Communication-Efficient Distributed Algorithms for Density Estimation

Abhiram Natarajan

Joint work with Ilias Diakonikolas (USC), Elena Grigorescu (Purdue), Jerry Li (MIT), Krzysztof Onak (IBM), and Ludwig Schmidt (MIT)

Outline

Motivation and Problem Definition

Learning Unstructured Distributions in ℓ_1

Learning k-Histograms in ℓ_2

Other Results

Conclusion

Can Distributed Cooks make Good Broth?

- too much data to store on one machine

Source: Google Images

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- distributed computation is necessary

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- distributed computation is necessary
- need communication-efficient distributed algorithms

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- communication-efficient algorithms vs instrinsic limits
- obtain optimal and near-optimal algorithms in a variety of settings
- time-complexity vs sample-complexity vs communication-complexity

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  error metric $d : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$
  $d(\hat{P}, P)$ must be low

Error in Density Estimation

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called Den-Est$(\mathcal{D}, \varepsilon, \ell_\rho)$ problem

Sample Complexity of Density Estimation

Definition

$m_1 = m_1(n, \varepsilon)$ is sufficient sample size for Den-Est$(\mathcal{D}, \varepsilon, \ell_\rho)$

- there exists algorithm $\mathcal{A}_\mathcal{D}$ takes m_1 samples and

$$\mathsf{E}\left[\|\hat{P} - P\|_\rho\right] \leqslant \varepsilon \qquad \forall\, P \in \mathcal{D}$$

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$m_2 = m_2(n, \varepsilon)$ is necessary sample size for Den-Est$(\mathcal{D}, \varepsilon, \ell_\rho)$

- any conceivable algorithm must take m_2 samples

Communication Complexity

- communication complexity introduced by [Yao, 1979]:
  - players contain information $X_1, \ldots, X_n$ known only to them
  - communicate to referee via a protocol to compute $f(X_1, \ldots, X_n)$
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- communication complexity - pratical and more!
- applications in seemingly unrelated complexity theory areas - turing machines, decision trees, geometric problems, etc.

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$\boxed{\text{called Dist-DE}(\mathcal{D}, m, \varepsilon, \ell_\rho) \text{ problem}}$

Communication Model – Simultaneous

referee

$$\boxed{\begin{matrix} X_1^{(1)} \\ \vdots \\ X_s^{(1)} \end{matrix}} \quad \boxed{\begin{matrix} X_1^{(2)} \\ \vdots \\ X_s^{(2)} \end{matrix}} \quad \cdots\cdots\cdots \quad \boxed{\begin{matrix} X_1^{(m)} \\ \vdots \\ X_s^{(m)} \end{matrix}}$$

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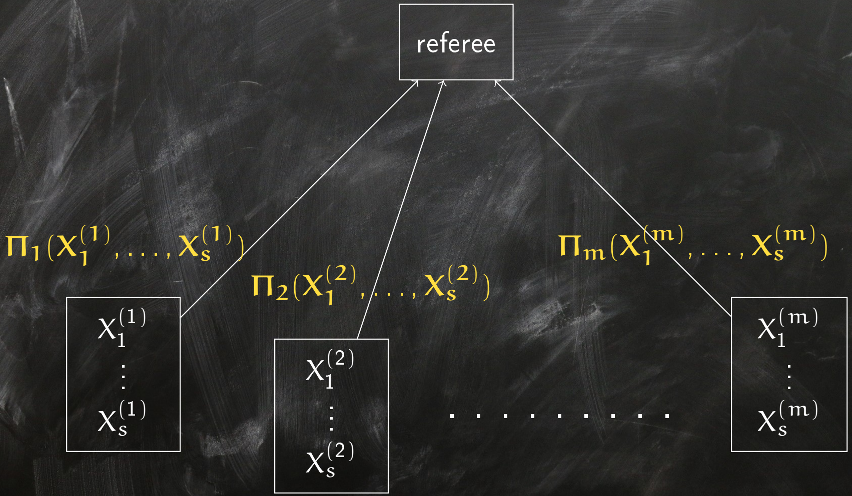

Communication Model – Interactive

blackboard (ξ)

referee

$X_1^{(1)}$
$\vdots$
$X_s^{(1)}$

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. . . . . .

