Séminaire Lotharingien de Combinatoire **XX** (2024)
Article #YY, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Gröbner bases in Hodge algebras, with applications in the bideterminant basis

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Abstract. We develop a theory of Gröbner bases in the basis of standard monomials in algebras with a straightening law (ASL, sometimes called Hodge algebras). This in turn gives a theory of Gröbner bases in the basis of standard bideterminants (called *bd*-Gröbner bases) in the polynomial ring. As an application in the polynomial ring, we show how our theory gives an elementary one-sentence proof of *bd*-Gröbner bases for the ideal of maximal minors, which is a well-studied classical problem.

Keywords: Hodge algebra, algebra with straightening law, standard monomial, Gröbner basis, bideterminants

1 Introduction

Algebras with straightening law (ASLs) were introduced by De Concini, Eisenbud, and Procesi [3] in the early 1980s, as a generalization of several well-studied concrete examples of finitely generated algebras in which a product of monomials in the generators get rewritten or "straightened" according to a system of equations satisfying some nice properties. ASLs include Grassmannians, flag varieties, Schubert varieties, determinantal and Pfaffian varieties, varieties of minimal degree, and varieties of complexes.

One of the advantages of an ASL structure, already highlighted in the original papers [3, 7], is that it facilitates computation with ideals whose generators are well-adapted to the generators of the ASL. For example, if $X = (x_{ij})$ is an $n \times m$ matrix of independent variables, then the bideterminant ASL structure on $\mathbb{F}[X]$ has as its generators the minors of X; ideals generated by various subsets of minors, which typically could take exponentially many monomials to write down in the usual monomial basis, instead become generated by single variables or monomials in the bideterminant basis! The price paid is that multiplication of monomials in such generators sometimes need to be rewritten to bring them back to a normal form. Indeed, the present work grew out of the authors' attempt to compute with and understand certain determinantal ideals in a Weyl algebra.

*jgrochow@colorado.edu. Partially supported US NSF CAREER grant CCF-2047756.
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Once nice feature of straightening rules in an ASL is that, when taken as generators of an ideal, the naturally form a Gröbner basis. Thus, if an ASL is presented as $A = \mathbb{F}[x_1, \ldots, x_n]/S$, where S is the ideal generated by the straightening relations, then the structure of the straightening relations already facilitates computation in A using ordinary Gröbner basis techniques. In essence, this is "offloading" the straightening process to the division process in Gröbner basis.

But in some cases, more efficient implementations of the straightening process may be available, as is the case of the bideterminant basis [14, 5, 16]. So while one might still wish to use Gröbner basis techniques on an ideal in A, one may wish to *circumvent* their use for the straightening relations themselves. Furthermore, given an ideal I in the ASL $A = \mathbb{F}[x_1, \ldots, x_n]/S$, rather than considering instead the ideal $I + S \subseteq \mathbb{F}[x_1, \ldots, x_n]$ and reverting to ordinary Gröbner basis techniques, it seems more natural to hope for a theory of Gröbner bases that is in some sense "native" to the algebra A.

In this paper, we develop exactly such a theory. This extended abstract includes all key definitions, main results, and a few proof sketches; full proofs will appear in the full version elsewhere.

To explain one of the key difficulties in developing such a theory, we must say a little more about ASLs. In an ASL $A = \mathbb{F}[x_1, \ldots, x_n]/S$, the structure of the straightening relations S is such that one may naturally choose a subset of the monomials in the x_i, called the *standard monomials*, which form an $\mathbb{F}$-basis for A, and possess several nice properties. However, because of the straightening relations, many products of standard monomials will not be standard monomials, but will get straightened into linear combinations of standard monomials. In attempting to define Gröbner bases in an ASL, this issue complicates what it means for one standard monomial to "divide" another.

Our resolution of this complication also offers certain tradeoffs in terms of both structure and algorithms. The high-level idea is that, in addition to a choice of term order (as in any theory of Gröbner bases), we also have a choice of "algebra of leading terms", which precisely governs which "division-like" relations among standard monomials we can use to cancel leading terms when it comes to Gröbner basis calculations. There are two natural possibilities that come to mind: (1) ordinary divisibility, that is, say that a standard monomial a divides another standard monomial b if there is a third standard monomial c such that $ac = b$; or (2) say that a divides b if there is a c such that the *leading term* (according to a chosen term order) of ac is precisely b. Each of these possibilities corresponds to a different choice of algebra of leading terms, and by reifying this choice in terms of an algebra, we open up even more possibilities and trade-offs.

