History : Four Realms of Mechanics

size ←

speed ↓

Classical Mechanics (Newton)	Quantum Mechanics (Bohr, Heisenberg, Schrödinger, *et al.*)
Special Relativity (Einstein)	Quantum Field Theory (Dirac, Pauli, Feynman, Schwinger, *et al.*)

Forces in order of decreasing strength :

- Strong : hold protons and neutrons together - extremely short range, we do not "feel" them, even though hundred times more powerful than electrical forces
- Electromagnetic
- Weak : account for certain kinds of radioactive decay, of short range and far weaker than electromagnetic forces
- Gravitational : extremely feeble - noticed only with huge masses like earth and sun

What about -

friction?
force that keeps one from falling through the floor?
chemical forces?
force of impact between colliding balls?

all these forces are electromagnetic

virtually every force we experience in everyday life except gravity is electromagnetic in origin

electrical repulsion between 2 electrons is $10^{42} \times$ gravitational attraction

if atoms were held together by gravitational forces
single hydrogen atom would be much larger than the known universe

Not only are electromagnetic forces overwhelmingly dominant
they are the only ones that are completely understood

There is a classical theory of gravity and a relativistic one (general relativity)
but no satisfactory quantum mechanical theory of gravity exists

there is a successful theory for weak interactions
and an attractive candidate (chromodynamics) for strong interactions

These theories draw their inspiration from electrodynamics

none can claim conclusive experimental verification

electrodynamics : beautifully complete and successful theory
- a paradigm : ideal model

$\sim$ 170 years old : Franklin, Coulomb, Ampère, Faraday, and others
- Maxwell put it in a compact and consistent form

Unification of Physical Theories

electricity: glass rods and cat's fur, pith balls, batteries, currents, electrolysis, lightning

magnetism: bar magnets, iron filings, compass needles, and North Pole

1820 : Oersted - electric current can deflect a magnetic compass needle
Ampère : all magnetic phenomena are due to electric charges in motion

1831 - Faraday : moving magnet generates electric current

Maxwell and Lorentz put finishing touches on the theory - electricity and magnetism are inextricably intertwined

- two aspects of a single subject: electromagnetism

Faraday speculated : light is electrical in nature

Maxwell's theory provided justification for this hypothesis

soon optics— study of lenses, mirrors, prisms, interference, and diffraction—was incorporated into electromagnetism

Hertz 1888 :decisive experimental confirmation for Maxwell's theory

By 1900 – electricity, magnetism, and optics – merged into a single unified theory

Einstein dreamed of a further unification of gravity and electrodynamics

His unified field theory was not particularly successful
but in recent years ambitious unification schemes have appeared

1960s : electroweak theory of Glashow, Weinberg, and Salam (joins the weak and electromagnetic forces)
1980s : superstring theory
(incorporates all four forces in a single "theory of everything")

At each step, mathematical difficulties mount
and gap between conjecture and experimental test widens

nevertheless, unification of forces is a major theme in progress of physics

Field Formulation

space around an electric charge is permeated by electric and magnetic fields

A second charge, in presence of these fields, experiences a force

fields transmit influence from one charge to other—they "mediate" the interaction

When a charge undergoes acceleration
part of the field "detaches" itself
and moves at the speed of light
carrying energy, momentum, and angular momentum
$\Longrightarrow$ electromagnetic radiation

fields are independent dynamical entities, as "real" as atoms or baseballs

1. Charge : two varieties

plus and minus $\Longrightarrow$ neutral : $+q$ and $-q$ at same point

extraordinary fact : plus and minus charges occur in exactly equal amounts in bulk matter, to fantastic precision - their effects are almost completely neutralized

Were it not for this, we would be subjected to enormous forces :
a potato would explode violently if cancellation is imperfect by one part in 10^{10}

2. Charge is conserved: it cannot be created or destroyed

total charge of the universe is fixed for all time : global conservation of charge

precise mathematical law expressing local conservation of charge : continuity eqn.

3. Charge is quantized

Coulomb law - experimental observn. :

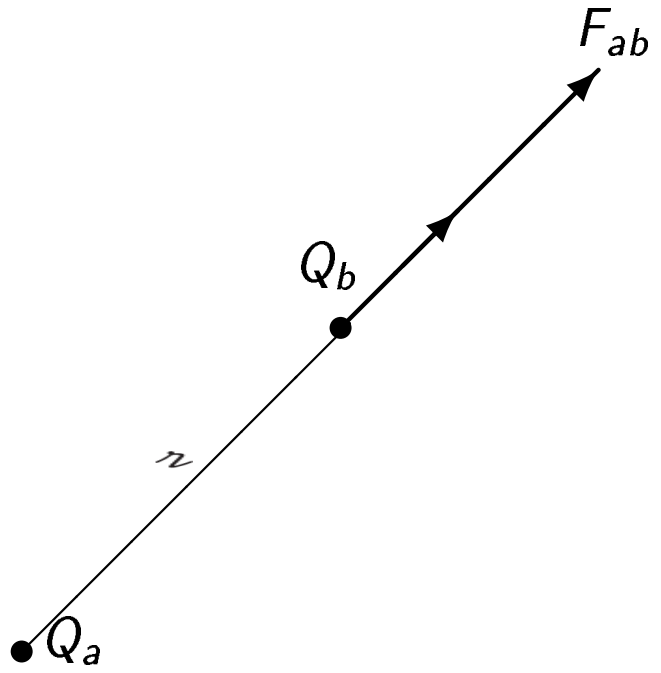
force between stationary electric **point** charges Q_a and Q_b

$\propto Q_a Q_b$; and $\propto \frac{1}{\mathscr{r}^2}$

acts along line joining the charges

$$\vec{F}_{ab} = \frac{1}{4\pi\varepsilon_0}\frac{Q_a Q_b}{\mathscr{r}^2}\hat{\mathscr{r}}$$

F measured in N, Q in C, and $\mathscr{r}$ in m



constant ε_0 =permittivity of free space= $8.8542x10^{-12}\mathrm{farad/m}$

$\mathrm{farad} \equiv \mathrm{C}^2(\mathrm{Nm})^{-1} \equiv \mathrm{C}^2\mathrm{J}^{-1}$

If charges are extended
"distance between the charges" has no definite meaning

presence of Q_b can modify the charge distribution within Q_a , and vice versa
$\Longrightarrow$ complicated variation of force with distance

Substituting value of ε_0 into Coulomb's law

$\vec{F}_{ab} = 8.988 \times 10^9 . \frac{Q_a Q_b}{r^2} \hat{r}$

Coulomb forces in nature are enormous when compared with gravitational forces

gravitational force between two masses m_a and m_b separated by a distance r,

$\vec{F} = 6.672 \times 10^{-11} . \frac{m_a m_b}{r^2} \hat{r}$

gravitational force on a proton at surface of sun (m=2.0x10^{30} kg, radius =7.0x10^8m)
= electric force between a proton and one μg of electrons, separated by a distance equal to sun's radius

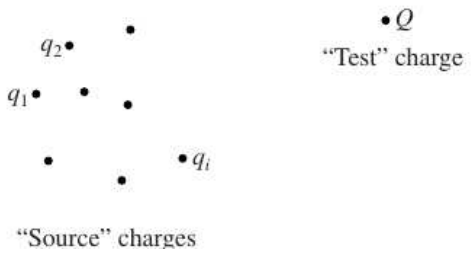
Ordinary matter is neutral, and enormous Coulomb forces prevent accumulation of appreciable quantity of charge of either sign

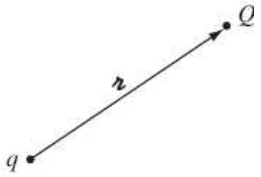
fundamental task : Given **source** charges, $q_1, q_2, q_3, \ldots$ at given positions
obtain force exerted on a **test** charge, Q

In general, both source charges and test charge are in motion
obtain trajectory of the test particle

PRINCIPLE OF SUPERPOSITION :

$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$
where $\vec{F}_i$ =force on Q due to
single source charge q_i





Electric Field :

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{r_1^2}\hat{r}_1 + \frac{q_2 Q}{r_2^2}\hat{r}_2 + \frac{q_3 Q}{r_3^2}\hat{r}_3 + \cdots\right)$$

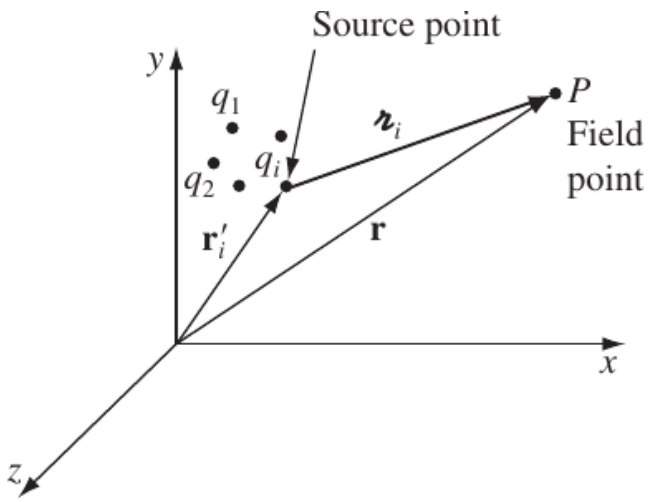
$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \frac{q_3}{r_3^2}\hat{r}_3 + \cdots\right) = Q\vec{E}$$

each charge produces its own field

resultant $\vec{E}$ = vector sum of individual $\vec{E}$'s

electric field, $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$

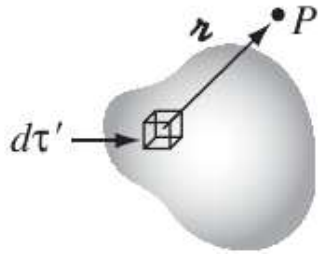
varies from point to point, determined by configuration of source charges



$\vec{E}(\vec{r})$= force per unit charge exerted on test charge

ith term is field at location of test charge due to charge q_i

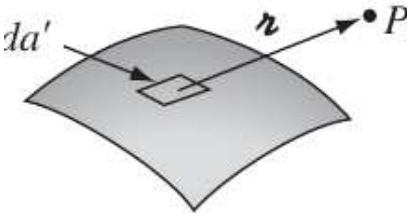
extended charge and field $\vec{E}(\vec{r})$:



volume charge density, ρ; $\quad dq \longrightarrow \rho d\tau$

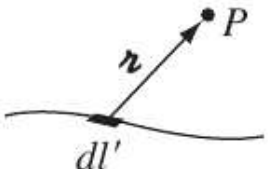
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\vec{r'})\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

For surface distributions of charge, use surface integral
Replace volume charge density ρ by surface charge density σ
and volume τ' by surface S'



surface charge density, σ; $\quad dq \longrightarrow \sigma da$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon}\int \frac{\sigma(\vec{r'})\hat{\mathscr{r}}}{\mathscr{r}^2} da'$$

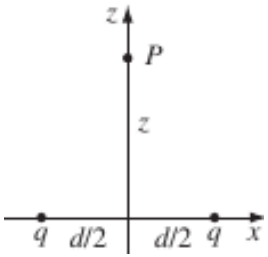


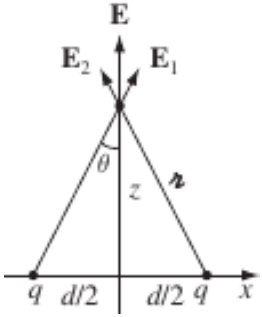
line charge density, λ; $\quad dq \longrightarrow \lambda dl$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon}\int \frac{\lambda(\vec{r'})\hat{\mathscr{r}}}{\mathscr{r}^2} dl'$$

applications of principle of superposition

electric field distance z above midpoint between two equal charges, q, distance d apart





horizontal components cancel

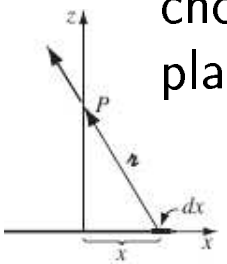
vertical components add

$$E_z = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left[z^2+\left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}}\hat{z}$$

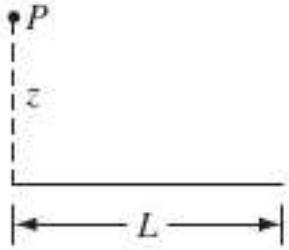
what happens if the charges are equal and opposite?

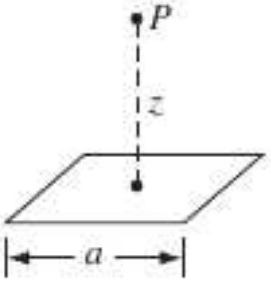
electric field a distance z above midpoint of a straight line segment of length $2L$ carrying uniform line charge density λ

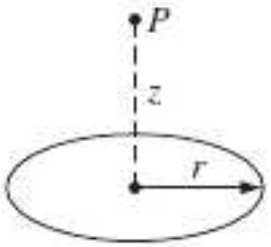


chop the line into symmetrically placed pairs at $\pm x$, and integrate $(x : 0 \to L)$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{z}$$



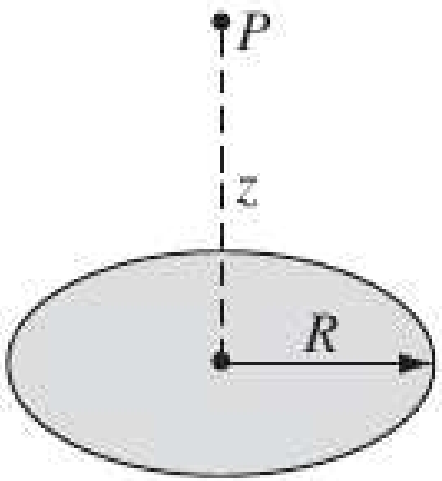


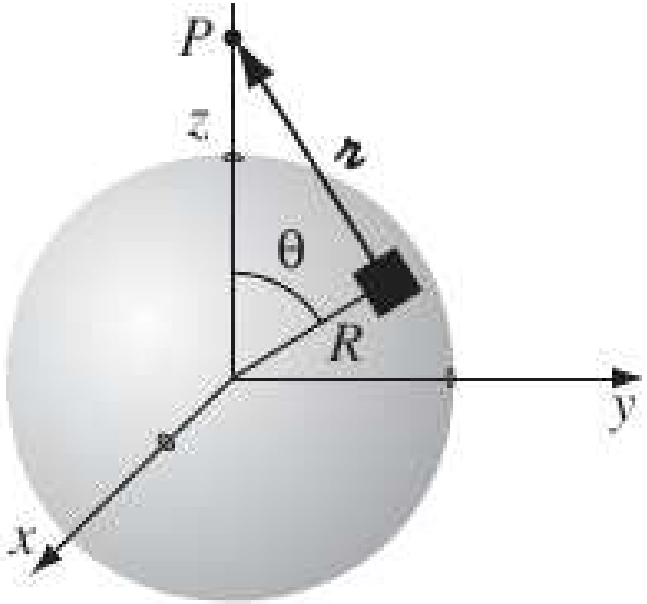


$$4\pi\varepsilon_0\vec{E} = \frac{\lambda}{z}\left[\left(-1+\frac{z}{\sqrt{z^2+L^2}}\right)\hat{x}+\frac{L\hat{z}}{\sqrt{z^2+L^2}}\right];$$

$$\frac{4\lambda a z\hat{z}}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}};$$

$$\frac{\lambda . 2\pi r\hat{z}}{[r^2+z^2]^{\frac{3}{2}}}$$





Break it into rings of radius r, and thickness dr, and integrate ($r : 0 \to R$)

