K-Nearest Neighbors

K Sri Rama Murty

IIT Hyderabad

ksrm@ee.iith.ac.in

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$$d_n = ||\mathbf{x}_n - \mathbf{x}_0||_p \qquad\qquad d_n = 1 - \frac{\mathbf{x}_n^T \mathbf{x}_0}{||\mathbf{x}_n|| \;\; ||\mathbf{x}_0||}$$

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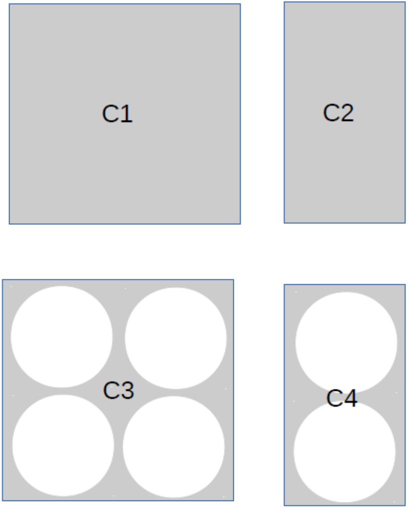
  - Assign output label y_0 for test point $\mathbf{x}_0$ from labels of y_{NN} - Voting

Geometry of KNN Decision Boundaries (k=1)



Steel-plate classification

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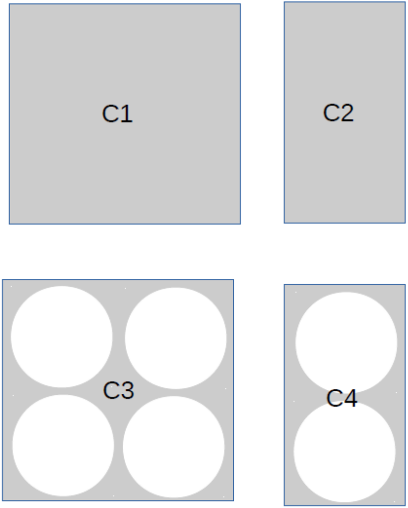

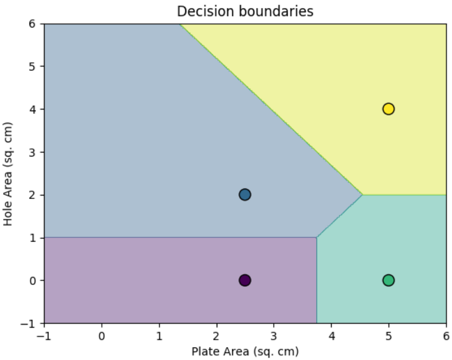

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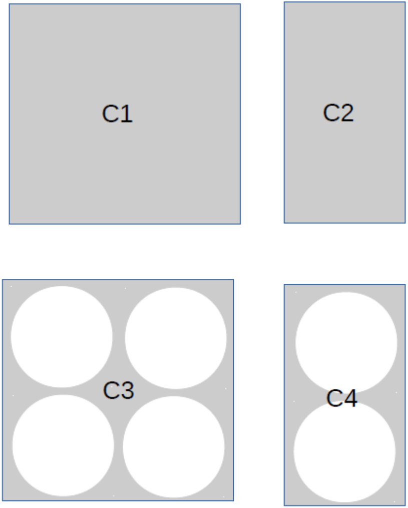

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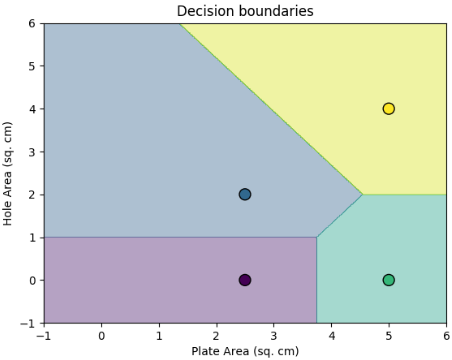
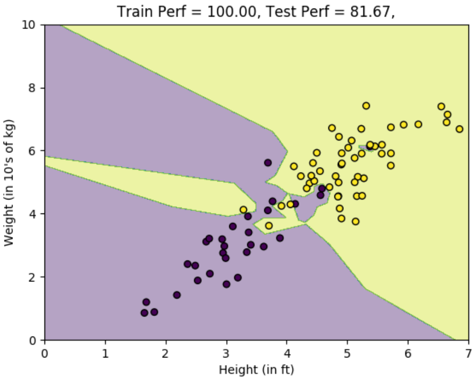


Decision boundaries are perpendicular bisectors of the lines joining nearest competitors!

