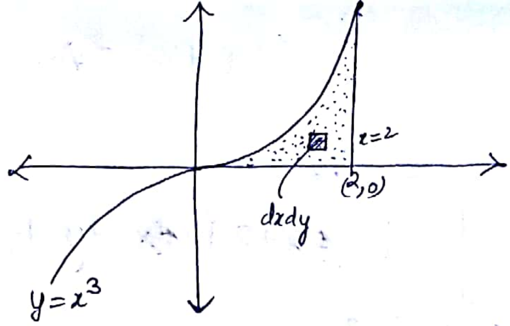
$$A = \int_{y=0}^{\infty}\int_{x=y}^{\infty} x\, e^{-x^2/y}\,dx\,dy$$

$x^2 = t$
$x\,dx = \frac{dt}{2}$

$$= \int_{y=0}^{\infty}\int_{y^2}^{\infty} e^{-t/y}\,\frac{dt}{2}\,dy = \frac{1}{2}\int_0^\infty \left[-\frac{e^{-t/y}}{1/y}\right]_{y^2}^{\infty} dy = -\frac{1}{2}\int_0^\infty y\left[0 - e^{-y}\right] dy$$

$$= 1/2$$

(d) $\int_0^8 \int_{y^{1/3}}^{2} \sqrt{x^4+1}\, dx\, dy$



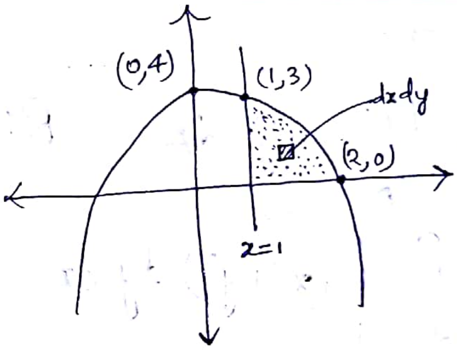
$y = 0, 8 \qquad x = y^{1/3}, 2$

$\qquad\qquad x^3 = y$

$$A = \int_{x=0}^{2}\int_{y=0}^{x^3} \sqrt{x^4+1}\; dy\, dx = \int_{x=0}^{2} x^3\sqrt{x^4+1}\,dx$$

$$= \frac{1}{4}\int_0^2 4x^3\sqrt{x^4+1}\,dx = \frac{1}{4}\left[\frac{2}{3}(x^4+1)^{3/2}\right]_0^2 = \frac{1}{4}\left[\frac{2}{3}\left(17^{3/2}-1\right)\right] = \frac{(17^{3/2}-1)}{6}$$

(e). $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$



$y = 0, 3 \qquad x = 1, \sqrt{4-y} \longrightarrow x^2 = 4-y$

$$A = \int_{x=1}^{2}\int_{y=0}^{4-x^2} (x+y)\, dy\, dx$$

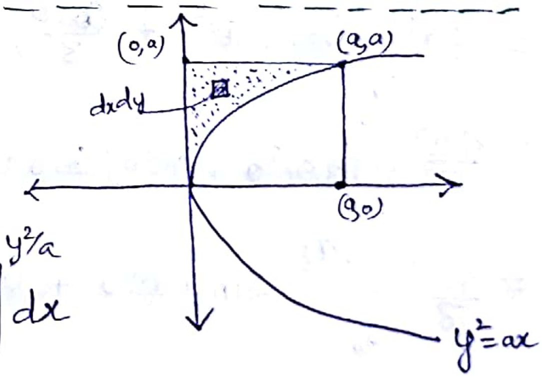
$$= \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2}\right] dx = \int_1^2 \left[4x - x^3 + \frac{(16+x^4-8x^2)}{2}\right] dx$$

$$= \frac{1}{2}\int_1^2 \left[8x - 2x^3 + 16 + x^4 - 8x^2\right] dx = \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^4}{2} - \frac{8x^3}{3} + 4x^2 + 16x\right]_1^2$$

$$= \frac{1}{2}\left[\left(\frac{32}{5} - \frac{16}{2} - \frac{64}{3} + 16 + 32\right) - \left(\frac{1}{5} - \frac{1}{2} - \frac{8}{3} + 4 + 16\right)\right]$$

$$= \frac{241}{60}$$

(f). $\int_0^a \int_{\sqrt{ax}}^{a} \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dy\, dx$



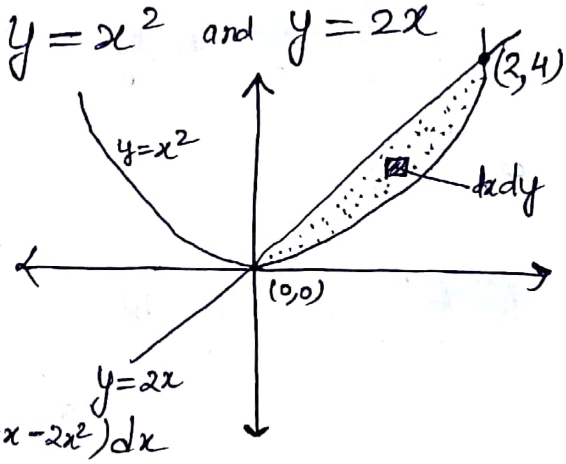
$x = 0, a \qquad y = \sqrt{ax}, a$

$\qquad\qquad y^2 = ax$

$$A = \int_{y=0}^{a}\int_{x=0}^{y^2/a} \frac{y^2}{\sqrt{y^4-a^2x^2}}\, dx\, dy = \int_{y=0}^{a}\left[\frac{y^2}{a}\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{y^2/a} dx$$

$$= \int_{y=0}^{a}\left[\frac{y^2}{a}\sin^{-1}(1) - 0\right] dx = \int_{y=0}^{a} \frac{\pi}{2a} y^2\, dx = \frac{\pi}{2a}\left[\frac{a^3}{3}\right] = \frac{\pi a^2}{6}$$