Break it into areas da at coordinates θ, ϕ and inegrate over the angular space

electrostatics

Stationary source charges (test charge may move) $\Rightarrow$ constant electric fields: $\frac{\partial \rho}{\partial t} = 0$

$\rho =$volume charge density

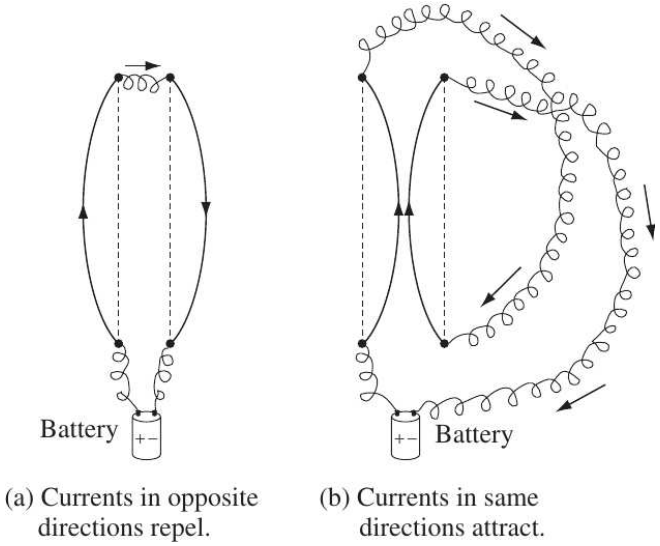
$\sigma =$surface charge density

magnetostatics

Steady currents $\Rightarrow$ constant magnetic fields : $\frac{\partial \vec{J}}{\partial t} = 0$

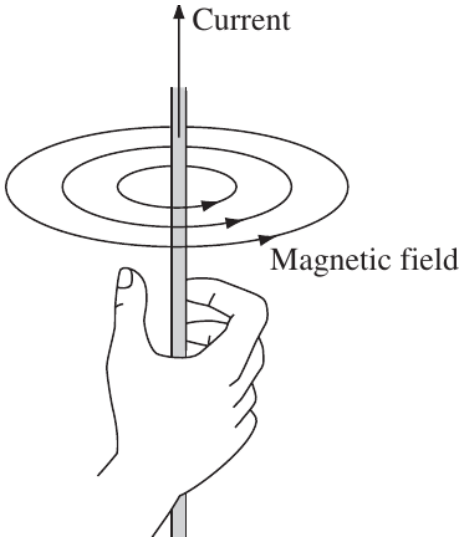
volume current density : $\vec{J} =$current per unit area (of cross-section)$= \rho \vec{v}$

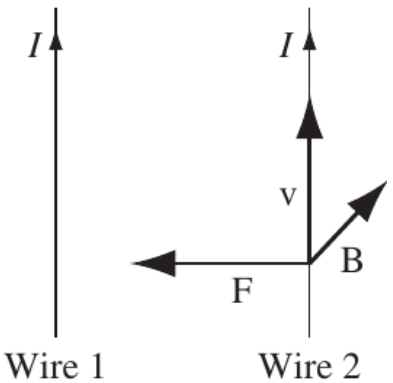
surface current density : $\vec{K} =$current per unit length (on a surface)$= \sigma \vec{v}$



(a) Currents in opposite directions repel.

(b) Currents in same directions attract.





LORENTZ FORCE LAW : $\vec{F}_{\text{mag}} = Q(\vec{v} \times \vec{B})$

In presence of both electric and magnetic fields, $\vec{F}_{\text{mag}} = Q(\vec{E} + \vec{v} \times \vec{B})$

Magnetic forces do no work

if Q moves an amount $d\vec{l} = \vec{v}dt$,

work done, $dW_{\text{mag}} = \vec{F}_{\text{mag}} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = 0$

in presence of electric field, $\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}) = \frac{d}{dt}[\gamma m \vec{v}]$; $\quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$; $\beta = \frac{v}{c}$

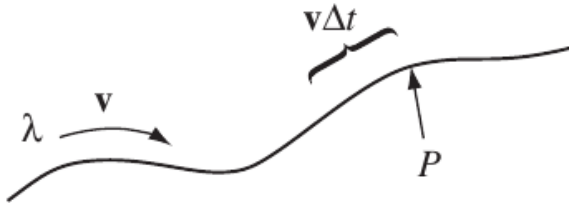
current :

line charge λ traveling down a wire at speed v

$\Longrightarrow$ segment of length $v\Delta t$, carrying charge $\lambda v \Delta t$

passes point P in time interval Δt

$\Longrightarrow$ current $\vec{I} = \lambda \vec{v}$



when both positive and negative charges move : $\vec{I} = \lambda_+ \vec{v}_+ + \lambda_- \vec{v}_-$

magnetic force on a segment of current-carrying wire

$$\vec{F}_{\text{mag}} = \int \vec{v} \times \vec{B} dq = \int \vec{v} \times \vec{B} \lambda dl = \int \vec{I} \times \vec{B} dl = \int I \vec{dl} \times \vec{B}$$

typically, with constant current, $\vec{F}_{\text{mag}} = I \int \vec{dl} \times \vec{B}$

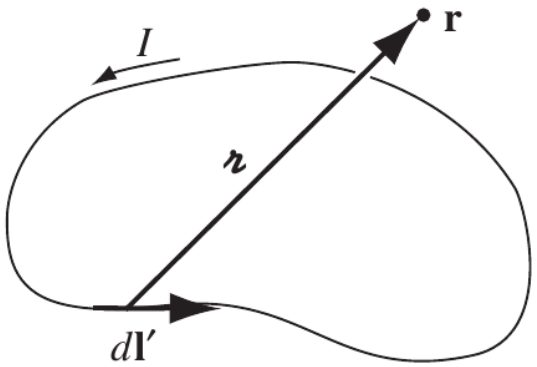
Eqn. of continuity

total current crossing a surface $\mathcal{S}$ enclosing a volume $\mathcal{V}$ is $\int\limits_{\mathcal{S}} \vec{J}.\vec{da} = \int\limits_{\mathcal{V}} \vec{\nabla}.\vec{J} d\tau$

$\because$ charge is conserved, whatever flows out through surface must come from inside

$$\therefore \int\limits_{\mathcal{V}} \vec{\nabla}.\vec{J} d\tau = -\frac{d}{dt} \int\limits_{\mathcal{V}} \rho d\tau = -\int\limits_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau \implies \boxed{\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t}}$$

MAGNETIC INDUCTION $\vec{B}$



Biot-Savart law: In a circuit carrying a current I magnetic induction at point P

$\vec{B} = \frac{\mu_0}{4\pi} \oint \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} dl = \frac{\mu_0}{4\pi} I \oint \frac{\vec{dl} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ teslas (kg/s^{2}A)

permeability of free space $\mu_0 = 4\pi \times 10^{-7}$tesla.m/amp

For surface and volume currents

$$\vec{B} = \frac{\mu_0}{4\pi} \oint \frac{\vec{K}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} da' \text{ and } \frac{\mu_0}{4\pi} \oint \frac{\vec{J}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

Note : There is no steady current for a point charge, $\therefore \vec{B} \neq \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$

electrostatics

Coulomb

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r'})\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

For surface distribution of charges

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r'})\hat{\mathscr{r}}}{\mathscr{r}^2}da'$$

magnetostatics

Biot-Savart

$$\vec{B} = \frac{\mu_0}{4\pi}\oint \frac{\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}dl = \frac{\mu_0}{4\pi}I\oint \frac{\vec{dl}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

For surface and volume currents,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r})\times\hat{r_s}}{r_s^2}da'$$ and

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r})\times\hat{r_s}}{r_s^2}d\tau'$$

When electric field produced by a charge distribution is disturbed by introduction of finite test charge Q

$\vec{E}$ = force per unit charge as magnitude of test charge $\rightarrow 0$: $\vec{E} = \lim_{Q' \rightarrow 0} \frac{\vec{F}}{Q}$

ELECTRIC POTENTIAL :

Consider test point charge Q that can be moved about in an electric field

work required to move it from point P_1 to a point P_2 along a given path

$$W = -\int_{P_1}^{P_2} \vec{E} Q . d\vec{l}$$

Note : negative sign $\implies$ work done against the field.

For closed path, total work $W = -\oint \vec{E}\, Q.d\vec{l}$

For single point charge Q

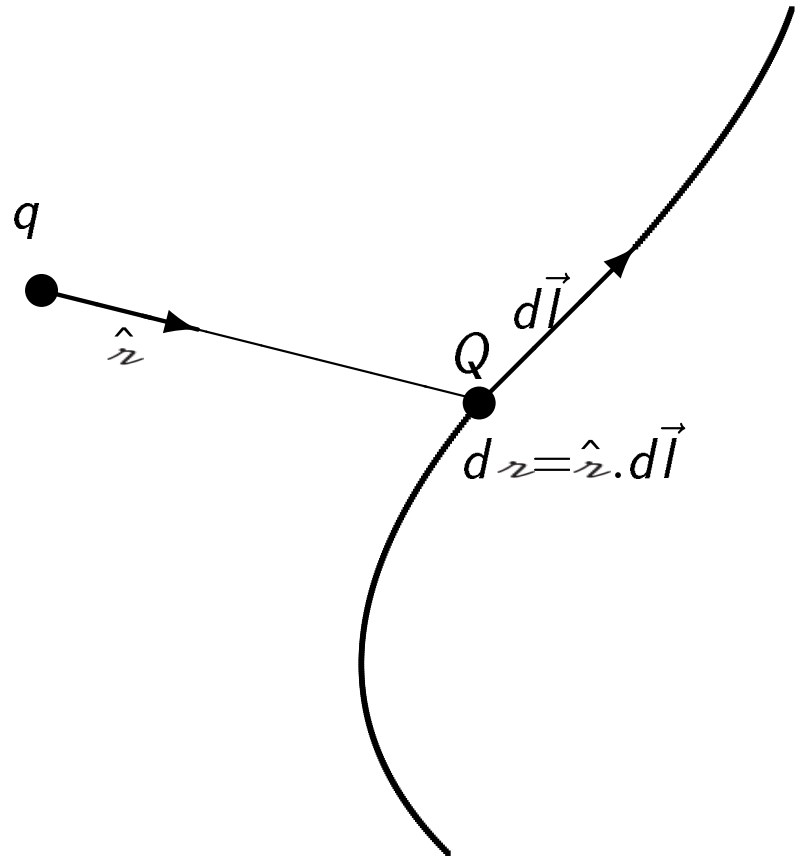
$$\oint \vec{E}\, Q.d\vec{l} = \frac{qQ}{4\pi\varepsilon_0} \oint \frac{\hat{r}.d\vec{l}}{r^2}$$

$$= \frac{qQ}{4\pi\varepsilon_0} \oint \frac{dr}{r^2} = -\frac{qQ}{4\pi\varepsilon_0} \oint d\left(\frac{1}{r}\right)$$

$\oint d\left(\frac{1}{r}\right) = 0$, since r has same value at the beginning and at the end of the path

$\therefore$ net work done in moving point charge Q around a closed path in the field of fixed point charge q is zero

For a fixed charge distribution line integrals corresponding to each individual charge are zero



$\therefore$ for any distribution of fixed charges, $\boxed{\oint \vec{E}.d\vec{l} = 0}$

$\oint \vec{E}.d\vec{l} = 0$

$\therefore$ An electrostatic field is conservative

- follows from the fact that Coulomb force is a central force

Stokes's theorem : $\oint \vec{E}.d\vec{l} = \int\limits_{\text{enclosed surface}} \left(\vec{\nabla} \times \vec{E}\right).\overrightarrow{da} = 0$

$$\therefore \boxed{\vec{\nabla} \times \vec{E} = 0} \implies \boxed{\vec{E} = -\vec{\nabla} V}$$ where V is a scalar function

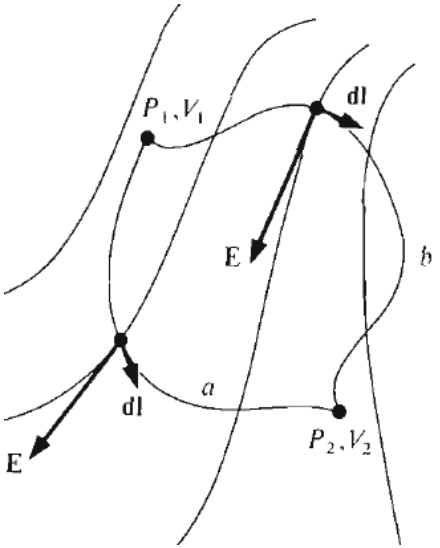
$\therefore$ electrostatic field is completely described by means of a function $V(x, y, z)$

$\boxed{\text{electric potential}}$

negative sign : electric field intensity $\vec{E}$ points toward decrease in potential

Note : V not uniquely defined; we can add to it any quantity independent of coordinates without affecting $\vec{E}$

work done in moving test charge from point P_1 to point P_2 is independent of path



$$\vec{E}.d\vec{l} = -\vec{\nabla}V.d\vec{l} = -dV$$

$$V_2 - V_1 = -\int_1^2 \vec{E}.d\vec{l} = \int_2^1 \vec{E}.d\vec{l}$$

Convention : potential at infinity$= 0$

$\therefore$ potential V at point 2 is $\int_2^\infty \vec{E}.d\vec{l}$

If charges are moving, $\vec{\nabla} \times \vec{E}$ is not necessarily zero

and $\vec{\nabla}V$ then describes only part of the electric field intensity $\vec{E}$

work required to bring charge Q from point at which the potential is defined as zero to the point considered, $W = VQ$

$\therefore V = \frac{W}{Q}$, work per unit charge, joules per coulomb$\equiv$ volts

- Potential and potenital energy are different
- If you know V, you can easily get $\vec{E} = -\vec{\nabla} V$
- potential formulation reduces a vector problem to a scalar one : no need to fuss with components
- reference : typically, potential at infinity$= 0$
- Potential obeys the superposition principle
- Units of Potential : Nm/C

