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Assignment : 2 Matrix : 2

1) a) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

eigenvalues : $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 10\lambda^2 + 28\lambda - 30 = 0$

$(\lambda-2)^2 (\lambda-6) = 0 \Rightarrow \lambda = 2, 2, 6$

eigenvectors : $(A - \lambda I) X = 0$

$\lambda = 2$: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$X = [1 \quad 0 \quad -1]^T$

$\lambda = 6$: $\begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$X = [1 \quad 2 \quad 1]^T$

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ eigenvalue = $\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$

$\Rightarrow (1-\lambda)^3 = 0$

$\lambda = 1, 1, 1$

eigen vector : $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$X = [j \quad 0 \quad k] \; ; \; j, k \in R$

c) $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \xrightarrow{\lambda} \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$

$\Rightarrow \quad (\lambda+2)^2 \cdot (\lambda-4) = 0$

$\Rightarrow \quad \lambda = -2, -2, 4$

eigenvector : $\lambda = 2$: $\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$X = [1, 0 -1]^T$.

$\lambda = 4$: $\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$X = [1 \ 1 \ 2]^T$

2) a) i) $AX = \lambda X \Rightarrow X = \lambda A^{-1} X$

$\Rightarrow \frac{1}{\lambda} X = A^{-1} X$

eigenvalue of $A^{-1} \rightarrow \frac{1}{\lambda}$

eigenvector of $A^{-1} \rightarrow X$

ii) Consider $AX = \lambda X$

$AX - kX = \lambda X - kX$

$(A - kI) X = (\lambda - k) X$

eigenvalue : $\lambda - k$

eigenvector : X

b) Let, $AX = \lambda X \Rightarrow (A - \lambda I) X = 0$

$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow A^T - \lambda I = 0$

Now $A^T Y = \lambda Y$ and $AX = \lambda X$

if $Y = X$ then $A = A^T$

possible only when $A \to$ symmetric

c) let $AX = \lambda X$

$P^{-1} A X = \lambda P^{-1} X$

$P^{-1} A P P^{-1} X = \lambda P^{-1} X$

$(P^{-1} A P) P^{-1} X = \lambda (P^{-1} X)$

$\Rightarrow$ eigenvalue of $P^{-1} A P \to \lambda$

eigenvector of $P^{-1} A P \to P^{-1} X$

3) a) $AX = \lambda X$

$\bar{X}^T A X = \lambda \bar{X}^T X \Rightarrow \lambda = \dfrac{\bar{X}^T A X}{\bar{X}^T X}$

$\bar{X}^T X = X \cdot X \geq 0$ and $\bar{X}^T AX \to$ scalar (complex or Real)

taking transpose of scalars makes no alternation to them hence,

$(\bar{X}^T A X)^T = \bar{X}^T \bar{A}^T X = \bar{X}^T A X \quad (\bar{A}^T = A)$

$\Rightarrow \overline{(\bar{X}^T A X)} = \bar{X}^T A X \Rightarrow$ Real

b) from (a) $\overline{(\bar{X}^T A X)} = \bar{X}^T \bar{A}^T X = -\bar{X}^T A X$

$\Rightarrow \bar{X}^T A X \to$ purely imaginary

c) let $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$

$\Rightarrow AX_1 \cdot X_2 = \lambda_1 X_1 \cdot X_2$

$= (\overline{AX_1})^T X_2$

$= \bar{X}_1^T \bar{A}^T X_2 = \bar{X}_1^T A X_2$

$= \bar{X}_1^T \lambda_2 X_2 = \lambda_2 (X_1 \cdot X_2)$

$\Rightarrow X_1 \cdot X_2 = 0$

d) Let $AX = \lambda X$

$\Rightarrow (\bar{A}\bar{X})^T = \bar{\lambda}\bar{X}^T$

$(\bar{A}\bar{X})^T(AX) = \lambda\bar{\lambda}\bar{X}^T X$

$\bar{X}^T\bar{A}^T A X = |\lambda|^2 \bar{X}^T X$

$\bar{X}^T X = |\lambda|^2 \bar{X}^T X$

$\Rightarrow |\lambda|^2 = 1 \quad \because (X \neq 0)$

e) Let $A \to n \times n$ and $A^T = -A$

$|A| = |A^T| = (-1)^n |A|$

$\Rightarrow |A|(1 - (-1)^n) = 0$

if $n \in 2k+1 \Rightarrow |A| = 0$

$n \in 2k \Rightarrow |A| \neq 0$

f) Let $AX = \lambda X$

$A^2 X = \lambda AX = \lambda^2 X$

$AX = \lambda^2 X \quad (A^2 = A)$

from quick induction it follows that $AX = \lambda^n X$

hence $\lambda = \lambda^n$

$\Rightarrow \lambda = 0, 1$

e) Let $AX = \lambda X$

$A^2 X = \lambda^2 X$

$0 = \lambda^2 X \quad (A^2 = 0)$

$\Rightarrow \lambda = 0 \quad (X \neq 0)$

4) Given : $ABX = \lambda X$

To Prove : $BAY = \lambda Y$

$ABX = \lambda X$

$(BA)BX = \lambda BX$

$BAY = \lambda Y \quad (Y = BX)$

5)

