Derivation of the branched solution with the QSA method

Part of the accompanying material for the manuscript titled "Quadratic Spline Approximation of the Contact Potential for Real-Time Simulation of Lumped Collisions in Musical Instruments" submitted to the DAFx-24 conference

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1 Preliminaries

The derivation assumes that the QSA $\widehat{V}(y)$ is convex (without proof). For convenience, let y^{n+1} and y^{n-1} be replaced by the shorthand symbols y_+ and y_-. From the definition of p (eq. (27) in the paper), the condition $p = 0$ is the same as $y_+ \leq 0$, and its complement $p \neq 0$ is the same as $y_+ > 0$. The nonlinear equation that needs to be solved (eq. (25) in the paper) can be re-written with the new notation as:

$$\underbrace{s + z + q\,\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}}_{G(s)} = 0, \tag{1}$$

where $s = y_+ - y_-$ is the update step, and $q > 0$. Listed in Table 1 are the forms that $G(s)$ takes in different branches based on the signs of y_- and y_+.

Table 1: *Forms of $G(s)$ for the branches based on the signs of y_- and y_+.*

Branch	$G(s)$
I: $\mathrm{y}_- \leq 0,\ \mathrm{y}_+ \leq 0$	$s + z$
II: $\mathrm{y}_- \leq 0,\ \mathrm{y}_+ > 0$	$s + z + q\,\frac{\widehat{V}(y_+)}{s}$
III: $\mathrm{y}_- > 0,\ \mathrm{y}_+ > 0$	$s + z + q\,\frac{\widehat{V}(y_+)-\widehat{V}(y_-)}{s}$
IV: $\mathrm{y}_- > 0,\ \mathrm{y}_+ \leq 0$	$s + z - q\,\frac{\widehat{V}(y_-)}{s}$

Due to convexity of $\widehat{V}(y)$, $G(s)$ has a unique real root, as described in section 4.2 of the paper. Therefore the quadratic $\lambda(s) := s\,G(s) = a_\lambda s^2 + b_\lambda s + c_\lambda$ must have real roots, one of which is also the root of $G(s)$, and hence the discriminant $b_\lambda^2 - 4a_\lambda c_\lambda$ must be non-negative in all branches. Recall from the paper that the solution in branches II, III and IV is obtained as one of the roots of $\lambda(s)$:

$$s_{\lambda,\pm} = \frac{-b_\lambda \pm \sqrt{b_\lambda^2 - 4a_\lambda c_\lambda}}{2a_\lambda}. \tag{2}$$

The solution is easier to derive for branches I and IV, and these are described first. The solution in branch k is denoted $s_{(k)}$, k = I, II, III, IV.

2 Branch IV: $y_- > 0, \ y_+ \le 0$

Multiplying the expression for $G(s)$ in the last row of Table 1 by s, it is not hard to see that the coefficients of $\lambda(s)$ are:

$$a_\lambda = 1, \quad b_\lambda = z, \quad c_\lambda = -q\, \widehat{V}(y_-), \tag{3}$$

which are the same as in the table in the paper. As mentioned in section 2.2 in the paper, convexity

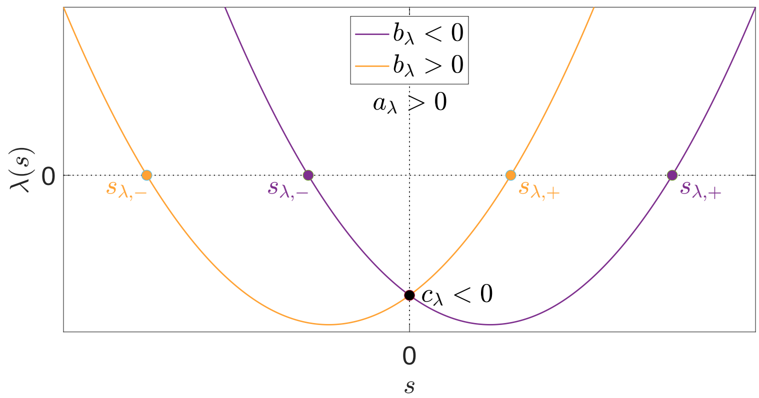

Figure 1: $\lambda(s)$ *with* $b_\lambda < 0$ *(purple) and* $b_\lambda > 0$ *(orange). In both cases,* $a_\lambda > 0$ *and* $c_\lambda < 0$*, as in branch IV. The positive root* $s_{\lambda,+}$ *and the negative root* $s_{\lambda,-}$ *are also shown for both cases.*

