UNIT- I (Q.B)

VSAQ's

1) Answer : [D] $y'' - 2y' + y = 0$

The general solution of such D.E is $y = c_1 e^x + c_2 x e^x$

$$y = (c_1 + c_2 x)e^x$$

Roots of A·E are $m = 1, 1$

$$AE : (m-1)^2 = 0 \Rightarrow m^2 - 2m + 1 = 0$$

$$\Rightarrow (D^2 - 2D + 1)y = 0$$

2) Answer : 3

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/3} = \left[\frac{d^2y}{dx^2}\right]^{1/2}$$

[$\because$ L.C.M of 2, 3 = 6]

Raising the power to 6, we get

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/3 \times 6} = \left(\frac{d^2y}{dx^2}\right)^{1/2 \times 6}$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2 = \left(\frac{d^2y}{dx^2}\right)^3$$

order = 2 ; degree = 3

3) Answer : $(D^2 - 1)y = 0$

$$y = ae^x + be^{-x} \quad \text{---} (1)$$

$$\frac{dy}{dx} = ae^x - be^{-x}$$

$$\frac{d^2y}{dx^2} = ae^x + be^{-x} = y \quad [\because \text{By } (1)]$$

$$\frac{d^2y}{dx^2} - y = 0$$

(or)

Short cut :

Roots of A·E are 1, −1

$$A \cdot E = (m-1)(m+1) = 0$$

$$m^2 - 1 = 0$$

$$D \cdot E : (D^2 - 1)y = 0$$

4) Answer : [A] $\frac{x}{y} + e^{x^3} = c$

$$y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$$

Multiplying with $\frac{1}{y^2}$

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}dx = 0$$

i.e., $d\left(\frac{x}{y}\right) + e^{x^3}\cdot 3x^2\,dx = 0$

Integrating

$$\int d\left(\frac{x}{y}\right) + \int e^{x^3}\cdot 3x^2\,dx = c$$

$$\frac{x}{y} + e^{x^3} = c. \qquad \left[\because \int e^{f(x)}\cdot f'(x)\,dx = e^{f(x)}\right]$$

5) Answer : $(e^y+1)\sin x = c$

$$(e^y+1)\cos x\,dx + e^y\sin x\,dy = c.$$

The general solution is

$$\int_{y=\text{const}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e., $\int_{y=\text{const.}} (e^y+1)\cos x\,dx + \int (0)\,dy = c$

$$\Rightarrow (e^y+1)\int \cos x\,dx = c$$

$$\Rightarrow (e^y+1)\sin x = c$$

6) Answer : [D] $\sin x$

I.F of $M\,dx + N\,dy = 0$ is $e^{\int f(x)\,dx}$

Given $f(x) = \cot x$

$$\text{I.F} = e^{\int \cot x\,dx} = e^{\log(\sin x)} = \sin x$$

7) Answer : [B] $\frac{1}{x^3}$

Among all the given options, [B] cannot be used as standard I.F for the given D.E

8) Answer : [B] $xe^{2/x}$

Given D.E = $x^2\frac{dy}{dx} + (x-2)y = x^2 e^{-2/x}$ — ①

Divide eq ① by x^2 we get $\Rightarrow \frac{dy}{dx} + \frac{(x-2)}{x^2}y = e^{-2/x}$

which is linear in y

$$I.F = e^{\int f(x)\,dx} = e^{\int\left(\frac{1}{x}-\frac{2}{x^2}\right)dx} = e^{\int\frac{1}{x}dx - \int\frac{2}{x^2}dx}$$

$$= e^{\log x + \frac{2}{x}} = e^{\log x}\cdot e^{2/x} = xe^{2/x}$$

9) Answer : [A] $\alpha = 2, \beta = 1$

The condition for exactness is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$M = \alpha xy^3 + y\cos x$ $\qquad$ $N = 3x^2y^2 + \beta\sin x$.

$\frac{\partial M}{\partial y} = \alpha x(3y^2) + \cos x$. $\qquad$ $\frac{\partial N}{\partial x} = 3y^2(2x) + \beta\cos x$.