3) (a). $\iint_D (4x+2)\,dA$ D: $y=x^2$ and $y=2x$

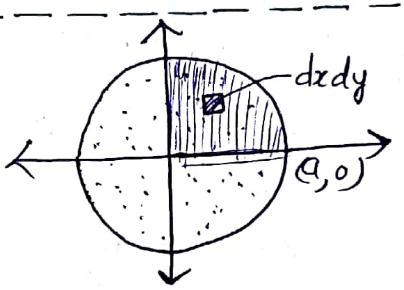


$$= \int_{x=0}^{2}\int_{y=x^2}^{2x} (4x+2)\,\cancel{dx}\,dy\,dx$$

$$= \int_{x=0}^{2} (4x+2)\left[2x-x^2\right]dx = \int_{0}^{2}\left(8x^2-4x^3+4x-2x^2\right)dx$$

$$= \int_{0}^{2}\left(6x^2-4x^3+4x\right)dx = \left[2x^3-x^4+2x^2\right]_0^2 = \left[16-16+8\right] = 8$$

(b). $\iint_R (x^2+y^2)\,dA$: R: $x^2+y^2 \le a^2$



$$=4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$$

$$= 4\int_{x=0}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}} dx = 4\int_{x=0}^{a}\left[x^2\sqrt{a^2-x^2}+\frac{(a^2-x^2)^{3/2}}{3}\right]dx$$

$x = a\sin\theta$ $dx = a\cos\theta\, d\theta$ $\theta \to 0$ to $\pi/2$

$$= 4\int_{0}^{\pi/2}\left[a^2\sin^2\theta\, a\cos\theta + \frac{a^3\cos^3\theta}{3}\right] a\cos\theta\, d\theta = 4a^4\int_{0}^{\pi/2}\left[\sin^2\theta\cos\theta+\frac{\cos^3\theta}{3}\right]\cos\theta\, d\theta$$

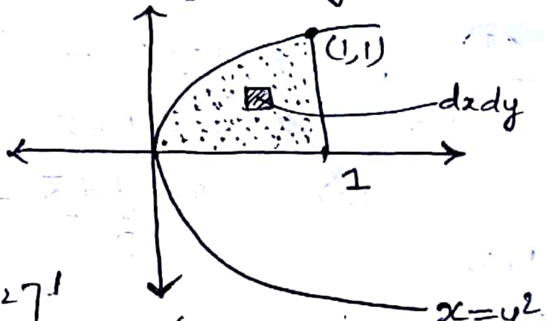
$$= 4a^4\int_{0}^{\pi/2}\left[\sin^2\theta\cos^2\theta+\frac{\cos^4\theta}{3}\right]d\theta = \frac{4a^4}{3}\int_{0}^{\pi/2}\left[3\sin^2\theta\cos^2\theta+\cos^4\theta\right]d\theta$$

$$= \frac{4a^4}{3}\int_{0}^{\pi/2}\left[3\sin^2\theta+\cos^2\theta\right]\cos^2\theta\, d\theta = \frac{4a^4}{3}\int_{0}^{\pi/2}\left[2\sin^2\theta+1\right]\cos^2\theta\, d\theta$$

$$= \frac{4a^4}{3}\int_{0}^{\pi/2}\left[2\sin^2\theta\cos^2\theta+\cos^2\theta\right]d\theta = \frac{4a^4}{3}\left[2\cdot\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{2+2+2}{2}}+\frac{\pi}{4}\right]$$

$$= \frac{4a^4}{3}\left[\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\pi}{\Gamma 3}+\frac{\pi}{4}\right] = \frac{4a^4}{3}\left[\frac{\pi}{8}+\frac{\pi}{4}\right] = \frac{4a^4}{3}\left[\frac{3\pi}{8}\right] = \frac{\pi a^4}{2}$$

(c) $\int_0^1 \int_{y^2}^1 y e^{x^2}\,dx\,dy$. Difficult to integrate e^{x^2} w.r.to x So changing the order of integration.



$y=0,1 \quad x=y^2,1$

$$\int_{x=0}^1 \int_{y=0}^{\sqrt{x}} y e^{x^2}\,dy\,dx = \int_{x=0}^1 \frac{x}{2} e^{x^2}\,dx$$

$$= \frac{1}{4}\int_0^1 e^{x^2} 2x\,dx = \frac{1}{4}\int_0^1 e^{x^2}_{2x}dx = \frac{1}{4}\left[e^{x^2}\right]_0^1 = \frac{(e-1)}{4}$$

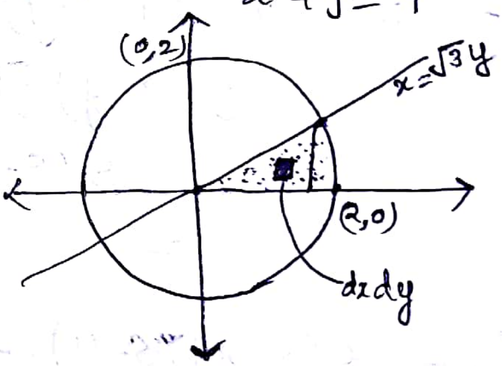
(d). $\int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\,dx\,dy$

