Theory Assignment - 1

1. $P(X=1)$

$$P = \frac{5 \times 9!}{10!} = \frac{1}{2}$$

$P(X=2)$

$$P = \frac{5^2 \times 8!}{10!} = \frac{5}{18}$$

$$P(X=3) = \frac{{}^5C_2 \times 7! \times {}^5C_1 \times 2!}{10!} = \frac{5}{36}$$

$$P(X=4) = \frac{{}^5C_3 \times 3! \times {}^5C_1 \times 6!}{10!} = \frac{5}{84}$$

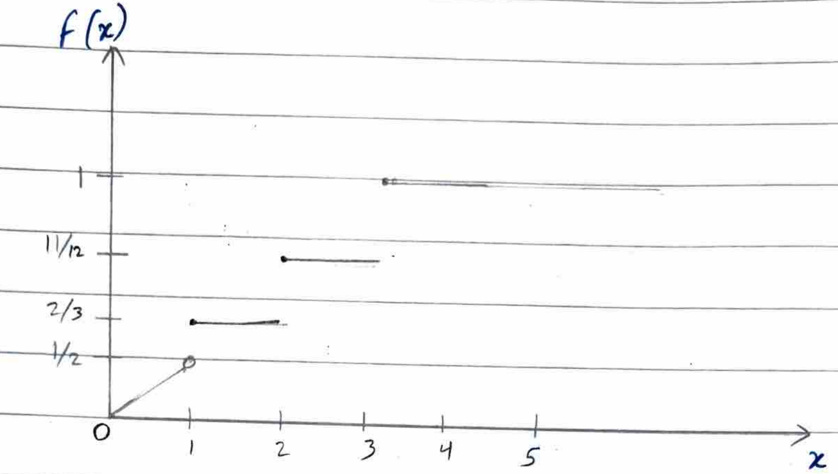
$$P(X=5) = \frac{{}^5C_4 \times 4! \times {}^5C_1 \times 5!}{10!} = \frac{5}{252}$$

$$P(X=6) = \frac{{}^5C_5 \times 5! \times {}^5C_1 \times 4!}{10!} = \frac{1}{252}$$

$P(X=7, 8, 9, 10) = 0$, because the Highest lowest position the girl can score is 6.

2. $$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x}{2}, & 0 \le x < 1 \\ 2/3, & 1 \le x < 2 \\ 11/12, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

a)



b) $P\left(X > \frac{1}{2}\right)$

$= P\left(X > \frac{1}{2}\right) = 1 - F\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$

c) $P(2 < x \leq 4)$

$= P(4) - P(2) = 1 - \frac{11}{12} = \frac{1}{12}$

d) $P(X < 3)$

$= F(3) = \frac{11}{12}$

e) $P(X = 1) = 0$, Probability at a point for a continuous function is 0.

3. $f(x) = \begin{cases} \lambda e^{-x/100}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

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Between 50 & 150 hours $\Rightarrow P(50 < x < 150)$

$$= \int_{50}^{150} \lambda e^{-x/100} \, dx$$

$$= -\lambda \int_{50}^{150} e^{-x/100} \times 100 \, dx = -100\lambda \left(e^{-1.5} - e^{-0.5}\right)$$

less than 100 hours $\Rightarrow P(x < 100)$

$$= \int_{-\infty}^{100} \lambda e^{-x/100} \, dx$$

$$= -100\lambda \left[e^{-x/100}\right]_0^{100} = -100\lambda\left(e^{-1} - 1\right)$$

4. X = Money paid to Customer

$$X = \begin{cases} 0 & , E \text{ does not occur} \\ A & , E \text{ occurs} \end{cases}$$

If company charges money B to customer,

profit $= B - X$

$$E(B-X) = 0.1A$$

$$= E(B) - E(X) = 0.1A$$

$$= B - 0 \times (1-p) - A \times p = 0.1A$$

$$\Rightarrow \boxed{B = A(p + 0.1)}$$

5. $$f(x) = \begin{cases} a + bx^2 & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$E(X) = \frac{3}{5}$$

$$E(X) = \int_0^1 \left(ax + bx^3\right) dx$$

$$F(x) = \frac{ax^2}{2} + \frac{bx^4}{4}\Big]_0^1$$

$$F(x) = \left(\frac{a}{2} + \frac{b}{4}\right)$$

$$\frac{a}{2} + \frac{b}{4} = \frac{3}{5} \qquad \left(\because F(x) = \frac{3}{5}\right)$$