With this setup, we prove many of the standard results from Gröbner bases for our notion of ASL Gröbner basis:

- existence of finite ASL Gröbner bases (and uniqueness of reduced Gröbner bases),
- a characterization of ASL Gröbner bases similar to Buchberger's criterion in terms

of S-remainders, but with some additional necessary ingredients,

- algorithms that halt in finite time for computing ASL Gröbner bases in the graded case
- uniqueness of remainders modulo an ASL Gröbner basis, and
- algorithms for computing remainders.

Finally, we apply this theory to the case of the bideterminant ASL structure mentioned above. Although we do not prove new results here, our theory allows us to give transparent, one-line proofs of several standard results about ideals generated by minors, for example, that the ideal generated by all minors of a given size has the set of minors as a Gröbner basis.

2 Preliminaries on ASLs

See [3] for a good reference on ASLs. Let R be a commutative ring and let A be a commutative R-algebra. Let $H \subseteq A$ be a finite set and a monomial on H is an element of the multiplicative monoid (called $\mathbb{W}^H$) generated by H. An *isotone set of monomials* is a subset $\Sigma \subseteq \mathbb{W}^H$ such that if $M \in \Sigma$ and $N \in \mathbb{W}^H$, then $MN \in \Sigma$. A *standard monomial* (w.r.t. Σ) is any monomial in $\mathbb{W}^H \setminus \Sigma$.

Definition 2.1 (Hodge algebra/ASL)**.** Let R, A, H, and Σ be as above, and let $\leq$ be a partial order on H. We say that A *is a R-Hodge algebra/ASL governed by* Σ *and generated by* H if the following axioms are satisfied.

(HA-1) A is a free R-module on the standard monomials with respect to Σ.

(HA-2) For a generator $N \in \Sigma$, where $N = \sum_i r_i M_i$, with $0 \neq r_i \in R$, is the unique expression for N (when looked at as an element of A) as a R-linear combination of distinct standard monomials (guaranteed by **(HA-1)**), then

$$x \in H, x \mid N \qquad \Longrightarrow \qquad \forall i,\ \exists y_i \in H \text{ such that } y_i \mid M_i \text{ and } y_i < x.$$

The expression of N in the standard monomial basis is called its straightening relation. For an $f \in A$, we say that a standard monomial m *appears* in f if m occurs with nonzero coefficient in the expression of f in the basis of standard monomials.

An ASL is a structure that has multiple elements to it, i.e. R, H, etc. Throughout this paper, whenever we refer to multi-element structures, we will sometimes omit certain elements if they are clear from context.

3 Definitions for ASL Gröbner bases

Given a total order on the standard monomials of A, let $\mathrm{LM}_{sm}(f)$ denote the largest standard monomial that appears in f.

Definition 3.1 (ASL term order)**.** Given an ASL A, an *ASL term order* on A is total ordering $\preceq$ on the standard monomials of A such that

(**ATO-1**) (Positivity) $1 \preceq m$ for all standard monomials $m \in A$.

(**ATO-2**) (Almost-multiplicativity) For all standard monomials $f, g, h \in A$, if $f \preceq g$, then $gh \neq 0 \Rightarrow fh \neq 0$ and in this case, $\mathrm{LM}_{sm}(fh) \preceq \mathrm{LM}_{sm}(gh)$.

Once an ASL term order $\preceq$ is fixed, we define $\mathrm{LC}_{sm}(f)$ to be the coefficient of $\mathrm{LM}_{sm}(f)$ in the expression of f in the basis of standard monomials, and we define the leading term $\mathrm{LT}_{sm}(f) := \mathrm{LC}_{sm}(f)\mathrm{LM}_{sm}(f)$. A *standard term* is defined the product of a standard monomial by a nonzero scalar.