KNN Classifier (Kid vs Adult) $k = 1$



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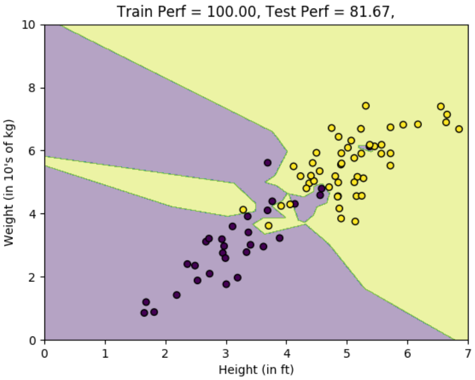

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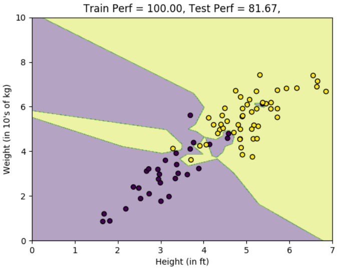
- 50 points from adult-class and 30 points from kid-class
- Big difference between train and test performances ($k = 1$ is a overfit)
- In a KNN-classifier, the decision regions are typically nonconvex

Effect of K on Performance



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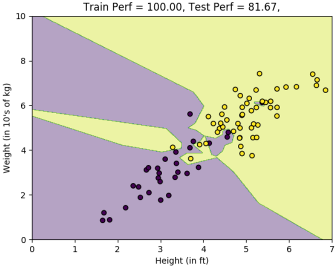

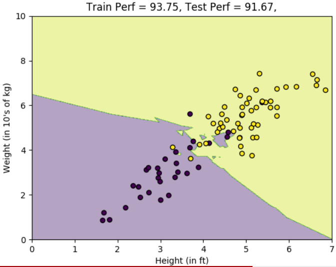

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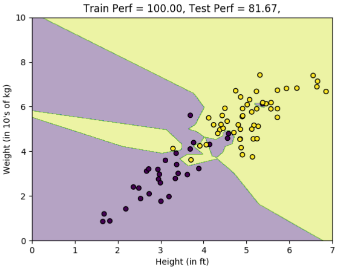

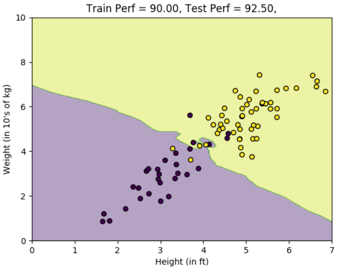

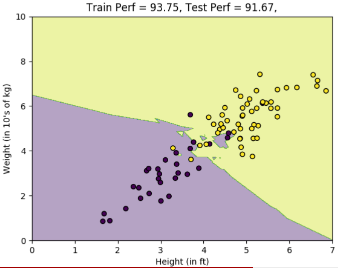

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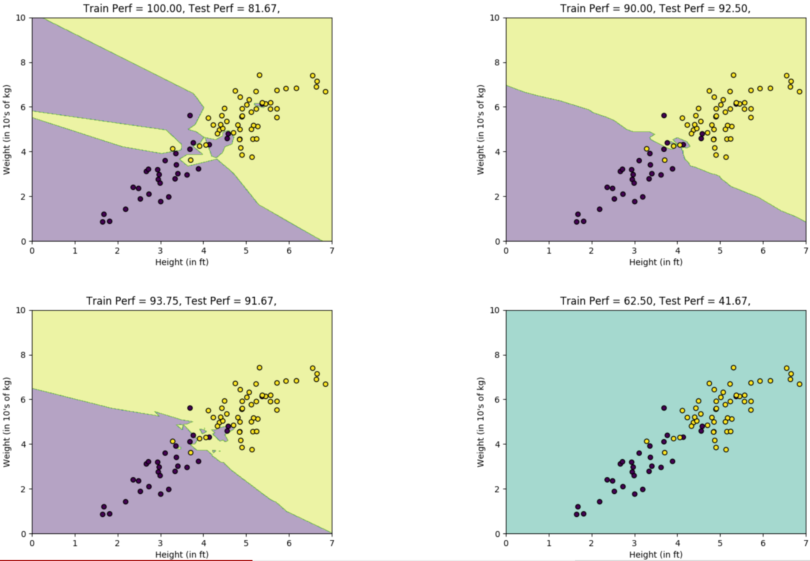
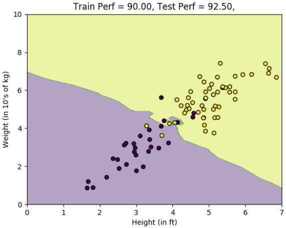


