Long Answer Questions

1. find the laplace transforms of the following:

(i) $e^{2t}+4t^3-2\sin 3t+3\cos 3t$

(ii) $\sin 2t \sin 3t$

(iii) $\cos^2 2t$

(iv) $\sin^3 2t$

Sol (i) $e^{2t}+4t^3-2\sin 3t+3\cos 3t$

$= L[e^{2t}]+4L[t^3]-2L[\sin 3t]+3L[\cos 3t]$

$= \frac{1}{s-2}+4\times\frac{3!}{s^4}-2\times\frac{3}{s^2+9}+3\times\frac{s}{s^2+9}$

$= \frac{1}{s-2}+\frac{24}{s^4}-\frac{6}{s^2+9}+\frac{3s}{s^2+9}$

$= \frac{1}{s-2}+\frac{24}{s^4}-\frac{(6-3s)}{s^2+9}$

(ii) $L[\sin 2t\cdot\sin 3t]$

$=\frac{1}{2}[2\sin 2t\cdot\sin 3t]$

$=\frac{1}{2}[\cos(2t-3t)-\cos(2t+3t)]$

$=\frac{1}{2}[\cos(-t)-\cos(5t)]$

$=\frac{1}{2}[\cos t-\cos 5t]$ $\quad (\because \cos(-\theta)=\cos\theta)$

$=\frac{1}{2}\left[\frac{s}{s^2+1}-\frac{s}{s^2+25}\right]$

$=\frac{1}{2}\left[\frac{s(s^2+25)-s(s^2+1)}{(s^2+1)(s^2+25)}\right]$

$=\frac{1}{2}\left[\frac{\cancel{s^3}+25s-\cancel{s^3}-s}{(s^2+1)(s^2+25)}\right]$

$= \frac{1}{\cancel{2}}\left[\frac{\overset{12}{\cancel{24}}s}{(s^2+1)(s^2+25)}\right]$

$= \frac{12s}{(s^2+1)(s^2+25)}$

(iii) $L[\cos^2 2t]$

$\cos^2 2t = \frac{1}{2}(1+\cos 4t)$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$

$= \frac{1}{2}\left[\frac{1(s^2+16)+s(s)}{(s)\ (s^2+16)}\right]$

$= \frac{1}{2}\left[\frac{s^2+16+s^2}{(s)\ (s^2+16)}\right]$

$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s^2+\overset{8}{\cancel{16}}}{(s)\ (s^2+16)}\right]$

$= \frac{s^2+8}{(s)\ (s^2+16)}$

(iv) $\sin^3 2t$

Wkt $\sin 6t = 3\sin 2t - 4\sin^3 2t$

$\frac{1}{4}[3\sin 2t - \sin 6t] = \sin^3 2t$

$L(\sin^3 2t) = \frac{1}{4}L[3\sin 2t - \sin 6t]$

$= \frac{1}{4}[3L(\sin 2t) - L(\sin 6t)]$

$= \frac{1}{4}\left[3\times\frac{2}{s^2+4} - \frac{6}{s^2+36}\right]$

$= \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right]$

$= \frac{1}{4}\left[\frac{6(s^2+36)-6(s^2+4)}{(s^2+4)\ (s^2+36)}\right]$

$= \frac{1}{4}\left[\frac{\cancel{6s^2}+216-\cancel{6s^2}-24}{(s^2+4)\ (s^2+36)}\right]$

$= \frac{1}{\cancel{4}}\left[\frac{\overset{48}{\cancel{192}}}{(s^2+4)\ (s^2+36)}\right]$

$= \frac{48}{(s^2+4)\ (s^2+36)}$

2(i) find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left[\frac{1}{s}\right]$

Sol We know that $L\left[\frac{f(t)}{t}\right] = \int_0^\infty \bar{f}(s)\,ds$

$L[\sin at] = \frac{a}{s^2+a^2} = \bar{f}(s)$

$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\,ds$

$= \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$

$= \cot^{-1}\left(\frac{s}{a}\right)$

(ii) If $L[f(t)] = \frac{1}{s}e^{-1/s}$ then prove that $L[e^{-t}\cdot f(3t)] = \frac{1}{s+1}\cdot e^{-\frac{3}{s+1}}$

Sol Given $L[f(t)] = \bar{f}(s) = \frac{1}{s} e^{-1/s}$

By change of scale property

$$L[f(3t)] = \frac{1}{3} \cdot \bar{f}\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \cdot \frac{1}{\frac{s}{3}} \cdot e^{-1/\frac{s}{3}}$$

$$= \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{s} \cdot e^{-3/s}$$

$$= \frac{1}{s} e^{-3/s}$$

Now Apply first shifting property

$$L[e^{-t} \cdot f(3t)] = L[f(3t)]_{s \to s+1}$$

$$= \left[\frac{1}{s} \cdot e^{-3/s}\right]_{s \to s+1}$$

$$= \frac{1}{s+1} \cdot e^{-3/s+1}$$

3. Find the laplace transforms of the following

(i) $e^{-3t}(2\cos 5t - 3\sin 5t)$

Sol $L[2\cos 5t - 3\sin 5t]$

$$2 \cdot L[\cos 5t] - 3L[\sin 5t]$$

$$2\left[\frac{s}{s^2+25}\right] - 3\left[\frac{5}{s^2+25}\right]$$

$$= \frac{2s}{s^2+25} - \frac{15}{s^2+25}$$

$$= \left[\frac{2s-15}{s^2+25}\right]_{s \to s+3} \quad (\because a=3)$$

$$= \left[\frac{2(s+3)-15}{(s+3)^2+25}\right]$$

$$= \frac{2s+6-15}{s^2+9+6s+25}$$

$$= \frac{2s-9}{s^2+6s+34}$$

(ii) $e^{-t}\sin^2 t$

Sol $L[e^{-t}\sin^2 t]$

$$\sin^{2t} = \frac{1}{2}[1 - \cos 2t]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{s^2}+\cancel{4}^{2} - \cancel{s^2}}{s(s^2+4)}\right]$$

$$= \left[\frac{2}{s(s^2+4)}\right]_{s \to s+1}$$

$$= \frac{2}{(s+1)(s+1)^2+4}$$

$$= \frac{2}{(s+1)(s^2+1+2s)+4}$$

$$= \frac{2}{s^3+s+2s^2+s^2+1+2s+4}$$

$$= \frac{2}{s^3+3s^2+3s+5}$$

(iii) $te^{2t}\cos t$

Sol $L\left[t^1 f(t)\right] = (-1)^n \frac{d}{ds} F(s)$

$\because L\left[\cos at\right] = \frac{s}{s^2+a^2}$

$\Rightarrow \frac{-d}{ds}\left[\frac{s}{s^2+a^2}\right]$

$\Rightarrow -\left[\frac{(s^2+a^2)\times 1 - s\times 2s}{(s^2+a^2)^2}\right]$

$\Rightarrow -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$

$\Rightarrow -\left[\frac{-s^2+a^2}{(s^2+a^2)^2}\right]$

$\Rightarrow \frac{s^2-a^2}{(s^2+a^2)^2}$

(iv) $t^2 e^{-3t} \sin 2t$

Sol $L[\sin 2t] = \frac{2}{s^2+4}$

$L\left[e^{-3t}\cdot \sin 2t\right] = \frac{2}{(s+3)^2+4}$

$= \frac{2}{s^2+6s+9+4}$

$= \frac{2}{s^2+6s+13}$

$L\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n} F(s)$

$L\left[t^2 e^{-3t} \sin 2t\right] = (-1)^2 \frac{d^2}{ds^2}\left[\frac{2}{s^2+6s+13}\right]$

$= \frac{d^2}{ds^2}\left[\frac{2}{s^2+6s+13}\right]$

$= \frac{d}{ds}\left[\frac{d}{ds}\left[\frac{2}{s^2+6s+13}\right]\right]$

$= \frac{d}{ds}\left[\frac{(s^2+6s+13)\frac{d}{ds}(2) - 2\frac{d}{ds}(s^2+6s+13)}{(s^2+6s+13)^2}\right]$

$= \frac{d}{ds}\left[\frac{-2(2s+6)}{(s^2+6s+13)^2}\right]$

$= \frac{-d}{ds}\left[\frac{4s+12}{(s^2+6s+13)^2}\right]$

$= -\left[\frac{(s^2+6s+13)^2 \frac{d}{ds}(4s+12) - (4s+12)\frac{d}{ds}(s^2+6s+13)^2}{\left[(s^2+6s+13)^2\right]^2}\right]$

$= -\left[\frac{(s^2+6s+13)^2 4 - (4s+12)\,2\,(s^2+6s+13)(2s+6)}{(s^2+6s+13)^4}\right]$

$= (-\cancel{s^2+6s+13})\left[\frac{(s^2+6s+13)4 - (4s+12)(4s+12)}{\cancel{(s^2+6s+13)}\;(s^2+6s+13)^3}\right]$

$= -\left[\frac{4s^2+24s+52 - [4s(4s+12)] + 12(4s+12)}{(s^2+6s+13)^3}\right]$

$$L\{t^2 e^{-3t}\sin 2t\} = -\left[\frac{4s^2+24s+52-[4s(4s+12)]+12(4s+12)}{(s^2+6s+13)^3}\right]$$

$$= -\left[\frac{4s^2+24s+52-(16s^2+48s+48s+144)}{(s^2+6s+13)^3}\right]$$

$$= \left[\frac{-4s^2-24s-52+16s^2+48s+48s+144}{(s^2+6s+13)^3}\right]$$

$$= \left[\frac{12s^2+72s+92}{(s^2+6s+13)^3}\right]$$

$$= 4\left[\frac{3s^2+18s+23}{(s^2+6s+13)^3}\right]$$

$$= 4\left[\frac{3(s^2+6s+9)-4}{[(s+3)^2+4]^3}\right]$$

$$= 4\left[\frac{3(s+3)^2-4}{[(s+3)^2+4]^3}\right]$$

$$= 4\left[\frac{3s^2+18s+27-4}{((s+3)^2+4)^3}\right]$$

4. Evaluate

(i) $L\left\{\frac{e^{-t}\sin t}{t}\right\}$

Sol $L[\sin t] = \frac{1}{s^2+1}$

By 1^{st} shifting theorem

$L(e^{-t}\sin t) = \frac{1}{\{s-(-1)\}^2+1}$ $\quad [\because L\{f(t)\} = f(p)]$

$$= \frac{1}{s^2+2s+2}$$

$$\therefore L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty \frac{1}{x^2+2x+c}\,dx$$

$$= \int_s^\infty \frac{1}{(x+1)^2+1}\,dx = \int_s^\infty \frac{1}{1+(x+1)^2}\,dx$$

$$= \int_{s+1}^\infty \frac{1}{1+z^2}\,dz \qquad \because \text{let } x+1 = z,\ dx = dz$$

$$= \left[\tan^{-1} z\right]_{s+1}^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) \qquad \because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

$$= \cot^{-1}(s+1) \qquad \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

(ii) $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Sol W.k.t $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

$$L\left[e^{-at}-e^{-bt}\right] = \frac{1}{s+a} - \frac{1}{s+b} = \bar{f}(s)$$

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty \frac{1}{s+a} - \frac{1}{s+b}\,ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \log\left(\frac{s+a}{s+b}\right)\Bigg]_s^\infty$$

$$= \log\left(\frac{s+a}{s+b}\right)\Bigg]_\infty - \left[\log\left(\frac{s+a}{s+b}\right)\right]_s$$

$= \underset{s\to\infty}{Lt} \log\left[\frac{s(1+\frac{a}{s})}{s(1+\frac{a}{s})}\right] - \log\left(\frac{s+a}{s+b}\right)$

$= \log\left[\frac{1+0}{1+0}\right] - \log\left[\frac{s+a}{s+b}\right]$

$= [\log 1] - \log\left[\frac{s+a}{s+b}\right]$

$= 0 - \log(s+a) - \log(s+b)$

$= \log(s+b) - \log(s+a)$

$= \log\left(\frac{s+b}{s+a}\right)$

(iii) $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

Sol $\left[\frac{s}{s^2+a^2} - \frac{1}{s^2+b^2}\right] = \bar{f}(s)$

W.K.T $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$

$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$

$= \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds$

$= \frac{1}{2}\log(s^2+a^2) - \log(s^2+b^2)\Big]_s^\infty$

$= \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\Big]_s^\infty$

$= \frac{1}{2}\left[\underset{s\to\infty}{Lt}\left[\log\left[\frac{s^2(1+a^2/s^2)}{s^2(1+b^2/s^2)}\right] - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right.\right.$

$= \frac{1}{2}\left[\log(1) - \log\left[\frac{s^2+a^2}{s^2+b^2}\right]\right.$

$= \frac{1}{2}\left[0 - \log(s^2+a^2) + \log(s^2+b^2)\right]$

$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

(iv) $L\left\{\frac{\sin t \sin 5t}{t}\right\}$

Sol $L[\sin t \cdot \sin 5t] = \frac{1}{2}L[2\sin t \cdot \sin 5t]$

$= \frac{1}{2}[\cos 4t - \cos 6t]$

$= \frac{1}{2}\left[\frac{s}{s^2+16} - \frac{s}{s^2+36}\right] = \bar{f}(s)$

$L\left[\frac{\sin t \cdot \sin 5t}{t}\right] = \frac{1}{2}\int_s^\infty \frac{1}{2}\left[\frac{2s}{s^2+16} - \frac{2s}{s^2+36}\right] ds$

$= \frac{1}{4}\left[\log(s^2+16) - \log(s^2+36)\right]_s^\infty$

$= \frac{1}{4}\left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]_s^\infty$

$= \frac{1}{4}\left[\underset{s\to\infty}{Lt} \log\left[\frac{s^2(1+\frac{16}{s^2})}{s^2(1+\frac{36}{s^2})}\right]\right] - \left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]_s$

$= \frac{1}{4}\left[\log(1) - \log\left(\frac{s^2+16}{s^2+36}\right)\right]$

$= \frac{1}{4}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]$

5. Using laplace transforms evaluate the following

(i) $\int_0^\infty t e^{-2t} \cos t\, dt$

Sol $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$L[t \cdot e^{-2t} \cos t]$

$= \int_0^\infty e^{-st} \cdot t \cdot e^{-2t} \cos t \, dt$ ——— ①

$L[t \cdot \cos t] = (-1) \frac{d}{ds} \cdot \frac{s}{s^2+1}$

$= -\left(\frac{-s^2+1}{(s^2+1)^2}\right)$

$= \frac{s^2-1}{(s^2+1)^2}$

First shifting

$L[e^{-2t} \cdot t \cdot \cos t] = \frac{(s+2)^2-1}{((s+2)^2+1)^2}$

To eliminate e^{-st} ——— ①

put $s=0$

① $\Rightarrow \int_0^\infty e^{-0(t)} \cdot t \, e^{-2t} \cdot \cos t \, dt$

$= \frac{2^2-1}{(2^2+1)^2}$

$= \frac{3}{25}$

(ii) $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \cdot dt$

Sol Wkt $L[f(t)] = \int_0^\infty e^{-st} f(t) dt = f(s)$ or $\bar{f}(s)$

$I = \int_0^\infty \left(\frac{e^{-at} - e^{-bt}}{t}\right) dt$

$L[\underbrace{e^{-at} - e^{-bt}}_{g(t)}] = L[e^{-at}] - L[e^{-bt}]$

$= \frac{1}{s+a} - \frac{1}{s+b} = g(s)$

Division by t:

$L\left[\frac{g(t)}{t}\right] = \int_s^\infty g(s) \, ds$

$L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$

$= [\log(s+a) - \log(s+b)]_s^\infty$

$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$

$= \left[\log \frac{\cancel{s}\left(1+\frac{a}{s}\right)}{\cancel{s}\left(1+\frac{b}{s}\right)}\right]_s^\infty$

$= \log 1 - \log\left(\frac{s+a}{s+b}\right)$

$= 0 - \log\left(\frac{s+a}{s+b}\right)$

$= \log\left(\frac{s+b}{s+a}\right)$

$L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right) = f(s)$

$\int_0^\infty e^{-st} f(t) \, dt = f(s)$

$\int_0^\infty e^{-st} \left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \log\left(\frac{s+b}{s+a}\right)$

To eliminate e^{-st} put $s=0$

$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log\left(\frac{b}{a}\right)$

(iii) $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$

Sol: $L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \int_s^\infty [L(\cos 6t) - L(\cos 4t)] ds$

$= \int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right) ds$

$= \int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right) ds$

$= \frac{1}{2}\left[\int_s^\infty \frac{2s}{s^2+36} ds - \int_s^\infty \frac{2s}{s^2+16} ds\right]$

$= \frac{1}{2}\left[\log(s^2+36) - \log(s^2+16)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{\cancel{s^2}\left(1+\frac{36}{s^2}\right)}{\cancel{s^2}\left(1+\frac{16}{s^2}\right)}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log 1 - \log\frac{(s^2+36)}{(s^2+16)}\right]$