It is natural to wonder in what way an ASL term order is "compatible" with the ASL structure; any such compatibility that we require with the straightening process is encapsulated in (**ATO-2**).

Remark 3.2. For all $f, g \in A$, such that $\mathrm{LT}_{sm}(f)\mathrm{LT}_{sm}(g) \neq 0$, we have $\mathrm{LT}_{sm}(fg) = \mathrm{LT}_{sm}(\mathrm{LT}_{sm}(f)\mathrm{LT}_{sm}(g)) = \mathrm{LT}_{sm}(f\mathrm{LT}_{sm}(g))$. In ordinary Gröbner bases, in contrast, one has $\mathrm{LT}(fg) = \mathrm{LT}(f)\mathrm{LT}(g)$, but this will typically not be true for ASL term orders. However, we will see below that "true multiplicativity of leading terms" reappears in the algebra of leading terms.

The rewriting rule in an ASL results in divisibility of standard monomials not always being as clear-cut as divisibility of ordinary monomials in a polynomial ring. We thus find it useful to have an auxiliary algebra associated to A, in which the leading terms will live, and where divisibility in the auxiliary algebra will govern what we mean for one standard monomial to "divide" another. This approach is similar to the "graded structures" approach developed by Robbiano [13] and Mora [10] for Gröbner bases in other settings. We encapsulate this idea in the following definition.

Definition 3.3 (Algebra of leading terms)**.** Given an R-ASL A defined by (H, Σ), we say that an algebra A_{lt} is an *algebra (ASL) of leading terms* for (A, H, Σ), if A_{lt} is also an R-ASL defined over the same H, Σ, and furthermore, if π_{lt} denotes the unique R-linear bijection $A \to A_{lt}$ that restricts to the identity map on standard monomials, and for all standard monomials m, m', if $\pi_{lt}(m)\pi_{lt}(m') \neq 0$, then $\pi_{lt}(\mathrm{LT}_{sm}(mm')) = \pi_{lt}(m)\pi_{lt}(m')$.

Lemma 3.4. *If A_{lt} is an algebra of leading terms for some $(A, \preceq)$, then for all standard monomials $m_1, m_2 \in A_{lt}$, m_1m_2 is either standard or 0.*

Definition 3.5 (Two key examples of A_{lt})**.** 1. ([3]) Given $(H, \leq, \Sigma)$, recall the *discrete ASL*, which we denote A_{disc}, is defined as $R[H]/\langle\Sigma\rangle$.

2. Given $(A, \preceq)$, the *generic algebra of leading terms*, denoted A_{gen}, is defined by the relations: if f, g are standard monomials in A, then $\pi_{lt}(f)\pi_{lt}(g) = \pi_{lt}(\mathrm{LT}_{sm}(fg))$. (Since the standard monomials are a basis for A_{gen} by assumption, and π_{lt} is bijective on standard monomials, this uniquely defines A_{gen}.)

Proposition 3.6. *Given $(A, \preceq)$, the discrete ASL A_{disc} and the generic algebra of leading terms A_{gen} are both algebras of leading terms for $(A, \preceq)$.*

Remark 3.7. In fact, A_{gen} is in some sense the "universal" algebra of leading terms, in that any algebra of leading terms for $(A, \preceq)$ is obtained from A_{gen} by further forcing certain products of standard monomials to be 0. Similarly, we see that A_{disc} is the "opposite extreme", in the sense that starting from any A_{lt}, one can recover A_{disc} by forcing certain additional products of standard monomials to be 0. In particular, any A_{lt} is a flat deformation of A_{gen} through algebras of leading terms, and A_{disc} is a flat deformation of any A_{lt} through algebras of leading terms.

The last ingredient we need towards defining our notion of ASL Gröbner basis is an appropriate analogue of the notion of monomial ideal. We call an ideal $I \subseteq A_{lt}$ a *standard monomial ideal* if it is generated by standard monomials; using Lemma. 3.4, this is equivalent to the property that for all $f \in A_{lt}$, f is in I iff every standard monomial appearing in f is in I. This equivalence may fail for ideals in A, rather than A_{lt}, hinting at one of the values of having A_{lt} in the first place.

