Abhiram Natarajan : https://abhiramnatarajan.github.io : abhiram.natarajan@warwick.ac.uk

Research Statement

1 Introduction

Two principles guide my work: first, the questions I pursue demand non-trivial development of algebraic tools; second, I focus on problems where I feel the solutions illuminate fundamental structure rather than incidental phenomena. My research lies in algebraic geometry (usually over the reals ($\mathbb{R}$), though sometimes in field-independent settings) and in o-minimal geometry, with a particular focus on quantitative topological aspects, often motivated by applications in combinatorics and computational complexity theory. A recurring leitmotif in my work is the use of *algebraic methods,*[1] *with a view towards mathematical applications.* The remainder of this section serves as an executive summary; more details can be found in subsequent sections.

Previous work: A unifying theme in my research has been the study of situations with underlying invariance or symmetry. These investigations have often led to more general developments (details in Section 2).

1. *Random algebraic geometry:* A major part of my work concerns the *homological complexity* of semi-algebraic sets and, more generally, *definable* sets in arbitrary o-minimal expansions of $\mathbb{R}$. I have undertaken statistical studies of the Betti numbers of random real algebraic sets, where the defining polynomials are sampled from the Kostlan distribution, which is invariant under the action of the orthogonal group on the space of variables. Under this model, we discovered a dichotomy (of a type that was previously unseen) in the topology of restrictions to definable hypersurfaces of arbitrary real algebraic hypersurfaces versus random real algebraic hypersurfaces [15].[2] In related work, our study of the Betti numbers of random arrangements revealed connections with spectral sequences and led to a new random graph model [14].

2. *Pfaffian polynomial partitioning:* The above questions in random algebraic geometry were motivated by efforts in incidence combinatorics to extend the seminal polynomial partitioning theorem of Guth–Katz [54, 52] to the o-minimal setting. The resulting insights later led us to a significant advance: a generalization of polynomial partitioning to the setting of semi-Pfaffian sets [68], which form a broad and natural class within o-minimal geometry. Using this, we produced new results in extremal combinatorics [74].

3. *Gröbner bases in Hodge algebras:* We have developed a comprehensive theory of Gröbner bases that preserves *determinant-like* symmetries (and more generally, extends to Hodge algebras) [49]. This framework yields universal Gröbner bases (in our sense) for ideals generated by minors of any fixed size t, previously known only for $t = 2$ and for maximal minors. This work was motivated by our attempts to compute, by hand, the algebraic de Rham cohomology of certain varieties central to computational complexity.

Current and future work: My current and forthcoming research (including projects with substantial progress) involves situations where algebraic methods interact deeply with other areas of mathematics. To me, this offers a vivid perspective on the unique vantage point that algebra affords. Below, I outline several main directions, balancing concrete problems with broader exploratory themes (details in Section 3).

1. *Pfaffian discrete geometry:* I intend to continue the line of inquiry we initiated in [68, 74] by extending classical incidence theorems to Pfaffian settings. Specifically, we aim to generalize known bounds for expanding polynomials [82, 81] to the Pfaffian case, requiring Pfaffian analogues of classical irreducibility results for bivariate polynomials, and to study the Zarankiewicz problem [39] on semi-Pfaffian graphs, likely necessitating new *multi-level partitioning* results. We are also applying tools in [68] to distinct distances and unit distances problems in ℓ_p norms for arbitrary $p \in \mathbb{R} \setminus \mathbb{Z}$ (for such p, the ℓ_p-sphere is a Pfaffian set, not algebraic). Finally, a foundational issue regarding projections of semi-Pfaffian sets also merits investigation.

2. *Bézout-type theorems in the o-minimal setting:* An important ingredient in [68] was Khovanskiĭ's Bézout-type theorem [59] for Pfaffian sets, which, unlike the result of Basu–Barone [8], applies only to non-degenerate intersections. I aim to broaden Khovanskiĭ's theorem to accommodate degenerate cases. I am also interested in understanding Bézout-type behaviour in $\mathbb{R}$ from a model-theoretic standpoint. Within the o-minimal universe, Bézout-type behaviour occurs in the algebraic and Pfaffian cases, but does not persist uniformly, suggesting a dividing line that I aim to characterize. This investigation is expected to interact with the emerging theory of sharp o-minimality [22] and has parallels to the theory of Zariski geometries.

[1] In the broad sense of including algebraic geometry, o-minimal geometry, algebraic topology, and commutative algebra.

[2] Citations in blue will be citations to my own work.

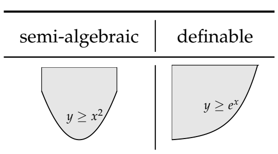
3. *Extensions and applications of our Gröbner basis theory:* First, in [49] we highlighted several open questions that are ripe for exploration, including a concrete conjecture on flat deformations of certain auxiliary algebras introduced in our theory. Second, our framework has strong potential impact on fundamental problems in computational complexity. One direction is to obtain bounds on the rank of the matrix multiplication tensor using Brent's equations; the presence of many minors in these equations suggests that our Gröbner basis theory could be salient. We also plan to explore instantiations of our theory in the algebra of bipermanents and, using our dimension-theoretic tools, aim to compute the codimension of the singular locus of the permanent hypersurface, an open problem in the Geometric Complexity Theory (GCT) program,[3] [64, Question 6.3.3.7]. Importantly, we have instantiated a 'determinant-symmetry-preserving' Gröbner basis theory similar to [49] in the (non-commutative) Weyl algebra [48]. This could enable using the algorithms of Oaku–Takayama and Walther [75, 96] to compute the cohomology of varieties with determinant-like symmetries, particularly those central to GCT such as tensor-rank varieties and the orbit closure of the determinant.

