DS-2(IC-252) Theory Assignment 2A

Submitted By – Abhinn Goyal

Course – B.Tech. 1st Year

Roll No. – B23108

Submitted To – Dr. Manoj Thakur

Session – (2023-2024)

_/_/__

Theory Assignment - 2a

1. $X \sim \text{Uniform}(-a, a)$

$$\underbrace{P(X = x)}_{f_X(x)} = \begin{cases} \frac{1}{2a} & , x \in [-a, a] \\ 0 & , \text{otherwise} \end{cases}$$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$= \begin{cases} \int_{-a}^{x} \frac{1}{2a}\, dx = \frac{x+a}{2a} & , x \in [-a, a] \\ 0 & , x < -a \\ 1 & , x > a \end{cases}$$

i) $P[x > 4] = \frac{1}{3}$

$\Rightarrow P[x \leq 4] = \frac{2}{3}$

$F_X(4) = \frac{2}{3}$

$\Rightarrow \frac{4+a}{2a} = \frac{2}{3}$

$\Rightarrow \boxed{a = 12}$ Ans

ii) $P(X < 1] = \frac{3}{4}$

$\frac{1+a}{2a} = \frac{3}{4}$

$\Rightarrow \boxed{a = 2}$ Ans

__/__/__

(iv) $P[|X|<2] = P[|X|>2]$
$= 1 - P[|X|<2]$

$\Rightarrow P[|X|<2] = \frac{1}{2}$

$P[-2 < X < 2] = \frac{1}{2}$

$F_X(2) - F_X(-2) = \frac{1}{2}$

$\frac{2+a}{2a} - \frac{(-2)+a}{2a} = \frac{1}{2}$

$\Rightarrow \boxed{a=4}$ Ans

2. X: Waiting Time.

$P[X \geq 3] = ?$

$f_X(x) = P(X=x) = \begin{cases} 1/5, & x \in (0,5) \\ 0, & \text{otherwise} \end{cases}$

$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x) = \begin{cases} 0, & x \leq 0 \\ x/5, & x \in (0,5) \\ 1, & x \geq 5 \end{cases}$

$P(x \geq 3) = 1 - P[X<3]$
$= 1 - F_X(3) = 1 - \frac{3}{5}$

$\Rightarrow \boxed{P(x \geq 3) = \frac{2}{5}}$ Ans

__/__/__

3. a) $P[X > 0.62] = 1 - P[X < 0.62]$

$= 1 - P\left[\frac{X - 0.5}{0.05} \leq \frac{0.62 - 0.5}{0.05}\right]$

$= 1 - P(Z \leq 2.4)$

$= 1 - 0.9918$

$= 0.0082$ Ans

b) $P(0.63 > x > 0.47)$

$= P(x \leq 0.63) - P(x \leq 0.47)$

$= P(z \leq 2.6) - P(x \leq 0.6)$

$= 0.99534 - 0.27425$

$= 0.72109$ Ans

c) $P(X \leq ?) = 0.9$

Let width be t.

$P\left(\frac{X - 0.5}{0.05} \leq \frac{t - 0.5}{0.05}\right) = 0.9$

$\Rightarrow P\left(Z \leq \frac{t - 0.5}{0.05}\right) = 0.9$

$\Rightarrow \frac{t - 0.5}{0.05} = 1.28$

$\Rightarrow \boxed{t = 0.564 \text{ u/n}}$ Ans

4. $P(10.20 > x > 10.05) = F_x(10.20) - F_x(10.05)$

$P\left(\frac{10.2 - 10.1}{0.2} > \frac{x - 10.1}{0.2} > \frac{10.05 - 10.1}{0.2}\right)$

$= P(0.5 > z > -0.25)$

$= F_z(0.5) - F_z(-0.25)$

$= 0.69146 - 0.40129$

$= 0.29017$

$= 29\%$ Ans

_/_/_

5. Assuming Normal Distribution:
$X \to$ Height of Soldiers $\quad X \sim \text{Normal}(68.22, \sqrt{10.8})$

$$P(X > 72) = P\left(\frac{X - 68.22}{\sqrt{10.8}} > \frac{72 - 68.22}{\sqrt{10.8}}\right) = P(Z > 1.15)$$
$$= 0.1251$$

No. of Soldiers with ht. above 6 feet in a group of 1000 soldiers $= 100 \times 0.1251$
$$= \boxed{125} \text{ Ans}$$

6. $$P(7.5 < x < 7.6) = P\left(\frac{7.5 - 7.55}{0.02} < \frac{x - 7.55}{0.02} < \frac{7.6 - 7.55}{0.02}\right)$$
$$= P(-2.5 < z < 2.5)$$
$$= F(2.5) - F(-2.5)$$
$$= 0.9876$$

% of rejected products $= (1 - 0.9876) \times 100$
$$= \boxed{1.24\%} \text{ Ans}$$

