

# AI1110

## Random Variables

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### CONTENTS

1	Uniform Random Numbers	1
2	Central Limit Theorem	2
3	From Uniform to Other	4
4	Triangular Distribution	4
5	Maximul Likelihood	6
6	Gaussian to Other	6
7	Conditional Probability	8
8	Two Dimensions	8

*Abstract*—This manual provides a simple introduction to the generation of random numbers

### 1 UNIFORM RANDOM NUMBERS

Let  $U$  be a uniform random variable between 0 and 1.

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat .

**Solution:** Download the following files.

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/uni_dat.c
```

Compile and execute the program as follows.

```
$ gcc uni_dat.c -lm -o uni_dat
$ ./uni_dat
```

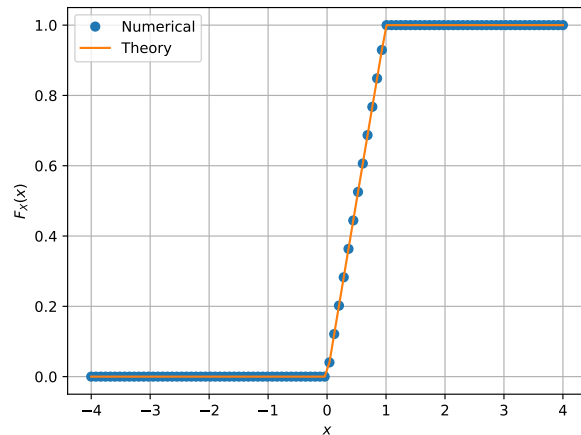


Fig. 1.2: The CDF of  $U$

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

**Solution:** Download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/
cdf_plot_uni.py
```

and run it as follows to get Figure 1.2

```
$ python cdf_plot_uni.py
```

- 1.3 Find a theoretical expression for  $F_U(x)$ .

**Solution:** The pdf of  $U$  is given by

$$P_U(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.2)$$

Now,

$$F_U(x) = \int_{-\infty}^x P_U(t) dt \quad (1.3)$$

$$F_U(x) = \begin{cases} 0, & x < 0 \\ \int_0^x 1, & 0 \leq x \leq 1 \\ \int_0^1 1 dx, & \text{otherwise} \end{cases} \quad (1.4)$$

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & \text{otherwise} \end{cases} \quad (1.5)$$

1.4 The mean of  $U$  is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.6)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.7)$$

Write a C program to find the mean and variance of  $U$ .

**Solution:** Download the following files.

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/
mean_var_uni.c
```

Compile and execute the program as follows.

```
$ gcc mean_var_uni.c -lm -o
mean_var_uni
$ ./mean_var_uni
```

After running the program, we get,

$$\mu = 0.500007 \quad (1.8)$$

$$\sigma = 0.083301 \quad (1.9)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.10)$$

**Solution:** Integrating, we get

$$E[U^k] = \int_{-\infty}^{\infty} x^k P_U(x) dx \quad (1.11)$$

$$= \int_0^1 x^k dx \quad (1.12)$$

$$= \frac{1}{k+1} \quad (1.13)$$

Now,

$$E[U] = \frac{1}{2} \quad (1.14)$$

This agrees with the experimental value of 0.500007

$$E[U^2] = \frac{1}{3} \quad (1.15)$$

$$E[(U - E(U))^2] = E[(U - 1/2)^2] \quad (1.16)$$

$$= E[U^2] + E(1/4) - 2E\left(\frac{U}{2}\right) \quad (1.17)$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \quad (1.18)$$

$$= \frac{1}{12} \quad (1.19)$$

This agrees with the experimental value 0.083301

## 2 CENTRAL LIMIT THEOREM

2.1 Generate  $10^6$  samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where  $U_i, i = 1, 2, \dots, 12$  are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

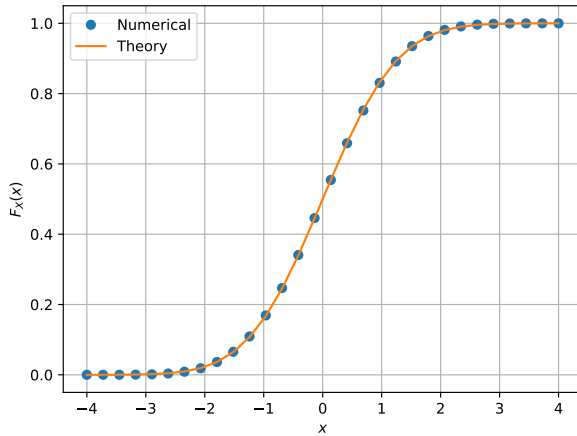
**Solution:** Download the following files.

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/gau_dat.c
```

