Chapter 6

Limulus linearis: An Image Processing Machine

Introduction of Linear Systems: (1) Additive; (2) Homogeneous. The Principle of Superposition. Unit step function, step response; Dirac Delta function, impulse response . Representation of signals as a linear combination of step functions. Convolution operator, the output of a linear system via convolution. Smoothing filter; sharpening filter.

This chapter builds upon two questions raised in the last chapter: (i) what are the right stimuli to be used to identify the properties of a system; and (ii) by what properties, if any, does the raw neural image differ from the original image. We illustrated the discrete nature of the neural image by the facet structure of the compound eye, but was this a special case? Given the prevalence of digital images, this suggests we must understand the discrete nature of these images by a general theory of image sampling. But to build this up we shall have to set quite a lot of background material in place.

In the series of lectures, we began with *A. coli's* behavior and built a theory "from the outside"; that is, by looking at the phenomenology of its behaviour and relating it to physical properties of its environment (the food potential function). We then started to look more closely at what's "inside" a behaving creature, with a concentration on neural circuity that might support very simple visually-mediated behaviors. Some success was achieved with the shadow-withdrawal reflex. *L. linearis* brought us to lateral inhibition, a fundamental motif in neural network design. Experiments with it revealed the enhancement of images in the neighborhood of edges, but did not yet satisfactorily answer the question of what *Limulus* uses lateral inhibition for.

In this lecture we try to relate the views "from the outside" and "from the inside" by taking a systems, or input/output modeling view.

We begin with more experiments with our lateral inhibitory network, to develop intuition.

6.1 Limulus linearis

In the last lecture we talked about the linear operating range of some neurons, for which there was a straight line relationship between the input variable (say intensity or, more realistically, log intensity) and the output variable (say, firing rate). But is there more to linearity than just a straight line in graphs? The very regular and repetitive nature of the wiring in the lateral plexus is rather intruiging ...

6.1.1 Idealized Experiments

Let's begin by doing some more experiments with the network derived in the last lecture; Fig. 6.1, except now with a slightly more mathematical bent. First, we will assume the network is balanced and we'll bring all the inputs into the summation:

- Suppose the excitatory synaptic coefficent $\alpha_{i,j} = 1, i = j$ while the inhibitory coefficents ($i \neq j$) are each = -0.5. Then we have the case that $\sum_{j \in \{\text{inputs to i}\}} \alpha_{i,j} = 0$ for all neurons i. In other words, the excitatory and the inhibitory synaptic strengths are balanced for each neuron.

And we will assume for this lecture that neurons can represent negative as well as positive numbers.

6.1.2 Remark: Coding of Numbers in Neurons

Input/output characterization of neurons is typically that they can only represent positive numbers; i.e., in terms of firing rate:

$$\text{firing rate} \propto \begin{cases} inputs & \text{if } inputs \geq 0 \\ 0 & \text{if } inputs < 0 \end{cases} \tag{6.1}$$

This would mean that neurons have a "resting" firing rate (with no inputs) of 0 spikes/second. In fact, there is a low resting firing rate which would imply that they can represent negative numbers with less precision than positive numbers, but it is not completely clear how to use this.

Instead, as we shall see in later lectures, visual systems can be modeled as separating the "positive part" from the "negative part" and representing these in separate systems, with -(negative part) also represented as a positive number.

This topic will be revisited in subsequent lectures.

6.1.3 The Step Input

We shall take this special version of the lateral inhibitory network as the definition of processing in "*Limulus linearis*", or *Limulus l.* for short.

6.1. LIMULUS LINEARIS

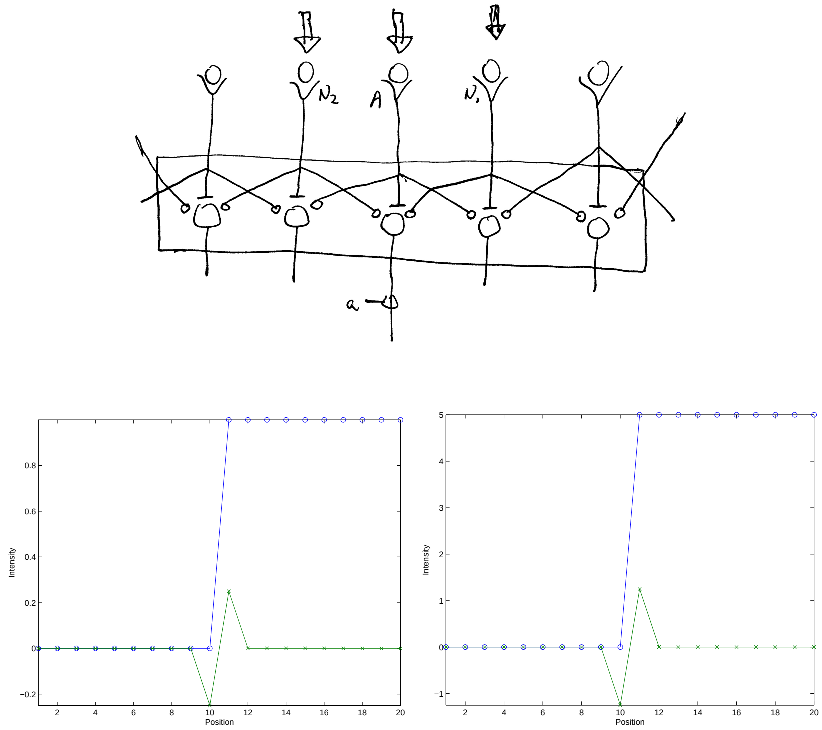

Figure 6.1: (TOP) The lateral inhibitory network with balanced coefficents. (BOTTOM) Examples of inputs and responses for a step function. Note how the form remains the same even though the amplitude of the step varies; look carefully at the scale of Intensity. What would happen in these experiments if synapsic weights were not balanced, as in the previous lecture? Can you guess any role for balanced weights in recurrent networks?

Our first experiments continue with a step function, in which it becomes clear that the form remains invariant regardless of the amplitude of the step. (Remember, please, that we're ignoring the fact that neurons can't signal negative values directly.)

position x	...	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_{0+}	x_1	x_2	x_3	...
input $I(x)$	...	0	0	0	0	10	10	10	10	...
output $O(x)$	...	0	0	0	-5	5	0	0	0	...

Now, it we put the step on a constant offset:

position x	...	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_{0+}	x_1	x_2	x_3	...
input $I(x)$	...	10	10	10	10	20	20	20	20	...
output $O(x)$	...	0	0	0	-5	5	0	0	0	...

we notice that it has no effect. Note that the response to a constant illumination for this system with balanced coefficents is always 0. But there's something else that's possibly indicated in this example: if we think of this elevated step as the sum of two functions, the "constant" function plus the "step" up from 0, then we can ask how separate they really are. In particular, we observe from above that:

$$\textsc{Response}(\textsc{step} + \textsc{constant}) = \textsc{Response}(\textsc{step}) + \textsc{Response}(\textsc{constant})$$

where RESPONSE(INPUT) is the response of our lateral inhibitory network to INPUT. Since the INPUT could be any image (in one or two dimensions), we see that this network implements a map from images $\rightarrow$ images.

We wonder if the above relationship might indicate a more general principle for such maps. It helps to think abstractly; consider an idealized network (system) L with input I and output 0:

$$I \longrightarrow \boxed{\mathbf{L}} \longrightarrow O.$$

6.1.4 Linear Systems, Superposition, and Convolution

The name *Limulus linearis* should have indicated to you that understanding linearity is at the heart of our endeavor. In this section we provide a brief introduction to aspects of linear systems theory. It is based on the following definition:

Let **L** denote a *time/space-invariant, linear system* with input $I(x)$ and output $O(x)$.