7. Let score be t.

$$P(X > t) = 0.01$$
$$P\left(\frac{X - 100}{14.2} > \frac{t - 100}{14.2}\right) = 0.01$$
$$P\left(Z > \frac{t - 100}{14.2}\right) = 0.01$$
$$P\left(Z \leq \frac{t - 100}{14.2}\right) = 0.99$$

_/_/__

$$\frac{k - 100}{14.2} = 2.33$$

$$\boxed{k = 133.086} \quad \underline{\underline{Ans}}$$

Top 1% of scores lie in range $[133.086, \infty]$

8. $X \sim \text{Normal}(6, \sqrt{9})$, $Y \sim \text{Normal}(7, \sqrt{16})$

$P(2X + Y \leq \lambda) = P(4X - 3Y \geq 4\lambda)$

Consider $Z_1 = 2X + Y$

Mean $(Z_1) = 2(\text{Mean } X) + \text{Mean } Y = 19$
Var $(Z_1) = 2^2 (\text{Var } X) + \text{Var } Y = 4(9) + 16 = 52$

$Z_1 \sim \text{Normal}(19, \sqrt{52})$

Similarly, $Z_2 = 4X - 3Y$

mean $Z_1 = 4\mu_x - 3\mu_y = 3$
Var $Z_1 = 16\sigma_x^2 + 9\sigma_y^2 = 2(144) = 288$

$Z_2 \sim \text{Normal}(3, \sqrt{288})$

$P(2X + Y \leq \lambda) = P(Z_1 \leq \lambda)$ $[Z_1 = 2X + Y]$
$\stackrel{?}{=} P((Z_1 - 19)/\sqrt{52}) \leq (\lambda - 19)/\sqrt{52})$

$\stackrel{?}{=} F\left[\frac{\lambda - 19}{\sqrt{52}}\right]$

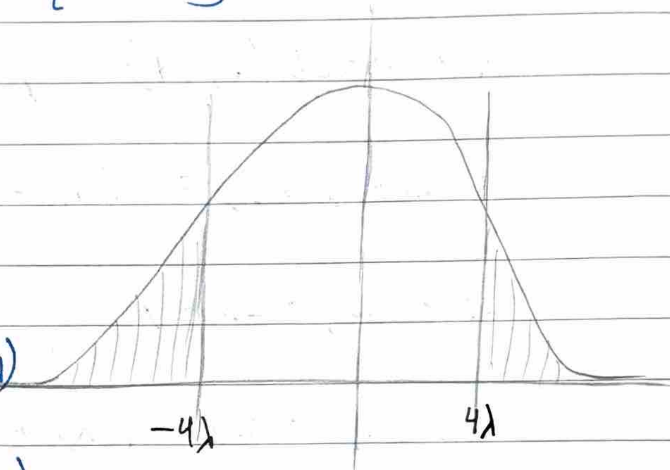
$$P(4X - 3Y \geq 4\lambda) = 1 - P(Z_2 < 4\lambda) \qquad [Z_2 = 4X - 3Y]$$

$$= 1 - P\left[\frac{Z_2 - 3}{12\sqrt{2}} < \frac{4\lambda - 3}{12\sqrt{2}}\right]$$

$$= 1 - F\left[\frac{4\lambda - 3}{12\sqrt{2}}\right]$$

Equating LHS & RHS,

$$F\left(\frac{\lambda - 19}{\sqrt{52}}\right) + F\left(\frac{4\lambda - 3}{12\sqrt{2}}\right) = 1$$



$$P(Z_2 \geq 4\lambda) = P(Z_2 \leq -4\lambda)$$

$$\therefore F\left(\frac{\lambda - 19}{\sqrt{52}}\right) = F\left(\frac{-4\lambda - 3}{12\sqrt{2}}\right)$$

$$\Rightarrow \frac{\lambda - 19}{\sqrt{52}} = \frac{-4\lambda - 3}{12\sqrt{2}}$$

$$\Rightarrow \boxed{\lambda = 6.5678} \text{ Ans}$$

9. a) No. of electrical Connectors = 1000

No. of Defective = 100

No. in selected Samples = 25

$$P(\text{No. of Defective Connector in sample}) = \left(\frac{900}{1000}\right)\left(\frac{899}{999}\right)\cdots\left(\frac{876}{976}\right)$$

$$= \left(1 - \frac{100}{1000}\right)\left(1 - \frac{100}{999}\right)\cdots\left(1 - \frac{100}{976}\right)$$

$$= \left(1 - \frac{100}{1000}\right)^{25} = \left(e^{\frac{-100}{1000}}\right)^{25} = 0.082 \text{ Ans}$$

b) Mean (μ) = $np = 2.5$

x: No. of defective Connectors

$\sigma = \sqrt{npq} = \sqrt{2.25} = 1.5$

$P(X=0) \rightarrow$ Using Continuity Correction

$P(X<0.5) = P\left(Z < \frac{0.5-2.5}{1.5}\right) = P\left(Z < \frac{-4}{3}\right) = 0.092$

This approx. is satisfactory as it has only 10 % variation from the result of binomial approx.

c) $P(\text{No defective connector in Sample}) = \left(\frac{400}{500}\right)\left(\frac{399}{499}\right)\cdots\left(\frac{376}{476}\right)$

$= \left(1 - \frac{100}{500}\right)^{25} = \left(e^{\frac{-100}{500}}\right)^{25} = e^{-5} = 0.0067$

X : No. of defective Connectors

Using normal approx.

