Linear Transform: $T(av_1 + bv_2) = aT(v_1) + bT(v_2)$

Null space: $N(T)$ $\{v \in V \mid T(v) = 0\}$

Range space: $R(T)$ $\{T(v) \mid v \in V\}$

Rank-Nullity Theoram $\Rightarrow$ $\dim(N(T)) + \dim(R(T)) = \dim(V)$

$A_{m\times n}$, $B_{n\times q}$ matrices $\quad \text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$

Division Algorithm: $f = dq + r$ $\{$either $r = 0$ or $\deg(r) < \deg(d)\}$

$f(x) = \phi(x)\, q(x) + r(x)$ $\qquad f(A) = r(A)$

characteristic equation. $\qquad f(\lambda_i) = r(\lambda_i) \Rightarrow$ get $\cancel{r}(x)$

★ when there are same eigen values, use differenciation, when λ_i is a root of order k then $\phi(\lambda_i) = 0$, $\phi'(\lambda_i) = 0$, ... $\phi^{k-1}(\lambda_i) = 0$ But $\phi^{k}(\lambda_i) \neq 0$

★ isse $\sin(A)$, $\tan(A)$, e^{A}, etc. sb calculate kr skte hai.

Theoram ✱: $z = x + iy$, $f(z)$ has a taylor series $\sum_{k=0}^{\infty} a_k z^k$, converges for $\boxed{|z| < k}$. Also for a square matrix A, with eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ s.t. $\boxed{|\lambda_i| < k}$ then $f(A) = \sum_{k=0}^{a} a_k A^k$, and $f(A)$ is said to be well defined.

Inner Product: $X \to$ vector space $x, y \in X$

(i) $\langle x, x\rangle = \|x\|^2 \geq 0$, $\langle x, x\rangle = 0$ iff $x = 0$

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(iii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \quad \forall\, x, y, z \in X$

$\langle x, y\rangle = y^{*} x = \sum_{i=1}^{n} x_i \bar{y}_i \qquad y^{*} = (\bar{y}^{T})$

$\langle x_i, x_j\rangle = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ $\begin{matrix} \to \text{orthogonal} \\ \to \text{ortho normalised.} \end{matrix}$

★ Every orthonormal set of vectors $x_1, \ldots, x_n$ in an inner product space linearly independent

Date ___ / ___ / ___

Gram-Schmidt Orthonormalization (GSO)

$$y_n = x_n - \frac{\langle x_n, y_1 \rangle}{\langle y_1, y_1 \rangle} y_1 - \frac{\langle x_n, y_2 \rangle}{\langle y_2, y_2 \rangle} y_2 - \cdots - \frac{\langle x_n, y_{n-1} \rangle}{\langle y_{n-1}, y_{n-1} \rangle} y_{n-1}$$

$$\langle E_i, E_i \rangle = \frac{1}{\|y_i\|^2} \langle y_i, y_i \rangle = 1 = \|E_i\|^2$$

Similar Matrices: two matrices A & B of order n are similar if $\exists$ a non-singular matrix P s.t. $P^{-1}AP = B$

A & B have same eigenvalues

IMP: $X_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$ $X_2 = \begin{bmatrix} x_{12} \\ \vdots \\ x_{n2} \end{bmatrix}$ ———— $X_n = \ldots$

$$[X_1, X_2, \ldots, X_n] = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & & x_{2n} \\ \vdots & & \vdots \\ x_{n1} & & x_{nn} \end{bmatrix}$$

Theorem: If A $\xrightarrow{\text{eigenvalues}}$ $\lambda_1, \lambda_2, \ldots, \lambda_n$ (all distinct)

D $\xrightarrow{\text{diagonal matrix}}$ of $\lambda_1, \ldots, \lambda_n$, then A is similar to D.

$$\underbrace{P^{-1} A P}_{} = \overbrace{\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)}^{D}$$

modal matrix (formed by eigenvectors) — Spectral matrix

Diagonalizable Matrix: A matrix 'A' is diagonalizable over field F if there exists non-singular matrix P over field F such that $P^{-1}AP$ is a diagonal matrix D over F.

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eigenvectors

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Theorem: $A \to$ eigen values $(\lambda_1, \lambda_2, \dots, \lambda_n)$, eigen vectors $(x_1, x_2, \dots, x_n)$, then $f(A)$ has $f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)$ as eigen values and $x_1, x_2, \dots, x_n$ as corresponding eigen vectors. But inverse is not true as there can be more eigenvalues other than these.

Lemma: $p(x) \to$ polynomial, then $\underline{\underline{|p(A)| = p(\lambda_1)\, p(\lambda_2) \dots p(\lambda_n)}} \to \lambda_i$'s are eigenvalues of A.

det $(p(A))$

Not all eigenvectors of $f(A)$ are eigenvectors of A.