$X_1^{(m)}$
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$\Pi_{1,1}(\xi, X_1^{(1)}, \ldots, X_s^{(1)})$

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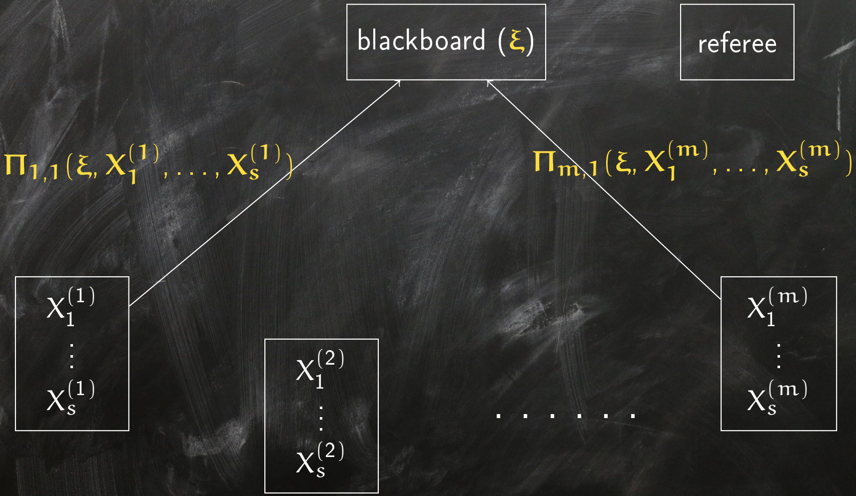

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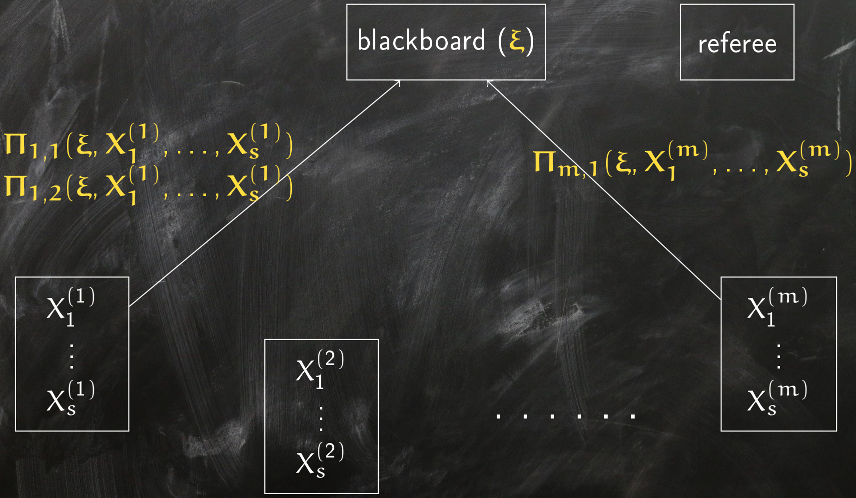

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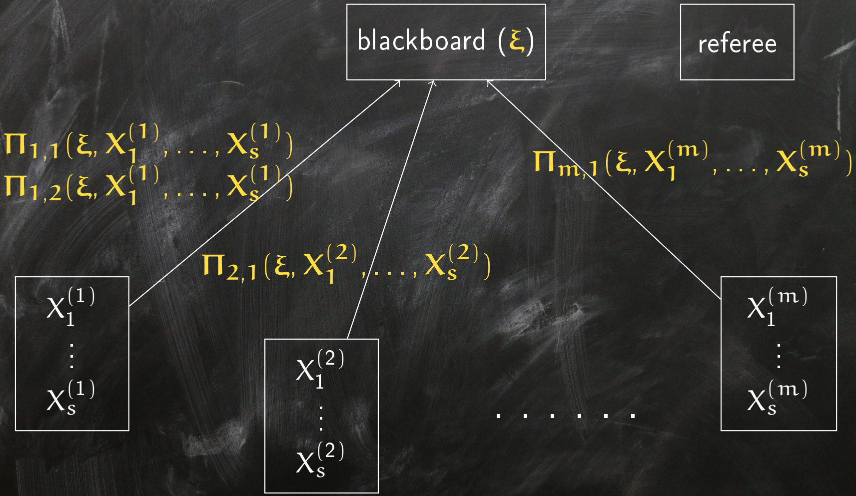