As given eq. is exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow 3\alpha xy^2 + \cos x = 6xy^2 + \beta\cos x$

$\Rightarrow 3\alpha = 6$ and $\beta = 1$

$\Rightarrow \alpha = 2$ and $\beta = 1$

10) Answer : [C] $\frac{1}{x^4}$

The given eq is non-exact,

Since $\frac{\partial M}{\partial y} = 2xy$; $\frac{\partial N}{\partial x} = -2xy$

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{-x^2y}(2xy+2xy)=-\frac{4\cancel{xy}}{x^{\cancel{2}}\cancel{y}}=-\frac{4}{x}=f(x)$

$\therefore$ I.F $= e^{\int f(x)} = e^{-4\int \frac{1}{x}dx} = e^{-4\log x} = e^{\log x^{-4}} = x^{-4} = \frac{1}{x^4}$

11) Answer : $\frac{\partial N}{\partial y}=\frac{\partial N}{\partial x}$

12) Answer : [B] $(\log y)^{-1}$

$$\frac{dy}{dx}+\left(\frac{y}{x}\right)\log y=\frac{y}{x}(\log y)^2$$

dividing with $y(\log y)^2$

$$\frac{1}{y(\log y)^2}\frac{dy}{dx}+\frac{y}{x}\frac{\log y}{y(\log y)^2}=\frac{\cancel{y}}{x}\frac{\cancel{(\log y)^2}}{\cancel{y(\log y)^2}}$$

$$\frac{1}{y(\log y)^2}\frac{dy}{dx}+\frac{1}{x}\frac{1}{\log y}=\frac{1}{x} \quad — ①$$

Put $z=\frac{1}{\log y}$ so that $\frac{dz}{dx}=\frac{-1}{(\log y)^2}\times\frac{1}{y}\frac{dy}{dx}$

eq ① becomes $-\frac{dz}{dx}+\frac{1}{x}z=\frac{1}{x}$

(or)

$$\frac{dz}{dx}-\frac{1}{x}z=\frac{-1}{x}.$$

13) Answer : [D] straight line.

14) Answer : $\frac{1}{y^2}$

given $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=-\frac{2}{y}$

$$g(y) = \frac{-2}{y}$$

$$I.F = e^{\int g(y)\,dy} = e^{\int \frac{-2}{y} dy} = e^{-2\int \frac{1}{y} dy} = e^{-2\log y}$$

$$= e^{\log y^{-2}} = \frac{1}{y^2}$$

15) Answer : $\frac{1}{y}$

$$\frac{dy}{dx} - y\tan x = -y^2 \sec x$$

divide with $-y^2$

$$\frac{-1}{y^2}\frac{dy}{dx} - \frac{y\tan x}{-y^{\cancel{2}}} = \frac{-\cancel{y^2}\sec x}{\cancel{-y^2}}$$

$$\frac{-1}{y^2}\frac{dy}{dx} - \frac{\tan x}{y} = \sec x \quad — (1)$$

put $\frac{1}{y} = z$

$$\frac{-1}{y^2} = \frac{dz}{dx}$$

substituting in (1), we get

$$\frac{dz}{dx} + z\tan x = \sec x$$

16) Answer : $\frac{1}{Mx - Ny}$ [By Method - III]

17) Answer : [D] $F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$

18) Answer : [A] $\frac{d\theta}{dt} = -k(\theta - \theta_0)$, $k > 0$

19) Answer : [D] $y^2 = 4b(x+b)$

Since the given family of parabolas is self orthogonal, the required family of O.T's is $y^2 = 4b(x+b)$.

20) Answer : [C] $r^n = b^n \sin n\theta$

UNIT - II

1) Answer : [C] $1, x, e^x, e^{-x}$

The A.E is $m^4 - m^2 = 0$

$$m^2(m^2-1) = 0$$

$$(m-0)^2 (m+1)(m-1) = 0$$

$$m = 0, 0, 1, -1$$

Linearly independent solutions are $e^{0x}, xe^{0x}, e^x, e^{-x}$

i.e., $1, x, e^x, e^{-x}$

2) Answer : $y = c_1 \cos 2x + c_2 \sin 2x$

Given $(D^2+4)y = 0$

$f(D)y = 0$; where $f(D) = D^2 + 4$

The A.E is $f(m) = 0$

i.e., $m^2 + 4 = 0$

$$m = \pm 2i$$

$$(\alpha = 0 ; \beta = 2)$$

∴ The general solution is $y = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