Theorem: Eigen vectors associated with distinct eigen values of an n-square matrix A are linearly independent.

Theorem: $\forall H \to$ Hermitian Matrix, there exists a unitary transformation S, such that $S^*HS = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ $\lambda_i \to$ eigen values of H not necessarily distinct and non-zero. $(S^*S = I)$

Corollary: $\forall A \to$ real symmetric matrix, there exists orthogonal ~~and~~ transformation P. such that $P^TAP = \mathrm{diag}(\lambda_1 \dots \lambda_n)$ $\lambda_1, \dots \lambda_n$ are eigenvalues of A not necessarily distinct and non-zero.

OR

Every real symmetric matrix A is orthogonally similar to a diagonal matrix whose diagonal elements are eigen values of A.

Corollary: For every square matrix A over C, there exists a unitary transform S such that S^*AS is an upper (or lower) triangular matrix (which is not unique) with the eigenvalues of A as diagonal elements.

(using the fact when H is not hermitian)

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Imp

1. Determine an orthogonal matrix S s.t. S^TAS is an upper triangular matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 2 \\ -3 & 3 & 4 \end{bmatrix} \qquad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 5 \end{matrix}$$

Ans. $\lambda = 1 \to$ eigenvector $\to (\alpha+\beta, \alpha, \beta)^T$

$x_1 = (1,1,0)^T$ $\quad x_3 = e_3 = (0,0,1)^T$ — x_1, x_2 & x_3 should be mutually orthogonal

$x_2 = e_2 = (0,1,0)^T$

x_1, x_2, x_3 are L.I. Applying Gram-Schmidt process to get orthonormal vectors.

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$U = [u_1 \; u_2 \; u_3]$$

$$U^TAU = \left[\begin{array}{c|cc} 1 & 3 & 3/\sqrt{2} \\ \hline 0 & 2 & 1/\sqrt{2} \\ 0 & 3\sqrt{2} & 4 \end{array}\right]$$

Take $G = \begin{bmatrix} 2 & 1/\sqrt{2} \\ 3\sqrt{2} & 4 \end{bmatrix}$ — eigenvalues $\to$ 1, 5

Eigenvector for $\lambda = 1 \Rightarrow \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$, orthonormal eigenvector is $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$

$$v_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \xrightarrow{\text{orthogonal}} v_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

$$T = [v_1 \; v_2] = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$$

$$T^TGT = \begin{bmatrix} 1 & -5/\sqrt{2} \\ 0 & 5 \end{bmatrix}$$

$$P = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{T}} \\ 0 & & \end{array}\right] = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix}$$

and the required transformation, $S = UP = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & -1 & -\sqrt{2} \\ \sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{bmatrix}$

Thus, $S^TAS = \begin{bmatrix} 1 & \sqrt{3}/2 & 3\sqrt{6}/2 \\ 0 & 1 & -5/\sqrt{2} \\ 0 & 0 & 5 \end{bmatrix}$ — upper triangular matrix

★ For lower triangular matrix, one has to start with transpose of A.

AX=0 has n-r linearly independant eigenvectors

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Saathi Notebooks

Theorem: The no. k of independant eigenvectors associated with an eigenvalue λ of multiplicity m of an n-square matrix A is equal to n-rank($\lambda I-A$)

$k = n - \text{rank}(\lambda I - A)$

Theorem $\text{rank}(AQ) = \text{rank}(Q'A') = \text{rank}(A') = \text{rank}(A)$

Also, if P, Q are non singular, $\text{rank}(A) = \text{rank}(PAQ) = \text{rank}(AQ)$

Theorem: Let λ be an eigenvalue of multiplicity m of a Hermitian Matrix H, then the no. of linearly independant eigenvectors associated with λ is m.

Corollary The no. of linearly independant eigenvectors associated with an eigenvalue λ of multiplicity m of a real symmetric matrix is m.

Theorem The eigenvectors associated with distinct eigenvalues of a Hermitian matrix constitute an orthogonal set. (Or real symmetric matrix).

(or real symmetric)

Theorem H → Hermitian Matrix eigenvalue, λ, multiplicity = m.
Then, there exists a set of m no. of orthogonal (as well as orthonormal) eigen vectors associated with λ.

Theorem H → Hermitian matrix of order n possesses an orthogonal (as well as orthonormal) set of n vectors.

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Theorem: $A \rightarrow$ Hermitian (or real symmetric) matrix of order n with positive eigen values.

$B \rightarrow$ Hermitian (or real symmetric) matrix of order n.

Then there exist a $n \times n$ non-singular matrix P (not unique) such that,

$$P^* A P = I \quad \text{and} \quad P^* B P = \text{diag}(c_1, c_2, \ldots, c_n)$$

where c_i are real.