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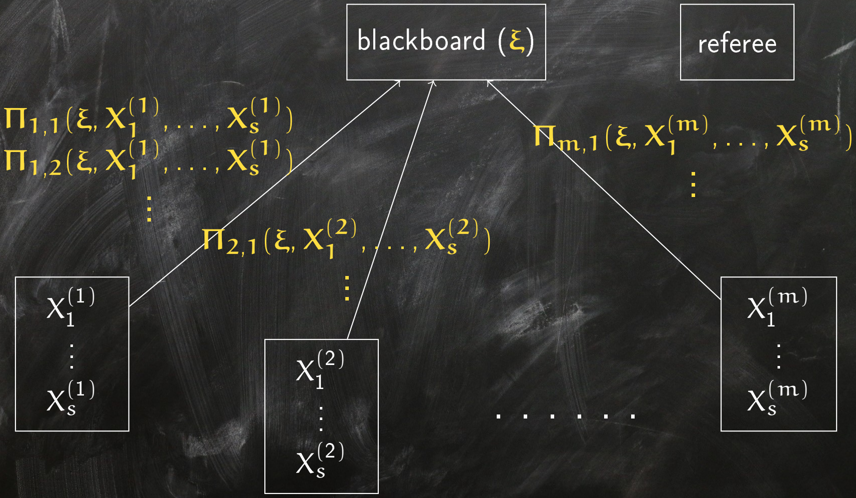

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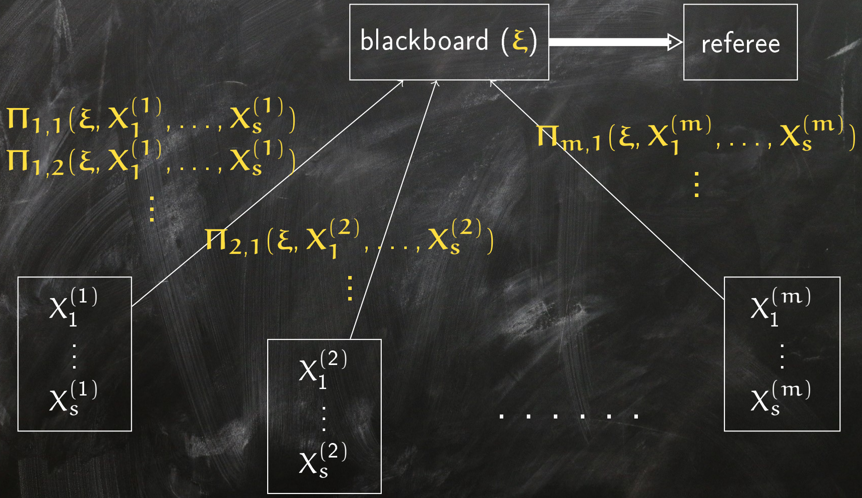

Main Conceptual Messages

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- when structure is present (k-histograms, monotone), can be exploited for non-trivial improvement

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Protocol $\Pi_{\mathcal{D}}$ solves Dist-DE$(\mathcal{D}, m, \varepsilon, \ell_\rho)$ with $\beta_1 := \mathcal{CC}(\Pi_{\mathcal{D}})$ bits

- in $\Pi_{\mathcal{D}}$ machines communicate at most β_1 bits and referee outputs hypothesis

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$\beta_2 := \mathcal{CC}(\text{Dist-DE}(\mathcal{D}, m, \varepsilon, \ell_\rho))$

- any conceivable protocol must take β_2 bits to solve Dist-DE$(\mathcal{D}, m, \varepsilon, \ell_\rho)$

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Folklore Result in Density Estimation

Theorem (Learning unstructured dists. in ℓ_1)

$\mathcal{D}_n$ - *unstructured distributions over* $[n]$. *For* Den-Est$(\mathcal{D}_n, \varepsilon, \ell_1)$

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Moreover, algorithm $\mathcal{A}_{\mathcal{D}_n}$ *outputs empirical distribution of samples*

$$\hat{P}(i) = \frac{\textit{number of } i \textit{ amongst samples}}{m_1} \qquad \forall i \in [n]$$

Communication Upper and Lower Bounds

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Theorem (Communication lower bound)

$$\mathcal{CC}(\text{Dist-DE}(\mathcal{D}, \alpha, \varepsilon, \ell_\rho)) = \Omega\left(\frac{n}{\varepsilon^2}\log n\right).$$

Lower Bound Proof Ideas

- construct family of *nearly uniform* distributions on $[n]$: for elements $2i-1$ and $2i$, probabilities are $\frac{1+100\delta_i\varepsilon}{n}$ and $\frac{1-100\delta_i\varepsilon}{n}$, δ_i uniform on $\{-1,1\}$