$1 \le y \le 1$

$\sqrt{3}y \le x \le \sqrt{4-y^2}$

$\sqrt{3}y = x \qquad x=\sqrt{4-y^2} \longrightarrow x^2 = 4-y^2$

$x^2+y^2=4$



~~Changing the order of integration~~

$\cancel{\int_{x=0}^{\sqrt{3}} \int_{y=0}^{x/\sqrt{3}} \sqrt{x^2+y^2}\,dy\,dx + \int_{x=\sqrt{3}}^{2} \int_{y=0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx}$

Changing to polar form

$x = r\cos\theta \; , \; y = r\sin\theta$

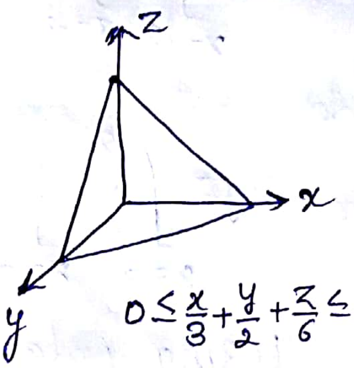
$dx\,dy = r\,dr\,d\theta$

$$\int_{\theta=0}^{\pi/6} \int_{r=0}^{2} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; r\,dr\,d\theta = \int_{\theta=0}^{\pi/6} \int_{r=0}^{2} r^2\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/6} \left[\frac{r^3}{3}\right]_0^2 d\theta = \frac{8}{3}\,\frac{\pi}{6} = \frac{4\pi}{9}$$

4 (a) $\iiint_E 2x\,dv$ E: $2x+3y+z=6$ first octant

ie $\frac{x}{3}+\frac{y}{2}+\frac{z}{6}=1$



$0 \le \frac{x}{3}+\frac{y}{2}+\frac{z}{6} \le 1$

take $\frac{x}{3}=u$, $\frac{y}{2}=v$, $\frac{z}{6}=w$

$dx=3du$, $dy=2dv$, $dz=6dw$

We have, from Dritchlet Integral

if $0 \le x+y+z \le 1$, then

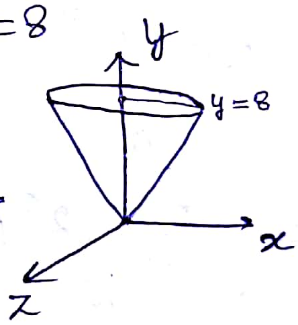
$$\iiint x^{l-1}y^{m-1}z^{n-1}\,dz\,dy\,dx = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

$$\iiint_E 2x\,dv = \iiint_E 2(3u).(3du)(2dv)(6dw)$$

$$= 216\iiint u^{2-1}v^{1-1}w^{1-1}\,du\,dv\,dw$$

$$= 216\frac{\Gamma 2\,\Gamma 1\,\Gamma 1}{\Gamma(2+1+1+1)} = 216\frac{1!\,0!\,0!}{4!} = \frac{216}{4\cdot3\cdot2} = 9$$

(b) $\iiint_E \sqrt{3x^2+3z^2}\,dv$ E: $y=2x^2+2z^2$, $y=8$



$$\int_{x=-2}^{2}\int_{z=-2}^{2}\int_{y=2x^2+2z^2}^{8}\sqrt{3x^2+3z^2}\,dy\,dz\,dx$$

$$\left.\begin{array}{l} 2x^2+2z^2 \le y \le 8 \\ -2 \le z \le 2 \\ -2 \le x \le 2 \end{array}\right\}$$

$$\int_{x=-2}^{2}\int_{z=-2}^{2}[y]_{2x^2+2z^2}^{8}\sqrt{3}\sqrt{x^2+z^2}\,dz\,dx$$

$$\sqrt{3}\int_{x=-2}^{2}\int_{z=-2}^{2}\left[(2x^2+2z^2)-8\right]\sqrt{x^2+z^2}\,dz\,dx$$

$$2\sqrt{3}\int_{x=-2}^{2}\int_{z=-2}^{2}(x^2+z^2-4)\sqrt{x^2+z^2}\,dz\,dx$$

$$= 2\sqrt{3}\int_{x=-2}^{2}\int_{z=-2}^{2}(x^2+z^2-4)\sqrt{x^2+z^2}\,dz\,dx$$

$\uparrow$ even function

so

$$= 8\sqrt{3}\int_{x=0}^{2}\int_{z=0}^{2}(x^2+z^2-4)\sqrt{x^2+z^2}\,dz\,dx$$

Using polar coordinates in x-z plane

$x = r\cos\theta \quad z = r\sin\theta$

$$= 8\sqrt{3}\int_{\theta=0}^{\pi/2}\int_{r=0}^{2}(r^2-4)\,r.\,r\,dr\,d\theta$$

$$= 8\sqrt{3}\int_{\theta=0}^{\pi/2}\int_{r=0}^{2}(r^4-4r^2)\,dr\,d\theta$$

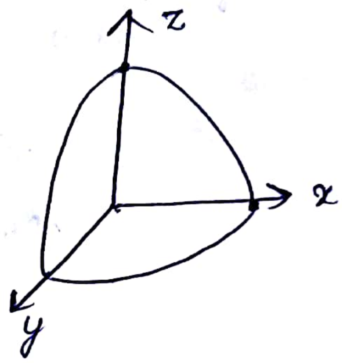
$$= 8\sqrt{3}\int_{\theta=0}^{\pi/2}\left[\frac{r^5}{5}-\frac{4r^3}{3}\right]_0^2 d\theta = 8\sqrt{3}\int_{\theta=0}^{\pi/2}2^5\left(\frac{1}{5}-\frac{1}{3}\right)d\theta$$

$$= \frac{8\sqrt{3}.\,2^5}{15}(-2)\left(\frac{\pi}{2}\right) = \frac{2^8\sqrt{3}\,\pi}{15}$$

