Mathematical Preliminaries

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1 Scalar and Vector Fields

In classical physics, fields describe distributions of physical quantities in space and time.

1. A scalar field assigns a single value (a scalar) to every point in space. This value can vary from point to point. They are used to represent quantities that have only magnitude and no direction. A scalar field ϕ in three-dimensional space is a function: $\phi : \mathbb{R}^3 \to \mathbb{R}$. $\phi(x, y, z)$ assigns a real number to every point (x, y, z) in space. They are visualized using contour lines or color maps. Each contour line or color represents points with the same scalar value. Below, we list some examples of scalar fields.

    - Temperature Distribution: The temperature at every point in a room can be represented as a scalar field. Each point in the room has a single temperature value. Mathematically, if T is the temperature, then T(x, y, z, t) gives the temperature at a point (x, y, z) at time t.
    - Pressure in a Fluid: The pressure at every point in a fluid (e.g., water in a lake) can be described by a scalar field. Each point in the fluid has a specific pressure value.

2. A vector field assigns a vector to every point in space. This vector can vary from point to point. Vector fields are used to represent quantities that have both magnitude and direction. A vector field $\mathbf{F}$ in three-dimensional space is a function: $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, where

$\mathbf{F}(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z))$ assigns a vector to every point (x, y, z) in space. The vector fields are visualized using arrows or vectors at various points in space. Each arrow's direction and length represent the vector's direction and magnitude at that point. We list below some examples of vector fields.

- Wind Velocity: The wind velocity at every point in the atmosphere can be represented as a vector field. Each point in the atmosphere has a velocity vector indicating the speed and direction of the wind. Mathematically, if $\mathbf{v}$ is the velocity, then $\mathbf{v}(x, y, z, t)$ gives the velocity vector at a point (x, y, z) at time t.
- Magnetic Field: The magnetic field ($\mathbf{B}(x, y, z)$) around a magnet can be described by a vector field. Each point in space has a magnetic field vector indicating the direction and strength of the magnetic field. Same can be said for the electric field ($\mathbf{E}(x, y, z)$) around a charged object.

Classical fields are governed by partial differential equations derived from physical laws. Some examples include 1. Maxwell's Equations for the electromagnetic field. 2. Einstein's Field Equations for the gravitational field in general relativity. 3. Wave Equation for sound waves or scalar fields. The electric field $\mathbf{E}(\mathbf{r}, t)$ in classical electromagnetism is a vector field described by Maxwell's equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}. \end{aligned}$$

1.1 Gradient of a scalar field

The gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field and whose magnitude is equal to that rate of increase in that direction. A large magnitude means a steeper slope. If the vector is resolved, its components represent the rate of change of the scalar field with respect to each directional component. For a scalar

field $\phi(x, y, z)$, the gradient of ϕ is denoted by $\nabla\phi$ and is defined as:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Consider a scalar field $\phi(x, y, z)$ in three-dimensional space. Suppose you move a small distance $d\mathbf{r}$ in space. The change in the scalar field $\delta\phi$ due to this small displacement can be approximated by the total derivative:

$$\delta\phi = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z$$

This can be written in vector form as $\delta\phi \approx \nabla\phi.\delta\mathbf{r}$, where $\delta\mathbf{r} = (\delta x, \delta y, \delta z)$ is the displacement vector. The gradient $\nabla\phi$ is thus a vector that describes the rate of change of the scalar field in space.

Consider the gravitational potential $\phi(x, y, z) = -\frac{GM}{\sqrt{x^2+y^2+z^2}}$, where G is the gravitational constant and M is the mass creating the gravitational field. The gradient of ϕ is

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

. Calculating the partial derivatives:

$$\frac{\partial\phi}{\partial x} = \frac{GMx}{(x^2+y^2+z^2)^{3/2}}, \frac{\partial\phi}{\partial y} = \frac{GMy}{(x^2+y^2+z^2)^{3/2}}, \frac{\partial\phi}{\partial z} = \frac{GMz}{(x^2+y^2+z^2)^{3/2}}$$

This vector field points towards the mass M and indicates the direction and rate of increase of the gravitational potential.

