ECE 476: Power System Analysis **Fall 2020**

Lecture 7: September 15

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- TRANSMISSION LINE MODELING (CONTINUED)
- ELECTROMAGNETISM INTERLUDE: MAXWELL'S LAWS

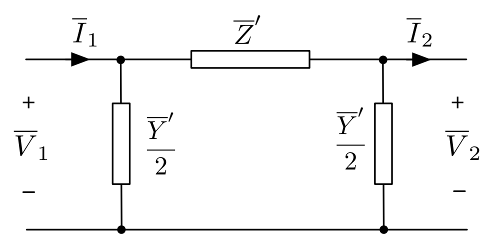


Figure 7.1: Transmission line π-equivalent circuit model.

Simplified Models

Recall that the series impedance and shunt admittance of the π-equivalent circuit model (see Fig. 7.1), denoted by $\overline{Z}'$ and $\overline{Y}'/2$, respectively, are given by

$$\begin{aligned}\overline{Z}' &= \overline{Z}\,\frac{\sinh(\gamma d)}{\gamma d},\\ \frac{\overline{Y}'}{2} &= \frac{\overline{Y}}{2}\,\frac{\tanh\left(\frac{\gamma d}{2}\right)}{\frac{\gamma d}{2}}.\end{aligned} \tag{7.1}$$

These expressions are exact for any d but they can be simplified as follows. First note that

$$\begin{aligned}\sinh(\gamma d) &= \frac{e^{\gamma d} - e^{-\gamma d}}{2},\\ \tanh\left(\frac{\gamma d}{2}\right) &= \frac{e^{\gamma d/2} - e^{-\gamma d/2}}{e^{\gamma d/2} + e^{-\gamma d/2}};\end{aligned} \tag{7.2}$$

then, for γd small enough, we have that $e^{\gamma d} = 1 + \gamma d$, $e^{-\gamma d} = 1 - \gamma d$, $e^{\gamma d} = 1 + \gamma d/2$, and $e^{-\gamma d} = 1 - \gamma d/2$; thus,

$$\frac{\sinh(\gamma d)}{\gamma d} \approx 1,$$
$$\frac{\tanh\left(\frac{\gamma d}{2}\right)}{\frac{\gamma d}{2}} \approx 1, \tag{7.3}$$

from where it follows that

$$\begin{aligned}\overline{Z}' &\approx \overline{Z} \\ &= \overline{z}d, \\ \frac{\overline{Y}'}{2} &\approx \frac{\overline{Y}}{2}; \\ &= \frac{\overline{y}\,d}{2}.\end{aligned} \tag{7.4}$$

Thus, the series impedance and shunt admittance elements in the π-model can be approximated the obvious way, i.e., the series impedance element can be approximated as the product of the transmission line per-unit length series impedance and the transmission line length, whereas the shunt admittance elements can be approximated as the product of the transmission line per-unit length shunt admittance and the transmission line half lenght. For typical values of $\overline{z}$ and $\overline{y}$, such approximation is valid when $d < 200$ miles. Furthremore, when $d < 50$ miles, the shunt admittance element can be completely neglected. The results above are summarized in Table 7.1.

Table 7.1: Parameters of the π-equivalent circuit model for different transmission line lengths.

length d	$\overline{Z}'$	$\frac{\overline{Y}'}{2}$
$d > 200$ miles	$\overline{Z}\frac{\sinh(\gamma d)}{\gamma d}$	$\frac{\overline{Y}}{2}\frac{\tanh(\frac{\gamma d}{2})}{\frac{\gamma d}{2}}$
50 miles$< d < 200$ miles	$\overline{Z}$	$\frac{\overline{Y}}{2}$
$d < 50$ miles	$\overline{Z}$	0