7)

assume $\lambda I - BA$ is invertible $\Rightarrow \lambda = 1$

given $(BA - \lambda I)X = 0$

above implies AB also have $\lambda = 1$

$\Rightarrow (AB - \lambda I)X = 0$ for $\lambda = 1$

hence $AB - I$ or $I - AB$ is invertible

5) $(\overline{A + iB})^T = \bar{A}^T - i\bar{B}^T = A + iB$ $\left(\begin{matrix} A^T = A \\ \cancel{B^T = -B} \end{matrix}\right)$

hence $A + iB$ is hermitian

7) a) Given,

$$A\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ put A in above

$$\left.\begin{aligned} -2a_{11} + a_{12} &= -4 \\ -a_{11} + a_{13} &= -2 \\ a_{11} + a_{13} &= 4 \end{aligned}\right\} \quad \begin{aligned} a_{11} &= 3 \\ a_{12} &= 2 \\ a_{13} &= 1 \end{aligned}$$

similarly, $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

b) i) $A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$

P will be column wise eigenvectors of A

eigenvalue of A : $\begin{vmatrix} 4-\lambda & -2 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -2 & 4-\lambda \end{vmatrix} = 0$

$\lambda(\lambda-6)(\lambda-4) = 0$

$\lambda = 0, 6, 4$

for $\lambda = 0$: $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$[x_1 \ x_2 \ x_3]^T = [1 \ 2 \ 1]$

for $\lambda = 6$: $[x_1 \ x_2 \ x_3]^T = [-1 \ 0 \ 1]$

for $\lambda = 4$: $[x_1 \ x_2 \ x_3]^T = [1 \ -1 \ 1]$

$\Rightarrow P = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

6) Given $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ → skew hermitian
→ use $A \cdot A^* = I$

To prove : $(I - A)(I + A)^{-1}$ → unitary

$I - A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix} \Rightarrow (I + A)^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$(I - A)(I + A)^{-1} = \frac{1}{6} \begin{bmatrix} -4 & -4i-2 \\ -4i+2 & -4 \end{bmatrix}$

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now apply dot product rule, if its unitary matrix then

$$a_j \cdot a_k = \bar{a}_j^T a_k = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

when a_i is column or row vector of matrix

for $j = k$: $\frac{16}{36} + \frac{4+16}{36} = 1$

for $j \neq k$: $-4\,(-4i-2) - 4\,\overline{(-4i+2)} = 0$

8) a) $A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

we have : $P^{-1} A P = D$

$\Rightarrow A = P D P^{-1}$

$\Rightarrow A^m = P D^m P^{-1}$

eigenvalue : $\begin{vmatrix} -4-\lambda & 1 & 0 \\ 0 & -3-\lambda & 1 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+4)(\lambda+3)(\lambda+2)=0$

$\Rightarrow \lambda = -2, -3, -4$

eigenvector for $\lambda = -2$: $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} X = 0$

$\Rightarrow X = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}^T$

$\lambda = -3$; $X = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$

$\lambda = -4$; $X = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$

$\Rightarrow P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$; $P^{-1} = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & -1 \\ 1 & -1 & 1/2 \end{bmatrix}$

$$A^m = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}^m \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & -1 \\ 1 & -1 & 1/2 \end{bmatrix}$$

$$= (-1)^m \begin{bmatrix} 4^m & 3^m - 4^m & 2^{2m-1} - 3^m + 2^{2m-1} \\ 0 & 3^m & 2^m - 3^m \\ 0 & 0 & 2^m \end{bmatrix}$$

b) e^{2A} : $e^{2A} = I + 2A + \frac{4A^2}{2!} + \frac{(2A)^3}{3!} + \cdots$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 4 & -1 & \\ & & \\ & & \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} \sum_{r=0}^{\infty} \frac{2^r(-4)^r}{r!} & \sum_{r=0}^{\infty} \frac{(3^r - 4^r)(-2)^r}{r!} & \sum \frac{(2^{2r-1} - 3^m + 2^{m-1})(-2)^r}{r!} \\ 0 & \sum \frac{(-2)^r \cdot 3^r}{r!} & \sum \frac{(2^m - 3^m)(-2)^r}{r!} \\ 0 & 0 & \sum \frac{2^r (-2)^r}{r!} \end{bmatrix}$$

$$= \begin{bmatrix} \sum (-8)^r / r! & \sum \frac{(-6)^r}{r!} - \sum \frac{(-8)^r}{r!} & - \\ 0 & - & - \\ 0 & 0 & - \end{bmatrix}$$

$$= \begin{bmatrix} e^{-8} & e^{-6} - e^{-8} & - \\ 0 & - & - \\ 0 & 0 & - \end{bmatrix}$$

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9) a) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ 1 & 3-\lambda & 4 \\ 1 & 4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda^3 + 7\lambda^2 + 7\lambda - 1 = 0$$