Mean (μ) = $np = 5$

$\sigma = npq = 2$

$P(X=0) \rightarrow$ Using Continuity Correction

$P(X < 0.5) = P\left(Z < \frac{0.5-5}{2}\right) = P(Z < -2.25)$

$= 0.0122$

Normal approx. is no longer satisfactory in this case.

_ / _ / _

10- $X \sim \text{Exponential}\left(\frac{1}{15}\right)$

pdf of $X = \lambda e^{-\lambda t} \Rightarrow P(X=t) = \frac{1}{15} \cdot e^{(-1/15)t}, \ (t>0)$

cdf of $X = 1 - e^{-\lambda t} \Rightarrow P(X \le t) = 1 - e^{(-1/15)t}, \ (t>0)$

a) P(no calls in 30 min. interval)

$P(X > 30)$

$\Rightarrow 1 - P(X \le 30)$

$\Rightarrow 1 - (1 - e^{-30/15}) = e^{-2}$

$= 0.135$ Ans

b) P(Atleast one call in 10 min. interval)

$= 1 - P(\text{no Call in 10 min. interval})$

$= 1 - P(X > 10)$

$= P(X \le 10)$

$= 1 - e^{-10/15} = 1 - e^{(-2/3)} = 0.486$ Ans

c) P(first call arrives within 5 & 10 min)

$= P(5 < X < 10)$

$= F(10) - F(5)$

$= 1 - e^{-10/15} - (1 - e^{-5/15})$

$= e^{-1/3} - e^{-2/3}$

$= 0.203$ Ans

d) $P(X < t) = 0.90$

$\Rightarrow 1 - e^{-t/15} = 0.90$

$\Rightarrow \boxed{t \approx 34.539}$

11. let X : Amount of time that appointment lasts

$X \sim \text{Exponential}(1/30) \Rightarrow E[X] = \frac{1}{\lambda} = 30$ min.

$f_X(t) = P(X=t) = \frac{1}{30} e^{(-t/30)}$

Expected time that 1:30 appointment spends

= Expected Time for only 1:30 appointment + Expected waiting time at office

$= 30 + \int_{30}^{\infty} e^{-(t/30)} \, dt$

$= 30 + \frac{30}{e}$

$= \boxed{41.04 \text{ min}}$ Ans

12. X: Time taken to repair a machine (in hrs)

$X \sim \text{Exponential}(1/2)$

$\Rightarrow f_X(x) = \frac{1}{2} e^{(-x/2)}$

a) $F_X(x) = \int_0^x f_X(x)\,dx = 1 - e^{-x/2}$

Required : $P(X > 2)$

$= 1 - P(X \leq 2)$

$= 1 - F_X(2)$

$= 1 - (1 - e^{-1})$

$= \frac{1}{e} = \boxed{0.368}$ Ans

b) $P(X \geq 10 \,/\, X > 9)$

$= \frac{P(X \geq 10 \text{ and } X > 9)}{P(X > 9)} = \frac{P(X \geq 10)}{P(X > 9)} = \frac{1 - F_X(10)}{1 - F_X(9)}$

$= \frac{e^{-10/2}}{e^{-9/2}} = e^{-1/2}$

$= 0.606$ Ans

__/__/__

13. Let X: Req. CPU time (in ms)

$X \sim \text{Exponential}\left(\frac{1}{140}\right)$

$\Rightarrow f_X(t) = \frac{1}{140} e^{-t/140}$ and $F_X(t) = 1 - e^{-t/140}$

P(arriving job is forced to wait for second quantum)

$= P(X > 100)$

$= 1 - P(X \le 100)$

$= 1 - F_X(100)$

$= e^{-100/140} = \boxed{0.489}$ Ans

Job expected to finish within first quantum $= 800(1 - 0.489)$

$= \boxed{409}$ Ans

14. $\lambda = \frac{1}{20}$, E[No. of accidents/working day) $= \frac{1}{20}$

let X: No. of days until first accident

$X \sim \text{Exponential}\left(\frac{1}{20}\right) \Rightarrow f_X(x) = \frac{1}{20} e^{-t/20}$

$F_X(x) = 1 - e^{-t/20}$

a) $P(X > 7)$

$= 1 - P(X \le 7) = e^{-7/20}$

$= \boxed{0.705}$ Ans

b) $P(1 < X < 3) = F_X(3) - F_X(1)$

$= 1 - e^{-3/20} - 1 + e^{-1/20}$

$= e^{-1/20} - e^{-3/20}$

$= \boxed{0.246}$ Ans