i.e., $y = c_1 \cos 2x + c_2 \sin 2x$

3) Answer : [D] $-x^2$

Given $(D^3 - D^2)y = 2$

$f(D) = D^3 - D^2$; $Q(x) = 2$

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D^3 - D^2} 2e^{0x} = 2x \frac{1}{3D^2 - 2D} e^{0x}$$

$$= 2x \cdot x \frac{1}{6D-2} e^{0 \cdot x} = 2x^2 \frac{1}{6(0)-2} e^{0 \cdot x}$$

$$= -x^2$$

4) Answer : [D] $(C_1 + C_2 x)e^x + C_3 e^{-2x}$

Given $(D+2)(D-1)^2 y = e^{-2x}$

A.E is $(m+2)(m-1)^2 = 0$

$m = 1, 1, -2$

$\therefore$ C.F $= (C_1 + xC_2)e^x + C_3 e^{-2x}$

5) Answer : [C] $x^3 e^x$

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{(D-1)^3} 6e^x = 6 \frac{1}{D^3 - 3D^2 + 3D - 1} e^x$$

$$= 6x \frac{1}{3D^2 - 6D + 3} e^x = 6x^2 \frac{1}{6D - 6} e^x = \cancel{6}x^3 \frac{1}{\cancel{6}} e^x$$

$$= x^3 e^x.$$

6) Answer : $(D^2 - 5D + 6)y = 0$

The A.E has roots 2, 3

$(m-2)(m-3) = 0$

$m^2 - 3m - 2m + 6 = 0$

$m^2 - 5m + 6 = 0$

D.E is $(D^2 - 5D + 6)y = 0$

7) Answer : $-\frac{x}{8}\sin 2x$

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D^3+4D}(\sin 2x)$$

$$\left\{ \begin{aligned} \because \phi(D^2) &= D^2.D + 4D \\ &= (D^2+4)D \\ \phi(-2^2) &= (-2^2+4)D = 0 \end{aligned} \right\}$$

$$= x.\frac{1}{3D^2+4}\sin 2x = x.\frac{1}{3(-2^2)+4}\sin 2x = x\frac{1}{-12+4}\sin 2x$$

$$= -\frac{x}{8}\sin 2x$$

8) Answer : $y = c_1e^{x} + c_2e^{3x} + c_3e^{2x}$

Given $(D^3-6D^2+11D-6)y = 0$

A.E is $f(m) = 0$

$m^3-6m^2+11m-6 = 0$

$m = 1, 3, 2.$

The general solution is $y = c_1e^{x} + c_2e^{3x} + c_3e^{2x}$.

9) Answer : [C] x^2-2

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{(D^2+1)}x^2 = \frac{1}{(1+D^2)}x^2 = (1+D^2)^{-1}x^2$$

$$= [1-D^2+D^4\ldots]x^2 = x^2 - D^2(x^2) = x^2-2.$$

10) Answer : [A] $c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$

The A.E is $m^2-4m+1 = 0$

$m = 2 \pm \sqrt{3}$

The general solution is $y = c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$

(or) $y = e^{2x}(c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x)$

11) Answer : [c] $\frac{x}{2a}\sin ax$

Given $(D^2+a^2)y = \cos ax$

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+a^2}\cos ax = x \cdot \frac{1}{2D+0}\cos ax$$

$$= \frac{x}{2}\cdot\frac{1}{D}(\cos ax) = \frac{x}{2}\left[\frac{1}{a}\sin ax\right] = \frac{x}{2a}\sin ax.$$

12) Answer : [c] $x^3 e^x$.

