AI1110
Random Variables

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/uni_dat.c
```

Compile and execute the program as follows.

```
$ gcc uni_dat.c −lm −o uni_dat
$ ./uni_dat
```

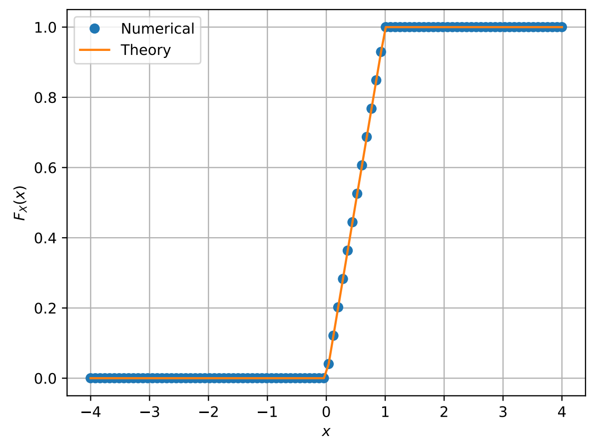

Fig. 1.2: The CDF of U

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: Download the following code

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/
    cdf_plot_uni.py
```

and run it as follows to get Figure 1.2

```
$ python cdf_plot_uni.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The pdf of U is given by

$$P_U(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

Now,

$$F_U(x) = \int_{-\infty}^{x} P_U(t)dt \tag{1.3}$$

$$F_U(x) = \begin{cases} \int_{-\infty}^{x} 0, & x < 0 \\ \int_{0}^{x} 1, & 0 \le x \le 1 \\ \int_{0}^{1} 1dx, & \text{otherwise} \end{cases} \tag{1.4}$$

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & \text{otherwise} \end{cases} \tag{1.5}$$

1.4 The mean of U is defined as

$$E\left[U\right] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.6}$$

and its variance as

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2 \tag{1.7}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/
    mean_var_uni.c
```

Compile and execute the program as follows.

```
$ gcc mean_var_uni.c -lm -o
    mean_var_uni
$ ./mean_var_uni
```

After running the program, we get,

$$\mu = 0.500007 \tag{1.8}$$

$$\sigma = 0.083301 \tag{1.9}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.10}$$

Solution: Integrating, we get

$$E[U^k] = \int_{-\infty}^{\infty} x^k P_U(x)dx \tag{1.11}$$

$$= \int_{0}^{1} x^k dx \tag{1.12}$$

$$= \frac{1}{k+1} \tag{1.13}$$

Now,

$$E[U] = \frac{1}{2} \tag{1.14}$$

This agrees with the experimental value of 0.500007

$$E[U^2] = \frac{1}{3} \tag{1.15}$$

$$E[(U - E(U))^2] = E[(U - 1/2)^2] \tag{1.16}$$

$$= E[U^2] + E(1/4) - 2E(\frac{U}{2}) \tag{1.17}$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \tag{1.18}$$

$$= \frac{1}{12} \tag{1.19}$$

This agrees with the experimental value 0.083301

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/gau_dat.c
```

Compile and execute the program as follows.

```
$ gcc gau_dat.c -lm -o gau_dat
$ ./gau_dat
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What

