Chapter 8

Limulus linearis in Frequency

The Fourier transform - Frequency domain representation of continuous signals. Sine and Cosine as eigen functions of linear systems. Modulation Transfer Function (MTF) as a characterization of linear systems in the frequency domain. Fourier transformation of convolution operation on functions. Low pass filter, band pass filter, and high pass filter. Acuity. Linear systems being represented in frequency domain. Sampling theory in a nutshell. Image Blurring vs Image Sharpening revisited. The human visual system is band pass. Nonlinearities in the human visual system.

8.1 Introduction

Sometimes you want to be found – for example a mating female Limulus whom we just studied – and sometimes you don't. That is, sometimes you just want to fade into the background and disappear, especially if a predator might be around. This is the case with the cuttlefish.

8.1.1 Cuttlefish and their Environment

The cephalopod molluscs, such as the octopus and cuttlefish, exhibit remarkably creative camouflage. Since this changes with the environment, there is no doubt that they are able to extract certain properties from their visual environment to set the surface pigmentation. This is an example in which the visual goal is to match an image up to the quality level detectable by preditors. In Fig. 8.1 we show the cuttlefish in context.

The variation in behaviour is fascinating; it goes from uniform, as would be appropriate on fine sand; to mottled, for reefs and variable envorinments; and finally, to disruptive, where large scale banding mimics possible large objects or shadows (Fig. 8.2, top). To appreciate the control of these effects, we turn to the laboratory

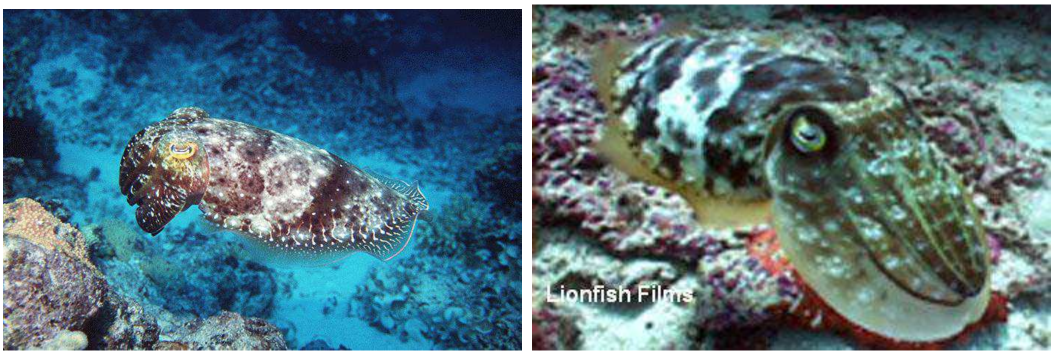

Figure 8.1: A cuttlefish in context. Note how well its skin coloration matches the reef. `http://amath.colorado.edu/faculty/martinss/diving/pics/cuttlefish.jpg`

environment. By placing the cuttlefish on a checkerboard pattern different levels of expression for the chromatophores are illustrated (Fig. 8.2, bottom).

8.1.2 Where we're going ..

In the last lecture we derived the most localized stimulus – the delta function – and showed how, for linear systems this was diagnostic. Basically, the action followed from the superposition principle and lead to the convolution integral. (Break up the input to small pieces and superimpose their results.) The impulse response function was a spatial analysis *in the small.* But a quick look at Fig. **??** suggests a very complementary viewpoint: work *in the large* by seeking some sort of spatially global input – that is, a light distribution that covers all the ommatidia. Might this reveal totally different properties of the network? We already know what happens if the same light is shined on every ommatidium; this is just a constant distribution. So instead we consider illuminating every other ommatidium, or every third one, and so on. These periodic functions will turn out to have an important role in understanding certain properties of sensors – we end with the Shannon sampling theorem – and are the subjects of this lecture. At the same time we'll introduce our first "abstract space" in which to work.

8.1.3 Experiments with Periodic Global Inputs

We start by abstracting the picket fence with the configuration shown in Fig. 8.4, of a square wave input on a slice through the compound eye, and consider the output from the balanced lateral plexus that we worked with in the last lecture. (Remember, the excitatory synapses have value +1 and the inhibitory synapses have value -0.5; and we are not worrying about negative numbers.)

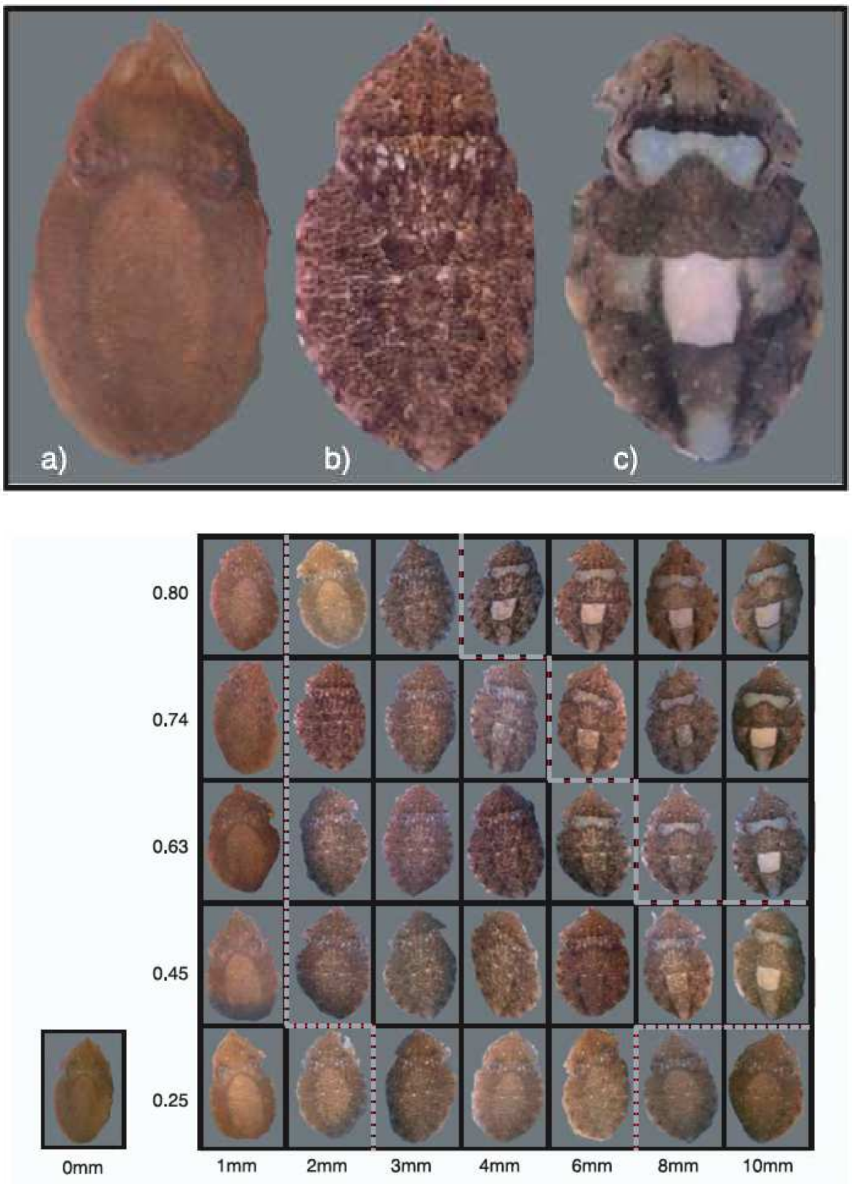

Figure 8.2: A cuttlefish in the laboratory. (Top) The three categories of coloration (from left to right): uniform, as would be appropriate on fine sand; mottle, for reefs and variable envorinments; and disruptive, where large scale banding mimics possible large objects. (Bottom) Cuttlefish response when placed on a checkerboard pattern. The size of the squares varies in the x-direction, and the contrast in the y-direction. Notice how the pattern proceeds from uniform to mottled to disruptive.

Figure 8.3: Some patterns have a natural repetition that extends across space; in an intuitive sense they complement our analysis in the last lecture. (a - b) Regular arrangements along one dimension, with some increasing complexity. (c - d) Even a bit more complexity. How complicated can patterns become before this periodicity no longer is apparent? Relevent? Applicable?

position x	...	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_{0^+}	x_1	x_2	x_3	...
input $I(x)$	...	0	10	0	10	0	10	0	10	...
output $O(x)$	...	-10	10	-10	10	-10	10	-10	10	...

This is quite encouraging, since the form remains the same and the contrast is increased (as one might expect). Now, let's take a big step closer to the picket fence, so the input is magnified by a factor of 2:

position x	...	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_{0^+}	x_1	x_2	x_3	...
input $I(x)$	...	0	10	10	0	0	10	10	0	...
output $O(x)$	...	-5	5	5	-5	-5	5	5	-5	...

This is super-encouraging; the output is again of the same form. But maybe this was too simple, so let's try separating the light pulses further to check whether this invariance will always occur:

position x	...	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_{0^+}	x_1	x_2	x_3	...
input $I(x)$	...	0	0	10	0	0	10	0	0	...
output $O(x)$	...	-5	-5	10	-5	-5	10	-5	-5	...

sort of encouraging, so let's push on:

position x	...	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_{0^+}	x_1	x_2	x_3	...
input $I(x)$	...	0	0	10	0	0	0	10	0	...
output $O(x)$	...	0	-5	10	-5	0	-5	10	-5	...

This is interesting. Somehow the output waveform has developed ripples, and these mini ripples oscillate even faster than the original pattern. These words – ripples, oscillations, repetitions – all suggest working with sinusoids (the ideal of a periodic pattern).

Replacing the square wave with a sine of the same frequency (Fig. 8.4b), we notice that the corners are not well fit. But since these are smaller, it is natural to try to fit them with a higher-frequency sinusoid. And so on.

8.1.4 Some Key Observations

1. In Fig. 8.5 we try a sinusoidal input at different frequencies to the lateral plexus and discover that (i) these inputs remain sin waves although (ii) their amplitude can vary. Moreover, this holds for a range of frequencies. Did you expect this? Don't be surprised if this is a surprise to you.

Two other thought experiments allow us to test the extremes of frequency – no surprises here.

2. If we use a very very slow sine, say 1 cycle/1000 ommatidia, then this looks essentially constant - there is hardly any change in the natural distance scale for the Limulus compound eye.