$= \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$

$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right) = \bar{f}(s)$

$\Rightarrow \int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s)$

put $s=0$

$\int_0^\infty e^0 \left(\frac{\cos 6t - \cos 4t}{t}\right) dt = \frac{1}{2}\log\frac{0+16}{0+36}$

$\int_0^\infty \left(\frac{\cos 6t - \cos 4t}{t}\right) dt = \frac{1}{2}\log\left(\frac{4}{6}\right)^2$

$= \log\frac{4}{6}$

$= \log\left(\frac{2}{3}\right)$

(iv) $\int_0^\infty \frac{e^{-t}\sin^2 t}{t} dt$

Sol: $L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right]$

$= L\left[\frac{1}{2} - \frac{\cos 2t}{2}\right]$

$= L\left[\frac{1}{2}\right] - \frac{1}{2}L[\cos 2t]$

$= \frac{1}{2}L[1] - \frac{1}{2}L[\cos 2t]$

$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2+4}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \bar{f}(s)$

$\left(\because L[f(t)] = \bar{f}(s)\right.$

$\left.L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds\right)$

$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$

$= \frac{1}{2}\left[\int_s^\infty \frac{1}{s} ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4} ds\right]$

$= \frac{1}{2}\left[(\log s)_s^\infty - \frac{1}{2}\left(\log(s^2+4)\right)_s^\infty\right]$

$= \frac{1}{2}\left[\log|s| - \log|s^2+4|^{1/2}\right]_s^\infty$

$= \frac{1}{2}\left[\frac{\log s}{(s^2+4)^{1/2}}\right]_s^\infty$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+4}\right)^{1/2}\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{1}{1+\frac{4}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log(1) - \log\left(\frac{1}{1+\frac{4}{s^2}}\right)\right]$$

$$= \frac{1}{4}\left[0 - \log\left(\frac{1}{1+\frac{4}{s^2}}\right)\right]$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

By definition:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{\sin^2 t}{t} \cdot dt$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s}\right)$$

Put $s=1$ on both sides

$$\int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)dt = \frac{1}{4}\log 5$$

6. Find the laplace transform of the triangular wave function of period $2a$ given by

$$f(t) = \begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases}$$

Sol $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-st}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\cdot t\,dt \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\,\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{s^2}\right]_0^a + \left[(2a-t)\frac{e^{-st}}{-s}\right]_a^{2a} + \left[\frac{e^{-st}}{s^2}\right]_a^{2a}\right]$$

$$= \frac{1}{1-(e^{-as})^2}\left[\frac{ae^{-as}-0}{-s} - \left[\frac{e^{-as}-1}{s^2}\right] + 0 - \left[\frac{ae^{-as}}{-s}\right] + \frac{e^{-2as}-e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-(e^{-as})^2}\cdot\frac{1}{s^2}\left[1-2e^{-as}+(e^{-as})^2\right]$$

$$= \frac{1}{(1+e^{-as})\cancel{(1-e^{-as})}s^2}\left[1-e^{-as}\right]^{\cancel{2}}$$

$$= \frac{1}{(1+e^{-as})s^2}(1-e^{-as})$$

$$= \frac{(1-e^{-as})}{s^2(1+e^{-as})}$$

$$= \frac{1}{s^2}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

7. Find the Inverse laplace transforms of the following

(i) $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

Sol Let $\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3}$ —— ①

$2s^2-6s+5 = A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$ —— ②

Put $s=1$ in ②

$1 = A(1-2)(1-3)+B(0)+C(0)$

$1 = 2A$

$\boxed{A=\frac{1}{2}}$

put $s=2$ in eq ②

$1 = A(0)+B(2-1)(2-3)+C(0)$

$1 = -B$

$\boxed{B=-1}$

Put $s=3$ in eq ③

$5 = A(0)+B(0)+C(2)$

$\boxed{C=\frac{5}{2}}$

Sub A, B, C in ①

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1}{2}\left[\frac{1}{s-1}\right]-\left[\frac{1}{s-2}\right]+\frac{5}{2}\left[\frac{1}{s-3}\right]$$

$$L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{s-2}\right]+\frac{5}{2}L^{-1}\left[\frac{1}{s-3}\right]$$

$$= \frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}.$$

(ii) $\frac{4s+5}{(s-1)^2(s+2)}$

Sol Let $\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$ —— ①

$4s+5 = A(s-1)(s+2)+B(s+2)+C(s-1)^2$ —— ②

Put $s=1$ in ②

$9 = A(0)+B(3)+C(0)$

$3B = 9$

$\boxed{B=3}$

Put $s=-2$ in ②

$-3 = A(0)+B(0)+C(9)$

$\boxed{C=\frac{-1}{3}}$

Equating s^2 on both sides

$0 = A(s^2+2s-s-2)+B(s+2)+C(s^2+1-2s)$

$0 = A+C$

$0 = A-\frac{1}{3}$

$\boxed{A=\frac{1}{3}}$

$$L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right]+3L^{-1}\left[\frac{1}{(s-1)^2}\right]-\frac{1}{3}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{3}e^{t}+3t\,e^{t}-\frac{1}{3}e^{-2t}$$

(iii) $\frac{5s+3}{(s-1)(s^2+2s+5)}$

Sol Let $\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$ —— ①

$5s+3 = A(s^2+2s+5)+Bs+C(s-1)$ —— ②

put $s=1$ in (2)

$$8 = 8A + (B(1) + C(0))$$

$$\boxed{A=1}$$

put $s=0$ in (2)

$$3 = 5A\,(B(0)+C)(-1)$$

$$3 = 5A - C$$

$$3-5 = C$$

$$\boxed{C=2}$$

put $s=2$ in (2)

$$13 = A(4+4+5) + (2B+C)(1)$$

$$13 = 13A + 2B + C$$

$$13 - 13 - 2 = 2B$$

$$\boxed{B=-1}$$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{-s+2}{s^2+2s+5}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{-s+1-1+2}{s^2+2s+1-1+5}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{-(s+1)+3}{(s+1)^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= e^{t} - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^{t} - e^{-t}\cos 2t + 3\cdot\frac{1}{2}\cdot e^{t}\sin 2t$$

$$= e^{t}\cdot e^{-t}\cos 2t + \frac{3}{2}e^{t}\cdot\sin 2t.$$

(iv) $\dfrac{s}{(s^2+1)(s^2+9)(s^2+25}$

sol Wkt $\dfrac{1}{(s^2+a)(s^2+b)} = \dfrac{1}{b-a}\left[\dfrac{1}{s^2+a} - \dfrac{1}{s^2+b}\right]$

let $\dfrac{1}{(s^2+1)(s^2+9)} = \dfrac{1}{9-1}\left[\dfrac{1}{s^2+1} - \dfrac{1}{s^2+9}\right]$

$$\frac{1}{(s^2+1)(s^2+9)(s^2+25)} = \frac{1}{8}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]\frac{1}{s^2+25}$$

$$= \frac{1}{8}\left[\frac{1}{(s^2+1)(s^2+25)} - \frac{1}{(s^2+9)(s^2+25)}\right]$$

$$= \frac{1}{8}\left[\frac{1}{25-1}\left[\frac{1}{s^2+1} - \frac{1}{s^2+25}\right] - \left[\frac{1}{25-9}\left(\frac{1}{s^2+9} - \frac{1}{s^2+25}\right)\right]\right]$$

$$= \frac{1}{8}\left[\frac{1}{24}\left(\frac{1}{s^2+1} - \frac{1}{s^2+25}\right) - \frac{1}{16}\left(\frac{1}{s^2+9} - \frac{1}{s^2+25}\right)\right]$$

$$= \frac{1}{8}\left[\frac{1}{24}\left(\frac{1}{s^2+1}\right) - \frac{1}{16}\left(\frac{1}{s^2+9}\right) + \left(\frac{1}{s^2+25}\right)\left(\frac{-1}{24} + \frac{1}{16}\right)\right]$$

$$= \frac{1}{8}\left[\frac{1}{24}\left(\frac{1}{s^2+1}\right) - \frac{1}{16}\left(\frac{1}{s^2+9}\right) + \frac{1}{48}\left(\frac{1}{s^2+25}\right)\right]$$

$$= \frac{1}{8}\left[\frac{1}{24\times 16}\left(\frac{16}{s^2+1}\right) - \left(\frac{24}{s^2+9}\right) + \left(\frac{8}{s^2+25}\right)\right]$$

$$= \frac{1}{8\times 24\times 16}\left[\frac{16}{s^2+1} - \frac{24}{s^2+9} + \frac{8}{s^2+25}\right]$$

Consider $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+9)(s^2+25)}\right] = \dfrac{1}{3072}\cdot 16L^{-1}\left[\dfrac{s}{s^2+1}\right] - 24L^{-1}\left[\dfrac{s}{s^2+9}\right] + 8L^{-1}\left[\dfrac{s}{s^2+25}\right]$

$$= \frac{1}{3072}(16\cos t - 24\cos 3t + 8\cos 5t)$$

8. Evaluate the following

(i) $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$

Sol $f(t) = L^{-1}\left[\log(s+1) - \log(s-1)\right]$

$t\cdot f(t) = -L^{-1}\left[\frac{d}{ds}\log(s+1) - \frac{d}{ds}\log(s-1)\right]$

$= -L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$= L^{-1}\left[\frac{-1}{s+1}\right] + L^{-1}\left[\frac{1}{s-1}\right]$

$= -e^{-t} + e^{t}$

$f(t) = \frac{e^{t} - e^{-t}}{t}$

(ii) $L^{-1}\left\{\log\left(\frac{s^2+4}{s^2+9}\right)\right\}$

Sol $f(t) = L^{-1}\left[\log(s^2+4) - \log(s^2+9)\right]$

$t\cdot f(t) = -L^{-1}\left[\frac{d}{ds}\log(s^2+4) - \frac{d}{ds}\log(s^2+9)\right]$

$= -L^{-1}\left[\frac{2s}{s^2+4} - \frac{2s}{s^2+9}\right]$

$= L^{-1}\left[\frac{-2s}{s^2+4}\right] + L^{-1}\left[\frac{2s}{s^2+9}\right]$

$= -2\cos 2t + 2\cdot\cos 3t$

$t\cdot f(t) = 2\cos 3t - \cos 2t$

$f(t) = \frac{2}{t}(\cos 3t - \cos 2t)$

(iii) $L^{-1}\left\{\cot^{-1}\frac{s}{2}\right\}$

Sol $f(t) = L^{-1}\left[\cot^{-1}\left(\frac{s}{2}\right)\right]$

$t\cdot f(t) = -L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right]$

$= -L^{-1}\left[\frac{-2}{s^2+4}\right]$

$= L^{-1}\left[\frac{2}{s^2+2^2}\right]$

$= \sin 2t$

9. Evaluate the following using convolution theorem

(i) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol $L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right]$

$L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f(t)$

$L^{-1}\left[\bar{g}(s)\right] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = g(t)$

By convolution theorem

$f(t) * g(t) = \int_0^t f(u)\cdot g(t-u)\,du$

$\cos at * \frac{1}{a}\sin at = \int_0^t \left[\cos au\cdot\frac{1}{a}\sin a(t-u)\right]du$

$= \frac{1}{a}\int_0^t \left[\cos au\cdot\sin(at-au)\right]du$

$= \frac{1}{2a}\int_0^t \left[2\cos au\cdot\sin(at-au)\right]du$

$= \frac{1}{2a}\left[\int_0^t \left[\sin(\cancel{au}+at-\cancel{au}) - \sin(au-at+au)\right]du\right.$

$= \frac{1}{2a}\left[\int_0^t \sin at - \sin(2au-at)\,du\right.$

$= \frac{1}{2a}\left[\sin at\cdot u + \frac{\cos(2au-at)}{2a}\right]_0^t$

$= \frac{1}{2a}\left[t\cdot \sin at + \frac{1}{2a}\cos(2at-at)\right] - \left[0+\frac{1}{2a}\cos(-at)\right]$

$= \frac{1}{2a}\left[t\cdot \sin at + \cancel{\frac{1}{2a}\cos at} - \cancel{\frac{1}{2a}\cos at}\right]$

$= \frac{1}{2a}\cdot t \sin at$

$= \frac{t}{2a}\sin at$

(ii) $L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}$

Sol $L^{-1}\left[\frac{s}{s^2+2^2}\cdot\frac{s}{s^2+3^2}\right]$

$\Rightarrow L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t = f(t)$

$L^{-1}\left[\frac{s}{s^2+3^2}\right] = \cos 3t = g(t)$

By convolution method

$\cos 2t \cdot \cos 3t = \int_0^t \cos 2u \cdot \cos 3(t-u)\,du$

$= \frac{1}{2}\int_0^t 2\cos 2u\cdot \cos 3(t-u)\,du$

$= \frac{1}{2}\int_0^t 2\cos 2u \cdot \cos(3t-3u)\,du$

$= \frac{1}{2}\int_0^t \cos(2u+3t-3u) + \cos(2u-3t+3u)\,du$

$= \frac{1}{2}\int_0^t \cos(3t-u) + \cos(5u-3t)\,du$

$= \frac{1}{2}\left[\frac{\sin(3t-u)}{-1} + \frac{\sin(5u-3t)}{5}\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin(5u-3t)}{5} - \sin(3t-u)\right]_0^t$

$= \frac{1}{2}\left[\frac{\sin(5t-3t)}{5} - \sin(3t-t)\right] - \frac{1}{2}\left[\frac{\sin(-3t)}{5} - \sin(3t)\right]$

$= \frac{1}{2}\left[\frac{\sin(2t)}{5} - \sin(2t)\right] - \frac{1}{2}\left[\frac{-\sin 3t}{5} - \sin 3t\right]$

$= \frac{1}{2}\left[\frac{\sin(2t) - 5\sin(2t)}{5}\right] - \frac{1}{2}\left[\frac{-\sin 3(t) - 5\sin(3t)}{5}\right]$

$= \frac{1}{2\times 5}\left[\sin(2t) - 5\sin(2t)\right] - \frac{1}{2\times 5}\left[-\sin 3(t) - 5\sin(3t)\right]$

$= \frac{1}{10}\left[-4\sin(2t)\right] - \frac{1}{2\times 5}\left[-6\sin(3t)\right]$

$= \frac{-2}{5}\sin(2t) + \frac{3}{5}\sin(3t)$

$= \frac{-1}{5}(2\sin 2t - 3\sin 3t).$

(iii) $L^{-1}\left\{\frac{s}{(s+2)(s^2+9)}\right\}$

Sol $L^{-1}\left\{\frac{s}{(s+2)(s^2+9)}\right\} = L^{-1}\left\{\frac{1}{s+2}\cdot\frac{s}{s^2+3^2}\right\}$

Let $\bar{f}(s) = \frac{1}{s+2}$ and $\bar{g}(s) = \frac{s}{s^2+3^2}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t} = f(t)$

$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos 3t = g(t)$

By convolution Theorem

$L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = f(t) * g(t) = \int_0^t f(x)\cdot g(t-x)\,dx = \int_0^t g(x)\cdot f(t-x)\,dx$ $\int_0^t \cos 3x \cdot e^{-2(t-x)}\,dx = \int_0^t \cos 3x \cdot e^{-2t+2x}\,dx = \int_0^t \cos 3x \cdot e^{-2t}\cdot e^{2x}\,dx$

$= e^{-2t}\int_0^t e^{2x}\cos 3x\,dx$

$= e^{-2t}\left[\frac{e^{2x}}{2^2+3^2}(2\cos 3x+3\sin 3x)\right]_{x=0}^{t}$

$= e^{-2t}\left[\frac{e^{2t}}{13}(2\cos 3t+3\sin 3t)-\frac{e^0}{13}(2\cos 0+3\sin 0)\right]$

$= e^{-2t}\left[\frac{e^{2t}}{13}(2\cos 3t+3\sin 3t)-\frac{2}{13}\right]$

$= \frac{1}{13}(2\cos 3t+3\sin 3t)-\frac{2}{13}e^{-2t}$.