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{(D-1)^2} 6xe^x = 6e^x \frac{1}{(D+1-1)^2} x$$

$$= 6e^x \frac{1}{D^2}(x)$$

$$= 6e^x \cdot \frac{1}{D}\left(\frac{x^2}{2}\right)$$

$$= \cancel{6}e^x \cdot \frac{x^3}{\cancel{3\cdot 2}}$$

$$= x^3 e^x$$

13) Answer : $\frac{e^x}{12}$

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+5D+6} e^x = \frac{1}{1^2+5(1)+6} e^x$$

$$= \frac{e^x}{12}$$

14) Answer : $(C_1 + xC_2)e^{x/2}$

Given $(4D^2-4D+1)y = 100$

The A.E is $4m^2-4m+1=0$

$$m = \frac{1}{2}, \frac{1}{2}$$

$$C.F = (C_1 + xC_2)e^{x/2}$$

15) Answer : $\sin x$

Given $(D^3 - D^2 + D)y = \sin x$

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D^3 - D^2 + D} \sin x$$

$$= \frac{1}{D \cdot D^2 - D^2 + D} \sin x$$

Put $D^2 = -1^2$

$$= \frac{1}{D(-1^2) - (-1^2) + D} \sin x = \frac{1}{-\cancel{D} + 1 + \cancel{D}} \sin x = \frac{1}{1} \sin x = \sin x$$

16) Answer : [B] e^{3x}

$y_1 = e^x \qquad y_2 = e^{2x}$

$y_1' = e^x \qquad y_2' = 2e^{2x}$

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{2x} e^x - e^x e^{2x} = 2e^{3x} - e^{3x} = e^{3x}$$

17) Answer : [B] $-e^{-x}\cos x$

$$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{(D+1)^2} e^{-x}\cos x = e^{-x} \frac{1}{(D-1+1)^2} \cos x$$

$$= e^{-x} \frac{1}{D^2} \cos x = e^{-x} \frac{1}{-1^2} \cos x = -e^{-x}\cos x.$$

18) Answer : $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

$(D^2+1)^2 y = 0$

The A.E is $(m^2+1)^2 = 0 \Rightarrow (m^2+1)(m^2+1) = 0$

$m = \pm i, \pm i$

The general solution is $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

19) Answer : [A] log 2

$$P.I = \frac{1}{D^2-2D+2}\log 4$$

$$= \log 4 \frac{1}{D^2-2D+2} e^{0.x} = \log 4 \frac{1}{0^2-2(0)+2} e^{0.x} = \frac{1}{2}\log 4$$

$$= \log (4)^{1/2} = \log 2$$

20) Answer : $C.F = e^{x}(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$

$$(D^2+4)y = x \sin x$$

A.E is $m^4 + 4 = 0 \Rightarrow m = 1 \pm i \, ; \, -1 \pm i$

$$\therefore C.F = e^{x}(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$$

UNIT - III

1) Answer : $\frac{1}{s+2}$

$$L\{e^{-at}\} = \frac{1}{s+a}, \quad s < a$$

$$\therefore L\{e^{-2t}\} = \frac{1}{s+2}$$

2) Answer : [A] $\frac{s}{s^2+9}$

$$L\{\cos 3t\} = \frac{s}{s^2+3^2} = \frac{s}{s^2+9} \quad \left[\because L\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

3) Answer : [B] $\frac{100!}{s^{101}}$

$$L\{t^{100}\} = \frac{100!}{s^{100+1}} \qquad \left[\because L\{t^n\} = \frac{n!}{s^{n+1}}\right]$$

$$= \frac{100!}{s^{101}}$$

4) Answer : [D] $\frac{b}{(s-a)^2+b^2}$

$$L\{e^{at}\sin bt\} = [L\{\sin bt\}]_{s=s-a} = \left[\frac{b}{s^2+b^2}\right]_{s=s-a}$$

$$= \frac{b}{(s-a)^2+b^2}$$

5) Answer : [B] $\frac{1}{s} + \frac{2}{s^2+4}$

$$L\{(\sin t + \cos t)^2\} = L\{\sin^2 t + \cos^2 t + 2\sin t\cos t\}$$

$$= L\{1 + 2\sin t\cos t\}$$

$$= L\{1\} + L\{\sin 2t\}$$

$$= \frac{1}{s} + \frac{2}{s^2+2^2} = \frac{1}{s} + \frac{2}{s^2+4}$$

6) Answer : $\frac{1}{\sqrt{s^2+9}}$

Given $L\{f(t)\} = \frac{1}{\sqrt{s^2+1}} = F(s)$

By change of scale property,

$$L\{f(3t)\} = \frac{1}{3}F\left(\frac{s}{3}\right) = \frac{1}{3}\frac{1}{\sqrt{\left(\frac{s}{3}\right)^2+1}} = \frac{1}{\cancel{3}}\cdot\frac{1}{\frac{\sqrt{s^2+9}}{\cancel{3}}} = \frac{1}{\sqrt{s^2+9}}$$