When the independent variable x = time, we think of Input(time); when x = (x,y), position coordinates, we think of image coordinates; i.e., $I(x, y)$ denotes image intensity at position (pixel) (x, y). To keep it simple, for now we'll work in 1 dimension.

Such systems are *additive* (the response to the sum of two inputs $I_1(x)$ and $I_2(x)$ equals the sum of the responses to each input taken individually):

$$\mathbf{L}(I_1 + I_2) = \mathbf{L}(I_1) + \mathbf{L}(I_2)$$

and *homogeneous* (the system can be scaled by the magnitude of the input α, where α is a scalar; think "volume control" on a radio or a "brightness control" on a TV):

$$\mathbf{L}(\alpha I) = \alpha \mathbf{L}(I)$$

Taken together, they specify the *Principle of Superposition:*

$$\mathbf{L}(\alpha I_1 + \beta I_2) = \alpha \mathbf{L}(I_1) + \beta \mathbf{L}(I_2)$$

This principle of superposition lies right at the heart of linear systems – we need to understand what it implies and how we can exploit it. There are some surprising consequences.

Our basic question for this lecture is whether we can break up an arbitrary input function into a collection of very simple input functions so that, if the response to these simple inputs were known then they could be combined to give the response to the arbitrary input. Several foundational questions are implied by this: (i) what are these simple functions, if they exist; (ii) when can an arbitrary function be decomposed into them; (iii) are they unique; and (iv) how can they be recombined?

Different aspects of these foundational questions will run through the remainder of these lectures. For now we start with an observation, that suggests there are limits to this first approach to them, followed by some additional background.

Here are some tangential points that become quite interesting with the above principle in mind. Let's recall a few calculations that you've done before.

Suppose $f(x) = \alpha f_1(x) + \beta f_2(x)$. We then have:

$$\int_a^b f(x)\,dx = \alpha \int_a^b f_1(x)\,dx + \beta \int_a^b f_2(x)\,dx \tag{6.2}$$

Here's another one:

$$\frac{d}{dx} f(x) = f'(x) = \alpha f_1'(x) + \beta f_2'(x) \tag{6.3}$$

These examples suggest something of a more abstract view around integration and differention: that they are a kind of "black box" operation. These OPERATORS have the form: $\int \Box\, dx$ and $\frac{d}{dx}\Box$ where they take as "input" some appropriate function (inserted at $\Box$) and "output" or return another function. It is common to think in these terms in mathematics. We shall see that this concept of operators that map from spaces of functions to spaces of functions is an important one. When the operators are linear, a lot can be said about them.

Saturation An important exception to a linear system is one in which the response saturates; can you see why this is no longer linear?

6.1.5 Idealized Input Functions

One of the input functions that we used in the last lecture was already seen to be important, because it resembles an 'edge' in intensity. Mathematically it is the *unit step function*, defined as:

$$u(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x > 0; \\ \text{undefined}, & \text{if } x = 0 \end{cases} \tag{6.4}$$

Following up on the question above, it will turn out that the step function achieves special importance in the analysis of linear systems because we *can* use it as a component in understanding more complex inputs. But something even more curious arises along the way. Let's start by considering the difference between two (displaced) steps.

input: step at x	...	0	0	0	10	10	10	...
input: step at $x+1$	...	0	0	0	0	10	10	...
output: step at x	...	0	0	-5	5	0	0	...
output: step at $x+1$	...	0	0	0	-5	5	0	...
diff in output:	...	0	0	-5	10	-5	0	...

Continuing our thinking about synthesizing a function as a sum of other functions, in this example we imagine the step at (x) to be composed of the step at (x+1) plus the extra bit at (x). Specifically, the difference between the two steps can be interpreted: start a step at x and start a negative step at $(x+1)$. The result is like shining a light only on the receptor at (x); it can be thought of as an impulse of light simulation. From the difference in the step reponses we compare this estimate of the response to an impulse to that obtained by shining the impulse directly (Fig. 6.2. Note how this reveals, immediately, the fundamental motif of the lateral inhibitory network!

Formally defining this impulse function is an interesting exercise that may seem illegial in a certain sense (although it can be made precise mathematically). In Chapter 2 we stated the classical definition of a derivative in calculus as

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

which we should not ordinarily be able to apply to the step function $u(x)$. However, what we did above is, in some sense, analagous to this, but not at the limit of h being small but limited (by *Limulus's* ommatidium) to a single, quantized distance.

Perhaps this gives us the courage to plow ahead and consider what it means to take the derivative of the step function. Formally, let's define it as:

$$\delta(x) = \frac{du(x)}{dx}$$

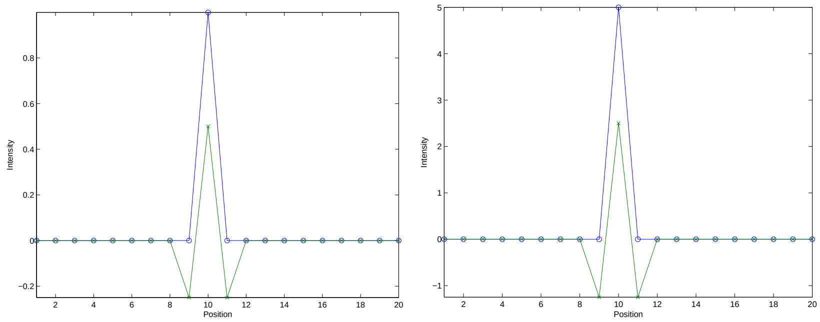

Figure 6.2: (TOP) The response of the lateral inhibatory network with balanced coefficents to an impulse of light on a single ommatidium. (Different amplitudes shown.)

It is called the *Dirac delta function* and is a special object indeed. This is to be expected, since the step function is not continuous (notice the value at 0), and perhaps the surprise is that one can make any sense of it at all. Clearly, one has to be careful in defining exactly what is meant by the above definition. It helps to set the stage, by reviewing material that you probably already know.

Integration and the Window Function

Before studying the delta function, it is useful to recall some aspects of integration from calculus. (Feel free to skip this section if you already know this material.)

While previously we had focussed on the derivative, the concept of integration came up informally several times. *A. coli*, for example, followed integral trajectories through the food potential function, and neurons integrated the current arriving from synapses with other neurons. Integration, in the classical sense, is the inverse operation to differentiation.

For functions of one variable, integration is introduced as the area under a curve. Drawing the right picture helps with evaluating simple cases, such as those in Fig. 6.3. Dividing the domain of the function into a number of regions bounded above by the function, below by the x-axis, and left and right by the limits of integration, the areas of these strips can be combined to give the area under the curve.

Two examples are instructive and can be calculated directly from this construction. For the simple case of $y = f(x) = 1$, we have

$$\int_a^b 1\,dx = 1\int_a^b dx = b - a \tag{6.5}$$

Breaking the interval $[a, b]$ into n pieces for more general functions, we have to take care in specifying what it means to form the strips (Fig. 6.3, bottom). They can be constructed from "above", by taking the height to be the maximum value of the function in the domain $(x, x + \frac{(b-a}{n})$, from "below" by taking it to be the minimum value, or by taking it somewhere in between. Of course, if the function being integrated is continuous and smooth, this limit from above and the limit from below converge to the same value. (Think also about the convergence of secants to the tangent.)