Imp

Q $A = \begin{bmatrix} 6 & 4 & -2 \\ 4 & 12 & -4 \\ -2 & -4 & 13 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ determine P such that $P^T A P = I$, $P^T B P$ is a diag. matrix.

$\lambda = 4, 9, 18$

Ans. $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ are eigenvectors (orthogonal)

Normalising them, $U = \dfrac{1}{3\sqrt{5}} \begin{bmatrix} -6 & 2 & \sqrt{5} \\ 3 & 4 & 2\sqrt{5} \\ 0 & 5 & -2\sqrt{5} \end{bmatrix}$ $\quad U^T A U = \text{diag}(4, 9, 18)$

Now, take Q s.t. $Q^{-1} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})$
$= \text{diag}(2, 3, 3\sqrt{2})$

so $Q = \text{diag}\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3\sqrt{2}}\right)$ $\quad Q^T = Q$

$(UQ)^T A (UQ) = Q^T (U^T A U) Q = I$.

Now, $C = (UQ)^T B (UQ) = \dfrac{1}{180} \begin{bmatrix} 81 & -18 & 9\sqrt{10} \\ -18 & 4 & -2\sqrt{10} \\ 9\sqrt{10} & -2\sqrt{10} & 10 \end{bmatrix}$

symmetric.

So $\exists$ an orthogonal matrix R s.t.

$R^T C R = \text{diag}(c_1, c_2, c_3)$ where c_1, c_2, c_3 are eigenvalues of C.

$c_1 = 0 \quad c_2 = 0 \quad c_3 = 19/36$

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Linearly independent eigen vectors $\Rightarrow \begin{pmatrix}-2\\-4\\\sqrt{10}\end{pmatrix}, \begin{pmatrix}1\\-1/2\\-\sqrt{10}\end{pmatrix}, \begin{pmatrix}9\\-2\\\sqrt{10}\end{pmatrix}$

orthonormalization $\longrightarrow y_1 = \frac{1}{\sqrt{30}}\begin{pmatrix}-2\\-4\\\sqrt{10}\end{pmatrix}; \; y_2 = \frac{-1}{\sqrt{285}}\begin{pmatrix}2\\-11\\-4\sqrt{10}\end{pmatrix}; \; y_3 = \frac{1}{\sqrt{95}}\begin{pmatrix}9\\-2\\\sqrt{10}\end{pmatrix}$

$R = [y_1 \; y_2 \; y_3]$

$$R^T C R = \operatorname{diag}[0, \; 0 \; \; 19/36]$$

$$P = UQR = \frac{1}{90\sqrt{57}}\begin{bmatrix}15\sqrt{38} & -120 & -160\sqrt{3}\\ -15\sqrt{38} & -150 & 85\sqrt{3}\\ -30\sqrt{38} & -30 & -40\sqrt{3}\end{bmatrix}$$

But P is not unique, another set of eigenvectors x, y give another P.

Definition: Vector x_m is generalised eigenvector of type p corresponding to matrix A and the eigen value λ if $(A-\lambda I)^p x_m = 0$ but $(A-\lambda I)^{p-1} x_m \neq 0$.

Bilinear Form

$$X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}_{x_m} \quad A = [a_1 \; a_2 \; a_3] = (a_{ij}) \quad Y = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}_{y}$$

~~X^TAY~~

$$X^TAY = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} x_i y_j$$

Remember to make A symmetric to obtain the correct solⁿ for A.

Quadratic Form $(Q = x^T A x)$

① Discriminant of quadratic form (det A) is called discriminant of quadratic form
② If A is singular ($|A|=0$), then quadratic form Q is called singular.
③ If A is non-singular, then quadratic form Q is called non-singular.

row operation → pre-multiplication. column operation → post-multiplication

Normal Form: Matrix A of order $m \times n$ can be reduced to any of these forms.

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad r < m, n \qquad \begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad r = n < m$$

$$\begin{bmatrix} I_r & 0 \end{bmatrix} \quad r = m < n \qquad \begin{bmatrix} I_r \end{bmatrix} \quad r = m = n$$

Congruent Form: Two matrices A & B of order n over a field F are said to be congruent over F iff $\exists$ a non-singular matrix P over F such that $B = P^T A P$.

Linear Transform of a quadratic form: Substitution of $x = Py$ in the quadratic form is called the linear transform of the quadratic form.

Singular and non-singular transformation: The linear transform $x = Py$ of a quadratic form $Q = x^T A x$ is singular or non-singular if P is singular or non-singular respectively.

Equivalent Quadratic form: Two quadratic form $x^T A x$ and $y^T B y$ are said to be equivalent over F if one can be obtained from the other by a non-singular transform over F defined by $x = Py$ such that $x^T A x = y^T B y$ $\quad B = P^T A P$.

