

ELLIPTIC CURVES IN GAME THEORY

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ABSTRACT. We investigate Spohn curves, the algebro-geometric models of dependency equilibria for 2×2 normal-form games. These curves arise as the intersection of two quadrics in \mathbb{P}^3 and are generically elliptic curves. We compute and verify the j -invariant for elliptic curves arising as the intersection of quadrics in \mathbb{P}^3 using two different implementations: by computing the Aronhold invariants and the discriminant (in Mathematica) and using algorithms for the arithmetic of elliptic curves (in-built in Pari/GP). We define an equivalency of generic 2×2 games based on the j -invariant of the Spohn curve. Additionally, we examine the reduction of Spohn curves to plane curves and analyze conditions under which they are reducible. Notably, we prove that the real points are dense on the Spohn curve in all cases. Our examples and computations are further supported by Macaulay2.

CONTENTS

1. Introduction	1
2. Dependency equilibria and the Spohn variety	2
3. Denseness of real points	5
3.1. Reducibility of the Spohn cubic	5
3.2. Finding real smooth points	10
4. Equivalent games from elliptic curve invariants	13
4.1. Invariants for Spohn curves as intersection of quadrics in \mathbb{P}^3	13
4.2. Aronhold invariants and the j -invariant	15
Appendix A. Continued fraction approximation of real numbers	18
References	19

1. INTRODUCTION

In game theory, one is usually interested in the best possible solution of a game. More precisely, one wants to reach an equilibrium such that the choice of strategy of each player is optimal for them. The concept of *Nash equilibria* fulfills this by giving strategies in which, if the players interact independently of each other, it makes no rational sense for any of them to deviate from the joint strategy while the others do not. Due to the assumption of independence, though, often these do not actually give desirable or ideal solutions. Wolfgang Spohn proposed a different concept of equilibria, the so-called *dependency equilibria*, in which it is assumed that the players have some form of communication and try to maximize their conditional expected payoff ([15],[16]). This type of equilibrium was first studied from a mathematical point of view in [13] and [14], which characterize it, up to a certain point, via the so-called *Spohn variety*. For the smallest possible games, i.e. games with two players in which both only have two choices, this variety generically takes the form of an *elliptic curve*.

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In Section 2, relevant background on dependency equilibria and the Spohn variety is summarized. We introduce a computational method to check whether the totally mixed Nash equilibria of a generic 2×2 game is Pareto dominated by a dependency equilibrium. In Section 3, we investigate when the planar model of such a curve coming from a 2×2 game is reducible (Lemma 3.2). This happens in precisely 12 cases with an interesting combinatorial structure, which is explained in Remark 3.3. We first use this to prove the denseness of real points for the planar cubic in Lemma 3.5. Then, in Theorem 3.7, we deduce the same result for the Spohn variety in \mathbb{P}^3 in the aforementioned 12 cases.

In Section 4, we provide an algorithm and code to compute the j -invariant of an elliptic curve. This applies not only to elliptic curves derived from the Spohn variety but also to any elliptic curve arising from two quadrics intersecting at an arbitrary but known rational point. A game theoretic equivalence between 2×2 games is defined based on the j -invariants of elliptic curves. In this section, we work with the cases when the payoff matrices, and thus the Spohn curve, are defined over \mathbb{Q} . In the case of real payoff matrices, Appendix A explains how to obtain rational approximations to real numbers.

All relevant code for this paper is hosted on the MathRepo page:

https://mathrepo.mis.mpg.de/elliptic_curves_game_theory/

2. DEPENDENCY EQUILIBRIA AND THE SPOHN VARIETY

We consider a single round game with n players $1, \dots, n$, where each player i can choose from d_i pure strategies $1, \dots, d_i$. The outcome of the game depends on the choices of the players and is represented by the payoff tables $X^{(1)}, \dots, X^{(n)}$. These are tensors of format $d_1 \times \dots \times d_n$, such that, for each player i , the entry $X_{j_1 \dots j_n}^{(i)} \in \mathbb{R}$ of the payoff table $X^{(i)}$ specifies their payoff in the case that every player l chooses pure strategy j_l . Formally, the game is then denoted by X and is said to be a $(d_1 \times \dots \times d_n)$ -game in normal-form. We may understand the players in such a game as probability variables with state space $[d_i]$ that decide the joint outcome. The probabilities of the joint decisions are recorded in the $d_1 \times \dots \times d_n$ -format tensor p . Its entries $p_{j_1 \dots j_n}$ are the probabilities that every player l chooses the strategy j_l . Coming from an algebraic perspective, we view p as an element in the projective space $\mathbb{P}^{d_1 \dots d_n - 1}$ over \mathbb{C} . Of course, p must have non-negative real entries that sum up to 1 and therefore lives in the projectivization $\overline{\Delta} \subset \mathbb{P}^{d_1 \dots d_n - 1}$ of the $(d_1 \dots d_n - 1)$ -dimensional probability simplex $\Delta_{d_1 \dots d_n - 1}$. We also define the open simplex $\Delta \subset \mathbb{P}^{d_1 \dots d_n - 1}$ of probability tensors with nonzero entries.

The *conditional expected payoff* of the i th player, conditioned on them choosing a certain pure strategy k with respect to $p \in \overline{\Delta}$, is the sum

$$\mathbb{E}_k^{(i)}(p) := \sum_{j_1=1}^{d_1} \dots \widehat{\sum_{j_i=1}^{d_i}} \dots \sum_{j_n=1}^{d_n} X_{j_1 \dots k \dots j_n}^{(i)} \frac{p_{j_1 \dots k \dots j_n}}{p_{+ \dots + k + \dots +}}.$$

Here,

$$p_{+ \dots + k + \dots +} := \sum_{j_1=1}^{d_1} \dots \widehat{\sum_{j_i=1}^{d_i}} \dots \sum_{j_n=1}^{d_n} p_{j_1 \dots k \dots j_n},$$

with k in the i -th position, is the probability that player i actually chooses the pure strategy k .

While for Nash equilibria, players maximize their expected payoff, dependency equilibria are defined by players maximizing their conditional expected payoff. This allows for some form of communication between the players as discussed by Spohn in [15, §2].

Definition 2.1 (Dependency equilibrium [13],[14]). A *dependency equilibrium* is a joint probability distribution $p \in \overline{\Delta}$ such that

(*) $p_{+...+k+...+} \neq 0$ for all players i and pure strategies k of player i and

$$\mathbb{E}_k^{(i)}(p) \geq \mathbb{E}_{k'}^{(i)}(p)$$

for all players i and all pure strategies k, k' of player i

or such that p is the limit of a sequence in $\mathbb{P}^{d_1 \cdots d_n - 1}$ with the property (*).

Defining dependency equilibria via a limit for some cases is necessary, since the denominators of the conditional expected payoffs may be zero. These boundary cases are studied to in detail in [14]. For now, we will focus on totally mixed dependency equilibria, which live in the open simplex Δ , i.e. for which the joint probabilities $p_{j_1 \cdots j_n}$ are neither 0 nor 1, and which can therefore simply be described by the equations $\mathbb{E}_k^{(i)}(p) = \mathbb{E}_{k'}^{(i)}(p)$ for all players i and all pure strategies k, k' of player i . By multiplying these equations by the denominators, one finds that the totally mixed dependency equilibria can be described via a determinantal variety.

Definition 2.2. The *Spohn variety* \mathcal{V} of a game X is defined as the vanishing set of the 2×2 minors of the matrices M_1, \dots, M_n , given by

$$M_i(p) := \begin{bmatrix} \vdots & & & \vdots \\ p_{+...+k+...+} & \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X_{j_1 \cdots k \cdots j_n}^{(i)} p_{j_1 \cdots k \cdots j_n} & & \vdots \\ \vdots & & & \vdots \end{bmatrix} \in \mathbb{R}^{d_i \times 2}.$$

In this paper, we will focus mainly on 2×2 games. In this case, for simplicity, we will denote the 2×2 payoff tables $X^{(1)}$ as A and $X^{(2)}$ as B . Then, the conditional expected payoffs are given by

$$\begin{aligned} \mathbb{E}_1^{(1)}(p) &= \frac{a_{11}p_{11} + a_{12}p_{12}}{p_{11} + p_{12}}, & \mathbb{E}_2^{(1)}(p) &= \frac{a_{21}p_{21} + a_{22}p_{22}}{p_{21} + p_{22}}, \\ \mathbb{E}_1^{(2)}(p) &= \frac{b_{11}p_{11} + b_{21}p_{21}}{p_{11} + p_{21}}, & \mathbb{E}_2^{(2)}(p) &= \frac{b_{12}p_{12} + b_{22}p_{22}}{p_{12} + p_{22}}. \end{aligned}$$