But the above example gives us something else to think about, because the difference $b - a$ brings the numerator of the definition of the derivative to mind.

We switch to the indefinitely integral (following the notation in Courant, p 111):

$$\Phi(x) = \int_a^x f(u)du \tag{6.6}$$

where $f(x)$ is a continuous function. The basic idea at the heart of calculus is that differentiation of the indefinite integral of a given *continuous* function always gives us back that function. In symbols, the indefinite integral above always possesses a derivative $\frac{d\Phi}{dx}(x) = \Phi'(x)$ such that

$$\Phi'(x) = f(x).$$

Recalling the difference quotient in the definition of the derivative,

$$\frac{\Phi(x+h) - \Phi(x)}{h} \tag{6.7}$$

and expanding the numerator:

$$\Phi(x+h) - \Phi(x) = \int_a^{x+h} f(u)du - \int_a^x f(u)du = \int_x^{x+h} f(u)du \tag{6.8}$$

This is the area of the rectangle delimited by the ordinate above x and the ordinate above $x + h$. The ratio of the limit converges to the derivative.

The Dirac δ Function

Now we're going to enter some new territory. I underlined the word *continuous* in the last section to underline that way students of first-year calculus are taught to think about things. Forget continuous: we need to move beyond that in a controlled fashion. Repeating the above expression:

$$\delta(x) = \frac{du(x)}{dx}$$

we are now ready to characterize it.

To start, let's return to the compound eye of *Limulus linearis* and take another look at how to think about illuminating just one facet. Instead of a tiny source, in Fig. 6.4 we place an opaque sheet over the facets of the compound eye, but with a

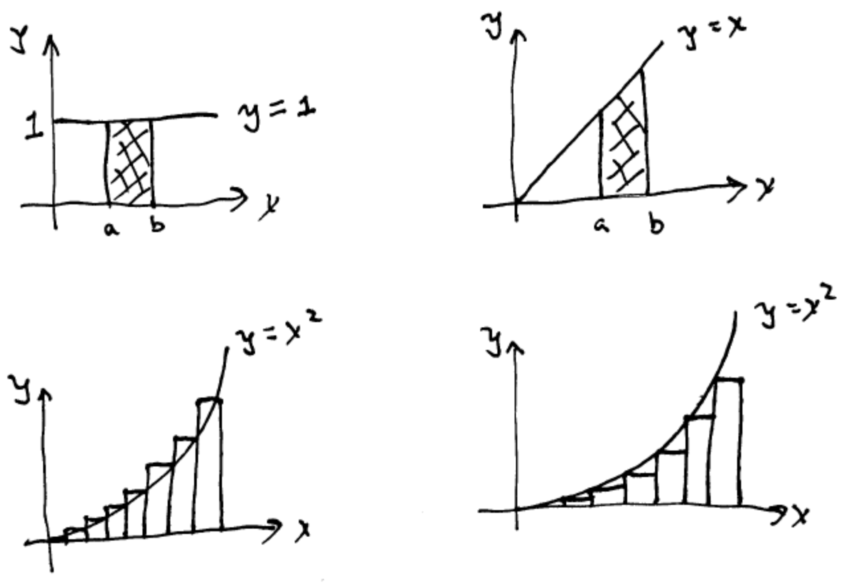

Figure 6.3: Integration as the area under a curve. (TOP) Two direct constructions for $y = 1$ and $y = x$. For the second of these we have $\int_a^b x\,dx = \frac{1}{2}(b-a)(b+a) = \frac{1}{2}(b^2 - a^2)$. BOTTOM) For general functions the integration strips can be developed "from above" or "from below." If the function is continuous and smooth, then these two constructions converge in the limit.

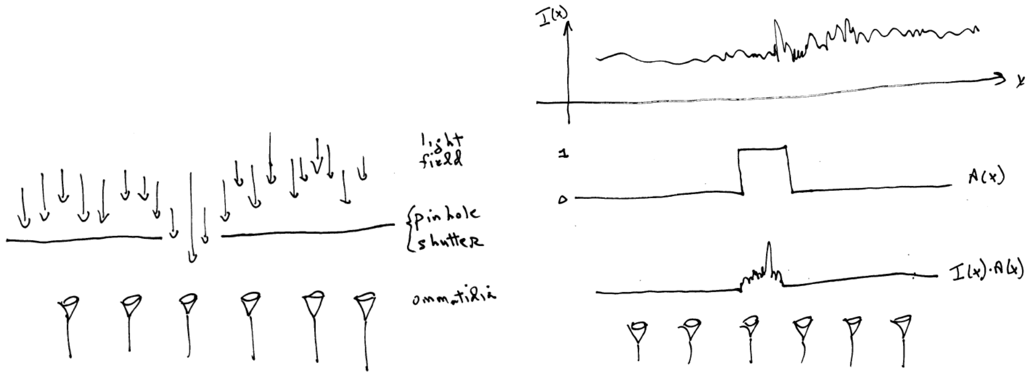

Figure 6.4: Characterizing an impulse of light onto a single ommatidium. (LEFT) An opaque sheet is placed over the lenses, blocking all the light except that through a small (pinhole) opening. (RIGHT) An aperture function $A(x)$ modifies the continuous light intensity $I(x)$, so that only those values where $A(x) = 1$ can "pass through." Thus the light field at the compound eye is given by the product $I(x) \cdot A(x)$. Theoretically we shall be interested in the case where the aperture function shrinks in width to an infinitesimal value.

tiny opening over only a single facet. This sheet blocks all incoming light except that on the facet in which we're interested.

We can model this situation with an aperture function $A_i(x)$, that has value 0 everywhere except over the facet i, where it has value 1:

$$A_i(x) = \begin{cases} 1, & \text{if } x \text{ over facet } i; \\ 0, & \text{if } x \text{ over facet } j \neq i \end{cases} \tag{6.9}$$

Now, the combined effect is that the light falling on the compound eye is the PRODUCT of the continuous light intensity and the aperture function, $I(x) \cdot A(x)$. We shall be interested in what happens to this as the size of the facet—the aperture—shrinks, in the limit, to 0. This gives rise to a beautiful mathematical generalization of the notion of function.

Formally, the Dirac $\delta(x)$ is a *distribution* that generalizes the notion of function in the following way. If we imagine it shrinking in width and increasing in height, in the limit we arrive at:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \tag{6.10}$$

This is a weird beast: it is infinitely large over a single point; under the integral this will be an area of measure 0, which further suggests that we define the above in such a manner that it keeps

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{6.11}$$

It is especially important to note that $\delta(x)$ is non-zero only at a *single* point, 0, among all numbers. Thus in informal, kind of picturesque terms, it is infinitely "large" over an infinitesimally "small step" h.

The second condition above indicates that this *generalized function* "lives" under an integral. Its important role is defined with the help of (suitably smooth) test functions. Here the product of functions under an integral returns. The integral equation

$$\int_{-\infty}^{\infty} v(x)\delta(x)dx = v(0).$$

says that, in effect, it "selects" the value of the test function at the 0 point of its argument. This relationship is extremely important and it provides just the kind of justification that we were seeking when we tried to understand how a (very very very small) ommatidium could "select" the value of the light ray impinging upon its center. Here the test function is the ambient light intensity and the aperture function is the delta function.

To understand the importance of unit area, one can think of a limiting process for a compound eye in which the ommatidia get smaller and smaller; in effect this would "cut out" smaller and smaller portions of the image. However, these smaller portions would in effect have so little light in them that they would eventually converge to just 0. So the idea of the selection function is to restrict the input diameter while "amplifying" the light so that the product of light amplification and capture width remains constant. In the limit this is a δ-function; see Fig. 6.5(bottom).