10. Solve the following differential equation by using laplace transforms

(i) $y'''+2y''-y'-2y=0$, $y(0)=y'(0)=0$ and $y''(0)=6$

Sol Given D.E is $y'''+2y''-y'-2y=0$

$L(y''')+2L(y'')-L(y')-2L(y)=L(0)$

i.e $[s^3\bar{y}(s)-s^2y(0)-sy'(0)-y''(0)]+2[s^2\bar{y}(s)-sy(0)-y'(0)]-[s\bar{y}(s)-y(0)]-2L(y)=0$

$[s^3y(s)-0-0-6]+2[s^2y(s)-0-0]-[s\bar{y}(s)-0]-2L(y)=0$

$[s^3L(y)-6]+2[s^2L(y)]-[sL(y)]-2L(y)=0$

$L(y)[s^3+2s^2-s-2]-6=0$

$L(y)[s^3+2s^2-s-2]=6$

$L(y)=\frac{6}{s^3+2s^2-s-2}$

Consider $\frac{6}{s^3+2s^2-s-2}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}$ — ①

$6=A(s+1)(s+2)+B(s-1)(s+2)+C(s-1)(s+1)$ — ②

put $s=-1$ in eqn ②

$6=B(-1-1)(-1+2)$

$6=B(-2)(1)$

$\boxed{B=-3}$

put $s=1$

$6=A(2)(3)$

$\boxed{A=1}$

put $s=-2$

$6=C(-3)(-1)$

$\boxed{C=2}$

Substitute A, B, C in eqn ①

$L\left[\frac{6}{s^3+2s^2-s-2}\right]=L\left[\frac{1}{s-1}-\frac{3}{s+1}+\frac{2}{s+2}\right]$

Inverse L.T on both sides

$y=L^{-1}\left[\frac{1}{s-1}\right]-3L^{-1}\left[\frac{1}{s+1}\right]+2L^{-1}\left[\frac{1}{s+2}\right]$

$y=e^{t}-3e^{-t}+2e^{-2t}$

(ii) $\frac{d^2x}{dt^2}-2\frac{dx}{dt}+x=e^{t}$ with $x=2$, $\frac{dx}{dt}=-1$ at $t=0$

Sol Given D.E is $x''-2x'+x=e^{t}$; $x(0)=2$; $x'(0)=-1$

$x''-2x'+x=e^{t}$

$L[x'']-2L[x']+L[x]=L[e^{t}]$

$[s^2\bar{x}(s)-sx(0)-x'(0)]-2[s\bar{x}(s)-x(0)]+L[x]=\frac{1}{s-1}$

$[s^2L(x)-2s+1]-2[sL(x)-2]+L[x]=\frac{1}{s-1}$

$L(x)[s^2-2s+1]-2s+1+4=\frac{1}{s-1}$

$L(x)[s^2-2s+1]=\frac{1}{s-1}+\frac{2s-5}{1}=\frac{1+(s-1)(2s-5)}{s-1}$

$L(x)[s^2-2s+1]=\frac{2s^2-7s+6}{s-1}$

$L(x)=\frac{2s^2-7s+6}{(s-1)(s^2-2s+1)}=\frac{A}{(s-1)}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}$ — (1)

$2s^2-7s+6=A(s-1)^2+B(s-1)+C$

$2s^2-7s+6=A(s^2-2s+1)+C$ — (2)

Equating the coefficient of s^2 & s

$s^2 \rightarrow \boxed{2=A}$

$s \rightarrow -7=-2A+B$

$-7=-2(2)+B$

$-7=-4+B$

$\boxed{B=-3}$

$C \rightarrow 6=A-B+C$

$6=2+3+C$

$\boxed{C=1}$

Substitute A, B, C in eqn (1)

$L(x)=\frac{2s^2-7s+6}{(s-1)(s^2-2s+1)}=\frac{2}{s-1}-\frac{3}{(s-1)^2}+\frac{1}{(s-1)^3}$

$x=L^{-1}\left[\frac{1}{s-1}\right]-3L^{-1}\left[\frac{1}{(s-1)^2}\right]+L^{-1}\left[\frac{1}{(s-1)^3}\right]$

$=2e^t-3e^t t+\frac{e^t t^{3-1}}{(3-1)!}$

$=2e^t-3te^t+\frac{t^2e^t}{2}$

$=2e^t-3te^t+\frac{1}{2}t^2e^t.$

(iii) $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=e^{-t}\sin t$ given that $y(0)=0$ and $y'(0)=1$

Sol: Given D.E is $y''+2y'+5y=e^{-t}\sin t$; $y(0)=0$ & $y'(0)=1$

$y''+2y'+5y=e^{-t}\sin t$

$L[y'']+2L[y']+5L[y]=L[e^{-t}\sin t]$

$[s^2\bar{y}(s)-sy(0)-y'(0)]+2L[s\bar{y}(s)-y(0)+5L[y]=$

$\frac{1}{(s^2+2s+1)+1}$

$L(y)[s^2+2s+5]-1=\frac{1}{s^2+2s+2}$

$L(y)[s^2+2s+5]=\frac{1}{s^2+2s+2}+1$

$L(y)[s^2+2s+5]=\frac{1+s^2+2s+2}{s^2+2s+2}$

$L(y)=\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}=\frac{As+B}{s^2+2s+2}+\frac{Cs+D}{s^2+2s+5}$ — (1)

$s^2+2s+3=(As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)$

$s^2+2s+3=A(s^3+2s^2+5s)+B(s^2+2s+5)+C(s^3+2s^2+2s)+$

$D(s^2+2s+2)$ — (2)

Equating s^3, s^2, s & constant terms

$s^3 \rightarrow 0=A+C$ — (3)

$s^2 \rightarrow 1=2A+B+2C+D$ — (4)

$s \rightarrow 2=5A+2B+2C+2D$ — (5)

$C \rightarrow 3=5B+2D$ — (6)

Solve (3) & (4)

$2(A+C)+B+D=1$

$2(0)+B+D=1$

$B+D=1$ ——— (7)

Solve (6) & (7)

$5B+2D=3$

$2B+2D=4$

$(-)\ (-)\ (-)$

$-3B=-1$

$\boxed{B=\frac{1}{3}}$

$B+D=1$

$\frac{1}{3}+D=1$

$D=1-\frac{1}{3}$

$\boxed{D=\frac{2}{3}}$

Substitute B & D in (4)

$2(A+C)+\frac{1}{3}+\frac{2}{3}=1$

$2(A+C)+\frac{3}{3}=1$

$2(A+C)+1=1$

$2(A+C)=1-1$

$2(A+C)=0$

$A+C=0$

$\boxed{A=0}$ & $\boxed{C=0}$

Substitute A, B, C, D in eqn (1)

$L(y)=\frac{1}{3}\cdot\frac{1}{s^2+2s+2}+\frac{2}{3}\cdot\frac{1}{s^2+2s+5}$

$y=\frac{1}{3}L^{-1}\left[\frac{1}{s^2+2s+2}\right]+\frac{2}{3}L^{-1}\left[\frac{1}{s^2+2s+5}\right]$

$=\frac{1}{3}e^{-t}\sin t+\frac{2}{3}e^{-t}\sin 2t$

$=\frac{e^{-t}}{3}(\sin t+2\sin 2t)$

(iv) $\frac{d^2x}{dx^2}+9x=\cos 2t$, if $x(0)=1$, $x\left(\frac{\pi}{2}\right)=-1$

Sol: Since, $x'(0)$ not given we assume $x'(0)=a$

$x''+9x=\cos 2t$; $x(0)=1$; $x'(0)=a$

$L[x'']+9L[x]=L[\cos 2t]$

$[s^2\bar{x}(s)-sx(0)-x'(0)]+9L[x]=\frac{s}{s^2+4}$

$[s^2L(x)-s-a]+9L[x]=\frac{s}{s^2+4}$

$L[x][s^2+9]-s-a=\frac{s}{s^2+4}$

$L[x][s^2+9]=\frac{s}{s^2+4}+s+a$

$L[x]=\frac{s}{(s^2+4)(s^2+9)}+\frac{s+a}{(s^2+9)}$

Consider $\frac{1}{(s^2+4)(s^2+9)}=\frac{1}{9-4}\left[\frac{1}{s^2+4}-\frac{1}{s^2+9}\right]$

$=\frac{1}{5}\left[\frac{1}{s^2+4}-\frac{1}{s^2+9}\right]$

$L[x]=\frac{s}{5}\left[\frac{1}{s^2+4}-\frac{1}{s^2+9}\right]+\frac{s}{s^2+9}+\frac{a}{s^2+9}$

$$x = \frac{1}{5} L^{-1}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + a\, L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= \frac{1}{5}\left[\cos 2t - \cos 3t\right] + \cos 3t + a\,\frac{1}{3}\sin 3t$$

$$= \frac{1}{5}\cos 2t + \cos 3t\left(\frac{-1}{5} + 1\right) + \frac{a}{3}\sin 3t$$

$$x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{a}{3}\,5\sin 3t$$

Given $x\left(\frac{\pi}{2}\right) = -1$

at $t = \frac{\pi}{2}$

$$x\left(\frac{\pi}{2}\right) = \frac{1}{5}\cos \cancel{2}\frac{\pi}{\cancel{2}} + \frac{4}{5}\cos 3\frac{\pi}{2} + \frac{a}{3}\sin\frac{3\pi}{2}$$

$$-1 = \frac{1}{5}(-1) + \frac{4}{5}(0) + \frac{a}{3}(-1)$$

$$-1 = \frac{-1}{5} - \frac{a}{3}$$

$$\frac{a}{3} = 1 - \frac{1}{5} \Rightarrow \frac{a}{3} = \frac{4}{5}$$

$$\boxed{a = \frac{12}{5}}$$

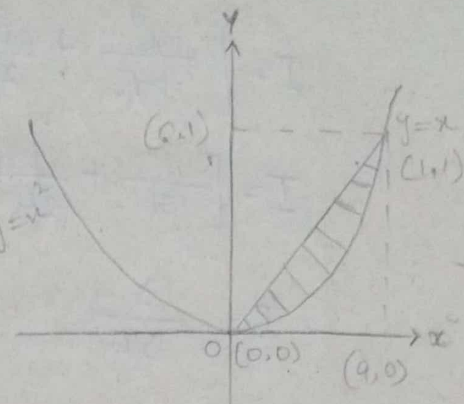
$$x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$$

$$= \frac{1}{5}\left(\cos 2t + 4\cos 3t + 4\sin 3t\right).$$

UNIT-04 Multiple Integrals & Vector Differentiation

Long Answer Questions

1. (i) Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$



Sol Now $y=x^2$ & $y=x$

$x^2=x \Rightarrow x^2-x=0$

$x(x-1)=0$

$x=0,\ x=1$

Sub in any above eqn

If $x=0;\ y=0$

$x=1;\ y=1$

Two Curves intersects at the points $(0,0), (1,1)$

$I=\iint xy(x+y)\,dx\,dy$

$x=0\longrightarrow 1$

$y=x^2\longrightarrow x$

$$=\int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\,dx$$

$$I=\int_{x=0}^{1}\left[\int_{y=x^2}^{x} xy(x+y)\,dy\right]dx$$

$$=\int_{x=0}^{1}\left[\int_{y=x^2}^{x} x^2y+xy^2\,dy\right]dx$$

$$=\int_{x=0}^{1}\left[x^2\frac{y^2}{2}+x\frac{y^3}{3}\right]_{x^2}^{x}dx$$

$$=\int_{x=0}^{1}\left[\frac{x^2(x^2)^2}{2}+\frac{x(x^2)^3}{3}-x^2\frac{x^2}{2}-x\frac{x^3}{3}\right]dx$$

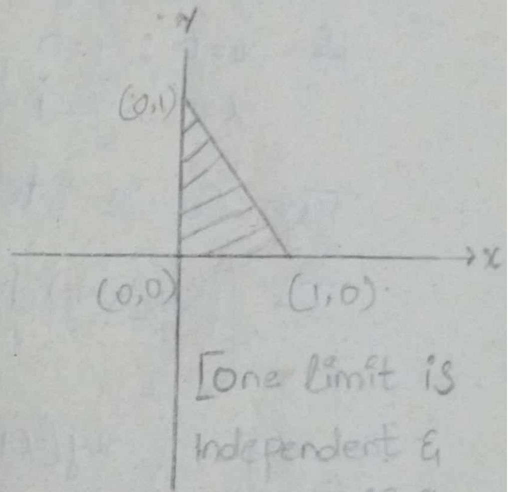
$$= \int_{x=0}^{1} \left[\frac{x^6}{2} + \frac{x^7}{3} - \frac{x^4}{2} - \frac{x^4}{3}\right] dx$$

$$I = \left[\frac{x^7}{14} + \frac{x^8}{24} - \frac{x^5}{10} - \frac{x^5}{15}\right]_0^1$$

$$I = \left[\frac{1}{14} + \frac{1}{24} - \frac{1}{10} - \frac{1}{15} - 0\right]$$

$$I = \frac{-3}{56}$$

(ii) Evaluate $\iint (x^2+y^2)\, dx dy$ in the positive quadrant for which $x+y \le 1$.



Sol $x+y=1$

put $y=0 \Rightarrow x=1$ i.e $(1,0)$

put $x=0 \Rightarrow y=1$ i.e $(0,1)$

[One limit is Independent & another limit Should be depend on remaining limit]

limits are

x	0	1
y	0	$1-x$

i.e $x: 0 \longrightarrow 1$

$x+y \le 1$

$y \le 1-x$

$$\int_{x=0}^{1}\int_{y=0}^{1-x} (x^2+y^2)\, dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] - \left[x^2(0) - \frac{(0)^3}{3}\right] dx$$

$$= \int_0^1 \left[x^2 - x^3 + \frac{1}{3}(1-x^3)\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\frac{(1-x)^4}{4}(-1)\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{12}(0)\right] - \left[0 - 0 - \frac{1}{2}\right]$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{4-3+1}{12}$$

$$= \frac{\cancel{2}^1}{\cancel{12}_6}$$

$$= \frac{1}{6}$$

(iii) Evaluate $\iint_R xy\,dx\,dy$, where R is the domain bounded by x-axis ordinate $x=2a$ and the curve $x^2 = 4ay$.

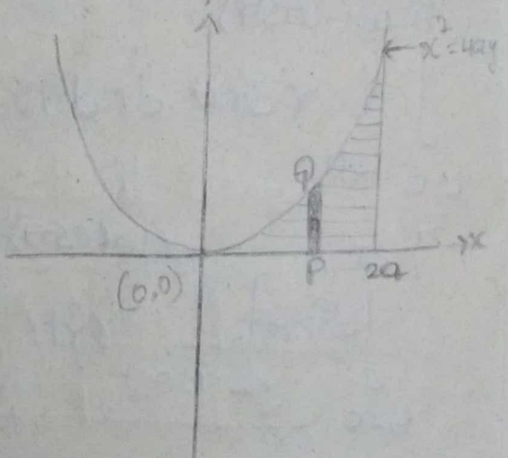
Sol

x	0	$2a$
y	0	$x^2/4a$

$x: 0 \to 2a$

$x^2 = 4ay$

$\frac{x^2}{4a} = y$



$$\int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx$$

$$= \int_0^{2a} \left[x\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \int_0^{2a} \left[\frac{1}{2} \cdot x \left(\frac{x^2}{4a}\right)^2 \right] dx$$

$$= \int_0^{2a} \left[\frac{1}{32a^2} \cdot x^5 \right] dx$$

$$= \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a}$$

$$= \frac{1}{32a^2} \cdot \frac{1}{6} \left[(2a)^6 - 0 \right]$$

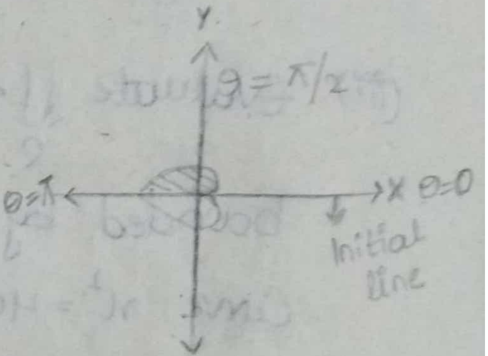
$$= \frac{1}{\cancel{32}_{1}\cancel{a^2}} \cdot \frac{\cancel{64}^{2} \cancel{a^6}^{4}}{\cancel{6}\,3}$$

$$= \frac{a^4}{3}$$

2(i) Evaluate $\iint r \sin\theta \, dr \, d\theta$ over the area Cardioid $r = a(1-\cos\theta)$ above the Initial line.

Sol

r	0	$a(1-\cos\theta)$
θ	0	π



$$\int_{\theta=0}^{\pi} \int_{r=0}^{a(1-\cos\theta)} r \sin\theta \, dr \, d\theta$$

$$\int_{\theta=0}^{\pi} \sin\theta \left[\int_{r=0}^{a(1-\cos\theta)} r \, dr \right] d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^2}{2} \right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \; a^2 (1-\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta \, (1-\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi}$$

$$= \frac{a^2}{6}\left[(1-\cos\theta)^3\right]_0^{\pi}$$

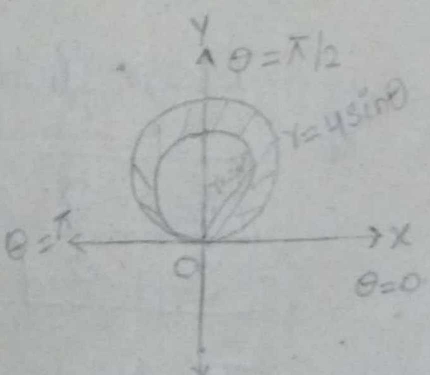
$$= \frac{a^2}{6}\left[(1-\cos\pi)^3 - (1-\cos 0)^3\right]$$

$$= \frac{a^2}{6}\left(1-(-1)^3\right) - (1-1)^3$$

$$= \frac{a^2}{\cancel{6}_3} \times \cancel{8}\,4$$

$$= \frac{4a^2}{3}$$

(ii) Evaluate $\iint r^3 dr d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.



