Lecture 4, 29/12/22

Gradiant of a scalar field, $\vec{\nabla}$: Considers a scalar function $\phi = \phi(x,y,z)$ and we want to know the derivative at a point say x, y, z. Example: let say scalar function $T(x,y,z)$ gives the temperature in a room. The derivative is supposed to tell how fast the temperature changes at a given point — obviously it changes differently in different directions — it means that the derivative of the scalar function must be a vector.

Using partial derivative

$$d\phi(x,y,z) = \frac{\partial \phi}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \cdot \left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \left(\vec{\nabla}\phi\right) \cdot d\vec{r}$$

$\left\{ \vec{r} = \hat{i}x + \hat{j}y + \hat{k}z \right.$ is the position vector at x, y, z

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$ is the gradiant operator

$\vec{\nabla}\phi$ is called the gradiant of scalar field ϕ.

$\vec{\nabla}$ is called "del" or "nabla"

Geometrical Interpretation of gradiant: For a change in $\vec{r}$ from $\vec{r}$ to $d\vec{r}$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} = |\vec{\nabla}\phi|\,|d\vec{r}|\cos\theta$$

Fix $|d\vec{r}|$ and search in various direction to find the maximum change.

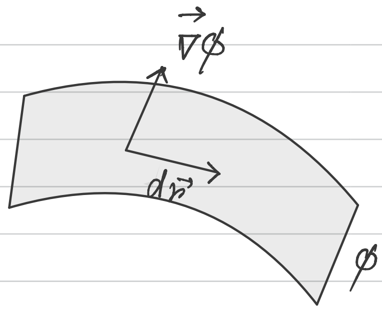
Maximum change happens when $\theta = 0$.

$\Rightarrow$ The gradiant $\vec{\nabla}\phi$ points to the maximum change of ϕ

The magnitude $|\vec{\nabla}\phi|$ gives the rate of change along the maximum direction.

* **Unit vector normal to a surface:** Let us consider a surface that is given by a scalar function $\phi \equiv \phi(x, y, z)$. From above

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r}$$



If $d\vec{r}$ is the infinitesimal separation between two points on the surface then $d\phi = 0$ { on the surface ϕ does not change

$\Rightarrow$ $\vec{\nabla}\phi \cdot d\vec{r}$ or $\vec{\nabla}\phi$ is perpendicular to $d\vec{r}$ ie, the surface at $\vec{r}$

$$\text{Unit vector } \hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}.$$

Example: Given the magnitude of position vector $|\vec{r}| = (x^2+y^2+z^2)^{1/2}$ find the gradiant

Soln: $$\vec{\nabla}|\vec{r}| = \hat{i}\frac{\partial|\vec{r}|}{\partial x} + \hat{j}\frac{\partial|\vec{r}|}{\partial y} + \hat{k}\frac{\partial|\vec{r}|}{\partial z}$$

$$= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$= \frac{\hat{i}x+\hat{j}y+\hat{k}z}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{|\vec{r}|} = \hat{r}$$

This example tells you that distance increases most rapidly along the radial direction

Example: For the function $\phi = x^2y + yz$ at the point $(1,2,-1)$, find the rate of change with distance in the direction $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$. At this point what is the greatest rate of change and in which direction does it occour?

Soln: At $(1,2,-1)$ the gradiant is

$$\vec{\nabla}\phi\Big|_{(1,2,-1)} = \left(2xy\,\hat{i} + (x^2+z)\hat{j} + yk\right)\Big|_{(1,2,-1)}$$

$$= 4\hat{i} + 2\hat{k}$$

Unit vector along $\vec{a}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}(\hat{i} + 2\hat{j} + 3\hat{k})$

Hence the rate of change

$$\frac{d\phi}{ds} = \vec{\nabla}\phi \cdot \hat{a} = \frac{1}{\sqrt{14}}(4+6) = \frac{10}{\sqrt{14}}.$$

The greatest rate is along the direction of $\vec{\nabla}\varphi$ and the value $|\vec{\nabla}\varphi| = \sqrt{20}$

* How Scalar field changes on a curve: ϕ is a scalar field and $\vec{r}(u)$ is a curve. The change of the field along the curve is given by

$$\frac{d\phi}{du} = \frac{\partial\phi}{\partial x} \cdot \frac{dx}{du} + \frac{\partial\phi}{\partial y} \frac{d\varphi}{dy} + \frac{\partial\varphi}{\partial z} \cdot \frac{dz}{du}$$

$$= \vec{\nabla}\phi \cdot \frac{d\vec{r}}{du}.$$

* Exact differential: $d\phi = \vec{\nabla}\phi \cdot d\vec{r}$ at any point $\vec{r}$

* Back to Integration: Consider a vector field $\vec{F} = \vec{\nabla}\phi$ where ϕ is scalar.
The line integral of $\vec{F}$ over a curve curve $C: \vec{r} \equiv \vec{r}(u)$; $u_1 < u < u_2$

$$\int \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla}\phi \cdot d\vec{r}$$

$$= \int_C \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \int_C \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right)$$

$$= \int_{u_1}^{u_2} \left(\frac{\partial \phi}{\partial x}\frac{dx}{du} + \frac{\partial \phi}{\partial y}\cdot\frac{dy}{du} + \frac{\partial \phi}{\partial z}\cdot\frac{dz}{du}\right) du$$

$$= \int_{u_1}^{u_2} \frac{d}{du}\phi(\vec{r}(u))\,du = \phi(\vec{r}(u))\Big|_{u_1}^{u_2}$$

$$= \phi(\vec{r}(u_2)) - \phi(\vec{r}(u_1))$$

So the result only depends on the end point and is independent of the path of the integration. If the curve is closed ie $u_1 = u_2$ then

$$\oint \vec{F}\cdot d\vec{r} = 0$$

The vector $\vec{F}$ that is equal to the gradiant of a Scalar field is called a conservative vector field.

* In a conservative vector field the line integral around a closed curve vanishes

* The opposite is also true: if the line integral in a vector field is independent of path then the field can be expressed as the gradiant of a scalar field.

* To check if a vector field can be constructed from a scalar:

Let $\vec{F} = F_x\hat{i} + \hat{j}F_y + \hat{k}F_z$ and assume that $\vec{F} = \vec{\nabla}\phi$

Then

$$F_x = \frac{\partial\phi}{\partial x}, \quad F_y = \partial\phi/\partial y, \quad F_z = \partial\phi/\partial z$$

For suitable well behaved function ϕ, the order of partial derivative does not matter

$$\Rightarrow \frac{\partial\phi}{\partial x\partial y} = \frac{\partial\phi}{\partial y\partial x} \quad \Rightarrow \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \text{or} \quad \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \begin{cases} x_i, x_j = x, y, z \\ F_i, F_j = F_x, F_y, F_z \\ \text{for } i, j = 1, 2, 3 \end{cases}$$