Definition 3.8 (Leading ideal, ASL Gröbner basis)**.** Given $(A, \preceq, A_{lt})$, and an ideal $I \subseteq A$, its *leading ideal* is $LI^{lt}_{sm}(I) := \langle \pi_{lt}(\mathrm{LT}_{sm}(f)) : f \in I\rangle \subseteq A_{lt}$. A set $G \subseteq I$ is an *ASL Gröbner basis* for I with respect to $(A, \preceq, A_{lt})$ if

$$LI^{lt}_{sm}(I) = \langle \pi_{lt}(\mathrm{LT}_{sm}(g)) : g \in G\rangle.$$

The leading ideal is always a standard monomial ideal.

4 Main Results

4.1 ASL Gröbner bases

Our central result establishes that ideals in ASLs admit finite Gröbner bases as per Definition 3.8. We also show an equivalent characterization of ASL Gröbner bases in terms of divisibility in A_{lt}.

Theorem 4.1. *1. Relative to $(A, \preceq, A_{lt})$, every ideal $I \subseteq A$ has a finite ASL Gröbner basis.*

2. *For any $f, g \in A$, let $g \mid_{lt} f$ denote that $\pi_{lt}(LT_{sm}(g)) \mid \pi_{lt}(LT_{sm}(f))$ in A_{lt}. G is an ASL Gröbner basis for I with respect to $(A, \preceq, A_{lt})$ if and only if for all $f \in I$, there exists $g \in G$ such that $g \mid_{lt} f$.*

Proof idea. Existence follows essentially from the fact that A_{lt} is Noetherian. □

4.2 Buchberger-like criterion for ASL Gröbner bases

Our next theorem will be a Buchberger-like criterion for ASL Gröbner bases. Two key respects in which our definitions go beyond those used in ordinary Gröbner bases are: (1) least common multiples need no longer be unique, so instead of a single S-polynomial for each pair, we get a set of S-polynomials; and (2) the following definition, as used in Def. 4.3(3), Def. 4.8, and subsequently.

Definition 4.2 (Compatible)**.** Given $(A, \preceq, A_{lt})$, two standard monomials $m, m' \in A$ are said to be *compatible (with respect to A_{lt})* if $mm' \neq 0 \iff \pi_{lt}(m)\pi_{lt}(m') \neq 0$. For standard monomials in A_{lt}, we also call them *compatible* if their product in A_{lt} is nonzero.

The assumption in Def. 3.3 can thus be equivalently rephrased as assuming that m, m' are compatible. Note that the $\Longleftarrow$ direction in Def. 4.2 follows directly from Def. 3.3, so the main additional criterion for compatibility is the $\Longrightarrow$ direction.

Definition 4.3 (Standard expression of $f \in A$)**.** Given $(A, \preceq, A_{lt})$ and a list of elements $g_1, \ldots, g_k, f \in A$, a *standard expression* for f relative to $g_1, \ldots, g_k$ is a list $r, h_1, \ldots, h_k \in A$ that satisfies $f = r + \sum_{i=1}^{k} h_i g_i$, which in turn satisfies the following conditions

1. for any standard monomial m occurring in r, $g_i \nmid_{lt} m$ for all $i \in [k]$,
2. $\mathrm{LM}_{sm}(f) \succeq \mathrm{LM}_{sm}(h_i g_i)$ for all i, and
3. for every $i = 1, \ldots, k$, every standard term appearing in h_i is compatible with $\mathrm{LT}_{sm}(g_i)$.

When this happens, we say that r is *a remainder* of f relative to $\{g_1, \ldots, g_k\}$, or that f *reduces to r relative to $\{g_1, \ldots, g_k\}$*.

Proposition 4.4 (Uniqueness of remainder)**.** *Given an ASL A, and an ideal $I \subseteq A$, if G is an ASL Gröbner basis for I, then every $f \in A$ reduces to a unique r relative to G.*

We now turn our attention to the necessary definitions to define S-polynomials, since in our context, although LCMs of standard monomials always exist, they need not be unique.