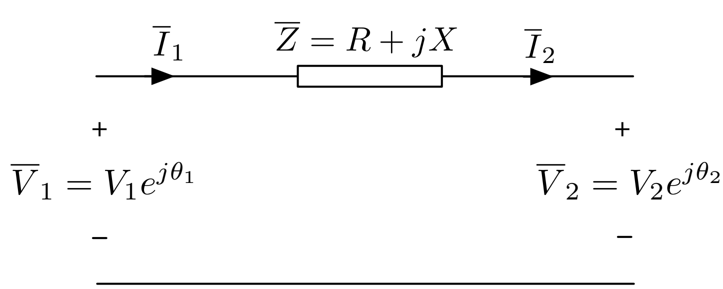


Figure 7.2: π-equivalent circuit model for a short transmission line

Power Transfer Through a Short Transmission Line

Now we will study the amount of active and reactive power that can be transferred from the sending end to the receiving end of a short transmission line, i.e., $d < 50$ miles; see Fig. 7.2 for the resulting π-equivalent circuit model. To this end, first, write the impedance of the transmission line, $\overline{Z} = R + jX$, in polar coordinates as follows:

$$\overline{Z} = Ze^{j\phi}, \tag{7.5}$$

where $Z = \sqrt{R^2 + X^2}$ and $\phi = \arctan(X/R)$. Let $\overline{S}_1$ denote the complex power that enters the transmission line via its sending end, then, we have that:

$$\begin{aligned}
\overline{S}_1 &= \overline{V}_1 \overline{I}_1^* \\
&= \overline{V}_1 \frac{\overline{V}_1^* - \overline{V}_2^*}{\overline{Z}^*} \\
&= V_1 e^{j\theta_1} \frac{V_1 e^{-j\theta_1} - V_2 e^{-j\theta_2}}{Z^{-j\phi}} \\
&= \frac{V_1^2}{Z} e^{j\phi} - \frac{V_1 V_2}{Z} e^{j(\theta_1 - \theta_2 + \phi)}.
\end{aligned} \tag{7.6}$$

Thus, the active and reactive power entering the transmission line via its sending end, denoted by P_1 and Q_1, respectively, are given by

$$\begin{aligned}
P_1 &= \frac{V_1^2}{Z}\cos(\phi) - \frac{V_1 V_2}{Z}\cos(\theta_1 - \theta_2 + \phi), \\
Q_1 &= \frac{V_1^2}{Z}\sin(\phi) - \frac{V_1 V_2}{Z}\sin(\theta_1 - \theta_2 + \phi).
\end{aligned} \tag{7.7}$$

Let $\overline{S}_2$ denote the complex power that enters the the transmission line via its receiving end; this can be computed as follows:

$$\begin{aligned}
\overline{S}_2 &= -\overline{V}_2 \overline{I}_2^* \\
&= -\overline{V}_2 \frac{\overline{V}_1^* - \overline{V}_2^*}{\overline{Z}^*} \\
&= -V_2 e^{j\theta_2} \frac{V_1 e^{-j\theta_1} - V_2 e^{-j\theta_2}}{Z^{-j\phi}} \\
&= \frac{V_2^2}{Z} e^{j\phi} - \frac{V_1 V_2}{Z} e^{j(\theta_2 - \theta_1 + \phi)}.
\end{aligned} \tag{7.8}$$

Thus, the active and reactive power entering the transmission line via its receiving end, denoted by P_2 and Q_2, respectively, are given by

$$\begin{aligned}
P_2 &= \frac{V_2^2}{Z}\cos(\phi) - \frac{V_1 V_2}{Z}\cos(\theta_2 - \theta_1 + \phi), \\
Q_2 &= \frac{V_2^2}{Z}\sin(\phi) - \frac{V_1 V_2}{Z}\sin(\theta_2 - \theta_1 + \phi).
\end{aligned} \tag{7.9}$$

Let $P_{\overline{Z}}$ and $Q_{\overline{Z}}$ respectively denote the active and reactive power consumed by the impedance $\overline{Z}$; then, since from the conservation of complex power principle studied earlier, we have that

$$\begin{aligned} P_{\overline{Z}} &= P_1 + P_2 \\ &= \frac{V_1^2 + V_2^2}{Z}\cos(\phi) - \frac{V_1 V_2}{Z}\big(\cos(\theta_2 - \theta_1 + \phi) + \cos(\theta_1 - \theta_2 + \phi)\big) \\ Q_{\overline{Z}} &= Q_1 + Q_2 \\ &= \frac{V_1^2 + V_2^2}{Z}\sin(\phi) - \frac{V_1 V_2}{Z}\big(\sin(\theta_2 - \theta_1 + \phi) + \sin(\theta_1 - \theta_2 + \phi)\big). \end{aligned} \tag{7.10}$$