We can define $\delta(x)$ as the limit of a sequence of functions such as;

$$\lim_{\epsilon \to 0} \delta_\epsilon(x) = \frac{1}{\pi x} \sin \frac{\pi x}{\epsilon}$$

See Fig. 6.2.

6.1.6 Impulse Response and Step Response

During this limiting process we have something that resembles a physical approximation to the unit impulse, the first test pattern that we used to evaluate the lateral inhibitory network. The result is called the *impulse response*

$$h(x) = \mathbf{L}[\delta(x)]$$

Similarly, since the step function is the integral of the impulse function, the *step response*, or the response to a unit step function

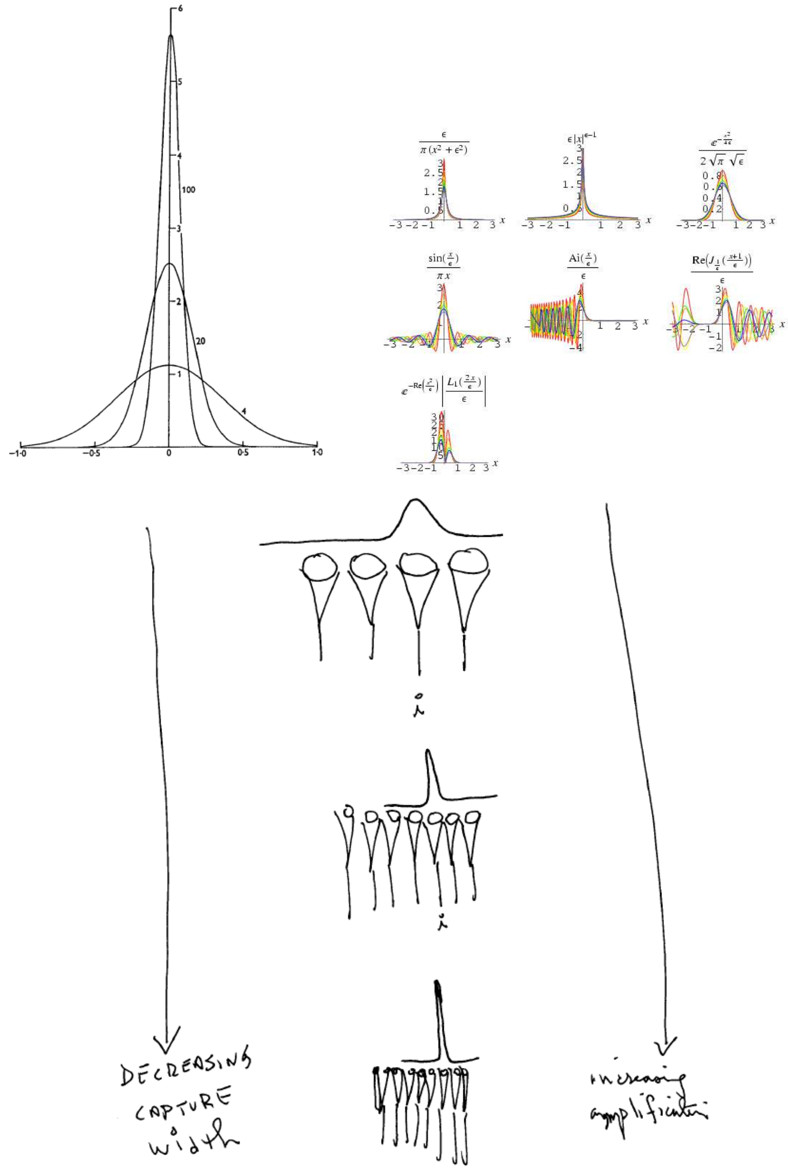

Figure 6.5: (**top**)The Dirac delta function $\delta(x)$ can be defined as the limit ($\epsilon \to 0$) of a sequence of ordinary functions. Notice how the width shrinks, the height increases, but the area remains constant. On the left is shown the sequence $(n/\pi)^{\frac{1}{2}} \exp^{-nx^2}$, which is reminiscent of a "blur spot" of different diameters and brightnesses. Also interesting is the $\frac{\sin(x/\epsilon)}{x}$; second row, right. (**bottom**) Imagining the limiting process for a delta function as a sequence of ommatidia, in which the diameter of the ommatidium gets smaller but the decrease in light is countered by an aperature function cleverly designed so that it picks out the right single value from the continuum. See text.

$$S(x) = \mathbf{L}(u(x))$$

is the integral of the impulse response function:

$$S(x) = \int_{-\infty}^{x} h(\lambda) d\lambda.$$

6.1.7 Approximation with Sequences of Steps –¿ Convolution

In general, of course, we are interested in the response of a system not to these special functions, but to an arbitrary input. The trick is to approximate this arbitrary input as a sequence of step functions, one following the other, and then to use superposition to put the response of the system to each of these steps together to obtain the overall response. In the limit that the steps become arbitrarily close to one another, the approximation should converge to the actual response.

It is necessary to *assume that the input function $I(x)$ is continuous* and physically possible, so that the approximation is meaningful. Now, suppose the input starts at $x = 0$, and the x-axis is discretized into bins Δ units apart. The first step in the approximation has height $I(0)$, a constant, and the input is approximated for small values of x by $I(0)u(x)$. By Δ units later the input will have deviated (in general) from this step approximation by a significant amount; in fact, by $I(\Delta) - I(0)$, so at Δ we must add a "correction" step of this amount. We're then OK for another Δ step. This step-correct cycle continues, with the additional step at $x = k\Delta$ having height $[x(k\Delta) - x(k-1)\Delta]$. Remembering that the step is unit height, $u(0) = 1$, by definition, we must multiply it by the right scalar amount each time. The signal $I(x)$ can thus be approximated by

$$\begin{aligned} \hat{I}(x) &= I(0)u(x) + [I(\Delta) - I(0)]u(x - \Delta) + [I(2\Delta) - I(\Delta)]u(x - 2\Delta) + \cdots && (6.12) \\ &= I(0)u(x) + \sum_{k=1}^{\infty} \{I(k\Delta) - I((k-1)\Delta)\} u(x - k\Delta) && (6.13) \end{aligned}$$

It is important to read this expression as a sum of scalars times steps. If we know the system, then we know its step response, so we can use the superposition principle to add the responses to these these individual steps together, properly weighted, to obtain the approximate output:

$$\begin{aligned} \hat{O}(X) &= I(0)S(x) + \sum_{k=1}^{\infty} \{I(k\Delta) - I((k-1)\Delta)\} S(x - k\Delta) && (6.14) \\ &= I(0)S(x) + \sum_{k=1}^{\infty} \{\frac{I(k\Delta) - I((k-1)\Delta)}{\Delta}\} S(x - k\Delta)\Delta && (6.15) \end{aligned}$$