4(c). $\iiint_E xyz\,dV \quad E:\ x^2+y^2+z^2=4 \quad x,y,z \geq 0$

$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2+\left(\frac{z}{2}\right)^2=1$$

$$u+v+w=1$$



$$\left(\frac{x}{2}\right)^2=u, \quad \left(\frac{y}{2}\right)^2=v, \quad \left(\frac{z}{2}\right)^2=w$$

$x=2\sqrt{u}$

$y=2\sqrt{v} \quad z=2\sqrt{w}$

$dx=u^{-1/2}du, \quad dy=v^{-1/2}dv, \quad dz=w^{-1/2}dw$

Now $\quad 0 \leq u+v+w \leq 1 \quad u,v,w \geq 0$

Apply Dirichlet theorem

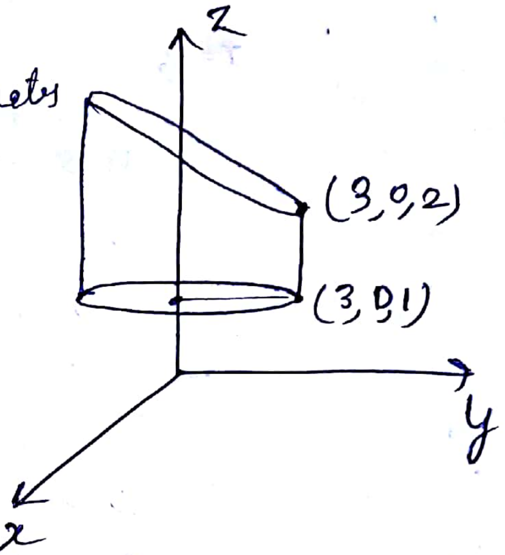
$$\iiint_V 2\sqrt{u}\,2\sqrt{v}\,2\sqrt{w}\,.u^{-1/2}du\,v^{-1/2}dv\,w^{-1/2}dw$$

$$=8\iiint_V du\,dv\,dw$$

$$=8\iiint_V u^{1-1}v^{1-1}w^{1-1}du\,dv\,dw = 8\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma 1+1+1+1} = 8\frac{1}{\Gamma 4}$$

$$=\frac{8}{\lfloor 3}=\frac{8}{6}=\frac{4}{3}$$

4(e). $\iiint_E dV \quad E : x^2+y^2=9 \quad z=1 \quad x+z=5$



Cylindrical coordinates

$x = r\cos\theta, \quad y = r\sin\theta$

$z = z$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=1}^{5-r\cos\theta} r\,dz\,d\theta\,dr$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{3} r\left[5-r\cos\theta-1\right] dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\left[4r - r^2\cos\theta\right] dr\,d\theta$$

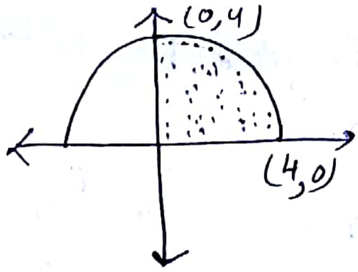
$$= \int_{\theta=0}^{2\pi}\left[2r^2 - \frac{r^3}{3}\cos\theta\right]_{r=0}^{3} d\theta = \int_{\theta=0}^{2\pi}\left[18 - 9\cos\theta\right] d\theta$$

$$= 9\int_{\theta=0}^{2\pi}\left[2-\cos\theta\right] d\theta = 9\left[2\theta - \sin\theta\right]_0^{2\pi}$$

$$= 9\left[(4\pi - 0) - (0)\right] = 36\pi$$

5 (a) $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{16-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$ $y=\sqrt{16-x^2}$ $0 \le x \le 4$

$x^2+y^2=16$



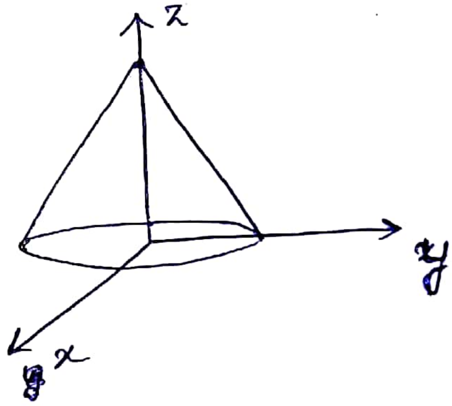
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{4} \int_{z=0}^{16-r^2} r\, dz\, r\,dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{4} r^2(16-r^2)\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[16\frac{r^3}{3} - \frac{r^5}{5} \right]_{r=0}^{4} d\theta = 4^5 \frac{2}{15} \cdot \frac{\pi}{2} = \frac{1024\pi}{15}$$

(b). $\iiint_E \sqrt{x^2+y^2}\, dV$ E: above xy-plane & below the cone $z = 4 - \sqrt{x^2+y^2}$



$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{4} \int_{z=0}^{4-r} r\; r\,dz\, d\theta\, dr$$