If $p \in \Delta$ (or more general, if $p \in \overline{\Delta}$ and $p_{1+}, p_{2+}, p_{+1}, p_{+2} \neq 0$), then p is a dependency equilibrium if and only if

$$p \in \mathcal{V} = \mathbb{V}(\det M_1, \det M_2) \subset \mathbb{P}_{\mathbb{C}}^3$$

for

$$M_1 = \begin{bmatrix} p_{11} + p_{12} & a_{11}p_{11} + a_{12}p_{12} \\ p_{21} + p_{22} & a_{21}p_{21} + a_{22}p_{22} \end{bmatrix}, \quad M_2 = \begin{bmatrix} p_{11} + p_{21} & b_{11}p_{11} + b_{21}p_{21} \\ p_{12} + p_{22} & b_{12}p_{12} + b_{22}p_{22} \end{bmatrix}.$$

More precisely,

$$\begin{aligned} \det M_1 &= (a_{21} - a_{11})p_{11}p_{21} + (a_{22} - a_{11})p_{11}p_{22} + (a_{21} - a_{12})p_{12}p_{21} + (a_{22} - a_{12})p_{12}p_{22}, \\ \det M_2 &= (b_{12} - b_{11})p_{11}p_{12} + (b_{22} - b_{11})p_{11}p_{22} + (b_{12} - b_{21})p_{12}p_{21} + (b_{22} - b_{21})p_{21}p_{22}. \end{aligned} \tag{1}$$

Example 2.3 (Prisoner's Dilemma). Consider the 2×2 game with payoff tables

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

The Spohn variety is defined by the determinants of the matrices

$$M_1 = \begin{bmatrix} p_{11} + p_{12} & 2p_{11} \\ p_{21} + p_{22} & 3p_{21} + p_{22} \end{bmatrix}, \quad M_2 = \begin{bmatrix} p_{11} + p_{21} & 2p_{11} \\ p_{12} + p_{22} & 3p_{12} + p_{22} \end{bmatrix},$$

namely

$$\mathcal{V} = \mathbb{V}(p_{11}p_{21} - p_{11}p_{22} + 3p_{12}p_{21} + p_{12}p_{22}, p_{11}p_{12} - p_{11}p_{22} + 3p_{12}p_{21} + p_{21}p_{22}).$$

Example 2.4. [14, Proposition 4.6] Consider the 2×2 game with payoff tables

$$A = \begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Spohn variety is defined by the determinants of the matrices

$$M_1 = \begin{bmatrix} p_{11} + p_{12} & -p_{11} - p_{12} \\ p_{21} + p_{22} & -2p_{21} + 3p_{22} \end{bmatrix}, \quad M_2 = \begin{bmatrix} p_{11} + p_{21} & -3p_{11} \\ p_{12} + p_{22} & 0 \end{bmatrix},$$

namely

$$\mathcal{V} = \mathbb{V}(-p_{11}p_{21} + 4p_{11}p_{22} - p_{12}p_{21} + 4p_{12}p_{22}, 3p_{11}p_{12} + 3p_{11}p_{22}).$$

The point $p = \begin{pmatrix} 0 & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{8} \end{pmatrix}$ lies on the Spohn variety and is a dependency equilibrium.

The point $p = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has $p_{1+} = 0$. It lies on the Spohn variety but is not a dependency equilibrium: For any sequence $p^{(r)}$ converging to p it is

$$-1 = \lim_{r \rightarrow \infty} \frac{-p_{11}^{(r)} - p_{12}^{(r)}}{p_{11}^{(r)} + p_{12}^{(r)}} = \lim_{r \rightarrow \infty} \frac{-2p_{21}^{(r)} + 3p_{22}^{(r)}}{p_{21}^{(r)} + p_{22}^{(r)}} = -1 + \frac{3}{2} = \frac{1}{2}.$$

Proposition 2.5. [13, Theorem 6] *If the payoff tables of a game are generic, then the Spohn variety \mathcal{V} is irreducible of codimension $d_1 + \dots + d_n - n$ and degree $d_1 \cdots d_n$. The intersection of \mathcal{V} with the Segre variety in the open simplex Δ is precisely the set of totally mixed Nash equilibria for the game X .*

The following result shows that the case of 2×2 games is unique and needs to be studied separately.

Proposition 2.6. [13, Theorem 8] *If $n = d_1 = d_2 = 2$ then the Spohn variety \mathcal{V} is an elliptic curve. In all other cases, the Spohn variety is rational, represented by a map onto $(\mathbb{P}^1)^n$ with linear fibers.*

It is important to bear in mind that the above Propositions 2.5 and 2.6 are proven for generic games and do not necessarily hold for specific games. For example, in the game Prisoner's dilemma from Example 2.3, the Spohn curve is reducible and singular.

While, generically, Nash equilibria consist of finitely many points, the Spohn variety clearly does not. It is natural to ask if this means that by looking at the dependency equilibria one can always find a better outcome than the one coming from Nash equilibria. The *payoff curve*, as defined in [13], describes the payoffs coming from totally mixed dependency equilibria. It can be computed via the determinant of the Konstanz matrix, the matrix that gives the parametrization in Proposition 2.6. For any two points on the curve, if one has higher payoffs for both players, it Pareto dominates the other one. For generic 2×2 games, there is only one totally mixed Nash equilibrium, which can be computed as in [19, Theorem 6.6]. Using quantifier elimination in **Mathematica** [20], we propose a method in [8] to check whether this Nash point is Pareto dominated by a dependency equilibrium.

3. DENSENESS OF REAL POINTS

When we are interested in dependency equilibria, we are automatically interested in the Spohn variety. This is, because, as seen above, dependency equilibria for which the denominators of the conditional expected payoff do not vanish can be completely described as the intersection of the Spohn variety with the open probability simplex Δ . From an algebraic viewpoint, it is much nicer to work with a variety, especially one we already know basic properties about, as seen above. The discrepancy between studying the Spohn variety and actual dependency equilibria arises not only from the boundary cases but also from the fact that we consider the Spohn variety as a projective variety over the complex numbers, while the probabilities making up the dependency equilibria of course have to be real. We can easily bridge this gap, and for example adopt statements of dimension and degree to the real part, if we can show that the real points lie dense within the Spohn variety. In general, this can be achieved via the parametrization in Proposition 2.6. We need to focus on the case of 2×2 games separately, in which the Spohn variety is a curve in \mathbb{P}^3 given by two quadrics.

3.1. Reducibility of the Spohn cubic. For a 2×2 game with payoff matrices $X^{(1)} = (a_{ij})$, $X^{(2)} = (b_{ij})$, the Spohn variety is defined by the two quadrics $\det(M_1)$ and $\det(M_2)$. We may eliminate $p_{2,2}$ in these equations while the elliptic curve remains the same up to isomorphy. After a relabeling of the variables to $x = p_{11}, y = p_{12}, z = p_{21}$ for simplicity, the resulting planar model is the ternary cubic $\mathcal{C} \subset \mathbb{P}^2$, given by

$$f = c_1x^2y + c_2x^2z + c_3xy^2 + c_4xz^2 + c_5y^2z + c_6yz^2 + c_7xyz,$$

where

$$\begin{aligned} c_1 &= (a_{11} - a_{22})(b_{11} - b_{12}), \\ c_2 &= (a_{11} - a_{21})(b_{22} - b_{11}), \\ c_3 &= (a_{12} - a_{22})(b_{11} - b_{12}), \\ c_4 &= (a_{11} - a_{21})(b_{22} - b_{21}), \\ c_5 &= (a_{12} - a_{22})(b_{21} - b_{12}), \\ c_6 &= (a_{12} - a_{21})(b_{22} - b_{21}), \\ c_7 &= (a_{12} - a_{21})(b_{22} - b_{11}) + (a_{11} - a_{22})(b_{21} - b_{12}). \end{aligned}$$

A ternary cubic of this form is called a *Spohn cubic*. For reasons that will become clearer later on, we are interested in the irreducible components of the Spohn cubic. According to 2.5, for generic payoff matrices, the Spohn variety, and therefore also the Spohn cubic \mathcal{C} , is irreducible. But what conditions must the entries of the payoff tables fulfill in order for \mathcal{C} to be reducible?