3. If we use a very fast sinusoid, say 1000 cycles/ommatidium, then this averages to 0. Why? (Think of the lens on the ommatidium as a blurring filter – what would the convolution theorem say?) See Fig. 8.6.

Are there certain frequences that really matter in characterizing our network? We need to take a different approach to approximating the input to answer this.

Size Units for Visual Stimuli

When displaying stimuli for a visual system, it is convenient to have a measure of size that is given in "retinal" units. Otherwise, distances from the observer, size of screen, etc, will all have to be known.
The typical unit scales with 1/(degrees of visual angle). 1 degree of visual angle is about the size of your thumbnail when held at arm's length. For sinusoidal inputs, then, the units would be cycles/degree. For a picket fence, it would be pickets/degree; consider how this will change as you move toward or away from the fence.

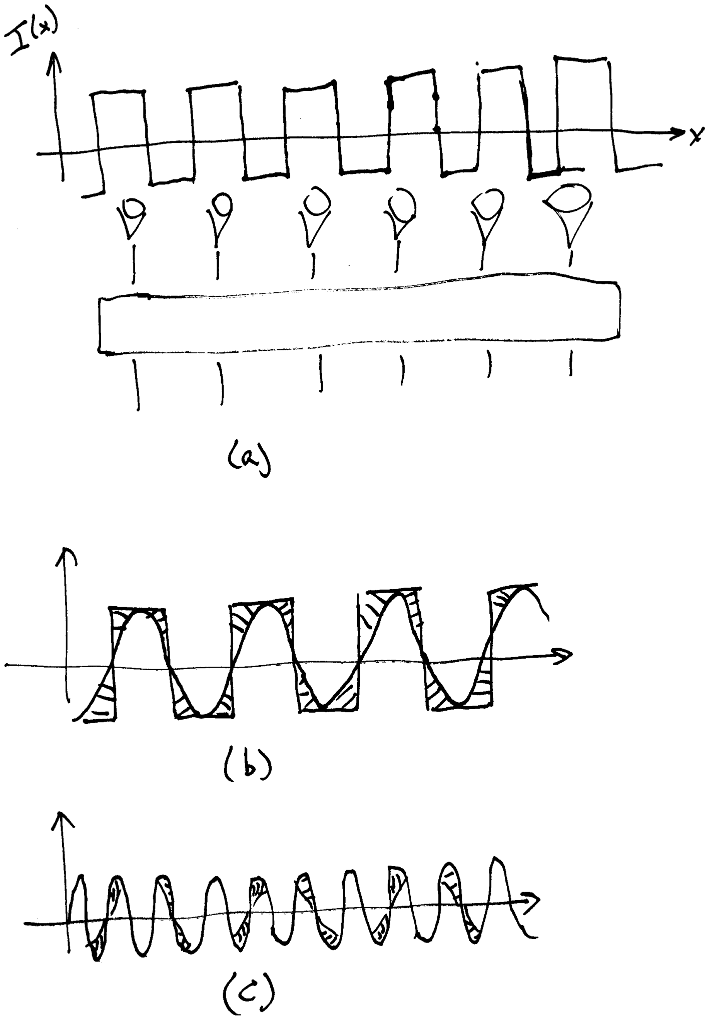

Figure 8.4: Periodic functions as inputs to the lateral plexus. (a) A square wave in which light is precisely focussed identically on every other ommatidium; output given in text. (b) Approximation of a square wave with a sinusoid of the same frequency. The match is pretty good at the centers, but not at the "corners (shown as hatched regions). (c) It is tempting to fit a higher-frequency sinusoid to the mismatched parts, but again it's not perfect; there are many (still smaller) bits that now don't match. Can these be covered if we take higher and higher frequency sinusoids?

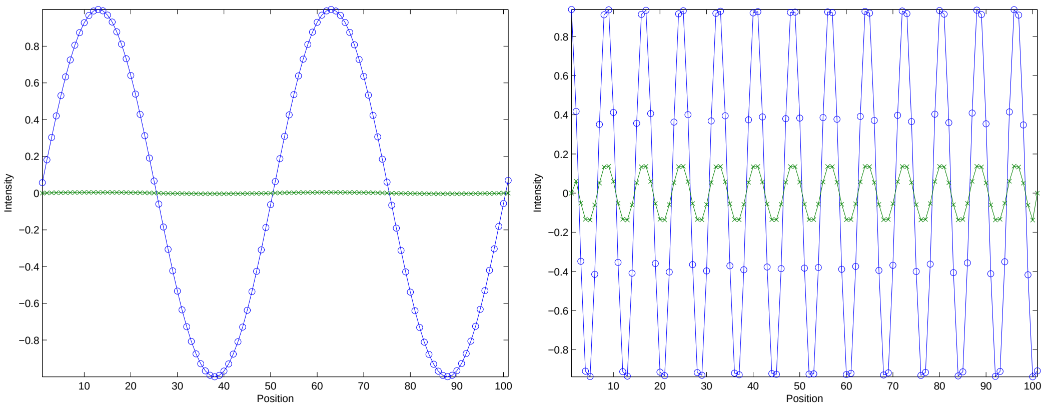

Figure 8.5: Sinusoidal inputs have a surprising property when acted upon by the linear lateral plexus: they remain sinusoids at the output. Notice the units: cycles/receptors; Blue curve: input; Green curve: output.

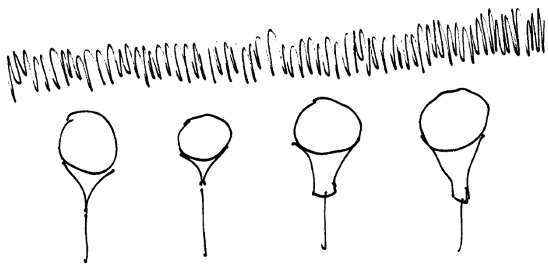

Figure 8.6: When the input is changing very rapidly with respect to a single ommatidium, it will average away and only the mean value will be reported. If this input is a sinusoid, it will average to 0.

8.2 Fourier, Frequency, and his Series

Our experiments with periodic stimuli suggest that it might be possible to construct an approximating series for our input light simulus in terms of sinusoids. Note that this is somewhat different from the manner in which we characterized the input as a sum over a staggered sequence of step functions in the previous lecture. That sequence developed over time or position (the x axis); this one would develop over frequency. And frequency-based notions are quite common; just think of the position of the sun during the day, or over a year, or a millenium ...

Following this motivation for a minute, think about the distribution of temperature at some position on the ground at different times of day. While this might be a complicated function, it will depend on where the sun is (time of day) and what season it is (winter vs summer). It will also depend on the material comprising the surface: whether it absorbs or reflects most of the sun's energy; and how well it stores it. It's going to be complicated. So let's simplify.

Since the motion of the sun is periodic, we expect the temperature $\Theta(x_0, t)$ at position x_0 to be a periodic function of time. (Since the specific position at which we do our analysis doesn't matter for this lecture, for now we shall supress it in our notation.)

$$\Theta(t) = \theta_1 \cos(\omega_1 t) + \theta_2 \cos(\omega_2 t) + \text{otherterms} \tag{8.1}$$

where the angular frequency $\omega_i = 2\pi/T_i$ is inversely proportional to the period, T, and $T_1 \approx 1\text{day}$ and $T_2 \approx 365\text{days}$.

This suggests that we think about whether a function of time, $T(t)$ (or of space, $I(x)$) can be represented as a sum of sinusiods. (In a few lectures we shall think about the distribution of temperature lower in the ground, so keep this example in mind.)

To start, we might just plunge in and write:

$I(x) = a_1 \sin(\omega x) + a_2 \sin(2\omega x) + \ldots + a_{365} \sin(365\omega x)$

where we have filled in the missing frequencies, or *harmonics* in the hope that they will capture the "other terms."

The mention of harmonics brings Pythagoras to mind, which appears as a slight diversion. However, it was he and his School who might be said to have introduced the study of *psychophysics*, to which we shall turn at the end of this lecture.

Pythagoras and the Music of the Spheres

Think of two strings on a musical instrument of different lengths but identical tensions. Pythagoras studied the sound when they were plucked together and observed that, when the lengths are in the ratio of two small integers, a pleasant sound is produced. 1:2, for example, is now called an octave; and 2:3 is a fifth.
The physical basis is now understood to be the vibrational modes of the strings, and the oscillatory pattern of air pressure that acts upon our ear drums. But for now it is the way in which these different oscillations combine that interests us: when they remain periodic, or close to it, the sound is quite musical and pleasant; when they combine into a very different waveform the sound can be rather unpleasant and noisy.
There are several aspects of Pythagoras' investigations that are appropriate lessons for us. First, he used observations of actual stimuli. Second, he provided an important *abstraction* of those stimuli into numerical relationships – that is, mathematical relationships – that could be further analyzed quantitatively. And finally, it was the percept with the brain–the *psycho*-–that was related to the physical variables. Together this is psycho-physics.
The mystical power behind such mathematical thinking was thought to extend to the heavens (the planets) themselves; hence the expression: "The music of the spheres."
But to bring us back to reality, it is important to realize that these relationships need only hold approximately; finding these approximations was what permitted keyboard instruments to be made *well tempered*. Pythagoras could never explain these approximations; this will require an understanding of our auditory systems far more modern than anything available to Pythagoras.

Several problems plague this straightforward approach, however. Is this program possible? At which frequency should the series be terminated? Does the series converge to our function of interest, or to anything at all? Do these terms suffice, or are others needed?

Returning to the compound eye of *Limulus*, we might convince ourselves that the program could makes sense if very high frequencies were included, because we already saw that–eventually–these had no effect. But how could the sinusoids be lined up exactly so maxima in light fell precisely at the centers of the ommatidia for all frequencies?

Clearly they cannot, so we introduce this offset with the phase angle, ϕ, or the starting position for each sinusoid: $\cos(\omega x + \phi)$. (The problem here is actually a very serious one: why would you expect all of the sinusoids to begin at the same location?) While this would seem to complicate our efforts enormously, the trig identity:

$$\sin(\omega x + \phi) = \cos\phi \sin(\omega t) + \sin\phi \cos(\omega t)$$

really helps. (Pythagoras's influence is still here, since trigonometry derives from his famous theorem about the lengths of the sides of a triangle.)