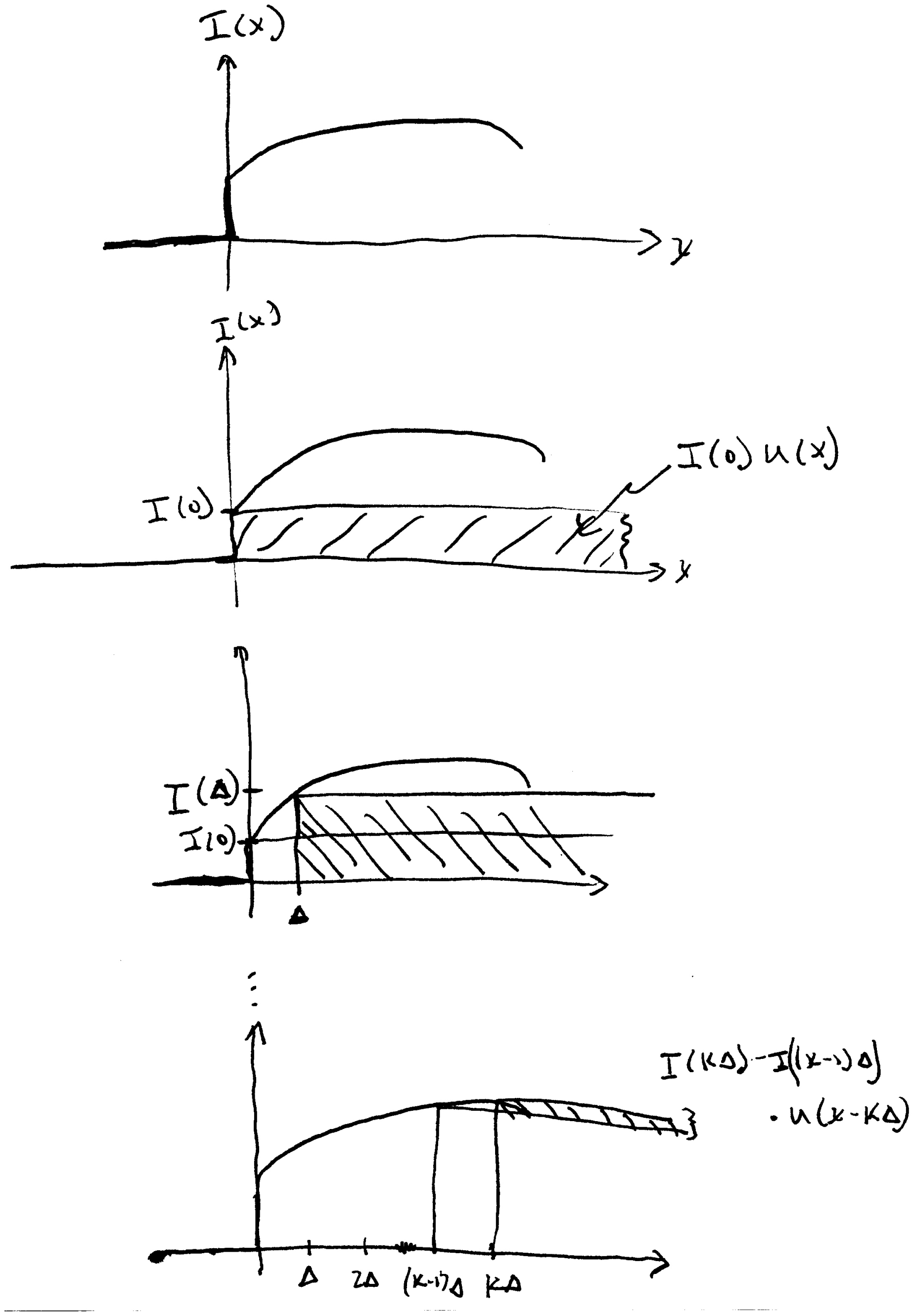

Figure 6.6: Approximation of a continuous function $I(x)$ by a sequence of step functions. See text.

Now, introduce a limiting process in which the Δ steps become vanishingly close, so that the "dummy" continuous variable τ can be introduced. Then $\Delta \to d\tau$, $k\Delta \to \tau$, the sum becomes an integral and we have:

$$\begin{aligned} \hat{O}(x) &\to O(x) && (6.16) \\ &= I(0)S(x) + \int_{0^+}^{\infty} \frac{dI(\tau)}{d\tau} S(x-\tau)d\tau. && (6.17) \end{aligned}$$

Note that the construction we used to draw Fig. 6.6 enters explicitly here, because we started with an $I(x)$ that may have been discontinuous at $x = 0$. (That is why we need a term at 0 and the integral starts at 0^+.) Going through the above again for a continuous input $I(x)$ starting at $\chi > -\infty$ and such that $I(\chi)S(x-\chi) \to 0$ as $\chi \to -\infty$ yields:

$$\begin{aligned} O(x) &= \int_{-\infty}^{\infty} \frac{dI(\tau)}{d\tau} S(x-\tau)d\tau && (6.18) \\ &= \int_{-\infty}^{\infty} I(\tau) \frac{dS(x-\tau)}{d\tau} d\tau && (6.19) \\ &= \int_{-\infty}^{\infty} I(\tau)h(x-\tau)d\tau && (6.20) \end{aligned}$$

Note that this shifting around of the order is a powerful consequence of linearity; we shall exploit this again in a couple of lectures.

Thus *the output of a linear network can be computed as an integral function of the impulse response.* The above construction illustrates an example in which an integral arises from an infinite number of sums. Such an integral is called a *convolution integral*, and the above expression is often abbreviated:

$$O(x) = I(x) * h(x).$$

6.2 Linear Filtering: Blurring and Sharpening

The convolution integral is a complicated expression, and it helps to step through the stages in evaluating it. These are illustrated in Fig. 6.7. The impulse response for our system $h(x)$ is just a rectangular-shaped function, and we take a very simple square shaped function as input.

The first step is to write these in terms of the dummy variable τ and then to "flip" h around the vertical axis and slide it completely off to the "left." The value of the convolution integral is the "overlap" area as h is moved back to the right.

(It's helpful to think of the square box as two edges facing "away" from one another.) *We immediately notice that for this system something very different from*

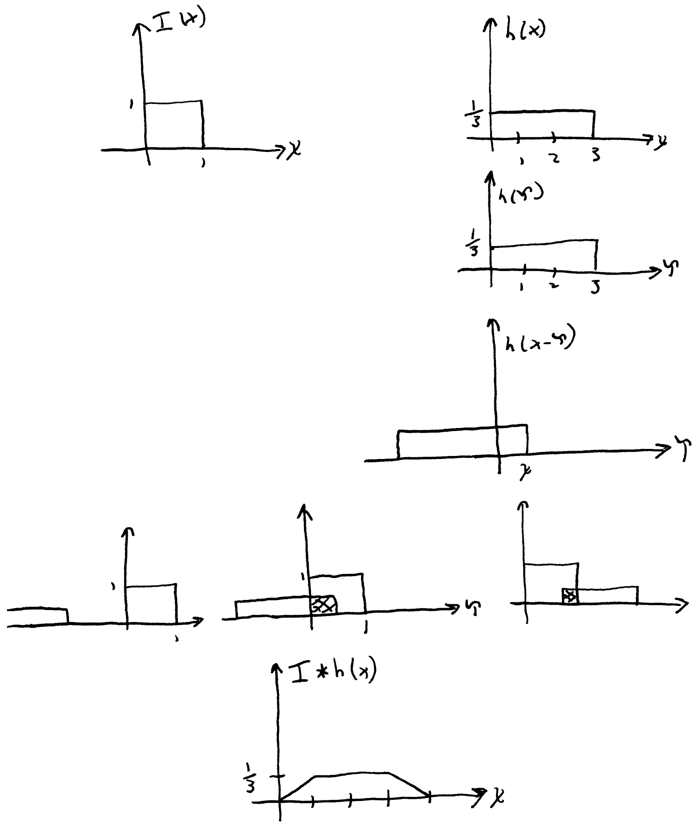

Figure 6.7: Stages in the graphical evaluation of the convolution integral. We start with an image, $I(x)$ that resembles a bright spot on a line, and an impulse response that spreads this out over 3 positions. We first rewrite $h(x)$ as $h(\tau)$, our 'dummy variable, then flip it and shift it across the τ axis. The result of the convolution is 0 except where the impulse response overlaps with non-zero values for the image. Notice how the input is spread out over position and lowered in amplitude; this is a blurring operation. In general, the image will be non-zero at many locations, so we'll have a superposition of this type of effect.

