All Pairs Shortest Paths (Contd.)
(APSP)

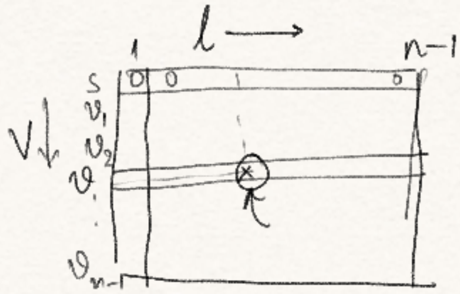
all pairs
→ Single source shortest paths: $O(mn)$.

all pairs
↳ Repeated single source shortest paths: $O(mn^2)$.

Fix a single source s:

$dist(s, v)$ $\forall$ $v \neq s$ in V.

$dist(u, v, \ell)$: Shortest distance between u and v with at most ℓ edges.



For every vertex u, For every vertex v:

$dist(u, v, \ell)$:

if $\ell = 1$:

return $w(u, v)$. // if $(u,v) \notin E$, then $w(u,v) = \infty$.

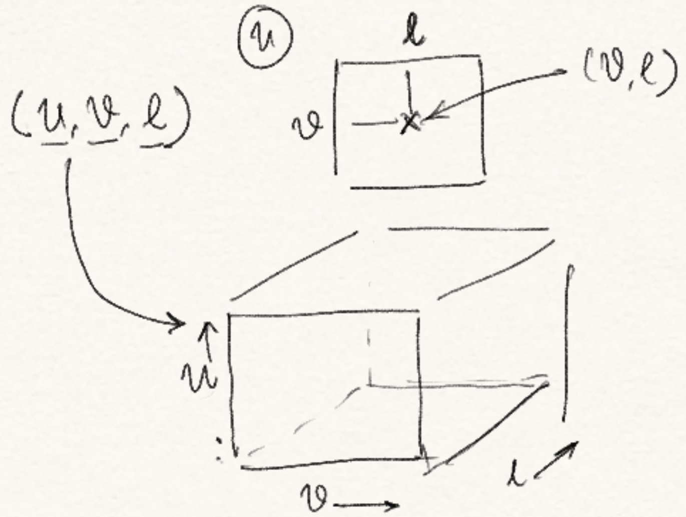
return $\min\limits_{\substack{v' \in V \\ (v', v) \in E}} \{ dist(u, v', \ell - 1) + w(v', v) \}$.

↑ # of lookups $= d_v$

# of lookups/row $= d_v (n-1)$

Total # of lookups $= \sum_v d_v (n-1)$

$= O(mn)$.



$u = s$

0

// Set all entries to ∞ at the start and $d[u, v, 0] = 0$.

For ℓ in 1 to $n-1$:
    For all vertices u: n^2
        For all vertices v:
            For all neighbours v' of v: $\sum_v d_v$
                if $dist[u,v,\ell] > dist[u,v',\ell-1] + w(v',v)$
                    $dist[u,v,\ell] \leftarrow dist[u,v',\ell-1] + w(v',v)$

$\boxed{n^2 m}$

Bellman-Ford.

$$dist[u,v,\ell] = \begin{cases} w(u,v) & \text{if } \ell = 1 \\ \min_{x \in V} \left\{ dist[u,x,\frac{\ell}{2}] + dist[x,v,\frac{\ell}{2}] \right\} & \text{o/w.} \end{cases}$$

$\ell = 2^i$

For i in 1 to $\lceil \log_2 n \rceil$:
    For all vertices u:
        For all vertices v:
            For all vertices x:
                if $dist[u,v,2^i] > dist[u,x,2^{i-1}] + dist[x,v,2^{i-1}]$
                    $dist[u,v,2^i] \leftarrow dist[u,x,2^{i-1}] + dist[x,v,2^{i-1}]$

$n^3 \log n$

$\boxed{\text{Fischer-Meyer}}$

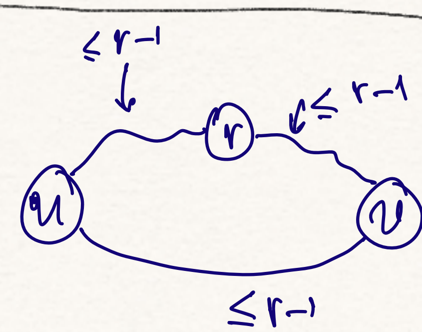
Williams:

APSP in $\frac{n^3}{2^{\Omega(\sqrt{\log n})}}$

Fine grained complexity

Let us take an arbitrary order on the vertices. $\{1, 2, \ldots, n\}$.

$dist[u,v,r]$: Shortest paths between u and v with all inter-mediate vertices $\leq r$.



$dist[u,v,r] \nearrow dist[u,v,r-1]$

$\rightarrow \text{dist}[u,r,r-1] + \text{dist}[r,v,r-1]$

$$\text{dist}[u,v,r] = \begin{cases} w(u,v) & \text{if } r=0 \\ \min \begin{cases} \text{dist}[u,v,r-1] \\ \text{dist}[u,r,r-1] + \text{dist}[r,v,r-1] \end{cases} & \text{o/w} \end{cases}$$

For r in 1 to n:
    For all vertices u:
        For all vertices v:
            if $\text{dist}[u,v,r-1] < \text{dist}[u,r,r-1] + \text{dist}[r,v,r-1]$:
                $\text{dist}[u,v,r] \leftarrow \text{dist}[u,v,r-1]$
            else:
                $\text{dist}[u,v,r] \leftarrow \text{dist}[u,r,r-1] + \text{dist}[r,v,r-1]$.

$O(n^3)$

Floyd-Warshall.