Equipotential surface and lines of force near a point charge

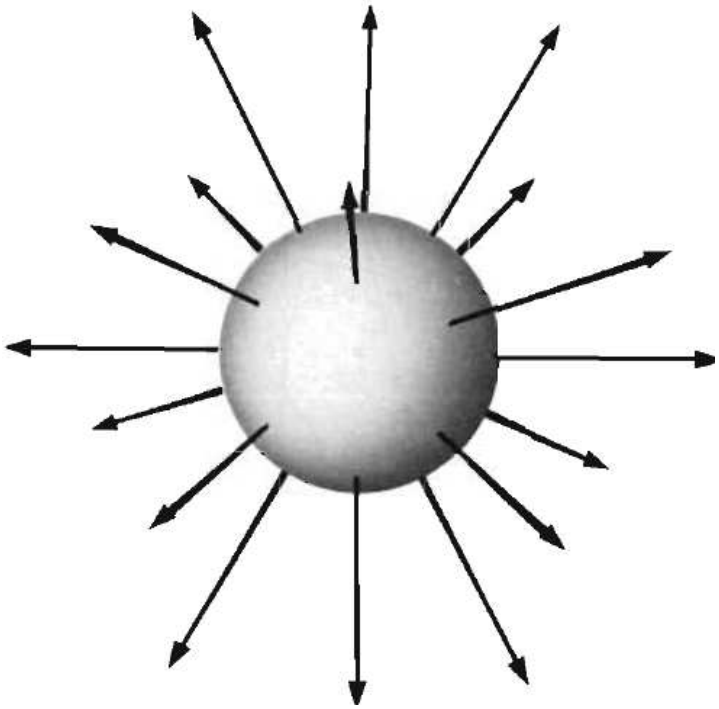
For field produced by a single point charge q located at $\vec{r}'$, potential at distance $\mathscr{r}$

$$V(\vec{r}) = \int_{\mathscr{r}}^{\infty} \frac{q}{4\pi\varepsilon_0} \frac{dr'}{\mathscr{r}^2} = \frac{q}{4\pi\varepsilon_0 \mathscr{r}}$$

sign of potential V is same as of q

principle of superposition
$\implies$ potential V at point P due to charge distribution of density ρ is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r'})d\tau'}{\mathscr{r}}$$



equipotential surface about a point charge

join end-to-end infinitesimal vectors representing $\vec{E}$
$\implies$ a curve in space— line of force

— everywhere normal to equipotential surfaces, $\vec{E} = -\vec{\nabla} V$

$\vec{E}$ is everywhere tangent to a line of force

V and $\vec{E}$ at distance r from ELECTRIC DIPOLE of charges separated by distance l

$r \gg l$

For field produced by a single point charge Q, potential at a distance r

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

$$r_a = r\sqrt{1+\left(\frac{l}{2r}\right)^2 + \frac{l}{r}\cos\theta} \approx r\sqrt{1+\frac{l}{r}\cos\theta}$$

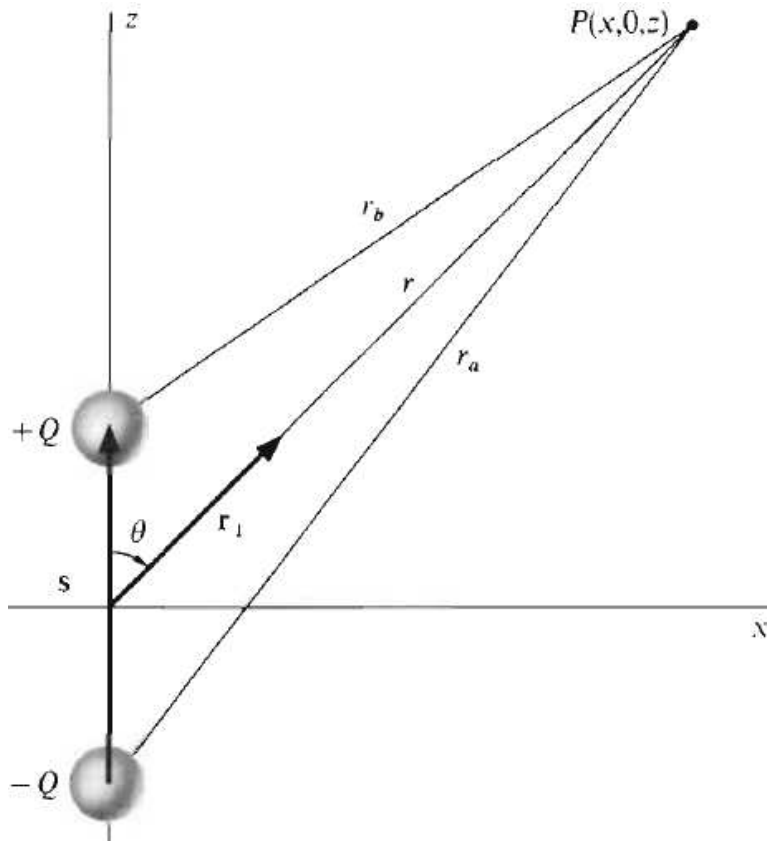
$$\approx r\left(1 + \frac{l}{2r}\cos\theta\right)$$

and similarly for r_b

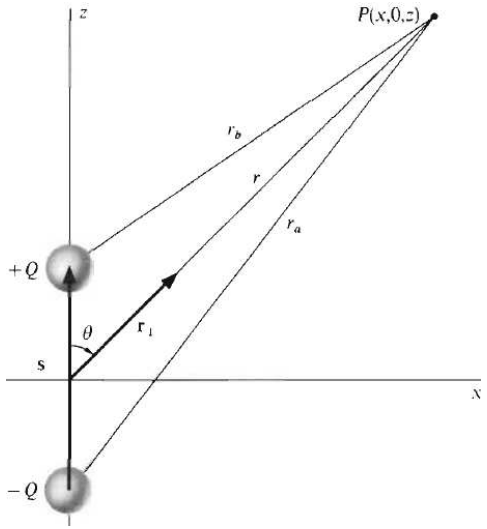
$$\therefore V = \frac{Q}{4\pi\varepsilon_0 r}\left(\frac{1}{1-\frac{l}{2r}\cos\theta} - \frac{1}{1+\frac{l}{2r}\cos\theta}\right)$$

$$= \frac{Ql}{4\pi\varepsilon_0 r^2}\cos\theta = \frac{\vec{\mu}.\hat{r}}{4\pi\varepsilon_0 r^2}$$

where $\vec{\mu} = Q\vec{l}$ =dipole moment



Lines of force (arrows) and equipotential lines for a dipole



at point P$(x, 0, z)$ in the plane $y = 0$

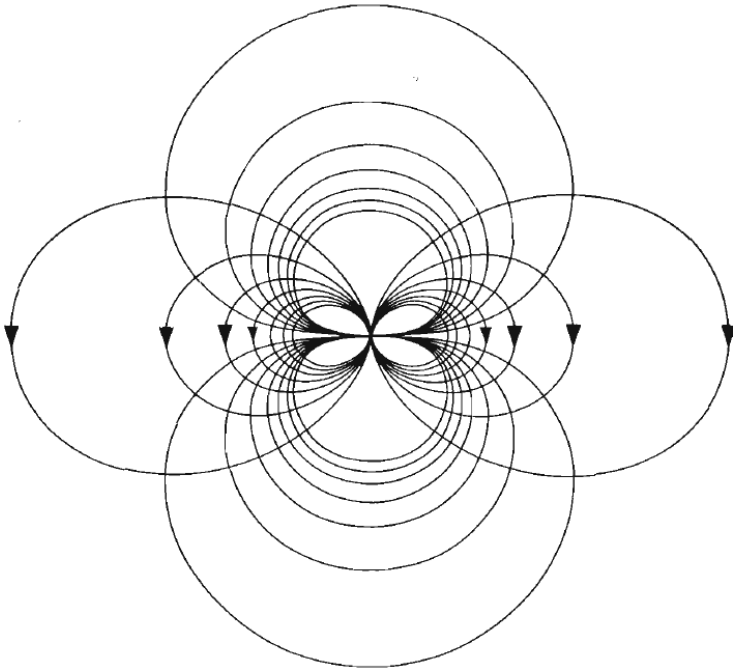
$$V = \frac{\mu}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} = \frac{\mu}{4\pi\varepsilon_0}\frac{z}{r^3}$$

and $E_x = -\frac{\partial V}{\partial x} = \frac{3\mu}{4\pi\varepsilon_0}\frac{xz}{r^5} = \frac{3\mu}{4\pi\varepsilon_0}\frac{\sin\theta\cos\theta\cos\phi}{r^3}$

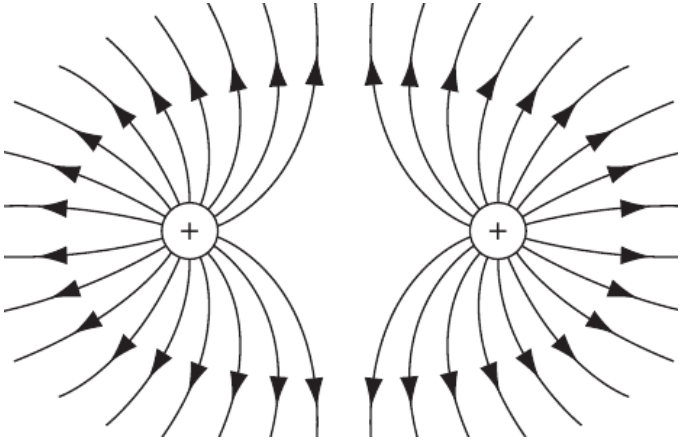
$$E_y = -\frac{\partial V}{\partial y} = \frac{3\mu}{4\pi\varepsilon_0}\frac{yz}{r^5} = 0$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\mu}{4\pi\varepsilon_0}\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right)$$

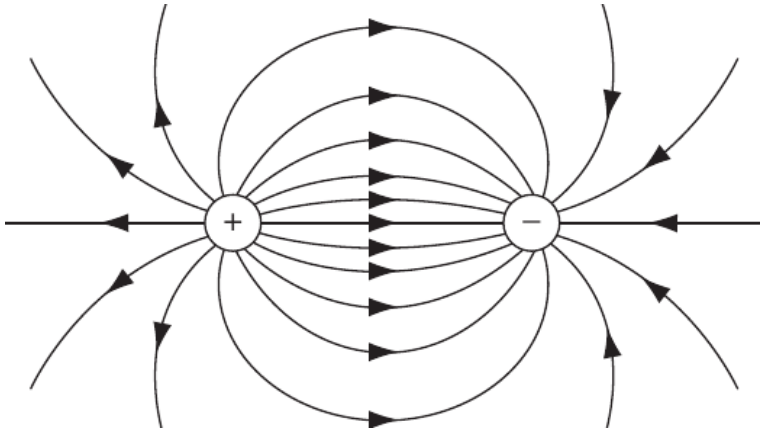
$$= \frac{\mu}{4\pi\varepsilon_0}\frac{3\cos^2\theta - 1}{r^3}$$



More generally

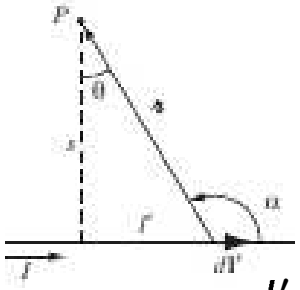


equal charges



opposite charges

Magnetic induction ex. : Long straight wire carrying a current



$$d\vec{l}' \times \hat{\mathscr{r}} = dl' \sin\alpha = dl' \cos\theta$$

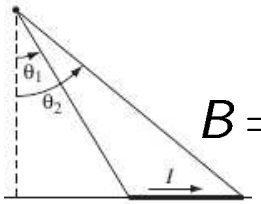
pointing outwards

$$l' = s\tan\theta \implies dl' = \frac{s}{\cos^2\theta} d\theta$$

$$\text{and } s = \mathscr{r}\cos\theta \implies \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

Lines of $\vec{B}$ in plane perpendicular to wire

For a line segment,



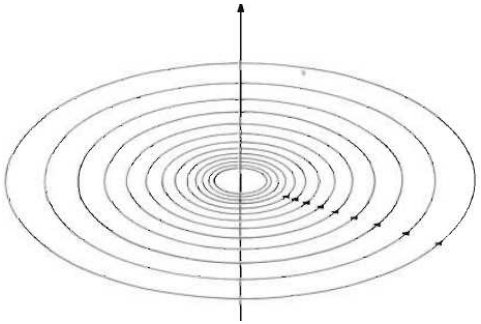
$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\cos^2\theta}{s^2} \frac{s}{\cos^2\theta} \cos\theta d\theta$$

$$= \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_1 - \sin\theta_2)$$

For an infinite wire,

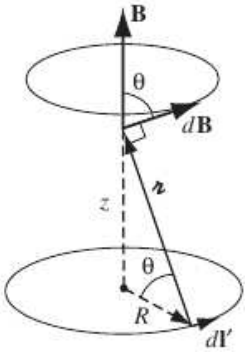
$$\therefore \vec{B} = \hat{\phi}\frac{\mu_0 I}{4\pi s} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta = \hat{\phi}\frac{\mu_0 I}{2\pi s}$$

Lines of $\vec{B}$ in plane perpendicular to wire

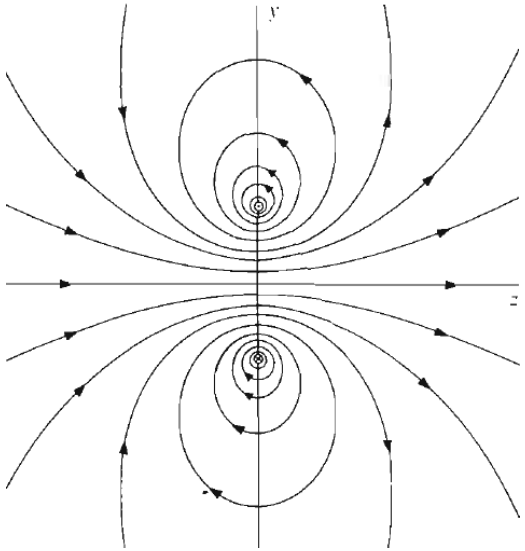


density of lines $\propto$ 1/distance to wire(s)

Magnetic induction ex. : Circular loop. magnetic dipole moment $\vec{m}$



Lines of $\vec{B}$ in plane containing axis of the loop



current element $Id\vec{l}'$ produces induction $d\vec{B}$

integrating around the loop, $d\vec{B}$ sweeps out a cone

horizontal components cancel, and vertical components combine

$$B_z = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{r^2} \cos\theta$$

By symmetry, the total $\vec{B}$ is along z-axis

$$B_z \overset{\int dl' = 2\pi R}{=} \frac{\mu_0 I}{4\pi} \frac{2\pi a}{r^2} \cos\theta = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{\frac{3}{2}}}$$

Far from loop, field same as that of electric dipole with $\frac{1}{4\pi\varepsilon_0}$ replaced by $\frac{\mu_0}{4\pi}$

magnetic dipole moment $\vec{m}$ is a vector of magnitude $\pi R^2 I$ and normal to plane of the loop

To what extent is a vector field determined by its divergence and curl?
if $\vec{\nabla}.\vec{F} = D$ and $\vec{\nabla} \times \vec{F} = \vec{C}$ with $\vec{\nabla}.\vec{C} = 0$, can we find $\vec{F}$?

