Indian Institute of Technology Mandi

DS-2(IC-252) Theory Assignment 3

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Theory Assignment-3

1.

	Y = 0	Y = 1	Y = 2
X = 0	1/6	1/4	1/8
X = 1	1/8	1/6	1/6

a) $P(X=0, Y \leq 1) = P(X=0, Y=0) + P(X=0, Y=1)$

$= 1/6 + 1/4$

$= \frac{5}{12}$

b) Marginal PMF of X

$P(X=x) = \sum_{y_i} P(Y=y_i \mid X=x)$

$\therefore P(X=0) = 1/6 + 1/4 + 1/8 = \frac{13}{24}$

$P(X=1) = 1/8 + 1/6 + 1/6 = \frac{11}{24}$

Marginal PMF of Y

$P(Y=y) = \sum_{x_i} P(X=x_i \mid Y=y)$

$\therefore P(Y=0) = 1/6 + 1/8 = 14/32$

$P(Y=1) = 1/4 + 1/6 = 10/24$

$P(Y=2) = 1/8 + 1/6 = 14/48$

c) $P(Y=1 \mid X=0) = \frac{P(Y=1 \cap X=0)}{P(X=0)}$

$= \frac{1/4}{13/24} = \frac{6}{13}$

d) As $P(X=0, Y=0) \neq P(X=0)\,P(Y=0)$

$$\frac{1}{6} \neq \frac{13}{24} \times \frac{7}{24}$$

We can clearly see X and Y are not independent.

2. let $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Bernoulli}(q)$ are independent

$P(X=0) = 1-p$
$P(X=1) = p$
$P(Y=0) = 1-q$
$P(Y=1) = q$

Joint PMF

	$P(Y=0)$	$P(Y=1)$
$P(X=0)$	$(1-p)(1-q)$	$(1-p)q$
$P(X=1)$	$p(1-q)$	pq

Joint CDF

$F_{XY}(0,0) = f_{XY}(0,0) = (1-p)(1-q)$

$F_{XY}(1,0) = f_{XY}(0,0) + f_{XY}(1,0) = (1-p)(1-q) + p(1-q)$
$= (1-q)$

$F_{XY}(0,1) = f_{XY}(0,0) + f_{XY}(0,1) = (1-p)(1-q) + q(1-p)$
$= (1-p)$

$F_{XY}(1,1) = f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0) + f_{XY}(1,1)$
$= 1$

	$P(Y=0)$	$P(Y=1)$
$P(X=0)$	$(1-p)(1-q)$	$(1-p)$
$P(X=1)$	$(1-q)$	1

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3. $P(X=x \mid X<5)$

If $X<5$, $x \in \{1,2,3,4\}$

$$P(X=1 \mid X<5) = \frac{P(X=1 \cap X<5)}{P(X<5)} = \frac{1/6}{4/6} = \frac{1}{4}$$

$$P(X=2 \mid X<5) = \frac{P(X=2 \cap X<5)}{P(X<5)} = \frac{1/6}{4/6} = \frac{1}{4}$$

$$P(X=3 \mid X<5) = \frac{P(X=3 \cap X<5)}{P(X<5)} = \frac{1/6}{4/6} = \frac{1}{4}$$

$$P(X=4 \mid X<5) = \frac{P(X=4 \cap X<5)}{P(X<5)} = \frac{1/6}{4/6} = \frac{1}{4}$$

4. Show that $E[X+Y] = E[X] + E[Y]$

for Discrete

$$E[X+Y] = \sum_x \sum_y (x+y) f_{XY}(x,y)$$

$$= \sum_x \sum_y x f_{XY}(x,y) + \sum_x \sum_y y f_{XY}(x,y)$$

$$= \sum_x x \sum_y f_{XY}(x,y) + \sum_y y \sum_x f_{XY}(x,y)$$

$$= \sum_x x f_X(x) + \sum_y y f_Y(y)$$

$$= E[X] + E[Y]$$

5. 3 Blue, 2 Red, 3 Green

2 are selected at Random

$X \sim$ No. of Blue Pens selected

$Y \sim$ No. of Red Pens Selected

a) $f_{XY}(x,y)$

$f_{XY}(0,0) = P(\text{both green pen})$

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$$= \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$$

$$f_{xy}(1,0) = P(\text{blue green}) + P(\text{green blue})$$
$$= \frac{3}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{3}{7}$$
$$= \frac{9}{28}$$

$$f_{xy}(2,0) = P(\text{blue blue})$$
$$= \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$$

$$f_{xy}(0,1) = P(\text{green red}) + P(\text{red green})$$
$$= \frac{3}{8} \times \frac{2}{7} + \frac{2}{8} \times \frac{3}{7}$$
$$= \frac{3}{14}$$

$$f_{xy}(0,2) = P(\text{red red})$$
$$= \frac{2}{8} \times \frac{1}{7} = \frac{1}{28}$$