Remark 3.1. The cubic equation f is zero and therefore $\mathcal{C} = \mathbb{P}^2$ if and only if one of the following holds

- (1) One of the payoff tables is constant.
- (2) $a_{11} = a_{21}, a_{12} = a_{22}, b_{11} = b_{12}, b_{21} = b_{22}$
- (3) $a_{11} = a_{12} = a_{22}, b_{11} = b_{21} = b_{22}$
- (4) $a_{11} = a_{12} = a_{21}, b_{11} = b_{12} = b_{21}$

The second case is exactly when $\det M_1 = \det M_2$.

The following result answers, to some extent, the questions posed in [14, Problem 4.3].

Lemma 3.2. *The cubic \mathcal{C} is reducible if and only if f is non-zero and one of the following cases holds:*

$$(1) \quad a_{11} = a_{12}$$

$$(4) \quad b_{11} = b_{12}$$

$$(7) \quad a_{12} = a_{22}, b_{21} = b_{22}$$

$$(2) \quad a_{11} = a_{21}$$

$$(5) \quad b_{11} = b_{21}$$

$$(8) \quad a_{12} = a_{21}, b_{12} = b_{21}$$

$$(3) \quad a_{21} = a_{22}$$

$$(6) \quad b_{12} = b_{22}$$

$$\begin{aligned} (9) \quad 0 &= a_{12}(b_{12} - b_{22}) + a_{21}(b_{22} - b_{21}) + a_{22}(b_{21} - b_{12}) \\ &= a_{11}(b_{22} - b_{12}) + a_{21}(b_{11} - b_{22}) + a_{22}(b_{12} - b_{11}) \\ &= a_{11}(b_{22} - b_{21}) + a_{12}(b_{11} - b_{22}) + a_{22}(b_{21} - b_{11}) \end{aligned}$$

$$\begin{aligned} (10) \quad 0 &= a_{11}(b_{12} - b_{21}) + a_{12}(b_{21} - b_{22}) + a_{21}(b_{22} - b_{12}) \\ &= a_{12}(b_{11} - b_{21}) + a_{21}(b_{12} - b_{11}) + a_{22}(b_{21} - b_{12}) \\ &= a_{11}(b_{11} - b_{21}) + a_{21}(b_{22} - b_{11}) + a_{22}(b_{21} - b_{22}) \end{aligned}$$

$$\begin{aligned} (11) \quad 0 &= a_{12}(b_{22} - b_{21}) + a_{21}(b_{12} - b_{22}) + a_{22}(b_{21} - b_{12}) \\ &= a_{11}(b_{22} - b_{21}) + a_{21}(b_{11} - b_{22}) + a_{22}(b_{21} - b_{11}) \\ &= a_{11}(b_{22} - b_{12}) + a_{12}(b_{11} - b_{22}) + a_{22}(b_{12} - b_{11}) \end{aligned}$$

$$\begin{aligned} (12) \quad 0 &= a_{11}(b_{12} - b_{21}) + a_{12}(b_{22} - b_{12}) + a_{21}(b_{21} - b_{22}) \\ &= a_{12}(b_{11} - b_{12}) + a_{21}(b_{21} - b_{11}) + a_{22}(b_{12} - b_{21}) \\ &= a_{11}(b_{11} - b_{21}) + a_{12}(b_{22} - b_{12}) + a_{21}(b_{21} - b_{11}) + a_{22}(b_{12} - b_{22}) \end{aligned}$$

Proof. The [detailed computations](#) can be found in [8]. The ternary cubic $\mathcal{C} \subset \mathbb{P}^2$ is reducible if and only if there exists a projective line completely contained in it. Take three lines L_1, L_2, L_3 which do not have a common point of intersection and consider their intersections $X_i = L_i \cap \mathcal{C}$ with the cubic. Clearly, if any of these intersections is the whole line, then \mathcal{C} is reducible. If this is not the case, then, since any two lines in \mathbb{P}^2 intersect, any line contained in \mathcal{C} passes through a pair of distinct points $(p_1, p_2) \in X_1 \times (X_2 \cup X_3)$.

Consider the projective lines $L_1 = \mathbb{V}(x)$, $L_2 = \mathbb{V}(y)$ and $L_3 = \mathbb{V}(z)$. We have

$$\begin{aligned} f(0, y, z) &= yz(c_5y + c_6z), \\ f(x, 0, z) &= xz(c_2x + c_4z), \\ f(x, y, 0) &= xy(c_1x + c_3y). \end{aligned}$$

If any of these is zero, then this means that the entire corresponding line is contained in the cubic. Hence, if $c_5 = c_6 = 0$, $c_2 = c_4 = 0$ or $c_1 = c_3 = 0$, the cubic is reducible and we are done. If not, we look at the points of intersection $X_i = \mathcal{C} \cap L_i$ of the cubic with these lines, which can be obtained through the zeros of the polynomials above. Namely,

$$\begin{aligned} X_1 &= \left\{ [0 : 0 : 1], [0 : 1 : 0], [0 : 1 : -\frac{c_5}{c_6}] \right\}, \\ X_2 &= \left\{ [0 : 0 : 1], [1 : 0 : 0], [1 : 0 : -\frac{c_2}{c_4}] \right\}, \\ X_3 &= \left\{ [0 : 1 : 0], [1 : 0 : 0], [1 : -\frac{c_1}{c_3} : 0] \right\}, \end{aligned}$$

where the last element of each set is only contained if the corresponding denominator is nonzero.

We now take pairs of distinct points $(p_1, p_2) \in X_1 \times ((X_2 \setminus X_1) \cup (X_3 \setminus X_1))$. By inserting both points into a line equation $ax + by + cz = 0$ we obtain the defining equation of the line going through them. For example, the first pair gives us $c = 0$ and $a - \frac{c_1}{c_3}b = 0$, which results in the given line. If one of these lines is contained in \mathcal{C} , then, under the given assumptions, \mathcal{C} is reducible; if not, it must be irreducible.

(p_1, p_2)	Line through p_1 and p_2
$[0 : 0 : 1], [1 : -\frac{c_1}{c_3} : 0]$	$V(\frac{c_1}{c_3}x + y)$
$[0 : 1 : 0], [1 : 0 : -\frac{c_2}{c_4}]$	$V(\frac{c_2}{c_4}x + z)$
$[0 : 1 : -\frac{c_5}{c_6}], [1 : 0 : 0]$	$V(\frac{c_5}{c_6}y + z)$
$[0 : 1 : -\frac{c_5}{c_6}], [1 : 0 : -\frac{c_2}{c_4}]$	$V(\frac{c_2}{c_4}x + \frac{c_5}{c_6}y + z)$
$[0 : 1 : -\frac{c_5}{c_6}], [1 : -\frac{c_1}{c_3} : 0]$	$V(\frac{c_1 c_5}{c_3 c_6}x + \frac{c_5}{c_6}y + z)$

Notice that the pairs only actually appear here if all denominators of coordinates are nonzero.

Inserting these lines into the conic equation and decomposing yields

$$\begin{aligned} f(x, -\frac{c_1}{c_3}x, z) &= \frac{-xz(a_{21} - a_{22})(a_{11} - a_{12})((b_{11} - b_{22})x + (b_{21} - b_{22})z)}{a_{12} - a_{22}}, \\ f(x, y, -\frac{c_2}{c_4}x) &= \frac{xy(b_{12} - b_{22})(b_{11} - b_{21})((a_{11} - a_{22})x + (a_{12} - a_{22})y)}{b_{21} - b_{22}}, \\ f(x, y, -\frac{c_5}{c_6}y) &= \frac{xy(d_3x + e_3y)}{(a_{12} - a_{21})^2(b_{21} - b_{22})}, \\ f(x, y, -\frac{c_2}{c_4}x - \frac{c_5}{c_6}y) &= \frac{-xy(d_4x + e_4y)}{(a_{12} - a_{21})^2(b_{21} - b_{22})}, \\ f(x, y, -\frac{c_1 c_5}{c_3 c_6}x - \frac{c_5}{c_6}y) &= \frac{x((a_{11} - a_{22})x + (a_{12} - a_{22})y)(d_5x + e_5y)}{(a_{12} - a_{21})^2(b_{21} - b_{22})}, \end{aligned}$$

where the d_l and e_l are very long polynomials in the entries of the payoff matrices with integer coefficients. Note that, under the given assumptions, the denominators are nonzero. The cubic is now reducible if, under the given assumptions, one of the numerators of these polynomials is zero everywhere or, more precisely, if one of the factors in their decompositions above is zero.