1.2 Divergence

In vector calculus, a vector operator is a differential operator. We can view ∇ as an object in its own right. It is called the gradient operator, $\nabla = e_i\partial/\partial x^i$ (e_i are the unit vectors). This is both a vector and an operator. The fact that ∇ is an operator means that it's just waiting for a function to come along (from the right) and be differentiated. In the case of a vector field, given two vectors, we can dot them together. This gives a derivative acting on vector fields known as the divergence

$$\nabla\cdot\mathbf{F} = \sum_i\sum_j\left(e_i\frac{\partial}{\partial x^i}\right)(e_jF_j) = \sum_i\sum_j\delta_{ij}\frac{\partial F_j}{\partial x^i} = \sum_i\frac{\partial F_i}{\partial x^i}, \qquad (1)$$

where we've used the orthonormality $e_i \cdot e_j = \delta_{ij}$. Note that the gradient of a scalar field gave a vector field. Now the divergence of a vector field gives a scalar field. For a three-dimensional vector field $\mathbf{F}(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z))$, the divergence is defined as:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \tag{2}$$

The divergence operator is a vector calculus operator that measures the magnitude of a vector field's source or sink at a given point, in other words, it quantifies how much a vector field spreads out (diverges) or converges at a point. The divergence of a vector field provides a scalar value that represents the net rate of flux expansion or compression at a given point. If the divergence is positive at a point, it indicates a source where the field is spreading out. If it is negative, it indicates a sink where the field is converging. If the divergence is zero, the field is said to be solenoidal, meaning there is no net flux expansion or compression at that point. In electromagnetism, the electric field $\mathbf{E}$ generated by a charge distribution ρ is described by Gauss's law, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$. Here, ϵ_0 is the permittivity of free space. This equation states that the divergence of the electric field is proportional to the local charge density. A positive charge density ρ results in a positive divergence, indicating a source of the electric field, while a negative charge density indicates a sink.

Consider a simple vector field $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$. The divergence of $\mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} \right) = 2x + 2y + 2z$$

This result shows that the divergence is not constant but depends on the position in space. For instance, at the point (1, 1, 1), the divergence is $\nabla \cdot \mathbf{F} = 2(1) + 2(1) + 2(1) = 6$. This positive divergence indicates that at the point (1, 1, 1), the vector field is spreading out.

1.3 Curl of a vector field

The curl of a vector field measures the rotation or the "twisting" force at a point in the field. It provides a vector that represents the axis of rotation and the magnitude of the rotational force such as the vorticity in fluid dynamics and the rotational electric and magnetic fields in electromagnetism. The curl

is denoted as $\nabla \times \mathbf{F}$ for a vector field $\mathbf{F}$. For a three-dimensional vector field $\mathbf{F}(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z))$, the curl is defined as:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \tag{3}$$

This can be conveniently expressed using the determinant of a matrix involving the unit vectors $\hat{i}, \hat{j}, \hat{k}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{4}$$

The curl of a vector field indicates the tendency of particles at a point to rotate around that point. If the curl at a point is zero, the field is irrotational at that point. If the curl is non-zero, it indicates the presence of a local rotational effect. In electromagnetism, the curl of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are described by Maxwell's equations. One of these equations states: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. This indicates that a time-varying magnetic field creates a rotational electric field. Conversely, another Maxwell equation states, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$. This shows that the curl of the magnetic field is related to the electric current density J and the time rate of change of the electric field, indicating that electric currents and changing electric fields create rotational magnetic fields. Consider the vector field $\mathbf{F}(x, y, z) = (0, z, -y)$. The curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial(-y)}{\partial y} - \frac{\partial z}{\partial z}, \frac{\partial 0}{\partial z} - \frac{\partial(-y)}{\partial x}, \frac{\partial z}{\partial x} - \frac{\partial 0}{\partial y} \right) = (-2, 0, 0) \tag{5}$$

This result indicates that the field F has a constant rotational component in the negative x-direction. The curl of a vector field is a measure of its local rotational tendency.

1.4 Laplacian Operator

The Laplacian operator is a second-order differential operator that is widely used in various fields of physics and engineering, particularly in the study of heat conduction, fluid dynamics, and electromagnetism. It is denoted by ∇^2

or Δ and operates on scalar fields and vector fields to measure the rate at which the average value of a field around a point differs from the value at that point.