Thus we write our Fourier Series:

$$\begin{aligned} I(x) = \quad & \qquad a_0 \\ & +a_1 \cos(\omega x) \qquad +b_1 \sin(\omega x) \\ & +a_2 \cos(2\omega x) \qquad +b_2 \sin(2\omega x) \\ & +a_3 \cos(3\omega x) \qquad +b_3 \sin(3\omega x) \\ & \quad + \ldots \qquad\qquad\quad + \ldots \\ & +a_N \cos(N\omega x) \qquad +b_N \sin(N\omega x) \\ & \quad + \ldots \qquad\qquad\quad + \ldots \\ = \quad & \underbrace{a_0}_{\text{constant}} + \underbrace{\sum_i a_i \cos(i\omega x)}_{\text{left}} + \underbrace{\sum_i b_i \sin(i\omega\, x)}_{\text{right}}. \end{aligned} \tag{8.2}$$

(8.3)

The first, constant term, is necessary to make the result general although for pressure waves from musical instruments it's usually 0. For image intensity distribution, however, it is not. In either case, it's interesting to think of a constant as having 0 frequency.

So far in our development we have simply asserted that there is a series like this, and that it 'makes sense.' This last term – 'makes sense' – has several important aspects. First, we're going to need to be able to find these different coefficients; this is a task that we shall be turning to. Second, we have to say how many terms there are. One might guess that, for periodic functions of the sort we have been considering, we might need a lot but the situation should remain under control. But since the error terms may become arbitrarily tiny (recall Fig. 8.4), this may require arbitrary many terms in the series; that is, the series could (in general) be infinite. This raises the final point: whether the series converge or diverge.

Although periodic functions have been behind our development thus far, the (more general) Fourier expansion exists for most of the functions that matter in these lectures; certainly for all of the physically-realizable ones. This amounts to saying that the signals are physically realizable images or, formally, that the series (or the integrals that lie behind them) converge.[1] We shall not deal with this aspect of analysis in these lectures.

[1] T. W. Korner, Fourier Analysis, Cambridge U.P., 1988; Feynman, vol I, 50.

We now have a very different way of expanding the input as a sum, but not one that evolves over position; rather one that evolves over frequency. This builds directly on the above experiments: if each sin passes through the system only with a change in amplitude, and if we can decompose the input into a collection of sinusoids, then we can apply superposition to characterize the output.

8.2.1 Even and Odd Functions

Before going into the analysis in more detail, we briefly comment on the curious fact that the groups of sine's and cosine's organize so perfectly; one is tempted to ask whether there is something going on here. This is our first mention of the concept of SYMMETRY, a concept that is fundamental to both mathematics and to vision. It shall emerge again.

The symmetry that matters here is given by looking at sine and cosine waves with respect to the ordinate: what happens if we reflect them about this axis? Two possibilities are important:

- EVEN SYMMETRY: $f(x) = f(-x)$ (flip positive x-part around y-axis)
- ODD SYMMETRY: $-f(x) = f(-x)$ (flip around y-axis then around x-axis)

That these are not isolated facts, we note from algebra that we can decompose any function[2] into an even and an odd part:

$$f(x) = f_{\text{even}} + f_{\text{odd}} \tag{8.4}$$

$$f_{\text{even}} = \frac{1}{2}[f(x) + f(-x)] \tag{8.5}$$

$$f_{\text{odd}} = \frac{1}{2}[f(x) - f(-x)] \tag{8.6}$$

Examples of even and odd symmetric functions are shown in Fig. 8.7.

8.2.2 Fourier Coefficents

How might the coefficents a_i and b_i be found? We shall discuss several approaches, the first of which involves a trick employed by Fourier. Since we are working with time, for now we'll switch the independent variable to t.

First, we observe that, over the period of a year the variation due to the daily cycle will average away (this was the second of our Key Observations):

$$\frac{1}{T}\int_0^T cos 365\omega t dt = 0.$$

[2] real-valued function of a real variable

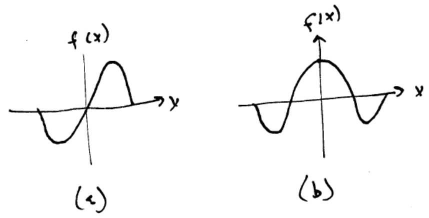

Figure 8.7: Odd (a) and even (b) functions. Note how these look like "windowed" portions of sine and cosine waves.

That is, the average of any sinusoid over many whole periods will be 0. Fourier exploited this – his trick was to multiply both side of eq. 8.3 by a cosine of high frequency and then average both sides, thus:

$$I(t) \cdot \cos(365\omega t) = \sum_i [a_i \cos(i\omega t) + b_i \sin(i\omega t)] \cos(365\omega t)$$

It looks like we've made a hard problem harder, until we recall the trig identity:

$$\cos(\alpha)\cos(\beta) = 1/2\cos(\alpha+\beta) + 1/2\cos(\alpha-\beta)$$

Now when we unpack the summation, we find terms such as (starting with the first one)

$$\frac{1}{2}a_1(\cos((365+1)\omega t) + \frac{1}{2}a_1(\cos(364\omega t)$$

both of which clearly average to 0 over T. The only a_i term that does not average to 0 for this multiplication is:

$$\frac{1}{2}a_{365}(\cos(730\omega t) + \frac{1}{2}a_{365}(\cos(0\omega t).$$

Since $\cos(0) = 1$ its average $= 1$. By similar calculations all the b_i terms average to 0. Thus

$$\frac{1}{T}\int_0^T I(t)\cos 365\omega t dt = a_{365}/2.$$

We can calculate the b terms by a multiplying both sides by $\sin 365\omega t$ and averaging from which we get the general formulas:

$$a_n = \frac{2}{T}\int_0^T I(t) \cdot \cos(n\omega t) dt \quad (8.7)$$

$$b_n = \frac{2}{T}\int_0^T I(t) \cdot \sin(n\omega t) dt \quad (8.8)$$

$$a_0 = \frac{1}{T}\int_0^T I(t) dt \quad (8.9)$$

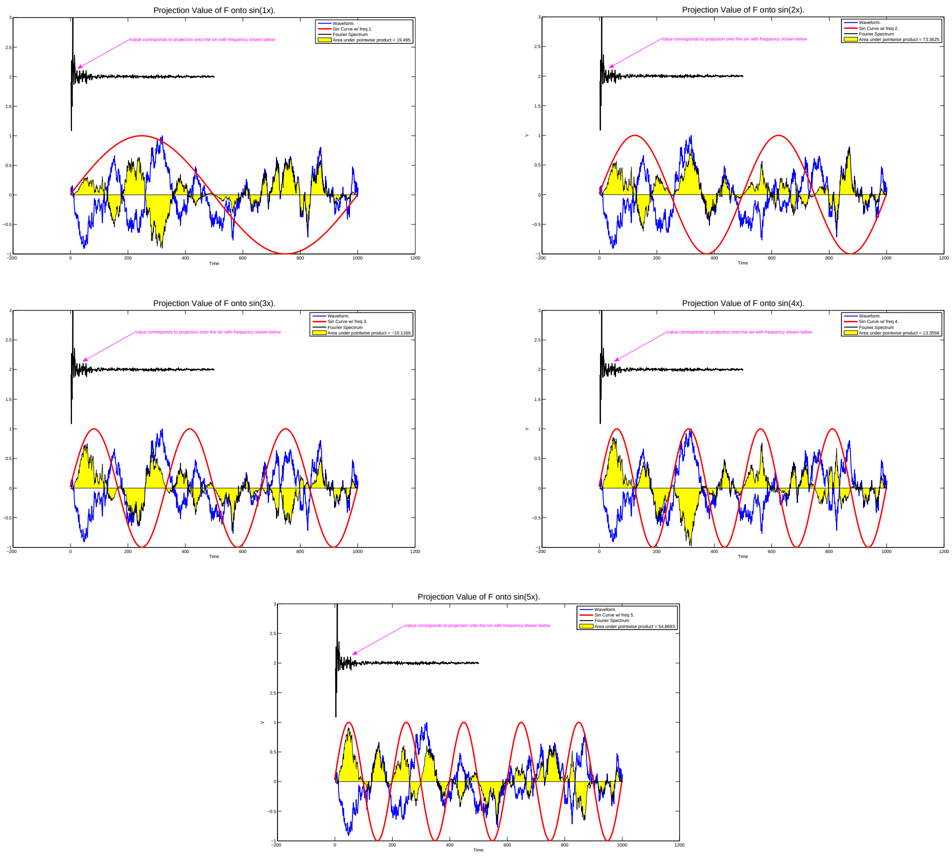

Figure 8.8: Evaluation of the Fourier coefficients. Blue waveform is (Ben) speaking "computational vision and biological perception." Red is the sin wave. Yellow is the product contribution to the integral. Arrow at top points to the coefficient in the "spectrum" or the plot of coefficient as a function of frequency.

These are the formulas for getting the Fourier coefficents.

Now, do these formulas make any sense? In Fig. 8.8 we show how a waveform "projects" onto each of the sinusoids in the series; notice how the lowest frequency captures the "fundamental" of the speech and how higher frequencies catch up the errors.

An example is shown in Fig. 8.10 for a waveform; the coefficents (power) in the waveform are shown in Fig. 8.14.

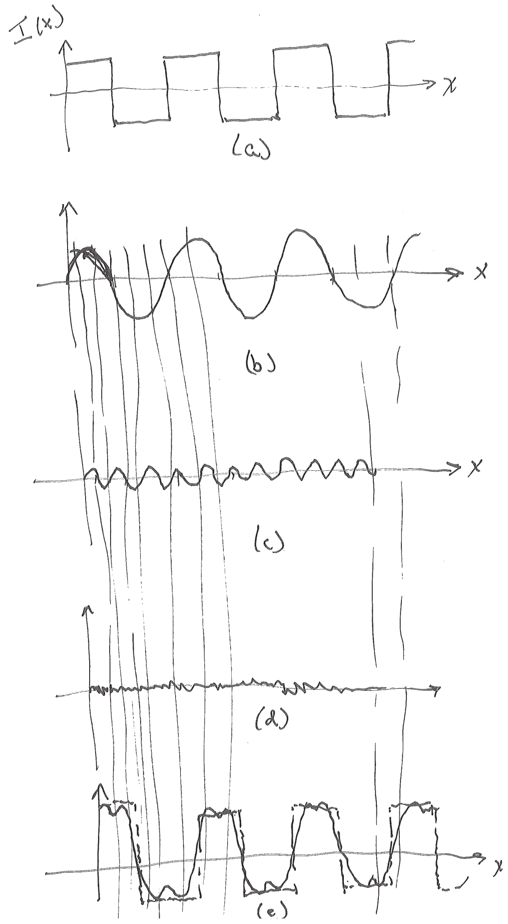

Figure 8.9: Illustration of Fourier components for a square wave and its reconstruction. (a) Original square wave (picture this as a slice through a pinstriped pattern. (b) First fundamental. Notice how the peaks match and how much of the waveform is captured. (c) Third (next significant) harmonic. Notice the difference in amplitude and frequency. (d) Fifth harmonic. (e) The Fourier reconstruction obtained by superimposing all of the above harmonics. This amounts to a pointwise addition along each of the (infinitely many) vertical lines. Notice how the result approximates the square wave, but not perfectly. Can you guess why the corners are rounded? Do you notice the ripples (oscillations)? The more such harmonics we include, the better the reconstruction. That is, the Fourier series, up to some number of terms, can be viewed as an approximation of the waveform. How many such terms should be required for the approximation to be a 'good' one? We'll see a more accurate version of this drawing in Fig. 8.17.