Definition 4.5 (Least common standard multiples)**.** Given any A_{lt} and standard monomials $m, m' \in A_{lt}$, let $CM_{sm}^{lt}(m, m')$ denote the set of their common (standard) multiples. The set of their least common standard multiples, denoted $LCM_{sm}^{lt}(m, m')$, consists of those common multiples $\ell \in CM_{sm}^{lt}(m, m')$ satisfying $\ell' \in CM_{sm}^{lt}(m, m'), \ell' \mid \ell$, implies $\ell' = \ell$. Also, for nonzero $\alpha, \beta \in \mathbb{F}$, define $LCM_{sm}^{lt}(\alpha m, \beta m') = LCM_{sm}^{lt}(m, m')$.

Proposition 4.6. *Given A_{lt}, and any standard monomials $m, m' \in A_{lt}$, $LCM^{lt}_{sm}(m, m')$ is finite.*

Proof sketch. Follows from the fact that CM^{lt}_{sm} spans a standard monomial ideal, with LCM^{lt}_{sm} as the unique minimal generators, and A_{lt} is Noetherian. □

Below we define a notion of S-pairs in an ASL similar to the notion of S-polynomials in polynomial rings. However, since there could be multiple LCMs, we could in turn have multiple S-pairs for just one single pair of elements $f, g \in A$.

Definition 4.7 (S-sets in an ASL)**.** Given $(A, \preceq, A_{lt})$ and $f, g \in A$, we define their S-set:

$$S^{lt}_{sm}(f, g) := \bigcup_{\ell \in LCM^{lt}_{sm}(f_{lt}, g_{lt})} \left\{ \pi^{-1}_{lt}\left(\frac{\ell}{f_{lt}}{}_{lt}\right) \cdot f - \pi^{-1}_{lt}\left(\frac{\ell}{g_{lt}}{}_{lt}\right) \cdot g \right\} \subseteq A,$$

where $f_{lt} := \pi_{lt}(\mathrm{LT}_{sm}(f))$ and $g_{lt} = \pi_{lt}(\mathrm{LT}_{sm}(g))$. Here we write $\frac{a}{b}{}_{lt}$ to highlight that this division is taking place in A_{lt}.

Definition 4.8 (Compatible generation)**.** Given $(A, \preceq, A_{lt})$ and $f, g_1, \dots, g_k \in A$, a *compatible expression (relative to A_{lt})* of f in terms of the $\{g_i\}$ is a list $h_1, \dots, h_k \in A$ such that $f = \sum_{i=1}^{k} h_i g_i$ and for all i, every standard monomial in h_i is compatible with $\mathrm{LT}_{sm}(g_i)$. An ideal $I \subseteq A$ is *compatibly generated by* $(g_1, \dots, g_k)$ if every $f \in I$ has a compatible expression in terms of the g_i, and in that case $\{g_i\}$ is a *compatible generating set* for I.

Definition 4.9 (S-closed)**.** Given $(A, \preceq, A_{lt})$, a subset $G \subseteq A$ is *S-closed* if for all pairs $g, g' \in G$, every element of $S^{lt}_{sm}(g, g')$ reduces to 0 relative to G.

For ordinary Gröbner bases, Buchberger's criterion says that a set is a Gröbner basis if and only if it is S-closed. For ASL Gröbner bases, we instead have the following.

Theorem 4.10 (ASL analogue of Buchberger's criterion)**.** *A set G is an ASL Gröbner basis if and only if it is S-closed and compatibly generates the ideal $\langle G \rangle$.*

Corollary 4.11. *In the special case when $A_{lt} = A_{gen}$ is the generic algebra of leading terms, a set G is an ASL Gröbner basis if and only if it is S-closed.*

4.3 Algorithms to compute ASL Gröbner bases

In full generality, for algorithms, we need to assume that remainders (Def. 4.3) and LCM^{lt}_{sm} (Def. 4.5) are both computable; we show that these are computable in cases where there is some convenient structure.