$P \rightarrow$ non-singular.

Orthogonal Reduction: Reduction of quadratic form to diagonal form using orthogonal ~~reduction~~ transformation is called orthogonal reduction.

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Theorem: Every quadratic $x^T A x$ (A is symmetric) can be reduced to a diagonal quadratic form $y^T D y$ by means of orthogonal transform P, where the diagonal elements of D are the eigen values of A.

Corollary: Every quadratic form $x^T A x$ of rank r (i.e. A of rank r) can be reduced by orthogonal transform P to $\lambda_1 y_1^2 + \cdots + \lambda_r y_r^2$ where $x = Py$, and $\lambda_1, \lambda_2, \ldots, \lambda_r$ are non-zero eigenvalues of A.

Langrange's reduction:

$$q(x_1, x_2, x_3) = 4x_1^2 + 10x_2^2 + 11x_3^2 - 4x_1x_2 + 12x_1x_3 - 12x_2x_3$$

$$= 4\left(x_1^2 - \tfrac{1}{4}\cdot 2x_1\left(\frac{x_2 - 3x_3}{2}\right) + \left(\frac{x_2 - 3x_3}{2}\right)^2\right) - (x_2 - 3x_3)^2 + 10x_2^2 + 11x_3^2 - 12x_2x_3$$

$$= (2x_1 - x_2 + 3x_3)^2 + (3x_2 - x_3)^2 + (x_3)^2$$

$y_1 = \frac{1}{2}(2x_1 - x_2 + 3x_3)$

$y_2 = \frac{1}{3}(3x_2 - x_3)$ $\quad q = 4y_1^2 + 9y_2^2 + y_3^2 = y^T D y \quad D = diag(4, 9, 1)$

$y_3 = x_3$

$y = Px$

$y = (y_1\ y_2\ y_3)^T \quad x = (x_1\ x_2\ x_3)^T \quad P = \begin{bmatrix} 1 & -1/2 & +3/2 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \quad |P| = 1$

$z_1 = 2y_1$

$z_2 = 3y_2 \quad q = z_1^2 + z_2^2 + z_3^2$

$z_3 = y_3$

Def: Diagonal and unit diagonal form: If the matrix of a quadratic form is diagonal (respective identity) then its ~~diagonal~~ is called diagonal (unit) quadratic form.

$x^T I_n x = x_1^2 + x_2^2 + \cdots + x_n^2$ → unit quadratic $\quad x^T(\mathrm{diag}(a_1 \ldots a_n))x = a_1x_1^2 + \cdots + a_nx_n^2$ → diagonal quadratic form.

Ek baar check
Rv.

rank r

Jacobi Theorem: A quadratic form $q = x^T A x = \sum_{i,j} a_{ij} x_i x_j$ with ~~x~~ rank r can be transformed by a non-singular transform $x = Py$ with $\det(P) = 1$ and P be upper triangular to a diagonal form $q = c_1 y_1^2 + c_2 y_2^2 + \dots + c_r y_r^2$ if and only if Δ_i's $i = 1, 2, \dots, r$ are non-zero, where

$$y_1 = x_1 + \alpha_{12} x_2 + \dots + \alpha_{1r} x_r + \alpha_{1r+1} x_{r+1} + \dots + \alpha_{1n} x_n$$
$$y_2 =$$
$$\vdots$$
$$y_n =$$

are n linearly independent & real linear forms in variables x_i's, α_i's are scalars completely determined by a_{ij}'s, $c_i = \frac{\Delta_i}{\Delta_{i-1}}$ & Δ_i's are leading principal minors of A defined as.

$$\Delta_0 = 1 ;\ \Delta_1 = a_{11} ;\ \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} ;\ \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \dots \Delta_r = \det(A)$$

Hermitian form: Let $X = (x_1, \dots, x_n)$ be an n-vector in C^n and $H = (h_{ij})$ be a hermitian matrix of order n over the complex field C, then the form $h(x_1, x_2, \dots, x_n) = x^* H x = \sum_{i,j=1}^{n} h_{ij} \bar{x}_i x_j$ is called the Hermitian form of order n. The matrix H is called matrix of the form and rank of H is called rank of the form.

$(x^* H x)^* = x^* H x$, as H is hermitian.

Thus $q = \sum_{i,j} h_{ij} \bar{x}_i x_j$ is always real.

Diagonal and unit hermitian form: Diagonal form can be represented by $h = c_1 \bar{x}_1 x_1 + c_2 \bar{x}_2 x_2 + \dots + c_n \bar{x}_n x_n$ where coefficient c_i's are real as h is real (since $h = x^* H x$, H is hermitian). This is called diagonal form and if c_i's can be reduced to one, the a Hermitian form is given by $h = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \dots + \bar{x}_n x_n = x^* x$ is called unit hermitian form.