If the line is lossless, i.e., $R = 0$, we have that $\overline{Z} = jX$; thus, $\phi = \pi/2$, therefore, the expressions in (7.7) simplify to

$$\begin{aligned} P_1 &= -\frac{V_1 V_2}{X}\cos(\theta_1 - \theta_2 + \pi/2) \\ &= \frac{V_1 V_2}{X}\sin(\theta_1 - \theta_2), \\ Q_1 &= \frac{V_1^2}{X} - \frac{V_1 V_2}{X}\sin(\theta_1 - \theta_2 + \pi/2) \\ &= \frac{V_1^2}{X} - \frac{V_1 V_2}{X}\cos(\theta_1 - \theta_2). \end{aligned} \tag{7.11}$$

Similarly, the expressions in (7.9) simplify to

$$\begin{aligned} P_2 &= \frac{V_1 V_2}{X}\sin(\theta_2 - \theta_1), \\ Q_2 &= \frac{V_2^2}{X} - \frac{V_1 V_2}{X}\cos(\theta_2 - \theta_1). \end{aligned} \tag{7.12}$$

Finally, the expressions in (7.10) simplify to

$$\begin{aligned} P_{\overline{Z}} &= 0, \\ Q_{\overline{Z}} &= \frac{V_1^2 + V_2^2}{X} - 2\frac{V_1 V_2}{X}\cos(\theta_1 - \theta_2). \end{aligned} \tag{7.13}$$

The fact that the calculation yields $P_{\overline{Z}} = 0$ is reassuring because we assumed the line to be lossless; therefore, it cannot dissipate any active power. The expression for $Q_{\overline{Z}}$ can also be easily obtained by noting that the reactive power consumed by $\overline{Z} = jX$ is given by

$$Q_{\overline{Z}} = \frac{V_{12}^2}{X}, \tag{7.14}$$

where

$$\begin{aligned} V_{12}^2 &= (\overline{V}_1 - \overline{V}_2)(\overline{V}_1^* - \overline{V}_2^*) \\ &= V_1^2 + V_2^2 - V_1 V_2 e^{j(\theta_1 - \theta_2)} - V_1 V_2 e^{j(\theta_2 - \theta_1)} \\ &= V_1^2 + V_2^2 - 2V_1 V_2 \cos(\theta_1 - \theta_2). \end{aligned} \tag{7.15}$$

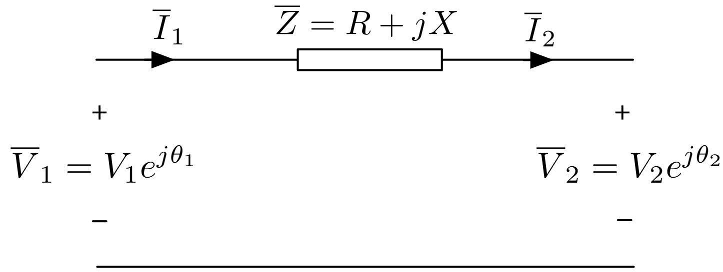


Figure 7.3: π-equivalent circuit model for a short transmission line

Lossless Transmission Lines and Surge Impedance Loading

When the ohmic losses are zero, we have that $\overline{z} = j\omega\ell$ and $\overline{y} = j\omega c$. Then, the propagation constant is given by

$$\begin{aligned}\gamma &= \sqrt{\overline{z}\,\overline{y}} \\ &= j\omega\sqrt{\ell c},\end{aligned} \tag{7.16}$$

i.e., it is a purely imaginary number, and the characteristic impedance is given by

$$\begin{aligned}\overline{Z}_c &= \sqrt{\frac{\overline{z}}{\overline{y}}} \\ &= \sqrt{\frac{j\omega\ell}{j\omega c}} \\ &= \sqrt{\frac{\ell}{c}},\end{aligned} \tag{7.17}$$

i.e., it is a real number, and it is typically refer to as the *surge impedance*.