In three-dimensional Cartesian coordinates (x, y, z), the Laplacian of a scalar field $\phi(x, y, z)$ is given by:

$$\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) \tag{6}$$

For a vector field $\mathbf{F}(x, y, z) = (F_x, F_y, F_z)$, the Laplacian is applied componentwise:

$$\nabla^2\mathbf{F} = (\nabla^2 F_x, \nabla^2 F_y, \nabla^2 F_z) \tag{7}$$

The Laplacian operator has significant physical implications in various contexts. In heat conduction, the temperature distribution T(x,y,z,t) evolves over time according to the heat equation $\partial T/\partial t = \alpha\nabla^2 T$, where α is the thermal diffusivity. The Laplacian $\nabla^2 T$ represents the net rate of heat flow into a point, influencing how the temperature changes over time. In fluid dynamics, the Laplacian appears in the Navier-Stokes equations, which describe the flow of incompressible fluids. The term $\nu\nabla^2 v$ (where ν is the kinematic viscosity and v is the velocity field) represents the diffusion of momentum, contributing to the viscous dissipation in the fluid. In Cartesian coordinates, the Laplacian is the sum of second partial derivatives. In polar and spherical coordinates, it includes terms that account for the geometry of the coordinate system.

Polar coordinates (r, θ) are commonly used in two-dimensional problems with circular symmetry. The relationship between Cartesian coordinates (x, y) and polar coordinates (r, θ) is given by:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Remember the gradient in Polar Coordinates:

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$$

The divergence then becomes:

$$\begin{aligned} \nabla \cdot \nabla f &= \nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \end{aligned}$$

Spherical coordinates (r, θ, ϕ) are used for three-dimensional problems with spherical symmetry. The relationship between Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) is given by:

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

where: r is the radial distance, θ is the polar angle (0 to π), and ϕ is the azimuthal angle (0 to 2π). The Gradient in Spherical Coordinates is:

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}.$$

The Divergence then becomes:

$$\nabla \cdot \nabla f = \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}.$$

2 Lagrangian and Hamiltonian

The Lagrangian and Hamiltonian are two fundamental formulations of classical mechanics that provide different perspectives on the dynamics of physical systems. They both yield the same equations of motion but use different approaches and variables.

2.0.1 Lagrangian Mechanics - Mathematical Formulation

The Lagrangian, denoted by L, is a function that summarizes the dynamics of a system. For a system with generalized coordinates q_i[1] and generalized

[1]Generalized coordinates are a set of parameters used to describe the configuration of a physical system. They are a more flexible and often simpler way to represent the state of a system compared to traditional Cartesian coordinates (x, y, z). They are independent parameters, meaning they can be chosen freely to describe the system's configuration. They can be any suitable parameters, like angles, lengths, or even abstract quantities, as long as they uniquely define the system's position. Often, they reduce the number of variables needed to describe a system, simplifying calculations. Consider a simple pendulum. Instead of using Cartesian coordinates (x, y) for the bob's position, we can use a single generalized coordinate, the angle θ it makes with the vertical. This simplifies the problem significantly.

velocities $\dot{q}_i$, the Lagrangian is defined as:

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i) - V(q_i)$$

where T is the kinetic energy of the system and V is the potential energy of the system. The equations of motion are derived from the Lagrangian using the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

for each generalized coordinate q_i. The Lagrangian formulation is particularly useful for systems with constraints and in generalized coordinates. It provides a powerful and elegant method to derive the equations of motion without needing to resolve the forces explicitly. This formulation is central in classical mechanics, field theory, and is foundational for quantum mechanics and quantum field theory.

Example: Simple Pendulum with Generalized Coordinates

A simple pendulum consists of a mass m suspended from a fixed point by a massless, inextensible string of length l. Instead of using Cartesian coordinates (x, y) for the mass, we use a single generalized coordinate, the angle θ the string makes with the vertical.

- Kinetic energy: The velocity of the mass is given by the arc length, which is $l\frac{d\theta}{dt}$. Therefore $T = \frac{1}{2}mv^2 = \frac{1}{2}m(ld\theta/dt)^2 = \frac{1}{2}ml^2(d\theta/dt)^2 = \frac{1}{2}ml^2\dot{\theta}^2$.
- Potential energy: The height of the mass above the lowest point is $l(1 - \cos\theta)$. $V = mgh = mgl(1 - \cos\theta)$.
- The Lagrangian L is the difference between kinetic and potential energy: $L = T - V = \frac{1}{2}ml^2(d\theta/dt)^2 - mgl(1 - \cos\theta) = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$.