Beyond these directions, I am also pursuing related problems that require further advances in real-algebraic and o-minimal geometry. In a project nearing completion, we analyze tubular neighbourhoods of Pfaffian sets, with applications to robustness phenomena in deep learning [66]. In another project we aim to extend polynomial partitioning from $\mathbb{R}$ to arbitrary real closed fields.

2 Previous Work

This section contains background and details of my major results. Section 2.1 addresses topological questions in real algebraic and o-minimal geometry, motivated by the quest for an o-minimal polynomial partitioning theorem. Section 2.2 presents topological and partitioning results for Pfaffian functions, and their implications in discrete geometry. Section 2.3 discusses work on Gröbner bases and its applications in commutative algebra and computational complexity.

2.1 Homological Complexity in Real Algebraic and O-minimal Geometry

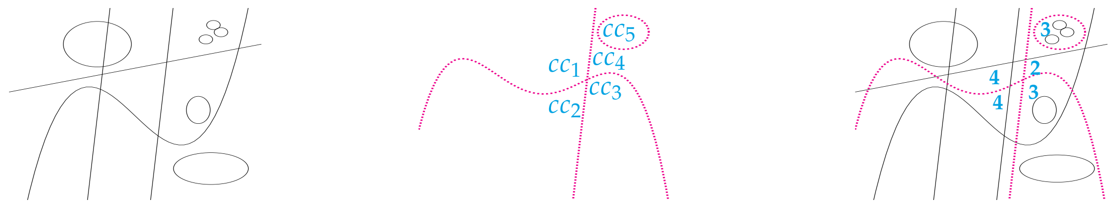


Real algebraic and o-minimal geometry: Real algebraic geometry studies polynomials with real coefficients and the subsets of $\mathbb{R}^n$ they define through polynomial equalities (*real algebraic sets*) and inequalities (*semi-algebraic sets*). By the Tarski–Seidenberg theorem, projections of semi-algebraic sets remain semi-algebraic, making them the natural objects of study; see [25] for an introduction. Semi-algebraic sets enjoy remarkable topological tameness properties, such as stratifiability and triangulability, and this makes their study feasible. Motivated by Grothendieck's call in the *Esquisse d'un Programme* [50] to "*...investigate other classes of sets with the tame topological properties of semi-algebraic sets...*", Knight, Pillay, and Steinhorn [80, 62] introduced *o-minimality*. Having its genesis in model theory, the notion of o-minimality isolates key axioms such that any collection of subsets of $\mathbb{R}^n$ satisfying these axioms — such a collection is called an *o-minimal structure*, and elements of an o-minimal structure are called *definable sets* — share the tame topological properties that semi-algebraic sets possess. Definable sets include not only semi-algebraic ones but also those involving exponential functions, Pfaffian functions, etc. [34, 99, 98]; see [69, 35] for an overview.

Topology of semi-algebraic and definable sets: Bounding the Betti numbers of semi-algebraic or definable sets provides a measure of their *homological complexity*, with deep implications across many areas such as real algebraic geometry, optimization, and learning theory; see [11, 16]. For a definable set S, let $b_i(S)$ denote its i^{th} Betti number (we only consider coefficients in $\mathbb{Q}$); specifically, $b_0(S)$ denotes the number of connected components of S. Given a (locally closed) semi-algebraic set $S \subset \mathbb{R}^n$, defined by at most m polynomials each of degree $\leq d$, Oleinik–Petrovski, Thom, and Milnor showed that $\sum_i b_i(S) = O_{n,m}(d)^n$ [77, 92, 71]. Subsequent refinements include [42, 18], and similar analogues in o-minimal geometry were obtained in [12].

Polynomial partitioning: Quantitative topological bounds are central to the *polynomial partitioning* method, which was introduced in the ground-breaking work of Guth and Katz [54, 52]. The polynomial partitioning theorem guarantees that, for any arbitrary collection Γ of k-dimensional semi-algebraic sets in $\mathbb{R}^n$ and for any $D \in \mathbb{N}$, there is a nonzero polynomial $P \in \mathbb{R}[X_1, \dots, X_n]$ of degree at most D, with zero set $Z(P)$, such that each connected component of $\mathbb{R}^n \setminus Z(P)$ is intersected by $\lesssim {|\Gamma|}/{D^{n-k}}$ elements of Γ. Below is an illustration.

