Derivation of the Quadratic Spline Coefficients

Part of the accompanying material for the manuscript titled "Quadratic Spline Approximation of the Contact Potential for Real-Time Simulation of Lumped Collisions in Musical Instruments" submitted to the DAFx-24 conference

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1 Re-parametrisation of the Quadratic Segments

This derivation largely adapts the recipe described for cubic splines in [1]. Recall that by sampling at the compression values $y_j = j\Delta_y$, the contact potential curve $V(y)$ is approximated by a piecewise-quadratic function $\widehat{V}(y)$ (eq. (6) in the paper), which is made up of quadratic segments $V_j(y)$ over intervals $y \in [y_{j-1}, y_j),\ j = 1, 2, ..., N-1$, and $V_N(y)$ over the interval $y \geq y_{N-1}$. Segments $V_j(y)$ and their derivatives with respect to y are given by:

$$V_j(y) = a_j\, y^2 + b_j\, y + c_j, \quad j = 1, 2, ..., N, \quad \text{(eq. (7) in the paper)}, \tag{1}$$

$$\partial_y V_j(y) = 2\, a_j\, y + b_j, \quad j = 1, 2, ..., N. \tag{2}$$

Following [1], here a re-parametrised (or normalised) compression variable is defined as

$$\overline{y} \coloneqq \frac{y - y_{j-1}}{y_j - y_{j-1}} = \frac{y - y_{j-1}}{\Delta_y}, \quad y \in [y_{j-1}, y_j], \quad j = 1, 2, ..., N. \tag{3}$$

Notice from (3) that $\overline{y}$ is nothing but a shifted and scaled version of y. Also, $\overline{y} = 0$ when $y = y_{j-1}$, and $\overline{y} = 1$ when $y = y_j$. Now, the segments can be locally re-parametrised over $y \in [y_{j-1}, y_j]$ as

$$\overline{V}_j(\overline{y}) \coloneqq V_j(y), \quad y \in [y_{j-1}, y_j], \quad j = 1, 2, ..., N. \tag{4}$$

Further, the re-parametrised segments $\overline{V}_j(\overline{y})$ and their derivatives with respect to $\overline{y}$ can be written as

$$\overline{V}_j(\overline{y}) = \overline{a}_j\, \overline{y}^2 + \overline{b}_j\, \overline{y} + \overline{c}_j, \quad j = 1, 2, ..., N, \tag{5}$$

$$\partial_{\overline{y}} \overline{V}_j(\overline{y}) = 2\, \overline{a}_j\, \overline{y} + \overline{b}_j, \quad j = 1, 2, ..., N. \tag{6}$$

Note that in this document, the "bar" above the variable indicates re-parametrisation, which is a flipped version of the notation in [1], where the absence of the bar indicates re-parametrisation.

1.1 Relation between coefficients of the regular and re-parametrised quadratic segments

From definition (4) and substitution of (3) in (5), we get

$$\overline{V}_j(\overline{y}) \coloneqq V_j(y) = \overline{a}_j \left[\frac{y - y_{j-1}}{\Delta_y}\right]^2 + \overline{b}_j \left[\frac{y - y_{j-1}}{\Delta_y}\right] + \overline{c}_j. \tag{7}$$

Expanding and re-arranging the terms in (7), and using $y_{j-1} = (j-1)\,\Delta_y$ yields

$$V_j(y) = \underbrace{\frac{\bar{a}_j}{\Delta_y^2}}_{a_j}\ y^2 + \underbrace{\left[\frac{\bar{b}_j - 2\,(j-1)\,\bar{a}_j}{\Delta_y}\right]}_{b_j}\ y + \underbrace{\left[\bar{a}_j(j-1)^2 - \bar{b}_j(j-1) + \bar{c}_j\right]}_{c_j}, \quad j = 1, 2, ..., N. \tag{8}$$

Thus the coefficients a_j, b_j, c_j of the regular segments can be obtained from the coefficients $\bar{a}_j, \bar{b}_j, \bar{c}_j$ of the re-parametrised segments.

