Indian Institute of Technology Mandi

DS-2 (IC – 252) Theory Assignment 4

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Theory Assignment - 4

1. $Z = 2X - Y \quad , \quad W = -X + Y$

$\Rightarrow \quad X = Z + W \quad , \quad Y = Z + 2W$

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1$$

$\boxed{J = 1}$ Ans

2. $V = \frac{X}{Y} \quad , \quad W = X + Y$

$\Rightarrow \quad X = \frac{WV}{1+V} \quad , \quad Y = \frac{W}{1+V}$

$$J = \begin{vmatrix} \partial X/\partial V & \partial X/\partial W \\ \partial Y/\partial V & \partial Y/\partial W \end{vmatrix} = \begin{vmatrix} \frac{W}{(1+V)^2} & \frac{V}{1+V} \\ -\frac{W}{(1+V)^2} & \frac{1}{1+V} \end{vmatrix}$$

$$\boxed{J = \frac{W}{(1+V)^2}}$$

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$f_X(x) = \lambda e^{-\lambda x}$

$f_{X,Y}(x, y) = f_X(x)\, f_Y(y) = (\lambda e^{-\lambda x})(\lambda e^{-\lambda y}) = \lambda^2 e^{-\lambda(x+y)}$

$f_{V,W}(v, w) = f_{X,Y}\left(\frac{vw}{1+v}, \frac{w}{1+v}\right) \times |J| = \lambda^2 e^{-\lambda w} \times \frac{w}{(1+v)^2}$

Marginal dist. of $W = \int_0^\infty f_{V,W}(v,w)\, dv = \lambda^2 w e^{-\lambda w} \int_0^\infty \frac{1}{(1+v)^2}\, dv$

$\Rightarrow f_W(w) = \lambda^2 w e^{-\lambda w} \times \frac{1}{2} = \frac{\lambda^2 w}{2} e^{-\lambda w}$

Marginal dist of $V = \int_0^\infty f_{V,W}(v,w)\, dw = \int_0^\infty \lambda^2 e^{-\lambda w} \frac{w}{(1+v)^2}\, dw$

$\Rightarrow \int_0^\infty \lambda^2 e^{-\lambda w} w\, dw = \frac{1}{\lambda}$

$f_V(v) = \frac{1}{(1+v)^2} \times \frac{1}{\lambda} = \frac{1}{\lambda(1+v)^2}$

$$\boxed{f_W(w) = \frac{\lambda^2 w e^{-\lambda w}}{2}, \quad f_V(v) = \frac{1}{\lambda(1+v)^2}}$$ Ans

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3. a) $X = R\cos\theta \quad , \quad Y = R\sin\theta$

$\frac{\partial X}{\partial R} = \cos\theta \quad , \quad \frac{\partial X}{\partial \theta} = -R\sin\theta$

$\frac{\partial Y}{\partial R} = \sin\theta \quad , \quad \frac{\partial Y}{\partial \theta} = R\cos\theta$

$$J = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R\cos^2\theta + R\sin^2\theta = R$$

$f_{R,\theta}(r,\theta) = f_{x,y}(r\cos\theta, r\sin\theta).\,|J| = f_{x,y}(r\cos\theta, r\sin\theta).\,r$

$$\boxed{f_{R,\theta}(r,\theta) = r.\,f_{x,y}(r\cos\theta, r\sin\theta)}$$

b) $f_X(x) = 2x \quad , \quad x \in (0,1)$
$f_Y(y) = 2y \quad , \quad y \in (0,1)$

$f_{x,y}(x,y) = f_x(x).\,f_y(y) = 4xy \quad , \quad x \in (0,1) \quad , \quad y \in (0,1)$

