Minimum Spanning Trees

Problem: We have a set of locations $V = \{v_1, \dots, v_n\}$ and we want to build a "wired" communication network.

$G = (V, E)$ with $c_e : E \to \mathbb{R}_{\geq 0}$.

Spanning tree.
↳ covers all nodes from G.

Minimum Spanning Tree: A spanning tree of G of minimum total edge cost.
wt

Claim: Let T be a minimum cost solution to the problem descr. above. Then (V, T) is a tree.

$e = (u, v)$

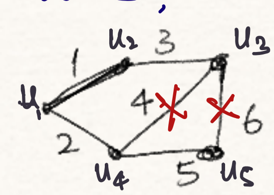
Suppose, T is not a tree; then $\exists$ a cycle in T.
Let e be the edge of maximal weight in that cycle.
Sub-claim: $T \setminus \{e\}$ is still spanning. And $wt(T \setminus \{e\})$ is strictly lower than $wt(T)$. This contradicts the fact that T is an optimal solution.

Qn: Can we modify Dijkstra's algorithm to get this?

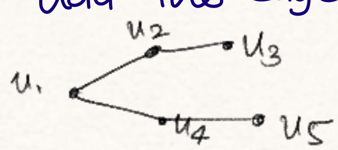
Start: From an arbitrary node, pick an edge of min wt incident on it. $S = \{s\}$, $T = T \cup (\text{min wt edge } (s, u))$; $S \leftarrow S \cup \{u\}$.

Prim's

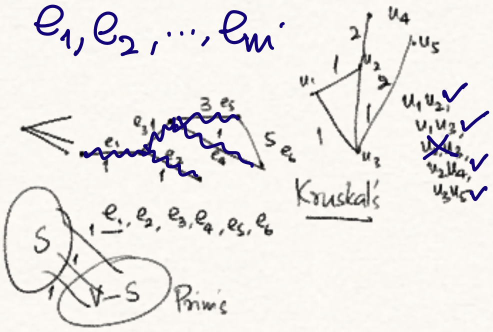
While $S \neq V$:
Look for a min wt edge between S and $V - S$.
and add the end point from $V - S$ to S,
add the edge to T.



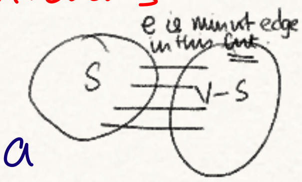
Kruskal's:



Sort your edges in non-decreasing order. $e_1, e_2, \ldots, e_m$.
Let $F \leftarrow \{\}$
For i in $[1, m]$:
Check if edge e_i forms a cycle in F.
↳ if not add e_i to F.



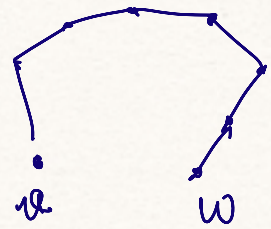
(Cut property)
Lemma: Assume that all edge costs are distinct.
Let $S \subseteq V$ s.t $|S| \notin \{0, |V|\}$, and $e = (v, w)$ be the min cost edge with one end in S and the other in $V-S$. Then every min spanning tree contains that edge e.



For the sake of contradiction, assume that a MST T does not contain the edge $e = (v, w)$

↳ Since T spans all the vertices, there should be a path from $v \rightsquigarrow w$ in T.

Sub-claim: $\exists$ edge (v', w') s.t $v' \in S$ and $w' \in V-S$ and (v', w') is on the path from v to w in T.



From the statement of the lemma, $wt(e) < wt(e')$.

↳ Sub-claim: $\underbrace{(T - \{e'\}) \cup \{e\}}_{T'}$ is still a spanning tree.

↳ We now should compare wts of trees.

$$wt(T') = wt(T) - wt(e') + wt(e) = wt(T) - (\underbrace{wt(e') - wt(e)}_{>0})$$

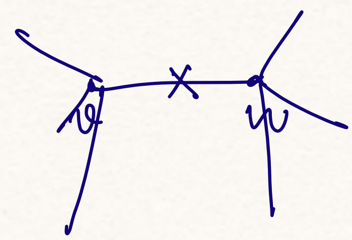
$$< wt(T).$$

This contradicts the optimality of T.

Lemma: Assume that all edge costs are distinct. Let C be any cycle in G, and let $e = (v, w)$ be the most expensive edge in C. Then e does not belong to any minimum spanning tree of G.

For the sake of contradiction, assume that there is a MST T with edge (v,w) in it. $G \setminus \{e\}$.

Let T' be a ST obtained from T by replacing e with an edge in the $v \rightsquigarrow w$ path in the cycle.

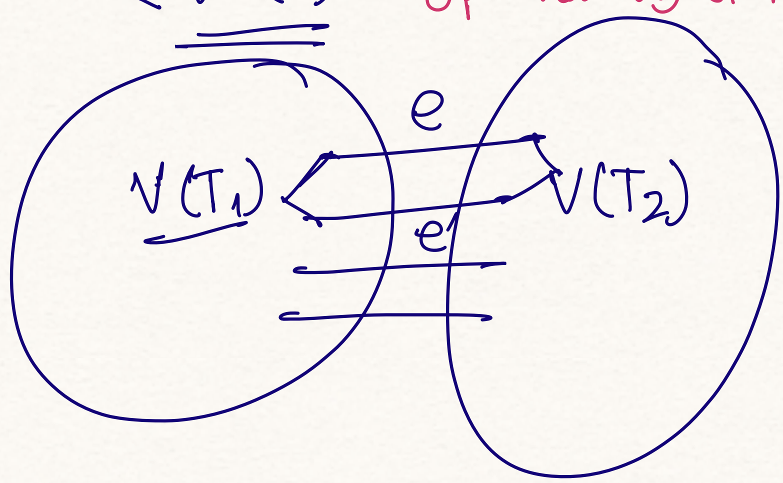


$$T' = \underline{(T \setminus \{e\})} \cup \{\underline{e'}\}.$$

$$wt(T') = wt(T) - wt(e) + wt(e')$$

$$< \underline{wt(T)}.$$ Contradicts the optimality of T.

$wt(e') < wt(e)$



Removing e from T breaks T into T_1 and T_2.

$V(T_1)$ $V(T_2)$

T_1 is connected

T_2 is connected.

and $\exists$ a path from $v \rightsquigarrow w$ in the G that does not include e.

Practice Problem:

If G has edges with distinct weights then there is a unique MST.