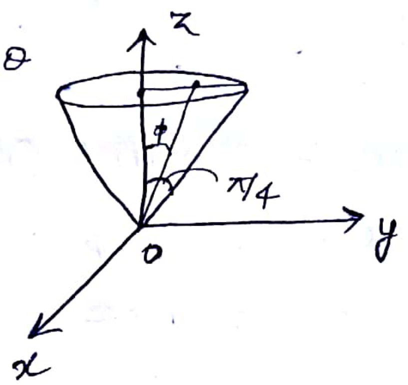
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} r^2 [4-r]\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{4r^3}{3} - \frac{r^4}{4} \right]_0^4 d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{4.4^4 - 3.4^4}{12} \right] d\theta = \frac{4^4}{12} \cdot 2\pi = \frac{64}{3}(2\pi)$$

6 (a). $\iiint_E (x^2+y^2+z^2)^{1/2}\, dV \qquad z=3$ and $z=\sqrt{x^2+y^2}$

$x = r\sin\phi\cos\theta\,,\ y = r\sin\phi\sin\theta\,,\ z = r\cos\theta$



$$|J| = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = r^2\sin\phi$$

$$r = \frac{3}{\cos\phi} = 3\sec\phi \qquad 0\le\phi\le\frac{\pi}{4}$$
$$0\le\theta = 2\pi$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{r=0}^{3\sec\phi} r(r^2\sin\phi)\, dr\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{r=0}^{3\sec\phi} r^3 \sin\phi\, dr\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\left[\frac{3^4\sec^4\phi}{4}\right]\sin\phi\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\frac{3^4}{4}\frac{\sin\phi}{\cos^4\phi}\, d\phi\, d\theta$$

$$= \frac{3^4}{4}\, 2\pi\int_{\phi=0}^{\pi/4}\frac{\sin\phi}{\cos^4\phi}\, d\phi$$

$$= \frac{3^4}{4}2\pi\left[\frac{1}{3\cos^3\phi}\right]_0^{\pi/4} = \frac{3^4 \cdot 2\pi}{5\ \ 3}\left[\frac{1}{(1/\sqrt{2})^3} - \frac{1}{1}\right]$$

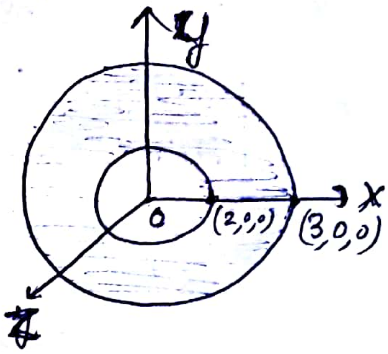
$$= \frac{3^4\, 2\pi}{4\cdot 3}\left[2\sqrt{2}-1\right] = \frac{3^3\cdot 2\pi}{4}\cdot\left(2\sqrt{2}-1\right)$$

$$= \frac{27}{2}\left(2\sqrt{2}-1\right)$$

(b). $\iiint_E (x^2+y^2+z^2)^{-3/2}\, dv$

E: $x^2+y^2+z^2=4$

$x^2+y^2+z^2=9$



$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=2}^{3} r^{-3}\, r^2\sin\phi\, dr\, d\theta\, d\phi$$

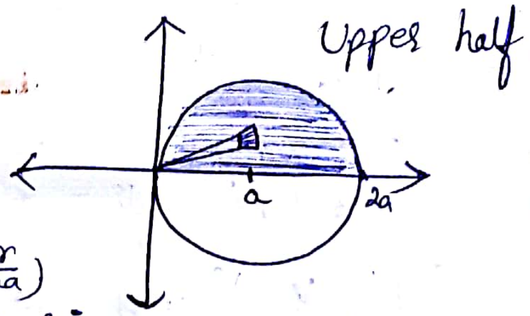
$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=2}^{3} r^{-1}\sin\phi\, dr\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\Big[\operatorname{Log} r\Big]_2^3 \sin\phi\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\left(\operatorname{Log}\frac{3}{2}\right)\sin\phi\, d\phi\, d\theta$$

$$= \left(2\pi\operatorname{Log}\frac{3}{2}\right)\Big[-\cos\phi\Big]_0^{\pi} = 4\pi\log\frac{3}{2}$$

(7) (a) $\int_0^{\pi/2} \int_0^{2a\cos\theta} f(r,\theta)\, dr\, d\theta$



$0 \le \theta \le \pi/2 \qquad 0 \le r \le 2a\cos\theta$

$\theta = 0 \qquad r = 0 \qquad r = 2a\cos\theta \longrightarrow \theta = \cos^{-1}\left(\frac{r}{2a}\right)$

$\theta = \pi/2$

Changing the order of integration

$\theta = 0$ to $\cos^{-1}\left(\frac{r}{2a}\right)$ and $r = 0$ to $2a$

$$\int_{r=0}^{2a} \int_{\theta=0}^{\cos^{-1}\left(\frac{r}{2a}\right)} f(r,\theta)\, d\theta\, dr.$$

(b) $\int_0^{\pi/3} \int_{a\sec^2(\theta/2)}^{\left(\frac{8a}{3}\right)\cos\theta} f(r,\theta)\, dr\, d\theta$