of $\widehat{V}$ and condition (8) in the paper together imply $\widehat{V}(y_-) > 0$ for $y_- > 0$, and therefore $c_\lambda < 0$ from (3). Thus, the graph of $\lambda(s)$ is an 'upright' (U-shaped) parabola ($a_\lambda > 0$) with a negative Y-intercept ($c_\lambda < 0$), like the ones in Figure 1. Hence, $\lambda(s)$ has one positive ($s_{\lambda,+}$) and one negative ($s_{\lambda,-}$) root. Because $s < 0$ from the branch condition, the solution of (1) is

$$s_{(\mathrm{IV})} = s_{\lambda,-}. \tag{4}$$

3 Branch I: $y_- \le 0, \ y_+ \le 0$

From the first row in Table 1, the solution of (1) trivially is

$$s_{(\mathrm{I})} = -z. \tag{5}$$

Though $\lambda(s)$ degenerates into a line and it is not necessary to solve a quadratic equation in this branch, one can still write the coefficients of $\lambda(s)$ as

$$a_\lambda = 1, \quad b_\lambda = z, \quad c_\lambda = 0, \tag{6}$$

as given in the table in the paper. Observe that the expression of $G(s)$ in branch I is the same as in branch IV with the 'history potential' $\widehat{V}(y_-) = 0$. Therefore, the solution for branch I can also be viewed as $s_{\lambda,-}$ with $c_\lambda = 0$.

4 Branch III: $y_- > 0,\ y_+ > 0$

4.1 Coefficients $(a_\lambda, b_\lambda, c_\lambda)$

In this branch, from Table 1, $\lambda(s)$ can be written as:

$$\lambda(s) = s\,G(s) = s^2 + z\,s + q\ \left\{\widehat{V}(y_+) - \widehat{V}(y_-)\right\}. \tag{7}$$

By definition, $\widehat{V}(y_+) = V_p(y_+)$ for $y_+ > 0$, and $y_+ = s + y_-$. Therefore (7) can be re-written as

$$\Rightarrow \lambda(s) = s^2 + z\,s + q\ \left\{V_p(s + y_-) - \widehat{V}(y_-)\right\}, \tag{8}$$

$$\Rightarrow \lambda(s) = s^2 + z\,s + q\ \left\{a_p(s + y_-)^2 + b_p(s + y_-) + c_p - \widehat{V}(y_-)\right\}. \tag{9}$$

Expanding and re-arranging the terms in the RHS, (9) can be written as

$$\lambda(s) = (1 + q\,a_p)\,s^2 + \{z + q\,(2\,a_p\,y_- + b_p)\}\,s + q\,(\underbrace{a_p\,y_-^2 + b_p\,y_- + c_p}_{V_p(y_-)} - \widehat{V}(y_-)), \tag{10}$$

$$\Rightarrow a_\lambda = (1 + q\,a_p), \quad b_\lambda = z + q\,(2\,a_p\,y_- + b_p), \quad c_\lambda = q\,(V_p(y_-) - \widehat{V}(y_-)), \tag{11}$$

which match the ones in the table in the paper. Notice that

$$b_\lambda = z + q\,\partial_y V_p(y_-). \tag{12}$$

4.2 Observations

Figure 2 aids in visualising some of the following relations between variables.