Why not Weight in kg



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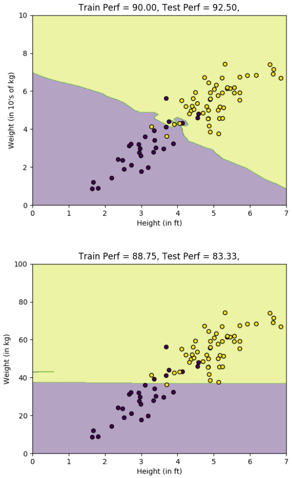

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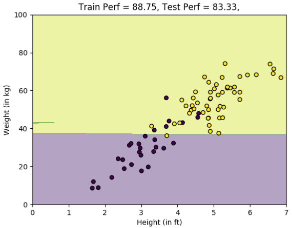

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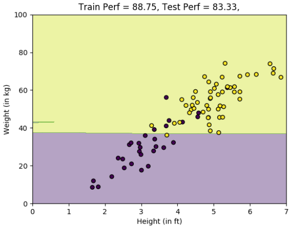

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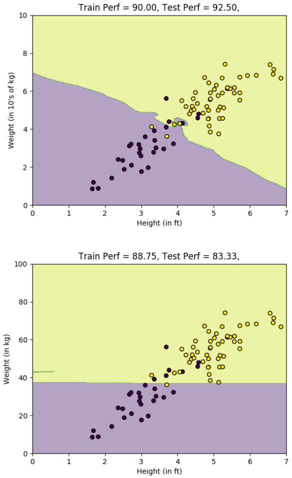

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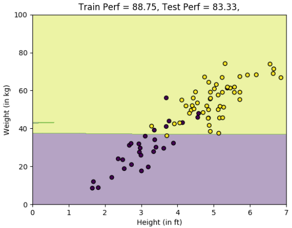

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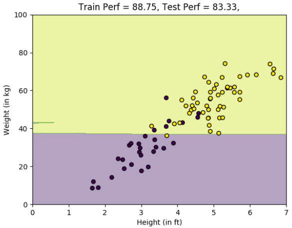

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- This process is called *data whitening*

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x	5.2	4.8	5.8	5.7	5.4	5.1	4.9	5.3	5.9	4.5	5.6
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- Assign higher weightage to closer neighbors

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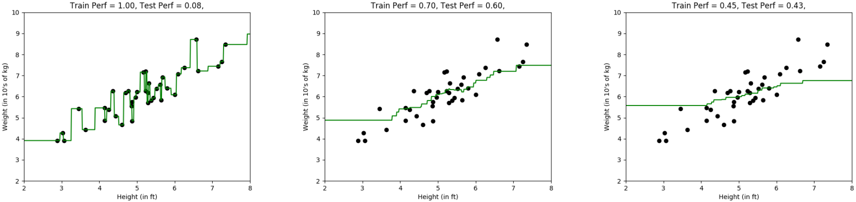
- Number of neighbors k is not very critical.
- Similarity measure $k(x_n, x_t)$ is an important design choice

KNN-Regression Illustration

Uniform weights for the neighbors



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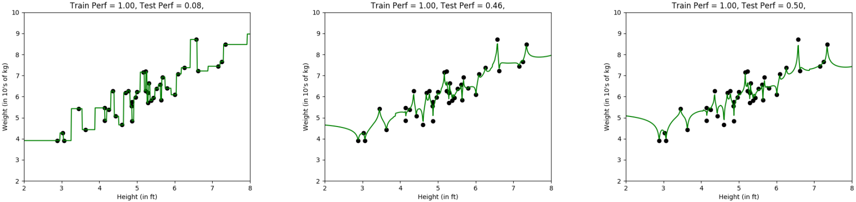
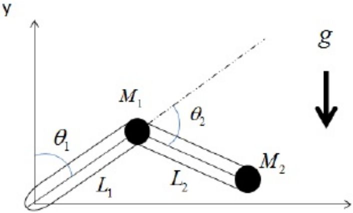