Helmholtz Theorem
To solve a differential equation we need boundary conditions
typically fields $\rightarrow 0$ at infinity (far away from all charges)
With that extra information, Helmholtz theorem guarantees that the field is uniquely determined by its divergence and curl

Potentials : If $\vec{\nabla} \times \vec{F} = 0$, then $\vec{F} = -\vec{\nabla} V$

Curl-less ('irrotational') fields : following conditions are equivalent :

- $\vec{\nabla} \times \vec{F} = 0$ everywhere
- $\int_a^b \vec{F}.d\vec{l}$ is independent of path, for any given end points
- $\oint \vec{F}.d\vec{l} = 0$ for any closed loop
- $\vec{F} = -\vec{\nabla} V$

Divergence-less (“solenoidal”) fields : following conditions are equivalent :

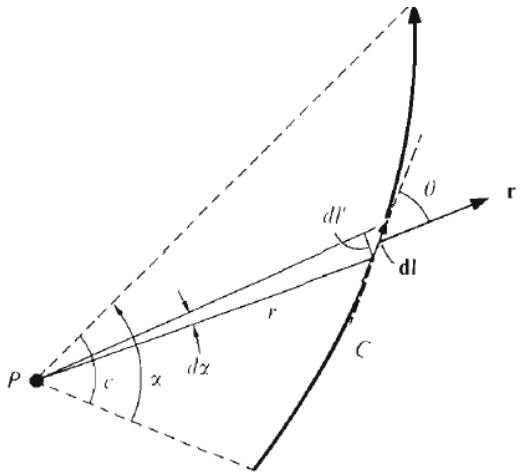
- $\vec{\nabla}.\vec{F} = 0$ everywhere
- $\int \vec{F}.d\vec{a}$ is independent of surface, for any given end points
- $\oint \vec{F}.d\vec{a} = 0$ for any closed surface
- $\vec{F} = \vec{\nabla} \times \vec{A}$ - curl of some vector function

vector potential is not unique
— gradient of any scalar function can be added to $\vec{A}$ without affecting the curl since the curl of a gradient is zero

in all cases (whatever curl and divergence may be) a vector field $\vec{F}$ can be written as the gradient of a scalar plus the curl of a vector:
$\vec{F} = -\vec{\nabla} V + \vec{\nabla} \times \vec{A}$

Angle subtended by a curve at a point

$$\alpha = \int_C \frac{dl'}{r} = \int_C \frac{dl \sin\theta}{r} \text{ radians}$$



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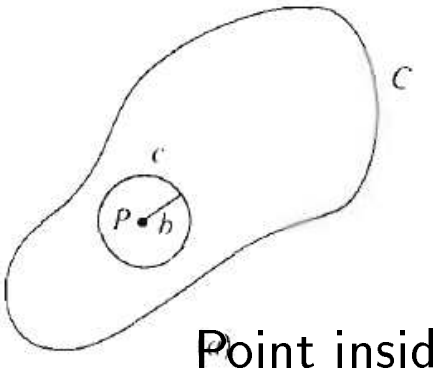
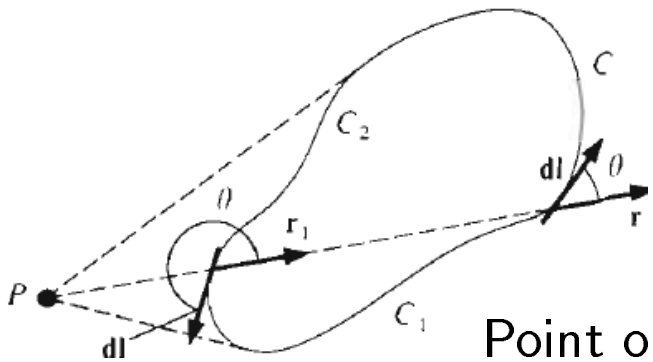
Angle subtended by a **closed** curve



Point inside

$$\alpha = \int_C \frac{dl'}{r} = \frac{2\pi b}{b} = 2\pi \text{ radians}$$

.

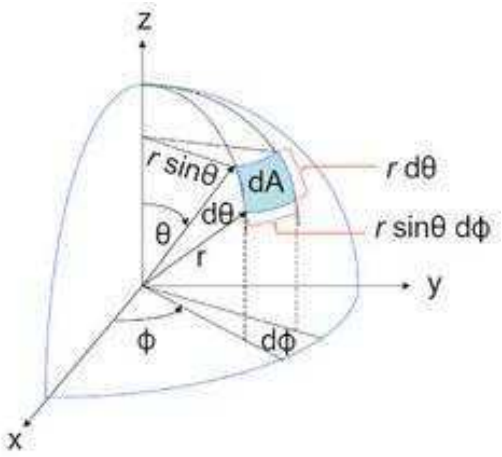


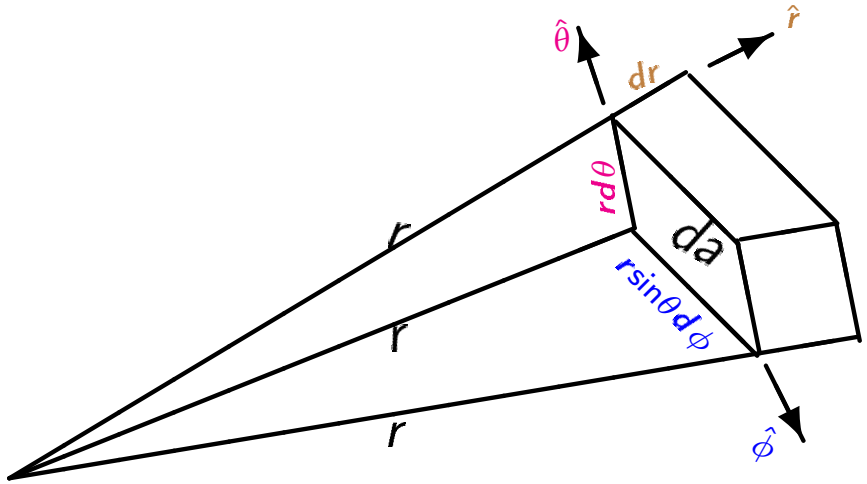
Point outside

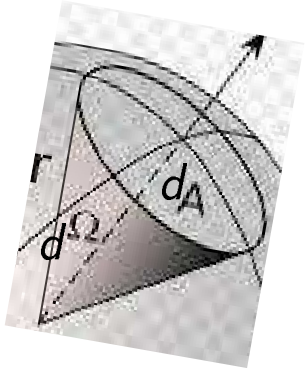
$$\alpha = \int_C \frac{dl'}{r} = \int_{C_1} \frac{dl \sin\theta}{r} + \int_{C_2} \frac{dl \sin\theta}{r} = 0$$

- terms equal in magnitude (same angle) and opposite in sign ($\sin\theta > 0 \,\forall\, C_1$ and $\sin\theta < 0 \,\forall\, C_2$)

solid angle







$d\tau = dr.rd\theta.r\sin\theta d\phi \qquad da = \frac{d\tau}{dr} = r^2 d\Omega; \qquad d\Omega = \sin\theta d\theta d\phi$

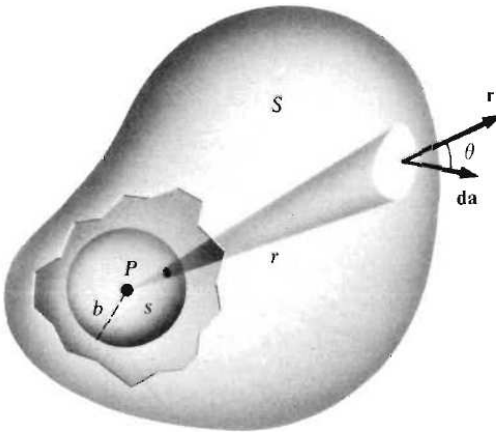
Defn. : $d\Omega = \frac{\hat{r}.\vec{da}}{r^2}$ steradians; $\int\limits_{\text{angular space}} d\Omega = \int\limits_0^{2\pi} d\phi \int\limits_0^{\pi} \sin\theta d\theta = 2\pi \int\limits_{-1}^{1} d(\cos\theta) = 2\pi.2 = 4\pi$

$\hat{r}.\vec{da} = da\cos\theta$, where θ = angle between radius vector $\vec{r}$ and vector $\vec{da}$, normal to surface and pointing outward (this θ is not the polar angle as in expression for $d\tau$)

Solid angle subtended by a closed surface at a point

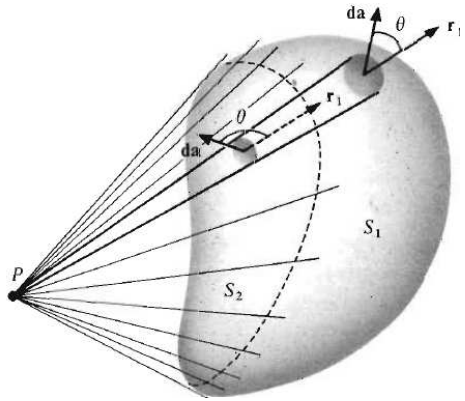
$$d\Omega = \frac{da\cos\theta}{r^2} = \frac{\hat{r}.\vec{da}}{r^2}$$

Point inside closed surface



$$\Omega = \int_S \frac{da\cos\theta}{r^2} = \frac{4\pi a^2}{a^2} = 4\pi \text{ steradians}$$

Point outside closed surface



$$\Omega = \int_S \frac{da\cos\theta}{r^2} = \int_{S_1} \frac{da\cos\theta}{r^2} + \int_{S_2} \frac{da\cos\theta}{r^2} = 0$$

- the two terms are equal in magnitude (same angle) but opposite in sign ($\cos\theta > 0 \,\forall\, S_1$ and $\cos\theta < 0 \,\forall\, S_2$)

situation remains unchanged if the surface is convoluted in such a way that a line drawn from P cuts the surface at more than one point

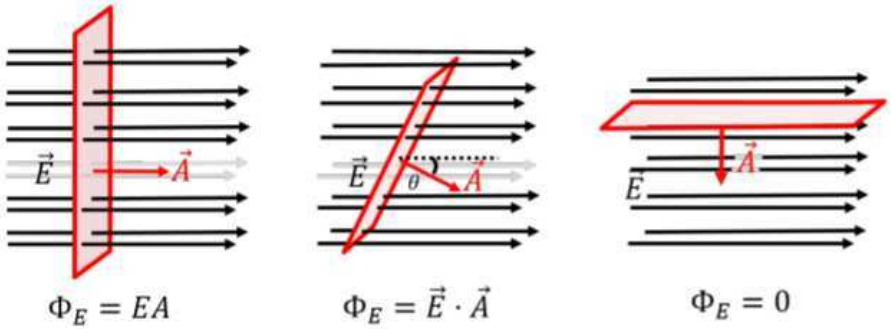
total solid angle subtended by closed surface$= 4\pi$ at a point inside

0 at a point outside

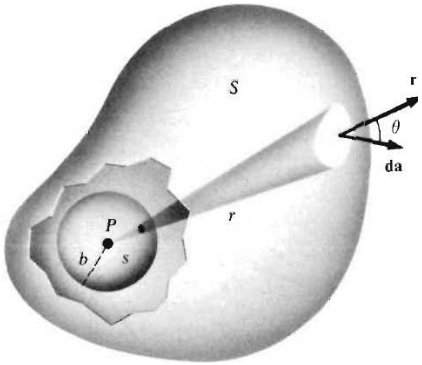
GAUSS'S LAW

relates flux of $\vec{E}$ through a closed surface to total charge enclosed within it

Flux :
measure of number of field lines from a vector field that cross a given surface



Take a point charge q located at P inside a closed surface



flux of $\vec{E}$ through element of area $\vec{da}$,

$\vec{E}.\overrightarrow{da} = \frac{q}{4\pi\varepsilon_0 r^2} r^2 d\Omega = \frac{q}{4\pi\varepsilon_0} d\Omega$, where Ω = solid angle subtended by $d\vec{a}$ at point P

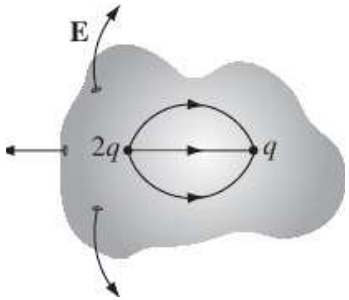
total flux across the surface

$$= \int_S \vec{E}.d\vec{a} = \frac{q}{4\pi\varepsilon_0} \int_S d\Omega = \frac{q}{4\pi\varepsilon_0} 4\pi = \frac{q}{\varepsilon_0}$$

$$\boxed{\int_S \vec{E}.d\vec{a} = \frac{q}{\varepsilon_0}}$$

flux through any surface enclosing charge q is $\frac{q}{\varepsilon_0}$

field lines originating on a positive charge either pass through the surface or else terminate on a negative charge inside



charge outside the surface contributes nothing to total flux, since its field lines pass in one side and out the other

This is essence of Gauss's law

$$\therefore \int \vec{E}.d\vec{a} = \int \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r^2} \hat{r}_E . r^2 d\Omega . \hat{n}$$

Notice : surface area goes up as r^2 and field intensity goes down as $\frac{1}{r^2}$

$\therefore$ radial variable cancels out - product is constant

same number of field lines pass through any sphere centered at the origin, regardless of its size

If more than one point charge resides within S, fluxes add algebraically

Total flux leaving the volume $= \frac{\text{total enclosed charge}}{\varepsilon_0}$

If charge enclosed by surface S is distributed over a finite volume
total enclosed charge $Q = \int \rho d\tau$
where $\rho =$charge density and integral is over volume enclosed by the surface S