Sol:

r	$2\sin\theta$	$4\sin\theta$
θ	0	π

$$\int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3 \, dr \, d\theta$$

$$\int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} \cdot d\theta$$

$$= \frac{1}{4}\int_0^{\pi} \left[4^4 \sin^4\theta - 2^4 \sin^4\theta\right] d\theta$$

$$= \frac{1}{4}\int_0^{\pi} \left[\sin^4\theta\,(256-16)\right] d\theta$$

$$= \frac{\overset{60}{\cancel{240}}}{\cancel{4}} \int_0^{\pi} \sin^4\theta\, d\theta$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta$$

$$= 60\times 2 \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 120\left[\frac{(4-1)(4-3)}{4(4-2)} \times \frac{\pi}{2}\right]$$

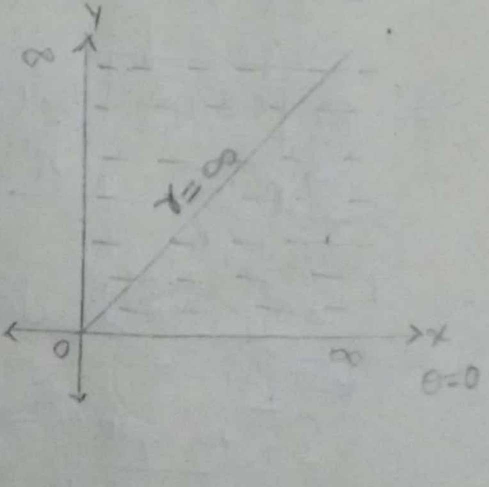
$$= \overset{60\ 15}{\cancel{120}}\left[\frac{3\times 1}{\cancel{4}\times\cancel{2}} \times \frac{\pi}{2}\right]$$

$$= \frac{45\pi}{2}$$

(iii) Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar coordinates. Hence show that $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

Sol: Given limits

x	0	∞
y	0	∞



To change the polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(1)$$

$$x^2+y^2 = r^2$$

$$dx\,dy = r\,dr\,d\theta$$

<u>New limits</u>

r	0	∞
θ	0	$\pi/2$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r\,dr\,d\theta$$

$$= \int_{r=0}^{\infty} e^{-r^2} r\,dr \int_{\theta=0}^{\pi/2} 1\,d\theta$$

$$= \int_{r=0}^{\infty} e^{-r^2} r\,[\theta]_0^{\pi/2}\,dr$$

$$= \int_0^\infty e^{-r^2} r\left(\frac{\pi}{2}-0\right)dr$$

$$= \frac{\pi}{2}\int_{r=0}^{\infty} e^{-r^2} r\,dr$$

put $r^2=t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

when $r=0$; $t=0$

$r=0$; $t=\infty$

$$= \frac{\pi}{2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}$$

$$= \frac{\pi}{4} \left[-e^{-t}\right]_0^{\infty}$$

$$= -\frac{\pi}{4}(0-1)$$

$$= \frac{\pi}{4} \quad \text{———} \ (1)$$

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\,dy = \int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy$$

$$= \left[\int_0^{\infty} e^{-x^2} dx\right]^2 \quad \text{———} \ (2)$$

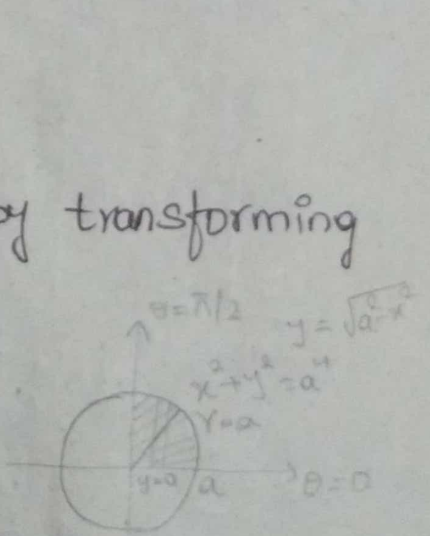
From (1) & (2)

$$\left[\int_0^{\infty} e^{-x^2} dx\right]^2 = \frac{\pi}{4}$$

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(iv) Evaluate $\int_0^{a}\int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\, dx\,dy$ by transforming into polar coordinates.

Sol Given



x	0	a
y	0	$\sqrt{a^2-x^2}$

Polar coordinate limits

r	θ	a
θ	0	π/2

$y = \sqrt{a^2 - x^2}$

S.O.B.S

$y^2 = a^2 - x^4 \Rightarrow x^2 + y^2 = a^2$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_0^a \int_0^{\pi/2} r\sin\theta \; r \cdot r\,dr\,d\theta$$

$$= \int_0^a \int_0^{\pi/2} r^2 \sin\theta \, dr\, d\theta$$

$$= \int_0^a r^3 \left[\int_0^{\pi/2} \sin\theta \, d\theta\right] dr$$

$$= \int_0^a r^3 \left[-\cos\theta\right]_0^{\pi/2} dr$$

$$= \int_0^a r^3 \left[-\cos\frac{\pi}{2} + \cos 0\right] dr$$

$$= \int_0^a r^3 [0+1]\, dr$$

$$= \left[\frac{r^4}{4}\right]_0^a.$$

$$= \frac{a^4}{4} - 0$$

$$= \frac{a^4}{4}$$

3(i) Change the order of Integration and hence evaluate

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$

Sol/

Actual limits	New limits
$x \to 0$ to $4a$	$x \to \frac{y^2}{4a}$ to $2\sqrt{ay}$
$y \to \frac{x^2}{4a}$ to $2\sqrt{ax}$	$y \to 0$ to $4a$

$0, 0, 4a, 4a$

$y = \frac{x^2}{4a}$; $y = 2\sqrt{ax}$

$x^2 = y4a$	$y^2 = 4ax$
$x = \sqrt{4ay}$	$\frac{y^2}{4a} = x$

Now

$$I = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx \cdot dy$$

$$= \int_{y=0}^{4a} \left[x\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} \cdot dy$$

$$= \int_{y=0}^{4a} \left(2\sqrt{a}\cdot\sqrt{y} - \frac{y^2}{4a}\right) dy$$

$$= \int_{y=0}^{4a} 2\sqrt{a}\cdot\sqrt{y}\,dy - \int_{y=0}^{4a} \frac{1}{4a} y^2 dy$$

$$= \int_{y=0}^{4a} 2\sqrt{a}\cdot y^{1/2}\,dy - \frac{1}{4a}\int_{y=0}^{4a} y^2\,dy$$

$$= 2\sqrt{a}\left(\frac{y^{1/2+1}}{1/2+1}\right)_{y=0}^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)_{y=0}^{4a}$$

$$= 2\sqrt{a}\left(\frac{y^{3/2}}{3/2}\right)_{0}^{4a} - \frac{1}{4a}\left(\frac{(4a)^3}{3}\right)$$

$$= 2\sqrt{a}\times\frac{2}{3}(4a)^{3/2} - \frac{1}{\cancel{4a}}\left(\frac{\overset{16}{\cancel{64}}a^{\cancel{3}}}{3}\right)$$

$$= 2\sqrt{a}\times\frac{2}{3}(2^{\cancel{2}})^{3/\cancel{2}}\cdot a^{3/2} - \frac{16a^2}{3}$$

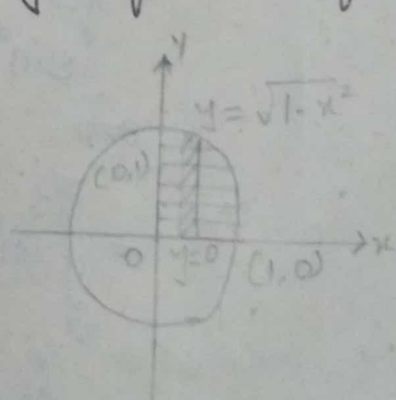
$$= \frac{2\times2\times2^3\cdot a^{3/2+\frac{1}{2}}}{3} - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{32a^2-16a^2}{3}$$

$$= \frac{16a^2}{3}$$

(ii) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$ by change of order of integration



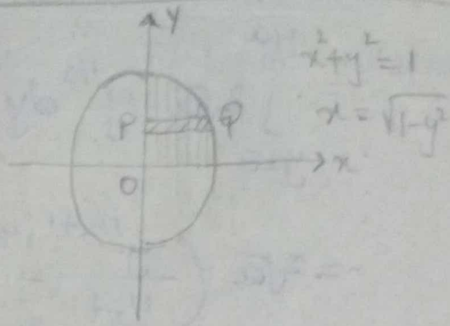
Sol $y=\sqrt{1-x^2}$

S.O.B.S

$x^2+y^2=1$ it is a circle

After changing the order of integration

New limits



x	0	$\sqrt{1-y^2}$
y	0	1

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2\,dy\,dx$$

$$= \int_{y=0}^{1} y^2\left[\int_{x=0}^{\sqrt{1-y^2}} dx\right]dy$$

$$= \int_{y=0}^{1} y^2\,[x]_0^{\sqrt{1-y^2}}\,dy$$

$$= \int_{y=0}^{1} y^2\left(\sqrt{1-y^2}\right)dy$$

To change the polar coordinates

$y=\sin\theta$

$dy=\cos\theta\,d\theta$

when $y=0 \Rightarrow 0=\sin 0 \Rightarrow \theta=0$

$y=1 \Rightarrow 1=\sin\frac{\pi}{2} \Rightarrow \theta=\frac{\pi}{2}$

$$\int_{\theta=0}^{\pi/2} \sin^2\theta\left(\sqrt{1-\sin^2\theta}\right)\cos\theta\,d\theta$$

$$= \int_{0}^{\pi/2} \sin^2\theta\,\cos^2\theta\,d\theta$$

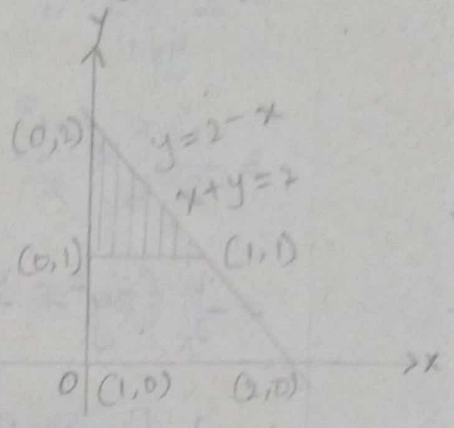
$$= \frac{(2-1)(2-1)}{(2+2)(2+2-2)} \times \frac{\pi}{2}$$

$$= \frac{1}{4(2)} \times \frac{\pi}{2}$$

$$= \frac{\pi}{16}$$

(iii) By changing the order of the Integration, evaluate

$$\int_0^1 \int_1^{2-x} xy \, dy \, dx$$

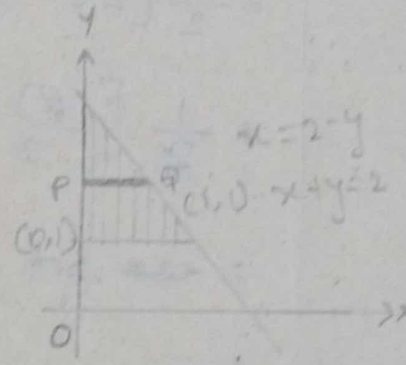


Sol

$y = 2 - x$

$x + y = 2$

x	0	1
y	1	$2-x$



New limits

x	0	$2-y$
y	1	2

$$\int_0^1 \int_1^{2-x} xy\,dy\,dx \Rightarrow \int_{y=1}^{2} \int_{x=0}^{2y} xy \, dy.dx$$

$$= \int_1^2 y \int_0^{2-y} x \, dx \, dy$$

$$= \int_1^2 y \left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_1^2 y \frac{1}{2}\left[(2-y)^2\right] dy$$

$$= \frac{1}{2} \int_{1}^{2} y(2-y)^2 \, dy$$

$$= \frac{1}{2} \int_{y=1}^{2} y(4+y^2-4y) \, dy$$

$$= \frac{1}{2} \int_{y=1}^{2} (4y+y^3-4y^2) \, dy$$

$$= \frac{1}{2} \left[\cancel{4} \frac{y^2}{\cancel{2}} + \frac{y^4}{4} - \frac{4y^3}{3} \right]_1^2$$

$$= \frac{1}{2} \left(2y^2 + \frac{y^4}{4} - \frac{4y^3}{3} \right)_1^2$$

$$= \frac{1}{2} \left[2(4-1) + \frac{1}{4}(16-1) - \frac{4}{3}(8-1) \right]$$

$$= \frac{1}{\cancel{2}} \left[(\cancel{6})_3 + \frac{1}{4}(15) - \frac{4}{3}(7) \right]$$

$$= 3 + \frac{15}{8} - \frac{\cancel{28}\,14}{\cancel{6}\,3}$$

$$\frac{24+15}{8} - \frac{14}{3}$$

$$\frac{39}{8} - \frac{14}{3}$$

$$\frac{117-112}{24}$$

$$\frac{5}{24}$$

4. Evaluate the following

(i) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz$

Sol $\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\, dy\, dx$

$= \int_{0}^{1}\int_{0}^{1-x} [1-x-y]\, dy\, dx$

$= \int_{0}^{1} \left(y - xy - \frac{y^2}{2}\right)_{0}^{1-x} dx$

$= \int_{0}^{1} \left[1-x - x(1-x) - \frac{(1-x)^2}{2}\right] dx$

$= \int_{0}^{1} \left(1-\cancel{x} - x + x^2 - \frac{1}{2} - \frac{x^2}{2} + \frac{\cancel{2}x}{\cancel{2}}\right) dx$

$= \int_{0}^{1} \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx$

$= \left[\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{3\times 2}\right]_{0}^{1}$

$= \left[\frac{\cancel{1}}{\cancel{2}} - \frac{\cancel{1}}{\cancel{2}} + \frac{1}{6}\right]$

$= \frac{1}{6}$

(ii) $\int_{0}^{1}\int_{y^2}^{1}\int_{0}^{1-x} x\, dz\, dx\, dy$

Sol $\int_{y=0}^{1}\int_{x=y^2}^{1}\int_{z=0}^{1-x} dy\, dx \cdot \cancel{dx} \cdot dz$

$\Rightarrow \int_{y=0}^{1}\int_{x=y^2}^{1} [y]_{0}^{1-x}\, x \cdot dx\, dy$

$$\int_{y=0}^{1}\int_{x=y^2}^{1} [1-x-0]\,dx\cdot x\cdot dy$$

$$\Rightarrow \int_{y=0}^{1}\left[\int_{y^2}^{1}(1-x)x\right]dx\cdot dy$$

$$\Rightarrow \int_{y=0}^{1}\left[\int_{y^2}^{1} x-x^2\right]dx\,dy$$

$$\Rightarrow \int_{y=0}^{1}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^{1} dy$$

$$\Rightarrow \int_{y=0}^{1}\left[\frac{1}{2}-\frac{1}{3}-\left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right]dy$$

$$\Rightarrow \int_{0}^{1}\left(\frac{3-2}{6}\right)-\frac{y^4}{2}+\frac{y^6}{3}\,dy$$

$$\Rightarrow \int_{0}^{1}\left(\frac{1}{6}-\frac{y^4}{2}+\frac{y^6}{3}\right)dy$$

$$\Rightarrow \left(\frac{1}{6}y-\frac{y^5}{10}+\frac{y^7}{21}\right)_0^1$$

$$\Rightarrow \frac{1}{6}-\frac{1}{10}+\frac{1}{21}$$

$$\Rightarrow \frac{4}{35}$$

(iii) $$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$$

Sol $$\int_{z=-1}^{1}\int_{x=0}^{z}\left[x(y)+\frac{y^2}{2}+z(y)\right]_{x-z}^{x+z} dx\cdot dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left[\int_{y=x-z}^{x+2}(x+y+z)\,dy\right]dx\,dz$$

$$\int_{z=1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z}dx\cdot dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left[x(x+z)+\frac{(x+z)^2}{2}+z(x+z)\right]$$

$$\left[x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right]dx\cdot dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\left[x(x+z)-(x-z)+\frac{1}{2}\left[(x+2)^2-(x-z)^2\right]+\left[z(x+z)-(x-z)\right]dx\,dz\right.$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\cancel{x}\left[\cancel{x}+z-\cancel{x}+z\right]+\frac{1}{2}\left[\cancel{x^2}+\cancel{z^2}+2xz-\cancel{x^2}-\cancel{z^2}+2xz\right]+z\left[\cancel{x}+z-\cancel{x}+z\right]dx\,dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}x(2z)+\frac{1}{\cancel{2}}\left[\cancel{4}^{2}xz\right]+z(2z)\,dx\,dz$$

$$=2\int_{z=-1}^{1}\int_{x=0}^{z}z(x+z)+xz\Big]dx\cdot dz$$

$$=2\int_{z=-1}^{1}\int_{x=0}^{z}\left[zx+z^2+xz\right]dx\,dz$$

$$=2\int_{z=-1}^{1}\left[z\frac{x^2}{2}+z^2x+\frac{x^2}{2}z\right]_0^z dz$$

$$=2\int_{z=-1}^{1}\left[\frac{z^3}{2}+z^3+\frac{z^3}{2}\right]dz$$

$$=2\left[\frac{z^4}{8}+\frac{z^4}{4}+\frac{z^4}{8}\right]_{-1}^{1}$$

$$=2\left[\left(\frac{1}{8}+\frac{1}{4}+\frac{1}{8}\right)-\left(\frac{1}{8}+\frac{1}{4}+\frac{1}{8}\right)\right]$$

$$=0$$

5(i) Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the vector $2\bar{i}-\bar{j}-\bar{k}$