$$f_{xy}(1,1) = P(\text{blue red}) + P(\text{red blue})$$
$$= \frac{3}{8} \times \frac{2}{7} + \frac{2}{8} \times \frac{3}{7}$$
$$= \frac{3}{14}$$

$$f_{xy}(1,2) = f_{xy}(2,1) = f_{xy}(2,2) = 0$$

	$P(Y=0)$	$P(Y=1)$	$P(Y=2)$
$P(X=0)$	$3/28$	$3/14$	$1/28$
$P(X=1)$	$9/28$	$3/14$	0
$P(X=2)$	$3/28$	0	0

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b) $P[(X,Y) \in A]$ where A is region $\{(x,y) \mid x+y \le 1\}$

$$= f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0)$$

$$= \frac{3}{28} + \frac{3}{14} + \frac{9}{28} = \frac{18}{28}$$

6. $X_1, X_2, \ldots, X_n \in$ Poisson with $\lambda =$ mean

Since $X_1, X_2, \ldots, X_n$ are random Samples, they are independent.

$$P(X_1 = k_1) = \frac{\lambda^{k_1} e^{-k_1\lambda}}{k_1!}, \quad P(X_2 = k_2) = \frac{\lambda^{k_2} e^{-k_2\lambda}}{k_2!}$$ and so on

$$P(X_1 = k_1, X_2 = k_2, X_3 = k_3, \ldots) = P(X_1 = k_1) \cap P(X_2 = k_2) \cap \ldots P(X_n = k_n)$$

$$= \frac{\lambda^{(k_1+k_2+k_3+\ldots+k_n)} \cdot e^{-n\lambda}}{k_1!\, k_2! \ldots k_n!}$$

7. X, Y are two jointly continuous random variable with joint PDF.

a) $$\int_0^1\int_0^1 (x+cy^2)\,dx\,dy = 1$$

$$\int_0^1 \left[\frac{x^2}{2} + cy^2x\right]_0^1 dy = 1$$

$$\int_0^1 \left(\frac{1}{2} + cy^2\right) dy = 1$$

$$\left.\frac{y}{2} + \frac{cy^3}{3}\right]_0^1 = 1$$

$$\frac{1}{2} + \frac{c}{3} = 1$$

$$\frac{c}{3} = \frac{1}{2}$$

$$\boxed{c = \frac{3}{2}}$$

b) $P\left(0 \le X \le \frac{1}{2}, \ 0 \le Y \le \frac{1}{2}\right)$

$$= \int_0^{1/2}\int_0^{1/2} \left(x + \frac{3}{2}y^2\right) dx \, dy$$

$$= \int_0^{1/2} \left. \frac{x^2}{2} + \frac{3}{2}xy^2 \right]_0^{1/2} dy$$

$$= \int_0^{1/2} \frac{1}{8} + \frac{3}{4}y^2 \, dy$$

$$= \cancel{\left.\frac{7y}{8}\right]_0^{1/2}} = \left. \frac{y}{8} + \frac{y^3}{4} \right]_0^{1/2}$$

$$= \cancel{\frac{7}{16}} = \frac{1}{16} + \frac{1}{32}$$

$$= \frac{3}{32}$$

c) Marginal PDF of $f_X(x)$ and $f_Y(y)$

$$f_X(x) = \int_0^1 f_{XY}(x,y) \, dy$$

$$= \int_0^1 (x + cy^2) \, dy$$

$$= \left. xy + \frac{cy^3}{3} \right]_0^1$$

$= x + \frac{c}{3}$

$= x + \frac{1}{2}$

$f_y(y) = \int_0^1 f_{xy}(x,y)\, dx$

$= \int_0^1 (x + cy^2)\, dx$

$= \left. \frac{x^2}{2} + cxy^2 \right]_0^1$

$= \frac{1}{2} + cy^2$

$= \frac{1}{2} + \frac{3}{2} y^2$

d) Joint CDF for X and Y

$CDF(X) = F_X(x') = \int_0^{x'} \int_0^1 \left(x + \frac{3}{2} y^2\right) dy\, dx$

$= \int_0^{x'} \left. xy + \frac{y^3}{2} \right]_0^1 dx$

$= \int_0^{x'} \left(x + \frac{1}{2}\right) dx$

$= \left. \frac{x^2}{2} + \frac{x}{2} \right]_0^{x'}$

$= \frac{(x')^2}{2} + \frac{x'}{2}$

$CDF(Y) = F_Y(y') = \int_0^{y'} \int_0^1 \left(x + \frac{3}{2} y^2\right) dx\, dy$

$$= \int_0^{y'} \left(\frac{x^2}{2} + \frac{3}{2} x y^2 \right)_0^1 dy$$

$$= \int_0^{y'} \left(\frac{1}{2} + \frac{3}{2} y^2 \right) dy$$

$$= \left. \frac{y}{2} + \frac{y^3}{2} \right]_0^{y'}$$

$$= \frac{y'}{2} + \frac{(y')^3}{2}$$

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8. $f_{xy}(x,y) = \begin{cases} a(e^{-x}+e^{-y}) & , 0 \le x, 0 \le y \\ 0 & , \text{otherwise} \end{cases}$