Limulus l. seems to be taking place: the input is "spread out" over position and reduced in amplitude. One might even say it has been blurred or smoothed. This is in direct contrast to Limulus l., where the input was sharpened at places of change.

Filtering is the operation of modifying one image (the input image) into another (output) image; we can represent this smoothing filter by the diagram in Fig. 6.8. This is why we called this lecture "image processing:" – we're studying linear maps between images.

6.2.1 Composition of Filters

The blurring and the sharpening filters are in some basic sense opposites of each other; what sharpening does by adding emphasis the blurring operation undoes by spreading the emphasis out. Now, here's something to think about: do blurring and sharpening have any relationship to integration and differentiation? (Remember, these also were like inverses of one another.)

Since filters map images to images, we can think of combining them like in algebra. In particular, different filters of this sort can be composed with one another, so that the output of one is taken as the input to the next:

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blur ( sharpen (I(x))) (x)
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What do you expect this to yield? Note: if perfect, then it should give an identity operation. Does it?

Also, does the order matter? Is:

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blur ( sharpen (I(x))) (x)  =?  sharpen ( blur (I(x))) (x)  =? identity
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The process can be seen as connected layers of the neural network. See Fig. 6.9.

Continuous vs. Discrete Evaluation of the Convolution Integral

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The slopes that appeared in the continuous evaluation of the convolution integral do not emerge in when the discrete circuit is used. This highlights the difference between the continuous mathematical view, with integrals and δ functions, and the finite, discrete view with summations and square pulses.

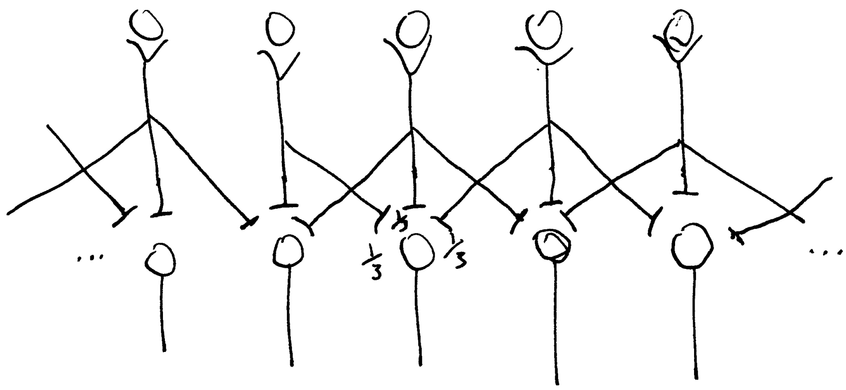

Figure 6.8: A variation of the lateral plexus network to implement a smoothing or blurring filter. Notice that all synaptic “weights” are now positive and sum to 1. What does this do to the “energy” in the signal?

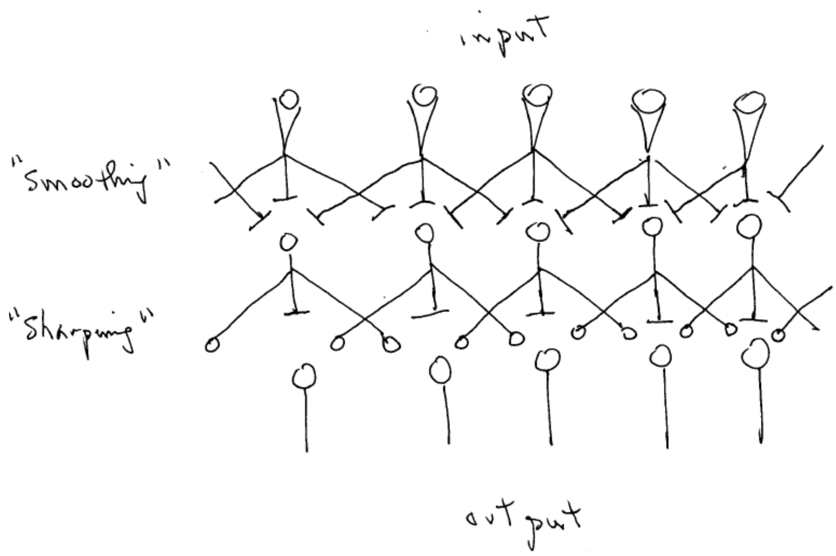

Figure 6.9: A network to compose the sharpening operation with the blurring operation. Notice how the inhibitory synapses “counter” the excitatory synapses from the previous stage. Do they cancel?

Example: RC Circuits

In engineering the most common examples are RC circuits, or circuits consisting of resistors and capacitors. These are also the classical (passive) models of neural membranes (recall lecture 3). Can you guess the circuit for which we have the following step response?

$$S(t) = (1 - e^{-t/RC})u(t)$$

Can you guess its impulse response?

6.3 Vectors, Matrices, and Linear Algebra

Looking back at the input/output diagram of the linear system L, it is natural to think of L as a map from inputs to outputs such that, when we graph this function it forms a straight line. This can be easily checked for numbers.

Pairs of numbers can be thought of as points in the plane, or as an arrow from the origin to this point, as we saw when we first used vectors in Sec. 2.3.2. Triples of numbers can be thought of a points in space. This immediately raises a whole series of questions: how do we combine vectors in any dimension, how do we transform vectors, what do we mean by dimension? The answers are given by LINEAR ALGEBRA, a few points of which we now review.

Vectors are columns of numbers:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} , \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{6.21}$$

that can be added together:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \tag{6.22}$$

and can be multiplied by a scalar α:

$$\alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} . \tag{6.23}$$

A LINEAR COMBINATION of vectors $\mathbf{v}$ and $\mathbf{w}$ is given by the sum $\alpha\mathbf{v} + \beta\mathbf{w}$, where α, β are both scalars. If $\mathbf{v}$ and $\mathbf{w}$ are chosen properly, e.g. as the orthogonal unit vectors i and j in sec:vector-anal, and if α, β take on all values, then the linear combination fills the plane.

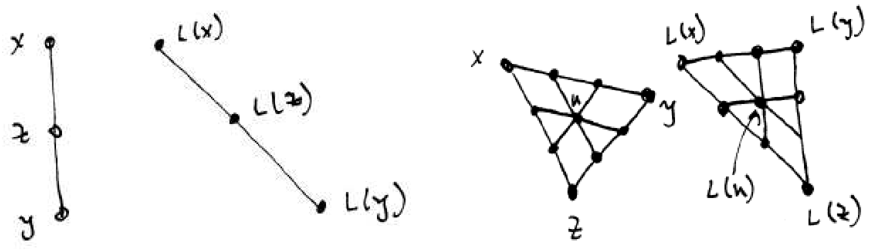

Figure 6.10: A linear transformation takes points in the plane to other points so that linearity and spacing are maintained. Notice how the central point along the line is transformed to a central point along the transformed line. Ditto for the triangle.