From Cayley - Hamilton theorem A satisfies its characteristic equation hence,

$$-A^3 + 7A^2 + 7A = I$$

$$A^{-1} = -A^2 + 7A + 7I$$

$$= \begin{pmatrix} -7 & -24 & -24 \\ -8 & -28 & -27 \\ -8 & -27 & -28 \end{pmatrix} + \begin{pmatrix} 7 & 21 & 21 \\ 7 & 21 & 28 \\ 7 & 28 & 21 \end{pmatrix} + 7I$$

$$= \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

b) $\begin{vmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 6 & 8 \end{vmatrix}$

$\Rightarrow -\lambda^3 + 17\lambda^2 - 62\lambda + 40 = 0$

$\Rightarrow A^{-1} = A^2 - 17A + 62I$

$$= \begin{pmatrix} -0.20 & -0.9 & 0.5 \\ 0.5 & 1.25 & -0.75 \\ -0.3 & -0.6 & 0.5 \end{pmatrix}$$

10) Given: $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 4 & 0 & 1 \end{pmatrix}$

its characteristic eq. $\equiv -\lambda^3 + 3\lambda^2 - 5\lambda + 11$

$A^3 = \begin{pmatrix} 3 & 2 & 6 \\ 11 & 3 & 1 \\ 4 & 27 & 9 \end{pmatrix}$; $A^2 = \begin{pmatrix} -1 & 4 & 2 \\ 2 & -1 & 2 \\ 8 & 8 & 1 \end{pmatrix}$

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Hence A satisfy its charactrist equation

$$\lambda^3 = 3\lambda^2 - 5\lambda + 11$$

$$\Rightarrow A^3 = 3A^2 - 5A + 11$$

$$A^4 = 3A^3 - 5A^2 + 11A$$

$$= \begin{pmatrix} 25 & 8 & 8 \\ 12 & 25 & 4 \\ 16 & 32 & 33 \end{pmatrix}$$

ii) Given : $A = \begin{pmatrix} 4 & \alpha & -1 \\ 2 & 5 & \beta \\ 1 & 1 & \gamma \end{pmatrix}$

eigenvalues : $3, 3, \delta$

$A \rightarrow$ diagonlizable

To find $\alpha, \beta, \gamma, \delta$.

We observe two values of eigenvalues are same still they have LI E.V as A is diagonlizable hence

Rank $[A - 3I] = 1$

$$\Rightarrow \begin{bmatrix} 4-3 & \alpha & -1 \\ 2 & 5-3 & \beta \\ 1 & 1 & \gamma-3 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & -1 \\ 0 & 2(\alpha-1) & \beta+2 \\ 0 & 0 & \gamma - \frac{\beta}{2} - 3 \end{bmatrix}$$

$$\Rightarrow \alpha = 1 \quad ; \quad \beta = -2 \quad ; \quad \gamma = 2$$

characteristic equation of A : $\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (\lambda-5)\cdot(\lambda-3)^2 = 0$

$\lambda = 3, 3, \underset{\underset{5}{\shortparallel}}{5}$

12) a) $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$ To make $P^* A P = D$

$P \rightarrow$ orthogonal / unitary.

as $P \rightarrow$ orthogonal / unitary

$\Rightarrow \quad P^* = P^{-1}$

hence, $P^* A P = P^{-1} A P = D$

A → symmetric matrix

eigenvalues of A : $\begin{vmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (\lambda-8)(\lambda+2) = 0$

$\Rightarrow \quad \lambda = 8, -2$

eigenvector of A for $\lambda = 8$: $\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$X = [3 \quad 1]^T$

for $\lambda = -2$: $X = [-1 \quad 3]^T$

hence $P = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ but to make P orthogonal we divide it by $\sqrt{3^2+1^2} = \sqrt{10}$

$P = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$

eigenvalues : $\begin{vmatrix} 1-\lambda & 1-i \\ 1+i & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda(\lambda-3) = 0$

$\Rightarrow \lambda = 0, 3$

$\lambda = 0$: $\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow X = [-2 \quad 1+i]^T$

$\lambda = 3$: $\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow X = [1 \quad 1+i]$

$\Rightarrow P = \begin{bmatrix} -2 & 1 \\ 1+i & 1+i \end{bmatrix}$

to make it unitary we divide its first column by $\sqrt{(-2)(-2) + (1+i)(1-i)} = \sqrt{6}$ and 2^{nd} column by $\sqrt{1^2 + (1+i)(1-i)} = \sqrt{3}$

$\Rightarrow P = \begin{bmatrix} -2/\sqrt{6} & 1/\sqrt{3} \\ (1+i)/\sqrt{6} & (1+i)/\sqrt{3} \end{bmatrix}$

5) Let C be hermitian $\Rightarrow$ $C = \underbrace{\frac{1}{2}(C+\bar{C})}_{A} + i\underbrace{\frac{1}{2i}(C-\bar{C})}_{B}$

we observe $A^T = A$ and $B^T = -B$

$\bar{A} = A$ $\qquad \bar{B} = \bar{B}$

A : Real and symmetric $\qquad$ B : Real and skew-symmetric