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Index & Signature of a quadratic form.

A quadratic form $Q = x^T A x$ can be reduced to diagonal form using non-singular transform. Let the reduced form be

$$Q = c_1 y_1^2 + c_2 y_2^2 + \cdots + c_r y_r^2 \quad - (1)$$

which we may write as,

$$Q = \alpha_1 y_1^2 + \alpha_2 y_2^2 + \cdots + \alpha_r y_r^2 \text{ where } \alpha_i\text{'s are +ve.} \quad - (2)$$

Then the no. of +ve terms in (1) is called the index and the difference between the no. of +ve and -ve terms which is, $k - (r-k) = 2k - r$ is called the signature of the quadratic form.

Theorem: Under a unitary transform ($x = Py$, P is a unitary matrix) a hermitian form $h = x^* H x$ can be ~~represented~~ reduced to diagonal form $h = d_1 \bar{y}_1 y_1 + d_2 \bar{y}_2 y_2 + \cdots + d_n \bar{y}_n y_n$, where d_i's are eigenvalues of H.

Theorem: Sylvester's law of inertia for Hermitian form:

Under all non-singular transforms, the rank r and the index k are invariants of a Hermitian form

does not change under transformations.

Defn: Equivalent Hermitian form: $x^* H x$ and $y^* Q y$ are ~~[illegible]~~ equivalent if $x^* H x = y^* P^* H P y = y^* Q y$.

or $x = Py$. $P \to$ non-singular

when same rank, index, they are ~~[illegible]~~ equivalent.

Corollary $q = x^T A x$ $\quad x = Py$ $\quad k \to$ index, $r \to$ rank of q.

then

$$q = y_1^2 + y_2^2 + \cdots + y_k^2 - y_{k+1}^2 - y_{k+2}^2 - \cdots - y_r^2$$ can be obtained

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Definition A real quadratic hermitian form $q(x)$ is said to be

(a) positive definite (PD) if $q(x) > 0$ for $x \neq 0$
(b) negative definite (ND) if $q(x) < 0$ for $x \neq 0$
(c) positive semidefinite (PSD) if $q(x) \geq 0$ for $x \neq 0$
(d) negative semidefinite (NSD) if $q(x) \leq 0$ for $x \neq 0$
(e) indefinite if $q(x) \gtrless 0$ for $x \neq 0$

IMP: space under consideration must be given. ($x_1^2 + x_2^2$ is PSD in R^3, and is PD in $R \times R$)

$x_1 = 0$
$x_2 = 0$
$x_3 = \text{any}$

Theorem: $q(x)$ → order n, rank r, signature p, real quadratic form of hermitian.

(a) PD iff $r = p = n$
(b) PSD iff $r = p < n$
(c) ND iff $r = -p = n$
(d) NSD iff $r = -p < n$
(e) indefinite iff $|p| < r$.

Defn Minor determinant of order p: Let $1 \leq i_1 < i_2 < \dots < i_p \leq n$ and $1 \leq j_1 < j_2 < \dots < j_p \leq n$, $1 \leq p \leq n$ be p row and column indices,

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ j_1 & j_2 & \dots & j_p \end{pmatrix} = \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \dots & a_{i_1 j_p} \\ a_{i_2 j_1} & \vdots & & \vdots \\ \vdots & & & \\ a_{i_p j_1} & \dots & & a_{i_p j_p} \end{vmatrix} \qquad \begin{pmatrix} 1 & 3 & 5 \\ 1 & 2 & 5 \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{31} & a_{32} & a_{35} \\ a_{51} & a_{52} & a_{55} \end{vmatrix}$$

Defn

Principal Minor

Selecting identical row & column matrices above $A = \begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix}$ Both are same

Leading-principal Minor

$i_1 = 1, i_2 = 2, \dots, i_p = p$ in principal minor is now leading-principal minor

$A = \begin{pmatrix} 1 & 2 & \dots & p \\ 1 & 2 & \dots & p \end{pmatrix}$

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Corollary: If the quadratic form $q = x^T A x$ (respective Hermitian form $x^* H x$) is PD, then $\exists$ a non-singular transformation $x = Py$ (respectively $x = Qy$) s.t. $x^T A x = y^T I_n y$ (resp. $x^* H x = y^* y$)

Theorem: If $q(x) = x^T A x$ is +ve definite,

ⓐ $\det A > 0$

ⓑ Every principal minor of A is +ve

ⓒ $a_{ii} > 0$, $i = 1, 2, \ldots, n$ where $A = (a_{ij})$

Theorem: If a real quadratic form $q(x) = x^T A x$ is psd then

ⓐ $|A| = 0$ ⓑ every principal minor of A is non-negative

ⓒ $a_{ii} > 0$ iff x_i^2 appears in $q(x)$.