Sol:

$$\nabla\phi = \left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\phi$$

$$\frac{\partial\phi}{\partial x} = 2xyz+4z^2$$

$$\frac{\partial\phi}{\partial y} = x^2z$$

$$\frac{\partial\phi}{\partial z} = x^2y+8xz$$

$$\nabla\phi = (2xyz+4z^2)i+(x^2z)j+(x^2y+8xz)k$$

$$\nabla\phi_{(1,-2,-1)} = \left[2(1)(-2)(-1)+4(-1)^2\right]i+\left[(1)^2(-1)\right]j+\left[(1)^2(-2)+8(1)(-1)\right]k$$

$$= (4+4)i+(-1)j+(-2-8)k$$

$$= 8i-j-10k$$

The unit vector of the given vector

$$\bar{e} = \frac{2i-j-k}{\sqrt{4+1+1}} \Rightarrow \frac{2i-j-k}{\sqrt{6}}$$

The $D.D = \nabla\phi \bar{e}$

$$= (8i-j-10k)\frac{(2i-j-k)}{\sqrt{6}}$$

$$= \frac{16+1+10}{\sqrt{6}}$$

$$= \frac{27}{\sqrt{6}}$$

(ii) Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1,1,1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$

Sol The given line is $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

The vector of the line is

$$a = 2i - 2j + k$$

$$\bar{a} = \frac{2i-2j+k}{\sqrt{4+4+1}}$$

$$= \frac{2i-2j+k}{\sqrt{9}} \Rightarrow \frac{2i-2j+k}{3}$$

$$\phi = 5x^2y - 5y^2z + 2.5z^2x$$

$$\nabla\phi = (10xy + 2.5z^2)i + (5x^2 - 10yz)j + (-5y^2 + 5zx)k$$

$$\nabla\phi_{(1,1,1)} = (10+2.5)i + (5-10)j + (-5+5)k$$

$$= 12.5i - 5j$$

Hence the DD along the given direction

$$= \frac{2i - 2j + k}{3} (12 - 5i - 5j)$$

$$= \frac{25 + 10}{3}$$

$$= \frac{35}{3}$$

6(i) find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ athe point $(1,-2,-1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1,2,1)$

Sol Given, $\phi(x^2yz + 4xz^2)$ $P(1,-2,-1)$

$f = x\log 2 - y^2 = -4$

$f = x\log 2 - y^2 + 4$ point $(-1,2,1)$

Here,

$\vec{a}$ = normal to the surface at $(-1,2,1)$

$= \nabla f$

$$= \bar{i}\frac{\partial f}{\partial x} + \bar{j}\cdot\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}\cdot\frac{\partial}{\partial x}(x\log 2 - y^2 + 4) + \bar{j}\frac{\partial}{\partial y}(x\log 2 - y^2 + 4) + \bar{k}\frac{\partial}{\partial z}(x\log 2 - y^2 + 4)$$

$$\vec{a} = \bar{i}(\log 2) + \bar{j}(-2y) + \bar{k}\left(\frac{x}{2}\right)$$

$$= \bar{i}(\log 1) + \bar{j}(-2(2)) + k\left(\frac{-1}{1}\right)$$

$$\vec{a} = 0 - 4\bar{j} - \bar{k} \Rightarrow -4\bar{j} - \bar{k}$$

$$|\bar{a}| = \sqrt{(-4)^2+(-1)^2}$$
$$= \sqrt{16+1}$$
$$= \sqrt{17}$$

$$D \cdot D = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$\nabla\phi = \bar{i}\,\frac{\partial}{\partial x}(x^2yz+4xz^2) + \bar{j}\,\frac{\partial}{\partial y}(x^2yz+4xz^2) + \bar{k}\,\frac{\partial}{\partial z}(x^2yz+4xz^2)$$

$$= \bar{i}(2xyz+4z^2) + \bar{j}(x^2z) + \bar{k}(x^2y+8xz)$$

$$= \bar{i}(2(1)(-2)(-1)+4(-1)^2) + \bar{j}((1)^2(-1)) + \bar{k}((1)^2(-2)+8(1)(-1))$$

$$\nabla\phi = \bar{i}(8) - \bar{j} - 10\bar{k}$$

$$D \cdot D = 8\bar{i} - \bar{j} - 10\bar{k} \cdot \frac{0-4\bar{j}-\bar{k}}{\sqrt{17}}$$

$$D \cdot D = \frac{1}{\sqrt{17}}(8(0)+(-1)(-4)+(-10)(-1))$$

$$D \cdot D = \frac{1}{\sqrt{17}}(4+10)$$

$$= \frac{14}{\sqrt{17}}$$

(ii) Find the directional derivatives of a scalar point function $\phi(x,y,z) = 4xy^2+2x^2yz$ at the point $A(1,2,3)$ in the direction of the line AB, where $B(5,0,4)$.

Sol $\phi = 4xy^2 + 2x^2yz$ —— ① $A(1,2,3), B(5,0,4)$

$\bar{a} = \bar{A \cdot B}$

$= \bar{i}(5-1) + \bar{j}(0-2) + \bar{k}(4-3)$

$\bar{a} = 4\bar{i} - 2\bar{j} + \bar{k}$

$|\bar{a}| = \sqrt{16+4+1}$

$= \sqrt{21}$

$\nabla\phi = \bar{i}\frac{\partial}{\partial x}(4xy^2 + 2x^2yz) + \bar{j}\frac{\partial}{\partial y}(4xy^2 + 2x^2yz) +$

$\bar{k}\frac{\partial}{\partial z}(4xy^2 + 2x^2yz)$

$\nabla\phi = \bar{i}(4y^2 + 4xyz) + \bar{j}(8xy + 2x^2z) +$

$\bar{k}(2x^2y)$

$\nabla\phi_{(1,2,3)} = \bar{i}(4(2)^2 + 4(1)(2)(3)) + \bar{j}(8(1)(2) + 2(1)^2(3)) +$

$\bar{k}(2(1)^2(2))$

$\nabla\phi = 40\bar{i} + 22\bar{j} + 4\bar{k}$

$D \cdot D = 40\bar{i} + 22\bar{j} + 4\bar{k} \cdot \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

$$= \frac{1}{\sqrt{21}} (160 - 44 + 4)$$

$$= \frac{120}{\sqrt{21}}$$

7 (i) Find the values of a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$

Sol $f = ax^2 - byz - (a+2)x$ —— ①

$g = 4x^2y + z^3 - 4 \quad (1,-1,2)$ —— ②

$\nabla f = [2ax - (a+2)]i + (-bz)j + (-by)k$

$= [2ax - (a+2)]i - (bz)j - (by)k$

$\nabla f_{(1,-1,2)} = (2a - (a+2)i - 2bj + bk)$

$= (2a - a - 2)i - 2bj + bk$

$= (a-2)i - 2bj + bk$ —— ③

$g = 4x^2y + z^3 - 4$

$\nabla g = (8xy)i + (4x^2)j + (3z^2)k$

$\nabla g_{(1,-1,2)} = [8(1)(-1)]i + [4(1)^2]j + [3(2)^2]k$

$= -8i + 4j + 12k$ —— ④

The given two surfaces meet at the point $P(1,-1,2)$ substitute in point ①

$f = ax^2 - byz - (a+2)x$ at $(1,-1,2)$

$\Rightarrow \cancel{a} + 2b - \cancel{a} - 2 = 0$

$\Rightarrow 2b - 2 = 0$

$\boxed{b = 1}$

Substitute 'b' value in eq^n ③

$$\nabla f_{(1,-1,2)} = (a-2)i - 2j + k \quad \text{———} \; (5)$$

The given two surface are orthogonal to each other.

$\therefore \nabla f \cdot \nabla g = 0$

$((a-2)i - 2j + k) \cdot (-8i + 4j + 12k) = 0$

$-8a + 16 - 8 + 12 = 0$

$-8a + 20 = 0$

$-8a = -20$

$a = \frac{20}{8}$

$$\boxed{a = \frac{5}{2}}$$

(ii) Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Sol $f = xy - z^2$

$\nabla f = (y)i + (x)j + (-2z)k$

$= yi + xj - 2zk$

$\nabla f_{(4,1,2)} = i + 4j - 4k \longrightarrow n_1$

$\nabla f_{(3,3,-3)} = 3i + 3j + 6k \longrightarrow n_2$

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$= \frac{(\bar{i}+4\bar{j}-4\bar{k})\,(3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{1+16+16}\;\sqrt{9+9+36}}$$

$$= \frac{3+12-24}{\sqrt{33}\;\sqrt{54}}$$

$$\cos\theta = \frac{-9}{\sqrt{33}\;\sqrt{54}}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{33}\cdot\sqrt{54}}\right)$$

$$\theta = 125.306$$

8(i) Find div $\bar{F}$ and curl $\bar{F}$ at the point $(1,-1,1)$, where $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$.

Sol $f_1 = xy^2$

$f_2 = 2x^2yz$

$f_3 = -3yz^2$

$$\text{div}\,\bar{f} = \nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla\cdot\bar{f}_{(1,-1,1)} = (-1)^2 + 2(1)^2(1) - 6(1)(1)$$

$$= 1+2+6$$

$$= 9$$

(ii) find div $\bar{F}$ and curl $\bar{F}$, where $\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol Let $\phi = x^3+y^3+z^3-3xyz$

$\bar{f} = \text{grad}\,\phi$

$= \nabla\phi$

$\cancel{= \left(i\frac{\partial}{\partial x} j \, j\frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)}$

$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$

$= i\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + j\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$

$+ k\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$

$\bar{f} = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$

$\bar{f} = 3(x^2-yz)i + 3(y^2-xz)j + 3(z^2-xy)k$

$f_1 = 3(x^2-yz)$

$f_2 = 3(y^2-xz)$

$f_3 = 3(z^2-xy)$

div $\bar{f} = \nabla\cdot\bar{f}$

$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= 3\left[\frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-xz) + \frac{\partial}{\partial z}(z^2-xy)\right]$

$= 3\,(2x+2y+2z)$

$= 6x + 6y + 6z$

$= 6(x+y+z).$

(iii) prove that $\operatorname{div}(r^n \cdot \bar{r}) = (n+3) r^n$. Hence show that $\frac{\bar{r}}{r^3}$ is solenoidal.

sol- let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{r} = |r| = \sqrt{x^2+y^2+z^2}$

$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}(\cancel{2}x) = \frac{x}{r}$

$\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\operatorname{div} \cdot \bar{f} = \nabla \cdot \bar{f}$

$= \nabla \cdot (r^n \cdot \bar{r})$

$\nabla \cdot (r^n \cdot \bar{r}) = \Sigma i \frac{\partial}{\partial x}(r^n \cdot \bar{r})$

$= \Sigma i \left[\frac{\partial}{\partial x} r^n \cdot \bar{r} + \frac{\partial \bar{r}}{\partial x} \cdot r^n\right]$

$= \Sigma i \left[n r^{n-1} \cdot \frac{\partial r}{\partial x} \cdot \bar{r} + i \cdot r^n\right]$

$= \Sigma \left[n r^{n-1} \cdot \frac{x}{r} \cdot \bar{r} + i \cdot r^n\right]$

$= \left[\Sigma i\, n r^{n-2} \cdot x \cdot \bar{r} + \Sigma i \cdot i\, r^n\right]$

$= \left[n r^{n-2}\, \Sigma x i \cdot \bar{r} + 3r^n\right]$

$= \left[n r^{n-2}(\bar{r} \cdot \bar{r}) + 3r^n\right]$

$= \left[n \cdot r^{n-2}(r^2) + 3r^n\right]$

$$= [n r^{n-2+2} + 3 r^n]$$

$$= [n r^n + 3 r^n]$$

$$\nabla (r^n \cdot \bar{r}) = (n+3) r^n$$

If $\bar{f}$ is a solenoidal

$$\text{div } \bar{f} = \nabla \cdot \bar{f} = 0$$

$$\Rightarrow (n+3) r^n = 0$$

$$n+3 = 0$$

$$\boxed{n = -3}$$

$$\nabla \left(\frac{\bar{r}}{r^3}\right) = \nabla (\bar{r}\, \bar{r}^{-3}) \qquad \left(\because \frac{1}{a^n} = a^{-n}\right)$$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$\cancel{2} r \frac{\partial r}{\partial x} = \cancel{2} x \quad \Bigg| \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \Bigg| \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= \nabla [(x\bar{i} + y\bar{j} + z\bar{k}) \bar{r}^{-3}]$$

$$= \nabla [x r^{-3} \bar{i} + y r^{-3} \bar{j} + z r^{-3} \bar{k}]$$

$$= \Sigma \left[x(-3r^{-4}) \frac{\partial r}{\partial x} + r^{-3}(1)\right] \bar{i}$$

$$= \Sigma \left[-3x\, r^{-4} \left(\frac{x}{r}\right) + r^{-3}\right] \bar{i}$$

$$= \Sigma [-3x^2 r^{-5} + r^{-3}] \bar{i}$$

$= [-3r^{-5} x^2 \bar{i} + r^{-3} \bar{i}] + [-3r^{-5} y^2 \bar{j} + r^{-3} \bar{j}] + [-3r^{-5} z^2 \bar{k} +$

$r^{-3} \bar{k}]$

$= -3r^{-5} [x^2 + y^2 + z^2] + 3r^{-3}$

$= -3r^{-5} (r^2) + 3r^{-3}$

$= -3r^{-3} + 3r^{-3}$

$= 0$

9. Show that the following vector point function are irrotational and find the scalar potential in each case:

(i) $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

Sol: $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ —— ①

Here $f_1 = x^2 - yz$

$f_2 = y^2 - zx$

$f_3 = z^2 - xy$

$$\text{curl } \bar{f} = \nabla \times \bar{f} \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$= i(-x + x) - j(-y + y) + k(-z + z)$

$= i(0) - j(0) + k(0)$

$= \Sigma i(-x + x)$

$\text{Curl} = \bar{0}$

$\therefore \bar{f}$ is irrotational then $\exists \phi \rightarrow \bar{f} = \nabla \phi$

$\bar{f} = \nabla \phi$

$(x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k = \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right)$ — (2)

Comparing the terms we get

$\frac{\partial \phi}{\partial x} = x^2 - yz \longrightarrow f_1$

$\frac{\partial \phi}{\partial y} = y^2 - zx \longrightarrow f_2$

$\frac{\partial \phi}{\partial z} = z^2 - xy \longrightarrow f_3$

$\phi = \int_{yz} f_1 \, dx + \int_{\text{not contain } x} f_2 \, dy + \int_{\text{not contain } xy} f_3 \, dz$

$= \int (x^2 - yz) dx + \int y^2 dy + \int z^2 dz + c$

$= \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c$

$= \frac{x^3 + y^3 + z^3}{3} - xyz + c$

$\therefore \phi = \frac{1}{3}(x^3 + y^3 + z^3) - xyz + c$

(ii) $\bar{F} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$

Sol $\bar{f} = (6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xz^2-y)\bar{k}$ —— ①

$f_1 = 6xy + z^3$

$f_2 = 3x^2 - z$

$f_3 = 3xz^2 - y$

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$= i(-1+1) - j(3z^2 - 3z^2) + k(6x - 6x)$

$= \bar{0}$

$\text{curl } \bar{f} = \bar{0}$

$\therefore$ $\bar{f}$ is irrotational then $\exists \phi \rightarrow \bar{f} = \nabla\phi$

$\bar{f} = \nabla\phi$

$(6xy+z^3)i + (3x^2-z)j + (3xz^2-y)\bar{k} = \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)$ —— ②

