AI1110
Random Variables

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/uni_dat.c
```

Compile and execute the program as follows.

```
$ gcc uni_dat.c −lm −o uni_dat
$ ./uni_dat
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: Download the following code

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/
    cdf_plot_uni.py
```

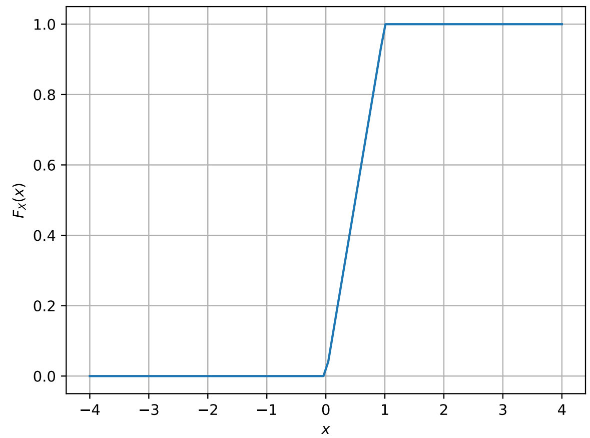

Fig. 1.2: The CDF of U

and run it as follows to get Figure 1.2

```
$ python cdf_plot_uni.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The pdf of U is given by

$$P_U(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

Now,

$$F_U(x) = \int_{-\infty}^{x} P_U(t)dt \tag{1.3}$$

$$F_U(x) = \begin{cases} \int_{-\infty}^{x} 0, & x < 0 \\ \int_{0}^{x} 1, & 0 \leq x \leq 1 \\ \int_{0}^{1} 1dx, & \text{otherwise} \end{cases} \tag{1.4}$$

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & \text{otherwise} \end{cases} \tag{1.5}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.6}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.7}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/
    mean_var_uni.c
```

Compile and execute the program as follows.

```
$ gcc mean_var_uni.c -lm -o
    mean_var_uni
$ ./mean_var_uni
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.8}$$

Solution: Integrating, we get

$$E[U^k] = \int_{-\infty}^{\infty} x^k P_U(x) dx \tag{1.9}$$

$$= \int_0^1 x^k dx \tag{1.10}$$

$$= \frac{1}{k+1} \tag{1.11}$$

Now,

$$E[U] = \frac{1}{2} \tag{1.12}$$

$$E[U^2] = \frac{1}{3} \tag{1.13}$$

$$E[(U - E(U))^2] = E[(U - 1/2)^2] \tag{1.14}$$

$$= E[U^2] + E(1/4) - 2E(\frac{U}{2}) \tag{1.15}$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \tag{1.16}$$

$$= \frac{1}{12} \tag{1.17}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/gau_dat.c
```

Compile and execute the program as follows.

```
$ gcc gau_dat.c -lm -o gau_dat
$ ./gau_dat
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Download the following code

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/
    cdf_plot_gau.py
```

and run it as follows to get Figure 2.2

```
    $ python cdf_plot_gau
```

The properties of a CDF are:

- $0 \le F_X(x) \le 1$
- $\forall x \in (-\infty, \infty)$, $\frac{d}{dx}F_X(x) \ge 0$
- $\forall a \in (-\infty, \infty)$,

$$\lim_{x \to a^+} F_X(x) = F_X(a)$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: Download the following code

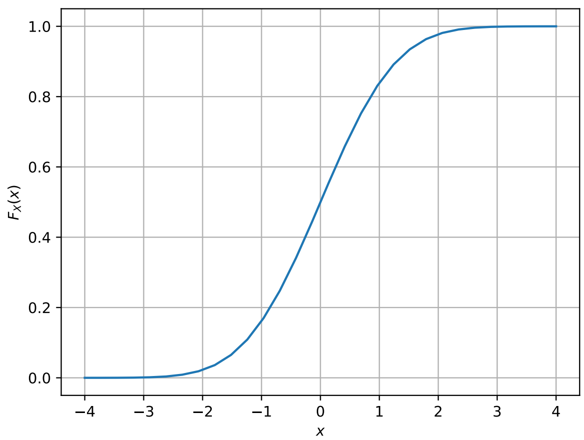

Fig. 2.2: The CDF of X

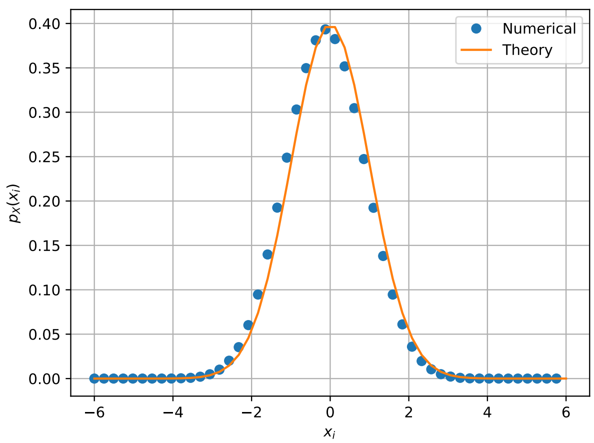

Fig. 2.3: The PDF of X

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/
    pdf_plot_gau.py
```