Robotic Arm Control - ME Approach



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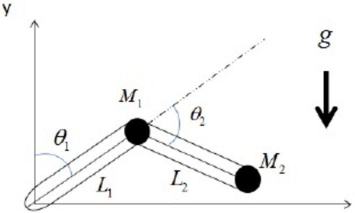

Solving for the forces:

$$F_{\theta_1} = ((M_1+M_2)L_1^2 + M_2L_2^2 + 2M_2L_1L_2cos\theta_2)\ddot{\theta}_1$$
$$+(M_2L_2^2 + M_2L_1L_2cos\theta_2)\ddot{\theta}_2 - M_2L_1L_2sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)$$
$$-(M_1+M_2)gL_1sin\theta_1 - M_2gL_2sin(\theta_1+\theta_2)$$

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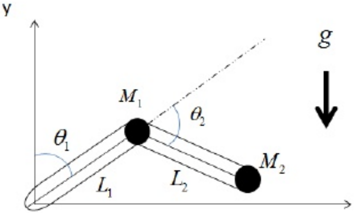

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- Force applied on robotic arm needs to be controlled
- Error between current and desired positions determine the forces
- F_{θ_1} and F_{θ_2} can be computing by solving Lagrangian using kinetic and potential energies
- Too complicated, and may not work
- However, we can conclude that

$$F = g(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, \ddot{\theta}_1, \ddot{\theta}_2)$$

Robotic Arm Control - KNN Approach

θ_1	θ_2	$\dot{\theta}_1$	$\dot{\theta}_2$	$\ddot{\theta}_1$	$\ddot{\theta}_2$	F_{θ_1}	F_{θ_2}

- From the current and desired positions, estimate the angles and their derivatives.
- Look for the k(=1)-nearest match in the table, and pick forces.
- If succeeded in pitching, append the current configuration to the table
- As the table size increases, the robot skill improves!

Robot Training - Testing

Speech Synthesis (Unit Selection)

- Regress speech waveform from the sequence of words
- Create a table of phonemes and their waveform from training data
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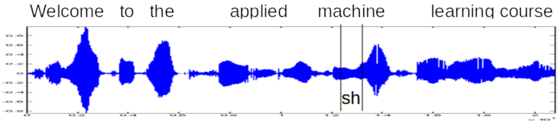
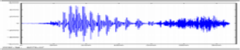


Phone	Prev. Phone	Next Phone	Phone position	Word Position	Duration	Waveform
/sh/	/a/	/i/	3/6	5/7	65 ms	

2h Data - 13h Data

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Homework - Data Generation Models

- Let the distribution of human height be given by

$$X \sim \mathcal{N}(5.3, 1)$$

Let the relationship between height and weight be given by

$$Y = 10X + 5 + \epsilon$$

where $\epsilon \sim \mathcal{N}(0, 3)$. Evaluate the joint density function $p_{XY}(x, y)$, marginal density functions $p_X(x)$ & $p_Y(y)$, conditional density functions p(x/y) & p(y/x), and their corresponding expectations.

Homework

- Let the joint density function of human heights and weights follow a multivariate Gaussian given by

$$[X, Y] \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$$

  where $\boldsymbol{\mu} \in \mathbb{R}^2$ is the mean vector and $\Sigma \in \mathbb{R}^{2\times 2}$ is the covariance matrix. Evaluate the joint density function $p_{XY}(x, y)$, marginal density functions $p_X(x)$ & $p_Y(y)$, conditional density functions p(x/y) & p(y/x), and their corresponding expectations.
- Is it related to the earlier model?