Comparing the term we get

$\frac{\partial\phi}{\partial x} = 6xy + z^3 \longrightarrow f_1$

$\frac{\partial\phi}{\partial y} = 3x^2 - z \longrightarrow f_2$

$\frac{\partial\phi}{\partial z} = 3xz^2 - y \longrightarrow f_3$

$$\phi = \underset{yz\,constant}{\int f_1\,dx} + \underset{\text{not contain } x}{\int f_2\,dy} + \underset{\text{not contain } x \,\&\, y}{\int f_3\,dz}$$

$$\phi = \int (6xy + z^3)\,dx + \int (-z)\,dy + \int 0\,dz$$

$$= {}^{3}\cancel{6}y\frac{x^2}{\cancel{2}} + xz^3 + (-yz) + c$$

$\cancel{\phi = 3x^2}$

$$\phi = 3x^2 - yz + xz^3 + c$$

10. Show that div $(\text{grad}\, r^n) = \nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: div $(\text{grad}\, r^n) = \nabla \cdot (\nabla r^n) = \nabla^2 r^n$

$$\nabla^2 r^n = \nabla \cdot (\nabla r^n)$$

Consider $\nabla r^n = \sum i \frac{\partial}{\partial x}(r^n)$

$$= \sum i\, n r^{n-1} \cdot \frac{\partial r}{\partial x}$$

$$= \sum i\, n r^{n-1} \left(\frac{x}{r}\right)$$

$$= n r^{n-2} \sum x i$$

$$\nabla r^n = (n r^{n-2}) \cdot \bar{r} \quad \text{—— (1)}$$

$$\nabla \cdot (\nabla r^n) = \sum i \frac{\partial}{\partial x}(n r^{n-2} \cdot \bar{r})$$

$$= \sum i \left[\frac{\partial}{\partial x}(n r^{n-2} \cdot \bar{r})\right]$$

$$= n \sum i \left[r^{n-2} \cdot \frac{\partial \bar{r}}{\partial x} + \frac{\partial}{\partial x}(r^{n-2}) \cdot \bar{r}\right]$$

$$= n \Sigma i \left[r^{n-2} \cdot i + (n-2) r^{n-3} \cdot \frac{\partial r}{\partial x} \cdot \bar{r} \right]$$

$$= n \Sigma i \left[r^{n-2} \cdot i + (n-2) r^{n-3} \left(\frac{x}{r} \right) \cdot \bar{r} \right]$$

$$= n \Sigma i \left[r^{n-2} \cdot i + (n-2) r^{n-4} \cdot x \cdot \bar{r} \right]$$

$$= n r^{n-2} \Sigma i \cdot i + n(n-2) r^{n-4} \Sigma x i \cdot \bar{r}$$

$$= n r^{n-2} (3) + n(n-2) r^{n-4} (\bar{r} \cdot \bar{r})$$

$$= 3n r^{n-2} + n(n-2) r^{n-4} \cdot r^2$$

$$= 3n r^{n-2} + n(n-2) r^{n-2}$$

$$= n r^{n-2} (3 + n - 2)$$

$$= n r^{n-2} (n+1)$$

$$= n(n+1) r^{n-2}.$$

UNIT-05 VECTOR Integration

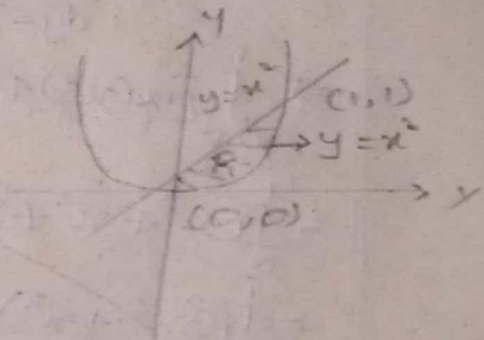
7. (i) Verify Green's theorem for $\oint_C (xy+x^2)\,dx + x^2\,dy$, where C is bounded by $y=x$ and $y=x^2$.

Sol: Given: $(xy+x^2)\,dx + x^2\,dy$

By comparing the given problem with the statement of green's theorem we have

$M = xy + x^2$ | $N = x^2$

$\frac{\partial M}{\partial y} = x$ | $\frac{\partial N}{\partial x} = 2x$



Solving $y=x^2$ & $y=x$, we get

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

when $x=0, y=0$

when $x=1, y=1$

For the region R: y varies from x^2 to x

x varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (2x - x)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} x\,dy\,dx$$

$$= \int_0^1 \left\{\int_{x^2}^{x} x\,dy\right\} dx$$

$$= \int_{x=0}^{1} \left\{x \int_{y=x^2}^{x} dy\right\} dx$$

$$= \int_{x=0}^{1} \left\{x\,(y)\right\}_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{1} \{x(x) - x(x^2)\} dx$$

$$= \int_0^1 (x^2 - x^3) dx \Rightarrow \left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 \Rightarrow \frac{1}{3} - \frac{1}{4} \Rightarrow \frac{1}{12} \quad \text{--- (1)}$$

verification:- $\oint M dx + N dy = \int_{C1} + \int_{C2} + \int_{C3}$

Along C_1 :- $y = x^2$

$dy = 2x\,dx \qquad x: 0 \to 1$

$$\Rightarrow \int_{x=0}^{1} [x(x^2) + x^2] dx + x^2(2x) dx$$

$$= \int_0^1 (x^3 + x^2 + 2x^3) dx$$

$$= \int_0^1 (3x^3 + x^2) dx \Rightarrow \left(3\frac{x^4}{4} + \frac{x^3}{3}\right)_0^1 \Rightarrow \frac{3}{4} + \frac{1}{3} = \frac{13}{12}$$

Along C_2 :- $y = x$

$dy = dx \quad ; \; x: 1 \to 0$

$$= \int_{x=1}^{0} (x^2 + x^2) dx + x^2 dx$$

$$= 3 \int_1^0 (x^2) dx$$

$$= \cancel{3} \left(\frac{x^3}{\cancel{3}}\right)_1^0$$

$$= 0 - 1$$

$$= -1$$

$\therefore \oint_C M dx + N dy = \int_{C1} + \int_{C2}$

$$= \frac{13}{12} - 1$$

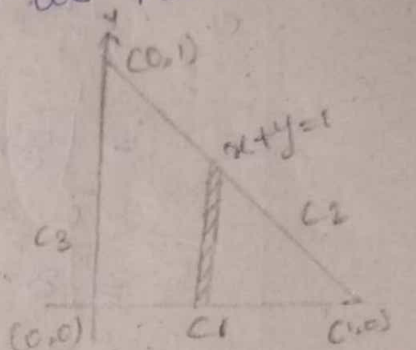
$$= \frac{1}{12}$$

Hence Green's theorem verified.

(ii) Verify Green's theorem for $\oint_C (3x-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region bounded by $x=0$, $y=0$ and $x+y=1$.

Sol Given $\oint_C (3x-8y^2) + (4y-6xy)\,dy$

By compare with Green's theorem we have



$M = 3x-8y^2$

$\frac{\partial M}{\partial y} = -16y$

$N = 4y-6xy$

$\frac{\partial N}{\partial x} = -6y$

$$\oint M\,dx \cdot N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

x	0	1
y	0	1-x

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy = \int_{x=0}^{1}\int_{y=0}^{1-x} (-6y+16y)\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} (10y)\,dy\,dx$$

$$= \int_{x=0}^{1} \cancel{10}^{5} \left[\frac{y^2}{\cancel{2}}\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1} 5\,(1-x)^2\,dx$$

$$= \left[5\,\frac{(1-x)^3}{3}(-1)\right]_0^1$$

$$= -\frac{5}{3}\left[(1-x)^3\right]_0^1$$

$$= -\frac{5}{3}(0-1) \Rightarrow \frac{5}{3} \quad \text{--- (1)}$$

Verification : $\oint_C M\,dx + N\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$

(i) Along C_1 :- $y = 0$

$dy = 0$ $\quad x : 0 \to 1$

$$\int_{C_1} = \int_0^1 [3x - 8(0)]dx + [4(0) - 6x(0)]0.$$

$$= \int_0^1 3x\,dx$$

$$= \left[3 \cdot \frac{x^2}{2}\right]_0^1$$

$$= \frac{3}{2}(1-0)$$

$$= \frac{3}{2}$$

(ii) Along C_2 = $x + y = 1$

$y = 1 - x$

$dy = -dx$

$$\int_{C_2} = \int_1^0 [3x - 8(1-x)^2]dx + [4(1-x) - 6x(1-x)(-dx)]$$

$$= \int_1^0 (3x - 8(1 + x^2 - 2x))dx + [4 - 4x - 6x + 6x^2](-dx)$$

$$= \int_1^0 (3x - 8 - 8x^2 + 16x - 4 + 4x + 6x - 6x^2)\,dx$$

$$= \int_1^0 (-14x^2 + 29x - 12)\,dx$$

$$= \left[-14\frac{x^3}{3} + 29\frac{x^2}{2} - 12x\right]_1^0$$

$$= -\frac{14}{3}(-1) + \frac{29}{2}(-1) - 12(-1)$$

$$= \frac{14}{3} - \frac{29}{2} + 12$$

$$= \frac{13}{6}$$

(iii) Along C_3 :- $x = 0$ $\quad$ ($\because$ y axis)

$dx = 0$

y varies from 1 to 0

$$\int_{C_3} = \int_1^0 0 + (4y - 0)dy$$

$$= \int_1^0 4y \cdot dy$$

$$= \left[\cancel{4}^2 \cdot \frac{y^2}{\cancel{2}}\right]_1^0$$

$$= 2(0-1)$$

$$= -2$$

$$\therefore \oint_C M\,dx + N\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= \frac{3}{2} + \frac{13}{6} - 2$$

$$= \frac{9 + 13 - 12}{6}$$

$$= \frac{10}{6}$$

$$= \frac{5}{3} \quad \text{--- (2)}$$

Hence Green's theorem is verified from the equality of (1) & (2)

(iii) Verify Green's theorem for $\oint_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$, where C is the square vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$

Sol Given $\oint_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$

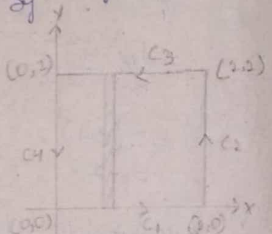
By compare with Green's theorem we have

$M = x^2-xy^3$	$N = y^2-2xy$
$\frac{\partial M}{\partial y} = -3xy^2$	$\frac{\partial N}{\partial x} = -2y$

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy$$

limits are

x	0	2
y	0	2



$$= \int_{x=0}^{2}\int_{y=0}^{2} (-2y+3xy^2)\,dy\,dx$$

$$= \int_{x=0}^{2} \left[-\cancel{2}\frac{y^2}{\cancel{2}} + \cancel{3}x\frac{y^3}{\cancel{3}}\right]_0^2 dx$$

$$= \int_{x=0}^{2} (-y^2+xy^3)_0^2\,dx$$

$$= \int_{x=0}^{2} (-4+8x)\,dx$$

$$= \left[-4x + \cancel{8}^{4}\frac{x^2}{\cancel{2}}\right]_0^2$$

$$= \left[-4x+4x^2\right]_0^2$$

$$= -8+16$$

$$= 8 \quad \text{—— (1)}$$

Verification : $\oint_C M\,dx + N\,dy = \int_{C1} + \int_{C2} + \int_{C3} + \int_{C4}$

(i) Along C_1 :- $y=0$ $(\because x\text{-axis})$

$dy = 0$

$x : 0 \to 2$

$$\int_{C1} = \int_{x=0}^{2} (x^2 - x(0)^3)\,dx + (0^2-2x(0))(0)$$

$$= \int_0^2 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{8}{3}$$

(ii) Along $x=2$

$dx = 0$

$y : 0 \to 2$

$$\int_{C2} = \int_0^2 (y^2-4y)\,dy$$

$$= \left[\frac{y^3}{3} - \frac{4y^2}{2}\right]_0^2$$

$$= \frac{8}{3} - 8$$

(iii) Along $y = 2$

$dy = 0$

$x : 2 \to 0$

$$\int_{C3} = \int_{x=2}^{0} (x^2 - 8x)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{\overset{4}{\cancel{8}} x^2}{\cancel{2}}\right]_2^0$$

$$= -\frac{8}{3} + 16$$

(iv) Along $C_4 \Rightarrow x = 0$ $(\because y\text{-axis})$

$dx = 0$

$$\int_{C4} = \int_{y=2}^{0} y^2\, dy$$

$$= \left[\frac{y^3}{3}\right]_2^0 \Rightarrow \frac{-8}{3}$$

$$\therefore \oint_C M\,dx + N\,dy = \int_{C1} + \int_{C2} + \int_{C3} + \int_{C4}$$

$$= \cancel{\frac{8}{3}} + \cancel{\frac{8}{3}} - 8 - \cancel{\frac{8}{3}} + 16 - \cancel{\frac{8}{3}}$$

$$= 8 \quad \text{—— (2)}$$

Hence Green's theorem is verified from the equality of (1) & (2)

1(i) If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$, where C is the curve $y = 2x^2$ in the xy-plane from $(0,0)$ to $(1,2)$.

Sol: $\bar{F} = 3xy\,i - y^2 j$

$\bar{a} = xi + yj + zk$

$d\bar{r} = dx\,i + dy\,j + dz\,k$

$y = 2x^2$

$dy = 4x\,dx$

Along the curve $y = 2x^2$

x limits 0 to 1

$$= \int_0^1 3xy\,i - y^2 j \cdot (dx\,i + dy\,j + dz\,k)$$

$$= \int_0^1 3xy\,dx - y^2\,dy$$

$$= \int_0^1 3x(2x^2)\,dx - (2x^2)^2\, 4x\,dx$$

$$= \int_0^1 6x^3\,dx - 16x^5\,dx$$

$$= \left[\overset{3}{\cancel{6}}\frac{x^4}{\cancel{4}_2} - \overset{8}{\cancel{16}}\frac{x^6}{\cancel{6}_3}\right]_0^1 \Rightarrow \left[\frac{3}{2}x^4 - \frac{8}{3}x^6\right]_0^1$$

$$= \cancel{\frac{6}{4}} - \cancel{\frac{16}{6}}$$

$$= \cancel{\frac{36 - 64}{24}}$$

$$= \frac{3}{2} - \frac{8}{3}$$

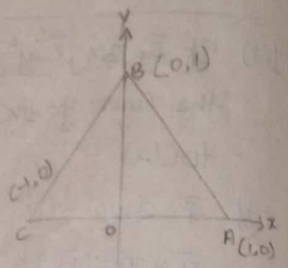
$$= \frac{9 - 16}{6}$$

$$= -\frac{7}{6}$$

(ii) Compute the line integral $\int_C y^2\,dx - x^2\,dy$ about the triangle whose vertices are $(1,0)$, $(0,1)$ & $(-1,0)$

Sol Given vertices are

A (1,0), B(0,1), C(-1,0)



$$\int_C \bar{F}\cdot\bar{dr} = \int_{AB} + \int_{BC} + \int_{CA}$$

(i) Along the line AB :-

$x+y=1$

$y=1-x$

$dy=-dx$

x limits are $1\to0$

$$\int_{AB} = \int_1^0 y^2dx - x^2dy$$

$$= \int_0^1 (1-x)^2dx - x^2(-dx)$$

$$= \int_1^0 (1-x)^2dx + x^2dx$$

$$= \left[\frac{(1-x)^3}{3}(-1) + \frac{x^3}{3}\right]_1^0$$

$$= \left[\frac{(1-0)^3}{3}(-1) + \frac{0}{3}\right] - \left[\frac{(1-1)^3}{3}(-1)\right]$$

$$= -\frac{1}{3} + 0 - \frac{1}{3}$$

$$= \frac{-2}{3}$$

(ii) Along the line BC :-

$$\frac{x}{-1} + \frac{y}{1} = 1$$

$-x+y=1$

$y=1+x$

$dy=dx$

x limits are 0 to -1

$$\int_{BC} = \int_0^{-1} y^2dx - x^2dy$$

$$= \int_0^{-1} (1+x)^2dx - x^2dx$$

$$= \left[\frac{(1+x)^3}{3}(1) - \frac{x^3}{3}\right]_0^{-1}$$

$$= \left(0+\frac{1}{3}\right) - \left(0+\frac{1}{3}\right)$$

$$= \frac{1}{3} - \frac{1}{3}$$

$$= 0$$

(iii) Along the line CA:-

$y=0$

$dy=0$

$$\int_{CA} = \int 0-0 \Rightarrow 0$$

$$\therefore \int_C \bar{F}\cdot\bar{dr} = \frac{-2}{3} + 0 + 0$$

$$= \frac{-2}{3}$$

2. If $\bar{A} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$, evaluate $\int \bar{A}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path $x=t,\ y=t^2,\ z=t^3$.

Sol Given $\bar{A} = (3x^2+6y)\bar{i} - 14yzk\bar{j} + 20xz^2\bar{k}$

$$\bar{r} = xi + yj + zk$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

Along the path

$x=t$	$y=t^2$	$z=t^3$
$dx=dt$	$dy=2tdt$	$dz=3t^2dt$

when $x=0 \Rightarrow t=0$

when $x=1 \Rightarrow t=1$

$$\int_c \bar{A}d\bar{r} = \int_c (3x^2+6y)dx - 14yz\,dy + 20xz^2dz$$

$$= \int_0^1 (3t^2+6t^2)dt - 14t^2\cdot t^3(2t)dt + 20t(t^3)$$

$3t^2dt$

$$= \int_0^1 9t^2dt - 28t^6dt + 60t^9dt$$

$$= \left[{}^{3}\cancel{9}\cdot\frac{t^3}{\cancel{3}} - {}^{4}\cancel{28}\cdot\frac{t^7}{\cancel{7}} + {}^{6}\cancel{60}\cdot\frac{t^{10}}{\cancel{10}} \right]_0^1$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= 3-4+6$$

$$= 5$$

8. Find the work done in moving a particle in the force $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along

(i) The straight line from $(0,0,0)$ to $(2,1,3)$

(ii) The curve defined by $x^2=4y,\ 3x^3=8z$ from $x=0$ to $x=2$.