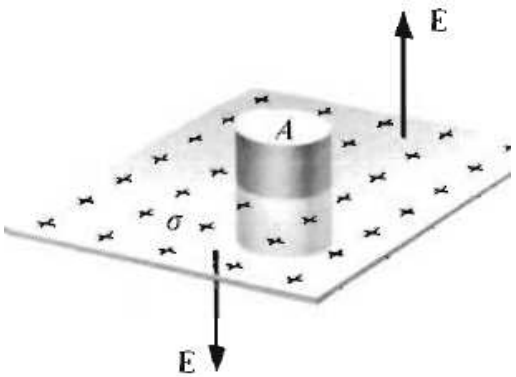
$\therefore \int\limits_S \vec{E}.\overrightarrow{da} = \frac{1}{\varepsilon_0} \int \rho d\tau \Leftarrow$ Gauss's law in integral form

$\therefore \int_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0}\int \rho d\tau \Leftarrow$ Gauss's law in integral form

Apply divergence theorem :

$\int \vec{\nabla}.\vec{E}\, d\tau = \frac{1}{\varepsilon_0}\int \rho d\tau \implies \boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}} \Leftarrow$Gauss's law in differential form

Ex. : infinite sheet of charge,



charge density $=\sigma$ C/m^2

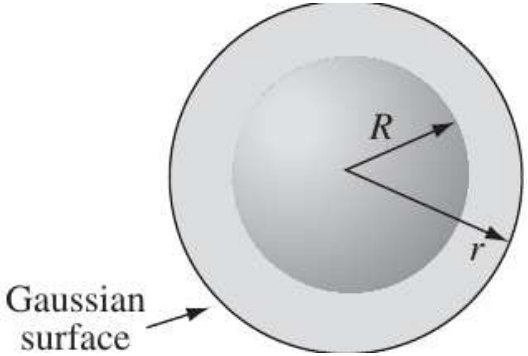
imaginary box of cross-section A encloses charge σA

flux of $\vec{E}$ leaving the box $=\int_A \vec{E}.\vec{da} = 2EA$

$\therefore 2EA = \frac{\sigma A}{\varepsilon_0}$ or, $E = \frac{\sigma}{2\varepsilon_0}$ on either side

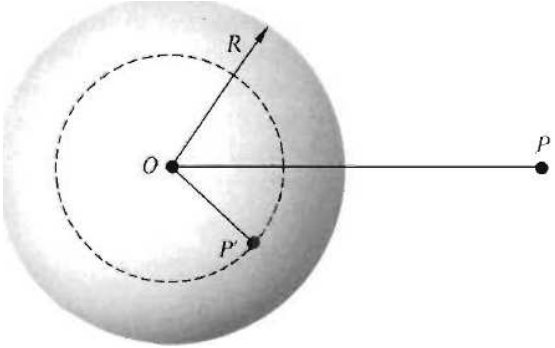
or, $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

Ex. : SPHERICAL CHARGE : field inside and outside sphere of radius R
charge density $=\rho$ C/m^3; total charge $Q = \frac{4}{3}\pi R^3 \rho$ C



imagine sphere of radius $r > R$
concentric with charged sphere
$\vec{E}$ must be radial
$\therefore$ using integral form of Gauss's law
$\int \vec{E}.\vec{da} = 4\pi r^2 E_{\text{out}} = \frac{Q}{\varepsilon_0} \Longrightarrow \vec{E}_{\text{out}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$
as if the total charge Q were situated
at the center of the sphere

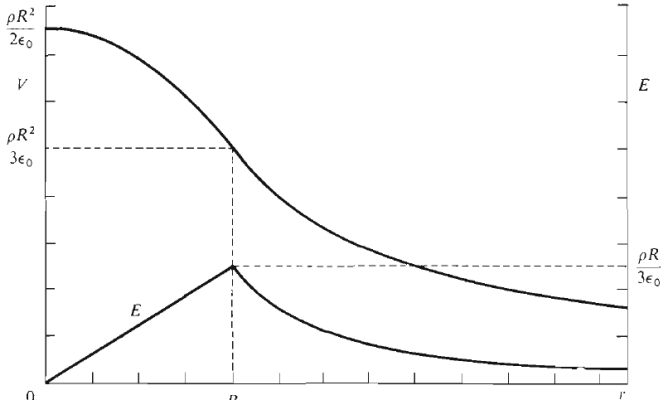
at an internal point, draw imaginary
sphere of radius $r < R$



Symmetry requires that $\vec{E}$ be radial

$\therefore 4\pi r^2 E_{\text{in}} = \frac{4}{3}\pi r^3 \frac{\rho}{\varepsilon_0} \Longrightarrow \vec{E}_{\text{in}} = \frac{\rho r}{3\varepsilon_0}\hat{r}$

uniform spherical charge distribution :

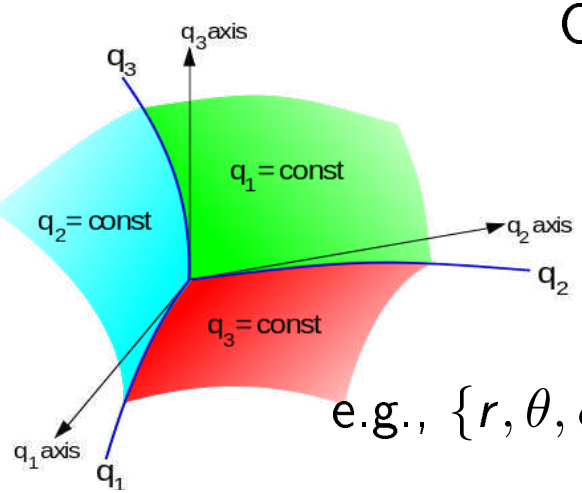


electric field intensity E rises linearly from the center to the surface of sphere and falls off as the inverse square of the distance outside the sphere

potential falls from a maximum at center in parabolic fashion inside the sphere, and as the inverse first power of the distance outside

Note $E_{\text{in}} = E_{\text{out}}$ and $V_{\text{in}} = V_{\text{out}}$ at $r = R$

mathematical digression - curvilinear coordinates q_1, q_2, q_3



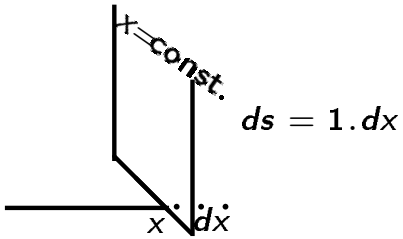
e.g., $\{r, \theta, \phi\}$

Consider differential increments $ds_i \perp$ surface q_i =constant

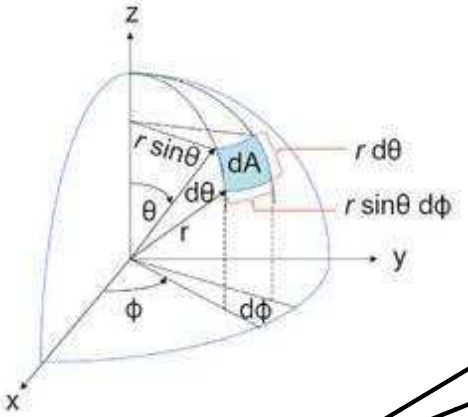
increments are related to dq_i as $ds_i = h_i.dq_i$

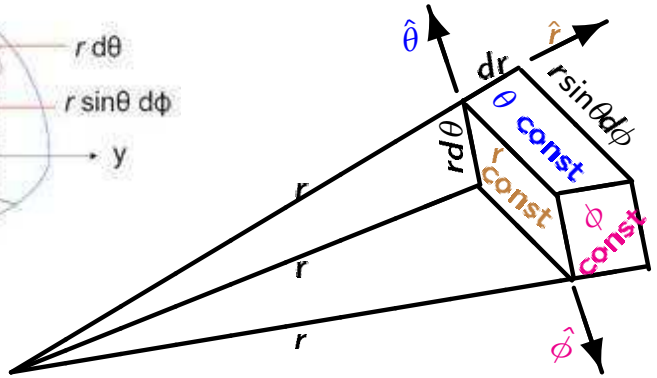
gradient : $\vec{\nabla} f = \hat{q}_1 \frac{1}{h_1}\frac{\partial f}{\partial q_1} + \hat{q}_2 \frac{1}{h_2}\frac{\partial f}{\partial q_2} + \hat{q}_3 \frac{1}{h_3}\frac{\partial f}{\partial q_3}$

Cartesian :, $ds_i = 1.dx_i$; $\therefore h_i = 1 \quad \forall i = 1, 2, 3$



$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$





$ds_1 = 1.dr$; $ds_2 = rd\theta$; $ds_3 = r\sin\theta d\phi$

$\therefore h_1 = 1$; $h_2 = r$; $h_3 = r\sin\theta$

$$\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

spherical polar

Divergence $\vec{\nabla}.\vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_3 h_1 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right]$

Cartesian : $\vec{\nabla}.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

spherical polar :
$h_1 = 1; h_2 = r$
$h_3 = r\sin\theta$

$$\vec{\nabla}.\vec{V} = \frac{1}{1.r.r\sin\theta}\left[\frac{\partial}{\partial r}(r.r\sin\theta V_r) + \frac{\partial}{\partial\theta}(r\sin\theta.1.V) + \frac{\partial}{\partial\phi}(1.rV_\phi)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta V_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}V_\phi$$

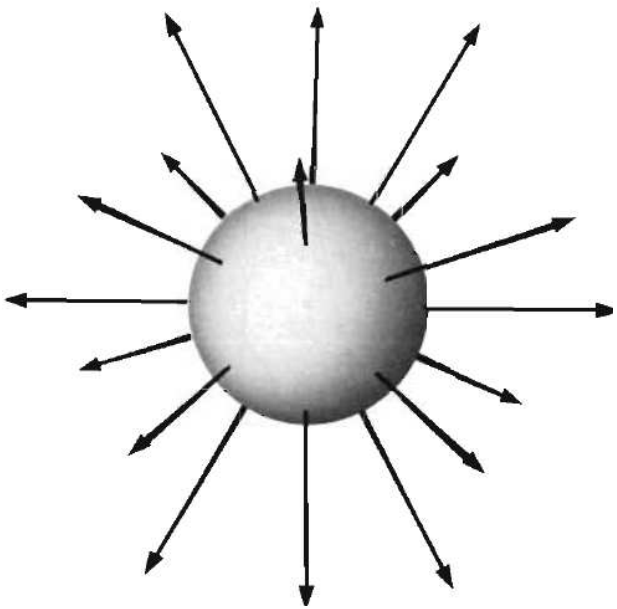
Laplacian $\nabla^2 f = \vec{\nabla}.\vec{\nabla} f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$

Cartesian : $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

spherical polar : $\nabla^2 f = \frac{1}{1.r.r\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{r.r\sin\theta}{1}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta.1}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1.r}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\right]$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

consider vector $\vec{V} = \frac{\hat{r}}{r^2}$
directed radially outward
with high divergence



for $r \neq 0$, $\vec{\nabla}.\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$

at $r = 0$, we have a discontinuity

Suppose we integrate over a sphere of radius R,
surface integral $\oint \vec{V}.\vec{da} = \oint\limits_{r=R} \frac{\hat{r}}{r^2}.r^2 d\Omega \hat{r} = \int d\Omega = 4\pi$

By Divergence theorem : $\oint \vec{V}.\vec{da} = \int \vec{\nabla}.\vec{V} d\tau$

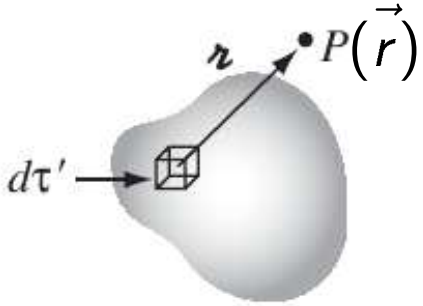
This is explained using $\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$, where $\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$

and $\delta(x)$ is defined as 0 everywhere else except at $x = 0$, s.t. $\int\limits_{-\infty}^{\infty} \delta(x) = 1$

$\delta(x) \Leftarrow$ Dirac delta function

$\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$

Alternate derivation of Gauss's law :



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r'})\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau';$$

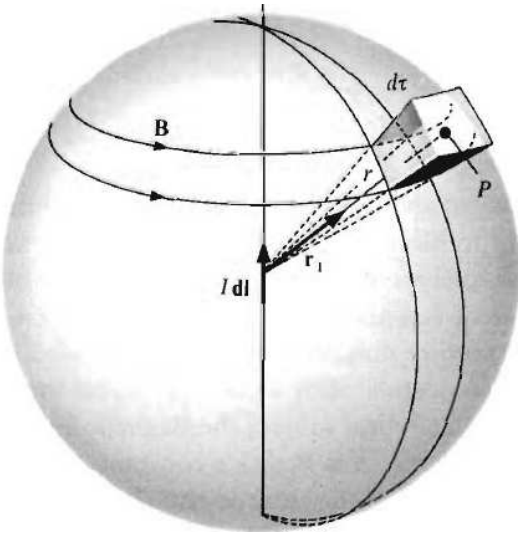
$$\vec{\nabla}.\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \vec{\nabla}.\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\rho(\vec{r'})d\tau' = \frac{4\pi}{4\pi\varepsilon_0}\int \delta^3(\vec{\mathscr{r}})\rho(\vec{r'})d\tau'$$

.

or, $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$

Divergence of $\vec{B}$

Consider current element $I\overrightarrow{dl}$ and volume $d\tau$ containing point P



magnetic induction at P, $d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\overrightarrow{l} \times \hat{r}}{r^2}$

lines of $\vec{B}$ due to element $Id\overrightarrow{l}$ are circles situated in planes perpendicular to a line through $d\overrightarrow{l}$ and centered on it

net outward flux of $\vec{B}$ through surface of volume $d\tau$ is 0

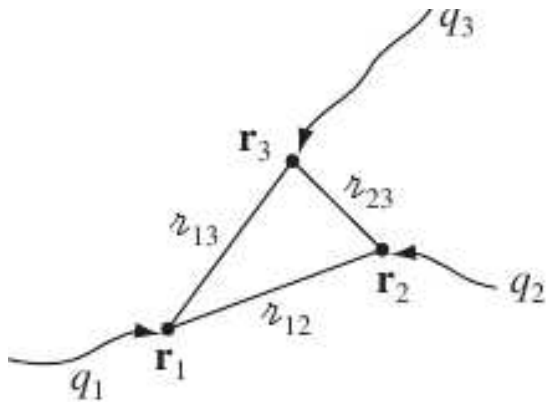
for volume bounded by surface S, $\int_S \vec{B}.\overrightarrow{da} = 0 = \int \vec{\nabla}.\vec{B} d\tau$

$\therefore \vec{\nabla}.\vec{B} = 0$: (no magnetic monopole)
magnetic fields are due solely to electric currents : magnetic 'charges' do not exist

magnetic monopoles : existence postulated by Dirac: 1931 - never observed
theoretical value of the magnetic monopole $= 2h/e$, or 8.27117×10^{-15} weber

Energy of a Point Charge Distribution

How much work would it take to assemble an entire collection of point charges?



bring in the charges, one by one, from far away

sequentially as we bring q_2, $q_3, \cdots$ close to an already placed charge q_1 at $\vec{r}_1$, each experiencing interactions , say, $q_j V_{ij}$,

work at each step, $W_2 = q_2 V_{12}$;

$W_2 = q_3 \left(V_{13} + V_{23}\right)$ and so on...

total work :

$$W = \sum_{i=1}^{n} \sum_{j>i}^{n} q_i V_{ij} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{\mathscr{r}_{ij}}$$

the last sum = potential due to all charges $j \neq i$ at where q_i is located, $\vec{r}_i$

$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$ = work it takes to assemble a configuration of point charges

= amount of work we get back if we dismantle the system

it represents energy stored in the configuration

Energy of a Continuous Charge Distribution

$W = \frac{1}{2}\int \rho V d\tau$

use Gauss's law : $W = \frac{\varepsilon_0}{2}\int \left(\vec{\nabla}.\vec{E}\right) V d\tau$

$$= \frac{\varepsilon_0}{2}\left[-\int \vec{E}.\vec{\nabla} V d\tau + \oint V\vec{E}.d\vec{a}\right] = \frac{\varepsilon_0}{2}\left(\int E^2 d\tau + \oint_S V\vec{E}.d\vec{a}\right)$$

integrate over the region where charge is located
any larger volume would do just as well:
'extra' territory contributes nothing to the integral, since $\rho = 0$ there.