$$\begin{aligned}\overline{Z}' &= \alpha\sinh(j\beta), \\ &= j\alpha\sin(\beta), \\ \frac{\overline{Y}'}{2} &=;\end{aligned} \tag{7.18}$$

see Fig. 7.3 for a graphical depiction. Let $\alpha = \sqrt{\frac{\ell}{c}}$ and $\beta = \omega\sqrt{\ell c}$; then, the tranmission line voltage-current terminal relations reduce to

$$\overline{V}_1 = \cos(\beta d)\overline{V}_2 + j\alpha\sin(\beta d)\overline{I}_2, \tag{7.19}$$

$$\overline{I}_1 = j\frac{1}{\alpha}\sin(\beta d)\overline{V}_2 + \cos(\beta d)\overline{I}_2. \tag{7.20}$$

Let us study now what happens when we load the receiving end of a transmission line with an impedance $\overline{Z}_l$ whose value is equal to that of the surge impedance, i.e., $\overline{Z}_l = \alpha$. Thus, in this case, the relation between $\overline{V}_2$ and $\overline{I}_2$ is given by

$$\begin{aligned} \overline{I}_2 &= \frac{\overline{V}_2}{\overline{Z}_l} \\ &= \frac{\overline{V}_2}{\alpha}, \end{aligned} \tag{7.21}$$

which by plugging into (7.19) yields

$$\begin{aligned} \overline{V}_1 &= \cos(\beta d)\overline{V}_2 + j\sin(\beta d)\overline{V}_2 \\ &= \big(\cos(\beta d) + j\sin(\beta d)\big)\overline{V}_2, \end{aligned} \tag{7.22}$$

from where it follows that

$$\overline{V}_2 = e^{-j\beta d}\overline{V}_1; \tag{7.23}$$

therefore,

$$V_2 = V_1. \tag{7.24}$$

In words, if we load a lossless transmission line loaded with its surge impedance, then the magnitude of the voltage at the receiving end of the line, V_2, is equal to the magnitude of the voltage at the sending end of the transmission line, V_1. Furthermore because the load at the receiving end of the transmission line is purely resistive (recall that the surge impedance is a real number), the active and reactive power consumed by it, P_l and Q_l, respectively, are

$$\begin{aligned} P_l &= \frac{V_2^2}{\alpha} \\ &= \frac{V_1^2}{\alpha}, \\ Q_l &= 0. \end{aligned} \tag{7.25}$$

The relation between the current in the sending and receiving ends can be obtained by using (7.20) and (7.21) as follows:

$$\begin{aligned} \overline{I}_1 &= j\frac{1}{\alpha}\sin(\beta d)\overline{V}_2 + \cos(\beta d)\overline{I}_2 \\ &= j\sin(\beta d)\overline{I}_2 + \cos(\beta d)\overline{I}_2, \end{aligned} \tag{7.26}$$

from where it follows that

$$\overline{I}_2 = e^{-j\beta d}\overline{I}_1; \tag{7.27}$$

therefore,

$$I_2 = I_1. \tag{7.28}$$

Thus, the magnitudes of the current at both sending and receiving ends are identical.

Let $\overline{S}_l$ denote the complex power delivered to the load at the receiving end of the transmission line; then, we have that

$$\begin{aligned}\overline{S}_l &= \overline{V}_2 \overline{I}_2^* \\ &= \overline{V}_1 \overline{I}_1^*,\end{aligned} \tag{7.29}$$

where the last line follows from (7.23) and (7.27). Now let $\overline{S}_1$ denote the complex power that enters the transmission line via the sending end; thus,

$$\overline{S}_1 = \overline{V}_1 \overline{I}_1^*. \tag{7.30}$$

Therefore, we have that

$$\overline{S}_1 = \overline{S}_l. \tag{7.31}$$

In words, the complex power consumed by the load is equal to the complex power entering the transmission line via its sending end. Furthermore, recall that the reactive power consumed by $\overline{Z}_l = \alpha$ is zero; therefore, $Q_1 = Q_l = 0$.