a) $\int_0^\infty \int_0^\infty f_{xy}(x,y) = 1$

$\Rightarrow \int_0^\infty \int_0^\infty a(e^{-x}+e^{-y})\, dx\, dy = 1$

$\Rightarrow a[e^{-x}]_\infty^0 + a[e^{-y}]_\infty^0 = 1$

$\Rightarrow a(2) = 1$

$\boxed{a = \frac{1}{2}}$

b) $P(X+Y \le 1)$

$= \int_0^1 \int_0^{1-x} \frac{1}{2}(e^{-x}+e^{-y})\, dy\, dx$

$= \int_0^1 \frac{1}{2}[ye^{-x} - e^{-y}]_0^{1-x}\, dx$

$= \int_0^1 \frac{1}{2}\{(1-x)e^{-x} - e^{x-1} + 1\}\, dx$

$= \frac{1}{2}[xe^{-x} - e^{x-1} + x]_0^1$

$= \frac{1}{e}$

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$$= \int_0^{y'} \left(\frac{x^2}{2} + \frac{3}{2} x y^2 \right)_0^1 dy$$

$$= \int_0^{y'} \left(\frac{1}{2} + \frac{3}{2} y^2 \right) dy$$

$$= \left. \frac{y}{2} + \frac{y^3}{2} \right]_0^{y'}$$

$$= \frac{y'}{2} + \frac{(y')^3}{2}$$

9. let X and ~~y~~ Y be two independent uniform $(0, 1)$. Random Variables. Find $F_{XY}(x, y)$

$$f_Y(y) = f_X(x) = \begin{cases} 1 & , \forall \; x \in (0,1) \\ 0 & , \text{otherwise} \end{cases}$$

$$f_{XY}(x, y) = f_X(x) \, f_Y(y)$$

$$= 1 \qquad \text{As independent RV} \; \forall \; x, y \in (0,1)$$

$$F_{XY}(x, y) = \int_0^x \int_x^y f_{XY}(x, y) \, dy \, dx$$

$$= \int_0^x \int_0^y 1 \, dy \, dx$$

$$= \int_0^x \left. y \right]_0^y dx$$

$$= \int_0^x y \, dx$$

$$= \left. xy \right]_0^x$$

$$= xy$$

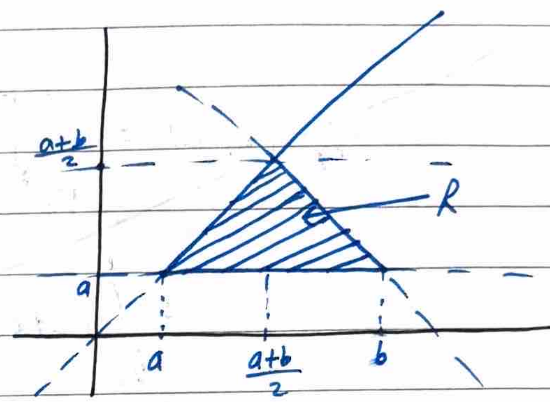
10. X, Y are two i.i.d random variable in interval $[a,b]$.

$$f_X(x) = f_Y(y) = \frac{1}{b-a}$$

$$f_{XY}(x,y) = \frac{1}{(b-a)^2} \qquad \forall\ x,y \in (a,b)$$

$$= f_X(x)\, f_Y(y)$$

$$P(X+Y \le a+b \cap Y \le X)$$



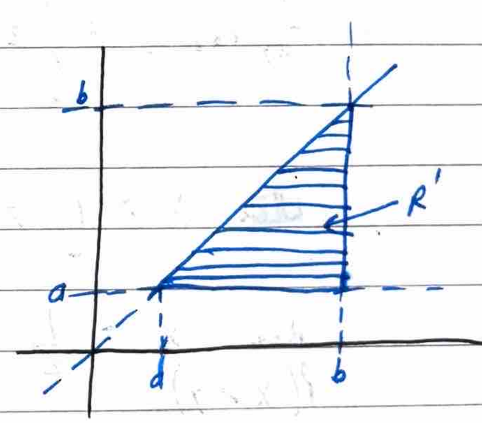
$$= \iint_R \frac{1}{(b-a)^2}\, dx\, dy$$

$$= \frac{1}{(b-a)^2} \times \text{Area of } \Delta R$$

$$= \frac{1}{(b-a)^2} \times \frac{1}{2} \times (b-a) \times \left(\frac{a+b}{2} - a\right)$$