As we enlarge volume beyond the minimum necessary, $\int E^2 d\tau$ increases ($E^2 > 0$);
evidently surface integral must decrease correspondingly to leave the sum intact.
At large distances, E goes like $\frac{1}{r^2}$ and V like $\frac{1}{r}$
while surface area grows like r^2
$\Longrightarrow$ surface integral goes down $\sim \frac{1}{r}$
integrate over all space $\Longrightarrow$ surface integral$\rightarrow 0$, and $W = \frac{\varepsilon_0}{2}\int E^2 d\tau$

perplexing "inconsistency" : $W = \frac{\varepsilon_0}{2} \int E^2 d\tau > 0$ but $\frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$ can be < 0

both correct, but they speak to different questions

$\sum_{i=1}^{n} q_i V(\vec{r}_i)$ does not include work necessary to make the point charges

in fact, energy of point charge is infinite: $W = \frac{\varepsilon_0}{2(4\pi\varepsilon_0^2)} \int \left(\frac{q^2}{r^4}\right) r^2 dr d\Omega = \frac{q^2}{8\pi\varepsilon_0} \int_0^\infty \frac{1}{r^2} dr = \infty$

$W = \frac{\varepsilon_0}{2} \int E^2 d\tau$ is complete: total energy stored in a charge configuration

$\frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$ is appropriate for point charges, because we leave out portion of total energy attributable to creating the point charges themselves

$\frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$ represents potential due to all other charges but not q_i
whereas $\frac{1}{2} \int \rho V d\tau$ is full potential

For continuous distribution, there is no distinction, since the amount of charge at point $\vec{r}$ is vanishingly small, and its contribution to potential is zero

Where is the energy stored?

in case of spherical shell $\Longrightarrow$ charge confined to surface
whereas electric field is everywhere outside this surface
unanswerable question

In radiation theory it is useful and in general relativity it is essential
to regard energy as stored in the field, with a density $\frac{\varepsilon_0}{2}E^2$ =energy/vol.

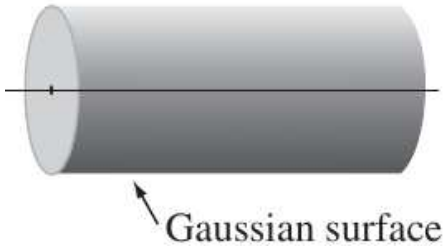
If ρ is not uniform (or, not spherically symmetrical)
or if some other shape is chosen as Gaussian surface

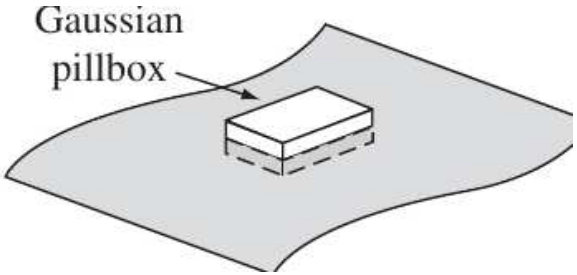
flux of $\vec{E}$ is still $\frac{q}{\varepsilon_0}$

but $\vec{E}$ does not point in the same direction as $\overrightarrow{da}$

and its magnitude is not constant over the surface

and without that $|\vec{E}|$ outside of the integral cannot be obtained





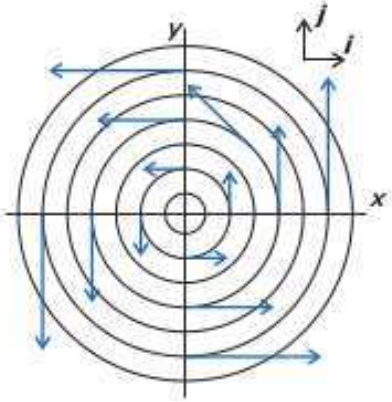
Symmetry is crucial to application of Gauss's law

1. Spherical symmetry : Gaussian surface a concentric sphere
2. Cylindrical symmetry : Gaussian surface a coaxial cylinder
3. Plane symmetry : Gaussian "pillbox" that straddles the surface

VECTOR POTENTIAL $\vec{A}$

$\because \vec{\nabla}.\vec{B} = 0$, it is reasonable to assume that $\exists\ \vec{A}$, s.t. $\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}$
because the divergence of a curl is identically equal to zero

Consider vectors in $x-y$ plane:
$\vec{W} = -y\hat{i} + x\hat{j}$



circulation$\neq 0$; curl: $\vec{\nabla} \times \vec{W} = 2\hat{k}$
(circulation vector $\perp$ plane)

Suppose $\overrightarrow{A} = \frac{1}{2}\mathcal{B}\vec{W} = \frac{1}{2}\mathcal{B}\left(-y\hat{i} + x\hat{j}\right)$

$\overrightarrow{\nabla} \times \frac{1}{2}\mathcal{B}\vec{W} = \mathcal{B}\hat{k} = \overrightarrow{B}$

$\therefore$ magnetic induction $\overrightarrow{B}$ in z-direction $\Longrightarrow$
corresponding vector potential $\overrightarrow{A} = \frac{1}{2}\overrightarrow{B} \times \vec{r}$
s.t.

$$\boxed{\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}}$$

flux outward in radial direction$= 0$
or, $\quad \vec{\nabla}.\vec{W} = 0$

vector potential $\frac{1}{2}\mathcal{B}\vec{W}$ in $x-y$ plane describes uniform magnetic field of induction $\overrightarrow{B}$ pointing in z-direction

generalise and obtain: $\overrightarrow{A} = \frac{1}{2}\overrightarrow{B} \times \overrightarrow{r}$ for the uniform induction $\overrightarrow{B}$

$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}}{r^2} d\tau$

$$\frac{\vec{J} \times \hat{r}}{r^2} = \vec{\nabla}\left(\frac{1}{r}\right) \times \vec{J} = \vec{\nabla} \times \frac{\vec{J}}{r} - \frac{1}{r}\vec{\nabla} \times \vec{J}$$

last term on the right$= 0$ (J function of source coordinates, while $\vec{\nabla}$ is for coordinates of target)

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \frac{\vec{J}}{r} d\tau = \vec{\nabla} \times \left[\frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r} d\tau\right] \implies \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r} d\tau$$

If the current is limited to a conducting wire, $\vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{\vec{dl}}{r}$

for surface current, $\vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{\vec{K}}{r} da$

any vector with zero curl can be added to this integral without affecting value of $\vec{B}$

$\therefore \vec{B}$ depends on space derivatives of $\vec{A}$, and not on $\vec{A}$ itself

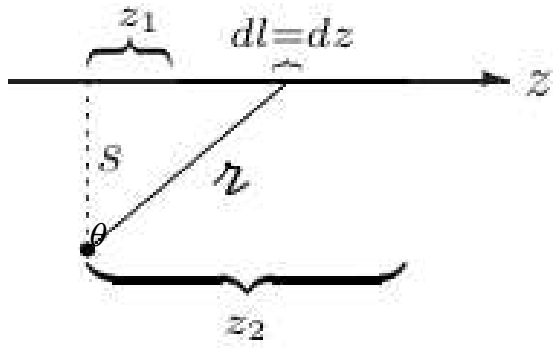
value of $\vec{B}$ at a given point can be calculated from $\vec{A}$, only if $\vec{A}$ is known in the region around the point

EX.: Long straight wire

We can calculate $\vec{B}$ starting from $\vec{A} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{d\vec{l}}{r}$

$\vec{A} \parallel$ wire, $\because$ $d\vec{l}$'s are all along the wire

task : calculate A for a current of finite length from z_1 to z_2



$$\vec{A} = \frac{\mu_0 I \hat{z}}{4\pi} \int_{z_1}^{z_2} \frac{dz}{\sqrt{s^2+z^2}}$$

or, $\vec{A} = \frac{\mu_0 I \hat{z}}{4\pi} \ln \left[z + \sqrt{s^2+z^2} \right]_{z_1}^{z_2} = \frac{\mu_0 I}{4\pi} \ln \frac{z_2 + \sqrt{s^2+z_2^2}}{z_1 + \sqrt{s^2+z_1^2}} \hat{z}$

if $z_1 = 0$; $z_2 = L \gg s$, then

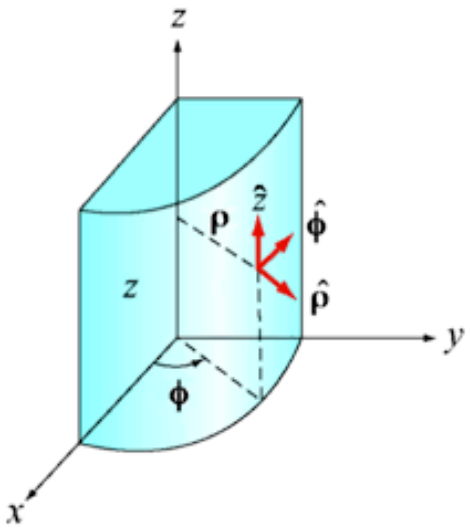
$$\vec{A} = \frac{\mu_0 I}{4\pi} \ln \frac{L + \sqrt{s^2+L^2}}{s} \hat{z} \approx \frac{\mu_0 I}{2\pi} \ln \frac{2L}{s} \hat{z}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A}{\partial s} \hat{\phi} = \frac{\mu_0 I}{4\pi s} \left[\frac{z_2}{\sqrt{s^2+z_2^2}} - \frac{z_1}{\sqrt{s^2+z_1^2}} \right] \hat{\phi}$$

$$= \frac{\mu_0 I}{4\pi s} (\sin\theta_1 - \sin\theta_2) \hat{\phi}$$

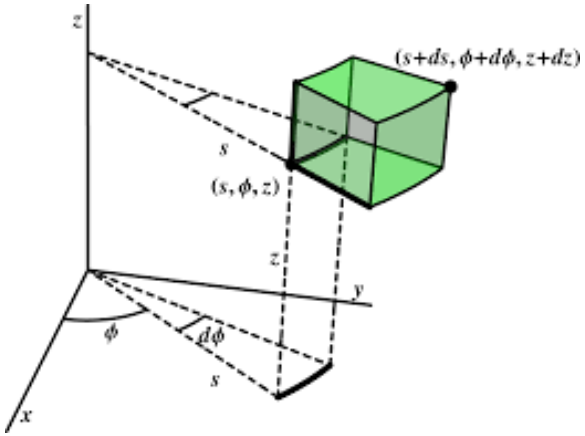
For infinitely long wire with $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$, $\quad \vec{B} = -\frac{\mu_0 I}{2\pi s} \hat{\phi}$

explanation for $\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A}{\partial s}\hat{\phi}$



cylindrical coordinates :

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$



$$\vec{\nabla} \times \vec{A} = \left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{z}A_z\right)$$

all derivatives of unit vectors vanish except $\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}$ and $\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}$

Also $A_\rho = A_\phi = 0$; $A_z = f(s) = A$; $\rho \equiv s$; $\therefore \vec{\nabla} \times \vec{A} = \frac{\partial A}{\partial \rho}(\hat{\rho} \times \hat{z}) = -\frac{\partial A}{\partial s}\hat{\phi}$

Vector potential $\vec{A}$ and the scalar potential ϕ

$\vec{A}$ of magnetic fields is similar to scalar potential ϕ of electric fields

(a) With electric fields, important quantity is the intensity $\vec{E}$

$\vec{E}$ gives stored energy and the forces on charged bodies and on polarized dielectrics

$\vec{E}$ causes breakdown in dielectrics

b) The quantity ϕ is related to $\vec{E}$ through $\vec{E} = -\vec{\nabla}\phi$

So one can add to ϕ any quantity that is independent of the coordinates without affecting $\vec{E}$ in any way

Also, value of $\vec{E}$ at a point can be calculated only if ϕ is known in the region around that point

c) Finally, integral for ϕ is simpler to evaluate than for $\vec{E}$

easier to find $\vec{E}$ by first calculating ϕ , and then $\vec{\nabla}\phi$, than to calculate $\vec{E}$ directly

However, if geometry of charge distribution is simple, use Gauss's law
Then $\vec{E}$ is easy to calculate directly and there is no need to go through ϕ

$\vec{A}$ versus ϕ

a) magnetostatics : quantity corresponding to $\vec{E}$ is magnetic induction $\vec{B}$
it is $\vec{B}$ that gives stored energy and the magnetic forces

b) $\vec{A}$ is related to $\vec{B}$ only through $\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}$

One can add to $\vec{A}$ any quantity whose curl is zero without changing $\vec{B}$
and one can deduce from $\vec{A}$ the value of $\vec{B}$ at a point only if one knows value of $\vec{A}$
in region around that point

c) integral for $\vec{A}$ is also simpler to calculate than for $\vec{B}$

However, with simple current distributions, it is much easier to use Ampere's
circuital law to find $\vec{B}$ than to calculate $\vec{A}$ first

Line integral of vector potential $\vec{A}$ over a closed curve

Using Stokes's theorem : $\oint_C \vec{A}.d\overrightarrow{l} = \int_S \overrightarrow{\nabla} \times \vec{A}.d\overrightarrow{a}$

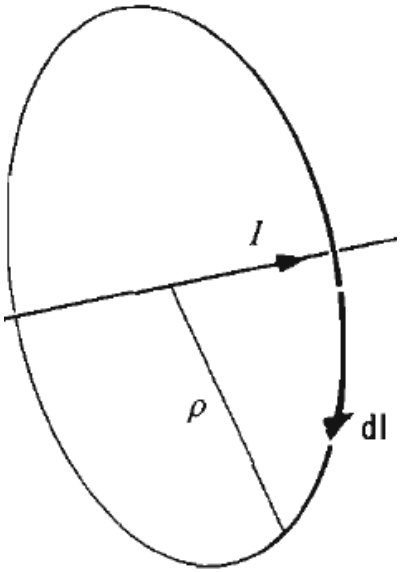
where S is any surface bounded by the closed curve C.