Electromagnetism Interlude: Maxwell's Laws

Maxwell's laws relate electric and magnetic fields to their sources, charge and current densities. They are important in power system modeling because they are the basis for obtaining transmission line per-unit length parameters as well as for developing transformer models. Next, we review two of these laws, namely Ampere's law, Gauss' law and Faraday's law and illustrate their application via examples.

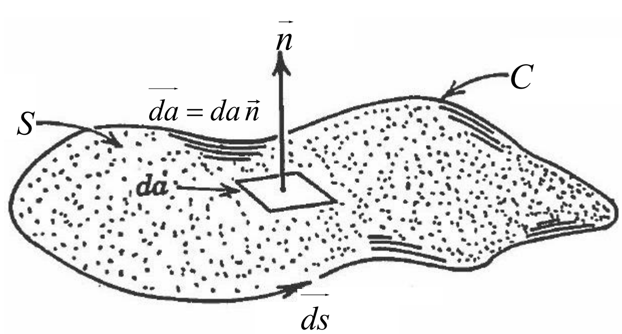


Figure 7.4: Surface Integral

Ampere's Law

Consider an open surface S enclosed by a contour C; see Fig. 7.4 for a graphical depiction. Let $\overrightarrow{ds}$ [m] denote a differential length vector tangent to the contour C and let $\overrightarrow{da}$ [m^2] denote a differential area vector normal to the surface S, with the direction of $\overrightarrow{ds}$ and $\overrightarrow{da}$ chosen according to the righthand rule. For example, choose the direction of $\overrightarrow{ds}$ as shown in Fig. 7.4. Then, place your right hand so that its fingers point in the direction of $\overrightarrow{ds}$. The positive direction of $\overrightarrow{da}$ is then determined by where the thumb is pointing to. Alternatively, you can place your right hand in such a way that the direction of $\overrightarrow{ds}$ is consistent with where the thumb is pointing to; then, where the fingers are pointing determines the positive direction of $\overrightarrow{da}$. Let $\overrightarrow{H}$ [A/m] and $\overrightarrow{E}$ [V/m] respectively denote the magnetic and electric field intensities at any point in space. Similarly, let $\overrightarrow{J}$ denote the current density at any point in space. Then, we have that

$$\oint_C \overrightarrow{H} \cdot \overrightarrow{ds} = \int_S \overrightarrow{J} \cdot \overrightarrow{da} + \frac{d}{dt} \int_S \epsilon \overrightarrow{E} \cdot \overrightarrow{da}, \tag{7.32}$$

with ϵ [F/m] denoting the permittivity of the medium ($\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$ if the medium is free space), and where the term $\epsilon\overrightarrow{E}$ is typically referred to as the electric displacement flux density. In words, the line integral of the magnetic field intensity around the contour C is equal to the net current passing through the surface S plus the rate at which the net electric displacement flux through the surface S changes with time.

Gauss' Law of Magnetic Field

The *magnetic flux density*, denoted by $\overrightarrow{B}$, is defined as

$$\overrightarrow{B} = \mu \overrightarrow{H}, \tag{7.33}$$

where μ denotes the permeability of the medium. The magnetic flux density is typically measured in Teslas [T] or Gauss [G], with the relation between them as follows: $1\,\mathrm{T} = 10^4\,\mathrm{G}$. It is typical to write μ in terms of the permeability of free space as follows:

$$\mu = \mu_0 \mu_r, \tag{7.34}$$

where μ_r denotes the relative permeability of the material (e.g., for magnetic materials, we have that $\mu_r \sim 10^3 - 10^4$). The net magnetic flux passing through any surface S, denoted by ϕ, can be then computed as follows

$$\phi = \int_S \overrightarrow{B} \cdot \overrightarrow{da} \tag{7.35}$$

where $\overrightarrow{da}$ denotes the differential area vector normal to S. Gauss' law of magnetic field states that the net magnetic flux through a closed surface S must be zero, i.e.,