Proposition 4.12. *Given $(A, \preceq, A_{lt})$, if A and A_{lt} are graded ASLs, then (1) remainders, (2) LCM sets, and (3) extending a generating set to a compatible generating set of the same ideal can all be computed in finite time.*

Theorem 4.13. *Assuming remainders and LCM_{sm}^{lt} are computable, ASL Gröbner bases relative to A_{gen} can be computed in finite time. More generally, for any fixed A_{lt}, suppose that given $G = \{g_1, \ldots, g_k\}$, a compatible generating set of $I = \langle G \rangle$ that contains G can be computed in finite time. Then ASL Gröbner bases relative to that A_{lt} can be computed finite time.*

Proof sketch. The algorithm for the first part uses a straightforward analogue of Buchberger's algorithm, using our S-sets (Def. 4.7) in place of S-polynomials. Finite termination follows from finiteness of LCMs (Prop. 4.6) and the fact that A_{lt} is Noetherian.

The algorithm for the second part is as follows. As in the first part, Buchberger's algorithm will compute an S-closed set containing a given set of elements of A. By assumption, we can then compute a compatible generating set extending the S-closed set. The latter need no longer be S-closed, so these two procedures must be iterated. Since each iteration strictly increases the corresponding ideal in A_{lt} and the latter is Noetherian, it must halt after finitely many iterations. □

Remark 4.14. We can now see some of the possible trade-offs when varying the algebra of leading terms. Relative to A_{gen}, finding a compatible generating set is unnecessary, but computing LCMs, S-sets, and remainders are all complicated by the straightening process. Relative to A_{disc}, the latter computations become much easier and more similar to their counterparts for ordinary monomials in a polynomial ring, but one must find a method of computing a compatible generating set.

4.4 Application to the basis of standard bideterminants

4.4.1 Preliminaries on bideterminants

In this section we shall demonstrate an application of our theory of ASL Gröbner bases. Specifically we will show that there is a theory of Gröbner bases in the polynomial ring in the basis of what are called standard bideterminants. We begin with some necessary background on the bideterminant basis for the polynomial ring, following [6, 4].

For any $n \in \mathbb{Z}_{\geq 0}$, $(\lambda) = (\lambda_1, \ldots, \lambda_p)$ is defined as a *partition* of n if $\sum_{i=1}^{p} \lambda_i = n$, and $\lambda_1 \geq \ldots \geq \lambda_p > 0$. For a partition of n, $(\lambda) = (\lambda_1, \ldots, \lambda_p)$, we define its *shape*, also denoted as (λ) by abuse of notation, as the set of integer points $(i, -j)$ in the plane, with $1 \leq i \leq p$ and $1 \leq j \leq \lambda_i$. We will say that the *length of the shape/partition* (λ) is p, and that the *size of the shape/partition* (λ) is n.

A *Young tableau* of shape (λ) is an assignment of values[1] to each point in the shape (λ). A *normal* (Young) tableau is one where the entries in each column are strictly increasing from top to bottom, and a *semi-standard* tableau is a normal tableau with entries in each row non-decreasing left to right. A *bitableau*, denoted as $[R \mid C]$, consists of a pair of two Young tableaux R, C of the same shape. For a bitableau $[R \mid C]$, R will be

[1] In our paper, the values are always from $\mathbb{N}$.

called its *row tableau*, and C its *column tableau*—named so because the entries of R (resp., C) index the rows (resp., columns) of X in the corresponding bideterminant. We define the *shape of a bitableau* $[R \mid C]$ to be the shape of the Young tableau R (or C).

Henceforth, we will refer to the combinatorial notions defined above in the context of an $n \times m$ matrix X, each of whose entries is a different indeterminate $x_{i,j}$. The entries in our Young tableaux will be integers which, in general, index rows/columns of X.

To a bitableau $[R \mid C]$ of shape $(\lambda_1, \ldots, \lambda_p)$, where the i^{th} column of R (resp., C) is $(r_1^{(i)}, \ldots, r_{\lambda_i}^{(i)})$ (resp., $(c_1^{(i)}, \ldots, c_{\lambda_i}^{(i)})$), we associate the *bideterminant* $(R \mid C) \in \mathbb{K}[X]$:

$$(R \mid C) := \prod_{i=1}^{p} \det \left(\begin{bmatrix} x_{r_1^{(i)}, c_1^{(i)}} & \cdots & x_{r_1^{(i)}, c_{\lambda_i}^{(i)}} \\ \vdots & \vdots & \vdots \\ x_{r_{\lambda_i}^{(i)}, c_1^{(i)}} & \cdots & x_{r_{\lambda_i}^{(i)}, c_{\lambda_i}^{(i)}} \end{bmatrix} \right).$$

A *standard* bitableau is one where both the row and column tableaux are semi-standard. The corresponding bideterminant will be called a *standard bideterminant*. Also, by convention, $[\ \mid\ ]$ will be a standard bitableau, and thus $(\ \mid\ ) = 1$ is a standard bideterminant.