[3]A proposed approach to the *P vs NP* Clay Millennium problem [72, 73].



a set Γ of 10 curves in $\mathbb{R}^2$ — $P \in \mathbb{R}[X_1, X_2]$ such that $\mathbb{R}^2 \setminus Z(P)$ has 5 connected components (cc) — each connected component intersects only few curves from Γ

At a high level, polynomial partitioning provides a divide-and-conquer framework: it breaks a problem into smaller subproblems whose local solutions can be combined to obtain the global answer. Polynomial partitioning has revolutionized discrete geometry, harmonic analysis, and several other fields [54, 57, 93, 51, 53, 2]. However, polynomial partitioning applies only to semi-algebraic sets. While there is growing work on discrete geometry over o-minimal and distal structures [19, 30, 5, 29], progress remains slow without a polynomial partitioning theorem for definable sets — a result that would mark a major advance. This issue guides several of my research directions.

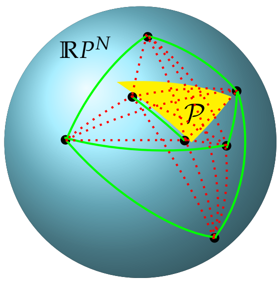
Pathological intersections of definable and algebraic hypersurfaces: The primary ingredient required for o-minimal polynomial partitioning is a uniform bound on the Betti numbers of the intersection of any fixed definable set with real algebraic hypersurfaces of growing degrees. However in [15] we show that for any sequence $\{Z_d\}_{d\in\mathbb{N}}$ of smooth compact hypersurfaces in $\mathbb{R}^{n-1}$, there exists a regular semi-analytic hypersurface $\gamma \subseteq \mathbb{R}P^n$ such that (after passing to a subsequence) each Z_d is diffeomorphic to a component of $\gamma \cap Z(P)$ (we also have that the intersection is transversal) for some degree d polynomial P. Thus, γ can intersect algebraic hypersurfaces in arbitrarily topologically complicated ways, unlike the semi-algebraic case where $\sum_i b_i(\gamma \cap Z(P)) \leq O(\deg P)^{O(1)}$ [8]. This shows a fundamental obstruction to extending polynomial partitioning to the o-minimal setting.

Random real algebraic geometry: While we established that there could be pathological behaviour, how typical is this? To answer this, we study random polynomials drawn according to the *Kostlan* measure on the space of homogeneous polynomials of degree d in $n+1$ variables. A Kostlan polynomial $p = \sum_{|\alpha|=d} \xi_\alpha x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ is such that the ξ_α's are independent random variables and $\xi_\alpha \sim \mathcal{N}(0, d!/\alpha_0!\ldots\alpha_n!)$. This measure, which is the restriction of the Fubini–Study measure to the reals, is invariant under orthogonal changes of variables, so it has *no preferred points or directions* in projective space. Thus it is an algebro-geometrically natural model of randomness, from which sampling is easy. Since the pioneering work of Edelman, Kostlan, Shub, and Smale [36, 85, 86], this model has become central in random real algebraic geometry.

Studying random polynomials is particularly meaningful in real algebraic geometry, where behaviour is far less predictable than in the complex case. For instance, a single real polynomial can define zero sets of varying dimensions, unlike complex polynomials, and while all non-singular complex projective hypersurfaces of a fixed degree are diffeomorphic, their real analogues can have many topological types. Random models therefore offer a natural way to capture *generic* real behaviour.

Pathologies are rare: Using Morse theory and the Kac–Rice formula, we prove in [15] that for any regular compact definable hypersurface $\gamma \subset \mathbb{R}P^n$, for every $0 \leq k \leq n-2$, $\mathbb{E}\left[b_k(\gamma \cap Z(p))\right]$ is at most $c_\gamma d^{n-1/2}$ for some constant c_γ depending on the volume of γ. Here the expectation is taken with respect to the Kostlan measure. Thus, while worst-case intersections can be arbitrarily complex, such behaviour is rare, giving some hope that an o-minimal version of polynomial partitioning might hold in some cases. This "average versus worst-case" dichotomy is dramatically sharper than the usual *square-root phenomenon* familiar in the Kostlan model (e.g. the expected number of real roots of a random degree-d bivariate homogeneous polynomial is $\sqrt{d}$, whereas it is d in the worst case [63]).

Random hypersurface arrangements: Continuing the study of the topology of random algebraic sets, in [14] we study the topology of *random algebraic arrangements*: given random Kostlan polynomials $P_1, \ldots, P_s$ of degrees d in $n+1$ variables, we ask about the Betti numbers of $\gamma = \bigcup_{j=1}^{s} Z(P_j) \subseteq \mathbb{R}P^n$. Beyond partitioning problems, understanding Betti numbers of such arrangements is relevant in applications like motion planning [3, 13]. Using a random spectral sequence arising from the Mayer–Vietoris exact sequence, we show that $\mathbb{E}\left[b_0(\mathbb{R}P^n \setminus \gamma)\right]$[4] is exactly $\binom{s}{n} d^{n/2} + O_{d,n}(s^{n-1})$, while in the worst case, $b_0(\mathbb{R}P^n \setminus \gamma)$ is bounded by $\frac{(2d)^n}{n!} s^n + O_{d,n}(s^{n-1})$. We also bound the higher Betti numbers of $\mathbb{R}P^n \setminus \gamma$ accordingly.