$$= \frac{1}{4}$$

$$P(Y \le X) = \iint_R \frac{1}{(b-a)^2}\, dx\, dy$$



$$= \frac{1}{(b-a)^2} \iint_R dx\, dy$$

$$= \frac{1}{(b-a)^2} \times \text{Area of } \Delta R'$$

$$= \frac{1}{(b-a)^2} \times \frac{1}{2} \times (b-a)(b-a)$$

$$= \frac{1}{2}$$

$$P(x+y \le a+b \mid y \le x)$$

$$= \frac{P(x+y \le a+b \cap y \le x)}{P(y \le x)}$$

$$= \frac{1/4}{1/2} = \frac{1}{2}$$

$$= \left(\int_0^2 e^{-x}\,dx\right)\left(\int_1^3 e^{-x}\,dx\right)\left(\int_2^\infty e^{-x}\,dx\right)$$

$$= (1-e^{-2})(e^{-1}-e^{-3})(e^{-2})$$

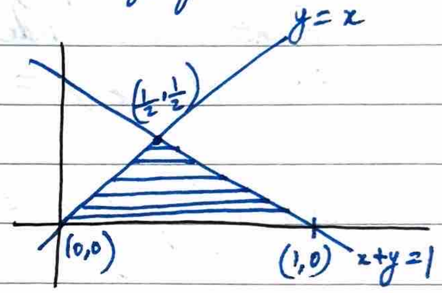
11. $f_{XY}(x,y) = \begin{cases} 24x(1-x-y), & x,y \geq 0, \quad x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

a) $P(X>Y)$

$$= \int_0^{1/2}\int_0^{x} 24x(1-x-y)\,dy\,dx + \int_{1/2}^{1}\int_0^{1-x} 24x(1-x-y)\,dy\,dx$$

$$= \frac{7}{16} + \frac{5}{16}$$

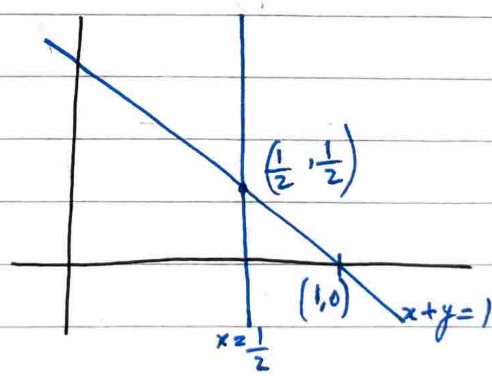
$$= \frac{12}{16} = \frac{3}{4}$$



b) $P\left(X > \frac{1}{2}\right)$

$$= \int_{1/2}^{1}\int_0^{1-x} 24x(1-x-y)\,dy\,dx$$

$$= \frac{5}{16}$$



12. $f_{XY}(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & x,y \in (0,1) \\ 0, & \text{otherwise} \end{cases}$

a) $P((x,y) \in A)$ where $A = \{(x,y) \mid x \in (0,\frac{1}{2}), y \in (\frac{1}{4}, \frac{1}{2})\}$

$$= \int_0^{1/2}\int_{1/4}^{1/2} \frac{2}{5}(2x+3y)\, dy\, dx$$

$$= \frac{2}{5}\int_0^{1/2} \left[2yx + \frac{3}{2}y^2\right]_{1/4}^{1/2} dx$$

$$= \frac{2}{5}\int_0^{1/2} \frac{x}{2} + \frac{9}{32}\, dx$$

$$= \frac{2}{5}\left[\frac{x^2}{4} + \frac{9x}{32}\right]_0^{1/2}$$

$$= \frac{2}{5}\left(\frac{1}{16} + \frac{9}{64}\right) = \frac{13}{64} \times \frac{2}{5}$$

$$= \frac{13}{160}$$

b) Marginal distribution of X and Y

Marginal dist of X

$$f_X(x) = \int_0^1 f_{XY}(x,y)\, dy$$

$$= \int_0^1 \frac{2}{5}(2x+3y)\, dy$$

$$= \frac{2}{5}\left[2xy + \frac{3y^2}{2}\right]_0^1 = \frac{2}{5}\left(2x + \frac{3}{2}\right) \cancel{= \frac{2}{5} \times \frac{7}{2} = \frac{7}{5}}$$

$$= \frac{4x}{5} + \frac{3}{5}$$

Marginal dist of Y

$$f_Y(y) = \int_0^1 f_{XY}(x,y)\,dx$$

$$= \int_0^1 \frac{2}{5}(2x+3y)\,dx$$

$$= \frac{2}{5}\left(x^2+3xy\right)\Big|_0^1$$

$$= \frac{2}{5}(1+3y)$$

$$= \frac{2}{5} + \frac{6}{5}y$$

c) $$P(X \mid Y=1) = \frac{P(X \cap (Y=1))}{P(Y=1)}$$

$$= \frac{\frac{2}{5}(2x+3)}{f_Y(1)}$$

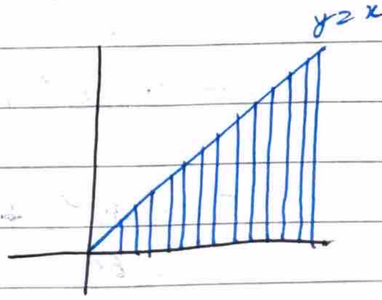
$$= \frac{\frac{2}{5}(2x+3)}{\frac{2}{5}+\frac{6}{5}(1)} = \frac{\cancel{\frac{2}{5}}(2x+3)}{\cancel{\frac{8}{5}}\,4}$$

$$= \frac{2x+3}{4}$$

$P(X=0 \mid Y=1)=0$ as at any point P is 0.