4.2.1 Relations directly from convexity

From convexity of $\widehat{V}(y)$, the following inequalities hold:

$$\widehat{V}(y_+) \geq \widehat{V}(y_-) + s\ \partial_y \widehat{V}(y_-), \tag{13}$$

$$\widehat{V}(y_-) \geq \widehat{V}(y_+) - s\ \partial_y \widehat{V}(y_+). \tag{14}$$

See top-left plot in Figure 2 for a visualisation of (13). Inequality (14) can also be similarly visualised (not shown in figure). Now, (13) can be split as the following two inequalities:

$$\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s} \geq \partial_y \widehat{V}(y_-), \quad s > 0, \tag{15}$$

$$\partial_y \widehat{V}(y_-) \geq \frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}, \quad s < 0. \tag{16}$$

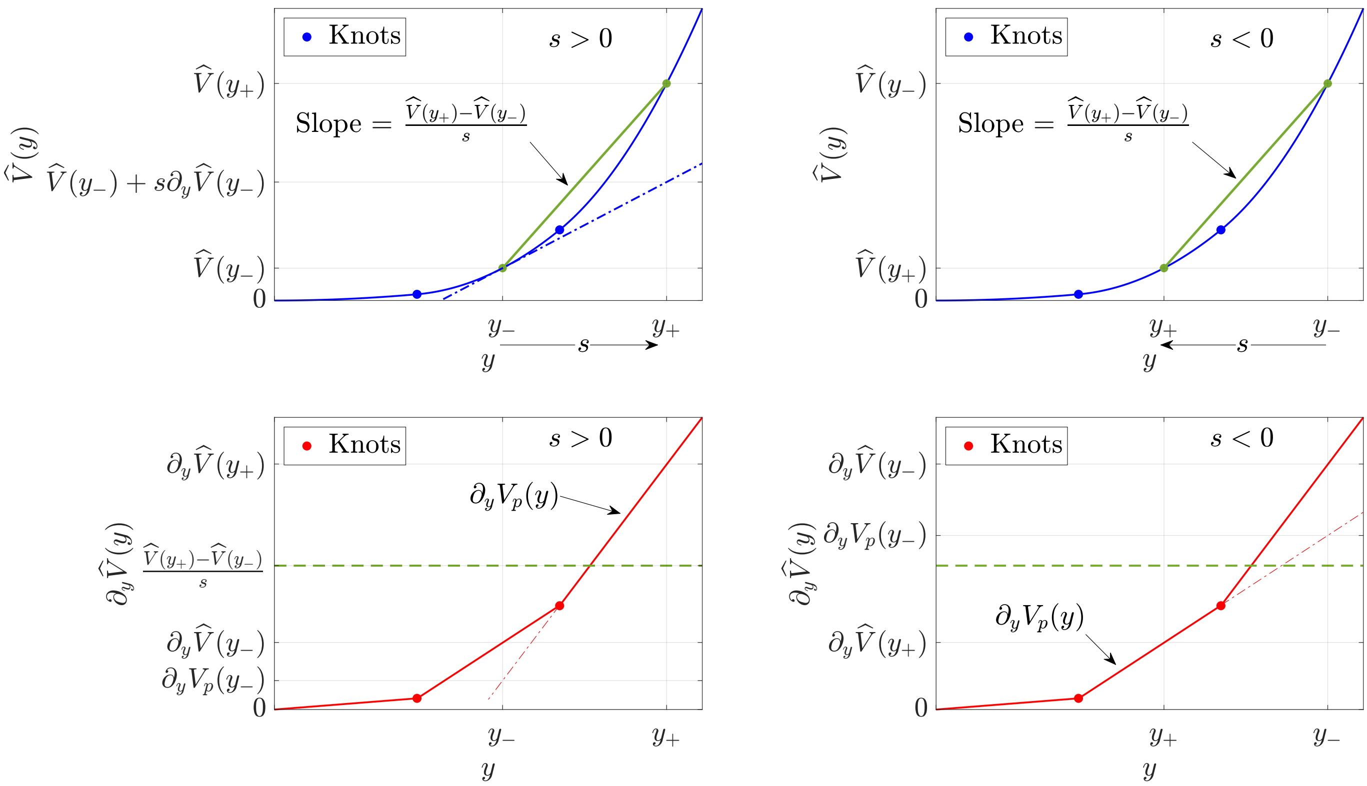

Figure 2: *QSA $\widehat{V}(y)$ (top, blue) and its derivative $\partial_y \widehat{V}(y)$ (bottom, red). The dashed-dotted line in the top-left plot is the tangent to the QSA curve at y_-, and the dashed-dotted lines in the bottom plots are the extensions of the segment $\partial_y V_p(y)$. In the top plots, the green line connects the points $(y_-, \widehat{V}(y_-))$ and $(y_+, \widehat{V}(y_+))$ and therefore its slope is the discrete gradient $(\widehat{V}(y_+) - \widehat{V}(y_-))/s$, which is also shown as the dashed green line in the bottom plots.*

Notice that the dashed-dotted extended lines in the bottom plots in Figure 2 are always below the derivative curve. Therefore, irrespective of the sign of s, the following inequality is satisfied:

$$\partial_y \widehat{V}(y_-) \geq \partial_y V_p(y_-). \tag{17}$$

4.2.2 Relation between $\partial_y V_p(y_-)$ and the discrete gradient

From (15) and (17), the following holds for $s > 0$:

$$\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s} \geq \partial_y V_p(y_-), \quad s > 0. \tag{18}$$

For $s < 0$, eqs. (16) and (17) do not readily result in the corresponding complementary inequality. However, observations from testing over multiple cases suggest that the following is true:

$$\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s} \leq \partial_y V_p(y_-), \quad s < 0. \tag{19}$$