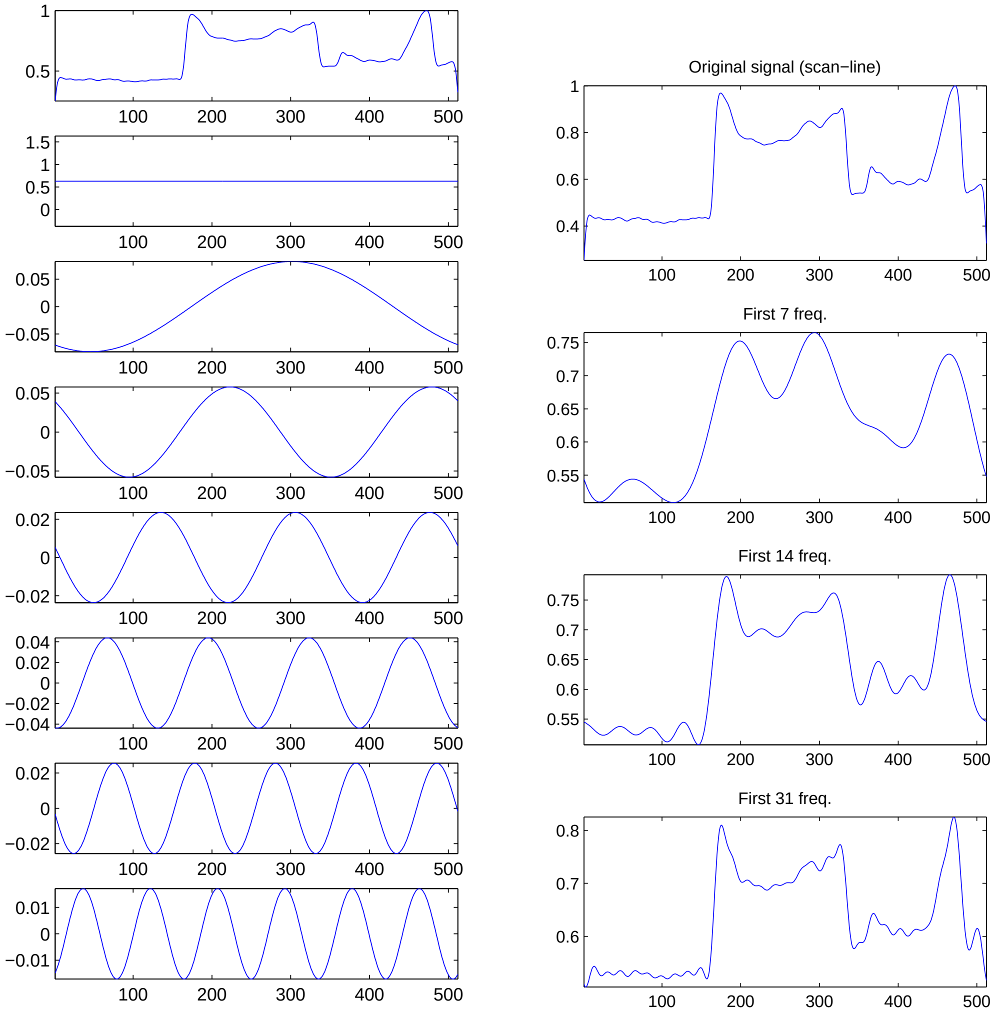

Figure 8.10: Illustration of Fourier reconstruction. (LEFT) The first 7 terms comprising a function $I(x)$, which is a portion of a slice through an image. Notice how the first one is a constant - the average intensity value. (RIGHT) The reconstruction after different numbers of harmonics. How many harmonics are necessary? Desirable? Why?

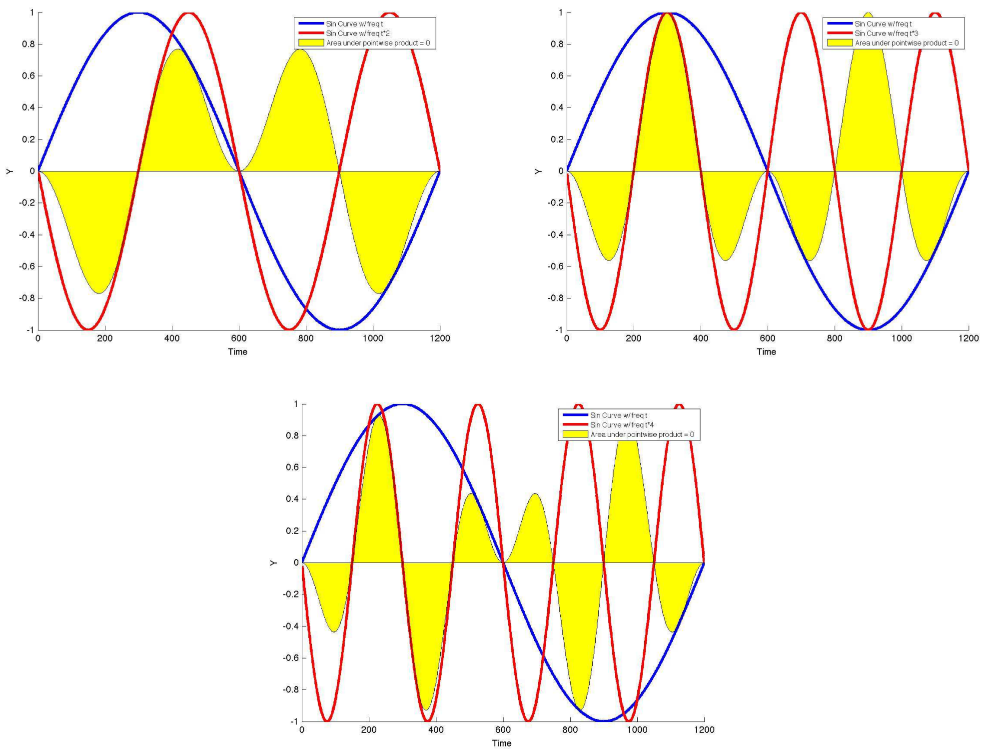

Figure 8.11: Projection of sins onto the sinusoidal basis. Sins of different frequencies are shown, with their overlap in yellow. The sum of these overlapping regions always cancels to 0, indicating that sin's of different freqencies are orthogonal. Of course, it is easy to see from these diagrams that a pair of sins of the same freqency would overlap indentically; with normalization this gives us a basis.

8.2.3 Orthonormal Families of Sinusoids

8.2.4 Toward Metric Spaces

For all of this to work, at a deep level, we needed a notion of length and of orthogonality that hold over all vectors. An important step in the development of mathematical analysis was the study of the convergence of Fourier series (question: when does the infinite series converge? To what? Is any combination of coefficients legal?)

What is a function? Does the Fourier series converge for any function?

What does a partial series mean – how well does it approximate the full series?

8.2.5 The Fourier Basis

Here is the big step. Suppose we label the axes of an abstract "space" with our trig functions; the axes would then be:

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos Nx, \sin Nx, \ldots$$

If we denote these as elements in a set, for short, as ϕ_i, they could be thought of as providing an orthonormal basis for the space provided:

- they were orthogonal: $\int_{-\pi}^{\pi} \phi_i \phi_j dx = 0 \forall i \neq j$;; see Fig. 8.11
- they were unit length: $\int_{-\pi}^{\pi} \phi_i^2 = 1 \forall i$.

The values of the above integrals gives us the necessary factors to create normalized bases:

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots$$

Note that unlike the 2D space of vectors that we started with, this space has infinite dimensions. That makes it hard to visualize directly, but not in analogy. In particular, just as we could add vectors in the plane, now we can add analagous "functions" in this more abstract space.

With an abstract notion of "bases" defined for our space, all we need to get the "coefficents" is to define the projection. (Remember, the components of a vector were obtained by projecting it onto the basis vectors. In abstract terms, $I \cdot \phi_i = (I, \phi_i) = \int I(x)\phi(x)dx$ so we can write out the Fourier expansion in exactly the same form that we developed for vectors:

$$I(x) = \sum_i (I \cdot \phi_i)\phi_i. \tag{8.10}$$

We shall have more to say about these abstract spaces in future lectures; for now just try to get a little familiar with them.

Since the sinusoids pass directly through linear systems with only a change in amplitude, these are clearly special for such systems. That they also label the "axes"

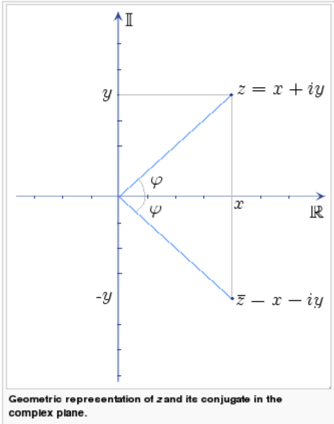

Figure 8.12: Complex numbers $z = (x + iy) = r \exp^{i\varphi}$ illustrated graphically. The magnitude r is applied at phase angle φ. An appreciation of Euler's formula: $\exp^{i\varphi} = \cos\varphi + i\sin\varphi$. can be obtained by allowing a unit r to rotate around.

in this description is also interesting. It says that, from an abstract perspective, an input that lies directly along these coordinates is special – it is called an EIGENFUNCTION of the system; the amount it is changed is an EIGENVALUE. Again, more on this in future lectures.