Def^n: A real symmetric matrix A (respectively Hermitian H) is called +ve definite matrix (resp. +ve definite Hermitian matrix) iff the corresponding form $x^T A x$ ($x^* H x$) is +ve definite and is denoted by $A > 0$ ($H > 0$).

Theorem: A real matrix A is symmetric and p.d. iff one of the following statements hold,

(i) $P^T A P$ is symmetric and PD for any real and non-singular P.

(ii) $A = P^T P$ for a real and non-singular P [i.e., $\exists$ a non-singular matrix P s.t. $A = P^T P$]

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Real Symmetric ⟶ Hermitian

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Theorem: Slyvester's Criteria for positive definiteness

A quadratic form $q(x_1, x_2, \ldots, x_n) = x^T A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ is p.d. iff the leading principal minors are +ve i.e. all the determinants, $\Delta_1 = a_{11}$, $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, ..., $\Delta_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$ are +ve

Theorem: (Jacobi) A quadratic form $q = x^T A x = \sum_{(i,j=1)}^{n} a_{ij} x_i x_j$ with rank r can be transformed by a non-singular transformation $x = Py$ with $|P| = 1$, and P upper triangular to the diagonal form $q = c_1 y_1^2 + \cdots + c_r y_r^2$ iff Δ_i's, $i = 1, \ldots, r$ are non-zero where $c_i = \frac{\Delta_i}{\Delta_{i-1}}$, $\Delta_0 = 1$, $\Delta_1 = a_{11}$, $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Now. $\Delta_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \cdots & \cdots & a_{nn} \end{vmatrix} > 0 \Rightarrow |A| > 0$, thus full rank of quadratic form ($\Delta_n = \det A$) all the terms will be there.

$q = x^T A x \Rightarrow q = \sum \frac{\Delta_i}{\Delta_{i-1}} y_i^2$ $\Delta_0 = 1$ $\underline{\Delta_i > 0} \Rightarrow q > 0$
$\Rightarrow q$ is p.d.

Theorem: $x^T A x$ is p.d iff all principal minors of A are +ve

Theorem: $q(x_1, x_2, \ldots, x_n) = \sum_{i,j}^{n} a_{ij} x_i x_j$ is n.d. iff $a_{11} < 0$; $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$;

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \ldots$

even order ⟶ +ve
odd order ⟶ −ve } principal minors of A.

If $\lambda_k = 0$, $\Rightarrow |A-\lambda I| = 0 \Rightarrow |A| = 0 \Rightarrow A$ is singular.
Hence PSD or NSD only when A is singular.

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Theorem: A real quadratic form $x^T A x$ is {real symmetric or Hermitian}
(a) p.d. iff all the eigenvalues of A are +ve.
(b) n.d. iff all the eigenvalues of A are −ve.
(c) p.s.d. iff all the eigenvalues of A are non-negative with atleast one zero eigenvalue.
(d) nsd iff all the eigenvalues of A are non-positive with atleast one zero eigenvalue.
(e) indefinite iff A has atleast one positive and one negative eigenvalue.

Theorem: A quadratic form $q(x) = x^T A x \rightarrow$ PSD iff $A \rightarrow$ singular + all principal minors are non-negative.

Theorem: $q(x) = x^T A x = \sum_{i,j} a_{ij} x_i x_j \rightarrow$ PSD if $|A| = 0$ and leading principal minors $A_1, A_2, \dots, A_{n-1}$ of A are +ve.

Norm/Vector Norm: Let X be a vector space & a real valued function $\|\cdot\|$ on X satisfies following properties,
(1) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$
(2) $\|\alpha x\| = |\alpha| \, \|x\|$
(3) $\|x+y\| \leq \|x\| + \|y\|$

Eg. $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \Rightarrow \|x\|_\infty = \max_i |x_i|$

$p \geq 1$

is basically a norm, satisfies its properties.

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Inequalities

(a) Let $p>1$ and define q by $\frac{1}{p}+\frac{1}{q}=1$ (then p & q are called conjugate exponents).

Let α, β be any +ve no. then $\alpha\beta \le \frac{\alpha^p}{p}+\frac{\beta^q}{q}$.