$$\oint_S \vec{B} \cdot \overrightarrow{da} = 0, \tag{7.36}$$

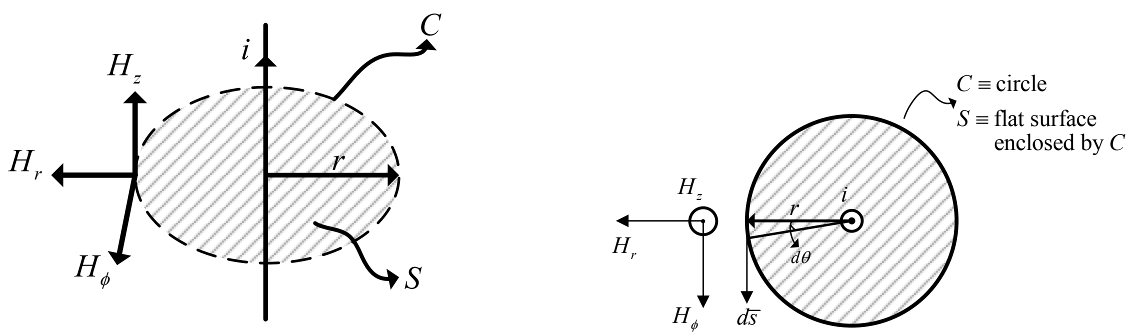
where $\overrightarrow{da}$ denotes the differential area vector normal to S.

Magnetostatic (MQS) Approximation of Ampere's Law

For the purpose of this course, and because we are dealing with low-frequency voltages and currents, e.g., 60 Hz, the term in (7.32) due to $\epsilon\vec{E}$ can be neglected. Thus, the expression in (7.32) reduces to

$$\oint_C \vec{H} \cdot \overrightarrow{ds} = \int_S \vec{J} \cdot \overrightarrow{da}; \tag{7.37}$$

this is referred to as the magnetostatic (MQS) approximation of Ampere's law. Next, we provide some examples on how to use the (MQS) approximation.



(a) Side view. (b) Top view.

Figure 7.5: single wire carrying current i and contour C (a circle with radius r) and surface S (a disk enclosed by C) used in the MQS approximation of Ampere's law for computing the azimuthal component of the magnetic field intensity.

Magnetic Field Created by a Current Flowing Along an Infinitely Long Wire

Consider an infinitely long wire carrying a current i [A]. We would like to characterize the magnetic field intensity, $\vec{H}$ for all points of space. Because of symmetry, it is convenient to choose a cylindrical coordinate system so that the wire is located along the z axis with the positive current flow coinciding with the positive direction of the z axis; see Fig. 7.5(a) for a graphical description. Let H_r, H_ϕ, and H_z respectively denote the radial, azimuthal, and vertical component of the vector $\overrightarrow{H}$. Because of symmetry, reversing the direction of the current does not reverse the direction of H_r; thus, we conclude that $H_r = 0$. Now, in order to compute H_ϕ, consider a disc S of radius r located on a plane perpendicular to the z axis and whose center is on the z axis; see Fig. 7.5(b) for a graphical description. Clearly, the contour C enclosing the disc S is a circle of radius r, also located on a plane perpendicular to the z axis and whose center is on the z axis. Then, the lefthand side of (7.37) can be computed as follows

$$\begin{aligned}\oint_C \overrightarrow{H} \cdot \overrightarrow{ds} &= \int_C H_\phi ds \\ &= \int_0^{2\pi} H_\phi r d\theta \\ &= 2\pi r H_\phi. \end{aligned} \tag{7.38}$$

Because in this case the wire crosses the surface S, we have that

$$\int_S \overrightarrow{J} \cdot \overrightarrow{da} = i; \tag{7.39}$$

thus, equating (7.38) and (7.39) yields

$$2\pi r H_\phi = i, \tag{7.40}$$

from where it follows that

$$H_\phi = \frac{i}{2\pi r}. \tag{7.41}$$

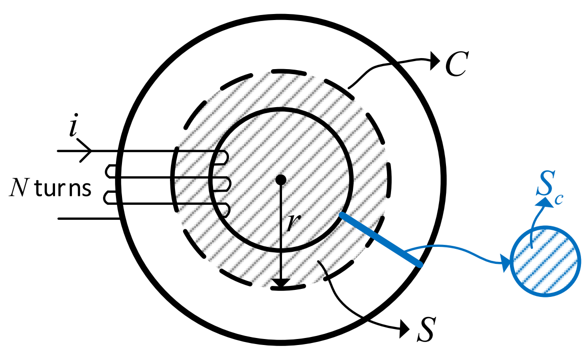


Figure 7.6: Toroidal magnetic Core. The contour C is a circle that passes through the center points of each cross-sectional surface (also a circle).