Since the growth of the Betti numbers of quadrics show different behavior compared to that of general semi-algebraic sets (see e.g. [10, 17, 65] in the random setting), we study the case $d = 2$. Using Calabi's theorem [28], we related the number of connected components of γ to the connectivity of a new random graph model we call 'obstacle random graphs': sample independent points $q_1, \ldots, q_s$ uniformly from $\mathbb{R}P^n$ and place an edge between vertices q_i and q_j if the great circle joining q_i and q_j does *not* intersect a semi-algebraic convex region $\mathcal{P} \subseteq \mathbb{R}P^N$. See illustration; solid lines denote edges present while dotted lines denote missing edges. We compute the expected number of connected components of this random graph (could be of independent interest) to deduce that for quadrics, $\mathbb{E}\left[b_0(\gamma)\right] = o(s)$, a sharper bound than the general case.

2.2 Pfaffians — Topology, Partitioning Theorems, and Discrete Geometry

Pfaffian functions: A sequence of functions $\vec{q} = (q_1, \ldots, q_r)$, where each $q_i \in \mathcal{C}^\infty(\mathbb{R}^n)$, is called a *Pfaffian chain of length* r if the functions satisfy $\frac{\partial q_i}{\partial X_j} = P_{i,j}(\mathrm{x}, q_1(\mathrm{x}), \ldots, q_i(\mathrm{x}))$, where $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_i]$. Then, a function $f : \mathbb{R}^n \to \mathbb{R}$, defined by $f(\mathrm{x}) := P_f(\mathrm{x}, q_1(\mathrm{x}), \ldots, q_r(\mathrm{x}))$ for some polynomial $P_f \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$, is a called *Pfaffian function defined with respect to* $\vec{q}$. f has *chain-degree* $\max_{i,j} \deg(P_{i,j})$, *degree* $\deg(P_f)$, and *order* r, collectively referred to as its *format*. The zero set $Z(f)$ is a called *Pfaffian set*, while loci defined by inequalities of Pfaffian functions are called *semi-Pfaffian sets*. Pfaffian functions of order $r = 0$ reduce to polynomials. Examples of semi-Pfaffian sets include $x^2 - e^y = 0$ and $x^{2\pi} - e^{e^y} \geq \sinh z$.

Pfaffian functions, introduced by Khovanskiĭ [60, 61] in connection with *fewnomials* [59] and the second part of Hilbert's sixteenth problem [56], form a remarkably rich analytic class encompassing many Liouvillian functions (with appropriate domain restrictions). They have deep connections across mathematics: in model theory, they play a key role in Wilkie's theorem [99], and the *Pfaffian structure*—the smallest structure containing all semi-Pfaffian sets and closed under structure operations—is known to be o-minimal [87]. Pfaffian functions naturally also arise even in areas such as machine learning theory (see [58]) and quantum field theory [46, 47].

Partitioning theorems in the Pfaffian world: In [68], we establish the first general partitioning theorem beyond the real algebraic world, generalizing polynomial partitioning to the case where Γ is a collection of semi-Pfaffian sets. This represents a substantial step toward an o-minimal polynomial partitioning, extending the reach of polynomial partitioning to a broad analytic context. Suppose that $\vec{q}$ is a Pfaffian chain of length r, and Γ is a collection of k-dimensional semi-Pfaffian sets in $\mathbb{R}^n$, where each $\gamma \in \Gamma$ is defined by a fixed number of fixed format Pfaffian functions defined with respect to $\vec{q}$. We show that for any $D \in \mathbb{N}$, there exists a polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most D such that each connected component of $\mathbb{R}^n \setminus Z(P)$ is intersected by $\lesssim {}^{|\Gamma|}/_{D^{n-k-r}}$ elements of Γ (note: setting $r = 0$ recovers the classical polynomial partitioning theorem). Moreover, under mild restrictions on $\vec{q}$, we prove a second partitioning theorem where the partitioning function f is itself Pfaffian (defined with respect to $\vec{q}$). In this case, each connected component of $\mathbb{R}^n \setminus Z(f)$ meets at most $\lesssim {}^{|\Gamma|}/_{D^{n-k}}$ elements of Γ, matching the semi-algebraic bound.

Discrete geometry with (semi-)Pfaffian sets: In [68], we apply these partitioning theorems and related topological tools to generalize classical results in discrete geometry. We extend the Szemerédi–Trotter theorem to count incidences between points and Pfaffian curves in the plane, and we obtain bounds on the number of joints formed by Pfaffian curves in $\mathbb{R}^n$, previously known only for algebraic curves [101].

Further, in [74], we use the results of [68] to study distinct distances: given point sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of sizes m and n, each on a fixed format Pfaffian curve, we show $|\{\|p - q\|_2 : p \in \mathcal{P}_1, q \in \mathcal{P}_2\}| = \Omega(\min\{m^{3/4}n^{3/4}, m^2, n^2\})$, unless the curves contain arcs of parallel lines, orthogonal lines, or concentric circles. This matches the best-known bounds in the algebraic case [79].

[4] Betti numbers of $\mathbb{R}P^n \setminus \gamma$ and γ are related by the –Pontraygin duality.

2.3 Symmetry Preserving Gröbner Bases

Computational complexity theory and Gröbner bases: While the field of computational complexity began as an effort to classify computational problems by their resource requirements, it has evolved over the past half-century into a profound inquiry about the limits of finite mathematical procedures [97]. Beyond its obvious practical relevance, it carries deep philosophical weight: Computational complexity theorems can be viewed as a kind of *quantitative epistemology* [1]. Computational complexity now interacts richly with nearly every branch of mathematics and, alongside its many conceptual insights, has given mathematics the **P** vs **NP** Clay Millennium question.