See the bottom plots in Figure 2 for an instance of these relations being satisfied. The green dashed line is above the value $\partial_y V_p(y_-)$ (18) in the bottom-left plot, and below $\partial_y V_p(y_-)$ (19) in the bottom-right plot.

4.2.3 Relation between $\partial_y V_p(y_+)$ and the discrete gradient

By definition, for $y \in [y_{p-1}, y_p)$, $\partial_y V_p(y) = \partial_y \widehat{V}(y)$. Because $y_+ \in [y_{p-1}, y_p)$, $\partial_y V_p(y_+) = \partial_y \widehat{V}(y_+)$. Using this relation, (14) can be re-written (in split-form) as

$$\partial_y V_p(y_+) \geq \frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}, \quad s > 0, \tag{20}$$

$$\partial_y V_p(y_+) \leq \frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}, \quad s < 0, \tag{21}$$

and these can be seen to hold in the bottom plots in Figure 2 as well.

4.3 Lemma: $b_\lambda < 0 \iff s > 0$

Case $b_\lambda < 0$

Suppose $s < 0$ in this case. Then,

$$\begin{aligned} b_\lambda < 0 &\Rightarrow z + q\,\partial_y V_p(y_-) < 0, && (22)\\ &\Rightarrow z + q\,\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s} < 0, \quad \text{(from (19))} && (23)\\ &\Rightarrow \underbrace{s + z + q\,\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}}_{G(s)} < s. && (24) \end{aligned}$$

For s to be a root of (1), $G(s) = 0$, and therefore (24) $\Rightarrow s > 0$, which is a contradiction. Thus,

$$b_\lambda < 0 \Rightarrow s > 0. \tag{25}$$

Case $b_\lambda > 0$

Taking a proof by contradiction approach as in the previous case, suppose $s > 0$. Then we have

$$\begin{aligned} b_\lambda > 0 &\Rightarrow z + q\,\partial_y V_p(y_-) > 0, && (26)\\ &\Rightarrow z + q\,\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s} > 0, \quad \text{(from (18))} && (27)\\ &\Rightarrow \underbrace{s + z + q\,\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}}_{G(s)} > s. && (28) \end{aligned}$$

For s to be a root of (1), $G(s) = 0$, and therefore (28) $\Rightarrow s < 0$, which is a contradiction. Thus,

$$b_\lambda > 0 \Rightarrow s < 0. \tag{29}$$

From (25) and (29), it follows that

$$b_\lambda < 0 \iff s > 0, \tag{30}$$

and hence the lemma is proven.

