Partitioning Theorems for Sets of Semi-Pfaffian Sets, with Applications

Abhiram Natarajan

Collaborators: Prof. Martin Lotz (Univ. of Warwick), Prof. Nicolai Vorobjov (Univ. of Bath)

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- Real Algebraic Set: The set of real zeros of a polynomial P

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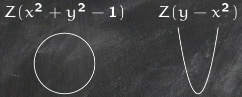

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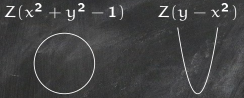

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- Algebro-geometric techniques have been very effective
- Technique called polynomial partitioning has helped solve several open problems in incidence geometry and other areas

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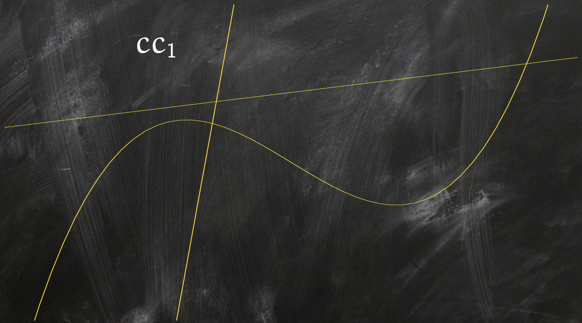

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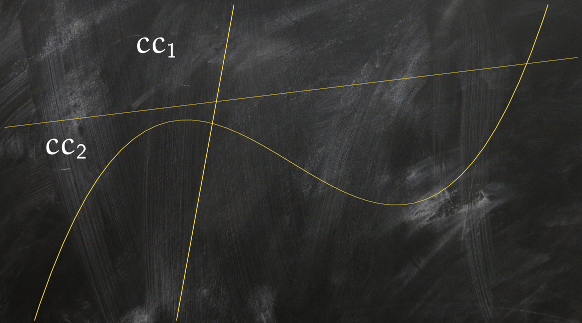

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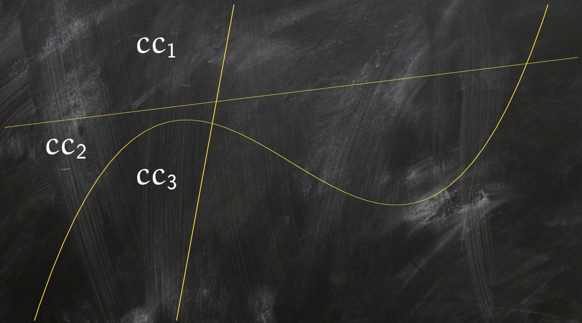

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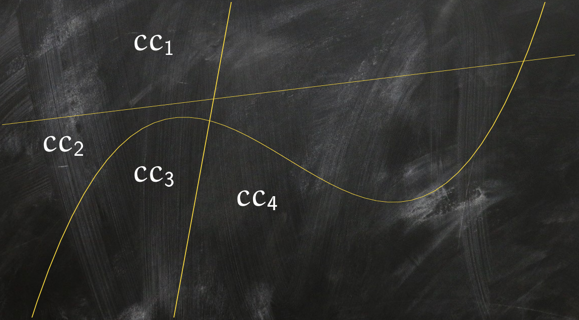

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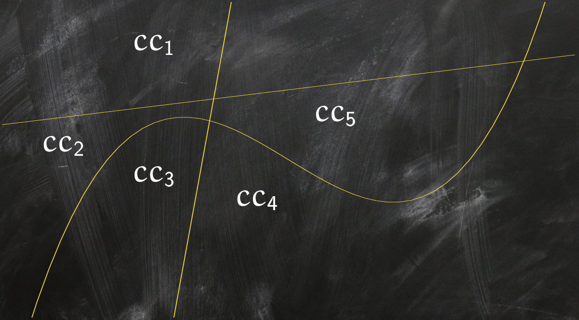

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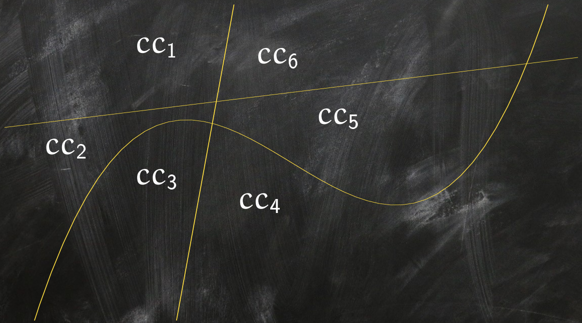

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Let Γ be a set of k-dimensional real algebraic sets in $\mathbb{R}^n$. For any $D \geqslant 1$, there is a polynomial P of degree $\leqslant D$, such that each cell induced by P intersects at most $\sim \frac{|\Gamma|}{D^{n-k}}$ elements of Γ.