Sol Given $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$

$$\bar{r} = xi + yj + zk$$

$$d\bar{r} = dxi + dyj + dzk$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_c 3x^2dx + (2xz-y)dy + zdz$$

(i) Along the straight line from $(0,0,0)$ to $(2,1,3)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t \text{ (say)}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$\frac{x}{2}=t$	$\frac{y}{1}=t$	$\frac{z}{3}=t$
$x=2t$	$y=t$	$z=3t$
$dx=2dt$	$dy=dt$	$dz=3dt$

when $y=0 \Rightarrow t=0$

$y=1 \Rightarrow t=1$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 3(2t)^2 2dt + \left[(2\cdot 2t\cdot 3t)-(t)\right]dt + (3t)3dt$$

$$= \int_0^1 24t^2 dt + (12t^2-t)dt + 9t\,dt$$

$$= \int_0^1 (36t^2+8t)\,dt$$

$$= \overset{12}{\cancel{36}}\frac{t^3}{\cancel{3}} + \overset{4}{\cancel{8}}\frac{t^2}{\cancel{2}}\Bigg]_0^1$$

$$= 12t^3 + 4t^2\Big]_0^1$$

$$\Rightarrow 12+4 = 16$$

(ii) The curve defined by $x^2=4y$ & $3x^3=8z$ from $x=0$ to 2

let $x=t$

$x^2=4y$

$t^2=4y$

$$\boxed{y=\frac{t^2}{4}}$$

$3x^3=8z$

$3t^3=8z$

$$\boxed{z=\frac{3}{8}t^3}$$

$\therefore$ t limits are 0 to 2

$x=t$	$y=t^2/4$	$z=\frac{3}{8}t^3$
$dx=dt$	$dy=\frac{\cancel{2}t}{\cancel{4}_2}dt$	$dz=\frac{3}{8}\cdot 3t^2dt$
	$dy=\frac{t}{2}dt$	$dz=\frac{9t^2}{8}dt$

$$\int_C \bar{F}\, d\bar{r} = \int_0^2 3t^2dt + \left(2t\cdot\frac{3}{8}t^3 - \frac{t^2}{4}\right)\frac{t}{2} + \frac{3}{8}t^3\cdot\frac{9t^2}{8}dt$$

$$= \int_0^2 3t^2dt + \left(\frac{6}{8}t^4 - \frac{t^2}{4}\right)\frac{t}{2}dt + \frac{27}{64}t^5dt$$

$$= \int_0^2 3t^2dt + \left(\frac{\overset{3}{\cancel{6}}t^5}{\cancel{16}_8} - \frac{t^3}{8}\right)dt + \frac{27}{64}t^5dt$$

$$= \int_0^2 3t^2 - \frac{t^3}{8} + \left(\frac{3}{8}+\frac{27}{64}\right)t^5dt$$

$$= \int_0^2 3t^2 - \frac{t^3}{8} + \left(\frac{51}{64}\right)t^5dt$$

$$= \cancel{3}\frac{t^3}{\cancel{3}} - \frac{t^4}{8\times 4} + \frac{\overset{17}{\cancel{51}}}{64}\times\frac{t^6}{\cancel{6}_2}\Bigg]_0^2$$

$$= t^3 - \frac{t^4}{32} + \frac{17}{64}\times\frac{t^6}{2}\Bigg]_0^2$$

$$= 8 - \frac{\cancel{16}^1}{\cancel{32}_2} + \frac{17}{\cancel{128}_2}\times\cancel{64}_1$$

$$= 8 - \frac{1}{2} + \frac{17}{2}$$

$$= 32/2$$

4. Find the total workdone in moving a particle in a force field given by $\bar{F} = 3xy\bar{i} - 5z\bar{j} + 10x\bar{k}$ along the curve $x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t=0$ to $t=2$.

Sol $\bar{F} = 3xyi + 5zj + 10xk$

$\bar{r} = xi + yj + zk$

$d\bar{r} = dxi + dyj + dzk$

The total work done :- $\int_c \bar{F}\cdot d\bar{r}$

$= \int_c (3xyi - 5zj + 10xk)\cdot d\bar{r}$

Along the curve :-

$x = t^2+1$	$y = 2t^2$	$z = t^3$
$dx = 2t\,dt$	$dy = 4t\,dt$	$dz = 3t^2\,dt$

$= \int_c 3(t^2+1)(2t^2)2t\,dt - 5(t^3)(4t\,dt) + 10(t^2+1)(3t^2\,dt)$

$= \int_0^2 6t(2t^4+2t^2)\,dt - 20t^4\,dt + 10(3t^4+3t^2)\,dt$

$= \int_0^2 (12t^5+12t^3)\,dt - (20t^4)\,dt + (30t^4+30t^2)\,dt$

$= \left[12\cdot\frac{t^6}{6} + 12\cdot\frac{t^4}{4} - 20\cdot\frac{t^5}{5} + 30\cdot\frac{t^5}{5} + 30\cdot\frac{t^3}{3}\right]_0^2$

$= \left[2t^6 + 3t^4 - 4t^5 + 6t^5 + 10t^3\right]_0^2$

$= [2(2)^6 + 3(2)^4 - 4(2)^5 + 6(2)^5 + 10(2)^3]$

$= \cancel{128} + 48 - \cancel{128} + 192 + 80 \Rightarrow 320$

~~6. Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$, where~~

5. Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the portion of the plane $2x+3y+6z=12$ in the first octant.

Sol Let $\phi = 2x+3y+6z-12$

grad $\phi = \nabla\phi = \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)(2x+3y+6z-12)$

$= 2i+3j+6k$

Unit normal $= \bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$= \frac{2i+3j+6k}{\sqrt{2^2+3^2+6^2}}$

$= \frac{2i+3j+6k}{\sqrt{4+9+36}}$

$= \frac{2i+3j+6k}{\sqrt{49}}$

$= \frac{2i+3j+6k}{7}$

Let R' be the projections of S on xy-plane then

$\int_c \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot k|}$

$$\iint \bar{F}\cdot\bar{n}\,dxy = \iint_R (18z\,i - 12j + 3yk)\frac{1}{7}(2i+3j+6k)\,dx\,dy$$

$$= \iint_R \frac{1}{7}(36z - 36 + 18y)\,dx\,dy$$

$$= \iint_R \frac{6}{7}(6z - 6 + 3y)\,dx\,dy$$

Now,

$$\bar{n}\cdot\bar{k} = \frac{(2i+3j+6k)}{7}\bar{k} = \frac{6}{7}$$

Given surface is $2x+3y+6z=12$

$2x+3y=12$ $\quad$ [xy-plane, $z=0$]

$3y = 12-2x$

$y = \frac{12-2x}{3}$

In first octant $x\geq 0$; $y\geq 0$

$y=0 \Rightarrow$ The limits of x varies from 0 to 6

y limits 0 to $\frac{12-2x}{3}$

$$\int \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \frac{\cancel{6}}{\cancel{7}}(6z-6+3y)\frac{\cancel{7}}{\cancel{6}}\,dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6z-6+3y)\,dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (12-2x-\cancel{3y}-6+\cancel{3y})\,dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6-2x)\,dy\,dx$$

$$= 2\int_0^6\int_0^{\frac{12-2x}{3}} (3-x)\,dy\,dx$$

$$= 2\int_0^6 (3-x)\int_0^{\frac{12-2x}{3}} dy\,dx$$

$$= 2\int_0^6 (3-x)\,[y]^{\frac{12-2x}{3}}\,dx$$

$$= 2\int_0^6 (3-x)\frac{2(6-x)}{3}\,dx$$

$$= \frac{4}{3}\int_0^6 (3-x)(6-x)\,dx$$

$$= \frac{4}{3}\int_0^6 18-9x+x^2\,dx$$

$$= \frac{4}{3}\left[18x - \frac{9x^2}{2} + \frac{x^3}{3}\right]_0^6$$

$$= \frac{4}{3}\left[18\times 6 - 9\frac{\overset{18}{\cancel{36}}}{\cancel{2}} + \frac{\overset{72}{\cancel{216}}}{\cancel{3}}\right]$$

$$= \frac{4}{3}[108 - 162 + 72]$$

$$= \frac{4}{3}(18)$$

$$= 24.$$

6. Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{F} = z\bar{i}+x\bar{j}-3y^2z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Sol. The closed surface 'S' in the curve is ABCDE

$\phi = x^2+y^2=16$

x^2+y^2-16

The unit normal to the surface is

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi+2yj}{\sqrt{4x^2+4y^2}} = \frac{2(xi+yj)}{2\sqrt{x^2+y^2}}$$

$$= \frac{xi+yj}{\sqrt{16}} \quad [\because x^2+y^2=16=\phi]$$

$$\bar{n} = \frac{xi+yj}{4}$$

let $\bar{R}$ be the projection of S on yz plane the R is the rectangle OBCD $\iint_S \bar{F}\cdot\bar{n}\,ds$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dx}{|n\cdot i|}$$

$$\bar{F}\cdot\bar{n} = (z\bar{i}+x\bar{j}+3y^2z\cdot\bar{k})\cdot\left(\frac{x\bar{i}+y\bar{j}}{4}\right) = \frac{1}{4}(xz+xy)$$

$$\bar{n}\cdot\bar{i} = \frac{1}{4}(x\bar{i}+y\bar{j})\bar{i}$$

$$= \frac{x}{4}$$

for the surface $x^2+y^2=16$ in the yz plane, $x=0$ then $y=4$

Hence in the I^{st} octant y varies from 0 to 4 & z varies from 0 to 5

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,dy\cdot dx$$

$$= \iint_R \left(\frac{xz+xy}{4}\right)\left(\frac{4}{x}\right) dy\,dx$$

$$= \iint_R \frac{x(y+z)}{x}\,dy\cdot dz$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} (y+z)\,dy\cdot dz$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} (y+z)\,dz\cdot dy$$

$$= \int_{y=0}^{4} \left[yz+\frac{z^2}{2}\right]_0^5 dy$$

$$= \int_{y=0}^{4} \left[y(5)+\frac{25}{2}\right] dy$$

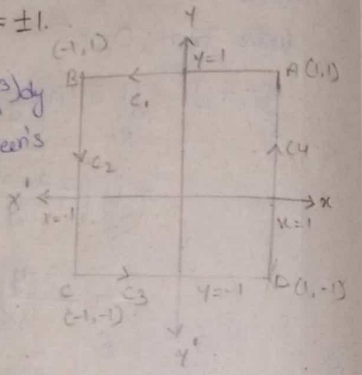
$$= \left[5\cdot\frac{y^2}{2}+\frac{25}{2}\cdot y\right]_0^4$$

$$= 5\cdot\frac{16}{2}+\frac{25}{2}\cdot 4$$

$$= 40+50$$

$$= 90$$

8(i) Apply Green's theorem to evaluate $\oint_C x^2(1+y)dx + (x^3+y^3)dy$, where C is the square formed by the lines $x=\pm1$ and $y=\pm1$.



Sol Given $\oint x^2(1+y)dx + (x^3+y^3)dy$

By compare with Green's theorem we have

$M = x^2+x^2y$	$N = x^3+y^3$
$\frac{\partial M}{\partial y} = x^2$	$\frac{\partial N}{\partial x} = 3x^2$

$$\oint_C Mdx+Ndy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

limits are

x	-1	1
y	-1	1

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{y=-1}^{1}\int_{x=-1}^{1} (3x^2-x^2)\,dx\,dy$$

$$= \int_{y=-1}^{1}\int_{x=-1}^{1} 2x^2\,dx\,dy$$

$$= \int_{y=-1}^{1} 2\left[\frac{x^3}{3}\right]_{-1}^{1} dy$$

$$= \frac{2}{3}\int_{y=-1}^{1} (x^3)_{-1}^{1}\,dy$$

$$= \frac{2}{3}(1+1)\int_{y=-1}^{1} dy$$

$$= \left[\frac{4}{3}y\right]_{-1}^{1}$$

$$= \frac{4}{3}(1-(-1)) \Rightarrow \frac{4}{3}(2) \Rightarrow \frac{8}{3} \quad \text{—(1)}$$

Verification: $\oint_C Mdx+Ndy = \int_{C1} + \int_{C2} + \int_{C3} + \int_{C4}$

(i) Along C1 :- $y=1$

$dy=0$

$x: 1$ to -1

$$\int_{C1} = \int_{1}^{-1} (x^2+x^2)\,dx$$

$$= \int_{+1}^{-1} 2x^2\,dx \Rightarrow \left[2\frac{x^3}{3}\right]_{1}^{-1} \Rightarrow \frac{2}{3}(-1-1)$$

$$= -\frac{4}{3}$$

(ii) Along C2 :- $x=-1$

$dx=0$

$y: 1$ to -1

$$\int_{C2} = \int_{1}^{-1} (-1+y^3)\,dy$$

$$= -y+\frac{y^4}{4}\Big]_{1}^{-1} \Rightarrow \left(1+\frac{1}{4}\right) - \left(-1+\frac{1}{4}\right)$$

$$= 1+\cancel{\frac{1}{4}}+1-\cancel{\frac{1}{4}}$$

$$= 2$$

(iii) Along C3 :- $y=-1$

$dy=0$

$x: -1$ to 1

$$\int_{C_3} = \int_{-1}^{1} (x^2 - x^2)\,dx = 0$$

(iv) Along C_4 :- $x = 1$

$dx = 0$

$y : -1$ to 1

$$\int_{C_4} = \int_{-1}^{1} (1+y^3)\,dy$$

$$= \left[y + \frac{y^4}{4}\right]_{-1}^{1} \Rightarrow \left(1+\frac{1}{4}\right) - \left(-1+\frac{1}{4}\right)$$

$$\Rightarrow 1 + \cancel{\frac{1}{4}} + 1 - \cancel{\frac{1}{4}}$$

$$= 2$$

$$\therefore \oint_C M\,dx + N\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

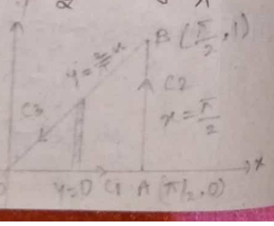
$$= \frac{-4}{3} + 2 + 0 + 2$$

$$= \frac{-4+6+6}{3} \Rightarrow \frac{-4+12}{3} \Rightarrow \frac{8}{3} \quad \text{— (2)}$$

Hence Green's theorem is verified from the equality of (1) & (2).

(ii) Using Green's theorem, evaluate $\oint_C (y - \sin x)\,dx + \cos x\,dy$, where C is the plane triangle enclosed by the lines $y=0$, $x = \frac{\pi}{2}$ and $y = \frac{2}{\pi}x$

Sol Given: $\oint_C (y - \sin x)\,dx + \cos x\,dy$



$M = y - \sin x$	$N = \cos x$
$\frac{\partial M}{\partial y} = 1$	$\frac{\partial N}{\partial x} = -\sin x$

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

x	0	$\pi/2$
y	0	$2\pi/x$

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_{x=0}^{\pi/2} \int_{y=0}^{2\pi/x} (-\sin x - 1)\,dy\,dx$$

$$= -\int_{x=0}^{\pi/2} \int_{y=0}^{2\pi/x} (\sin x + 1)\,dy\,dx$$

$$= -\int_{x=0}^{\pi/2} (\sin x + 1)\,[y]_0^{2\pi/x} \cdot dx$$

$$= -\frac{2}{\pi} \int_{x=0}^{\pi/2} x\,(\sin x + 1)\,dx \qquad \left[\because \int uv = uv_1 - u'v_2 + u''v_3 - \ldots\right]$$

$$= -\frac{2}{\pi} \left[x(-\cos x + x) - 1\left(-\sin x + \frac{x^2}{2}\right)\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left[\frac{\pi}{2}\left(0 + \frac{\pi}{2}\right) + 1 - \frac{\pi^2}{8}\right]$$

$$= -\frac{2}{\pi} \left(\frac{\pi^2}{4} + 1 - \frac{\pi^2}{8}\right)$$

$$= -\frac{2}{\pi} \left(\frac{\pi^2}{8} + 1\right)$$

$$= -\frac{\cancel{2}}{\cancel{\pi}} \times \frac{\pi^{\cancel{2}}}{\cancel{8}\,4} - \frac{2}{\pi}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi}$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right) \text{ —— (1)}$$

Verification: $\oint_C M\,dx + N\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$

(i) Along C_1 :- $y=0$

$dy=0$

$x: 0$ to $\pi/2$

$$\int_{C_1} = \int_0^{\pi/2} -\sin x \, dx$$

$$= [\cos x]_0^{\pi/2}$$

$$= 0-1$$
$$= -1$$

(ii) Along C_2 :- $x=\pi/2$

$dx=0$

$y: 0$ to 1

$$\int_{C_2} = \int_0^1 0 \Rightarrow 0$$

(iii) Along C_3 :- $y=\frac{2}{\pi}x$

$dy=\frac{2}{\pi}dx$

$x: \frac{\pi}{2}$ to 0

$$\int_{C_3} = \int_{\pi/2}^{0} \left(\frac{2}{\pi}x - \sin x\right)dx + \cos x\left(\frac{2}{\pi}dx\right)$$

$$= \left[\frac{\cancel{2}}{\pi}\cdot\frac{x^2}{\cancel{2}} + \cos x\right]_{\pi/2}^{0} + \left[\frac{2}{\pi}\sin x\right]_{\pi/2}^{0}$$

$$= (0+1) - \left(\frac{\pi^{\cancel{2}}}{4}\times\frac{1}{\cancel{\pi}} + 0\right) + \left(0 - \frac{2}{\pi}(1)\right)$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

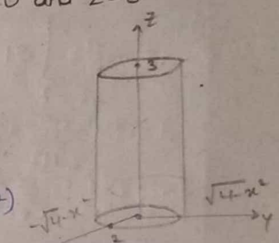
$$\therefore \oint_C M\,dx + N\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= \cancel{-1} + \cancel{1} - \frac{\pi}{4} - \frac{2}{\pi}$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right) \text{ — (2)}$$

$\therefore$ Hence Green's theorem is verified.