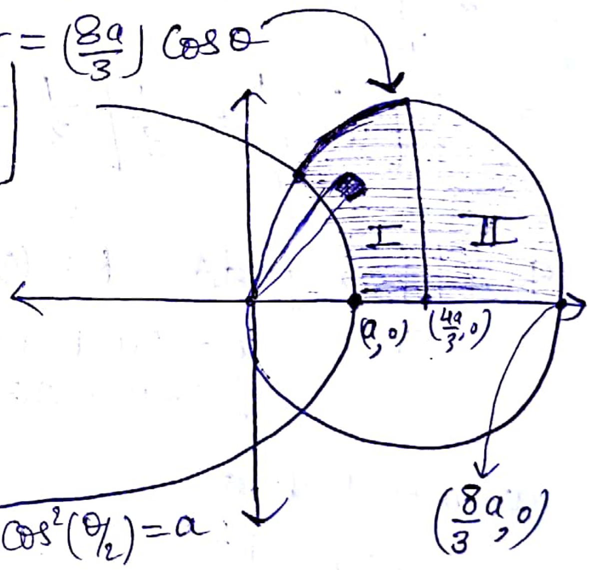
to $\theta = 0$ $\qquad r = a\sec^2(\theta/2) \qquad r = \frac{8a}{3}\cos\theta$

$\theta = \pi/3$ $\qquad \hookrightarrow r\cos^2(\theta/2) = a \qquad r = \left(\frac{8a}{3}\right)\cos\theta$

$\theta = 2\cos^{-1}\left(\sqrt{\frac{a}{r}}\right)$

and

$\theta = \cos^{-1}\left(\frac{3r}{8a}\right)$



For region I:

θ varies from 0 to $2\cos^{-1}\left(\sqrt{\frac{a}{r}}\right)$

So

$$I = \int_{r=a}^{\frac{4a}{3}} \int_{\theta=0}^{2\cos^{-1}\left(\sqrt{\frac{a}{r}}\right)} f(r,\theta)\, \cancel{dr}\, d\theta\, dr$$

For region II

θ varies from 0 to $\cos^{-1}\left(\frac{3r}{8a}\right)$

So

$$II = \int_{r=\frac{4a}{3}}^{\frac{8a}{3}} \int_{\theta=0}^{\cos^{-1}\left(\frac{3r}{8a}\right)} f(r,\theta)\, d\theta\, dr$$

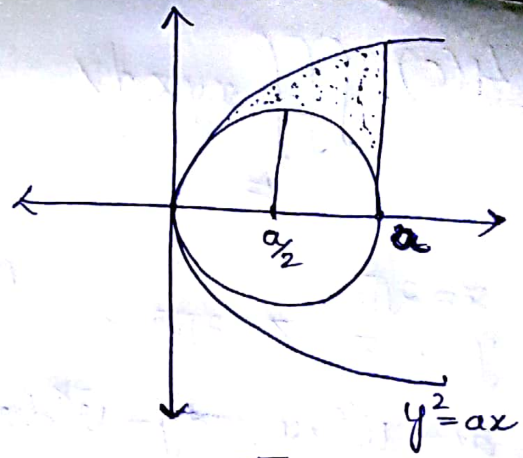
thus

$$\int_0^{\pi/3} \int_{a\sec^2\theta/2}^{\left(\frac{8a}{3}\right)\cos\theta} f(r,\theta)\, dr\, d\theta = I + II$$

(c). $\displaystyle\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x,y)\,dy\,dx$

$x=0, a$ $\qquad y=\sqrt{ax-x^2}$ $\qquad y=\sqrt{ax}$

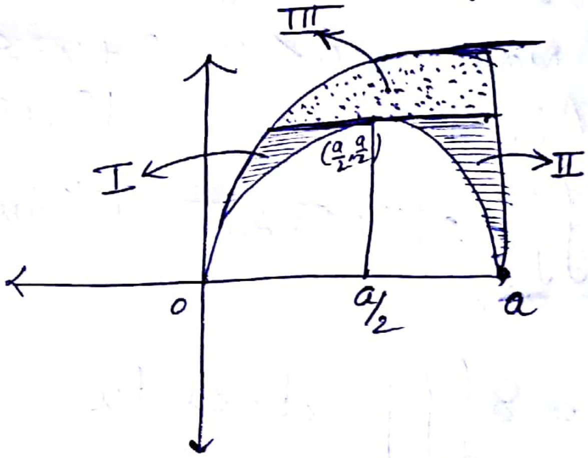
$x^2+y^2=ax$ $\qquad y^2=ax$



$x^2-ax+y^2=0$

$$x=\frac{a\pm\sqrt{a^2-4y^2}}{2}$$

$$x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-y^2}$$



for region

I: $\quad x=\dfrac{a}{2}-\sqrt{\dfrac{a^2}{4}-y^2}$

II: $\quad x=\dfrac{a}{2}+\sqrt{\dfrac{a^2}{4}-y^2}$

Integration over region I

$$I=\int_{y=0}^{\frac{a}{2}}\int_{x=\frac{y^2}{a}}^{\frac{a}{2}-\sqrt{\frac{a^2}{4}-y^2}} f(x,y)\,dx\,dy$$

Integration over region II

$$II=\int_{y=0}^{a/2}\int_{x=\frac{a}{2}+\sqrt{\frac{a^2}{4}-y^2}}^{a} f(x,y)\,dx\,dy$$

Integration over the region III

$$III=\int_{y=a/2}^{a}\int_{x=\frac{y^2}{a}}^{a} f(x,y)\,dx\,dy$$

thus

$$\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x,y)\,dy\,dx = I + II + III$$