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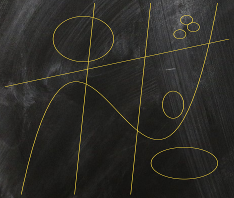

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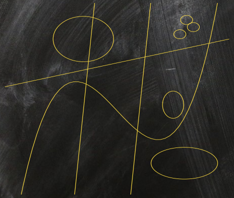

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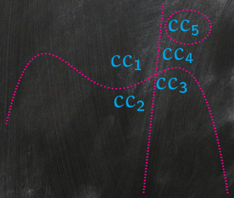

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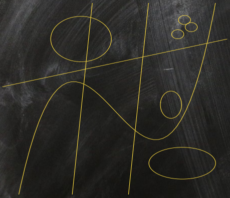

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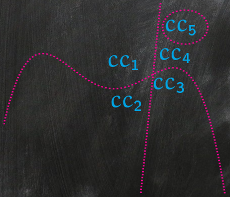

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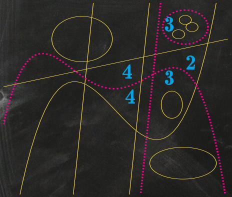

each cell intersects only few curves from Γ

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Takeaway

Polynomial partitioning and basic arguments worked!

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- O-minimal geometry (geometry of definable sets) is an axiomatic generalization of real algebraic geometry

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Pfaffian Functions

- $\vec{f} = (f_1, \ldots, f_r)$ is a Pfaffian chain if there exist polynomials $P_{i,j}$ verifying

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- e.g. e^x, e^{e^x}, ..., $\tan x$, $\ln x$, x^π

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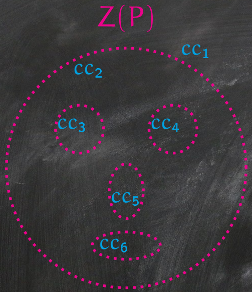
Takeaway

1. *Generalization of Polynomial Partitioning to Pfaffians*
2. *New technique of Pfaffian Partitioning*

Proof - Main Technical Step



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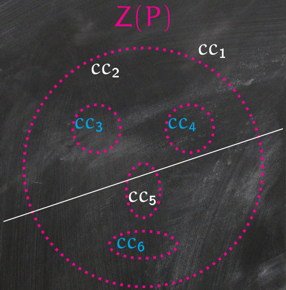

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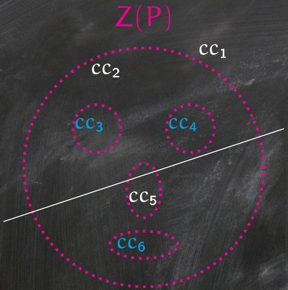

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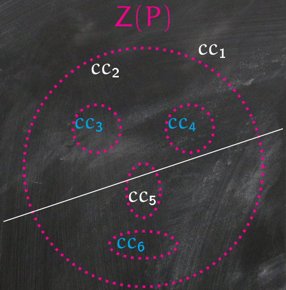

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- Poly. P of deg. D in n variables induces at most D^n cells
- We show for a k-dimensional Pfaffian set γ of order r

  γ intersects at most D^{k+r} cells induced by P

Applications of Partitioning Pfaffians

- (Pfaffian Szemerédi-Trotter) m points and n Pfaffian curves in $\mathbb{R}^2$: $\mathcal{O}(m^{\frac{2r+2}{2r+3}+\varepsilon}n^{\frac{(r+2)}{2r+3}} + m + n)$ incidences

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- Our technique lends itself to generalizing to other o-minimal structures (caveat [Basu-Lerario-N, 2019])

References

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