Key topics at the intersection of algebra and computational complexity include matrix multiplication, the **VP** $\neq$ **VNP** conjecture (algebraic analogue of **P** $\neq$ **NP**) and the associated GCT program, and algebraic decision/computation trees [27, 64]. Gröbner bases are naturally suited to questions in these topics: in matrix multiplication, for small r, n, compute the Gröbner basis of the r^{th} secant variety of the Segre variety to test if the $n \times n$ matrix multiplication tensor lies in it, or in GCT, for small $m \ll n$, compute the Gröbner basis of the orbit closure of the $n \times n$ determinant to check if it contains the $m \times m$ padded permanent. Also, since lower bounds for algebraic computation trees rely on topological invariants like Betti numbers [20, 102, 43], one natural approach is to use the algorithms of Oaku—Takayama and Walther [75, 76] to compute algebraic de Rham cohomology of varieties of interest. Such computations involve obtaining Gröbner bases in the (non-commutative) Weyl algebra. Thus Gröbner bases are a theoretically attractive approach for major open problems at the heart of computational complexity.

Gröbner bases in Hodge algebras: Our initial goal was to apply the Oaku-–Takayama and Walther algorithms to tensor rank varieties and the orbit closure of the determinant. However, even in very simple instances, the required Gröbner bases computations proved intractable. We observed that while the ideals were rich in determinantal structure, standard Gröbner bases computations in the Weyl algebra [84] ignore this structure. This suggested that one might gain leverage by developing a theory of Gröbner bases that *preserves and exploits* the abundance of determinantal structure, rather than treating the minors as ordinary polynomials.

We thus turn to the algebra of bideterminants (products of minors). Bideterminants correspond to pairs of Young tableaux of the same shape, e.g. $\left[\begin{array}{cc|c} \boxed{a_1} & \boxed{b_1} \\ \boxed{a_2} & \end{array}\middle| \begin{array}{cc} \boxed{c_1} & \boxed{d_1} \\ \boxed{c_2} & \end{array}\right]$ corresponds to the minor $(x_{a_1,c_1}x_{a_2,c_2} - x_{a_1,c_2}x_{a_2,c_1}) \cdot x_{b_1,d_1}$. *Standard bideterminants*, which correspond to tableaux with strictly increasing columns and non-decreasing rows, form an $\mathbb{F}$-basis of the coordinate ring $A^{bd} = \mathbb{F}\left[\left\{X_{ij}\right\}\right]$ of generic matrices [33, 32]. Equivalently, A^{bd} is $\mathbb{F}$-algebra isomorphic to a polynomial ring with one variable for each minor of a generic matrix, quotiented by what are called *straightening relations*, i.e. expressions of nonstandard bideterminants as $\mathbb{F}$-linear combinations of standard ones. Since minors correspond to single variables, A^{bd} is well adapted to deal with polynomials that have many minors.

More generally, A^{bd} is an example of an *algebra with straightening law* (ASL, or Hodge algebra): a finitely generated commutative algebra equipped with a distinguished set of *standard monomials* forming a basis. In A^{bd}, the standard monomials are precisely the standard bideterminants. Thus our goal became to develop a theory of Gröbner bases in ASLs that operates with standard monomials. Since every finitely generated commutative algebra A is a quotient $\mathbb{F}[X_1, \ldots, X_n] \big/ J$ for some ideal J, one could just lift an ideal $I \subseteq A$ to $I + J \subseteq \mathbb{F}[X_1, \ldots, X_n]$ and use standard Gröbner bases there. However, in many situations it is desirable to have a Gröbner basis theory *native* to A, allowing computations within A itself without reference to J. This can be advantageous both computationally, since multiplication intrinsically in A may be more efficient than lifting, and structurally, since symmetries of A and I may be obscured upon lifting.

A central difficulty in developing such a theory is that unlike ordinary monomials, products of standard monomials are not necessarily standard, but could be linear combinations of standard monomials. To handle this, we introduce an auxiliary "algebra of leading terms" that encodes which standard monomials can divide others. This notion allows us to define *standard monomial ideals* in the algebra of leading terms, which behave analogously to monomial ideals in polynomial rings.

In [49], we introduce *pseudo-ASLs*, a generalization of ASLs, and whenever there is a suitable term order in a pseudo-ASL, we establish a theory of Gröbner bases, termed *pseudo-ASL Gröbner bases* (pAGb). In pAGb, standard monomials of the pseudo-ASL play a role akin to monomials in the polynomial ring. We prove existence of finite pAGb and finite universal pAGb, uniqueness of reduced pAGb, pAGb for syzygies when our pseudo-ASL is a domain (analogue of Schreyer's Theorem), algorithms for computing pAGb, and tools to calculate Krull dimension in pseudo-ASLs using pAGb. Our notion of universal pAGb is different from the classical one - it is universal with respect to term orders on our pseudo-ASL as well as choices of algebras of leading terms. The key advantage of our theory of pAGb is that you have *smaller expressions for polynomials adapted to the p-ASL basis*.