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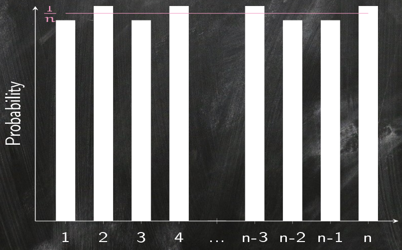

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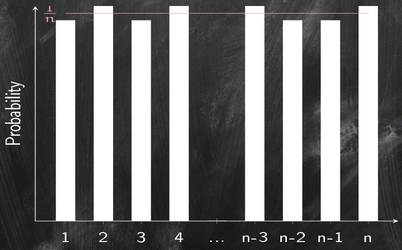

- learning distribution is equivalent to learning $\{\delta_i\}$

Lower Bound Proof Ideas

- contradiction: there is protocol sends $o\left(\frac{n}{\varepsilon^2}\log n\right)$ bits
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- less information means more error (Fano's inequality)

Other Regimes of Unstructured Dists. in ℓ_1

samp. per mach.	lower bound	upper bound
1	$\Omega\left(\frac{n}{\varepsilon^2}\log n\right)$	$O\left(\frac{n}{\varepsilon^2}\log n\right)$
$s=\Theta\left(\frac{n}{\varepsilon}\right)$	$\Omega\left(n\log\frac{1}{\varepsilon}\right)$	$O\left(\frac{n}{\varepsilon}\log\frac{1}{\varepsilon}\right)$
$s=\Theta\left(\frac{n}{\varepsilon^2}\right)$	$\Omega\left(n\log\frac{1}{\varepsilon}\right)$	$O\left(n\log\frac{1}{\varepsilon}\right)$

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- when partition known, reduces to unstructured $\Theta(\frac{k}{\varepsilon^2}\log k)$ bits
- when partition unknown, trivial protocol uses too much communication $\Theta(\frac{k}{\varepsilon^2}\log n)$ bits

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Key idea to approximate error:

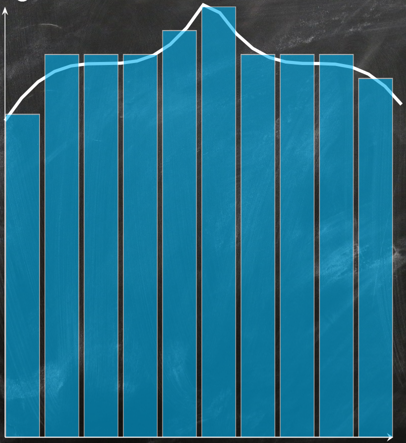
- server i has access to $x_i \in \mathbb{R}^n$ (vector of counts)
- using [Johnson and Lindenstrauss, 1984] lemma, get accurate estimate of $\|x\|_2^2$, where $x = \sum_i x_i$

Some Regimes for k–Histograms in ℓ_2

ε	samp. per mach.	lower bound	upper bound
$\Theta\left(\frac{1}{\sqrt{k}}\right)$	$\leqslant \tilde{O}(k \log n)$	$\Omega(k \log \frac{n}{k} + \sqrt{k} \log k)$	$O(k \log n)$
	$> \tilde{O}(k \log n)$	$\Omega(k \log \frac{n}{k} + \sqrt{k} \log k)$	$\tilde{O}(\frac{k^2}{s} \log n)$

Histogram Approximation

Our algorithm is agnostic!



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  - k-histograms in ℓ_1 (quite different from ℓ_2, need to approximate $\mathcal{A}_k$ distance)
  - monotone distributions in ℓ_1 (uses Birgé oblivious decomposition [Birgé, 1987b, Birgé, 1987a])

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  - monotone distributions in ℓ_1 (uses Birgé oblivious decomposition [Birgé, 1987b, Birgé, 1987a])
- our algorithms are agnostic
- can be extended to a huge class of distributions - unimodal, O(1)-modal, log-concave, monotone hazard rate (MHR) distributions, certain piecewise-polynomial continuous distributions, etc.

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- go from univariate to multivariate
- study distribution testing in this model
- generalize bounds in terms of entropy of distribution
- more than sufficient sample, also unequal number of samples

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- we provide first near-optimal bounds for a huge class of discrete distributions
- communication complexity of learning tasks - can it shed fundamental insights on the nature of learning?

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