$\therefore \oint_C \vec{A}.d\overrightarrow{l} = \int_S \vec{B}.d\overrightarrow{a} = \Phi$

line integral of $\vec{A}.\overrightarrow{dl}$ over a closed curve C is equal to magnetic flux through C

AMPERE'S CIRCUITAL LAW

magnetic induction $\vec{B}$ near a long straight wire is azimuthal
$B = \frac{\mu_0 I}{2\pi\rho}$, where $\rho =$ distance from wire to the point considered



circle of radius ρ centered on a wire

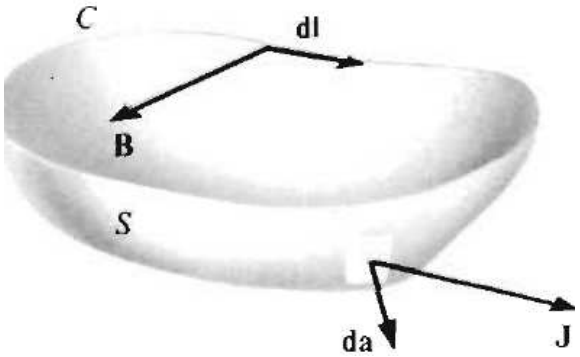
$$\oint \vec{B}.d\vec{l} = \frac{\mu_0 I}{2\pi\rho}.2\pi\rho = \mu_0 I$$

a general result for any closed curve C (amperian loop)

$$\boxed{\oint_C \vec{B}.d\vec{l} = \mu_0 I}$$

where $I =$ current enclosed by C

positive direction for integration related to positive direction for I by right-hand screw rule

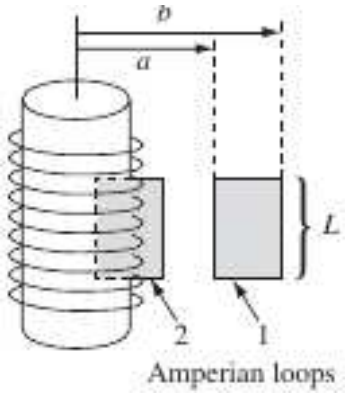


For a volume distribution of current

$$\oint_C \vec{B}.d\vec{l} = \mu_0 \int_S \vec{J}.d\vec{a}$$

where $\vec{J} =$ current density through any surface S bounded by curve C

Solenoid : (1)$\vec{B} \parallel$ axis. (2) outside the solenoid $B = 0$. Why?



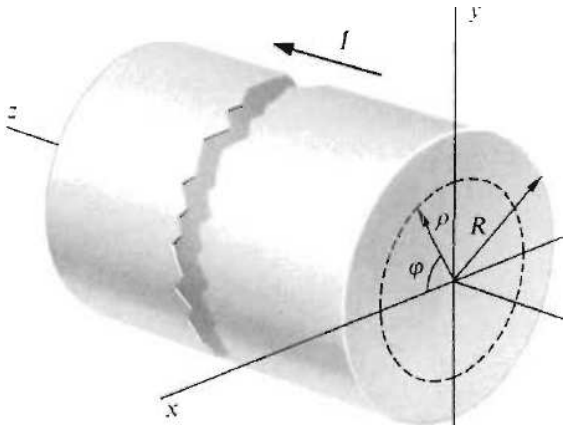
If $B_r \perp$axis, say $B_r > 0$ (r: radial varable)
if we reversed the direction of the current, $B_r < 0$
switching I is physically equivalent to turning the solenoid upside down, and that should not alter the radial field
$\therefore B_r = 0$ and $\vec{B} \parallel$ axis

we expect that $\vec{B}$ points upward inside the solenoid and downward outside
asymptotically, at large distances $B \overset{r\to\infty}{\Longrightarrow} 0$
consider amperian loop 1,
$\oint_C \vec{B}.d\vec{l} = [B(a) - B(b)L = \mu_0 I_{\text{enclosed}} = 0 \implies B(a) = B(b)$,
or, B is independent of r, But $B \overset{r\to\infty}{\Longrightarrow} 0$, $\therefore B(r) = 0 \; \forall \, r$

same current crosses the surface bounded by the loop 2 several times
total current crossing surface= current in each turn $\times$ #turns, or #ampere-turns

$$\oint_C \vec{B}.d\vec{l} = \mu_0 I_{\text{enclosed}} = n\mu_0 IL \implies B = \begin{matrix} n\mu_0 I\hat{z} \text{ inside} \\ 0 \text{ outside} \end{matrix}$$

EX.: Long cylindrical conductor carrying a current I uniformly distributed over its cross-section with density $J = \frac{I}{\pi r^2}$

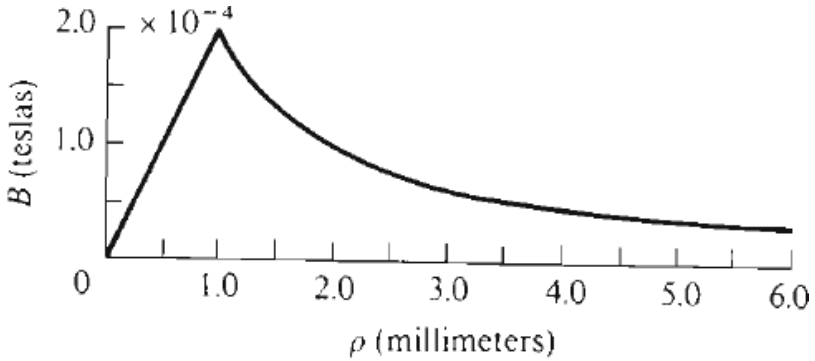


Outside conductor, $\vec{B}$ is azimuthal and magnitude independent of ϕ

$$\oint_C \vec{B}.d\vec{l} = \mu_0 \int_S \vec{J}.d\vec{a} \implies$$

$$B_{\text{in}} = \frac{\mu_0 J \pi \rho^2}{2\pi\rho} = \frac{\mu_0 j \rho}{2} = \frac{\mu_0 I \rho}{2\pi R^2}$$

$\therefore$ B increases linearly with ρ inside conductor
Outside the conductor, B decreases as $\frac{1}{\rho}$



Curl of magnetic induction

$$\oint_C \vec{B}.d\vec{l} = \mu_0 \int_S \vec{J}.d\vec{a}$$

Applying Stokes's theorem : $\oint_S \left(\vec{\nabla} \times \vec{B}\right).d\vec{a} = \mu_0 \int_S \vec{J}.d\vec{a}$

$$\therefore \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Electrostatics	Magnetrostatics
Maxwell's equations :	Maxwell's equations :
$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$(Gauss); $\vec{\nabla} \times \vec{E} = 0$	$\vec{\nabla}.\vec{B} = 0$; $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$(Ampere)
$\vec{E} \to 0$ far from all charges	$\vec{B} \to 0$ far from all currents
equations determine electric field (given source charge density ρ)	equations determine magnetic field
$\equiv$ Coulomb's law plus superposition	$\equiv$ Biot-Savart law (plus superposition)

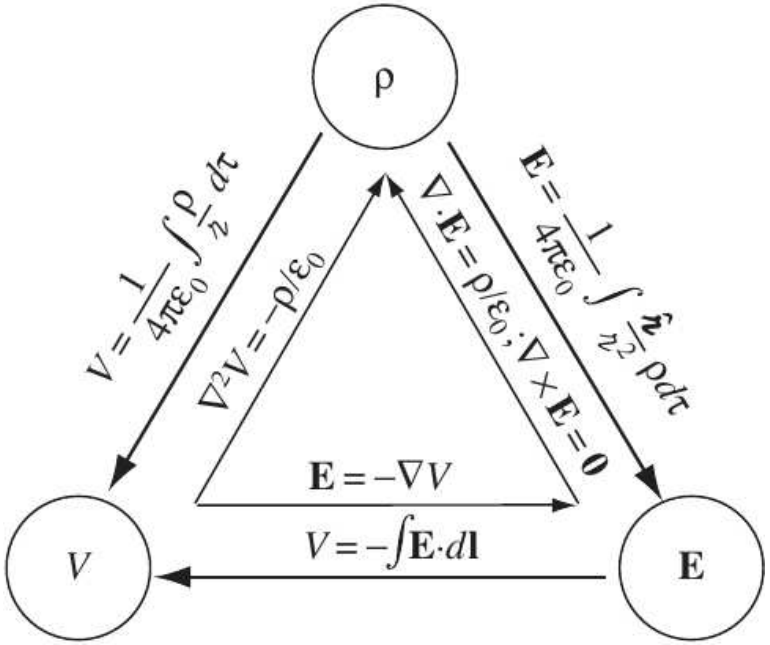
Maxwell's equations and force law , $\vec{F}_{\text{mag}} = Q(\vec{E} + \vec{v} \times \vec{B})$ constitute most elegant formulation of electrostatics and magnetostatics

electric field diverges away from a (positive) charge
magnetic field line curls around a current

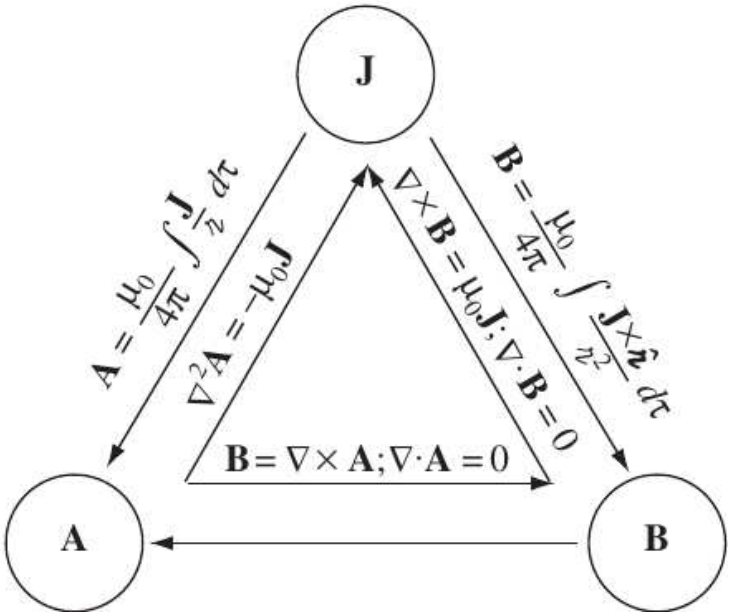
$\vec{E}$ lines originate on positive charges and terminate on negative ones
$\vec{B}$ lines do not begin or end anywhere—that would require nonzero divergence : typically closed loops or extend out to infinity

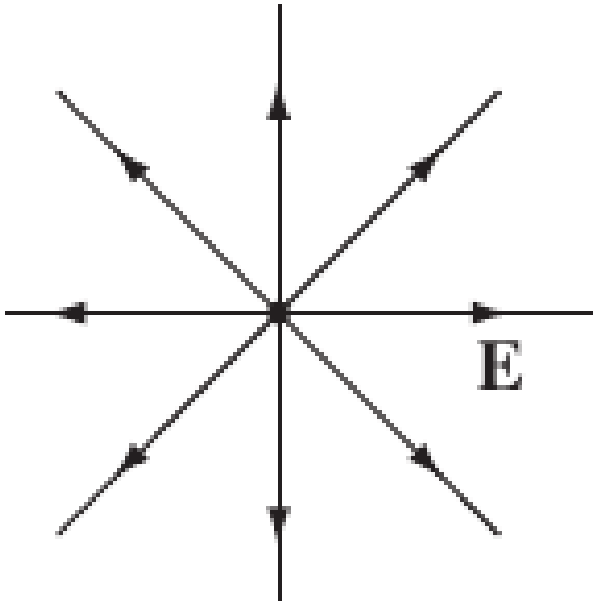
no point sources for $\vec{B}$, as there are for $\vec{E}$- no magnetic analog to electric charge

Electrostatics

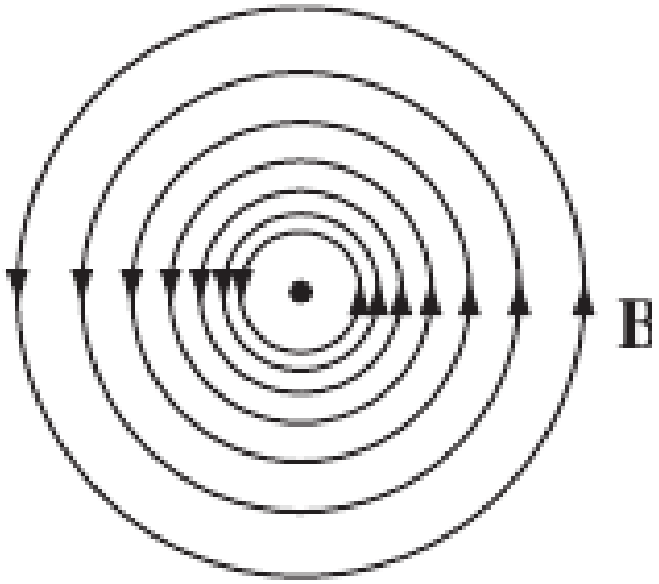


Magnetrostatics





Electric fied of a point charge



magnetic field of a long wire

Gradient, divergence and curl in curvilinear coordinates :

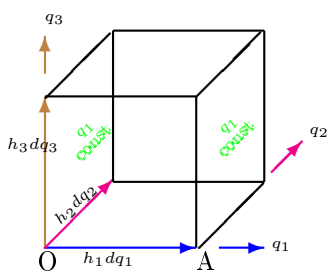
Consider an infinitesimally small cube with edges parallel to the local curvilinear coordinate directions with faces satisfying $q_i =$ constant, $\forall i = 1, 2, 3$ for the three pairs of faces.

The sides are $h_1 dq_1$, $h_2 dq_2$, and $h_3 dq_3$, where h_1, h_2, and h_3 are in general functions of q_1, q_2, and q_3.