Compile and execute the program as follows.

```
$ gcc gau_dat.c -lm -o gau_dat
$ ./gau_dat
```

2.2 Load gau.dat in python and plot the empirical CDF of  $X$  using the samples in gau.dat. What

Fig. 2.2: The CDF of  $X$ 

properties does a CDF have?

**Solution:** Download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/
cdf_plot_gau.py
```

and run it as follows to get Figure 2.2

```
$ python cdf_plot_gau
```

The properties of a CDF are:

- $0 \leq F_X(x) \leq 1$
- $\forall x \in (-\infty, \infty), \frac{d}{dx} F_X(x) \geq 0$
- $\forall a \in (-\infty, \infty),$

$$\lim_{x \rightarrow a^+} F_X(x) = F_X(a)$$

- 2.3 Load `gau.dat` in python and plot the empirical PDF of  $X$  using the samples in `gau.dat`. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

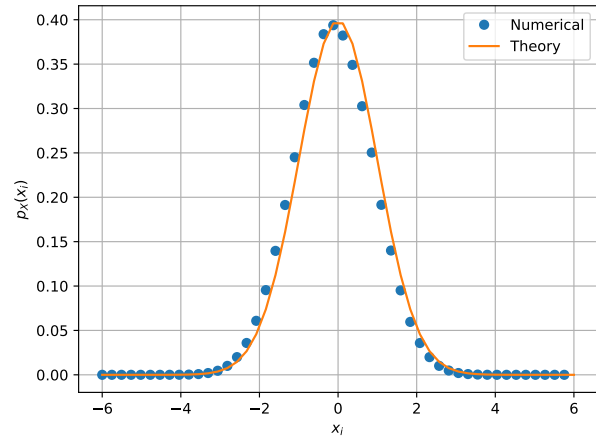
What properties does the PDF have?

**Solution:** Download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/
pdf_plot_gau.py
```

and run it as follows to get Figure 2.3

```
$ python pdf_plot_gau.py
```

Fig. 2.3: The PDF of  $X$ 

The PDF has following properties:

- $P_X(x) \geq 0, \forall x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} P_X(x) = 1$

- 2.4 Find the mean and variance of  $X$  by writing a C program.

**Solution:** Download the following files.

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/
mean_var_gau.c
```

Compile and execute the program as follows.