Matrices can be built from vectors. Following the example from Strang, take the three vectors:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{6.24}$$

and form their linear combinations:

$$c\mathbf{u} + d\mathbf{v} + e\mathbf{w} = c \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \tag{6.25}$$

Bundling the three vectors into a MATRIX gives us the same result:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix} \tag{6.26}$$

or, in general terms, represents the linear system of equations $\mathbf{Ax} = \mathbf{b}$;

$$\begin{aligned} x_1 &= b_1 \\ -x_1 + x_2 &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

If you use the standard way of thinking about matrix-vector multiplication, which is to take the row - column product as learned in high school, you get the same result.

We think of the matrix as OPERATING on the vector. If the vector represents a point, then the matrix transforms it to another point. If many points are arranged along a line, operating on all of them with the same matrix transforms it to another line. Triangles go to triangles, although they may be skewed. See Fig. 6.10.

The solution for this system of linear equations is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b-1+b_2 \\ b_1+b_2+b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{6.27}$$

Notice, most importantly, how the matrix on the right is the inverse of **A**:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Identity Matrix}}. \tag{6.28}$$

The matrix on the far right (past the = sign) is very special – it's the IDENTITY MATRIX.

The **A** matrix has a very special structure, with 1's along the diagonal and -1's below it; it is a first difference matrix and hence is related to (a discrete approximation to) the derivative. It resembles ("half") of a matrix version of our lateral inhibitory matrix for *Limulus linearis:*

$$\begin{bmatrix} \ddots & & & & & \\ -.5 & 1 & -.5 & 0 & 0 & 0 \\ 0 & -.5 & 1 & -.5 & 0 & 0 \\ 0 & 0 & -.5 & 1 & -.5 & 0 \\ 1 & -1 & 0 & -.5 & 1 & -.5 \\ & & & & & \ddots \end{bmatrix} \tag{6.29}$$

where the ellipsis $\ddots$ denotes that we're only looking at a portion of our matrix (to avoid "end effects"). This, we may now surmise, is a second-difference matrix, and is related to the second derivative.

6.4 Receptive Fields

Neural networks can be explored in two ways: (i) by shining a spot (impulse) of light and checking the output at different places; and (ii) moving the light around and checking the output at a single place (axon). For the networks we have considered these are mostly equivalent.

In neurobiology it is common to use (ii) more directly. The RECEPTIVE FIELD of a neuron is the locus of (image, or input) positions that effect the output, for a given type of stimulus.

For the network of *Limulus l.* the receptive field has two domains: an EXCITATORY center and an INHIBITORY surround. In 2-D this looks like a "Mexican hat" function; see fig. 6.11. For the blurring filter, the receptive field is purely excitatory and has a single-lobed shape, more like a Gaussian mountain. Viewing this as a convolution is shown in Fig. 6.12. The above concepts will become fundamental.

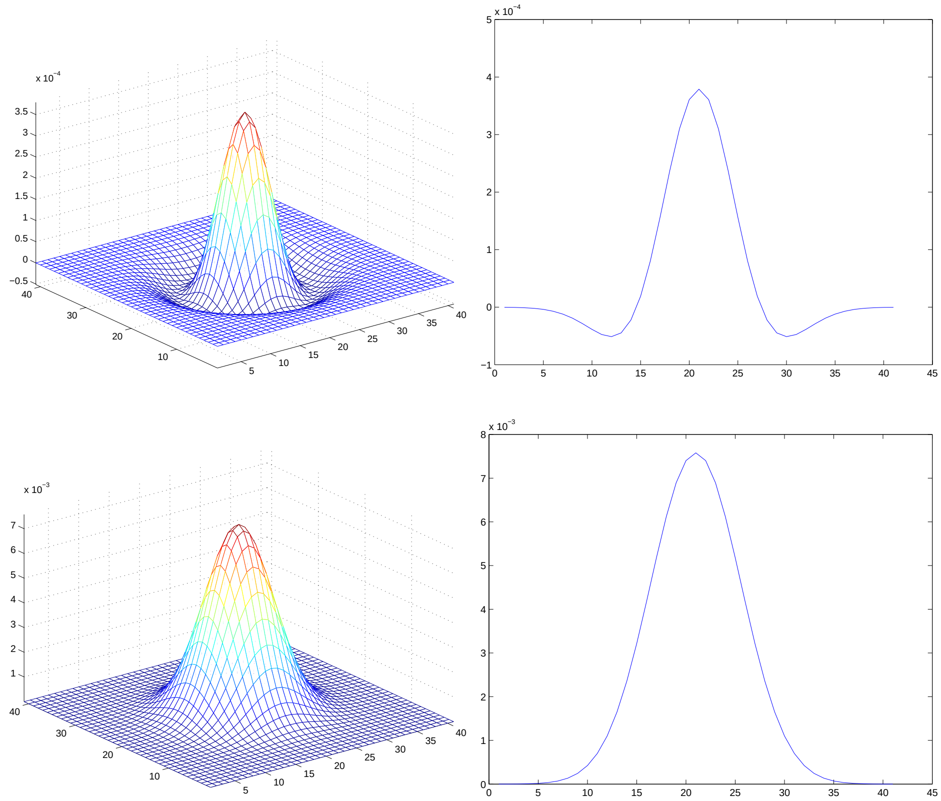

Figure 6.11: Versions of the two filters discussed in this lecture. (TOP) "Mexican hat" filter with an excitatory center and an inhibitory surround. This is the receptive field for an output neuron in the lateral plexus *Limulus l.*. (BOTTOM) Gaussian blurring filter; a smooth version of the "box" filter developed in the text. Note how the Mexican hat can be viewed as the difference between two Gaussians (with different spatial extents).

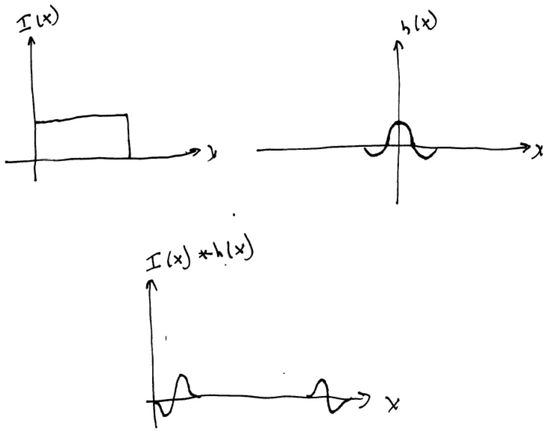

Figure 6.12: Convolution of a "Mexican hat" filter with an excitatory center and an inhibitory surround against an image that is a bright bar. How would you describe the result? Note: not to scale.

6.5 Convolution, Lenses, and Photography

Lenses and geometrical optics indicate when light rays are focussed on a plane; see fig. 6.13. If you think of this plane as the plane of the film in a camera (or a CCD array in a digital camera) and imagine it is displaced then a point of light is no longer focussed. Rather it is "blurred" into a circle. See Fig. 6.14.

6.5.1 Blur as an Abstraction

It is interesting to compare the physical causes of blur in a compound eye with the neural cause of blur in the above example. This shows the importance of taking an abstract view. The key idea of a sharp image is that the rays from a single point in the scene are collected to a single point in the image. When this fails, either because we move the image plane or we do not focus the image or the ommatidia are too wide, then in effect we are "adding" rays together from different points on the surface.

In the neural analog of this, we "add" light-based signals from nearby ommatidia together. Thus the causes differ at a physical or neural level, but in an abstract sense the effect is the same. Abstraction has therefore provided a common explanation for both physical phenomena, so that what is learned about one can, by analogy, be carried to the other via abstraction.