First, let us take a look at the simpler factors. The first polynomial, $f(x, -\frac{c_1}{c_3}x, z)$, is zero if and only if $a_{21} = a_{22}$, $a_{11} = a_{12}$ or $b_{11} = b_{21} = b_{22}$. We assume here that $c_3 \neq 0$, but actually reducibility follows from these cases even if this is not satisfied: Indeed, if $c_3 = 0$, then there are two possibilities. If $b_{11} = b_{12}$, then $c_1 = c_3 = 0$. If $a_{12} = a_{22}$, then $a_{12} = a_{21} = a_{22}$ implies $c_5 = c_6 = 0$ and $a_{11} = a_{12} = a_{22}$ implies $c_1 = c_3 = 0$. The case that $b_{11} = b_{21} = b_{22}$ implies $c_2 = c_4 = 0$ anyways and can therefore actually be omitted. We are left with the cases (1) and (3) from the classification.

Very similarly, the second polynomial gives us the cases (5) and (6). The case $a_{11} = a_{12} = a_{22}$ coming from the last factor cannot occur, since we assume $c_3 \neq 0$ here.

We just used that \mathcal{C} is reducible if $c_1 = c_3 = 0$, $c_2 = c_4 = 0$ or $c_5 = c_6 = 0$. Given the simpler cases we just obtained, this now reduces to $b_{11} = b_{12}$, $a_{11} = a_{21}$, $(a_{12} = a_{22} \wedge b_{22} = b_{21})$ or $(a_{12} = a_{21} \wedge b_{21} = b_{12})$, namely the cases (2), (4), (7) and (8).

For the cases under which the long factors vanish we consider the ideals $J_i = (d_i, e_i)$, since we are interested in the conditions on the payoff matrix entries that guarantee $d_i = e_i = 0$.

Let us decompose for example J_5 :

$$\begin{aligned} J_5 &= (b_{21} - b_{22}, b_{12} - b_{22}) \cap (b_{12} - b_{22}, a_{11} - a_{21}) \cap (b_{12} - b_{21}, b_{11} - b_{21}) \\ &\quad \cap (b_{12} - b_{21}, a_{12} - a_{21}) \cap (b_{11} - b_{21}, a_{11} - a_{21}) \cap (a_{12} - a_{21}, a_{11} - a_{21}) \\ &\quad \cap I_{11} \cap I_{12} \end{aligned}$$

Here, I_{11} and I_{12} are the ideals corresponding to the cases (11) and (12). One can check that the vanishing of all the other components is already covered by the cases (1) to (8). In order to actually obtain (11) and (12) from I_{11} and I_{12} we need to show that the assumption that $c_3 \neq 0$ and $c_6 \neq 0$ is not necessary. Let for instance $c_3 = 0$. If $b_{11} = b_{12}$, then that is case (4). To see what happens if $a_{12} = a_{22}$ we look at the decomposition of the two ideals

$$\begin{aligned} &I_{11} + (a_{12} - a_{22}) \\ &= (b_{12} - b_{22}, a_{12} - a_{22}, a_{21}b_{11} - a_{22}b_{11} - a_{11}b_{21} + a_{22}b_{21} + a_{11}b_{22} - a_{21}b_{22}) \\ &\quad \cap (a_{21} - a_{22}, a_{12} - a_{22}, a_{11} - a_{22}), \\ \\ &I_{12} + (a_{12} - a_{22}) \\ &= (b_{11} - b_{21}, a_{12} - a_{22}, a_{11}b_{12} - a_{22}b_{12} - a_{11}b_{21} + a_{21}b_{21} - a_{21}b_{22} + a_{22}b_{22}) \\ &\quad \cap (a_{21} - a_{22}, a_{12} - a_{22}, a_{11} - a_{22}). \end{aligned}$$

The second components cannot vanish, since f is nonzero (Remark 3.1). The first components vanishing is already covered by the existing cases. Hence, it is not necessary to assume that $c_3 \neq 0$ and, similarly, neither is $c_6 \neq 0$. In this case we also note that $f(x, y, -\frac{c_1c_5}{c_3c_6}x - \frac{c_5}{c_6}y)$ also vanishes if $a_{11} = a_{12} = a_{22}$, but this is already covered by case (1).

For J_3 and J_4 , we can proceed similarly. Decomposing J_3 gives us the cases (9) and (10), and decomposing J_4 gives us the cases (11) and (12) again. Assumptions are required for none of both. \square

Given the cases from Lemma 3.2 and assuming that the payoff tables are otherwise generic, one can compute the actual **irreducible components of the Spohn cubic**, as is done in [8]. We see that here, the Spohn cubic always decomposes into a line and a conic.

Let us take a closer look at what these cases look like. For the first 8 cases, it is quite clear that the Spohn cubic is reducible. The longer cases are not as obvious but there is still some pattern to them. In fact, the game Prisoner's Dilemma from Example 2.3 falls under case (9). Notice that the equalities in the cases are invariant under operations on the payoff tables that preserve the Spohn curve (see also Example 4.7).

Remark 3.3. Denote by $I_9, I_{10}, I_{11}, I_{12}$ the ideals spanned by the (quadratic) polynomials in the cases (9) to (12) of Lemma 3.2. First we notice **computationally** ([8]), that

$$\begin{aligned} I_9 &= I_9 + (a_{11}(b_{12} - b_{21}) + a_{12}(b_{11} - b_{12}) + a_{21}(b_{21} - b_{11})) \\ I_{10} &= I_{10} + (a_{11}(b_{11} - b_{12}) + a_{12}(b_{22} - b_{11}) + a_{22}(b_{12} - b_{22})) \\ I_{11} &= I_{11} + (a_{11}(b_{12} - b_{21}) + a_{12}(b_{21} - b_{11}) + a_{21}(b_{11} - b_{12})) \\ I_{12} &= I_{12} + (a_{11}(b_{11} - b_{21}) + a_{12}(b_{22} - b_{11}) + a_{22}(b_{21} - b_{22}), \\ &\quad a_{11}(b_{11} - b_{12}) + a_{21}(b_{22} - b_{11}) + a_{22}(b_{12} - b_{22})). \end{aligned} \tag{2}$$

The original generators of the ideals I_9, I_{10}, I_{11} (i.e., the polynomials in Lemma 3.2) are interchangeable with these new polynomials that have been added in (2): any choice of three of the four generators (i.e. of the three original generators and the generator added in (2)) is sufficient to generate the entire ideal I_m , for $m = 9, 10, 11$. For case (12), the third generator of I_{12} does not fit in with the others. Here, any choice of three of the first two original generators of I_{12} and the two additional generators in (2) is sufficient to generate the entire ideal. We can therefore forget about the long term in (12) and replace it with the shorter ones that fit in with the other cases.

For each of the generating four polynomials of I_m , there is exactly one a_{ij} that does not occur in the polynomial. These a_{ij} are pairwise different within the four generators. Denote by $f_{lk}^{(m)}$ the polynomial among the generators of I_m that does not contain a_{ij} . These 4-element sets of polynomials can be represented in the payoff tables A and B as follows:

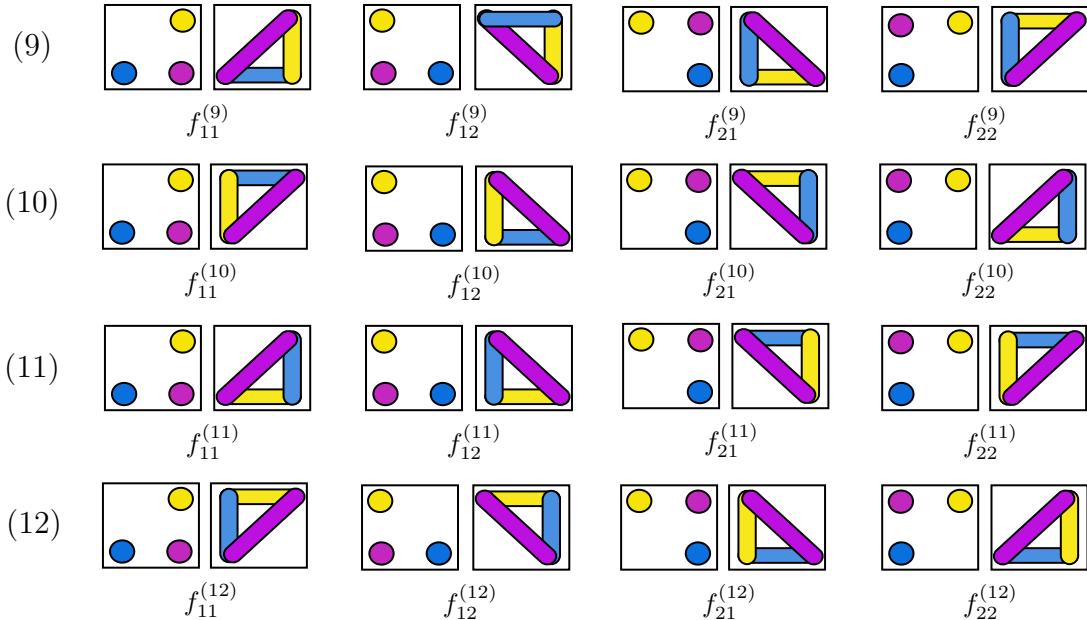


FIGURE 1. For a polynomial $f_{lk}^{(m)}$, the dots on the left side in A represent the variables a_{ij} appearing in $f_{lk}^{(m)}$. The lines on the right side of the same color in B represent the two variables b_{ij} that occur in the same monomial as a_{ij} within $f_{lk}^{(m)}$. For example, the yellow line in B for $f_{11}^{(9)}$ represents the monomials $a_{12}b_{12}$ and $a_{12}b_{22}$ in $f_{11}^{(9)}$.