$$4a + 2b = \frac{24}{5} \Rightarrow \boxed{20a + 10b = 24} - ①$$

We know, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty} a + bx^2 \; dx = 1$$

$$\Rightarrow \int_0^1 (a + bx^2)\,dx = 1$$

$$ax + \frac{bx^3}{3}\Big]_0^1 = 1$$

$$a + \frac{b}{3} = 1$$

$$\boxed{3a + b = 3} - ②$$

$$30a + \cancel{10b} = 30$$

$$-\;20a \pm \cancel{10b} = -24$$

$$10a = 6$$

$$\boxed{a = \frac{3}{5}}$$

$$3\left(\frac{3}{5}\right) + b = 3$$

$$b = 3 - \frac{9}{5}$$

$$\boxed{b = \frac{6}{5}}$$

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6. a) $f(x) = e^{-x}, \quad x \geq 0$

$$F(x) = \int_{-\infty}^{x} f(x) \cdot dx$$

for $F(m) = \frac{1}{2}$,

$$\Rightarrow \int_{-\infty}^{m} e^{-x} \, dx = \frac{1}{2}$$

$$\Rightarrow -\left(e^{-x}\right)_0^m = \frac{1}{2}$$

$$\Rightarrow -\left(e^{-m} - 1\right) = \frac{1}{2}$$

$$\Rightarrow \boxed{e^{-m} = \frac{1}{2}}$$

$$\Rightarrow \frac{1}{e^m} = \frac{1}{2}$$

$$e^m = 2$$

$$\boxed{m = \ln(2)}$$

b) $f(x) = 1, \quad 0 \leq x \leq 1$

$$\int_0^m 1 \cdot dx = \frac{1}{2}$$

$$\Rightarrow \boxed{m = \frac{1}{2}}$$

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7. $X =$ No. of Heads

$R_x = \{0, 1, 2, 3)$

$$E(X) = \sum_{i=0}^{3} x_i \; P(X = x_i)$$

$$= 0 \times P(X=0) + 1 \times P(X=1) + 2 \times P(X=2) + 3 \times P(X=3)$$

$$= 0 + \frac{3}{8} + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right)$$

$$E(X) = \frac{3+6+3}{8} = \frac{12}{8} = \frac{3}{2}$$

$$\boxed{E(X) = \frac{3}{2}}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{i=0}^{3} x_i^2 \; P(X = x_i)$$

$$= \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(4 \times \frac{3}{8}\right) + \left(9 \times \frac{1}{8}\right)$$

$$E(X^2) = 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$$

$$V(X) = 3 - \frac{9}{4} = \frac{12-9}{4}$$

$$\boxed{V(X) = \frac{3}{4}}$$

8. $f(z) = \begin{cases} z-8 & , 8 \le x \le 9 \\ 10-z & , 9 < x \le 10 \\ 0 & , \text{otherwise} \end{cases}$

a) $E(x) = \int_8^9 z(z-8)\, dz + \int_9^{10} z(10-z)\, dz$

$$= \left[\frac{z^3}{3} - \frac{8z^2}{2}\right]_8^9 + \left[\frac{10z^2}{2} - \frac{z^3}{3}\right]_9^{10}$$

$$= \left[\left(\frac{729-972}{3}\right) - \left(\frac{512-768}{3}\right)\right] + \left[\left(\frac{1500-1000}{3}\right) - \left(\frac{1215-729}{3}\right)\right]$$

$$= \left[\left(\frac{-243}{3}\right) - \left(\frac{-256}{3}\right)\right] + \left[\frac{500}{3} - \frac{486}{3}\right]$$

$$= \frac{13}{3} + \frac{\cancel{2}14}{3} = \frac{27}{3}$$

$\boxed{E(x) = 9}$

$$E(x^2) = \int_8^9 z^2(z-8)\, dz + \int_9^{10} z^2(10-z)$$

$$= \left[\frac{z^4}{4} - \frac{8z^3}{3}\right]_8^9 + \left[\frac{10z^3}{3} - \frac{z^4}{4}\right]_9^{10}$$

$$= \frac{6561-4096}{4} - \frac{8}{3}(729-512) + \frac{10}{3}(-729+1000) - \left(\frac{10000-6561}{4}\right)$$