Using θ simplifies the problem significantly compared to using Cartesian coordinates. The constraint (length of the string) is automatically satisfied by using θ. The Euler-Lagrange equation for θ is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

which simplifies to the equation of motion:

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

2.1 Hamiltonian Mechanics - Mathematical Formulation

The Hamiltonian, denoted by H, is another function that describes the dynamics of a system. It is related to the Lagrangian through a Legendre transformation. The Hamiltonian is defined in terms of generalized coordinates q_i and generalized momenta p_i:

$$H(q_i, p_i, t) = \sum_i p_i\dot{q}_i - L(q_i, \dot{q}_i, t)$$

where the generalized momenta are defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

The equations of motion in Hamiltonian mechanics are given by Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

The Hamiltonian often corresponds to the total energy of the system (kinetic plus potential energy). Hamiltonian mechanics provides a clear and structured approach to the study of dynamical systems, and it is especially useful in the context of phase space analysis and in the formulation of quantum mechanics. It allows for the application of powerful mathematical techniques from symplectic geometry and can be used to study the properties of systems in terms of conserved quantities and symmetries.

For the same simple pendulum, the generalized momentum is:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

The Hamiltonian is:

$$H = p_\theta\dot{\theta} - L = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta)$$

Hamilton's equations are:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$$

These yield the same equations of motion as the Lagrangian formulation. Lagrangian Mechanics is often more straightforward for systems with constraints, as it directly incorporates generalized coordinates and velocities. It is also the starting point for many field theories and forms the basis of the principle of least action. Hamiltonian Mechanics is particularly useful for systems where energy conservation is central and for analyzing the behavior of systems in phase space. It is the natural framework for the transition to quantum mechanics and is valuable in statistical mechanics and dynamical systems theory. Both formulations are mathematically equivalent and can be used interchangeably depending on the problem at hand and the preferences of the physicist.

3 Dirac Delta Function

The Heaviside step function $H(x)$ is defined as:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

The Dirac delta function $\delta(x)$ is defined such that:

1. $\delta(x) = 0$ for all $x \neq 0$,
2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

The Dirac delta function, $\delta(x)$, and the Heaviside step function, $H(x)$, are closely related in the context of distribution theory and mathematical physics. The Dirac delta function is often used to model an idealized point source, while the Heaviside function models a step discontinuity. The Dirac delta function is not a function in the traditional sense but rather a distribution. It is often visualized as an infinitely high, infinitely narrow spike at $x = 0$

with unit area under the curve. Mathematically, the Dirac delta function can be seen as the derivative of the Heaviside step function. In other words:

$$\delta(x) = \frac{dH(x)}{dx}$$

.

To understand this relationship, consider the integral of $\delta(x)$ from $-\infty$ to x:

$$\int_{-\infty}^{x} \delta(t)\, dt = H(x)$$

This integral is zero for $x < 0$ and one for $x \geq 0$, which matches the definition of the Heaviside step function. Therefore, differentiating both sides of this equation with respect to x gives us:

$$\frac{d}{dx}\left(\int_{-\infty}^{x} \delta(t)\, dt\right) = \frac{dH(x)}{dx}$$

By the Fundamental Theorem of Calculus, the left-hand side simplifies to $\delta(x)$, leading to:

$$\delta(x) = \frac{dH(x)}{dx}$$

Dirac Delta Function represents an idealized point source or impulse. It is used to model situations where an instantaneous change or impulse occurs, such as a point charge in electromagnetism or a sudden force applied in mechanics. Heaviside Step Function represents a step change or switch. It is used to model situations where a quantity changes abruptly from one value to another, such as the turning on of an electric current at a specific time.
Consider a charge density $\rho(x)$ given by a point charge at $x = 0$. The charge density can be represented by the Dirac delta function:

$$\rho(x) = q\delta(x)$$

where q is the total charge. The electric field $E(x)$ generated by this charge distribution can be found by integrating $\rho(x)$. If we integrate $\delta(x)$ from $-\infty$ to x, we get:

$$\int_{-\infty}^{x} q\delta(t)\, dt = qH(x)$$

This integral shows how the charge distribution leads to a step change in the potential or field at the location of the charge.