13. $$f_{XY}(x,y) = \begin{cases} c(e^{-2x-3y}), & 0 \le y \le x \\ 0, & \text{otherwise} \end{cases}$$

a) $$\int_0^\infty \int_y^\infty c(e^{-2x-3y})\,dx\,dy = 1$$



$$\Rightarrow \int_0^\infty \int_y^\infty e^{-2x} \cdot e^{-3y} \, dx \, dy = \frac{1}{c}$$

$$\Rightarrow \int_0^\infty \left(\frac{e^{-2x}}{-2} e^{-3y} \right)_y^\infty dy = \frac{1}{c}$$

$$\Rightarrow \int_0^\infty \frac{e^{-2y} e^{-3y}}{2} dy = \frac{1}{c}$$

$$\Rightarrow \int_0^\infty e^{-5y} \, dy = \frac{2}{c}$$

$$\Rightarrow \left[\frac{e^{-5y}}{-5} \right]_0^\infty = \frac{2}{c}$$

$$\Rightarrow \frac{1}{5} = \frac{2}{c}$$

$$\Rightarrow \boxed{c = 10}$$

b) $P(X<1, Y<2) = P(X<1, Y<1)$

$$= \int_0^1 \int_y^1 10 \left(e^{-2x} e^{-3y} \right) dx \, dy$$

$$= \int_0^1 10 \left(\frac{e^{-2x}}{-2} e^{-3y} \right]_y^1 dy$$

$$= \int_0^1 10 \left[\frac{e^{-2}}{-2} e^{-3y} + \frac{e^{-5y}}{2} \right] dy$$

$$= 10 \left[\frac{e^{-2} e^{-3y}}{6} + \frac{e^{-5y}}{-10} \right]_0^1$$

$$= 10 \left[\frac{e^{-5}}{6} + \frac{e^{-5}}{-10} - \frac{e^{-2}}{6} + \frac{1}{10} \right]$$

$$= 10 \left(\frac{4e^{-5}}{60} + \frac{1}{10} - \frac{e^{-2}}{6} \right) = \cancel{10 \left(\frac{4e^{-5}}{30} - \frac{e^{-2}}{6} \right)}$$

$$= 0.778$$

c) $P(1 < x < 2)$

$$= \int_1^2 \int_0^x 10\, e^{-2x} e^{-3y}\, dy\, dx$$

$$= \int_1^2 \left[10\, e^{-2x} \frac{e^{-3y}}{-3}\right]_0^x dx$$

$$\cancel{= \int_1^2 10\left(\frac{e^{-5x}}{-3} - \frac{e^{-3x}}{-3}\right) dx} = \int_1^2 10\left(\frac{e^{-5x}}{-3} - \frac{e^{-2x}}{-3}\right) dx$$

$$\cancel{= 10\left(\frac{e^{-5x}}{15} - \frac{e^{-3x}}{-3}\right]_1^2} = 10\left(\frac{e^{-5x}}{15} - \frac{e^{-2x}}{6}\right)_1^2$$

$$\cancel{= 10\left[\frac{e^{-10}}{15} - 2e^{-3y}\right.}$$

$$= 10\left(\frac{e^{-10}}{15} - \frac{e^{-4}}{6} - \frac{e^{-5}}{15} + \frac{e^{-2}}{6}\right)$$

$$= 0.1905$$

d) $P(Y > 3) = P(X > 3,\ Y > 3)$

$$= \int_3^\infty \int_3^x 10\, e^{-2x} e^{-3y}\, dy\, dx$$

$$= 10 \int_3^\infty \left[e^{-2x} \frac{e^{-3y}}{-3}\right]_3^x dx$$

$$= 10 \int_3^\infty \left(\frac{e^{-5x}}{-3} - \frac{e^{-2x} \cdot e^{-9}}{-3}\right) dx$$

$$= 10\left[\frac{e^{-5x}}{15} - \frac{e^{-2x} e^{-9}}{6}\right]_3^\infty$$

$$= 10\left[-\frac{e^{-15}}{15} + \frac{e^{-15}}{6}\right] = \cancel{10^2\left(\frac{9e^{-15}}{15}\right) = 6e^{-15}}$$

$$= 10 \times \frac{9e^{-15}}{90} = e^{-15}$$ Ans

e) $E[X] = \int_0^\infty x \, f_X(x) \, dx$

$= \int_0^\infty x \int_0^x f_{XY}(x,y) \, dy \, dx$

$= \int_0^\infty x \int_0^x 10 e^{-2x} e^{-3y} \, dy \, dx$

$= \int_0^\infty x \left(10e^{-2x}\right)\left(\frac{e^{-3y}}{-3}\right)_0^x dx$

$= \int_0^\infty x\left(10e^{-2x}\right)\left(\frac{e^{-3x}}{-3} - \frac{1}{-3}\right) dx$