Exponential Version

Since the cosine and sine terms in the Fourier expansion come in pairs, there is another way to write them out; this leads to easier algebraic manipulation.
A complex number can be written $z = x + iy$, where x is the *real* part and y is the imaginary part. $i = \sqrt{-1}$. Although this looks quite formidable, the advantage starts to emerge when we write:

$$x + iy = r \exp^{i\varphi}$$

where $r^2 = x^2 + y^2$ and φ is the phase angle. See Fig. 8.12.

Our study of how *Limulus* finds a mate suggests a second view toward thinking of these abstract spaces. The operation of convolution against a "mate function" will be generalized substantially in the next lecture to more general templates. But properly interpreted, one might also think of these as bases. More on this interpretation in the next lecture.

8.2.6 Hilbert Space

In abstract mathematics huge advantages have accrued from considering the abstraction notion of a SPACE, or a set of objects and some operators defined on that set. In this lecture we've considered vectors as the objects, and also functions (that admit a Fourier expansion). We could perhaps more easily have thought about real numbers, or complex ones.

An extremely important example for us is Euclidean 3-space, objects, $\Re^3$. Elements of this space are vectors $x = (x_1, x_2, x_3) \in \Re^3$ with length (or norm):

$$||x|| = (|x_1|^2 + |x_2|^2 + |x_3|^2) = \left(\sum_{i=1}^{i=3} |x_i|^2 \right)^{\frac{1}{2}}$$

From vector algebra, we have the inner product (with another vector $y \in \Re^3$)

$$x \cdot y = (x, y) = \sum_{i=1}^{i=3} x_i y_i$$

from which we observe the relationship between the norm and the inner product:

$$(x, x) = ||x||^2.$$

For vectors in a complex space we simply need to incorporate the complex conjugate to keep this true:

$$x \cdot y = (x, y) = \sum_{i=1}^{i=3} x_i \overline{y}_i$$

A NORMED LINEAR SPACE is a linear space (such as the space of vectors) in which each vector has a corresponding NORM that is a real number, $||x||$ such that

1. $||x|| \geq 0$, and $||x|| = 0$ implies $x = 0$ and vice versa;
2. $||x + y|| \leq ||x|| + ||y||$; and
3. $||\alpha x|| = |\alpha| ||x||$.

When this space is complete, it is called a Banach space (more on what this means in a later lecture).

A HILBERT SPACE is a complex Banach space in which the inner product is fundamental and gives rise to the norm. Think of this inner product (x, y) as a complex function of the two vectors x and y such that (for another vector z) we have:

1. $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$;
2. $\overline{(x, y)}(y, x)$;
3. $(x, x) = ||x||^2$.

α and β are scalars.

8.2.7 Complex Exponentials

In the history of mathematics imaginary numbers arise in the solution to equations such as $x^2 = -1$, because $\sqrt{-1} = \imath$ is required; it is stressed that they extend the algebraic theory of equations beyond those that can be solved by integer, rational, or irriational *real numbers*. Combining real and imaginary numbers into a *complex number* $z = x + \imath y$, with both x and y real (denoted $x, y \in \Re$), they can be represented in the complex plane (Fig. 8.13).

A complex number can be located in the complex plane in Cartesian or in polar coordinates. The relationship between them is revealing. If $z = x + \imath y$, we have

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \arctan y/x \tag{8.11}$$

Or we could go the other way around;

$$x = \cos(\phi), \qquad y = \sin(\phi) \tag{8.12}$$

The modulus, (r), measures the distance from the origin and the angle ϕ measures the angular rotation. How should these be put together? Choosing $r = 1$ and looking at the axes provides insight. Think of a point on the circle, and consider its projection onto the real and the imaginary axes: they're both sinusoids and they're precisely out of phase with each other. Looking at the projections, then, we can think of the number as:

$$1 \cdot (\cos\phi + \imath \sin\phi) = e^{\imath\phi}. \tag{8.13}$$

This last step is curious – it's our old friend the exponential but now raised to an imaginary power. A quick check shows that some of the important properties of the exponential are preserved, e.g., $e^{\imath(\phi+\psi)} = e^{\imath\phi}e^{\imath\psi}$. But exactly what is this beast?

A little history about the exponential function helps. Newton and Leibniz, before the modern notion of function was completely developed, were concerned with algebraic curves: those curves that could be defined as ratios of polynomials.[3] The exponential *transcends* this class of elementary algebraic curves, or the purvue of elementary algebra.

We saw in Fig. 3.6 that the exponential function arises because the weight of molecules *compounds* like interest on an investment: if you invest \$1 for a year at 10%, compounded monthly, you would have $(1 + 1/10)^{12}$ dollars; in general,

$$(1 + \frac{x}{n})^n.$$

The important step is letting $n \to \infty$, so the expression is defined as the limit of an infinite *power series*:

$$\lim_{n\to\infty} (1 + \frac{x}{n})^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = e^x \tag{8.14}$$

[3]Recall: A polynomial is an expression of the form $\alpha_n x^n + \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} + ... + \alpha_1 x^1 + \alpha_0$.

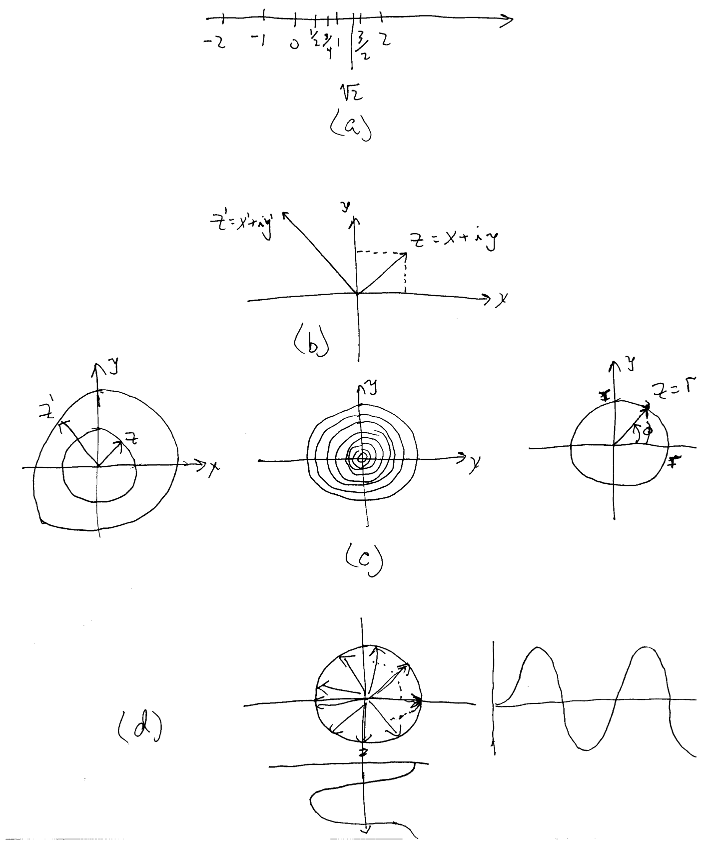

Figure 8.13: The arrangement of real and complex numbers. (a) Real numbers can be arranged along the real number line. This includes the integers (1,2, ...), the rational numbers (1/2, 1/4, .005, 6.02, etc.), and the irrational numbers ($\sqrt{2}, \pi$, etc.). Taken together they populate the full line, which you can think of as a set as well as a continuum, although the integers, say, have gaps between them. These gaps are filled in part with the rationals, which still have gaps between them, until they are "filled" with the irrational numbers. (b) Complex numbers, because they have a real and an imaginary component, can be thought of occupying the plane. Just as with 2-vectors, any complex number $z = x + iy$ can be thought of as a point in this plane with coordinates (x, y). We refer to the x-axis as the real axis, and the y-axis as the imaginary axis. (c) In polar coordinates, think of covering the plane with nested circles. Then a point can be localized by picking the right circle (r) and measuring the angular rotation: $z = re^{\imath\phi}$. (d) Choosing the unit circle ($r = 1$), imagine a vector rotating around it. The tip of this vector sweeps out coordinates $x(t)$ and $y(t)$; they repeat periodically and a little trigonometry shows that these are sinusoidal functions of "time."

We've already seen what can be taken as another definition of the exponential: $f(x) = e^x$ is the unique solution of the differential equation $\frac{d}{dx} f(x) = f(x)$, with the initial condition $f(0) = 1$. (Certain objects in mathematics have multiple definitions like this: seemingly completely different but actually, after lots of work, reducable to each other.)

Now we're ready for Euler's discovery: he substituted the imaginary number $z = \imath x$ for the real variable x in the power series and reorganized the terms:

$$\begin{aligned} e^{\imath x} &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...) + \imath(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...) \\ &= \cos(x) + \imath \sin(x). \end{aligned} \tag{8.15}$$

since the power series for $\sin(x)$ and $\cos(x)$ were already known. After a ton of work to show that the above steps are rigorous, this formalizes the Fig. 8.13.

8.2.8 Filters in the Frequency Domain

Our first calculation using the complex exponential relates to the previous lecture. We saw that the impulse response $h(x)$ defines the linear, time-invariant system L via the convolution operation. Thus we wonder what happens if we use this as input:

$$Le^{\imath\omega t} = \int_{-\infty}^{+\infty} h(u) e^{\imath\omega(t-u)} du \tag{8.16}$$

$$= e^{\imath\omega t} \int_{-\infty}^{+\infty} h(u) e^{\imath\omega(u)} du \tag{8.17}$$

$$= \mathcal{H}(\omega) e^{\imath\omega t}. \tag{8.18}$$

This is exactly what we saw in the earlier experiments: sinusoids maintain their from when they pass through linear systems. Working in the complex plane let's us understand that they may change in magnitude or in phase.

These complex sinusoids thus seem like the ideal "atoms" with which to describe arbitrary functions, and we're now ready to define the Fourier Transform ($\omega = 2\pi f$):

$$\boxed{\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-\imath\omega t} dt} \tag{8.19}$$

and the inverse Fourier Transform:

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}(\omega) e^{\imath\omega t} d\omega.} \tag{8.20}$$

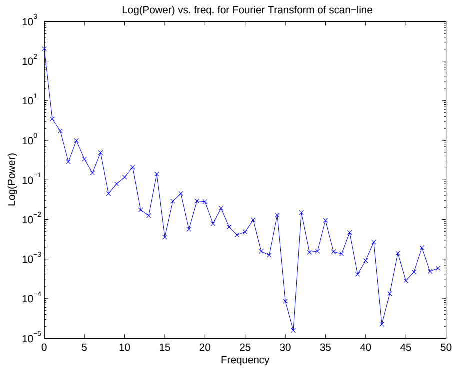

Figure 8.14: The power spectrum for the waveform in Fig. 8.10. The power spectrum is the magnitude term squared.