$g(R,\theta) = R.4\,(R\cos\theta)(R\sin\theta)$
$g(R,\theta) = 4R^3\sin\theta\cos\theta$
$g(R,\theta) = 2R^3\sin 2\theta \quad , \quad \theta \in \left(0, \frac{\pi}{2}\right)$

$$\int_0^1\int_0^{\pi/2} 2R^3\sin 2\theta\, d\theta\, dR = \int_0^1 2R^3\, dR = \left[\frac{R^4}{2}\right]_0^1$$

$$= \boxed{\frac{1}{2}} \quad \underline{\text{Ans}}$$

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4. $f_X(x) = \lambda e^{-\lambda x} \quad , \quad x > 0$

$f_Y(y) = \lambda e^{-\lambda y} \quad , \quad y > 0$

X and Y are independent

$$f_{xy}(x,y) = \lambda^2 e^{-\lambda(x+y)} \quad , \quad x,y > 0$$

$U = X + Y$

$V = X - Y$

$\Rightarrow X = \frac{U+V}{2} \quad , \quad Y = \frac{U-V}{2}$

$$J = \begin{vmatrix} \partial X/\partial U & \partial X/\partial V \\ \partial Y/\partial U & \partial Y/\partial V \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

a) $g_{UV}(U,V) = |J| \, f_{xy}(x,y)$

$$\boxed{g_{UV}(U,V) = \frac{\lambda^2}{2} e^{-\lambda y} \quad , \; x+y > 0 \; , \; x-y > 0}$$

b) $f_U(U) = \int_{-U}^{U} \frac{\lambda^2}{2} e^{-\lambda U} \, dV = \frac{\lambda^2}{2} e^{-\lambda U}(2U) = \boxed{\lambda^2 U e^{-\lambda U}} \; , \; U > 0$

$$f_V(V) = \begin{cases} \int_{V}^{\infty} \frac{\lambda^2}{2} e^{-\lambda U} \, dU , & V \geq 0 \\ \int_{-V}^{\infty} \frac{\lambda^2}{2} e^{-\lambda U} \, dU , & V < 0 \end{cases} \Rightarrow \begin{cases} \frac{\lambda}{2}\left(e^{-\lambda U}\right) , & U \geq 0 \\ \frac{\lambda}{2}\left(e^{\lambda U}\right) , & V < 0 \end{cases}$$

c) No, they are not independent.

$$f_U(U) f_V(V) \neq f_{UV}(U,V)$$

5. $f_{XY}(x,y) = \begin{cases} \frac{xy}{96}, & 0<x<4,\ 1<y<5 \\ 0, & \text{otherwise} \end{cases}$

$U = X + 2Y$

$$f_X(x) = \int_1^5 f_{XY}(x,y)\,dy = \frac{x}{96}\left[\frac{y^2}{2}\right]_1^5 = \frac{x}{8}$$

$$f_Y(y) = \int_0^4 f_{XY}(x,y)\,dx = \frac{y}{96}\left[\frac{x^2}{2}\right]_0^4 = \frac{y}{12}$$

$$f_U(u) = \int_1^5 f_Y(y)\cdot\left(f_X(u-2y)\right)dy$$

$$= \int_1^5 \frac{y}{12} \times \frac{u-2y}{8}\,dy = \frac{1}{96}\left[\frac{uy^2}{2} - \frac{2y^3}{3}\right]_1^5$$

$$\cancel{= \frac{1}{96}\left[12u - 8u\right] = \frac{1}{96} \times 4u}$$

$$= \frac{1}{96}\left[12u - \frac{248}{3}\right]$$

$$\boxed{f_U(u) = \frac{u}{8} - \frac{31}{36}}$$

$U = X + 2y$

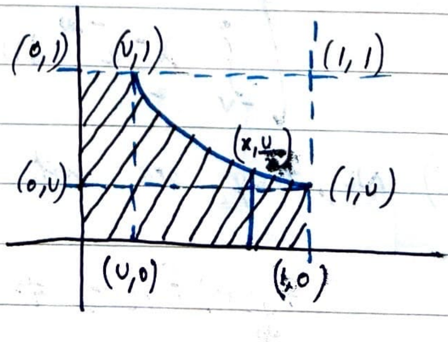
$x \in (0,4),\ y \in (1,5)$

6. $f_{XY}(x,y) = \begin{cases} x+y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