$$= \frac{2465}{4} - \frac{1736}{3} + \frac{2710}{3} - \frac{3439}{4}$$

$$= \frac{-974}{4} + \frac{974}{3}$$

$$E(x^2) = \frac{974}{12} = \frac{487}{6}$$

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$$Var(x) = E(x^2) - (E(x))^2$$
$$= \frac{487}{6} - 81$$

$$\boxed{Var(x) = \frac{1}{6}}$$

b) let X represent selling price per article

$$X = \begin{cases} 0, & x < 8.25 \\ 2, & x \geq 8.25 \end{cases}$$

Cost price of each article $= \frac{x}{15} + 0.35$

Profit made per article $= -\left(\frac{x}{15} + 0.35\right) + X = X - \frac{x}{15} - 0.35$

$$E(\text{Profit}) = E[X] - \frac{E(x)}{15} - E(0.35)$$

$$E(X) = \int_{8.25}^{9} z(z-8)\,dz + \int_{9}^{10} z(10-z)\,dz$$

$\cancel{E(x) = 1.94}$ $E[X] = \left[z^2 - 16z\right]_{8.25}^{9} + \left[20z - z^2\right]_{9}^{10}$

~~E[Profit] =~~ $= [(81-144) - (68.06-132)] + [(200-100) - (180-81)]$

$$= (-63) - (-63.94) + [100 - 99]$$
$$= 0.94 + 1$$

$$\boxed{E[X] = 1.94}$$

$$E[\text{Profit}] = 1.94 - \frac{1.94}{15} - 0.35 = 1.94 - 0.95$$

$$\boxed{E[\text{Profit}] = 0.99}$$

$\cancel{E(\text{Profit}) = 1.47}$

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9. P(System functions) = P(Atleast 2 Components work) = 1 - P(none work) - P(1 component works)

$P(\text{none work}) = {}^4C_0\,(0.6)^0(1-0.6)^4$

$= (0.4)^4 = 0.0256$

$P(1\text{ works}) = {}^4C_1\,(0.6)(0.4)^3$

$= 4 \times 0.6 \times (0.4)^3 = 0.1536$

$P(\text{System Works}) = 1 - 0.0256 - 0.1536$

$= 0.82$

10. $E[X] = 7$, $Var(X) = 2.1$

$E[X] = np$

$Var(X) = np(1-p)$

$\Rightarrow \quad np(1-p) = 2.1$

$1 - p = 0.3$

$\boxed{p = 0.7 \;,\; n = 10}$

a) $P(X=4)$

$= {}^{10}C_4 \times (0.7)^4 (0.3)^6$

$= \frac{10!}{4!\,6!} \times 0.2401 \times 0.000729$

$= 210 \times 0.000175$

$= 0.036$

b) $P(X > 12)$

Since, $n = 10 \Rightarrow P(X > 12) = 0$

11. 50 Lotteries , $P = \frac{1}{100} = 0.01$

a) $1 - P(\text{Never Winning}) = 1 - {}^{50}C_0 (0.01)^0 (0.99)^{50}$

$= 1 - 0.60$

$= 0.4$

b) $P(\text{Exactly One}) = {}^{50}C_1 (0.01)^1 (0.99)^{49}$

$= \frac{50}{100} \times 0.61 = 0.5 \times 0.61$

$= 0.305$

c) $P(\text{Atleast Twice}) = 1 - (P(\text{Never Winning}) + P(\text{Exactly One}))$

$= 1 - (0.6 + 0.305)$

$= 1 - (0.905)$

$= 0.095$

12. $\lambda = 121.95$

$f(k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$F(k) = e^{-\lambda} \sum_{k=0}^{x} \frac{\lambda^k}{k!}$

a) $P(\text{130 Attritions or more}) = 1 - P(\text{Less than 130 Attritions})$

$= 1 - F_x(k = 129)$

$F(k=129) = e^{-121.95} \sum_{k=0}^{129} \frac{(121.95)^k}{k!}$

≈ 0.7555

$P = 1 - 0.7555$

$\boxed{P = 0.2455}$

b) $P(\text{100 Attritions or less}) = \sum_{k=0}^{100} e^{-121.95} \frac{(121.95)^k}{k!} \approx 0.02338532$

$= 0.0234$