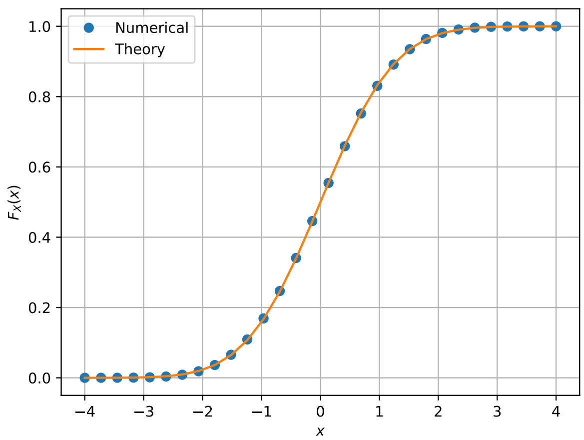

Fig. 2.2: The CDF of X

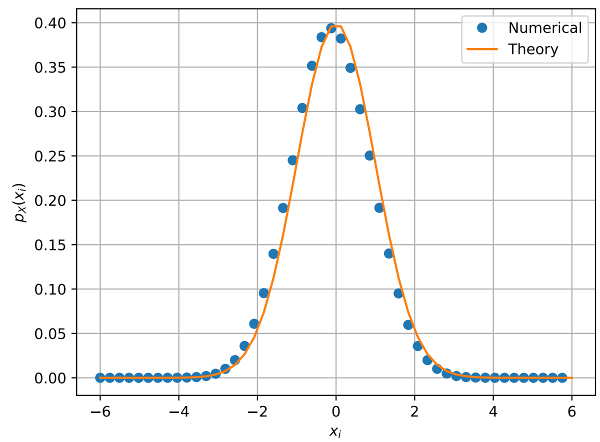

Fig. 2.3: The PDF of X

properties does a CDF have?
Solution: Download the following code

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/
    cdf_plot_gau.py
```

and run it as follows to get Figure 2.2

```
$ python cdf_plot_gau
```

The properties of a CDF are:

- $0 \le F_X(x) \le 1$
- $\forall x \in (-\infty, \infty), \frac{d}{dx} F_X(x) \ge 0$
- $\forall a \in (-\infty, \infty)$,

$$\lim_{x \to a^+} F_X(x) = F_X(a)$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: Download the following code

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/
    pdf_plot_gau.py
```

and run it as follows to get Figure 2.3

```
$ python pdf_plot_gau.py
```

The PDF has following properties:

- $P_X(x) \ge 0, \forall x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} P_X(x) = 1$

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/
    mean_var_gau.c
```

Compile and execute the program as follows.

```
$ gcc mean_var_gau.c -lm -o
    mean_var_gau
$ ./mean_var_gau
```

After running the program, we get,

$$\mu = 0.000294 \tag{2.3}$$
$$\sigma = 0.999560 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.

Solution: The mean is given by

$$\mu = E(X) \tag{2.6}$$
$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.7}$$
$$= \frac{1}{\sqrt{2\pi}} \times 0 = 0 \tag{2.8}$$

Variance is given by

$$\sigma^2 = E(X^2) \tag{2.9}$$
$$= \int_{-\infty}^{\infty} x \times x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.10}$$

Integrating by parts, we get

$$= \frac{1}{\sqrt{2\pi}} \left(\left[-x \exp\left(-\frac{x^2}{2}\right) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \right) \tag{2.11}$$
$$= \frac{1}{\sqrt{2\pi}} \left(0 + \sqrt{2\pi}\right) \tag{2.12}$$
$$= 1 \tag{2.13}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: To generate samples of V download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/exp_dat.c
```

and then compile and execute the program as follows.

```
$ gcc exp_dat.c -lm -o exp_dat
$ ./exp_dat
```

To plot the CDF download the following code

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/
    cdf_plot_exp.py
```

and run it as follows to get Figure 3.1

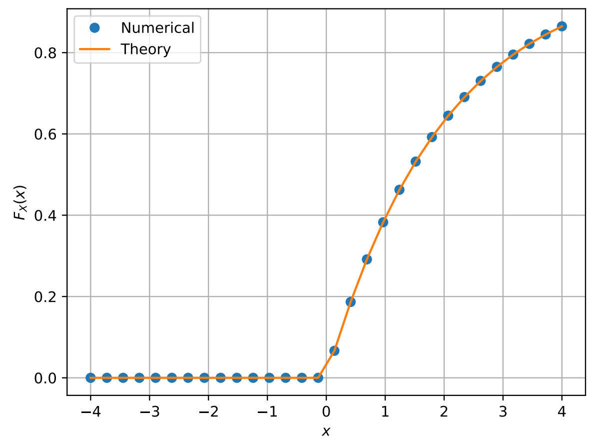

Fig. 3.1: The PDF of V

```
$ python cdf_plot_exp.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(\ln(1-U) \le -\frac{x}{2}\right) \tag{3.4}$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$= F_U(1 - \exp\left(-\frac{x}{2}\right)) \tag{3.6}$$
$$= \begin{cases} 0, & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right), & x \ge 0 \end{cases} \tag{3.7}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files.

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/coeffs.h
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110-Assignments/master/
    RandomVariablesManual/codes/tri_dat.c
```

Compile and execute the program as follows.