4.4 Root Selection Based on the Sign Of b_λ

From (2), it follows that

$$s = \begin{cases} s_{\lambda,+}: & 2a_\lambda s + b_\lambda > 0, \\ s_{\lambda,-}: & \text{otherwise.} \end{cases} \tag{31}$$

Expanding the term $(2a_\lambda s + b_\lambda)$, we have

$$2a_\lambda s + b_\lambda = 2\,(1 + q a_p)\, s + z + q\,(2 a_p y_- + b_p), \quad \text{(from the first two equations in (11))} \tag{32}$$

$$\Rightarrow 2a_\lambda s + b_\lambda = 2s + z + 2\,q\,a_p(s + y_-) + q\,b_p, \tag{33}$$

$$\Rightarrow 2a_\lambda s + b_\lambda = 2s + z + q\,\underbrace{(2\,a_p\,y_+ + b_p)}_{\partial_y V_p(y_+)}, \quad (\because\ s + y_- = y_+). \tag{34}$$

Case $b_\lambda < 0$

From the lemma (30), $s > 0$ if $b_\lambda < 0$. Therefore, from (34) the following can be derived:

$$\Rightarrow 2a_\lambda s + b_\lambda \geq 2s + z + q\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}, \quad \text{(from (20))} \tag{35}$$

$$\Rightarrow 2a_\lambda s + b_\lambda \geq s + \underbrace{s + z + q\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}}_{G(s)}. \tag{36}$$

For s to be a root of (1), $G(s) = 0$, and hence from (36), we have

$$2a_\lambda s + b_\lambda \geq s > 0. \tag{37}$$

Therefore, for this case, we have

$$b_\lambda < 0 \Rightarrow 2a_\lambda s + b_\lambda > 0. \tag{38}$$

Case $b_\lambda > 0$

Again from lemma (30), $s < 0$ if $b_\lambda > 0$. Thus, from (34) the following can be derived:

$$\Rightarrow 2a_\lambda s + b_\lambda \leq 2s + z + q\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}, \quad \text{(from (21))} \tag{39}$$

$$\Rightarrow 2a_\lambda s + b_\lambda \leq s + \underbrace{s + z + q\frac{\widehat{V}(y_+) - \widehat{V}(y_-)}{s}}_{G(s)}. \tag{40}$$

For s to be a root of (1), $G(s) = 0$, and hence from (40), we have

$$2a_\lambda s + b_\lambda \leq s < 0. \tag{41}$$

Thus, for this case, we have

$$b_\lambda > 0 \Rightarrow 2a_\lambda s + b_\lambda < 0. \tag{42}$$

The statements (38) and (42) together result in the logical equivalence

$$b_\lambda < 0 \iff 2a_\lambda s + b_\lambda > 0. \tag{43}$$

Finally, using (43) in (31), the roots can be selected as

$$s_{(\mathrm{III})} = \begin{cases} s_{\lambda,+} : & b_\lambda < 0, \\ s_{\lambda,-} : & \text{otherwise.} \end{cases} \tag{44}$$

5 Branch II: $y_- \leq 0, y_+ > 0$

From Table 1, observe that $G(s)$ in branch II is the same as in branch III with $\widehat{V}(y_-) = 0$. Therefore, the coefficients $(a_\lambda, b_\lambda, c_\lambda)$ in this case are:

$$a_\lambda = (1 + q\, a_p), \quad b_\lambda = z + q\,(2\, a_p\, y_- + b_p), \quad c_\lambda = q\, V_p(y_-), \tag{45}$$

which are the same as in the table in the paper. From the branch condition, $s > 0$, and thus from (30), (38) and (31), as in the case of branch III it follows that

$$s_{(\mathrm{II})} = s_{\lambda,+}. \tag{46}$$