With this said, we can now think of a neural system to counter the effects of a physical system? Does the lateral plexus undo the effects of the poor quality Limulus eye?

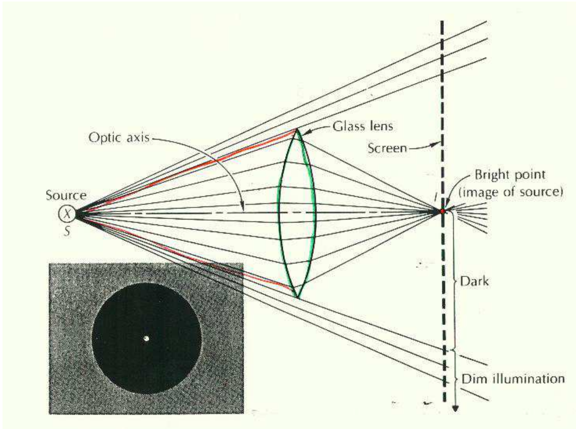

Figure 6.13: Geometric optics and image formation. Need short description from Feynman here of how optics works (I-26). (Image from www.cogsci.bme.hu/ ikovacs/SandP/prepII_2_files/d_fig7.jpg)

6.6 Summary

The main point of this lecture was the principle of superposition, and the manner in which it leads to the convolution integral. This is one way to view filters, or operators that modify images. They can smooth structure, which might be relevant to removing noise; and they can sharpen structure, which raises contrast and emphasizes change.

In the process we reviewed basic linear algebra, and now close with two observations. First: when is it the case that the response of a linear system to a sum of inputs can be obtained from the sum of responses to them individually. The answer is when they can be arranged as a matrix.

Note: a shift is not linear for the transformations we're working with now. Let's revisit the question raised at the opening of this lecture. Consider the map T that adds a constant vector $\mathbf{a}$ to another vector. Then $T(\mathbf{v}) = \mathbf{v}+\mathbf{a}$ and $T(\mathbf{w}) = \mathbf{w}+\mathbf{a}$ but $T(\mathbf{v}+\mathbf{w}) \neq T(\mathbf{v})+T(\mathbf{w})$ unless $\mathbf{a} = \mathbf{0}$. (Our example worked because the coefficients were balanced.)

Since filters can sharpen structure, they could be relevant to finding edges. But what does *Limulus p.* actually use this machinery for?

Why would one want to do image sharpening? Image blurring? When is this possible?

We shall address the above two questions in the next lecture.

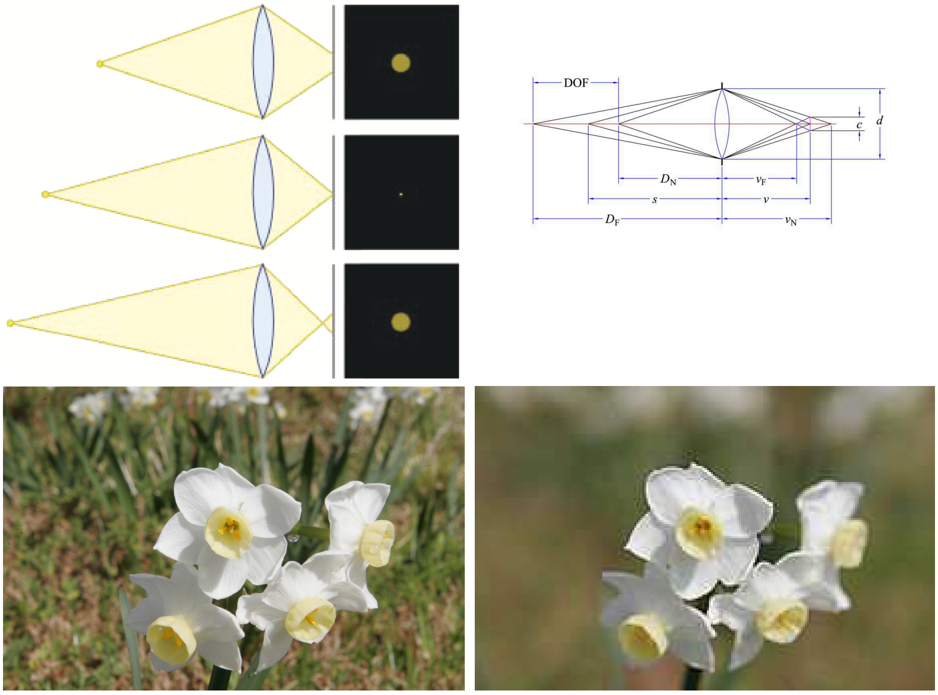

Figure 6.14: Depth of field and blur in a camera. (TOP) Blur arises in cameras when the image plane is not in the correct location. By geometric optics, this corresponds to points at the wrong depth from the lens. The effect of the wrong plane, either too close or too far from the lens results in the point of light being blurred into a smear of some diameter. Often the blur is Gaussian in shape. (BOTTOM) Two examples of images taken at different aperatures of the camera. Left f32; right f5. Notice how the background has been blurred by the large aperature but remains in focus for the small one. This blur can be modelled as a convolution. For these examples, what determines the size of the blurring filter? (Images from Wikimedia)

6.7 Notes

reference Frederick, D., and Carlson, A., *Linear Systems in Communication and Control*, Wiley, New York, 1971.

Lighthill, J., Fourier Analysis and Generalized Functions, Cambridge UP 1958.

There is an interesting literature on the image forming properties of the Limulus eye and their relationship with actual physical devices (as well as fossils): see M. F. LAND The optical mechanism of the eye of Limulus Nature 280, 396 - 397 (02 August 1979); doi:10.1038/280396a0

Here's his abstract:

> THE compound eyes of many marine and some terrestrial arthropods have smooth surfaces with no convex curvature to the facets. This means that image formation by ordinary spherical refraction is impossible. Nevertheless, apposition eyes of this kind form images behind each surface facet1,2 (Fig. 1b) and the question has repeatedly arisen as to how these images are produced. Exner chose the king crab, Limulus, as providing the definitive example of such an eye, and from a mixture of observation and theory came up with the idea of a lens-cylinder, a flat-ended optical device in which the refractive index decreases from the central axis to the periphery in a roughly parabolic manner1. This arrangement will form an image (Fig. 2a), and devices of this kind have actually been manufactured3,4. Exner's theory has long been considered to be the correct account of image formation in this type of eye5. However, Levi-Setti, Park and Winston2 have recently produced a fundamentally different explanation of image formation in Limulus eyes (Fig. 2b). This was based on the idea that each optical element, or crystalline cone, concentrated light by reflection, not refraction, and that it was the shape of the crystalline cone's reflective surface, and not its internal refractive index gradient, that led to the image-forming properties of the structure. This principle had already been used in another optical device, the 'ideal light collector' (ref. 6) which was invented as a tool for concentrating faint radiation. In this report I discuss the two suggested optical mechanisms for the eye of Limulus, and present evidence from interference microscopy that Exner's theory is probably the correct one.

Exner, S. Die Physiologie der facettierten Augen von Krebsen and Insecten (Leipzig, Deuticke, 1891).

Levi-Setti, R., Park, D. A. & Winston, R. Nature 253, 115116 (1975).

The linear algebra material comes from G. Strang, Introduction to Linear Algebra, 4-th edition, Wellesley-Cambridge Press, 2009. Strang's crisp and direct presentation of this material should not be missed.