Magnetic Field Inside a Torus

Consider the setting in Fig. 7.6 comprised of a coil with N turns wound around a torus made of some magnetic material with magnetic permeability μ. The radius of revolution of the toroid is R and cross-sectional area is A. We would like to compute the magnetic field intensity, $\overrightarrow{H}$, at any point inside the torus. We assume that $\overrightarrow{H}$ is uniform throughout the core, i.e., it does not depend on the spatial coordinates; thus, when applying the MQS approximation to Ampere's law to obtain $\overrightarrow{H}$, we can use the contour C in the figure and the disk S enclosed by it. Note that the contour C is a circle whose points are the centers of the torus cross-sectional areas; thus, its radius, r, is equal to the distance between the torus geometric center and the center of any cross-sectional area. In this case, the lefthand side of the MQS approximation to Ampere's law is given by

$$\begin{aligned}\oint_C \overrightarrow{H} \cdot \overrightarrow{ds} &= \int_0^{2\pi} Hrd\theta \\ &= 2\pi r H \\ &= lH,\end{aligned}$$

where $l = 2\pi R$, whereas the righthand side is given by

$$\int_S \overrightarrow{J} \cdot \overrightarrow{da} = Ni,$$

(because the current i crosses N times the surface S), thus,

$$lH = Ni,$$

from where it follows that

$$H = \frac{Ni}{l}. \tag{7.42}$$

Magnetic Equivalent Circuits

The electromagnetic behavior of the torus example above can be conveniently captured by an equivalent magnetic circuit by using the notions of magnetomotive force and reluctance, as defined next. To this end, first recall the expression in (7.35) for calculating the net magnetic flux passing through a surface S:

$$\phi = \int_S \overrightarrow{B} \cdot \overrightarrow{da}. \tag{7.43}$$

If $\vec{B}$ is uniform over the surface S and perpendicular to it, then we have that

$$\begin{aligned}\phi &= \int_S \vec{B} \cdot \vec{da} \\ &= BA,\end{aligned}$$

where A is the area of the surface S. Now, by using the expression in (7.42), we can compute the magnetic flux density inside the toroid as follows

$$B_c = \mu \frac{Ni}{l}, \tag{7.44}$$

where μ denotes the permeability of the material of which the torus is made. Then, the magnetic flux across the torus cross section S_c with area A_c shown in Fig. 7.6 is given by

$$\begin{aligned}\phi_c &= B_c A_c \\ &= \frac{\mu A_c}{l} Ni.\end{aligned} \tag{7.45}$$

Now define $\mathcal{F} := Ni$, which we refer to as magnetomotive force (mmf) and $\mathcal{R} := \frac{l}{\mu A_c}$ as the reluctance of the torus; then, we can rewrite (7.45) as follows:

$$\mathcal{F} = \mathcal{R} \phi_c; \tag{7.46}$$

Note that the relation above resembles Ohm's law for DC circuits if one thinks of the mmf, $\mathcal{F}$, as the analogous quantity to a voltage across a resistor and the reluctance, $\mathcal{R}$, as the analogous quantity to a resistance. For this reason, it is convenient to graphically describe the torus-winding system by the equivalent magnetic circuit model in Fig. 7.7

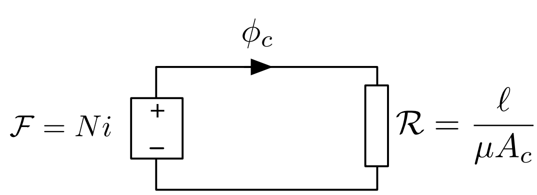


Figure 7.7: Equivalent magnetic circuit model for torus-winding setting.