There is only one way (up to sign) to arrange monomials in $f_{lk}^{(m)}$ such that each variable occurs exactly in two monomials and one monomial that it occurs in has a negative sign and one has a positive sign. Therefore, Figure 1 combinatorially determines the polynomials $f_{lk}^{(m)}$.

The triangles in B either agree in their placement with the triangles in A , or they are mirrored on the hypotenuse of A . For a fixed $m = 9, 10, 11, 12$, either both triangles in A with the hypotenuse on the diagonal are mirrored on it in B , or they both remain the same in B . The same holds for the anti-diagonal. We also notice that, for the (purple) point a_{ij} in the triangle in A that is not on the hypotenuse, the corresponding line in B is always the hypotenuse of B . For the other (blue and yellow) points a_{ij} in

the triangle, either the corresponding edge in B goes through the point b_{ij} in the same position or through the point that lies opposite of b_{ij} in B . Within each case (m), for both diagrams in which the triangles have an edge on the diagonal, exactly one of the two choices explained above holds for all blue and yellow points. The same holds for the anti-diagonal.

3.2. Finding real smooth points. As hinted, knowing about the irreducible components of a variety is useful in determining the denseness of its real part. A pertinent comment to explicitly state here is that not all quadrics defined over \mathbb{R} have real points, and not all quadrics with dense real points have a common smooth real point. This is different from the case of irreducible plane cubics with a distinguished point for which a smooth real point always exists.

Theorem 3.4. [10, Theorem 2.2.9] *Let V be an irreducible complex affine variety defined by real polynomials. If V has a smooth point with real coordinates, then $V(\mathbb{R})$, the set of real points of V , lies Zariski dense in V .*

By covering the ambient projective space with dense open affine subsets, we may apply this to the Spohn cubic, as well as the Spohn variety.

Lemma 3.5. *Given that the defining equation f is non-zero, every irreducible component of the Spohn cubic \mathcal{C} contains a smooth point with real coordinates. In particular, the set of real points $\mathcal{C}(\mathbb{R})$ lies Zariski dense in \mathcal{C} .*

Proof. If f is zero, the real points trivially lie dense in \mathcal{C} . Hence assume that $f \neq 0$. If f is irreducible, then the cubic is an elliptic curve and has real points. If the cubic \mathcal{C} is reducible, it is either the union of a line and a conic or the union of three lines. All lines in \mathbb{P}^2 are smooth and contain real points. While all irreducible conics are projectively equivalent to $x^2 + yz$ and therefore smooth, it can happen that they contain no real smooth points. Hence, it suffices to take a closer look at those conics that we obtain from the cases in Lemma 3.2, as shown in [8].¹ We may assume that the defining polynomials are always irreducible, as otherwise we obtain a union of two (possibly identical) lines.

(1) & (3) The conic in this case is of the form $g = d_1xy + d_2xz + d_3yz + d_4z^2$ and its Jacobian at a point p is given by

$$J(p) = (d_1y + d_2z \quad d_1x + d_3z \quad d_2x + d_3y + 2d_4z).$$

Clearly, if $d_1 = 0$, then g is reducible. Hence, we can assume that $d_1 \neq 0$, under which circumstance the point $[0 : 1 : 0]$ is a smooth point.

(2) The conic in this case is of the form

$$\begin{aligned} g &= (a_{21} - a_{22})(b_{11} - b_{12})x^2 + (a_{12} - a_{22})(b_{11} - b_{12})xy \\ &\quad + ((a_{21} - a_{12})(b_{11} - b_{22}) + (a_{21} - a_{22})(b_{21} - b_{12}))xz \\ &\quad + (a_{22} - a_{12})(b_{12} - b_{21})yz + (a_{21} - a_{12})(b_{21} - b_{22})z^2 \\ &= d_1x^2 + d_2xy + d_3xz + d_4yz + d_5z^2 \end{aligned}$$

and its Jacobian at a point p is given by

$$J(p) = (2d_1x + d_2y + d_3z \quad d_2x + d_4z \quad d_3x + d_4y + 2d_5z).$$

Assume for contradiction that $d_2 = d_4 = 0$. This implies that $a_{12} = a_{22}$, in which case $g = (a_{21} - a_{12})(x + z)((b_{11} - b_{12})x + (b_{21} - b_{22})z)$, or that $b_{11} = b_{12} = b_{21}$, in which case $d_1 = 0$ and g is divisible by z . Therefore, either d_2 or d_4 is non-zero and hence $[0 : 1 : 0]$ is a smooth point.

¹These 12 cases are presented in the section titled *Decomposition of the Spohn Cubic* in [8].

The cases (4),(5) and (6) follow in the same way as the previous three.

(7), (11) & (12) The conic in this case is of the form $g = d_1xy + d_2xz + d_3yz$ and its Jacobian at a point p is given by

$$J(p) = \begin{pmatrix} d_1y + d_2z & d_1x + d_3z & d_2x + d_3y \end{pmatrix}.$$

Since g is irreducible, d_1 must be non-zero and thus $[1 : 0 : 0]$ is a smooth point.

(8) The conic in this case is of the form $g = d_1xy + d_2y^2 + d_3xz + d_4z^2$ and its Jacobian at a point p is given by

$$J(p) = \begin{pmatrix} d_1y + d_3z & d_1x + 2d_2y & d_3x + 2d_4z \end{pmatrix}.$$

If $d_1 = d_3 = 0$, then g is reducible over \mathbb{C} , hence we may assume that one of them is non-zero, under which circumstance the point $[1 : 0 : 0]$ is smooth.

(9) & (10) The conic in this case is of the form $g = d_1x^2 + d_2xy + d_3xz + d_4yz$ and its Jacobian at a point p is given by

$$J(p) = \begin{pmatrix} 2d_1x + d_2y + d_3z & d_2x + d_4z & d_3x + d_4y \end{pmatrix}.$$

Since g is irreducible, d_4 must be non-zero and thus $[0 : 1 : 0]$ is a smooth point. \square

Our aim is to extend these properties to the Spohn variety $\mathcal{V} \subseteq \mathbb{P}^3$. For generic games, this has already been done in [14, Proposition 4.1], but we are interested in the non-generic cases from Lemma 3.2. We start by decomposing \mathcal{V} into its **irreducible components** in the twelve cases and under the assumption that all the other entries are generic. This can be found in [8].

Proposition 3.6. [1, Proposition 5.8] *Let $V \subset \mathbb{C}^m$ be an irreducible variety defined over \mathbb{R} of dimension d and $\pi : \mathbb{C}^m \rightarrow \mathbb{C}^{d+1}$ be a generic projection defined over \mathbb{R} . If $\overline{\pi(V)}$ contains a real smooth point, then so does V .*

Theorem 3.7. *For each of the cases from Lemma 3.2, assuming that all the other entries of the payoff tables are generic, the irreducible components of the Spohn variety \mathcal{V} each contain a smooth point with real coordinates, hence the set of real points of \mathcal{V} lies Zariski dense in \mathcal{V} .*

Proof. Consider an irreducible component $V = \mathbb{V}_{\mathbb{P}^3}(P)$ of \mathcal{V} , where P is prime. If $W := \mathbb{V}_{\mathbb{P}^2}(P \cap \mathbb{C}[p_{11}, p_{12}, p_{21}])$ is an irreducible component of \mathcal{C} , then, under our assumptions, it contains a smooth real point p . By covering \mathbb{P}^2 with the affine open spaces $\tilde{U}_{p_{11}=1}$, $\tilde{U}_{p_{12}=1}$ and $\tilde{U}_{p_{21}=1}$, for illustration we assume here that $p \in \tilde{U}_{p_{11}=1}$. Let $U_{p_{11}=1}$ be the corresponding affine open space of \mathbb{P}^3 and $\pi : \mathbb{C}^3 \rightarrow \mathbb{C}^2$ the generic projection between affine spaces. Then

$$V \cap U_{p_{11}=1} \simeq \mathbb{V}_{\mathbb{C}^3}(g(1, \cdot) \mid g \in P)$$

and

$$\overline{\pi(\mathbb{V}_{\mathbb{C}^3}(g(1, \cdot) \mid g \in P))} = \mathbb{V}_{\mathbb{C}^2}(g(1, \cdot) \mid g \in P \cap \mathbb{C}[p_{11}, p_{12}, p_{21}]) \simeq W \cap \tilde{U}_{p_{11}=1}.$$

If now $\dim V = 1$, then by Proposition 3.6, the preimage $V \cap U_{p_{11}=1}$ also contains a smooth real point.