Previously we had introduced the Fourier Series, which is important in the analysis of periodic functions. The Fourier transform arises as the period extend to ∞ and the sum becomes an integral.

The power spectrum gives a sense of the distribution of energy in a waveform as a function of frequency. Clearly the bass keys have their energy at low frequencies, while the treble have their energy at high ones. The result in Eq 8.16 suggests that we should look at how this happens at each frequency; See Fig. 8.14.

If we now pass an input through a system, it is to be expected that not all frequencies will be treated equally. (If they were treated identically then the system would be very uninteresting – can you identify it?)

The importance of the convolution operation and the impulse response was already established. There is an important dual to it in the frequency domain. Recall: If an input $I(x)$ is passed through a system with impulse response $h(x)$ then the output $O(x) = I * h(x)$. Denoting their Fourier transforms $\mathcal{I}(\omega), \mathcal{O}(\omega)$, and $\mathcal{H}(\omega)$, we have to calculate the Fourier transform of the convolution:

$$\mathrm{FT}\{I * h\} = \int_{-\infty}^{+\infty} e^{-(\imath t\omega)} \underbrace{\left(\int_{-\infty}^{+\infty} \underbrace{I(t-u)}_{\text{input}} h(u) du \right)}_{\text{convolution}} \quad dt. \tag{8.21}$$

This is a tricky integral, but if we apply Fubini's theorem and the change of variable

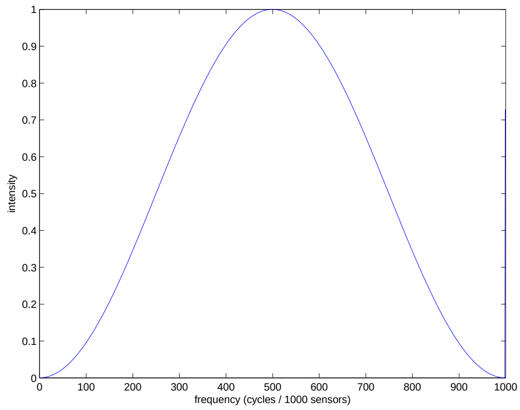

Figure 8.15: The MTF for the lateral inhibitory network in *Limulus l.* The units for the vertical axis are relative gain.

$(t,u) \mapsto (v = t-u, u)$ we get:

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\imath(u+v)\omega} I(v)h(u)dudv \tag{8.22}$$

$$= \left(\int_{-\infty}^{+\infty} e^{-\imath v\omega} I(v)dv\right)\cdot\left(\int_{-\infty}^{+\infty} e^{-\imath u\omega} h(u)du\right) \tag{8.23}$$

$$= \mathcal{I}(\omega)\cdot\mathcal{H}(\omega) \tag{8.24}$$

from which we have the remarkable "duals:"

$$I(x) * h(x) \iff \mathcal{I}(\omega)\cdot\mathcal{H}(\omega) \tag{8.25}$$

$$I(x)\cdot h(x) \iff \mathcal{I}(\omega) * \mathcal{H}(\omega) \tag{8.26}$$

This duality is fundamental; sometimes it's called Fourier's theorem.

The Fourier transform of the impulse response is the *modulation transfer function* (MTF) of the system; it indicates the multiplicative factor at each frequency component of the input.

The MTF might favor low-frequencies (low-pass); high frequencies (high-pass); or middle-range frequencies (band pass); these are the most common types of filters. Low-pass filters tend to blur the image, while high-pass sharpens it. The MTF for *Limulus l.* is shown in Fig. 8.15; it is clearly band-pass.

8.2.9 Filters in the Frequency Domain

In the previous lecture, we derived 'sharpening' and 'blurring' filters in the time/space domain. What is their frequency domain counterpart? The duality theorems just discussed suggests that there is a dual version.

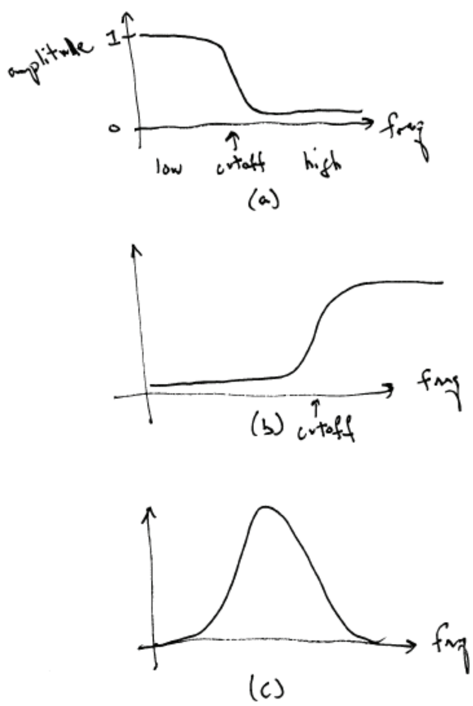

Figure 8.16: Different classes of filters in the frequency domain. (a) low pass; (b) high pass; (c) band pass;

Convolution in time is multiplication in frequency, so our job is to design filters in frequency so that, when multiplied against their time/space domain counterparts, they 'amplify' the relevant features in the desired fashion (either extending or compressing them). Two basic possibilities immediately suggest themselves (Fig. 8.16:

- LOW-PASS FILTER Inhibit high frequency components and allow low-frequency components to pass. This resembles a blurrring operation.

- HIGH-PASS FILTER Inhibit low frequency components and allow high-frequency components to pass. This resembles a sharpening operation.

Of course, the natural case is to inhibit both very low and very high frequencies; this is called a BAND-PASS filter. What is the lowest frequency that you can hear? The highest? How does this relate to the qualities of your music equipment?

8.2.10 Experiments with Square Waves

The Fourier transform of a square wave function is an interesting example. Working with a single period, we define the square wave:

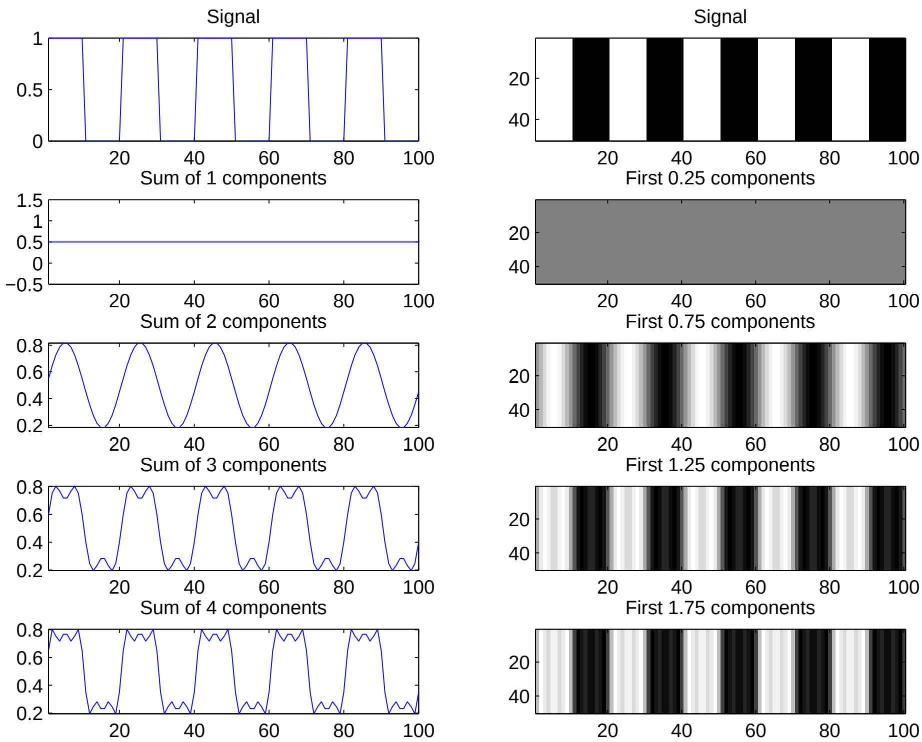

Figure 8.17: Constructing a square wave from its Fourier components. Again, the first term is a constant. The first 4 components (including the 5-th harmonic) yields an image that starts to resemble the square wave but with some "ringing", as shown on right. (Please ignore the incorrect headings on the right side.

$$I(x) = \begin{cases} 1 & \text{if } 0 < x < T/2 \\ -1 & \text{if } T/2 < x < T \end{cases} \tag{8.27}$$

It's Fourier series expansion ($\omega = 2\pi/T$) is:

$$\mathcal{I} = \frac{4}{\pi}\{\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \ldots + \frac{1}{n}\sin(n\omega t) + \ldots \tag{8.28}$$

The Fourier expansion for a square wave has all odd harmonics; their amplitude is inversely proportional to frequency, and, most importantly, to get the "sharp" edges of the square wave very high frequency components are required. See Fig. 8.17.

If a square wave is blurred, or low-pass filtered, the sharp edges are "rounded" by reducing the high-frequency content. Examples are shown in Figs. **??**.

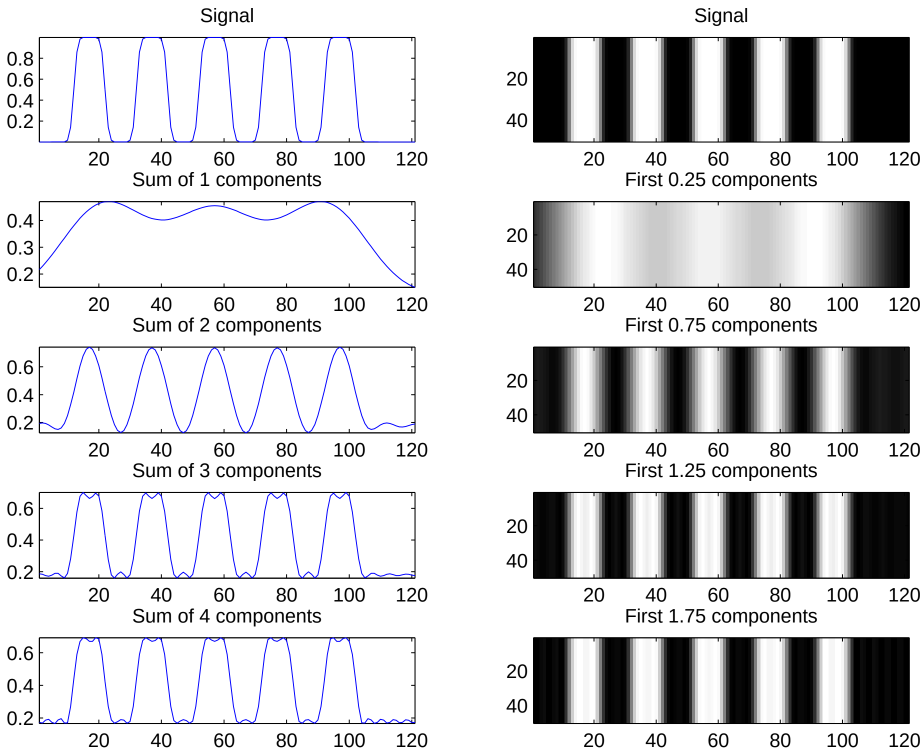

Figure 8.18: Convolution of a Gaussian ($\sigma = 2$ pixels) against the square wave. Notice how the edges are rounded but it's still fairly similiar to the original square wave. (Ignore boundary effects; only look at the middle of the displays.