7) Answer : [B] $-\frac{d}{ds}[F(s)]$ [Multiplication by t]

8) Answer : [A] $\frac{1}{a}F\left(\frac{s}{a}\right)$ [Change of scale property]

9) Answer : [D] $\int_s^\infty F(s)\,ds$ [Division by t]

10) Answer : [B] $\cot^{-1}s$.

We have $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}\,ds$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\,ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \tan^{-1}s\Big|_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s$$

UNIT - III

VSAQs 9) [D] 10) [B]

11) $L\left\{\frac{\cos 3t - \cos 2t}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$ then

$\int_0^\infty \frac{\cos 3t - \cos 2t}{t}\, dt = \int_0^\infty e^{-at}\left(\frac{\cos 3t - \cos 2t}{t}\right) dt$

$= \left[\int_0^\infty e^{-st}\left(\frac{\cos 3t - \cos 2t}{t}\right) dt\right]_{s=0} = \left[L\left\{\frac{\cos 3t - \cos 2t}{t}\right\}\right]_{s=0}$

$= \left[\frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)\right]_{s=0} = \frac{1}{2}\log\left(\frac{0+4}{0+9}\right) = \log\left(\frac{4}{9}\right)^2$

$= \log\frac{2}{3}$ [D]

12) $L\{u(t-a)\} = \frac{e^{-as}}{s}$, where $u(t-a)$ is unit step function.

13) [C] e^{-as}

14) $L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0) = s^2 F(s) - sf(0) - f'(0)$

15) $L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\, dt = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\, dt$

($\because$ Here $T = 2\pi$) [C]

16) Given $L\{f(t)\} = \frac{1}{s^{3/2}}$ & $f(0) = 0$

w.k.T $L\{f'(t)\} = sL\{f(t)\} - f(0)$

$= s \cdot \frac{1}{s^{3/2}} - 0 = \frac{1}{\sqrt{s}}$ [A]

17) $L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = e^t L^{-1}\left\{\frac{1}{s^2+1}\right\}$ [$\because$ By First shifting theorem]

$= e^t \sin t.$

18) $L^{-1}\left\{\frac{s^2-s+2}{s^3}\right\} = L^{-1}\left\{\frac{1}{s}-\frac{1}{s^2}+\frac{2}{s^3}\right\} = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\} + 2L^{-1}\left\{\frac{1}{s^3}\right\}$

$= 1-t+2\frac{t^2}{2} = 1-t+t^2$. [B]

19) By convolution theorem,

$$L^{-1}\{F(s)\cdot G(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

20) $L\left\{\int_0^t\int_0^t \cosh au\,du\,du\right\} = \frac{1}{s^2}L\{\cosh at\} = \frac{1}{s^2}\cdot\frac{s}{s^2-a^2}$

$= \frac{1}{s(s^2-a^2)}$.

UNIT- IV

VSAQs

1) $\int_0^2\int_0^3 xy\,dy\,dx = \int_{x=0}^{2}\int_{y=0}^{3} xy\,dx\,dy = \left(\int_0^2 x\,dx\right)\left(\int_0^3 y\,dy\right) = \underline{9}$

2) $\int_0^1\int_0^x e^x\,dy\,dx = \int_{x=0}^{1}\left\{\int_{y=0}^{x} e^x\,dy\right\}dx = \int_{x=0}^{1}\left\{e^x(y)\right\}_{y=0}^{y=x}dx$

$= \int_0^1 e^x(x-0)\,dx = \int_0^1 e^x\cdot x\,dx = \left[e^x(x-1)\right]_0^1$

$= e^1(1-1) - e^0(0^0-1) = 0-1(-1) = 1$ [B]