$= \int_0^\infty \frac{10x e^{-5x}}{-3} + \frac{10x e^{-2x}}{3} \, dx$

$= \frac{10}{3} \int_0^\infty \left(x e^{-2x} - x e^{-5x}\right) dx$

$= \frac{10}{3}\left[x\left(\frac{e^{-2x}}{-2} + \frac{e^{-5x}}{5}\right) - \int \left(\frac{e^{-2x}}{-2} + \frac{e^{5x}}{5}\right) dx \right]_0^\infty$

$= \frac{10}{3}\left[x\left(\frac{e^{-2x}}{-2} + \frac{e^{-5x}}{5}\right) - \frac{e^{-2x}}{4} + \frac{e^{-5x}}{25} \right]_0^\infty$

$= \frac{10}{3}\left(-\left(-\frac{1}{4} + \frac{1}{25}\right)\right)$

$= \frac{10}{3} \times \frac{21}{100} = \frac{7}{10}$

$\boxed{E[X] = 0.7}$

$E[Y] = \int_0^\infty y \, f_Y(y) \, dy$

$= \int_0^\infty y \int_y^\infty 10 e^{-2x} e^{-3y} \, dx \, dy$

$$= \int_0^\infty y \left[\frac{-10\, e^{-2x}}{-2} \; e^{-3y} \right]_y^\infty dy$$

$$= \int_0^\infty y \left(-5e^{-5y}\right) dy$$

$$= 5 \int_0^\infty y e^{-5y}\, dy$$

$$= 5 \left[\frac{y e^{-5y}}{-5} - \frac{e^{-5y}}{25} \right]_0^\infty$$

$$= \frac{e^{-0}}{5} = \frac{1}{5}$$

$$\boxed{E[Y] = 0.2}$$

f) $$P(X \mid Y=1) = \frac{P(X \cap Y=1)}{P(Y=1)}$$

$$= \frac{\cancel{10}\, e^{-2x}\, \cancel{e^{-3}}}{\int_1^\infty \cancel{10}\, e^{-2x}\, \cancel{e^{-3}}\, dx} = \frac{e^{-2x}}{\int_1^\infty e^{-2x}\, dx}$$

$$= \frac{e^{-2x}}{\left[\frac{e^{-2x}}{-2x} \right]_1^\infty} = \frac{2e^{-2x}}{e^{-2}}$$

g) $$E[X \mid Y=2] = \int_2^\infty \left(x\; 10e^{-2x}\; e^{-3}\right) dx$$

$$= 10e^{-3} \int_2^\infty x e^{-2x}\, dx$$

$$= 10e^{-3} \left[\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4} \right]_2^\infty$$

$$= 10e^{-3} \left(- \left(-e^{-4} - \frac{e^{-4}}{4} \right) \right)$$

$$= \frac{50e^{-7}}{4}$$

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14. $X \sim \text{Exponential}(1)$

e^{-x}

a) Conditional PDF given $X>1$

$$P(X>1) = \int_1^{\infty} e^{-x}\,dx = \frac{1}{e}$$

If $X>1$

$$f_{X|X>1}(x) = \frac{f(x)}{P(X>1)} = e\,e^{-x} = e^{-x+1}$$

$$F_{X|X>1}(x') = \int_1^{x'} e^{-x+1}\,dx = \left(-e^{-x+1}\right)_1^{x'}$$

$$= -e^{-x+1} - (-1)$$

$$= 1 - e^{-x+1}$$

b) $$E[X|X>1] = \int_1^{\infty} x\, f_{X|X>1}(x)\,dx$$

$$= \int_1^{\infty} x e^{-x+1}\,dx$$

$$= \left(-x e^{-x+1} - \int -e^{-x+1}\,dx\right]_1^{\infty}$$

$$= \left(-xe^{-x+1} - e^{-x+1}\right)_1^{\infty}$$

$$= -(-1-1) = 2$$

c) $Var(X \mid X>1)$

$$E[X^2 \mid X>1] = \int_1^{\infty} x^2 f_{X\mid X>1}(x)\,dx$$

$$= \int_1^{\infty} x^2 e^{-x+1}\,dx$$

$$= \left(-x^2 e^{-x+1} + \int 2x + e^{-x+1}\,dx\right)_1^{\infty}$$

$$= \left(-x^2 e^{-x+1} + 2(-xe^{-x+1} - e^{-x+1})\right)_1^{\infty}$$

$$= -(-1 + 2(-1-1))$$

$$= 5$$

$$Var(X \mid X>1) = E[X^2 \mid X>1] - E[X \mid X>1]$$

$$= 5 - 2^2$$

$$= 1$$

15. a) Are X and Y independent?