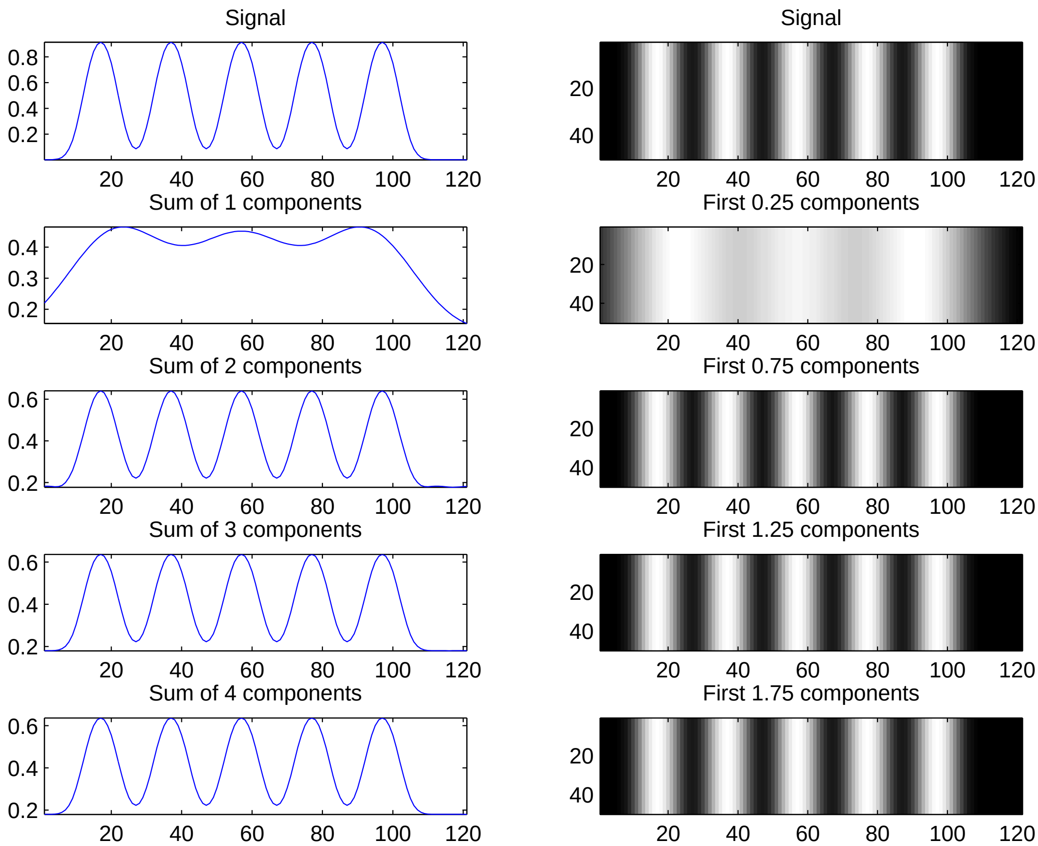

Figure 8.19: Convolution of a broader Gaussian ($\sigma = 6$ pixels) against the square wave. Notice how the edges are much more rounded. (Ignore boundary effects; only look at the middle of the displays.

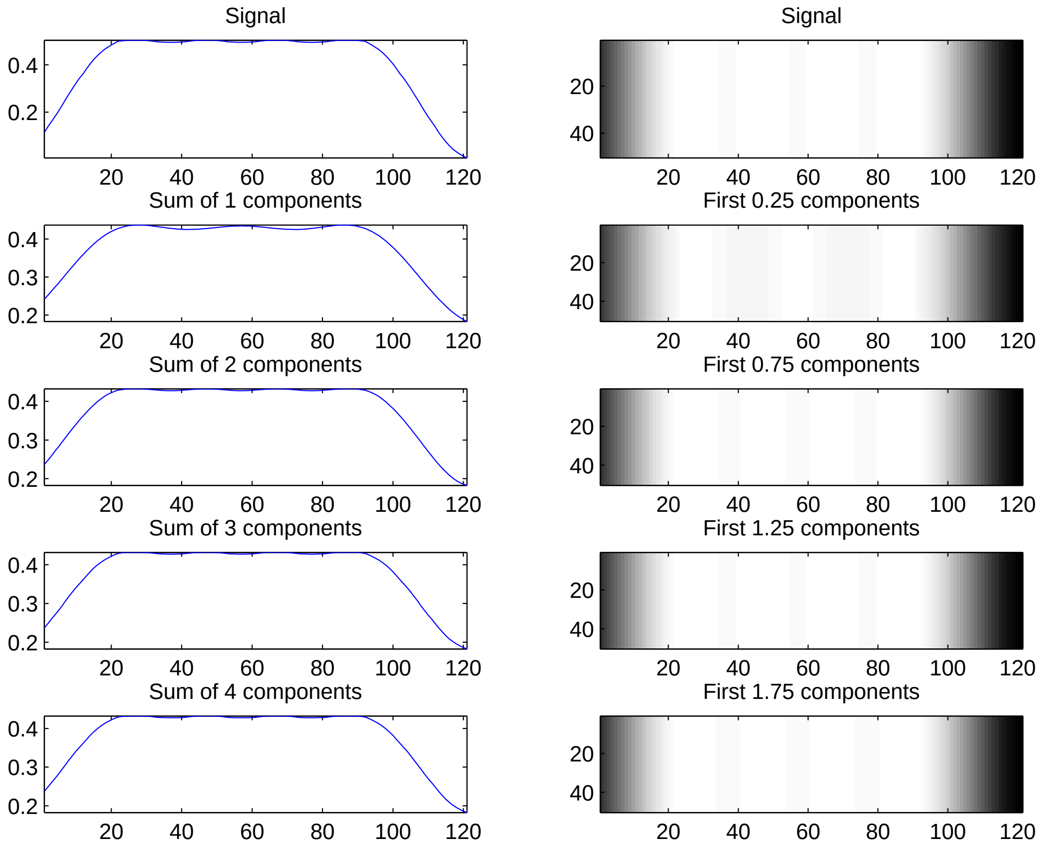

Figure 8.20: Convolution of a broad Gaussian ($\sigma = 20$ pixels) against the square wave. Now much of the square wave structure is blurred away. Again ignore boundary effects.

CHAPTER 8. LIMULUS IN FREQUENCY

8.3 Spatial Frequencies and Self Psychophysics

To generalize the sinusoid to two dimension, we have to worry about two spatial frequencies, one being the period that projects along the x-axis and the other a projection along y. Their inversion is the two spatial frequencies.

To visualize the Fourier basis in 2D, See Fig. 8.21. We start with a SPATIAL FREQUENCY GRATING oriented along the x-axis. This has sinusoidal variation in x and is constant in y. So the inverse of the period in x gives the first spatial frequency, ζ; the second, η, is 0. As it rotates the two frequencies co-vary.

Demonstration of your own MTF; see Fig. 8.22.

8.4 Fourier Transform to Fourier Integral

(need to write this section.)

8.5 Cuttlefish revisited

So, we now return to the cuttlefish, and observe that it's estimating its coloration pattern by pigmentation controlled by the first 3 harmonics; see Fig. 8.23.

8.6 The Design of an Eye

The bandlimited nature of sampled signals imposes an issue: are the natural signals in the world band-limited? We saw in the last section that the answer to this in a pure sense is negative: the step function, which is a model for an ideal edge, has infinite harmonic content. However, since the amplitude for each of these harmonics decreases, if we could filter it properly then a bandlimited signal would result. The lens on our eye is key here; it bandlimits the signals. This role is played by both the eye of Limulus and our own; see Fig. 8.24.

Simple model for the lens: Gaussian blur. This is a pretty good model for the human eye. In particular, this Gaussian blur filter, which we already saw as a smoothing filter, limits the frequency content of light distributions falling on our retina (where the sampling occurs).

8.6.1 Acuity

Any design for an eye imposes an acuity limit. One way to express this: the closest that two very tiny points of light can be without *blurring* into a single one. See Fig. 8.25.