3) $\int_1^3\int_0^{\log y} \frac{1}{\log y}\,dx\,dy = \int_{y=1}^{3}\left\{\int_{x=0}^{\log y}\frac{1}{\log y}\,dx\right\}dy$

$= \int_{y=1}^{3}\left\{\frac{1}{\log y}(x)\right\}_{x=0}^{\log y}dy = \int_{y=1}^{3}\left\{\log y(\log y-0)\right\}dy$

$= \int_1^3 dy = (y)_1^3 = \underline{2}$

4) $\int_0^1\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dx\,dy = \left(\int_0^1 \frac{dx}{\sqrt{1-x^2}}\right)\left(\int_0^1 \frac{dy}{\sqrt{1-y^2}}\right)$

$= \left[\sin^{-1}(x)\right]_0^1 \left[\sin^{-1}(y)\right]_0^1$

$= \left[\sin^{-1}(1) - \sin^{-1}(0)\right]\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$

$= \left(\frac{\pi}{2} - 0\right)\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{4}$ [D]

5) $\int_0^{\pi/2}\int_0^2 \sin\theta\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2} \sin\theta\,.\,d\theta\,dr = \left(\int_{\theta=0}^{\pi/2} \sin\theta\,d\theta\right)\left(\int_{r=0}^{2} dr\right)$

$= (-\cos\theta)_0^{\pi/2}\,(r)_0^2$

$= \left(-\cos\frac{\pi}{2} + \cos 0\right)(2-0)$

$= (-0+1)(2) = 2$ [B]

6) In polar coordinates, $dx\,dy = \underline{r\,dr\,d\theta}$ [C]

7) [A]

8) [B]

9) $\int_0^1\int_0^2\int_0^4 xyz\,dx\,dy\,dz = \int_{z=0}^{1}\int_{y=0}^{2}\int_{x=0}^{4} xyz\,dz\,dy\,dx.$

$= \left(\int_0^1 z\,dz\right)\left(\int_0^2 y\,dy\right)\left(\int_0^4 x\,dx\right)$

$= \left(\frac{1}{2}\right)\left(\frac{2^2}{2}\right)\left(\frac{4^2}{2}\right) = \underline{8}$

10) $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 e^x.e^y.e^z\,dx\,dy\,dz$

$= \left(\int_0^1 e^x\,dx\right)\left(\int_0^1 e^y\,dy\right)\left(\int_0^1 e^z\,dz\right)$

$$= (e^x)_0^1 (e^y)_0^1 (e^z)_0^1 = (e^1 - e^0)(e^1 - e^0)(e^1 - e^0) = (e-1)^3 \text{ [C]}$$

11) Let $\phi = xy - z^2$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \hat{i}(y) + \hat{j}(x) + \hat{k}(-2z)$$

The vector normal to the given surface $= \nabla\phi(-2,1,3)$

$$= \hat{i} - 2\hat{j} - 6\hat{k} \quad \text{[A]}$$

12) Given $\phi = xyz$

$$\nabla\phi = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\nabla\phi(1,1,1) = \hat{i} + \hat{j} + \hat{k}$$

$\therefore$ The maximum value of DD $= |\nabla\phi|(1,1,1)$

$$= \sqrt{1+1+1} = \sqrt{3} \quad \text{[B]}$$

13) $\nabla\phi = y\hat{i} + x\hat{j} - 2z\hat{k}$

$$\therefore \nabla\phi_{(1,2,3)} = \underline{2\hat{i} + \hat{j} - 6\hat{k}}$$

14) $\text{div}\, \vec{r} = \underline{3}$

15) $\text{curl}(\text{grad}\, \phi) = \underline{\vec{0}}$ [D]

16) $$\text{curl}\, \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$$\therefore \text{curl}\, \vec{F}(1,2,3) = \underline{\vec{0}}$$

17) $\text{div}\, \vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = \underline{0}$ [A]