Homework

- Let V be a 2-D random vector drawn from a standard Gaussian, i.e.,

$$\mathbf{V} \sim \mathcal{N}(\boldsymbol{O}, \mathbf{I})$$

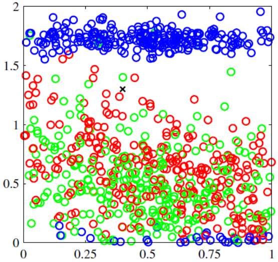
Let height and weight be obtained by an affine transformation of **V** as

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{V} + \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$

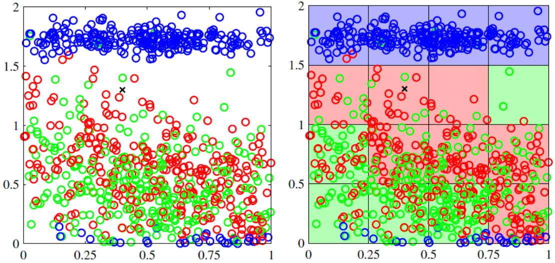
Evaluate the joint density function $p_{XY}(x, y)$, marginal density functions $p_X(x)$ & $p_Y(y)$, conditional density functions p(x/y) & p(y/x), and their corresponding expectations.

- Is it related to two earlier models?

Curse of Dimensionality



Curse of Dimensionality



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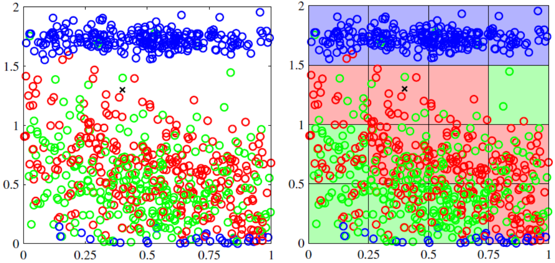
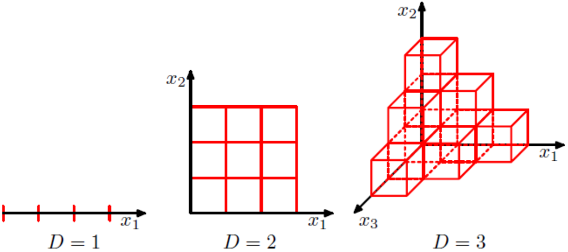
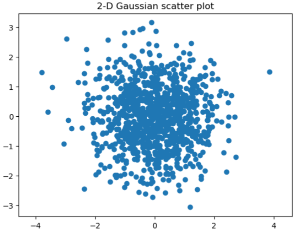




Understanding High-Dimensional Space



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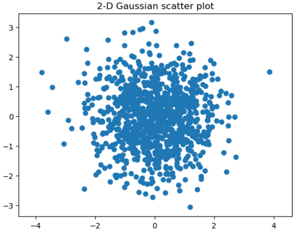

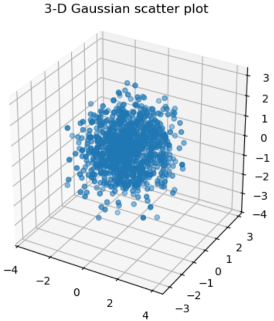

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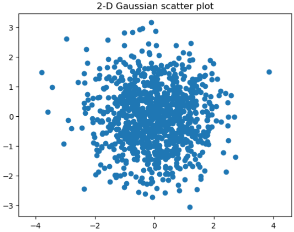

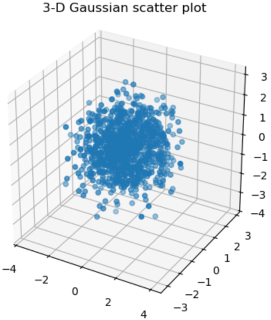

1000-D Gaussian

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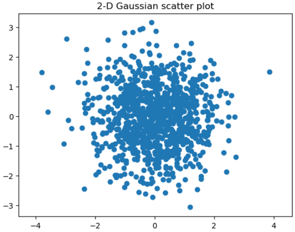

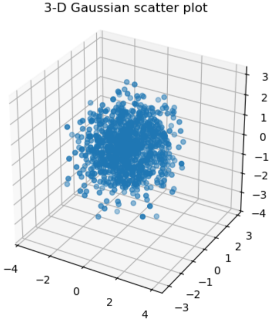

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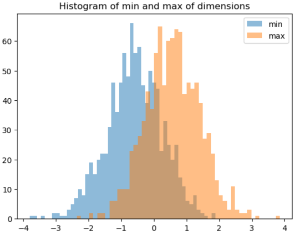