and run it as follows to get Figure 2.3

```
$ python pdf_plot_gau.py
```

The PDF has following properties:

- $P_X(x) \geq 0, \forall x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} P_X(x) = 1$

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/
    mean_var_gau.c
```

Compile and execute the program as follows.

```
$ gcc mean_var_gau.c −lm −o
    mean_var_gau
$ ./mean_var_gau
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: The mean is given by

$$\mu = E(X) \tag{2.4}$$
$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.5}$$
$$= \frac{1}{\sqrt{2\pi}} \times 0 = 0 \tag{2.6}$$

Variance is given by

$$\sigma^2 = E(X^2) \tag{2.7}$$
$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.8}$$
$$= 2 \int_{0}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.9}$$

substitute $t = \frac{x^2}{2}$ and $dt = xdx$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{2t} e^{-t} dt \tag{2.10}$$
$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \tag{2.11}$$
$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \frac{1}{2} \tag{2.12}$$

put $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we get

$$= 1 \tag{2.13}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

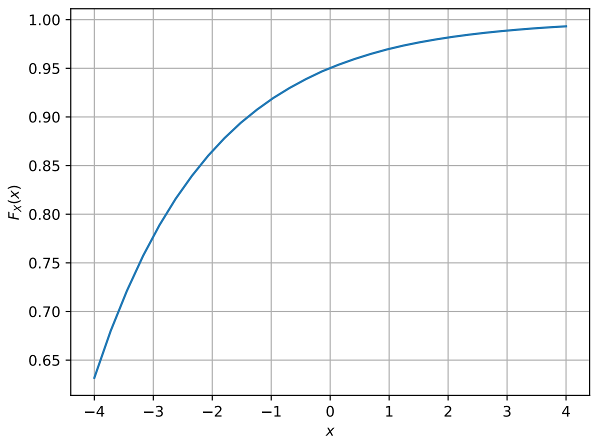

Fig. 3.1: The PDF of X

and plot its CDF.
Solution: To generate samples of V download the following files.

```
wget https://raw.githubusercontent.com/
     abhinavydv/AI1110-Assignments/master/
     RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
     abhinavydv/AI1110-Assignments/master/
     RandomVariablesManual/codes/exp_dat.c
```

and then compile and execute the program as follows.

```
$ gcc exp_dat.c -lm -o exp_dat
$ ./exp_dat
```

To plot the CDF download the following code

```
wget https://raw.githubusercontent.com/
     abhinavydv/AI1110-Assignments/master/
     RandomVariablesManual/codes/
     cdf_plot_exp.py
```

and run it as follows to get Figure 3.1

```
$ python cdf_plot_exp.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$\begin{aligned}
F_V(x) &= \Pr\left(V \le x\right) && (3.2)\\
&= \Pr\left(-2\ln(1-U) \le x\right) && (3.3)\\
&= \Pr\left(\ln(1-U) \le -\frac{x}{2}\right) && (3.4)\\
&= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) && (3.5)\\
&= F_U(1 - \exp\left(-\frac{x}{2}\right)) && (3.6)\\
&= \begin{cases} 0, & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right), & x \ge 0 \end{cases} && (3.7)
\end{aligned}$$