9(i) Use divergence theorem to evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$.



Sol $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \text{div}\,\bar{F}\,dv$

$\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$

$$\text{div}\,\bar{F} = \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z$$

$dv = dx\,dy\,dz$

x	-2	2
y	$-\sqrt{4-x^2}$	$\sqrt{4-x^2}$
z	0	3

$$\int_V \text{div}\,\bar{F}\cdot dv = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)\,dz\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + \frac{\cancel{2}z^2}{\cancel{2}}\right]_0^3 dy\,dx$$

$$= \int_{x=-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12-12y+9)\,dy\,dx$$

$$= \int_{x=-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\, dy\, dx$$

$$= \int_{x=-2}^{2} \left[21y - \cancel{12}^{6} \frac{y^2}{\cancel{2}} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2} 21 \left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) - 6\left(\cancel{4} - \cancel{x^2} - \cancel{4} + \cancel{x^2}\right) dx$$

$$= \int_{x=-2}^{2} 21 \left(2\sqrt{4-x^2}\right) dx$$

$$= 42 \int_{x=-2}^{2} \sqrt{2^2 - x^2}\, dx$$

$$\because \left[\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= 42 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2} \sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} + c$$

$$= 42 \left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right]$$

$$= 42 \left[2\sin^{-1}(1) + 2\sin^{-1}(1)\right]$$

$$= 42 \left[\cancel{2} \cdot \frac{\pi}{\cancel{2}} + \cancel{2} \frac{\pi}{\cancel{2}}\right]$$

$$= 42 (2\pi)$$

$$= 84\pi.$$

(ii) Verify divergence theorem for $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ taken over the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

Sol $\int_S \bar{F} \cdot \bar{n}\, ds = \int_V \text{div}\, \bar{F} \cdot dv$

$\text{div}\, \bar{F} = 2x + 2y + 2z$

$$\int_V \text{div}\, \bar{F} \cdot dv = \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} 2(x+y+z)\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} 2\left(xz + yz + \frac{z^2}{2}\right)_0^c dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} 2\left(xc + yc + \frac{c^2}{2}\right) dy\, dx$$

$$= 2\int_{x=0}^{a} \left[x c y + \frac{y^2}{2} c + \frac{c^2}{2} y\right]_0^b dx$$

$$= 2\int_{x=0}^{a} \left[x cb + \frac{b^2}{2} c + \frac{bc^2}{2}\right] dx$$

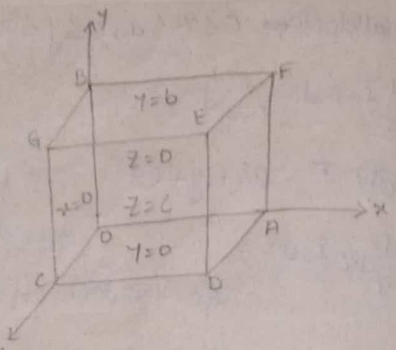
$$= 2\left[\frac{x^2}{2} bc + \frac{x b^2 c}{2} + x \frac{bc^2}{2}\right]_0^a$$

$$= 2\left[\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2}\right]$$

$$= \cancel{2}\left[\frac{a^2bc + ab^2c + abc^2}{\cancel{2}}\right]$$

$$= abc(a+b+c).$$

Verification :



$$\int_S \bar{F}\cdot\bar{n}\,ds$$

$$\bar{F} = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$$

face (i) OBGC :-

$x=0 \quad \bar{n} = (-i) \quad ds = dy\,dz$

$$\bar{F}\cdot\bar{n} = \left[(x^2-yz)i + (y^2-zx)j + (z^2-xy)k\right](-i)$$

$$= -(x^2-yz)$$

$$= yz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} yz\,dy\,dz$$

$$= \int_{z=0}^{c} z\,dz \cdot \int_{y=0}^{b} y\,dy$$

$$= \frac{z^2}{2}\Big]_0^c \cdot \frac{y^2}{2}\Big]_0^b$$

$$= \frac{c^2}{2}\cdot\frac{b^2}{2}$$

$$= \frac{b^2c^2}{4} \quad \text{--- (1)}$$

face (ii) :- ABDEF

$x=a, \ \bar{n}=i, \ d\bar{s} = \dfrac{dy\cdot dz}{|\bar{n}\cdot i|}$

$$\bar{F}\cdot\bar{n} = \left[(x^2-yz)i + (y^2-zx)j + (z^2-xy)k\right](i)$$

$$= x^2 - yz$$

$$= a^2 - yz$$

$$\int_{z=0}^{c}\int_{y=0}^{b} (a^2-yz)\,dy\cdot dz \Rightarrow \int_{z=0}^{c}\left[a^2y - \frac{zy^2}{2}\right]_0^b dz$$

$$= \int_{z=0}^{c}\left[a^2b - z\frac{b^2}{2}\right]dz \Rightarrow \left[a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= \left[a^2bc - \frac{b^2}{2}\frac{c^2}{2}\right] \Rightarrow a^2bc - \frac{b^2c^2}{4} \quad \text{--- (2)}$$

face (iii) :- GBFE

$y=b \ ; \ \bar{n}=j, \ ds = \dfrac{dx\cdot dz}{|\bar{n}\cdot j|}$

$$\bar{F}\cdot\bar{n} = \left[(x^2-yz)i + (y^2-zx)j + (z^2-xy)k\right](j)$$

$$= y^2 - zx$$

$$\Rightarrow b^2 = zx$$

$$\int_{x=0}^{a}\int_{z=0}^{c} (b^2-zx)\,dz\cdot dx \Rightarrow \int_{x=0}^{a}\left[b^2z - x\frac{z^2}{2}\right]_0^c dx$$

$$= \int_{x=0}^{a} \left[b^2c - x\frac{c^2}{2}\right] dx$$

$$= \left[b^2cx - \frac{c^2}{2}\frac{x^2}{2}\right]_0^a$$

$$= \left[b^2ca - \frac{c^2}{2}\frac{a^2}{2}\right]$$

$$= ab^2c - \frac{a^2c^2}{4} \quad \text{--- } 3$$

Face (iv) :- OADC

$y=0$; $\bar{n} = -j$, $ds = \frac{dx \cdot dz}{|\bar{n} \cdot -j|}$

$$\bar{F} \cdot \bar{n} = \left[(x^2-yz)i + (y^2-zx)j + (z^2-xy)k\right](-j)$$

$$= -(y^2 - zx)$$

$$= -(-zx)$$

$$\int_{x=0}^{a}\int_{z=0}^{c} (zx)\, dz\, dx = \int_0^a \left[x\frac{z^2}{2}\right]_0^c dx$$

$$= \int_{x=0}^{a} \left[x\frac{c^2}{2}\right] dx = \left[\frac{c^2}{2}\cdot\frac{x^2}{2}\right]_0^a$$

$$= \left[\frac{c^2}{2}\frac{a^2}{2}\right]$$

$$= \frac{a^2c^2}{4} \quad \text{--- } 4$$

Face (v) :- GEDC

$z = c$, $\bar{n} = k$, $ds = \frac{dx \cdot dy}{|\bar{n} \cdot k|}$

$$= \bar{F} \cdot \bar{n} = \left[(x^2-yx)i + (y^2-zx)j + (z^2-xy)k\right](k)$$

$$= z^2 - xy$$

$$= c^2 - xy$$

$$\int_{x=0}^{a}\int_{y=0}^{b} (c^2-xy)\, dy \cdot dx \Rightarrow \int_{x=0}^{a} \left(c^2y - \frac{xy^2}{2}\right)_0^b dx$$

$$= \int_{x=0}^{a} \left(c^2b - x\frac{b^2}{2}\right) dx$$

$$= \left[c^2bx - \frac{b^2}{2}\frac{x^2}{2}\right]_0^a$$

$$= \left[c^2ba - \frac{b^2}{2}\frac{a^2}{2}\right]$$

$$= \frac{abc^2}{1} - \frac{a^2b^2}{4} \quad \text{--- } 5$$

Face (vi) : OBAF

$z = 0$, $\bar{n} = -k$, $ds = \frac{dx \cdot dy}{|\bar{n} \cdot -k|}$

$$\bar{F} = \left[(x^2-yz)i + (y^2-zx)j + (z^2-xy)k\right](-k)$$

$$\int_{x=0}^{a}\int_{y=0}^{b} (xy)\, dy \cdot dx \Rightarrow \int_{x=0}^{a} \left[x\frac{y^2}{2}\right]_0^b dx$$

$$= \int_{x=0}^{a} \left[x\frac{b^2}{2}\right] dx \Rightarrow \left[\frac{b^2}{2}\cdot\frac{x^2}{2}\right]_0^a$$

$$= \frac{b^2}{2}\frac{a^2}{2} \Rightarrow \frac{a^2b^2}{4} \quad \text{--- } 6$$

Add ①, ②, ③, ④, ⑤ & ⑥

$$\frac{\cancel{b^2c^2}}{\cancel{4}} + a^2bc - \frac{\cancel{b^2c^2}}{\cancel{4}} + ab^2c - \frac{\cancel{a^2c^2}}{\cancel{4}} + \frac{\cancel{a^2c^2}}{\cancel{4}} + abc^2 - \frac{\cancel{a^2b^2}}{\cancel{4}} + \frac{\cancel{a^2b^2}}{\cancel{4}}$$

$= a^2bc + ab^2c + abc^2$

$= abc(a+b+c)$

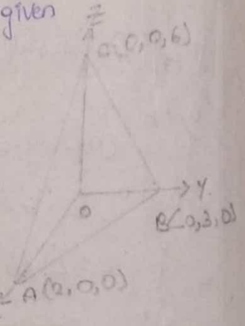
Hence divergency theorem is proved.

10. (i) Apply stoke's theorem to evaluate $\oint_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$, where C is the boundary of the triangle with vertices (2,0,0) (0,3,0) and (0,0,6)

Sol: stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\cdot ds$

Given $\oint_C \bar{F}\cdot d\bar{r} = \oint_C (x+y)dx + (2x-z)dy + (y+z)dz$

Eqn of Triangular surface is given by $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$



$3x + 2y + z = 6 = \phi$ let

Vector normal to this plane is

$\nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(3x+2y+z-6)$

$= 3i + 2j + k$

$\therefore \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3i+2j+k}{\sqrt{9+4+1}} = \frac{3i+2j+k}{\sqrt{14}}$

$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds \quad \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\cdot\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2xz & y+z \end{vmatrix}$$

$= i(1+1) - j(0-0) + k(2-1)$

$= 2\bar{i} + \bar{k}$

$\text{curl}\,\bar{F}\cdot\bar{n} = (2i+k)\cdot\frac{(3i+2j+k)}{\sqrt{14}} = \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$

$|\bar{n}\cdot\bar{k}| = \left(\frac{3\bar{i}+2\bar{j}+\bar{k}}{\sqrt{14}}\right)\cdot\bar{k} = \frac{1}{\sqrt{14}}$

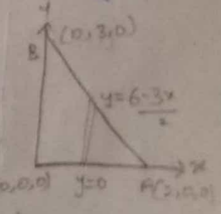
Consider R.H.S $\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \iint_S \frac{7}{\sqrt{14}}\cdot\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

Triangular surface is $3x+2y+z=6$

The projection 'S' on xy plane then $3x+2y=6$ $(\because z=0)$

$y = \frac{6-3x}{2}$. So y limits are 0 to $\frac{6-3x}{2}$

x : 0 to 2



x	0	2
y	0	6-3x/2

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} \frac{7}{\cancel{\sqrt{14}}}\cdot\cancel{\sqrt{14}}\,dy\,dx$$

$$= \int_{x=0}^{2} 7y\Big]_{0}^{\frac{6-3x}{2}} . dx$$

$$= \int_{x=0}^{2} 7\left(\frac{6-3x}{2}\right) dx$$

$$= \frac{7}{2}\int_{0}^{2} (6-3x)\, dx$$

$$= \frac{7}{2}\left(0-\frac{3x^2}{2}\right)_0^2$$

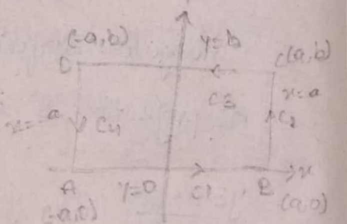
$$= \frac{21}{4}\left(x^2\right)_0^2$$

$$= \frac{21}{\cancel{4}} \times \cancel{4}$$

$$= 21$$

(ii) Verify stoke's theorem for $\bar{F} = (x^2+y^2)\bar{i} - 2xyj$ taken around the reltangle bounded by the lines $x = \pm a$, $y=0$ and $y=b$.

Sol Stoke's theorem



$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds$$

Given $\bar{F} = (x^2+y^2)i - 2xyj$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + \bar{k}(2y-2y)$$

$$= -4y\hat{k}.$$

Since the rectangle lies in xy-plane

$\therefore \bar{n} = \bar{k}$ [i.e Unit normal $\bar{n} = \bar{k}$]

$ds = dx\, dy$

Limits are

x	$-a$	a
y	0	b

$$\text{curl}\,\bar{F}\cdot\bar{n} = (-4y\bar{k})\cdot\bar{k}$$

$$= -4y$$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds = \int_{x=-a}^{a}\int_{y=0}^{y} -4y\, \frac{dx\cdot dy}{|\bar{k}\cdot\bar{k}|}$$

$$= \int_{x=-a}^{a} -4\left[\int_{y=0}^{b} y\, dy\right] dx$$

$$= \int_{-a}^{a} -\overset{2}{\cancel{4}}\, \frac{y^2}{\cancel{2}}\Big]_0^b dx$$

$$= \int_{-a}^{a} -2b^2\, dx$$

$$= -2b^2 x\Big]_{-a}^{a}$$

$$= -2b^2(a+a)$$

$$= -4ab^2 \text{ --- (1)}$$

Verification : $\oint_C \bar{F}\cdot d\bar{r} = \int_{C1} + \int_{C2} + \int_{C3} + \int_{C4}$

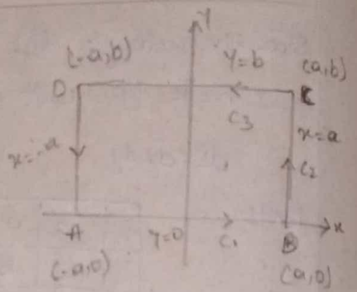
$$\bar{F}\cdot d\bar{r} = ((x^2+y^2)i - 2xyj)\cdot(dxi + dyj + dzk)$$

$$\bar{F}\cdot d\bar{r} = (x^2+y^2)\,dx - (2xy)\,dy$$

(i) Along C_1 : $y=0$

$dy=0$

$x: -a$ to a



$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{x=-a}^{a} x^2\,dx$$

$$= \frac{x^3}{3}\Big]_{-a}^{a} \Rightarrow \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3}$$

(ii) Along C_2 : $x=a$

$dx=0$

$y: 0$ to b

$$\int_{C_2} = \int_{y=0}^{b} -2ay\cdot dy$$

$$= -2a\int_{y=0}^{b} y\cdot dy$$

$$= -\cancel{2}a\cdot\frac{y^2}{\cancel{2}}\Big]_0^b$$

$$= -ab^2$$

(iii) Along C_3 :- $y=b$

$dy=0$

$x: a$ to $-a$

$$\int_{C_3} = \int_{x=a}^{-a} (x^2+b^2)\,dx = \frac{x^3}{3} + b^2x\Big]_a^{-a}$$

$$= \left(-\frac{a^3}{3} - ab^2\right) - \left(\frac{a^3}{3} + ab^2\right)$$

$$= \frac{-2a^3}{3} - 2ab^2$$

(iv) Along C_4 :- $x=-a$

$dx=0$

$y: b$ to 0

$$\int_{C_4} = \int_{y=b}^{0} -2(-a)y\cdot dy$$

$$= \int_{y=b}^{0} 2ay\cdot dy$$

$$= \cancel{2}a\frac{y^2}{\cancel{2}}\Big]_b^0$$

$$= ay^2\Big]_b^0$$

$$= 0 - ab^2$$

$$= -ab^2$$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= \frac{\cancel{2a^3}}{\cancel{3}} - ab^2 - \frac{\cancel{2a^3}}{\cancel{3}} - 2ab^2 - ab^2$$

$$= -4ab^2 \text{ ——— } (2)$$

Hence stokes theorem is verified from the equality of (1) & (2).