```
$ gcc mean_var_gau.c -lm -o
mean_var_gau
$ ./mean_var_gau
```

After running the program, we get,

$$\mu = 0.000294 \quad (2.3)$$

$$\sigma = 0.999560 \quad (2.4)$$

- 2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.

**Solution:** The mean is given by

$$\mu = E(X) \quad (2.6)$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= \frac{1}{\sqrt{2\pi}} \times 0 = 0 \quad (2.8)$$

Variance is given by

$$\sigma^2 = E(X^2) \quad (2.9)$$

$$= \int_{-\infty}^{\infty} x \times x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

Integrating by parts, we get

$$= \frac{1}{\sqrt{2\pi}} \left( \left[ -x \exp\left(-\frac{x^2}{2}\right) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \right) \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} (0 + \sqrt{2\pi}) \quad (2.12)$$

$$= 1 \quad (2.13)$$

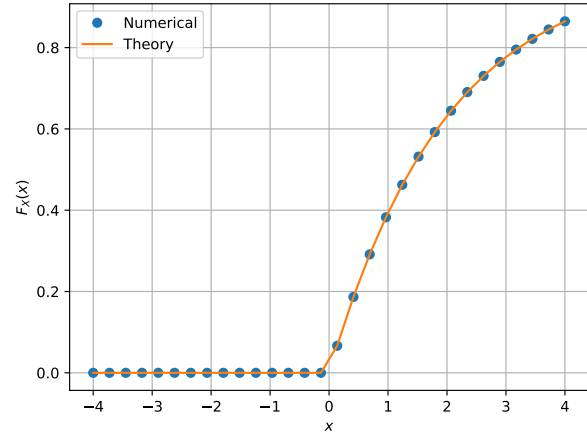


Fig. 3.1: The PDF of  $V$

```
$ python cdf_plot_exp.py
```

3.2 Find a theoretical expression for  $F_V(x)$ .

**Solution:**

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(\ln(1 - U) \leq -\frac{x}{2}\right) \quad (3.4)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$= \begin{cases} 0, & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right), & x \geq 0 \end{cases} \quad (3.7)$$

## 4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

**Solution:** Download the following files.

```
wget https://raw.githubusercontent.com/
abhinavdyv/AI1110-Assignments/master/
RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
abhinavdyv/AI1110-Assignments/master/
RandomVariablesManual/codes/tri_dat.c
```

Compile and execute the program as follows.

```
$ gcc tri_dat.c -lm -o tri_dat
$ ./tri_dat
```

## 3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

**Solution:** To generate samples of  $V$  download the following files.

```
wget https://raw.githubusercontent.com/
abhinavdyv/AI1110-Assignments/master/
RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
abhinavdyv/AI1110-Assignments/master/
RandomVariablesManual/codes/exp_dat.c
```

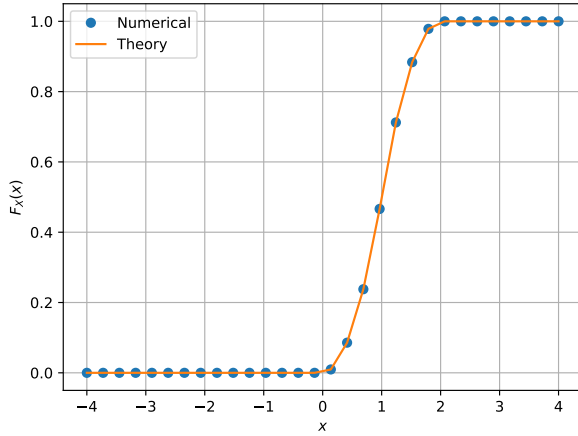
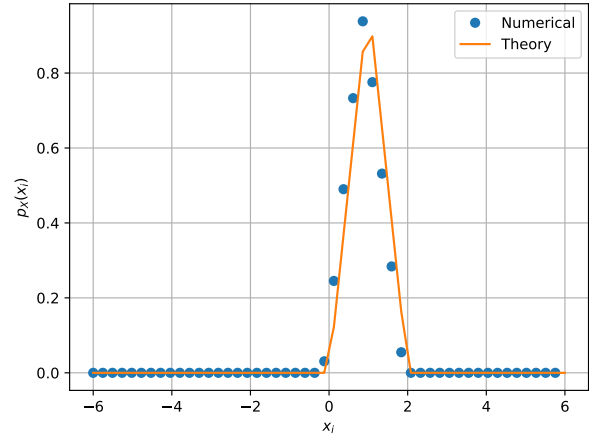
and then compile and execute the program as follows.