Properties of the Dirac Delta Function

1. Sifting Property: The Dirac delta function is defined such that it "shifts out" the value of a function at a specific point. For a continuous function $f(x)$:
$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

2. Normalization: The integral of the Dirac delta function over the entire real line is 1:
$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

3. Localization: The Dirac delta function is zero everywhere except at $x = 0$:
$$\delta(x) = 0 \quad \text{for} \quad x \neq 0$$

4. Representation: The Dirac delta function can be formally understood as the limit of a sequence of functions that become increasingly peaked around zero and integrate to 1. For example, one common representation is:
$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/\epsilon}$$

The Dirac delta function can be extended to higher dimensions. For a vector $\mathbf{r} = (x_1, x_2, \ldots, x_n)$ in n-dimensional space, the n-dimensional delta function $\delta^n(\mathbf{r})$ is defined as:

$$\delta^n(\mathbf{r}) = \delta(x_1)\delta(x_2)\cdots\delta(x_n)$$

Properties of the Higher-Dimensional Delta Function

- Normalization: The integral over all space is 1:
$$\int_{\mathbb{R}^n} \delta^n(\mathbf{r})\,d^n\mathbf{r} = 1$$

- Sifting Property: For a continuous function $f(\mathbf{r})$:
$$\int_{\mathbb{R}^n} f(\mathbf{r})\delta^n(\mathbf{r}-\mathbf{a})\,d^n\mathbf{r} = f(\mathbf{a})$$

2-Dimensional Delta Function

The 2-dimensional Dirac delta function, $\delta^2(\mathbf{r})$, where $\mathbf{r} = (x, y)$, is defined as:

$$\delta^2(\mathbf{r}) = \delta(x)\delta(y)$$

In two dimensions, for a vector $\mathbf{r} = (x, y)$ and $\mathbf{a} = (a, b)$, the 2D delta function is:

$$\delta^2(\mathbf{r} - \mathbf{a}) = \delta(x - a)\delta(y - b)$$

Properties

- Normalization:
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta^2(\mathbf{r})\, dx\, dy = 1$$

- Sifting Property: For a continuous function $f(x, y)$:
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta^2(\mathbf{r} - \mathbf{a})\, dx\, dy = f(a_x, a_y)$$
where $\mathbf{a} = (a_x, a_y)$.

3-Dimensional Dirac Delta Function: The 3-dimensional Dirac delta function, $\delta^3(\mathbf{r})$, where $\mathbf{r} = (x, y, z)$, is defined as:

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

- Normalization:
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta^3(\mathbf{r})\, dx\, dy\, dz = 1$$

- Sifting Property: For a continuous function $f(x, y, z)$:
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\delta^3(\mathbf{r} - \mathbf{a})\, dx\, dy\, dz = f(a_x, a_y, a_z)$$
where $\mathbf{a} = (a_x, a_y, a_z)$

4-Dimensional Dirac Delta Function

The 4-dimensional Dirac delta function, $\delta^4(\mathbf{r})$, where $\mathbf{r} = (x_1, x_2, x_3, x_4)$, is defined as:

$$\delta^4(\mathbf{r}) = \delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4)$$

- Normalization:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta^4(\mathbf{r})\,dx_1\,dx_2\,dx_3\,dx_4 = 1$$

- Sifting Property: For a continuous function $f(x_1, x_2, x_3, x_4)$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x_1, x_2, x_3, x_4)\delta^4(\mathbf{r}-\mathbf{a})\,dx_1\,dx_2\,dx_3\,dx_4 = f(a_1, a_2, a_3, a_4)$$

  where $\mathbf{a} = (a_1, a_2, a_3, a_4)$

In spacetime physics, particularly in the context of general relativity or quantum field theory, events can be described in a 4-dimensional spacetime. A point event occurring at (t_0, x_0, y_0, z_0) can be represented using the 4-dimensional delta function:

$$\delta^4(t - t_0, x - x_0, y - y_0, z - z_0)$$

In electromagnetism, the charge density $\rho(\mathbf{r})$ of a point charge q located at position $\mathbf{a}$ can be represented using the Dirac delta function:

$$\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{a})$$

In the study of differential equations, particularly in boundary value problems, the Dirac delta function is used to construct Green's functions, which are fundamental solutions to linear differential operators. For example, the Green's function $G(\mathbf{r}, \mathbf{r}')$ for the Poisson equation in three dimensions satisfies:

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

Next, the Fourier transform of the Dirac delta function $\delta(x)$ is discussed next. The Dirac delta function is a distribution and is defined by its sifting property:

$$\int_{-\infty}^{\infty}\delta(x)f(x)\,dx = f(0)$$

for any well-behaved function $f(x)$. The Fourier transform $\mathcal{F}\{f(x)\}$ of a function $f(x)$ is given by:

$$\mathcal{F}\{f(x)\} = \tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

Applying this definition to the Dirac delta function, we have:

$$\mathcal{F}\{\delta(x)\} = \tilde{\delta}(k) = \int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx$$

Using the sifting property of the Dirac delta function:

$$\tilde{\delta}(k) = \int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx = e^{-ik\cdot 0} = 1$$

Thus, the Fourier transform of the Dirac delta function $\delta(x)$ is:

$$\tilde{\delta}(k) = 1$$

To verify the result, we can also consider the inverse Fourier transform. The inverse Fourier transform $\mathcal{F}^{-1}\{\tilde{f}(k)\}$ of a function $\tilde{f}(k)$ is given by:

$$\mathcal{F}^{-1}\{\tilde{f}(k)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\,dk$$

Applying this to the constant function $\tilde{\delta}(k) = 1$:

$$\mathcal{F}^{-1}\{1\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

The integral:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

is known to be the Dirac delta function $\delta(x)$. Thus,

$$\mathcal{F}^{-1}\{1\} = \delta(x)$$

For the higher-dimensional Dirac delta function, $\delta^n(\mathbf{r})$, where $\mathbf{r} = (x_1, x_2, \ldots, x_n)$ in n-dimensional space, the Fourier transform is similarly straightforward. The Fourier transform in n-dimensional space is given by:

$$\mathcal{F}\{\delta^n(\mathbf{r})\} = \tilde{\delta}^n(\mathbf{k}) = \int_{-\infty}^{\infty} \delta^n(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^n\mathbf{r}$$

Using the sifting property:

$$\tilde{\delta}^n(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{0}} = 1$$

Thus, the Fourier transform of the n-dimensional Dirac delta function is:

$$\tilde{\delta}^n(\mathbf{k}) = 1$$

- The Fourier transform of the Dirac delta function $\delta(x)$ is a constant: $\tilde{\delta}(k) = 1$.
- The inverse Fourier transform of this constant returns the original Dirac delta function: $\mathcal{F}^{-1}\{1\} = \delta(x)$.
- In higher dimensions, the Fourier transform of the n-dimensional Dirac delta function $\delta^n(\mathbf{r})$ is also a constant: $\tilde{\delta}^n(\mathbf{k}) = 1$.

4 Special Relativity

We consider a four-dimensional world, which is parameterized by coordinates $x^\mu = (x^0, x^i)$, where $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$. The concept of four-vectors was introduced such that the scalar product of any two four-vectors is invariant under Lorentz transformations. It is similar to the concept that the scalar product of any two three-vectors in the three-dimensional space is invariant under rotation of the coordinate system. We will often denote x^0 as time t, and the spatial three vector x^i as x. We will work in units in which the speed of light is set to c = 1. In order to define the scalar product of two four-vectors, which is invariant under Lorentz transformation, a metric tensor $g^{\mu\nu}(g_{\mu\nu})$ is introduced.

$$g^{\mu\nu}(\eta^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}(\eta_{\mu\nu}).$$

so that $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$. We use $g^{\mu\nu}(g_{\mu\nu})$ to raise (lower) indices on vectors and tensors. The introduction of the metric tensor leads to two distinct types of four-vectors: contravariant and covariant four-vectors. Let us consider a four vector A. A contravariant four vector is defined by a superscript $A^\mu =$

$(A^0, A^1, A^2, A^3) = (A_t, A_x, A_y, A_z)$. The covariant four vector[2] is defined by the subscript $A_\mu = (A_0, A_1, A_2, A_3) = (A_t, -A_x, -A_y, -A_z)$. One can convert a contravariant four-vector into a covariant four-vector and vice versa, using the metric tensor.