Considering the specific cases, we start by looking at the **decompositions of \mathcal{V}** ([8]). One can check in all cases that every irreducible component has codimension 2. For all cases except (7), one can also check that the elimination ideals of the minimal primes are exactly the minimal primes of \mathcal{C} for that specific case. By the above reasoning, this implies the existence of a real smooth point in the components of \mathcal{V} . In case (7), the components $\mathbb{V}(p_{21}, p_{11})$ and $\mathbb{V}(p_{12}, p_{11})$ remain the same after eliminating p_{22} . They are proper subsets of the component $\mathbb{V}(p_{11})$ of \mathcal{C} . However, since they are lines, they must contain a smooth

real point. Elimination from the third component gives the remaining component of \mathcal{C} in that case, so as before, this gives us a real smooth point in said component of \mathcal{V} . \square

If the cubic $\mathcal{C} \subseteq \mathbb{P}^2$ is reducible, then so is the Spohn variety $\mathcal{V} \subseteq \mathbb{P}^3$. There are, however, cases in which \mathcal{V} is reducible and \mathcal{C} is not.

Example 3.8. Assume that $a_{12} = a_{22}$ and that all other entries are generic. This is not among the cases from Lemma 3.2, hence, $\mathcal{C} = \mathbb{V}(f)$ is irreducible. We can, however, compute that

$$\begin{aligned}\mathbb{I}(\mathcal{V}) = (p_{21}, p_{11}) \cap & (\det M_1, \det M_2, ((a_{21} - a_{22})b_{11} + (a_{22} - a_{21})b_{21})p_{12}^2 \\ & + (a_{21} - a_{11})b_{12} + (a_{11} - a_{21})b_{21})p_{12}p_{21} \\ & + ((a_{21} - a_{22})b_{11} + (a_{22} - a_{11})b_{12} + (a_{11} - a_{22})b_{21} + (a_{22} - a_{21})b_{22})p_{12}p_{22} \\ & + ((a_{11} - a_{21})b_{21} + (a_{21} - a_{11})b_{22})p_{21}p_{22} \\ & + ((a_{11} - a_{22})b_{21} + (a_{22} - a_{11})b_{22})p_{22}^2)\end{aligned}$$

Eliminating p_{22} from these gives the ideals (p_{21}, p_{11}) and (f) . It is $V := \mathbb{V}(p_{21}, p_{11}) \subsetneq \mathbb{V}(f) = \mathcal{C}$. This corresponds to \mathcal{C} being irreducible and also, one can observe that we cannot simply obtain a real smooth point in V by pulling back a real smooth point in the planar model. Since V is a line, it contains a real smooth point anyways; therefore, the real points also lie dense in \mathcal{C} in this case.

Similarly, \mathcal{V} is reducible if $b_{21} = b_{22}$. There are no other known cases where \mathcal{V} is reducible but \mathcal{C} is not.

Under the cases of Lemma 3.2, the cubic \mathcal{C} is reducible if and only if the defining equation f is non-zero. Similarly, the conditions for \mathcal{V} being reducible are also not actually closed. For example, if $a_{11} = a_{21} \neq a_{12} = a_{22}$ and $b_{11} = b_{12} \neq b_{21} = b_{22}$, then $\mathcal{V} = V(p_{11}p_{22} + p_{12}p_{21})$ is irreducible. Notice that this also results from f being zero in this case.

Tracing back from the reducibility and the denseness of real points of the Spohn curve to what this means in practice, it is natural to ask if in the cases when the Spohn curve is reducible, the probabilities that lie on it are actually dependency equilibria.

Remark 3.9. Following [14], denote

$$\mathcal{W} := \mathbb{V}((p_{11} + p_{12})(p_{21} + p_{22})(p_{11} + p_{21})(p_{12} + p_{22})).$$

Then, any point $p \in \overline{\Delta} \cap \overline{(\mathcal{V} \setminus \mathcal{W})}^{\text{Zar}}$ is a dependency equilibrium. For any point contained in the union of hyperplanes \mathcal{W} , that is not necessarily the case. [14, Theorem 4.9] proves that, assuming all other entries of the payoff tables are generic, in the cases (1) to (7) from Lemma 3.2, it is $\overline{\Delta} \cap \overline{(\mathcal{V} \setminus \mathcal{W})}^{\text{Zar}} \subsetneq \overline{\Delta} \cap \mathcal{V}$, which means we cannot make general statements about dependency equilibria from the Spohn variety in these cases. It also shows that in the cases (8) to (12), equality holds and that, therefore, any point $p \in \overline{\Delta} \cap \mathcal{V}$ is a dependency equilibrium. Indeed, one can see in the decomposition of \mathcal{V} in [8] that, in the cases (1) to (7), some components of \mathcal{V} are contained in \mathcal{W} and that, in the cases (8) to (12), no component of \mathcal{V} is entirely contained in \mathcal{W} . Furthermore, in these latter cases, [14, Corollary 3.19] also guarantees that every Nash equilibrium of the game is also a dependency equilibrium.

4. EQUIVALENT GAMES FROM ELLIPTIC CURVE INVARIANTS

We have studied in detail when the Spohn curves of 2×2 games are reducible and therefore not smooth. We will now take a closer look at Spohn curves from generic 2×2 games, for which we already know that real points lie dense in them ([14, Proposition 4.1]) and that the points in $\overline{\Delta} \cap \mathcal{V}$ are dependency equilibria ([14, Corollary 3.19]). Also, by Proposition 2.6, these Spohn varieties are elliptic curves and we may use properties of elliptic curves to make statements about dependency equilibria of generic 2×2 games.

It is natural to ask when two generic games are equivalent with respect to their Spohn curves. This can be addressed using invariants of elliptic curves, particularly the j -invariant (9). The j -invariant of an elliptic curve classifies endomorphism algebras of elliptic curves and demonstrates isomorphisms between elliptic curves (Theorem 4.6). Two 2×2 games are *equivalent with respect to the Spohn curve* if and only if their j -invariants are equal. This implies that they have the same totally mixed dependency equilibria. Elliptic curves obtained from 2×2 games have a very specific form but we review the calculation of the j -invariant of an elliptic curve given by the intersection of two quadrics in \mathbb{P}^3 in the general case where the quadrics are defined over \mathbb{Q} . We restrict to elliptic curves over \mathbb{Q} since in many known examples of games, the payoffs are rational. Should this not be the case, payoffs can be approximated by rational numbers (for example, via the method of continued fractions [5, § 1.8]). See Appendix A for more details.

Definition 4.1 (Elliptic curve over K). An elliptic curve over a field K , ($\text{char}(K) \neq 2$) is a smooth (projective) curve of genus 1 with at least one K -rational point.

As mentioned above, we restrict ourselves to the case of $K = \mathbb{Q}$. Elliptic curves over \mathbb{Q} can be expressed in the form of a cubic equation known as the *Weierstrass form*.

Theorem 4.2 ([17], Chapter III.2.3). *Every elliptic curve over \mathbb{Q} can be expressed as a cubic in \mathbb{P}^2 with the following form:*

$$E_{\mathbb{Q}} := y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3, \quad a_{1,2,3,4,6} \in \mathbb{Q} \quad (3)$$

which upon dehomogenization gives the more recognizable long Weierstrass form of the curve:

$$E_{\mathbb{Q}} := y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_{1,2,3,4,6} \in \mathbb{Q}.$$

Remark 4.3 (Short Weierstrass form, [17], Chapter III.1). Every elliptic curve over \mathbb{Q} can be reduced further into the short Weierstrass form (via appropriate coordinate transformations) which is given by

$$E_{\mathbb{Q}} := y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{Q}.$$

Equation (3) can be obtained from the intersection of two quadrics in \mathbb{P}^3 , with intersection at a *known* \mathbb{Q} -point.