```
$ gcc exp_dat.c -lm -o exp_dat
$ ./exp_dat
```

To plot the CDF download the following code

```
wget https://raw.githubusercontent.com/
abhinavdyv/AI1110-Assignments/master/
RandomVariablesManual/codes/
cdf_plot_exp.py
```

and run it as follows to get Figure 3.1

Fig. 4.2: The CDF of  $T$ Fig. 4.3: The PDF of  $T$ 

4.2 Find the CDF of  $T$ .

**Solution:** Download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/
cdf_plot_tri.py
```

and run it as follows to get Figure 4.2

```
$ python cdf_plot_tri.py
```

4.3 Find the PDF of  $T$ .

**Solution:** Download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/master/
RandomVariablesManual/codes/
pdf_plot_tri.py
```

and run it as follows to get Figure 4.3

```
$ python pdf_plot_tri.py
```

4.4 Find the theoretical expressions for the PDF and CDF of  $T$ .

**Solution:** We have,

$$T = U_1 + U_2 \quad (4.2)$$

PDF can be found as follows

$$f_T(t) = f_{U_1+U_2}(t) \quad (4.3)$$

$$= f_{U_1}(t) * f_{U_2}(t) \quad (4.4)$$

$$= \int_{-\infty}^{\infty} f_{U_1}(\tau) f_{U_2}(t - \tau) d\tau \quad (4.5)$$

$$= \int_0^1 f_{U_2}(t - \tau) d\tau \quad (4.6)$$

$$= \begin{cases} \int_0^1 0 d\tau, & t < 0 \\ \int_0^t 1 d\tau, & 0 \leq t \leq 1 \\ \int_{t-1}^1 d\tau, & 1 < t \leq 2 \\ \int_0^1 0 d\tau, & t > 2 \end{cases} \quad (4.7)$$

$$= \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases} \quad (4.8)$$

Now, The CDF can be determined as follows,

$$F_T(x) = \int_{-\infty}^x f_T(t)dt \quad (4.9)$$

$$= \begin{cases} 0, & x < 0 \\ \int_0^x tdt, & 0 \leq x \leq 1 \\ \int_0^1 tdt + \int_1^x (2-t)dt, & 1 < x \leq 2 \\ 1, & x > 2 \end{cases} \quad (4.10)$$

$$= \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & 0 \leq x \leq 1 \\ 2x - \frac{x^2}{2} - 1, & 1 < x \leq 2 \\ 1, & x > 2 \end{cases} \quad (4.11)$$

4.5 Verify your results through a plot.

**Solution:** Already plotted in Figure 4.2 and Figure 4.3

## 5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable  $X \in \{1, -1\}$ .

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where  $A = 5$  dB, and  $N \sim \mathcal{N}(0, 1)$ .

5.3 Plot  $Y$  using a scatter plot.

5.4 Guess how to estimate  $X$  from  $Y$ .

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.3)$$

5.6 Find  $P_e$  assuming that  $X$  has equiprobable symbols.

5.7 Verify by plotting the theoretical  $P_e$  with respect to  $A$  from 0 to 10 dB.

5.8 Now, consider a threshold  $\delta$  while estimating  $X$  from  $Y$ . Find the value of  $\delta$  that maximizes the theoretical  $P_e$ .

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.4)$$

5.10 Repeat the above exercise using the MAP criterion.

## 6 GAUSSIAN TO OTHER

6.1 Let  $X_1 \sim \mathcal{N}(0, 1)$  and  $X_2 \sim \mathcal{N}(0, 1)$ . Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

**Solution:**

### • Generate data

Download the following files.

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/
master/RandomVariablesManual/
codes/coeffs.h
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/
master/RandomVariablesManual/
codes/chi_dat.c
```

Compile and execute the program as follows to generate data for the distribution

```
$ gcc chi_dat.c -lm -o chi_dat
$ ./chi_dat
```

### • Plot CDF:

To plot the CDF download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/
master/RandomVariablesManual/codes/
cdf_plot_chi.py
```

and run it as follows to get Figure 6.1