Bideterminants, universal Gröbner bases of determinantal ideals: In [49], we instantiate the pAGb framework in the bideterminant algebra A^{bd} by showing that it admits a suitable term order on standard bideterminants. Consequently, one can perform Gröbner computations using only standard bideterminants, never reverting to monomials. Thus, our theory is exactly what one needs to 'preserve' determinantal structure. In this setting, we prove that minors of size $\geq t$ form a universal pAGb. In contrast, in the classical setting, universal Gröbner bases are known only for maximal [90, 21] and minimal (i.e. 2×2) minors [89], with intricate proofs. Within our framework, the corresponding argument becomes remarkably short and simple.

3 Future Directions

I now outline my current and future research, emphasizing substantial open problems over incremental ones.

3.1 Khovanskiĭ's theorem, and model-theoretic investigations of Bézout-type behaviour

Refining Khovanskiĭ's Bézout-type theorem: Using the classical Bézout theorem naively fails over the reals, as illustrated by the well-known example [41]: $\left\{P_1 = X_3,\ P_2 = X_3,\ P_3 = \prod_{i=1}^{d}(X_1 - i)^2 + \prod_{i=1}^{d}(X_2 - i)^2\right\}$. This system has d^2 real zeros in $\mathbb{R}^3$, while the polynomial degrees are $1, 1$, and $2d$, which would incorrectly predict $O(d)$ zeros if the complex Bézout bound were applied directly. Hence, over $\mathbb{R}$, Bézout-type results either require non-degeneracy assumptions (e.g., nonsingular complete intersections) [25, Chapter 11], or have bounds more intricate than a simple product of degrees [9, 8]. In the above example, the zero set of P_3 causes a dimension drop of 2 rather than 1, suggesting that one should raise d to the power of the actual dimension drop, giving d^2. This principle underlies the general results of [9, 8].

Khovanskiĭ's Bézout-type theorem bounds the number of isolated zeros of a system of Pfaffian equations. Its proof, using a modified Rolle's theorem, reduces a Pfaffian system to an algebraic one by replacing a system of n Pfaffian functions in $\mathbb{R}^n$ with $n + r$ polynomials in $\mathbb{R}^{n+r}$ and then uses Bézout over the reals. Obtaining bounds in the Pfaffian case as general as those in Barone–Basu [8] remains an open problem. A plausible approach is to follow [8]: approximate the given Pfaffian sets by ones with controlled properties, and establish a Pfaffian analogue of Smith's inequality [40, Section 5.2].

Also, currently all Pfaffian functions in Khovanskiĭ's theorem share the same domain and chain. If instead different functions belong to different chains, one could combine them into a single larger chain, but this may be inefficient. Specifically, for n Pfaffian functions of degrees $d_1, \ldots, d_n$ defined with respect to a chain of length r, Khovanskiĭ's bound is $O\left(\prod_{i=1}^{n} d_i \cdot \left(\sum_{i=1}^{n} d_i\right)^r\right)$. If the functions were defined with respect to chains of different lengths, one might hope for a bound where each d_i is raised only to its corresponding chain length. Such a refinement, especially when some of the functions are polynomial, would have immediate implications for our work on Pfaffian partitioning [68] and robustness in deep learning [66].

Open Question 1. *Generalize Khovanskiĭ's Bézout-type bound by removing any assumed non-degeneracy in the intersection of Pfaffian sets. Also, investigate other refinements.*

Bézout-type behaviour from a model-theoretic viewpoint: Bézout-type results are known for semi-algebraic sets [8] and Pfaffian sets [59], but not uniformly across all o-minimal structures [15]. This suggests a dividing line within the o-minimal world.

Open Question 2. *Develop a theory of 'Bézout structures' which exhibits necessary and/or sufficient conditions for an o-minimal structure to exhibit Bézout-type behaviour, particularly of the kind required for polynomial partitioning.*

Discovering additional o-minimal structures with Bézout-type bounds could open new directions in discrete geometry. The theory of Zariski geometries [55, 103] admits Bézout-type theorems, but it is only instantiated over $\mathbb{C}$. The emerging notion of sharp o-minimality [22, 23] appears especially promising, but it is unclear if the Bézout-type behaviour required for polynomial partitioning can be derived from the axioms of sharply o-minimal structures.

Open Question 3. *Establish polynomial partitioning in sharply o-minimal structures by proving the required Bézout-type behaviour.*

Both algebraic and Pfaffian sets come with notions of 'complexity'. Resolving the above question would first require defining a meaningful notion of 'complexity' of sets definable in sharply o-minimal structures.

3.2 Pfaffian Discrete Geometry

Continuing our work from [68, 74], I aim to study central problems in discrete geometry in the Pfaffian setting. One direction is a Pfaffian analogue of the Zarankiewicz problem, which asks for the maximum number of edges in a bipartite graph with given vertex counts that avoids a complete bipartite subgraph of prescribed size. The best general bound is the Kővári–Sós–Turán theorem [70, Theorem 4.5.2], but [39] proved tighter bounds for semi-algebraic graphs. The analogous semi-Pfaffian case remains open.