$$f_{XY}(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & x,y > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_0^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^{\infty} 2e^{-x}e^{-2y}\,dy$$

$$= \left[\cancel{2}e^{-x}\frac{e^{-2y}}{\cancel{2}}\right]_0^{\infty} = -(-e^{-x})$$

$$= e^{-x}$$

$$f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx$$

$$= \int_0^\infty 2e^{-x} e^{-2y}\,dx$$

$$= \left[-2e^{-x} e^{-2y}\right]_0^\infty$$

$$= -(-2e^{-2y})$$

$$= 2e^{-2y}$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$$2e^{-x}e^{-2y} = 2e^{-2y}e^{-x}$$

$\therefore$ Independent

b) $f_{XY}(x,y) = \begin{cases} 8xy & ; \quad 0 < x < y < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

$$f_X(x) = \int_{x}^{1} 8xy\,dy$$

$$= 8x\left(\frac{y^2}{2}\right)_{x}^{1}$$

$$= 4x(1-x^2)$$

$$f_Y(y) = \int_0^{y} 8xy\,dx$$

$$= 8y\left(\frac{x^2}{2}\right]_0^{y}$$

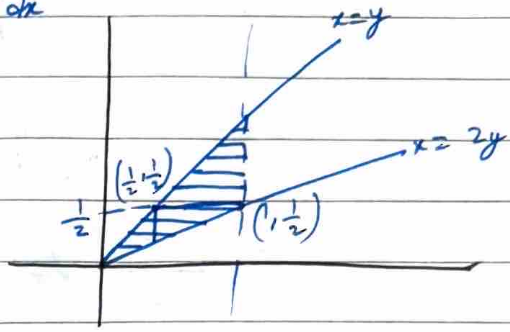
$$= \frac{8y^3}{2} = 4y^3$$

As $f_{XY}(x,y) \neq f_X(x)\, f_Y(y)$

$\therefore$ Not Independent

16. $f_{XY}(x,y) = \begin{cases} 1/x, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$$P(X \le 2Y) = \int_0^1 \int_{x/2}^{x} \frac{1}{x}\, dy\, dx$$



$$= \int_0^1 \left(\frac{y}{x}\right)_{x/2}^{x} dx$$

$$= \int_0^1 \left(1 - \frac{1}{2}\right) dx$$

$$= \frac{x}{2}\Big]_0^1 = \frac{1}{2}$$

17.a) $f_{XY}(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$

$$g(x) = f_X(x) = \int_x^1 10xy^2\, dy$$

$$= 10x\frac{y^3}{3}\Big]_x^1$$

$$= \frac{10x}{3} - \frac{10x^4}{3}$$

$$h(y) = f_Y(y) = \int_0^y 10xy^2\, dx$$

$$= 5x^2y^2\Big]_0^y = 5y^4$$

$$f(y|x) = \frac{f_{YX}(y,x)}{f_X(x)} = \frac{10xy^2}{\frac{10}{3}(x - x^4)}$$

$$= \frac{3y^2}{(1 - x^3)}$$

b) $P(Y > \frac{1}{2} \mid X = 0.25) = \dfrac{P(Y > \frac{1}{2} \wedge X = 0.25)}{P(X = 0.25)}$

$$= \frac{\int_{1/2}^{1} 10\left(\frac{1}{4}\right) y^2 \, dy}{f_X\left(\frac{1}{4}\right)}$$

$$= \frac{\frac{10}{4}\left[\frac{y^3}{3}\right]_{1/2}^{1}}{\frac{10}{3}\left[\frac{1}{4} - \frac{1}{256}\right]} = \frac{\frac{7}{1\cancel{8} \times 4}}{\frac{63}{\cancel{256}\, \cancel{64}\, 8}} = \frac{8 \times 7}{63}$$

$$= \frac{56}{63} = \frac{8}{9}$$

18. a) $f_{XY}(x,y) = \begin{cases} \dfrac{2}{(1+x+y)^3}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$$P(X > 1, Y > 1) = \int_1^\infty \int_1^\infty f_{XY}(x,y) \, dx \, dy$$

$$= \int_1^\infty \int_1^\infty \frac{2}{(1+x+y)^3} \, dx \, dy$$

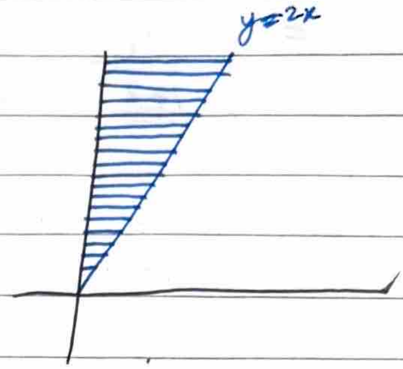
$$= \int_1^\infty \left[\frac{2}{-2}\left(\frac{1}{1+x+y}\right)^2\right]_1^\infty dy$$