$U = XY$

$F_U(u) = P(U \le u)$

$= P(XY \le u)$



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$$= \int_0^U \int_0^1 (x+y)\, dy\, dx + \int_U^1 \int_0^{U/x} (x+y)\, dy\, dx$$

$$= \int_0^U \left(xy + \frac{y^2}{2}\right]_0^1 dx + \int_U^1 \left[xy + \frac{y^2}{2}\right]_0^{U/x} dx$$

$$= \left[\frac{x^2}{2} + \frac{x}{2}\right]_0^U + \left[Ux - \frac{U^2}{2x}\right]_U^1$$

$$= \left(\frac{U^2}{2} + \frac{U}{2}\right) + \left(U - \frac{U^2}{2} - \left(U^2 - \frac{U}{2}\right)\right]$$

$$= \frac{U^2}{2} + \frac{U}{2} - \frac{3U^2}{2} + \frac{3U}{2}$$

$$= 2U - U^2$$

$F_U(U) = U(2-U)\,, \quad 0 \le U \le 1$

$$f_U(U) = \frac{\partial}{\partial U}(2U - U^2) = 2 - 2U$$

$$\boxed{f_U(U) = 2(1-U)\,, \quad 0 \le U \le 1}$$

7.

Symbol	$p(x)$	$q(x)$
a	1/2	1/3
b	1/4	1/3
c	1/4	1/3

a) Entropy of $P(x)$

$$H_p(X) = \Sigma P(x = x_i) \log_2 \left(\frac{1}{P(X = x_i)}\right)$$

$$= \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{4}\log_2(4)$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$= 1.5$$

Entropy of $q(x)$

$$H_{q(x)} = \Sigma q(x = x_i) \log_2\left(\frac{1}{q(x=x_i)}\right)$$

$$= \frac{1}{3}\log_2(3) + \frac{1}{3}\log_2(3) + \frac{1}{3}\log_2(3)$$

$$= \log_2 3$$
$$\approx 1.585$$

Cross Entropy

$$CE(p,q) = \Sigma p(i) \log\left(\frac{1}{q(i)}\right)$$

$$= \frac{1}{2}\log(3) + \frac{1}{4}\log(3) + \frac{1}{4}\log(3)$$

$$= \log 3$$

b) $$D(p\|q) = \Sigma p(i) \log\left(\frac{p(i)}{q(i)}\right)$$

$$= \frac{1}{2}\log\left(\frac{3}{2}\right) + \frac{1}{4}\log\left(\frac{3}{4}\right) + \frac{1}{4}\log\left(\frac{3}{4}\right)$$

$$= \frac{1}{2}\log\left(\frac{3}{2} \times \frac{3}{4}\right)$$

$$= \frac{1}{2}\left(2\log 3 - 3\log 2\right)$$

$$= \boxed{\log_2 3 - \frac{3}{2}}$$

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$$D(q\|p) = \Sigma q(i)\log\left(\frac{q(i)}{p(i)}\right)$$

$$= \frac{1}{3}\log\left(\frac{2}{3}\right) + \frac{1}{3}\log\left(\frac{4}{3}\right) + \frac{1}{3}\log\left(\frac{4}{3}\right)$$

$$= \frac{1}{3}\log\left(\frac{2}{3}\right) + \frac{2}{3}\log\left(\frac{4}{3}\right)$$

c) No, $D(p\|q) \neq D(q\|p)$

i.e. $\log_2 3 - \frac{3}{2} \neq \frac{1}{3}\log\left(\frac{2}{3}\right) + \frac{2}{3}\log\left(\frac{4}{3}\right)$