4.1. Invariants for Spohn curves as intersection of quadrics in \mathbb{P}^3 . Consider two quadric equations $P_1, P_2 \in \mathbb{Q}[x, y, z, t]$ in \mathbb{P}^3 . In the context of 2×2 games, these quadrics are the exact same equations that appear in (1) after renaming the variables. We require that these quadrics intersect in \mathbb{P}^3 at an arbitrary rational point $k = [x_0 : y_0 : z_0 : t_0]$. It is important to know what this point is, and in all the cases at hand we are given this rational point. Finding rational points on elliptic curves is an interesting and challenging endeavor in its own right. We do not make any comments on finding a common rational solution since for Spohn curves, we know that e.g. $k = [0 : 0 : 0 : 1]$, i.e. the point at infinity in \mathbb{P}^3 , is a rational solution. Now, given two such intersecting quadrics with a common rational point, we wish to compute the j -invariant.

The first step is representing quadrics in \mathbb{P}^3 . Let the coordinates in \mathbb{P}^3 be $V = (x, y, z, t)^\top$.² We represent the quadrics as

$$P_1 = V^\top \cdot A \cdot V, \quad A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad A \in \text{Mat}_{4 \times 4}(\mathbb{Q})$$

$$P_2 = V^\top \cdot B \cdot V, \quad B = \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad B \in \text{Mat}_{4 \times 4}(\mathbb{Q}).$$

Given $k = [x_0 : y_0 : z_0 : t_0]$, the common rational solution to the two quadrics P_1, P_2 , we define

$$Q = \begin{pmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & t_0 \end{pmatrix}. \quad (4)$$

When the common rational solution is $[0 : 0 : 0 : 1]$, the matrix Q as in (4) is just the 4×4 identity matrix. This is always the case we have when considering the Spohn curve. Having the common rational point at infinity is necessary to project the curve to lower dimensions and compute its j -invariant.

In case the common rational solution of the intersection of the two quadrics is *not* the point at infinity i.e. Q is not the 4×4 identity matrix, one needs to perform coordinate transformations on P_1 and P_2 such that the common rational solution becomes the point at infinity. This straightforward algorithm is explained below and also in [9]. To transform the two quadrics such that the common solution is the point at infinity:

(Step I) Normalize solutions: Choose a normalization of the solutions such that $t_0 \neq 0$. One may also assume that $t_0 = 1$ as in [9].

(Step II) Coordinate change: Now, let $V = (x, y, z, 1)^\top$ be the coordinates of \mathbb{P}^3 . Transform to coordinates $W = (X, Y, Z, T)^\top$ such that the common rational point in the new coordinates is at $[X_0 : Y_0 : Z_0 : T_0] = [0 : 0 : 0 : 1]$. This is done by the following transformation:

$$V = \begin{pmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot W, \quad W = \begin{pmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot V.$$

(Step III) Change of quadrics: Under the coordinate change, the quadrics transform as follows

$$P_1(x, y, z, t) \mapsto P'_1(X, Y, Z, T) = W^\top \cdot Q^\top \cdot A \cdot Q \cdot W$$

$$P_2(x, y, z, t) \mapsto P'_2(X, Y, Z, T) = W^\top \cdot Q^\top \cdot B \cdot Q \cdot W.$$

²These are projective coordinates but for the sake of explaining the algorithm, we represent them as a vector.

(Step IV) Eliminating terms: This follows from [4, § 7.2.2]. We now have two quadrics $P'_1[X, Y, Z, T], P'_2[X, Y, Z, T] \subseteq \mathbb{P}^3$ with a common rational point at $[0 : 0 : 0 : 1]$. Both these quadrics can be expressed as

$$P'_i(X, Y, Z, T) = K_i T^2 + L_i(X, Y, Z) T + M_i(X, Y, Z), \quad i = 1, 2$$

where K_i is a constant, $L_i(X, Y, Z)$ are linear in X, Y, Z and M_i are quadratic in X, Y, Z . Since the common rational point is already at $[0 : 0 : 0 : 1]$, $K_i = 0$ for $i = 1, 2$. This means that the two intersecting quadrics take the form

$$P'_1(X, Y, Z, T) = L_1(X, Y, Z) T + M_1(X, Y, Z)$$

$$P'_2(X, Y, Z, T) = L_2(X, Y, Z) T + M_2(X, Y, Z).$$

It is important to note that L_1 and L_2 above are linearly *independent*. The proof of this statement can be found in [4, § 7.2.2]. To state it succinctly, if L_1, L_2 are linearly *dependent*, then there is a linear combination which will make the intersecting quadric a genus 0 curve, which is false for elliptic curves.

(Step V) Reduction to cubic in \mathbb{P}^2 . The equation

$$C(X, Y, Z) = L_1(X, Y, Z)M_2(X, Y, Z) - L_2(X, Y, Z)M_1(X, Y, Z), \quad C \subseteq \mathbb{P}^2$$

is homogeneous of degree 3, and represents an elliptic curve in homogeneous, projective coordinates. One may wish to proceed to represent the curve in Weierstrass form from here, however it is not necessary for the purposes of this paper.

Remark 4.4 (Defining elliptic curves from cubics in Pari/GP). In Pari/GP [12], one can use in-built commands `ellfromeqn` and `ellinit` to define elliptic curves directly from the cubic in \mathbb{P}^2 . The file `getJ.gp` in [8] contains the relevant code to do this in Pari/GP.

Remark 4.5. To re-iterate, since $[0 : 0 : 0 : 1]$ is always a common rational solution for Spohn curves, one may proceed straight to (Step IV) above.

4.2. Aronhold invariants and the j -invariant. Starting with two quadrics that intersect at an arbitrary rational point, we now have a cubic in \mathbb{P}^2 . This cubic depends on the monomials in the two quadrics, as well as the common rational point in \mathbb{P}^3 , which has been transformed to $[0 : 0 : 0 : 1]$. A generic rational cubic equation in 3 variables (x, y, z) ³ will have the following form

$$C(x, y, z) := ax^3 + by^3 + cz^3 + 3dx^2y + 3ey^2z + 3fz^2x + 3gxy^2 + 3hyz^2 + 3izx^2 + 6jxyz, \quad (5)$$

where the coefficients $a, b, c, d, e, f, g, h, i, j \in \mathbb{Q}$. From the coefficients of the cubic (and as a result, the coefficients of the monomials of P_1, P_2 and the common rational point $[x_0 : y_0 : z_0 : t_0]$), we may derive the two Aronhold (S and T) invariants of the cubic $C(x, y, z)$. We wish to point out that the Aronhold invariants as in (6) and (7) are equivalent to the expressions given in [3] and [18] under the rescaling of the coefficients of the cubic.⁴

For the cubic equation of the form (5), the S invariant is given as:

$$\begin{aligned} S = & agec - agh^2 - ajbc + ajeh + afbh - afe^2 - d^2ec + d^2h^2 + dibc \\ & - dieh + dgjc - dgfh - 2dj^2h + 3dje - df^2b - i^2bh + i^2e^2 - ig^2c \\ & + 3igjh - igfe - 2ij^2e + ijfb + g^2f^2 - 2gj^2f + j^4, \end{aligned} \quad (6)$$

³We reset to lowercase coordinates for ease of readability henceforth. For the case of Spohn curves, the coordinates $(x, y, z, t) = (X, Y, Z, T)$.

⁴Since in these references, the cubic is represented as $C(x, y, z) = ax^3 + by^3 + cz^3 + dx^2y + ey^2z + fz^2x + gxy^2 + hyz^2 + izx^2 + jxyz$, i.e. without the factors of 3's and 6.