```
$ python cdf_plot_chi.py
```

### • plot PDF To plot the PDF download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/
master/RandomVariablesManual/codes/
pdf_plot_chi.py
```

and run it as follows to get Figure 6.1

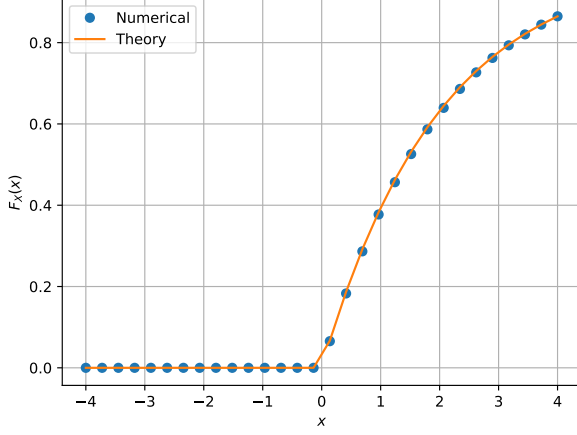
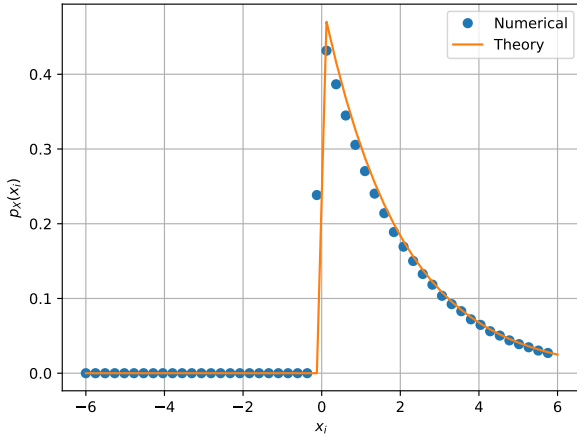
```
$ python pdf_plot_chi.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find  $\alpha$ .

**Solution:** Since  $X_1$  and  $X_2$  are i.i.d Normal

Fig. 6.1: The CDF of  $V$ Fig. 6.1: The PDF of  $V$ 

Random Variables

Let  $X_1 = R \sin \theta$  and  $X_2 = R \cos \theta$

Jacobian matrix is given as follows

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{pmatrix} \quad (6.3)$$

$$J = \begin{pmatrix} \cos \theta & -R \sin \theta \\ \sin \theta & R \cos \theta \end{pmatrix} \quad (6.4)$$

$$|J| = R \quad (6.5)$$

Also,

$$f_{r,\theta}(r, \theta) = |J| f_{X_1, X_2}(x_1, x_2) \quad (6.6)$$

Since,  $X_1$  and  $X_2$  are independent,

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) \quad (6.7)$$

$$(6.8)$$

$$= \frac{1}{2\pi} e^{-\frac{(x_1^2 + x_2^2)}{2}} \quad (6.9)$$

$$= \frac{1}{2\pi} e^{-\frac{r^2}{2}} \quad (6.10)$$

Put in Equation 6.6, we get,

$$f_{R,\theta}(r, \theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \quad (6.11)$$

Now,

$$f_R(r) = \int_0^{2\pi} f_{R,\theta}(r, \theta) d\theta \quad (6.12)$$

$$= \int_0^{2\pi} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d\theta \quad (6.13)$$

$$= r e^{-\frac{r^2}{2}} \quad (6.14)$$

CDF is given by,

$$F_R(r) = \int_0^r r e^{-\frac{r^2}{2}} dr \quad (6.15)$$

$$= 1 - e^{-\frac{r^2}{2}} \quad (6.16)$$

Since,

$$V = X_1^2 + X_2^2 \quad (6.17)$$

$$= R^2 \quad (6.18)$$

Now,

$$F_V(x) = \Pr(V \leq x) \quad (6.19)$$

$$= \Pr(R^2 \leq x) \quad (6.20)$$

$$= \Pr(R \leq \sqrt{x}) \quad (6.21)$$

$$F_V(x) = F_R(\sqrt{x}) \quad (6.22)$$

$$= \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}}, & \text{otherwise} \end{cases} \quad (6.23)$$

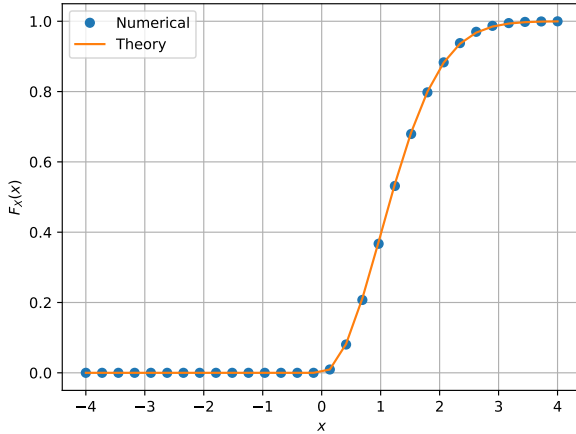
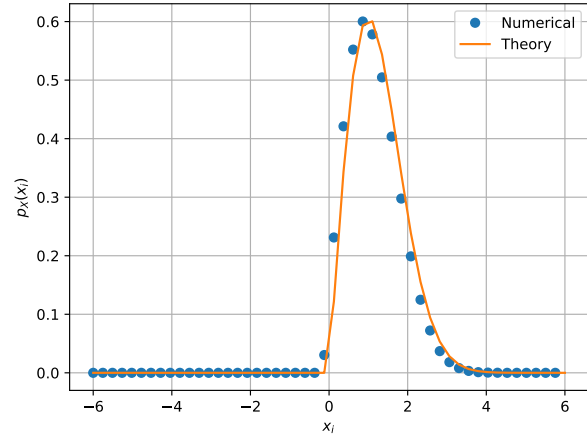
Therefore,

$$\alpha = \frac{1}{2} \quad (6.24)$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.25)$$

**Solution:**

Fig. 6.3: The CDF of  $\sqrt{V}$ Fig. 6.3: The PDF of  $\sqrt{V}$ 

### • Generate data

Download the following files.

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/
master/RandomVariablesManual/
codes/coeffs.h
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/
master/RandomVariablesManual/
codes/rayleigh_dat.c
```