To make a precise statement of the straightening law (Theorem 4.18), we need to introduce a few more definitions.

Definition 4.15 (Partial ordering on minors)**.** We define a partial order $(\leq)$ on minors as follows. Given two minors $(a_1, \ldots, a_m \mid b_1, \ldots, b_m)$ and $(c_1, \ldots, c_n \mid d_1, \ldots, d_n)$, we will say $(a_1, \ldots, a_m \mid b_1, \ldots, b_m) \leq (c_1, \ldots, c_n \mid d_1, \ldots, d_n)$ if $m \geq n$ and $\forall i \in [n], a_i \leq c_i, b_i \leq d_i$.

Definition 4.16 (Total order on shapes, standard bitableaux/bideterminants)**.** Given two shapes (λ) and (μ), we say (λ) is *smaller than* (μ) if (λ) is of smaller size than (μ), or if (λ) and (μ) have the same size, and (λ) is lexicographically larger than (μ).

To obtain an ordering on standard bitableaux, for a standard bitableau $[R \mid C]$, let $\theta([R \mid C])$ denote the sequence obtained by first reading off R row by row, from left to right within each row, followed by the same for C. Given two bitableaux f and g, of shape (λ) and (μ) respectively, we write $f \prec g$ if $(\lambda) \prec (\mu)$, or when $(\lambda) = (\mu)$, we have $\theta(f) >_{lex} \theta(g)$.

Remark 4.17. Naturally, the order $\prec$ in Definition 4.16 gives a total order on standard bideterminants as well. The correspondence between bideterminants and bitableaux is an order preserving bijection when restricted to standard bitableaux and standard bideterminants.

Theorem 4.18 (Straightening law [6, 4])**.** *Let $\mathbb{K}$ be any arbitrary commutative ring, and let $\mathbb{K}[X] := \mathbb{K}[\{X_{ij} : i \in [n], j \in [m]\}]$. Every bideterminant $(R \mid C)$ has a unique finite expression: $(R \mid C) = \sum a_{R_i,C_i}(R_i \mid C_i)$, where $a_{R_i,C_i} \in \mathbb{K}$ and $(R_i \mid C_i)$ are standard. All*

$(R_i \mid C_i)$ in the above expression are of same or shorter shape. As a consequence, any polynomial $f \in \mathbb{K}[X]$ can be expressed as a K-linear combination of standard bideterminants.

Additionally [5], for a normal bitableau $[R \mid C]$, let $[{}^{(s)}R \mid {}^{(s)}C]$ be the bitableau obtained from $[R \mid C]$ by writing the entries of each row of each of the tableaux in ascending order. Then $[{}^{(s)}R \mid {}^{(s)}C]$ is the largest standard bitableau whose bideterminant appears with a co-efficient of 1 *in the expression of* $(R \mid C)$

4.4.2 *bd*-Gröbner Bases with an application

Our main theorem of this section (Thm. 4.19) establishes that there is a theory of ASL Gröbner bases in the basis of standard bideterminants in the polynomial ring.

Theorem 4.19. *Let $H \subseteq \mathbb{K}[X]$ be the set of minors of X, and let $\Sigma \subseteq \mathbb{W}^H$ be the set of non-standard bideterminants. Finally, let $\leq$ be the order from Definition 4.15. Then:*

(I) $A^{bd} := (\mathbb{K}[X], \mathbb{K}, H, \leq, \Sigma)$ is a $\mathbb{K}$-ASL, and the order $\preceq$ in Definition 4.16 is an ASL term order on A^{bd}, thus implying that $\mathbb{K}[X]$ admits a theory of Gröbner bases in the basis of standard bideterminants (henceforth called bd-Gröbner bases).