and the T invariant is given by:

$$\begin{aligned}
T = & a^2b^2c^2 - 3a^2e^2h^2 - 6a^2behc + 4a^2bh^3 + 4a^2e^3c - 6adgbc^2 \\
& + 18adgehc - 12adgh^3 + 12adjbhc - 24adje^2c + 12adjeh^2 \\
& - 12adfjh^2 + 6adfbec + 6adfe^2h + 6aigbhc - 12aige^2c + 6aige^2h^2 \\
& + 12aijbec + 12aije^2h - 6aifb^2c + 18aifbeh - 24ag^2jhc - 24aijbh^2 \\
& - 12aife^3 + 4ag^3c^2 - 12ag^2fec + 24ag^2fh^2 + 36agj^2ec + 12agj^2h^2 \\
& + 12agjfbc - 60agjfeh - 12agf^2bh + 24agf^2e^2 - 20aj^3bc - 12aj^3eh \\
& + 36aj^2fbh + 12aj^2fe^2 - 24ajf^2be + 4af^3b^2 + 4d^3bc^2 - 12d^3ehc \\
& + 8d^3h^3 + 24d^2ie^2c - 12d^2ieh^2 + 12d^2gjhc + 6d^2gfec - 24d^2j^2h^2 \\
& - 12d^2ibhc - 3d^2g^2c^2 - 24g^2j^2f^2 + 24gj^4f - 12d^2gfh^2 + 12d^2j^2ec \\
& - 24d^2jfb - 27d^2f^2e^2 + 36d^2jfeh + 24d^2f^2bh + 24di^2bh^2 \\
& - 12di^2bec - 12di^2e^2h + 6dig^2hc - 60digjec + 36digjh^2 + 18digfbc \\
& - 6digfeh + 36dij^2bc - 12dij^2eh - 60dijfbh + 36dijfe^2 + 6dif^2be \\
& + 12dg^2jfc - 12dgj^3c - 12dgj^2fh + 36dgjf^2e - 12dgf^3b + 24dj^4h \\
& + 12dj^2f^2b + 4i^3b^2c + 24i^2g^2ec - 27i^2g^2h^2 - 36dj^3fe - 12i^3beh \\
& + 8i^3e^3 - 24i^2gjbc + 36i^2gjeh + 6i^2gfbh + 12i^2j^2bh - 3i^2f^2b^2 \\
& - 12dg^2f^2h - 12i^2gfe^2 - 24i^2j^2e^2 + 12i^2jfe - 12ig^3fc + 12ig^2j^2c \\
& + 36ig^2jfh - 12ig^2f^2e - 36igj^3h - 12igj^2fe + 12igjf^2b + 24ij^4e \\
& - 12ij^3fb + 8g^3f^3 - 8j^6.
\end{aligned} \tag{7}$$

These invariants can also be found at [2]. The discriminant of the elliptic curve is

$$\Delta = \frac{64S^3 - T^2}{1728}. \tag{8}$$

The j -invariant of the elliptic curve is

$$j = \frac{64S^3}{\Delta}. \tag{9}$$

The j -invariant and discriminant as above are again equal to the corresponding expressions in [18] under the change in notation for the coefficients of the cubic. The j -invariant in terms of the coefficients of the two quadrics in \mathbb{P}^3 that intersect at an arbitrary rational point is available in `Generic_J_invt.txt`⁵ at [8]. The j -invariant for Spohn curves as in Section 2 is given in `SpohnCurveJInv.txt`, also available at [8]. It corresponds to the result in [13, Proposition 12]

Theorem 4.6. *Two elliptic curves are isomorphic if and only if their j -invariants are equal.*

Proof. See [17, Chapter III, Proposition 1.4]. □

Example 4.7. Consider the quadrics

$$\begin{aligned}
P_1 &= -xz + 2xt - 2yz + yt = t(2x + y) - (xz + 2yz) \\
P_2 &= -5yx - 6xt - 3yz - 4zt = -t(6x + 4z) - (5xy + 3yz)
\end{aligned}$$

⁵This file does not explicitly set $t_0 = 1$ as is assumed in Step 1 of the algorithm to transform the quadrics to have a common rational solution at infinity.

coming from the 2×2 game with payoff tables $X^{(1)} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ and $X^{(2)} = \begin{pmatrix} 6 & 1 \\ 4 & 0 \end{pmatrix}$. The point $[0 : 0 : 0 : 1]$ is a common rational solution, so we have no terms quadratic in t in the quadrics. To reduce to a cubic in \mathbb{P}^2 , we have

$$\begin{aligned} L_1 &= (2x + y), & L_2 &= -(6x + 4z) \\ M_1 &= -(zx + 2zy), & M_2 &= -(5yx + 3zy), \end{aligned}$$

and we recall that L_i is the term in P_i that is linear in t , and M_i is the term in P_i that is independent of t . The reduced cubic in \mathbb{P}^2 is

$$C(x, y, z) = L_1 M_2 - L_2 M_1,$$

which is explicitly given by

$$C := (-10y - 6z)x^2 + (-5y^2 - 18zy - 4z^2)x + (-3zy^2 - 8z^2y).$$

Comparing with the coefficients of (5), we can compute the two Aronhold invariants S (6) and T (7), and use them to compute the j -invariant as:

$$j = \frac{2810381476}{227025}.$$

This can be computed using the `IntersectionQuadricsJ.nb` in **Mathematica**. One can also initialize the curve in **Pari/GP** using the `getJ.gp` script and find the same value for the j -invariant ([8]). An immediate observation here is that for any $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, the games defined by the following two payoff matrices

$$\begin{pmatrix} \lambda_1 + \alpha_1 & 2\lambda_1 + \alpha_1 \\ \alpha_1 & 3\lambda_1 + \alpha_1 \end{pmatrix} \text{ and } \begin{pmatrix} 6\lambda_2 + \alpha_2 & \lambda_2 + \alpha_2 \\ 4\lambda_2 + \alpha_2 & \alpha_2 \end{pmatrix}$$

have the same (elliptic) Spohn curve and thus the same j -invariant.

Example 4.8 (Singular curve). Consider the quadrics

$$\begin{aligned} P_1(x, y, z, t) &= xz + yz - xt - yt = -(x + y)t + (xz + yz) \\ P_2(x, y, z, t) &= -5xy - yz + xt + 5zt = t(x + 5z) - (5xy + yz). \end{aligned}$$

coming from the 2×2 game with payoff tables $X^{(1)} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ and $X^{(2)} = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$, for which $[0 : 0 : 0 : 1]$ is a common solution. By following the algorithm above, we can eliminate t to get a cubic:

$$C(x, y, z) = -(xz + yz)(x + 5z) + (x + y)(5xy + yz).$$

The j -invariant of this cubic is infinite i.e. the discriminant Δ as in (8) is zero and the cubic, being singular, therefore does not define an elliptic curve. This is also apparent from the fact that the game is not generic and falls under case (1) of Lemma 3.2.

Example 4.9. We now consider a case of intersecting quadrics in \mathbb{P}^3 for which $[0 : 0 : 0 : 1]$ is not a common rational point. Consider the quadrics

$$\begin{aligned} P_1(x, y, z, t) &= x^2 + y^2 - z^2 - t^2 \\ P_2(x, y, z, t) &= xz - yz + yt - zt. \end{aligned}$$

A common rational solution in this case is $[1 : 1 : 1 : 1]$. By following the algorithm above, we can reduce the two quadrics to a cubic in \mathbb{P}^2 which has the form

$$C(x, y, z) = -x^3 - xy^2 + 3x^2z - y^2z - xz^2 + 2yz^2 - z^3.$$

This represents an elliptic curve and the j -invariant of this curve is

$$j = \frac{65536}{37}.$$

APPENDIX A. CONTINUED FRACTION APPROXIMATION OF REAL NUMBERS

In Pari/GP, rational approximations can be implemented using the command `contfrac`. A script to compute rational approximations using continued fractions (`contfrac.gp`) can also be found on [8]. A short tutorial and example are presented below:

```
\\" Continued fraction representation of a number x to k convergents
>> contfrac(x,k)

\\ Find rational approximation of the number
>> eval_contfrac(V) = { local(x=0);
    forstep(i = #V, 1, -1, x = V[i] + if(x,1/x,0));
    return(x);
}
\\ To get decimal representation, use .'
```

LISTING 1. Evaluating Continued Fractions

Example A.1 (Rational approximation to $\zeta(3)$). Irrationality of $\zeta(3)$ was established in the 1970's by R. Apéry, meaning that its continued fraction representation is infinitely long. However, one can approximate $\zeta(3)$ as a rational number to any finite precision using continued fractions.

```
\\" Examples: Consider zeta(3) which is an irrational number
>> zeta(3)
output = 1.2020569031595942853997381615114499908
\\ Continued fraction of zeta(3) up to 15 convergents
>> V = contfrac(zeta(3), 15)
output = [1, 4, 1, 18, 1, 1, 1, 4, 1, 9, 9, 2, 1, 2]
\\ Evaluate the continued fraction
>> eval_contfrac(V)
output = 1479821/1231074
\\ Decimal representation of above
>> 1479821/1231074.
output = 1.2020569031593551646773467720055821177
```

LISTING 2. Rational approximation of $\zeta(3)$

For 15 convergents,

$$\zeta(3) \approx \frac{1479821}{1231074}$$

is correct up to 12 decimal places, where as

$$\zeta(3) \approx \frac{461424925}{383862797}$$

which is the continued fraction approximation of $\zeta(3)$ to 20 convergents gives the correct approximation to 18 decimal places. In Mathematica, the command to construct a continued fraction is `ContinuedFraction`, while the command to evaluate a continued fraction is `FromContinuedFraction`.

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