Open Question 4. *Let $G = (P, Q, E)$ be a bipartite graph with $P \subseteq \mathbb{R}^{d_1}$, $Q \subseteq \mathbb{R}^{d_2}$, and a semi-Pfaffian set $V \subseteq \mathbb{R}^{d_1+d_2}$ defined by a constant number of Pfaffian functions of constant degree, order, and chain length, such that for all $(p, q) \in P \times Q$, $(p, q) \in E \iff (p, q) \in V$. If G contains no $K_{s,t}$ subgraph, obtain an upper bound on $|E|$.*

A natural approach is to follow the strategy of [39]. A challenge in applying polynomial partitioning on a collection Γ is that the partitioning hypersurface $Z(P_1)$ may fully contain a large subset $\Gamma_1 \subseteq \Gamma$. This is addressed by repeatedly partitioning such subsets with new polynomials $P_2, P_3, \ldots$ chosen so that $P_{i+1} \notin I(Z(P_i))$, a process known as *multi-level* polynomial partitioning [39, 95]. For Open Question 4, developing a Pfaffian analogue of multi-level polynomial partitioning is essential - a problem I am investigating with Adam Sheffer.

Our lower bound on the number of distinct distances between points on plane Pfaffian curves [74] generalizes the algebraic case of [79]. The latter is itself a special case of the bounds for expanding polynomials [82, 81], which serve as a unifying framework for several problems in extremal and additive combinatorics. Extending those expansion bounds to the Pfaffian setting would thus advance multiple directions in Pfaffian discrete geometry. Since a key ingredient in these bounds is Stein's result [88, 7] asserting that a generic bivariate polynomial is irreducible, the following question naturally arises.

Open Question 5. *Extend the irreducibility results of Stein [88, 7] to bivariate Pfaffian functions in order to generalize current bounds on expanding polynomials to the Pfaffian setting.*

Also in progress with Adam Sheffer is an attempt to study unit distances and distinct distances problems in ℓ_p norms specifically when $p \in \mathbb{R} \setminus \mathbb{Z}$ (note that such ℓ_p-circles are Pfaffian sets). Progress on Open Question 1 promises to have an impact on these questions.

A more foundational question concerns the behavior of *Pfaffian projections*. A serious obstacle to higher-dimensional Pfaffian generalizations is that the projection of a semi-Pfaffian set need not be semi-Pfaffian (unlike in the semi-algebraic case), as demonstrated by an example due to Osgood [78]. However, projections of semi-analytic curves are known to remain semi-analytic [67]. An analogous result for semi-Pfaffian curves would immediately yield higher-dimensional versions of results in [68, 74].

Open Question 6. *Prove that projections of semi-Pfaffian curves are semi-Pfaffian.*

3.3 Gröbner bases that preserve symmetry, and applications

Ongoing work in extending pAGbs to the Weyl algebra: Recall from Section 2.3 that our attempts to compute algebraic de Rham cohomology using the algorithms of Oaku-–Takayama and Walther were hindered by the intractability of existing Gröbner basis methods in the Weyl algebra. In fact, this was what motivated the development of our 'symmetry-preserving' Gröbner bases theory (pAGb) in the first place [49]. We have also since been able to establish a 'determinant-symmetry-preserving' Gröbner bases theory in the Weyl algebra [48]. This uses the fact that the Weyl algebra, although non-commutative, is isomorphic as a module (but not as an algebra) to the tensor product of polynomial rings - the n^{th} Weyl algebra over a ring R is $W_n(R) \cong_{R\text{-module}} R[X_1, \ldots, X_n] \otimes_R R[\partial X_1, \ldots, \partial X_n]$. While we have already computed Gröbner bases (in our sense) of some small ideals in the Weyl algebra, it remains to be seen how far we can take this.

Open Question 7. *Obtain Gröbner bases (in our theory) for key annihilating D-ideals in the Weyl algebra, and adapt the Oaku–Takayama–Walther algorithms to compute algebraic de Rham cohomology using these bases. Apply the resulting methods to varieties arising in computational complexity theory that exhibit strong determinantal symmetries, such as tensor-rank varieties, secant varieties of the Segre variety, and the orbit closure of the determinant.*

Flat deformations and term orders in pAGb: In [49], although we established the foundations of our theory of pseudo-ASL Gröbner bases (pAGb), several promising directions remain. A natural next step is to implement our algorithms for computing pAGb in standard computer algebra systems, ideally integrating efficient straightening procedures. It would also be worthwhile to analyze these algorithms from the perspective of computational complexity theory.

Open Question 8. *Can our algorithms for computing pseudo-ASL Gröbner bases be implemented efficiently in full generality? What is their computational complexity, and can they always be realized using at most exponential space?*

In our pAGb theory, we introduced an auxiliary *algebra of leading terms* that determines when one standard monomial divides another. Since the product of two standard monomials in a pseudo-ASL A may not be standard, there are two extreme choices for such algebras: the *discrete* algebra A_{disc}, where $m \mid m'$ if there exists a standard monomial t with $mt = m'$ in A, and the *generic* algebra A_{gen}, where m' equals the leading term of the straightening of mt. In classical Gröbner theory, the leading ideal forms a flat deformation of the original ideal, preserving many algebraic invariants. We conjecture that a similar geometric picture holds for pseudo-ASLs.