$$A_\mu = \sum_\nu g_{\mu\nu} A^\nu, \quad \mu = 0, 1, 2, 3$$

$$A^\mu = \sum_\nu g^{\mu\nu} A_\nu, \quad \mu = 0, 1, 2, 3$$

The scalar product of any two four vectors can be written as:

$$\begin{aligned} A \cdot B &= A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \\ &= A_t B_t - A_x B_x - A_y B_y - A_z B_z = A_t B_t - \vec{A} \cdot \vec{B} \end{aligned}$$

You can also write the scalar product as:

$$\begin{aligned} A \cdot B &= A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu \\ &= g_{00} A^0 B^0 + g_{11} A^1 B^1 + g_{22} A^2 B^2 + g_{33} A^3 B^3 \\ &= A_t B_t - A_x B_x - A_y B_y - A_z B_z = A_t B_t - \vec{A} \cdot \vec{B} \end{aligned}$$

A note on indices: throughout this course, the summation convention will be employed in which repeated indices are summed over. For the spacetime indices μ it will be crucial to keep track of whether they're up or down. Take note that you should never encounter expressions that look like $A^\mu B^\mu$. Remember that the repeated indices are dummy indices – it doesn't matter what you call them. But take care of not using the same pairs of dummy indices twice. For example, the expression $(A^\mu B_\mu)(C^\mu D_\mu)$ makes no sense – even with the brackets! Avoid this confusion by writing as $A^\mu B_\mu C^\nu D_\nu$ to show which pairs are summed over. Denoting multiple pairs of dummy indices with the same label will be a very easy trap to fall into. Avoid this mistake.

5 Position Space, Momentum Space

In quantum field theory (QFT), position space and momentum space are two different but related ways of describing the state of a quantum field. They

[2]The components A_t, A_x, A_y, A_z of the four-vector A have the same signs as the metric tensor $g_{\mu\nu}$ and so, it is called the covariant four-vector. The components A_t, A_x, A_y, A_z of the four-vector A have signs that are not in conformity with the signs of the metric tensor $g_{\mu\nu}$ and so it is called the contravariant four-vector.

provide complementary perspectives on the properties and behavior of fields and particles.

Position space refers to the description of fields and particles in terms of their spatial coordinates. In position space, the field is represented as a function of space and time coordinates. For a scalar field ϕ, the field in position space is denoted as $\phi(x)$, where $x = (t, \mathbf{r})$ includes both the time t and spatial coordinates $\mathbf{r} = (x, y, z)$. The Lagrangian density $\mathcal{L}$ and other physical quantities are typically expressed in terms of $\phi(x)$ and its derivatives with respect to space and time. The Klein-Gordon field, which describes a free scalar particle, has a Lagrangian density in position space:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2$$

where ∂^μ represents the spacetime derivative.

Momentum space refers to the description of fields and particles in terms of their momentum coordinates. In momentum space, the field is represented as a function of energy and momentum. For a scalar field ϕ, the field in momentum space is denoted as $\tilde{\phi}(p)$, where $p = (E, \mathbf{p})$ includes both the energy E and the spatial momentum $\mathbf{p} = (p_x, p_y, p_z)$. The momentum space representation is obtained via the Fourier transform of the position space field:

$$\tilde{\phi}(p) = \int d^4x\, e^{ip\cdot x}\phi(x)$$

Conversely, the inverse Fourier transform allows us to recover the position space field from its momentum space representation:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\tilde{\phi}(p)$$

The Klein-Gordon equation in momentum space is:

$$(p^\mu p_\mu - m^2)\tilde{\phi}(p) = 0$$

where $p^\mu p_\mu = E^2 - \mathbf{p}^2$. Position Space is useful for understanding local interactions and the behavior of fields at specific points in space and time. It is the natural setting for formulating the Lagrangian and action principles in QFT, as well as for defining local operators and examining causality. Momentum Space is useful for analyzing the energy and momentum distribution of particles and fields. It simplifies the treatment of translationally invariant

systems and is particularly convenient for performing calculations involving propagators, Feynman diagrams, and scattering amplitudes.

The Fourier transform provides a bridge between position space and momentum space. This transformation is essential in QFT because it allows us to switch between these complementary descriptions depending on the problem at hand.

6 Bibliography

You can go through the books listed here in this site for a brief ideas of all the assumed prerequisites. https://fliptomato.wordpress.com/2006/12/30/from-griffiths-to-peskin-a-lit-review-for-beginners/