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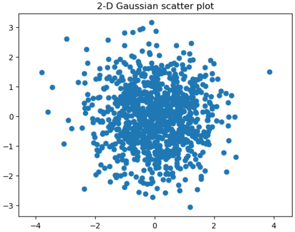

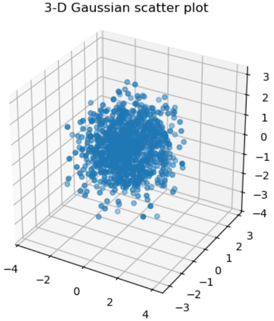

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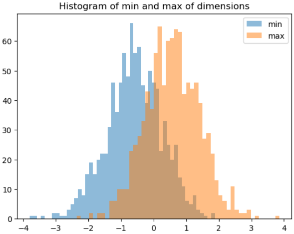

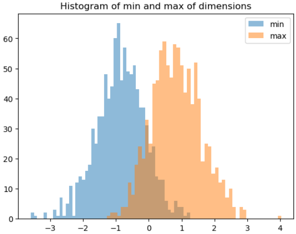

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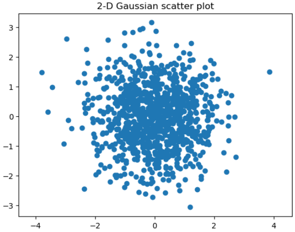

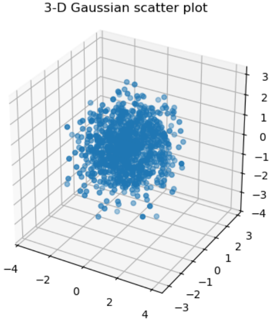

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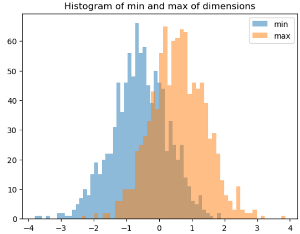

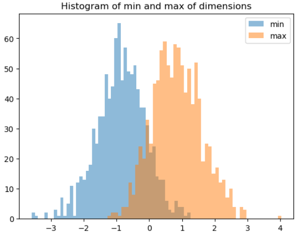

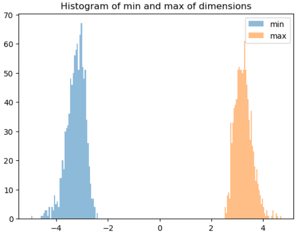

Gaussian in High Dimensional Space

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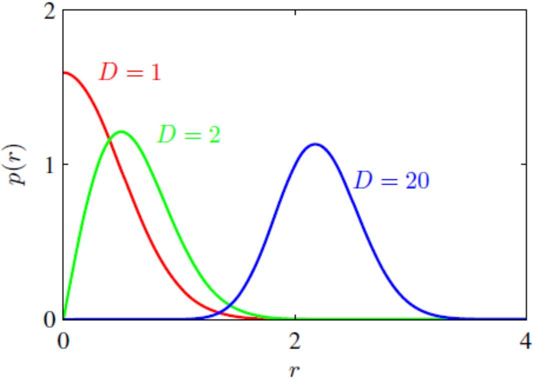

Thank You !

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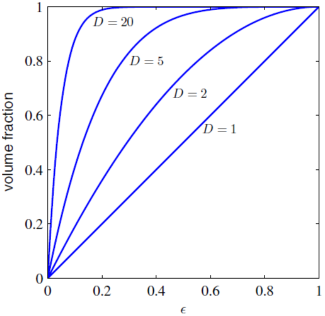

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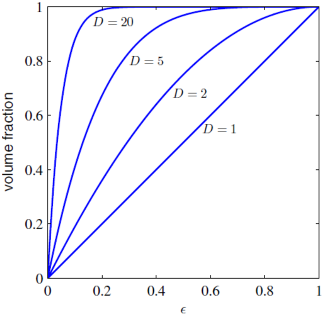

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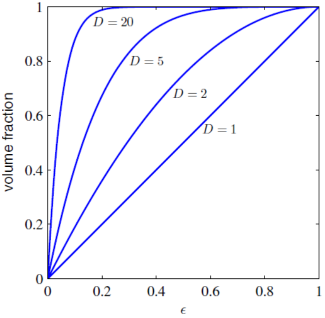

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