Open Question 9 (Flatness conjecture for pAGb)**.** *Investigate the conjecture:*

1. *For any pseudo-ASL A, the algebra A_{gen} is a flat deformation of A.*
2. *For any algebra of leading terms A_{lt}, A_{lt} is a flat deformation of A_{gen}, and A_{disc} is a flat deformation of A_{lt}.*
3. *If two algebras of leading terms are flat deformations of each other, then the natural bijection between their standard monomials is an isomorphism of algebras.*

The classical proof of flatness for Gröbner degenerations relies on the fact that every term order is a lexicographic product of weighted-degree orders [83]. Hence, a characterization of admissible term orders on standard monomials in pseudo-ASLs may be crucial for resolving the conjecture.

Open Question 10. *Characterize term orders on standard monomials in pseudo-ASLs. In particular, is there a weighted-degree characterization analogous to Robbiano's theorem [83] for polynomial rings?*

pAGb in the algebra of bipermanents: Despite the superficial similarity between the determinant and permanent polynomials, determinantal varieties and ideals are far better understood than their permanental counterparts (see [45, 26]). Since instantiating our pAGb theory in the algebra of bideterminants A^{bd} yielded universal Gröbner bases effortlessly, one might hope that a parallel theory for *bipermanents* could shed light on permanental ideals. Although straightening laws for bipermanents are known [31, 4], it remains unclear whether they endow an ASL or even a pseudo-ASL structure.

Open Question 11. *Does the coordinate ring of $n \times m$ matrices admit a pseudo-ASL structure whose standard monomials are standard bipermanents, and which supports a pseudo-ASL term order?*

A positive answer would yield a Gröbner basis theory adapted to bipermanents, potentially enabling progress on the long-standing problem of determining the codimension of the singular locus of the permanent hypersurface (see [64, Open Question 6.3.3.7], originating in [44]). Specifically, one could attempt to compute a pAGb for the ideal of $(n-1)$-subpermanents (corresponding to the singular locus) and follow our technique to obtain Krull dimension from pAGb [49]. This question connects deeply to computational complexity, including notions such as Ulrich complexity [24] and Valiant's $\mathbf{VP} \neq \mathbf{VNP}$ conjecture [94].

Applications of pAGb to matrix multiplication and $\mathbf{VP} \neq \mathbf{VNP}$: For a fixed $n \times n$ matrix multiplication tensor T, one can attempt to find a rank-r decomposition by introducing variables $u_{i,\ell}$, $v_{j,\ell}$, and $w_{k,\ell}$ and studying the classical Brent equations $\sum_{\ell=1}^{r} u_{i,\ell} v_{j,\ell} w_{k,\ell} = T_{ijk}$ for all i, j, k. The ideal generated by these equations naturally contains many determinantal relations, giving rise to an abundance of minors. Thus, our Gröbner basis theory in the algebra of bideterminants, which preserves and exploits determinantal structure, could be particularly effective for these ideals. A similar approach is also viable for analyzing the determinantal complexity of the permanent polynomial.

Open Question 12. *Form the ideals generated by Brent's equations for tensor rank decompositions, and for determinantal complexity, and solve them by computing Gröbner bases in the basis of bideterminants.*

3.4 Miscellaneous problems

Polynomial partitioning over arbitrary real closed fields: Real algebraic geometry is usually done over arbitrary real closed fields, which were introduced in [6]; e.g. Levi-Civita field, surreal numbers. Here there is the additional challenge of not being able to rely on Archimedean properties of $\mathbb{R}$. Since many topological bounds that are used in discrete geometry are already formulated over arbitrary real closed fields, a polynomial partitioning theorem that works over arbitrary real closed fields would immediately lead to a plethora of generalizations.

Open Question 13. *Prove a polynomial partitioning theorem over arbitrary real closed fields.*

The main ingredients in the proof of polynomial partitioning [52] are a continuous approximation of indicator functions and certain Borsuk–Ulam-type fixed point theorems. Existing work on fixed point theorems [38, 91] and on definable continuous approximations [100, 37] could prove useful.

Topology of random sign conditions: For an arrangement of s Kostlan polynomials of degree d, in [14], we show that on average, $\mathbb{R}P^n \setminus \Gamma$ has about $s^n d^{n/2}$ connected components. Since there are 2^s possible sign conditions, this shows that many of them are unrealizable. It is therefore natural to ask: what is the probability that a given sign condition on an arrangement of Kostlan polynomials is realizable? A further question is to determine the expected Betti numbers of those sign conditions that are realizable. A separate question is - since we show that the expected sum of Betti numbers of $\mathbb{R}P^n \setminus \Gamma$ has the same order as its zeroth Betti number, this suggests that a random maximally connected component is, on average, homotopy equivalent to a point.

Open Question 14. *Study the expected topology of sign conditions of an arrangement of Kostlan polynomials.*

4 Conclusion

This research program engages with algebraic geometry, o-minimal geometry and model theory, commutative algebra, random algebraic geometry, discrete geometry and combinatorics, and computational complexity theory. I have gravitated toward ambitious problems, many of which I have seen through to completion, guided by the conviction that they yield enduring insight and illuminate how algebraic structure pervades the mathematical universe.

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