8. $p(x) = N(\mu_1, \sigma^2)$, $q(x) = N(\mu_2, 1)$

$$KL(p(x)\|q(x)) = \int p(x)\log\left(\frac{p(x)}{q(x)}\right)dx$$

$$= \underbrace{\int p(x)\log(p(x))\,dx}_{I} - \underbrace{\int p(x)\log(q(x))\,dx}_{II}$$

$$I = \int p(x)\log\left(e^{\frac{-(x-\mu_1)^2}{2\sigma^2}}\right)dx - \int p(x)\log(\sigma\sqrt{2\pi})\,dx \quad \left[\because p(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu_1)^2}{2\sigma^2}}\right]$$

$$\boxed{I = \frac{-1}{2} - \log(\sigma\sqrt{2\pi})}$$

$$II = \int p(x)\log\left(e^{\frac{-(x-\mu_2)^2}{2}}\right)dx - \int p(x)\log(\sqrt{2\pi})\,dx \quad \left[\because q(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-(x-\mu_2)^2}{2}}\right]$$

$$\boxed{II = -\log(2\pi) - \frac{\sigma^2}{2} - \frac{(\mu_1-\mu_2)^2}{2}}$$

$$KL(p(x)\|q(x)) = I - II$$

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$$\boxed{KL(p\|q) = -\log(\sigma) + \frac{\sigma^2 - 1}{2} + \frac{(\mu_1 - \mu_2)^2}{2}}$$ Ans

$$\frac{\partial KL(p\|q)}{\partial \mu_1} = 0 + 0 + \frac{2(\mu_1 - \mu_2)}{2} = 0$$

$$\therefore \boxed{\mu_1 = \mu_2}$$

Value of KL is minimum when $\mu_1 = \mu_2$.

$$\boxed{KL(p\|q) = \frac{\sigma^2 - 1}{2} - \log(\sigma)}$$ $(\because \mu_1 = \mu_2)$ (Min. Value)

9. $$f = \begin{cases} \left(\frac{2}{x}\right)^3, & x > 2 \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \int_2^x \left(\frac{2}{x}\right)^3 dx, \quad x > 2$$

$$= 8\left[-\frac{1}{2x^2}\right]_2^x$$

$$= 4\left[\frac{1}{4} - \frac{1}{x^2}\right]$$

$$\boxed{F(x)} = \begin{cases} \boxed{1 - \frac{4}{x^2}}, & x > 2 \\ 0, & \text{otherwise} \end{cases}$$

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let $V = 1 - \frac{4}{x^2}$

$$\frac{4}{x^2} = 1 - V$$

$$\frac{4}{1-V} = x^2$$

$$\boxed{x = \left(\frac{4}{1-V}\right)^{1/2} = f(V)}$$

Acc. to inverse Transform Sampling, $V \sim V[0,1]$

$$X = \left(\frac{4}{1-V}\right)^{1/2}$$

$$1 - V[0,1] \sim V[0,1]$$

$$\therefore \quad X = \left[\frac{4}{V}\right]^{1/2}$$

$$\boxed{X = \frac{2}{\sqrt{V}}}$$ Ans

10.- Define → mean (μ), standard deviation (σ)
- Define Intervals → (a, b)

- Cal. Z for a and b

$$Z_a = \frac{a-\mu}{\sigma}, \quad Z_b = \frac{b-\mu}{\sigma}$$

- Cal. value of CDF function at lower and upper bounds, i.e., ϕ_a and ϕ_b.

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- Use inverse Transform Sampling Method

i) Generate ($v \in U \sim U[\phi_a, \phi_b]$)
ii) Calculate Z
$$Z_v = f^{-1}(v)$$

iii) $x = Z_v \cdot \sigma + \mu$
iv) Check if $x \in [a, b]$
v) If True, x is req. number.
If false, repeat the process.

- Repeat the steps until req. numbers are obtained.