Compile and execute the program as follows to generate data for the distribution

```
$ gcc rayleigh_dat.c -lm -o
rayleigh_dat
$ ./rayleigh_dat
```

### • Plot CDF:

To plot the CDF download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/
master/RandomVariablesManual/codes/
cdf_plot_rayleigh.py
```

and run it as follows to get Figure 6.3

```
$ python cdf_plot_rayleigh.py
```

### • plot PDF To plot the PDF download the following code

```
wget https://raw.githubusercontent.com/
abhinavydv/AI1110-Assignments/
```

```
master/RandomVariablesManual/codes/
pdf_plot_rayleigh.py
```

and run it as follows to get Figure 6.3

```
$ python pdf_plot_rayleigh.py
```

## 7 CONDITIONAL PROBABILITY

### 7.1

### 7.2 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where  $A$  is Raleigh with  $E[A^2] = \gamma$ ,  $N \sim \mathcal{N}(0, 1)$ ,  $X \in (-1, 1)$  for  $0 \leq \gamma \leq 10$  dB.

### 7.3 Assuming that $N$ is a constant, find an expression for $P_e$ . Call this $P_e(N)$

### 7.4 For a function $g$ ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx \quad (7.3)$$

Find  $P_e = E[P_e(N)]$ .

### 7.5 Plot $P_e$ in problems 7.2 and 7.4 on the same graph w.r.t $\gamma$ . Comment.

## 8 TWO DIMENSIONS

Let

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (8.1)$$



where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols  $\mathbf{s}_0$  and  $\mathbf{s}_1$ .

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for  $P_e$ . Verify this by comparing the theory and simulation plots on the same graph.