Assignment0 Solutions

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1 Question 1

Oddtown

Let the number of people in the town be n. Define vectors $v_1, v_2, \ldots.., v_m$ in $\mathbb{F}_2$ such that these are the incidence vectors of the clubs $C_1, C_2, \ldots..C_m$.

This means that v_i has entry $j = 1$ of $j \in C_i$ and 0 otherwise.

$$\therefore v_i \cdot v_j = \sum_{k=1}^{n} (v_i)_k (v_j)_k = |C_i \cap C_j| \tag{1}$$

Hence, for $i \neq j, v_i \cdot v_j = 0$ (since the intersections of the clubs are even) and $v_i \cdot v_i = 1$ (since the size of clubs is odd).

Assume $\sum_{k=1}^{n} a_k \cdot v_k = 0$. Taking dot product with each v_i,

$$v_i \cdot \sum_{k=1}^{n} a_k \cdot v_k = 0 \implies a_i v_i \cdot v_i = 0 \implies a_i = 0 \tag{2}$$

$\therefore$ all a_i are zero and hence, $v_1, v_2, \ldots..v_m$ are linearly independent.Therefore, dimension of vector space V spanned by m linearly independent vectors is m. $\therefore m = n$ and maximum number of clubs is m.

Eventown

Using the same framework used in the Oddtwon problem, here, $v_i \cdot v_j = 0$ for all ij since each club has even number of members and the intersections are also even.

Say, V is a vector space spanned by $v_1, v_2, \ldots.v_n$, then any vectors $a = a_1 v_1 + a_2 v_2 + \ldots..a_n v_n$ and $a = a_1 v_1 + a_2 v_2 + \ldots..a_n v_n$ in V, $a \cdot b = 0$. This implies that V is a subspace of $V^\perp$, since the same results holds for some a in V and some b in $V^\perp$.

$$dimV^\perp \geq dimV \implies 2dimV \leq dimV + dimV^\perp = n \implies dimV \leq n/2 \tag{3}$$

$\implies V$ has at most $2^{(n/2)}$ elements. Hence, a maximum of $2^{(n/2)}$ clubs can be formed.

2 Question 2

For forward implication, if A is Hermitian,

$$\langle x, Ax\rangle = \langle A^{\dagger}x, x\rangle = \langle Ax, x\rangle. \tag{4}$$

For backward implication,

$$\langle x, Ax\rangle = \langle Ax, x\rangle \implies \langle A^{\dagger}x, x\rangle = \langle Ax, x\rangle \tag{5}$$

If V is finite dimensional with dimension n, then V is isomorphic to R^n.
In R^n, for any vector u,

$$\langle Bu, u\rangle = \langle Au, u\rangle \implies A = B. \tag{6}$$

This is because,

$$U^{\dagger}B^{\dagger}U = U^{\dagger}A^{\dagger}U \implies U^{\dagger}(B^{\dagger} - A^{\dagger})U = 0 \implies A = B. \tag{7}$$

Since V is isomorphic to R^n, the result is extendable to V as well. Hence, $A^{\dagger} = A$ and A is Hermitian.

3 Question 3

For any two matrices A and B with eigenvalues $\lambda and \mu$ corresponding to eigenvectors $x and z$ respectively, $x \otimes z$ are the eigenvectors of $A \otimes B$ corresponding to eigenvalues $\lambda\mu$. This can be shown as,

$$(A \otimes B)(x \otimes z) = Ax \otimes Bz = \lambda\mu(x \otimes z) \tag{8}$$

if $A and B$ are diagonalizable, then all eigenvectors of $A \otimes B$ correspond to $x \otimes z$.

(a) $\begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The eigenvalues of $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} are \frac{7 \pm \sqrt{57}}{2}$.

The eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} are \pm 1$.

Therefore, the eigenvalues of the given matrix are $\pm(\frac{7+\sqrt{57}}{2}), \pm(\frac{7-\sqrt{57}}{2})$.

(b) $\begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$

The eigenvalues of $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} are \frac{7 \pm \sqrt{57}}{2}$.

The eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} are \pm 1$.

Therefore, the eigenvalues of the given matrix are $\pm(\frac{7+\sqrt{57}}{2}), \pm(\frac{7-\sqrt{57}}{2})$.

(c) $\begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 12 & 10 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$

The eigenvalues of $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} are \frac{7 \pm \sqrt{57}}{2}$.

Therefore, the eigenvalues of the given matrix are $\pm(\frac{53+7\sqrt{57}}{2}), \pm(\frac{53-7\sqrt{57}}{2}), -2$.

4 Question 4

A norm on a vector space X is a real valued function $p : X \to R$ such that,

1. $p(x+y) \leq p(x) + p(y)$
2. $p(sx) = |s|p(x)$
3. If $p(x) = 0$, then $x = 0$
4. $p(x) \geq 0$

$\sqrt{\langle x|x \rangle}$ is a valid norm since it satisfies all the four conditions.

1. $\langle x|y \rangle \leq \sqrt{\langle x|x \rangle \langle y|y \rangle}$
   $\langle x|x \rangle + \langle y|y \rangle + 2\langle x|y \rangle \leq \langle x|x \rangle + \langle y|y \rangle + 2\sqrt{\langle x|x \rangle \langle y|y \rangle}$
   $\sqrt{\langle x+y|x+y \rangle} \leq sqrt\langle x|x \rangle + sqrt\langle y|y \rangle$
2. $sqrt\langle sx|sx \rangle = ssqrt\langle x|x \rangle$
3. Trivial.
4. Trivial.

A metric is a function $d : M \times M \to R$ such that,

1. $d(x,x) = 0$
2. $d(x,y) > 0$
3. $d(x,y) = d(y,x)$
4. $d(x,z) \leq d(x,y) + d(y,z)$

$|x-y|$ quite evidently satisfies all these conditions.

If $n > m, \langle f_n|f_m \rangle = 1 + \frac{1}{2n} - \frac{m}{6n^2}$.

To show that f_n is Cauchy for any $\epsilon > 0$, there must exist an N such that for $n, m \geq N, |f_n(x) - f_m(x)| < \epsilon$ for any x.

$$|f_n(x) - f_m(x)| = \begin{cases} 0 & x \in [-1, 0] \\ (n-m)x & x \in [0, \frac{1}{n}] \\ (1-m)x & x \in [\frac{1}{n}, \frac{1}{m}] \\ 0 & x \in [\frac{1}{m}, 1] \end{cases} \tag{9}$$

Take n as ϵ. $(n-m)x, \epsilon$ and $(1-m) < (n-m)$, therefore, $(1-m)x < \epsilon$. Hence, f_n is a Cauchy sequence.

The sequence converges to

$$f(x) = \begin{cases} 1 & x \in [-1, 0] \\ 0 & x \in [0, 1] \end{cases} \tag{10}$$

which is not part of the vector space since it is discontinuous. Therefore, inner product space is not complete.

$\mathbb{C}^{\ltimes}$ is a complete inner product space as any element in $\mathbb{C}$ $x + \iota y$ can be represented as $(x, y) \in \mathbb{R}^{\nvDash}$ which is a complete inner product space. Therefore, complex space is complete.

Any finite dimensional vector space V over field $\mathbb{F}$ is isomorphic to $\mathbb{F}^{\ltimes}$ where $n = dim V$. Therefore, it can be said that any finite dimensional inner product space is complete and hence a Hilbert space.