(II) Given $(A^{bd}, \preceq)$, and any corresponding algebra of leading terms A^{bd}_{lt}, then for any ideal $I \subseteq A^{bd}$, there is an algorithm to compute a bd-Gröbner basis of I w.r.t. $(A^{bd}, \preceq, A^{bd}_{lt})$.

Proof idea. (I) follows from Theorem 4.18 and the nature of the chosen orders. (II) follows from Theorem 4.13 and uses the combinatorial nature of the definitions for bideterminants to show that all required notions for ASL Gröbner bases are computable. □

Let $m > n$, and let $I \subseteq \mathbb{K}[X]$ be the ideal of maximal minors, i.e. minors of size n. Gröbner bases of I is a classically studied problem [1, 15]. Below we shall see how easy it is to study I within our framework. Our one-sentence proof should be contrasted with the proofs of analogous results on this ideal using ordinary Gröbner bases, e.g., *ibid.*

Theorem 4.20. *For any $S = (s_1, \ldots, s_n)$, with $s_1 < \ldots < s_n$, define* $B_S := \left(\begin{array}{|c|} \hline s_1 \\ \hline \vdots \\ \hline s_n \\ \hline \end{array} \;\middle|\; \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline n \\ \hline \end{array} \right)$.
Then for any A_{lt}, $G_I := \{B_S\}_{S \subseteq [m], |S|=n}$ is a bd-Gröbner basis of I.

Proof. By Theorem 4.18, the expression of $B_S B_T$ in the basis of standard bideterminants will contain bideterminants whose corresponding bitableaux either contain columns of height exactly equal to n, in which case they are divisible by some element of G_I, or columns of height strictly greater than n, in which case they evaluate to 0. □

5 Conclusion and Future Work

While we have laid much of the groundwork for this theory, much also remains to be done. We highlight here a few avenues we find particularly interesting and think should be approachable.

As one of the motivations of our theory is potential computational advantages—both for theoretical, "by-hand" computations, and algorithms implemented in software—it would be natural, interesting, and useful to implement our ideas in a standard computer algebra system. This would be especially interesting if implemented in a way that allowed one to incorporate more efficient straightening algorithms for the underlying Hodge algebra in a modular fashion.

Open Question 5.1. *Characterize ASL term orders. In particular, is there a characterization in terms of weights similar to Robbiano's characterization [12] in the case of term orders on polynomial rings?*

In the full version, we prove that it suffices to characterize ASL term orders on A_{gen}, which may be easier since the standard monomials form a multiplicative basis in A_{gen}.

Open Question 5.2. *What is the complexity of computing ASL Gröbner bases? Can they always be computed using only an exponential amount of space?*

Because ordinary Gröbner bases are a special case of ASL Gröbner bases, computing ASL Gröbner bases in general is at least as hard as computing ordinary Gröbner bases. In particular, it is $\mathsf{EXPSPACE}$-hard [9]. A positive answer to the above question would show that this complexity bound gives a tight characterization. Nonetheless, it is also possible that for particular ASLs, computing ASL Gröbner bases might be easier or harder than this; are there nontrivial examples of this phenomenon?

In the case of the ASL structure given by the bideterminant basis, our work highlights the following question:

Open Question 5.3. *What is the complexity of straightening bideterminants? We conjecture that it is* $\#\mathsf{P}$*-hard. Is it in* GapP*? Is it in* $\#\mathsf{P}$*?*

Our conjecture of $\#\mathsf{P}$-hardness is based partly on intuition from computing many examples and seeing the complicated structure, and partly on $\#\mathsf{P}$-hardness results for other related problems on Young tableaux [11, 2, 8]. While we are not aware of detailed worst-case analysis of the complexity of improved algorithms for straightening bideterminants [14, 5, 16], the algorithms we are aware of do not appear to work in less than exponential time in the worst case.

As with ordinary Gröbner bases, we expect that the theory here should be extendible to ASLs in certain mildly noncommutative settings. In fact, this work first arose out of the authors' attempt to calculate the algebraic de Rham cohomology of certain determinantal varieties using Weyl algebras; the authors are thus working to extend the theory developed in this paper to ASL Weyl algebras.

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