(B) Holder's inequality: Let $x=(\xi_1 \xi_2 \dots \xi_n)^T$ $(p>1)$

$y=(\eta_1 \eta_2 \dots \eta_n)^T$

$$\sum_{i=1}^{n} |\xi_i \eta_i| \le \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{n} |\eta_i|^q\right)^{1/q}$$ where p & q are any +ve no. satisfying $\frac{1}{p}+\frac{1}{q}=1$

(C) Cauchy Schwarz Inequality: $p=q=2$; $\sum_{i=1}^{n} |\xi_i \eta_i| \le \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |\eta_i|^2\right)^{1/2}$

(d) Minkowski's Inequality: $\|x+y\|_p \le \|x\|_p + \|y\|_p$

$p\ge 1$

or, $$\left(\sum_{i=1}^{n} |\xi_i+\eta_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |\eta_i|^p\right)^{1/p}$$

Metric: A metric in a vector space V is a real valued f^n d of ordered pairs of elements of V which satisfies following,

1. $d(u,v) \ge 0$ and $d(u,v)=0$ iff $u=v$
2. $d(u,v) = d(v,u)$ (symmetry)
3. $d(u,v) \le d(u,w) + d(w,v)$ (triangular inequality)

- A vector space for which a metric is defined is called a metric space.

$\Rightarrow$ d induced by the norm $\|x\|_p$ means, $d(x,y) = \|x-y\|_p$

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Matrix Norm:

① $\|A\| \geq 0$ and $\|A\| = 0$ iff $A = 0$

② $\|\alpha A\| = |\alpha| \, \|A\| \quad \forall \alpha \in C \,\&\, R$

③ $\|A+B\| \leq \|A\| + \|B\|$ ④ $\|AB\| \leq \|A\| \cdot \|B\|$

ⓐ $\|A\|_a = \sum_{i,j} |a_{ij}|$ ⓒ $\|A\|_\infty = \max_i \left(\sum_j |a_{ij}| \right)$ ← row sum

ⓑ $\|A\|_1 = \max_j \left(\sum_i |a_{ij}| \right)$ ⓓ $\|A\|_e = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}$

Column sum

① $\|I\| \geq 1$ ③ $\|A^n\| \leq \|A\|^n$

② $\|A^{-1}\| \geq \dfrac{1}{\|A\|}$

→ **Cholesky factorisation:** For every symmetric +ve definite matrix, there exists a lower triangular matrix L s.t $A = LL^T$

$$A = \left[\begin{array}{c|c} a_{11} & A_{21}^T \\ \hline A_{21} & A_{22} \end{array}\right] ; \quad L = \left[\begin{array}{c|c} l_{11} & 0 \\ \hline L_{21} & L_{22} \end{array}\right] \quad l_{11} = \sqrt{a_{11}} \quad L_{21} = \frac{1}{\sqrt{a_{11}}} A_{21}$$

$$L_{22} L_{22}^T = \underbrace{A_{22} - \frac{1}{a_{11}} A_{21} A_{21}^T}_{\text{symmetric \& +ve definite.}}$$

Def^n Matrix Norm compatible with vector norm: Let A be matrix in $C^{m\times n}$ ($V^{m\times n}$) over $\mathbb{C}$ and x be any vector in $\mathbb{C}^n$ then a matrix norm is said to be compatible with the vector norm iff $\forall x \quad \|Ax\|_a \leq \|A\|_b \|x\|_a$

so $\|A\|_b$ is compatible with $\|x\|_a$.

Theorem: ⓐ Let x & y be in $V(n)$. If $x \to y$ then real no. $\|x\| \to \|y\|$ and $\|x-y\| \to 0$

ⓑ Let A & B be in $V_{n\times n}$. If $A \to B$, then real no. $\|A\| \to \|B\|$ and $\|A-B\| \to 0$

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Def^n Equivalent Vector Norm: Let $\|x\|_a$ and $\|x\|_b$ be two vector norms in V_n then $\|x\|_a$ is said to be equivalent to $\|x\|_b$ if there exist two +ve no. M & m s.t.

$$m\|x\|_a \le \|x\|_b \le M\|x\|_a \quad \forall\, x \text{ in } V_n$$

Theorem: Equivalence of vector norm satisfy (i) reflexive (ii) symmetric (iii) transitive properties.

Theorem: All norms on $\mathbb{C}^n$ are equivalent.

Theorem: All norms on $\mathbb{C}^{n\times n}$ are equivalent, i.e. if $\|\cdot\|_a$ and $\|\cdot\|_b$ are two different norms on $\mathbb{C}^{n\times n}$, then there exists real no. m and M s.t. $m\|A\|_a \le \|A\|_b \le M\|A\|_a$ for any $A \in \mathbb{C}^{n\times n}$

$\Rightarrow$ $\|x\|_p$, $p = 1, 2, \ldots, \infty$ satisfy the following inequalities.

(a) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$, $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$

(b) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$, $\frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2 \le \|x\|_1$

(c) $\frac{1}{n}\|x\|_1 \le \|x\|_\infty \le \|x\|_1$, $\frac{1}{\sqrt{n}}\|x\|_2 \le \|x\|_\infty \le \|x\|_2$

House holder Transform

Def^n. If w is a unit vector in $\mathbb{C}^n$, then a matrix P in $\mathbb{C}^{n\times n}$ of the form $P = I - 2ww^*$ $\quad w^*w = 1$ (norm) is called the householder transform, elementary hermitian matrix or an elementary reflector.