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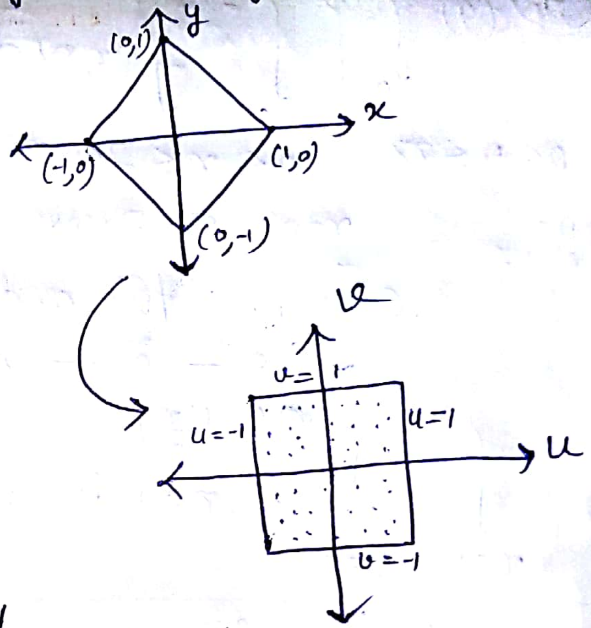
8 $\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy$ R: $x+y=\pm1$, $x-y=\pm1$



$u = x+y \qquad v = x-y$

$u=\pm1 \qquad v=\pm1$

$$\iint \left(\frac{v}{u+2}\right)^2 J\,du\,dv$$

Now $u=x+y$, $v=x-y$ $\Rightarrow$ $x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

So

$$\int_{u=-1}^{1}\int_{v=-1}^{1} \frac{v^2}{(u+2)^2}\left(-\frac{1}{2}\right)dv\,du = -\frac{1}{2}\int_{u=-1}^{1}\frac{1}{(u+2)^2}\left[\frac{v^3}{3}\right]_{-1}^{1}du$$

$$= -\frac{1}{2}\int_{u=-1}^{1}\frac{1}{(u+2)^2}\left[\frac{2}{3}\right]du = -\frac{1}{3}\int_{u=-1}^{1}(u+2)^{-2}du = -\frac{1}{3}\left[\frac{(u+2)^{-1}}{-1}\right]_{-1}^{1}$$

$$= \frac{1}{3}\left[\frac{1}{u+2}\right]_{-1}^{1} = \frac{1}{3}\left[\frac{1}{3}-\frac{1}{1}\right] = -\frac{2}{9}$$

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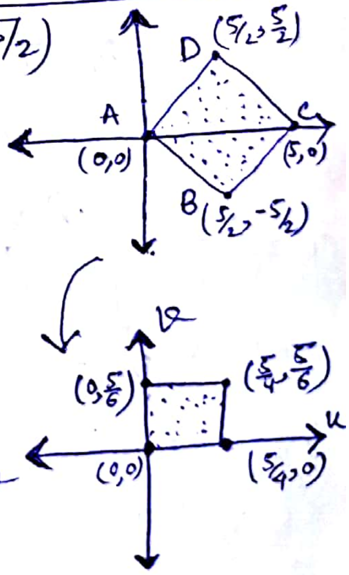
9 $\iint_R (x+y)\,dA$ R: $(0,0)$, $(5,0)$, $(\frac{5}{2},\frac{5}{2})$, $(\frac{5}{2},-\frac{5}{2})$



$x = 2u+3v \qquad u=\frac{x+y}{4}$

$y = 2u-3v \qquad v=\frac{x-y}{6}$

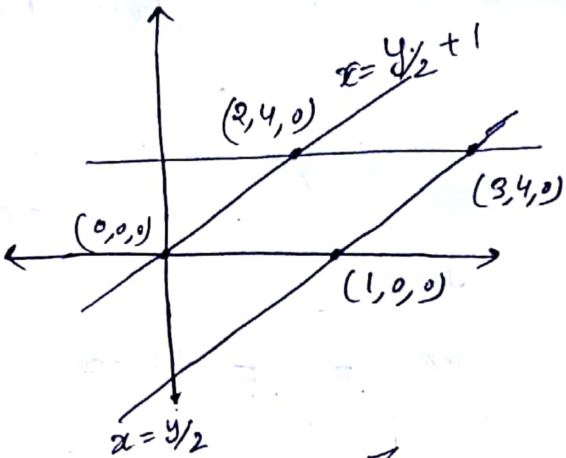
$dx\,dy = J\,du\,dv$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12$$

$$\iint_R (x+y)\,dx\,dy = \int_{v=0}^{5/6}\int_{u=0}^{5/4} 4u\,(-12)\,du\,dv = (-12)\cdot\int_{v=0}^{5/6}\left[2u^2\right]_{u=0}^{5/4}dv$$

$$= (-12)\,2\int_{v=0}^{5/6}\left(\frac{5}{4}\right)^2 dv = -\frac{24\cdot 25}{16}\cdot\frac{5}{6} = -\frac{125}{4}$$

(11) $\displaystyle\int_0^3\int_0^4\int_{x=y/2}^{x=y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)$



(u,v,w)	(x,y,z)
(0,0,0)	(0,0,0)
(1,0,0)	(1,0,0)
(1,2,0)	(3,4,0)
(0,2,0)	(2,4,0)
(1,0,1)	(1,0,3)
(1,2,1)	(3,4,3)
(0,2,1)	(2,4,3)
(0,0,1)	(0,0,3)