```
$ gcc tri_dat.c -lm -o tri_dat
$ ./tri_dat
```

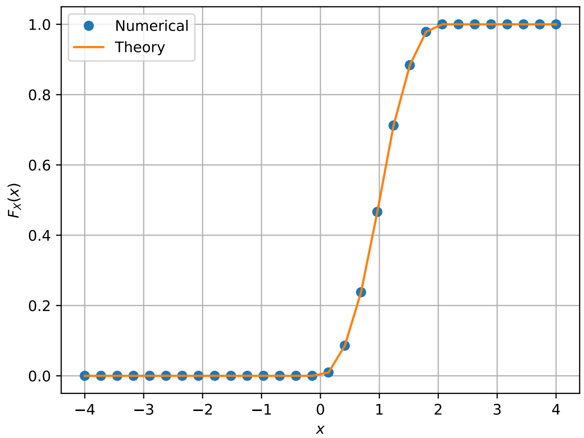

Fig. 4.2: The CDF of T

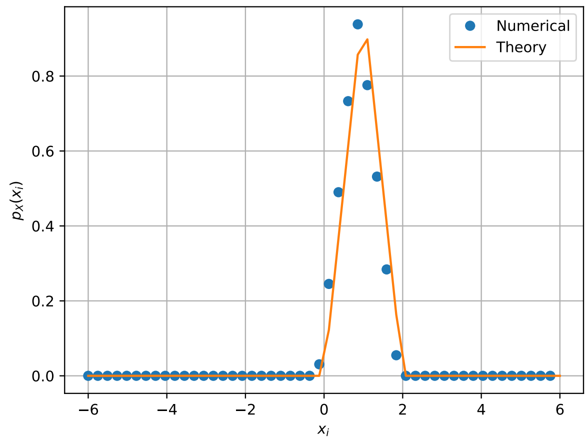

Fig. 4.3: The PDF of T

4.2 Find the CDF of T.
Solution: Download the following code

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/
    cdf_plot_tri.py
```

and run it as follows to get Figure 4.2

```
$ python cdf_plot_tri.py
```

4.3 Find the PDF of T.
Solution: Download the following code

```
wget https://raw.githubusercontent.com/
    abhinavydv/AI1110−Assignments/master/
    RandomVariablesManual/codes/
    pdf_plot_tri.py
```

and run it as follows to get Figure 4.3

```
$ python cdf_plot_tri.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: We have,

$$T = U_1 + U_2 \tag{4.2}$$

PDF can be found as follows

$$\begin{aligned}
f_T(t) &= f_{U_1+U_2}(t) && (4.3)\\
&= f_{U_1}(t) * f_{U_2}(t) && (4.4)\\
&= \int_{-\infty}^{\infty} f_{U_1}(\tau) f_{U_2}(t-\tau) d\tau && (4.5)\\
&= \int_0^1 f_{U_2}(t-\tau) d\tau && (4.6)\\
&= \begin{cases} \int_0^1 0 d\tau, & t < 0 \\ \int_0^t 1 d\tau, & 0 \le t \le 1 \\ \int_{t-1}^1 d\tau, & 1 < t \le 2 \\ \int_0^1 0 d\tau, & t > 2 \end{cases} && (4.7)\\
&= \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases} && (4.8)
\end{aligned}$$

Now, The CDF can be determined as follows,

$$F_T(x) = \int_{-\infty}^{x} f_T(t)dt \tag{4.9}$$

$$= \begin{cases} 0, & x < 0 \\ \int_0^x tdt, & 0 \le x \le 1 \\ \int_0^1 tdt + \int_1^x (2-t)dt, & 1 < x \le 2 \\ 1, & x > 2 \end{cases} \tag{4.10}$$

$$= \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & 0 \le x \le 1 \\ 2x - \frac{x^2}{2} - 1, & 1 < x \le 2 \\ 1, & x > 2 \end{cases} \tag{4.11}$$

4.5 Verify your results through a plot.
Solution: Already plotted in Figure 4.2 and Figure 4.3

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X \in \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

5.2 Plot Y.

5.3 Guess how to estimate X from Y.

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

5.5 Find P_e.

5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \ge 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1

7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.4 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E[P_e(N)]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.