18) $\text{div}\, \vec{V} = 0 \Rightarrow 2+4-p = 0 \Rightarrow \boxed{p = 6}$

19) $\nabla u = \hat{i}(2xyz) + \hat{j}(x^2 z) + \hat{k}(x^2 y)$

$$\nabla u_{\text{at }(1,4,1)} = 8\hat{i} + \hat{j} + 4\hat{k}$$

$\therefore$ the greatest rate of increase of $u = |\nabla u|(1,4,1)$

$$= \sqrt{64+1+16} = \underline{9}$$

20) $\nabla(r^n) = n r^{n-2}\vec{r}$ [D]

$$\nabla(r^n) = \sum \hat{i}\frac{\partial}{\partial x}(r^n)$$

$$= \sum \hat{i}(n r^{n-1})\frac{\partial r}{\partial x} = \sum \hat{i}(n r^{n-1})\frac{x}{r}$$

$$= n\sum r^{n-2}(x\hat{i}) = n r^{n-2}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= n r^{n-2}\vec{r}.$$

UNIT-V

VSAQs

1) πab

2) Along C: $y = x^2 \Rightarrow dy = 2x\,dx$ & $0 \le x \le 1$

$$\therefore \int_C y\,dx + x\,dy = \int_0^1 x^2 dx + x(2x)\,dx = \int_0^1 3x^2 dx = \cancel{3}\left(\frac{x^3}{\cancel{3}}\right)_0^1$$

$$= 1$$

3) By divergence theorem, $\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv = \int_V 3\,dv$

$= 3\int_V dv = 3(V)$, where V is the volume of unit sphere

$= \cancel{3}\times\frac{4\pi}{\cancel{3}}\times 1^3 = 4\pi$ [A]

4) $\oint_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot\vec{n}\,ds$ [B] (statement of Stoke's theorem)

5) Green's theorem in a plane

6) $\oint_C M\,dx + N\,dy = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$ (statement of Green's theorem)

7) By divergence theorem, $\int_S \vec{F}\cdot\hat{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv$

i.e., $\int_S (ax\hat{i} + by\hat{j} + cz\hat{k})\cdot\hat{n}\,ds = \int_V (a+b+c)\,dv$ [$\because \operatorname{div}\vec{F} = a+b+c$]

$= (a+b+c)\int_V dv$

$= (a+b+c)V = (a+b+c)\frac{4\pi}{3}$ [B]

8) Gauss's divergence theorem :

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{F}\cdot dv$$

9) $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$ [B]

10) $$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dx\,dy\,dz$$

(divergence theorem) $= \iiint_V (1+1+1)\,dx\,dy\,dz$

$= 3\int_V dv = 3V = \cancel{3}\times\frac{4\pi}{\cancel{3}} = 4\pi$

11) By Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{— (1)}$$

Here $M = y$; $N = -x$ so that $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$

For R : $-1 \le x \le 1$ and $-1 \le y \le 1$

using (1), $\oint_C y\,dx - x\,dy = \iint_R (-1-1)\,dx\,dy$

$= -2\int_{-1}^{1}\int_{-1}^{1} dx\,dy = -2\left(\int_{-1}^{1} dx\right)\left(\int_{-1}^{1} dy\right)$

$= -2(x)_{-1}^{1}\,(y)_{-1}^{1} = -2(1+1)(1+1)$

$= -8$ [C]

12) By divergence theorem,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv$$

i.e, $\int_S \vec{r}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{r}\,dv = \int_V 3\,dv = 3V$

13) Gauss's divergence theorem

14) Along c: $x = t^2$, $y = 2t$ so that $dx = 2t\,dt$, $0 \le t \le 1$

$$\int_c (x+y^2)\,dx = \int_0^1 (t^2+4t^2)(2t)\,dt$$

$$= 10\int_0^1 t^3 dt = 10\left(\frac{t^4}{4}\right)_0^1 = \cancel{10}\left(\frac{1}{\cancel{4}}\right) = \frac{5}{2}$$

15) By Green's theorem,

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad -①$$

Here $M = 2xy - x^2$; $N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = 2x \quad ; \quad \frac{\partial N}{\partial x} = 2x$$

using ①, $\oint_c (2xy - x^2)\,dx + (x^2+y^2)\,dy = \iint_R (2x - 2x)\,dx\,dy$

$$= \iint_R (0)\,dx\,dy$$

$$= 0$$