Properties:
1. P is hermitian, $P^* = P$
2. P is unitary, $P^*P = PP^* = I$
3. P is involuntary, $P^2 = I$
4. If $Px = y$, then $\|x\|_2 = \|y\|_2$ $\{x^*x = y^*y\}$

   Thus, norm is preserved

   $y^*y = x^*P^*Px = x^*x$ as $P^*P = I$

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Theorem: Given a non-zero vector $x = (x_1 \dots x_k)^T$ in R^k, there exists a householder transformation $P_x = I - 2ww^T$ of order k such that $P_x x = \alpha e$, where α is real no., $e = (1, 0 \dots 0)^T$ is an elementary vector in R^k and I is identity matrix of order k.

Orthogonal reduction to a triangular form and Q-R decomposition by householder transform.

$Q \Rightarrow$ orthogonal matrix $\qquad R \Rightarrow$ upper-triangular matrix.

$$A = QR$$

Algorithm: Given $x = (x_1 \dots x_n)^T$, compute

ⓐ $m = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$ $\qquad$ ⓑ $2q^2 = m^2 + m x_1 \operatorname{sgn} x_1$

$\|x\|_2$

$\operatorname{sgn} x = 1 \quad 1 \quad -1$

$x>0 \quad x=0 \quad x<0$

ⓒ $u = [x_1 + m \operatorname{sgn} x_1, x_2, \dots, x_n]^T$ $\qquad$ ⓓ $P_m = I - \dfrac{uu^T}{2q^2}$

$P_{m-1} = I - \dfrac{uu^T}{2q^2}, \dots$ $\qquad Q^T = V_r V_{r-1} \dots V_1$ $\qquad (Q^T)^{-1} = Q.$

ek less order ke $\qquad$ and $Q^T A = R$ $\qquad$ orthogonal

$$\underset{V_2}{\begin{bmatrix} 1 & 0 \\ 0 & P_{m-1} \end{bmatrix}}; \underset{V_3}{\begin{bmatrix} I_2 & 0 \\ 0 & P_{m-2} \end{bmatrix}}; \underset{V_4}{\begin{bmatrix} I_3 & 0 \\ 0 & P_{m-3} \end{bmatrix}} \dots$$

Induced Matrix Norms

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|y\|=1} \|Ay\|$$

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Theorem: Every induced matrix norm is a compatible matrix norm. In other words,

(a) $\|A\| = \max_{\|x\|=1} \|Ax\|$ is a matrix norm.

(b) The induced matrix norm defined in (a) is compatible with the vector norm $\|x\|$, i.e. $\|Ax\| \leq \|A\| \cdot \|x\|$.

Axiom 1: $\|A\|_i \geq 0$ always, $A=0 \Rightarrow \|A\|_i = 0$.

Axiom 2: $\|\alpha A\| = |\alpha| \|A\|$ $\left\{ \max_{\|x\|=1} \|\alpha Ax\| = |\alpha| \max_{\|x\|=1} \|Ax\| \right\}$.

Axiom 3: $\|A+B\| \leq \|A\| + \|B\|$.

Weierstrass theorem: A continuous f^n defined on closed and a bounded set of points attains a max. and a min. in its domain. $\|Ax\|$ is a continuous f^n and set of points for $\|x\|=1$ is a closed and bounded set then $\exists$ $x_0 \in \mathbb{R}^n$ s.t. $\|x_0\|=1$ and

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \|Ax_0\|$$

So

$$\|AB\| \leq \|A\| \|B\|$$

$$\|AB\| = \max_{\|x\|=1} \|ABx\| = \|ABx_0\| \quad ; \quad \|x_0\|=1$$

$$= \|A(Bx_0)\| \leq \|A\| \|Bx_0\|$$

$$\leq \|A\| \|B\| \|x_0\|$$

★ A maxima should occur in the given domain for this theorem to hold true.

★ Every induced matrix norm is a matrix norm.

$$\|x\|_2 = \left(\sum |x_i|^2 \right)^{1/2}$$

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notation to show eigenvalues of A^*A.

Q. Show that the matrix norms.

(a) $\|A\|_1 = \max_j \sum_i |a_{ij}|$

(c) $\|A\|_2 = \max_i \left[\lambda_i (A^*A) \right]^{1/2}$

(b) $\|A\|_\infty = \max_i \sum_j |a_{ij}|$

are induced respectively by the vector norms.

(a) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

(b) $\|x\|_\infty = \max_i |x_i|$

$\|A\|_m = n \max_{i,j} |a_{ij}|$

$\|A\|_e = \left(\sum |a_{ij}|^2 \right)^{1/2}$

$\|A\|_a = \sum_{i,j} |a_{ij}|$

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