$$= -\int_1^\infty \left(\frac{1}{2+y}\right)^2 dy$$

$$= \left[\frac{-1}{2+y}\right]_1^\infty = \frac{1}{3}$$

b) $P\left(X < \frac{Y}{2}\right) = P(Y > 2X)$

$$= \int_0^\infty \int_{2x}^\infty \frac{2}{(1+x+y)^3}\, dy\, dx$$



$$= \int_0^\infty \left[\frac{2}{-2} \frac{1}{(1+x+y)^2} \right]_{2x}^{\infty} dx$$

$$= \int_0^\infty \frac{1}{(1+3x)^2}\, dx$$

$$= \left[\frac{-1}{(1+3x)} \times \frac{1}{3} \right]_0^\infty$$

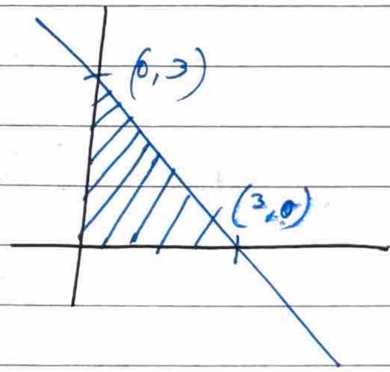
$$= \frac{1}{3}$$

c) CDF for Total Probability of C_1 and C_2

let $Z = X + Y$

$$CDF(z) = F_Z(z) = P(X+Y < z)$$

$$= \int_0^z \int_0^{z-x} \frac{2}{(1+x+y)^3}\, dy\, dx$$



$$= \int_0^z \left[\frac{2}{-2} \frac{1}{(1+x+y)^2} \right]_0^{z-x} dx$$

$$= -\int_0^z \frac{1}{(z+1)^2} - \frac{1}{(1+x)^2}\, dx$$

$$= \int_0^z \frac{1}{(1+x)^2} - \frac{1}{(1+z)^2}\, dx$$

$$= \left[\frac{-1}{1+x} - \frac{x}{(1+z)^2} \right]_0^2$$

$$= \frac{-1}{1+z} - \frac{z}{(1+z)^2} - (-1)$$

$$= 1 - \left(\frac{1-2z}{(1+z)^2} \right)$$

$$= \frac{\cancel{1} + z^2 + \cancel{2z} \cancel{-1} - \cancel{2z}}{(1+2z)^2}$$

$$= \frac{z^2}{(1+2z)^2} = \frac{z}{1+2z}$$

19. $f_{XY}(x,y) = \begin{cases} \frac{x(1+3y^2)}{4} & , x \in (0,2),\ y \in (0,1) \\ 0 & , \text{otherwise} \end{cases}$

$$g(x) = f_X(x) = \int_0^1 \frac{x(1+3y^2)}{4}\, dy$$

$$= \frac{x}{4} \int_0^1 1 + 3y^2\, dy$$

$$= \frac{x}{4} \left[y + y^3 \right]_0^1$$

$$= \frac{x}{2}$$

$$h(y) = f_Y(y) = \int_0^2 \frac{x(1+3y^2)}{4} \, dx$$

$$= \left(\frac{1+3y^2}{4}\right) \int_0^2 x \, dx$$

$$= \left(\frac{1+3y^2}{4}\right) \left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{1+3y^2}{2}$$

As $f_X(x)\, f_Y(y) = f_{XY}(x,y)$

Events are independent

$\therefore \quad f(x|y) = f_X(x) = \frac{x}{2}$

Also, $P\left(\frac{1}{4} < X < \frac{1}{2} \middle| Y = \frac{1}{3}\right) = P\left(\frac{1}{4} < X < \frac{1}{2}\right)$

$$= \int_{1/4}^{1/2} f_X(x) \, dx$$

$$= \int_{1/4}^{1/2} \frac{x}{2} \, dx = \left[\frac{x^2}{4}\right]_{1/4}^{1/2}$$

$$= \frac{1}{16} - \frac{1}{64} = \frac{3}{64}$$

20. Shelf Life in Years is R.V. X

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

As all three containers will have their shelf lives independent

$P(X_1 < 2,\ 1 < X_2 < 3,\ X_3 > 2)$

$= P(X_1 < 2)\, P(1 < X_2 < 3)\, P(X_3 > 2)$

$$= \left(\int_0^2 e^{-x}\,dx\right)\left(\int_1^3 e^{-x}\,dx\right)\left(\int_2^\infty e^{-x}\,dx\right)$$

$$= (1-e^{-2})(e^{-1}-e^{-3})(e^{-2})$$