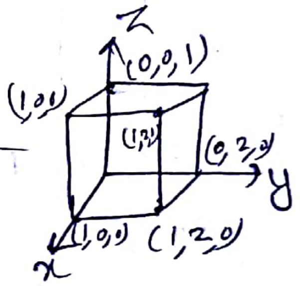
$u=\frac{2x-y}{2}$

$v=y/2$

$w=z/3$

$x=u+v$

$y=2v$ $\quad z=3w$



$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}1&1&0\\0&2&0\\0&0&3\end{vmatrix}=6$$

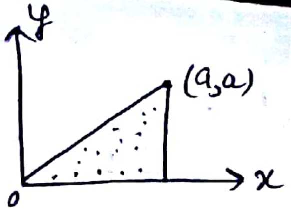
$$\int_{w=0}^{1}\int_{v=0}^{2}\int_{u=0}^{1}(u+w)6\,du\,dv\,dw$$

$$=6\int_{w=0}^{1}\int_{v=0}^{2}\left[\frac{u^2}{2}+wu\right]_0^1 dv\,dw$$

$$=6\int_{w=0}^{1}\left[\frac{v}{2}+wv\right]_0^2 dw=6\int_{w=0}^{1}[1+2w]\,dw$$

$$=6\left[w+w^2\right]_0^1=6\left[1+1^2\right]=12$$

12 (a) $\int_0^a \int_0^x \frac{f'(y)\,dy\,dx}{\sqrt{(a-x)(x-y)}} = \pi(f(a)-f(0))$



Changing the order of integration, we get

$$= \int_{y=0}^{a} \int_{x=y}^{a} \frac{f'(y)\,dx\,dy}{\sqrt{(a-x)(x-y)}} \longrightarrow \sqrt{ax-ay-x^2+xy} \to \sqrt{-x^2+(a+y)x-ay}$$

$$= \int_{y=0}^{a} \int_{x=y}^{a} \frac{f'(y)}{\sqrt{\left(\frac{a-y}{2}\right)^2 - \left(x-\left(\frac{a+y}{2}\right)\right)^2}}\,dx\,dy$$

$$= \int_{y=0}^{a} f'(y) \int_{x=y}^{a} \frac{dx}{\sqrt{\left(\frac{a-y}{2}\right)^2 - \left(x-\left(\frac{a+y}{2}\right)\right)^2}}\,dy$$

$$= \int_{y=0}^{a} f'(y) \left[\sin^{-1}\left(\frac{x-\left(\frac{a+y}{2}\right)}{\left(\frac{a-y}{2}\right)}\right)\right]_y^a dy$$

$$= \int_{y=0}^{a} f'(y) \left[\sin^{-1}(1) - \sin^{-1}(-1)\right] dy$$

$$= \int_{y=0}^{a} f'(y) \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] dy$$

$$= \pi \int_{y=0}^{a} f'(y)\,dy$$

$$= \pi \left[f(y)\right]_0^a$$

$$= \pi \left[f(a) - f(0)\right]$$

(b). $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{b\,dy\,dx}{(x^2+y^2+b^2)^{3/2}(x^2+y^2+a^2)^{1/2}} = \frac{2\pi}{a+b}$$

$x = r\cos\theta \quad y = r\sin\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}\frac{b\,r\,dr\,d\theta}{(r^2+b^2)^{3/2}(r^2+a^2)^{1/2}}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}\frac{b\,r\,dr\,d\theta}{\frac{(r^2+b^2)^{3/2}}{(r^2+a^2)^{3/2}}\cdot(r^2+a^2)^2}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}\frac{b\,r\,dr\,d\theta}{\left(\frac{r^2+b^2}{r^2+a^2}\right)^{3/2}\cdot(r^2+a^2)^2}$$

Let $\frac{r^2+b^2}{r^2+a^2} = t \implies \frac{(r^2+a^2)2r-(r^2+b^2)2r}{(r^2+a^2)^2}dr = dt$

$$\implies \frac{2r(a^2-b^2)}{(r^2+a^2)^2}dr = dt$$

limits $r=0$ to ∞
$t=\frac{b^2}{a^2}$ to 1

$$= \int_{\theta=0}^{2\pi}\int_{t=\frac{b^2}{a^2}}^{1}\frac{b\,dt\,d\theta}{2(a^2-b^2)\,t^{3/2}}$$

$$= \frac{1}{2(a^2-b^2)}\int_{\theta=0}^{2\pi} b\left[\frac{t^{-3/2+1}}{-3/2+1}\right]_{t=b^2/a^2}^{1} d\theta$$

$$= \frac{b}{2(a^2-b^2)}\int_{\theta=0}^{2\pi}\left[\frac{t^{-1/2}}{-1/2}\right]_{t=b^2/a^2}^{1} d\theta$$

$$= \frac{b}{2(a^2-b^2)} 2\pi \left[\frac{t^{-1/2}}{-1/2}\right]_{t=b^2/a^2}^{1}$$

$$= \frac{b}{2(a^2-b^2)} \frac{2\pi}{(-1/2)} \left[1 - \frac{1}{\sqrt{b^2/a^2}}\right]$$

$$= \frac{b\,(-2\pi)}{(a^2-b^2)} \left[1 - \frac{1}{b/a}\right]$$

$$= \frac{b}{(a^2-b^2)} (-2\pi) \left[\frac{b-a}{b}\right]$$

$$= \frac{-2\pi b}{(a+b)(a-b)} \left(\frac{b-a}{b}\right)$$

$$= \frac{2\pi b}{(a+b)} \frac{1}{b}$$

$$= \frac{2\pi}{(a+b)}$$