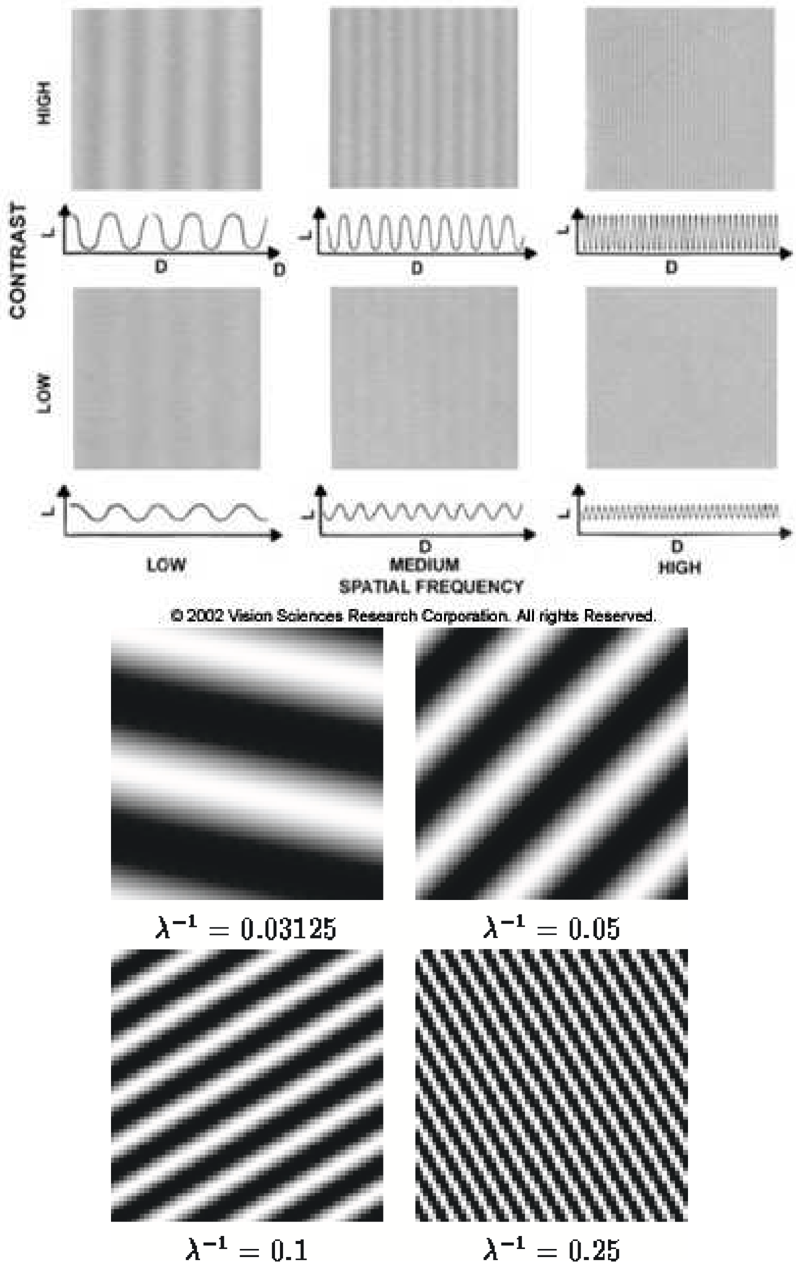

Figure 8.21: Generalizing the spatial frequency sinusoids to two dimensions. (Top) A 2-dimensional spatial frequency grating is a sin in one direction and a constant in the other. Different examples are shown at different frequencies and contrasts. (Bottom) To cover the full spectrum, we need to consider gratings at all frequencies and at all angles. The frequency is given by the inverse of the period, or wavelength, in units such as cycles per degree of visual angle.

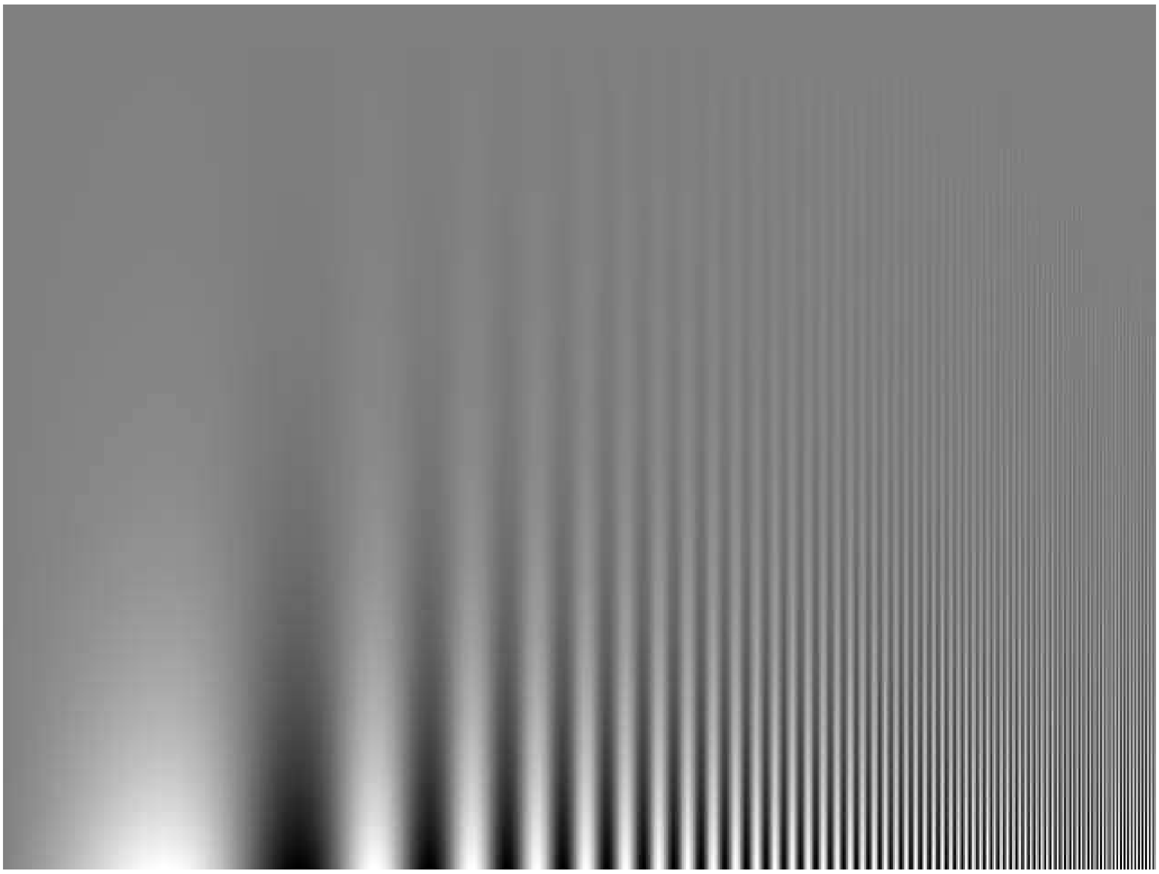

Figure 8.22: Estimate your own MTF. In this display, freqency varies along the x-axis and contrast along the y-axis. The just-noticable-difference (JND) in contrast values is a (crude) measure of your sensitivity to contrast. In this display you should be able to read off something like an "MTF" for your own visual system. Does it make sense? Why? Why not?

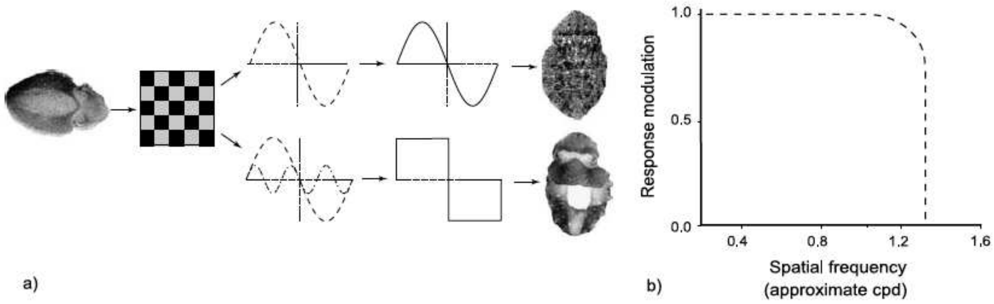

Figure 8.23: Returning to the cuttlefish, we can now see that it's approximating the "square" texture of the background with the first new harmonics. The precise mechanism by which it estimates these harmonics remains to be discovered. see Phil. Trans. R. Soc. B 2009 364, 439-448, S Zylinski, D Osorio and A.J Shohet, Perception of edges and visual texture in the camouflage of the common cuttlefish, Sepia officinalis.

http://webvision.med.utah.edu/imageswv/draweye.jpeg

Figure 8.24: The design of an eye consists of a lens to focus the image and a sensor array to sample it. This basic set of design constraints is employed by a wide range of eyes in nature; see ref??

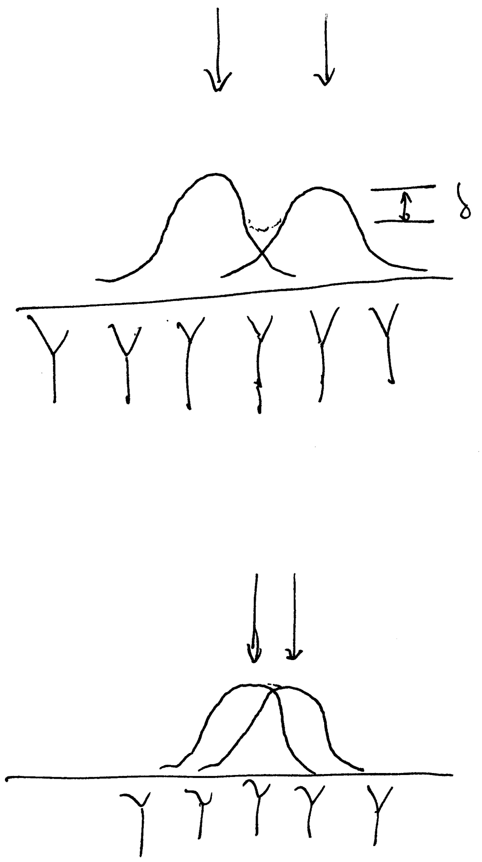

Figure 8.25: Illustration of visual acuity. Notice how, when two very fine spots of light are shown on the receptor array through a lens, the lens blurs the points. To complete the test, one receptor must be able to detect the lower value from its surround. In the top illustration two points are visible, but not in the bottom.

8.7 Summary

Fourier analysis provides an alternative view of linear processing of images. The concept of filtering is everywhere (even though it might be non-linear in many cases where you encounter it.) For example, what distortions to an image are introduced by a xerox copier? Are all xerox copiers the same? Some experiments here could be fun.

However, the units of which the visual world is composed do not fall nicely and naturally into the Fourier decomposition - just think about occlusion and what this implies for the principle of superposition. See Figs.**??**. What other possible units for expansion are there?

8.8 Notes

General history and informal background: **e: The story of a number** by Eli Maor, Princeton U. P. , 1994.

The calculation of the Fourier coefficients comes from the Feynman Lectures on Physics, vol 1, lecture 50.

Nice discussion of the Fourier Transform is in Stephane Mallat, **A Wavelet Tour of Signal Processing**, Academic Press, 2009, or your favorite book on signal processing.

For a taste of the mathematician's view, see the entry of the Fourier Transform by Terry Tao in **The Princeton Companion to Mathematics**, p 204.

More advanced material on primate vision is in De Valois, R. L., & De Valois, K. K. (1988). Spatial vision. New York: Oxford University Press, but we'll get to this later in this term.