2 Quadratic Spline Conditions In Re-parametrised Notation

2.1 Sampling the contact potential $V(y)$

From the notation in the paper, $V_j(y_j) = V(y_j),\ j = 1, 2, ..., N$ are the sampled values of the contact potential $V(y)$. These samples could be written in re-parametrised terms as

$$\overline{V}_j(1) = V(y_j), \quad j = 1, 2, ..., N. \tag{9}$$

2.2 One-sided nonlinearity conditions

For $\widehat{V}(y)$ and its derivative $\partial_y \widehat{V}(y)$ to represent a one-sided nonlinearity, $V_1(y_0) = 0$ and $\partial_y V_1(y_0) = 0$. Because $y_0 = 0$, from (1) these result in $b_1 = c_1 = 0$ (conditions (8) in the paper). The equivalent conditions in terms of the re-parametrisation are derived here as follows:

$$\overline{V}_1(0) = 0 \implies \bar{c}_1 = 0 \quad \text{(from (5))}, \quad \text{and} \tag{10}$$

$$\partial_y \overline{V}_1(0) = 0 \implies \bar{b}_1 = 0 \quad \text{(from (6))}. \tag{11}$$

2.3 Continuity condition

For continuity of $\widehat{V}(y)$, successive quadratic segments need to meet at their knots. Therefore, the condition $V_j(y_j) = V_{j+1}(y_j),\ j = 1, 2, ..., N-1$ (eq. (9) in the paper) should hold. In re-parametrised notation, this is equivalent to

$$\overline{V}_j(1) = \overline{V}_{j+1}(0), \quad j = 1, 2, ..., N-1. \tag{12}$$

Note that, from (9) and (12), we also have

$$\overline{V}_{j+1}(0) = V(y_j), \quad j = 1, 2, ..., N-1. \tag{13}$$

2.4 Smoothness condition

$\widehat{V}(y)$ is smooth only if the first derivatives of successive segments are equal at their knots. This means $\partial_y V_j(y_j) = \partial_y V_{j+1}(y_j),\ j = 1, 2, ..., N-1$ (eq. (10) in the paper) must be satisfied. The equivalent condition in re-parametrised notation is

$$\partial_{\overline{y}} \overline{V}_j(1) = \partial_{\overline{y}} \overline{V}_{j+1}(0), \quad j = 1, 2, ..., N-1. \tag{14}$$

3 Quadratic Spline Coefficients

From (5) and (13), it can be seen that

$$\bar{c}_{j+1} = V(y_j), \quad j = 1, 2, ..., N-1. \tag{15}$$

From (5) and (9), it follows that

$$\overline{a}_j + \overline{b}_j + \bar{c}_j = V(y_j), \tag{16}$$

$$\Rightarrow \overline{a}_j = V(y_j) - \bar{c}_j - \overline{b}_j, \tag{17}$$

$$\Rightarrow \overline{a}_j = V(y_j) - V(y_{j-1}) - \overline{b}_j, \quad j = 1, 2, ..., N \quad \text{(from (15) and (10))}. \tag{18}$$

Eqs. (5) and (14) give

$$2\,\overline{a}_j + \overline{b}_j = \overline{b}_{j+1}, \quad j = 1, 2, ..., N-1. \tag{19}$$

Substituting (18) in (19), we have

$$2\left[V(y_j) - V(y_{j-1}) - \overline{b}_j\right] + \overline{b}_j = \overline{b}_{j+1}, \tag{20}$$

$$\Rightarrow \overline{b}_j + \overline{b}_{j+1} = 2\left[V(y_j) - V(y_{j-1})\right], \quad j = 1, 2, ..., N-1. \tag{21}$$

which is a recursion in $\overline{b}_j$ and can be expressed in the form of a linear system of equations involving a bidiagonal matrix as follows:

$$\begin{bmatrix} 1 & 0 & & 0 \\ 1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 1 \end{bmatrix} \begin{bmatrix} \overline{b}_2 \\ \overline{b}_3 \\ \vdots \\ \overline{b}_N \end{bmatrix} = 2 \begin{bmatrix} V(y_1) \\ V(y_2) - V(y_1) \\ \vdots \\ V(y_{N-1}) - V(y_{N-2}) \end{bmatrix}. \tag{22}$$

There are $3N$ coefficients $\{\overline{a}_j, \overline{b}_j, \bar{c}_j\},\ j = 1, 2, ..., N$ to solve for. From (11) and (10) there are 2 constraints, and from each of (15) and (22) there are (N-1) constraints. Further, from (18) there are N constraints. Thus, the conditions together give $2 + 2(N-1) + N = 3N$ constraints, and thereby we can obtain a unique quadratic spline approximation of the contact potential from solving these equations. $\bar{c}_j$ and $\overline{b}_j$ are obtained first for $j = 2, 3, ...N$ from (15) and (22), respectively. These values of $\overline{b}_j$ are then used in (18) to solve for $\overline{a}_j$. Finally, the regular quadratic coefficients $\{a_j, b_j, c_j\},\ j = 1, 2, ..., N$ are obtained from the conversion formulas in (8).

References

[1] R. H. Bartels, J. C. Beatty, and B. A. Barsky, *An introduction to splines for use in computer graphics & geometric modeling*, Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1987.