The diagonal distance is, $ds^2 = h_1 dq_1^2 + h_2 dq_2^2 + h_3 dq_3^2 = dx^2 + dy^2 + dz^2$

A useful relation is : $h_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}$,

e.g., in spherical polar coordinaes, $h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} = r\sqrt{\left(\frac{\partial[\sin\theta\cos\phi]}{\partial\phi}\right)^2 + \left(\frac{\partial[\sin\theta\sin\phi]}{\partial\phi}\right)^2} = r\sin\theta$



Component of gradient in the q_1 direction: $(\nabla f)_1 = \lim_{dq_1 \to 0} \frac{f(A) - f(O)}{h_1 dq_1} = \frac{1}{h_1}\frac{\partial f}{\partial q_1}$

complete gradient : $\vec{\nabla} f = \hat{q_1}\frac{1}{h_1}\frac{\partial f}{\partial q_1} + \hat{q_2}\frac{1}{h_2}\frac{\partial f}{\partial q_2} + \hat{q_3}\frac{1}{h_3}\frac{\partial f}{\partial q_3}$

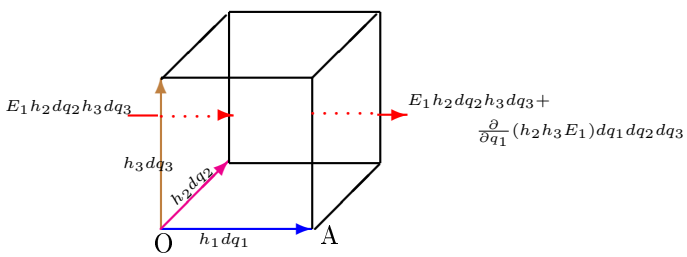
The divergence of a vector field $\vec{E}$ in curvilinear coordinates is found using Gauss' theorem :

total vector flux through the six sides of the cube $= \vec{\nabla}.\vec{E} \times$ volume of the cube (higher order terms ignored)

The face with sides $h_2 dq_2$, and $h_3 dq_3$ has area $= h_2 dq_2 h_3 dq_3$

The flux entering the face is $-E_1 h_2 dq_2 h_3 dq_3$

To find the flux across the opposite (parallel) face of the cube, corresponding to an increase in q_1 by dq_1, note that h_2, h_3 and E_1 all vary with q_1.



flux $= E_1 h_2 dq_2 h_3 dq_3 + \frac{\partial}{\partial q_1}(h_2 h_3 E_1) dq_2 dq_3 \times dq_1$ (first order Taylor expansion)

Net flux in the direction of q_1: $-E_1 h_2 dq_2 h_3 dq_3 + E_1 h_2 dq_2 h_3 dq_3 + \frac{\partial}{\partial q_1}(h_2 h_3 E_1) dq_1 dq_2 dq_3 = \frac{\partial}{\partial q_1}(h_2 h_3 E_1) dq_1 dq_2 dq_3 = \frac{\partial}{\partial q_1}(h_2 h_3 E_1) \times \frac{\text{vol. of cube}}{h_1 h_2 h_3}$

Then with all faces taken into account : net flux $= \int\limits_{\text{all faces}} \vec{E}.\vec{da} = \left[\frac{\partial}{\partial q_1}(h_2 h_3 E_1) + \frac{\partial}{\partial q_2}(h_3 h_1 E_2) + \frac{\partial}{\partial q_3}(h_1 h_2 E_3)\right] \times \frac{\text{vol. of cube}}{h_1 h_2 h_3}$

Applying Gauss's law : $\vec{\nabla}.\vec{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 E_1) + \frac{\partial}{\partial q_2}(h_3 h_1 E_2) + \frac{\partial}{\partial q_3}(h_1 h_2 E_3)\right]$

Deriving the relation for curl is more complex. Given below are the expressions for components of curl of a vector function $\vec{F}$:

$\left(\vec{\nabla} \times \vec{E}\right)_i = \frac{1}{h_j h_k}\left(\frac{\partial(h_k E_k)}{\partial q_j} - \frac{\partial(h_j E_j)}{\partial q_k}\right)$, the indices i, j, k ae used in cyclic permutation.

Conductor: material inside which charges can flow freely
Insulator : e.g., glass or rubber, each electron tied to a particular atom

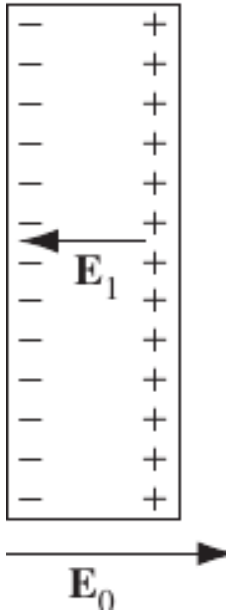
Perfect conductor : contains unlimited supply of free charges
- metals come pretty close

electrostatics : charges in equilibrium positions and fixed in space

- inside a conductor, $\vec{E} = 0$; all points are at same potential

if there were any field, free charges would move, and it wouldn't be electrostatics any more

If a conductor is placed in an electric field $\vec{E}_0$, charges flow temporarily within it piling up on the edges (induced charges) producing a second field, $\vec{E}_1$,
opposite to $\vec{E}_0$
and it cancels the first one at every point inside it



Coulomb's law applies within conductors, even though net field $= 0$

Gauss's law is also valid

- $\because E = 0$ inside $\implies$ charge density $\rho = 0$

There is still charge around, plus and minus equally, so the net charge density in the interior is zero

corollary : net static charge on conductor must reside on its surface

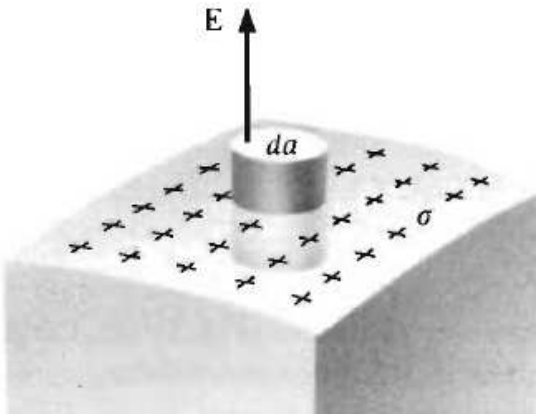
- conductor is an equipotential

for two points and b within (or at the surface of) a given conductor,

$$V(b) - V(a) = -\int_a^b \vec{E}.d\vec{l} = 0$$

- At the surface $\vec{E}$ must be normal,

for if there were a tangential component, charges will flow along the surface, contrary to our hypothesis



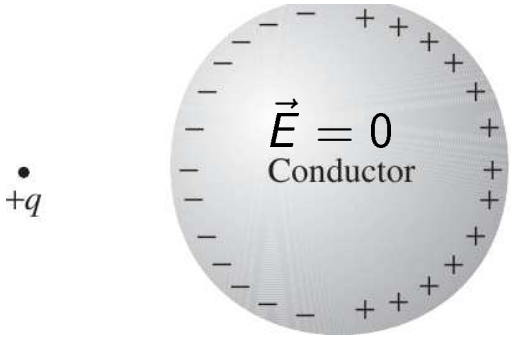
by Gauss's law

just outside the surface, $E = \frac{\sigma}{\varepsilon_0}$

where $\sigma =$ surface charge density

paradox : electric field intensity at the surface of a conductor is expressed in terms of local surface charge density σ alone, despite the fact that field is due to all charges, whether on the conductor or elsewhere

Induced charge



If we hold a charge $+q$ near an uncharged conductor

- the two attract one another

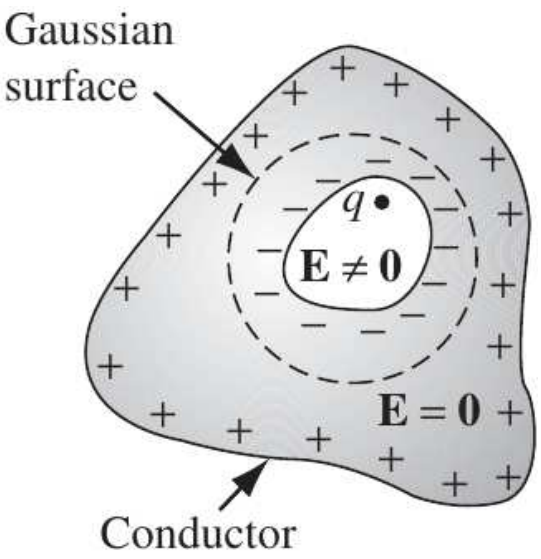
q pulls minus charges to the near side and repels plus charges to the far side

The charge moves around in such a way as to kill off the field of q for points inside the conductor where the total field $= 0$

Since the negative induced charge is closer to q, there is a net force of attraction

EX.: HOLLOW CONDUCTOR

charge on inner surface equal in magnitude and opposite in sign to any charge enclosed within the hollow



consider a Gaussian surface lying within the conductor enclosing the hollow

$\because$ everywhere on this surface, $E = 0$

$\therefore \oint \vec{E}.d\vec{a} = 0$

$\therefore$ total enclosed charge$= q + q_{\text{induced}} = 0$

compensating charge left over on outer surface of conductor effectively "communicates" presence of q to the outside world

total charge induced on cavity wall $= -$ charge inside

Isolated charged conducting plate of infinite extent

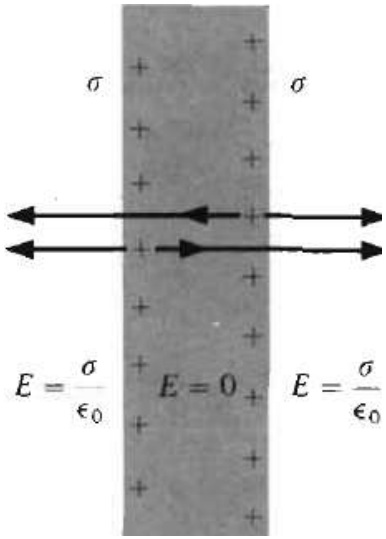
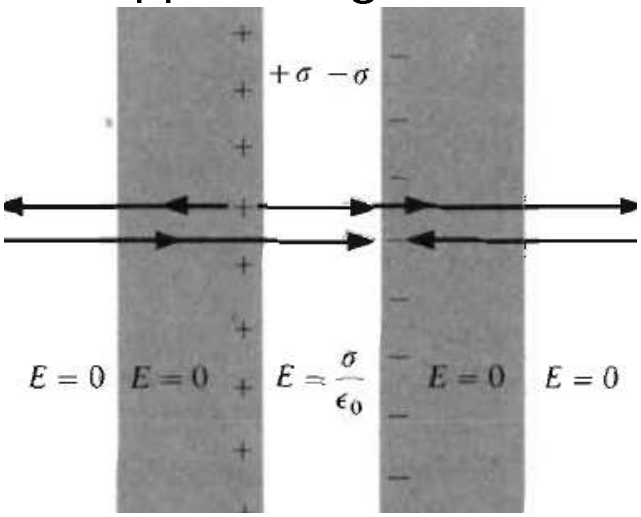


plate must carry equal charge densities σ on its two faces

outside : $E = 2.\frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$

inside : the two surface distributions cancel and $E = 0$

Pair of parallel conducting plates carrying charges of equal magnitudes and opposite signs



surface charge densities on plates : σ and $-\sigma$

field inside the plates and outside$= 0$

It seems surprising that field of an infinite plane is independent of how far one is - What about $\frac{1}{r^2}$ in Coulomb's law?

as we move farther away from the plane, more charge comes into "field of view" - this compensates for the diminishing influence of any particular piece

electric field falls off as :

sphere	infinite line	infinite plane
$\frac{1}{r^2}$	$\frac{1}{r}$	does not fall off at all

$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ and $\vec{E} = -\vec{\nabla}V$

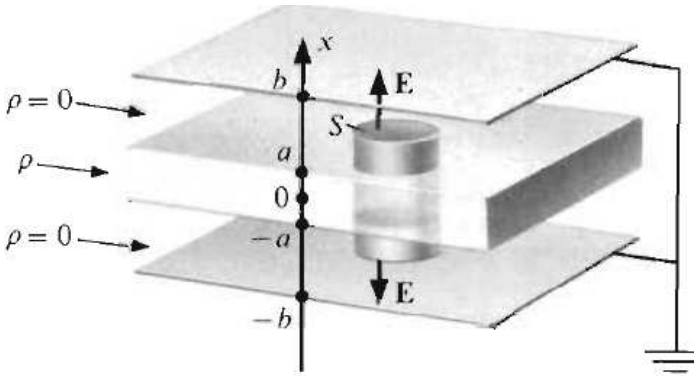
$\Longrightarrow \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}} \Leftarrow$ POISSON'S EQUATION

often the starting point for calculating electrostatic fields

integrals $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\vec{r'})\hat{r}}{r^2}d\tau'$ and $V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r'})d\tau'}{r}$ are tough to handle

it is fruitful to recast the problem in differential form
using Poisson's equation with appropriate boundary conditions

Ex. : Flat ion beam ($-a < x < a$) between 2 grounded parallel conducting plates ($x = \pm b$)



disregard ionic motion(magnetic effects)

ρ uniform inside beam
- edge effects negligible

inside : $\nabla^2 V = \frac{d^2 V}{dx^2} = -\frac{\rho}{\varepsilon_0}$

$E_{\text{in}} = -\frac{dV}{dx} = \frac{\rho x}{\varepsilon_0} + A(\text{const.})$

symmetry : at $x = 0$, $E_{\text{in}} = 0$

$\therefore$ inside beam, $A = 0$

$E_{\text{in}} = \frac{\rho x}{\varepsilon_0}$; $V_{\text{in}} = -\frac{\rho x^2}{2\varepsilon_0} + B(\text{const})$

consider cylindrical volume with upper and lower faces outside the beam at equal distances from the plane $x = 0$

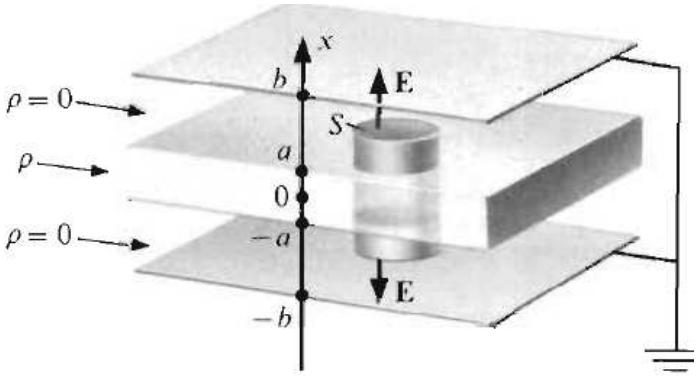
magnitude of E_{out} is same at both faces

Let S =area of one face

Gauss's law $\implies 2E_{\text{out}}S = 2aS\frac{\rho}{\varepsilon_0}$

or, $E_{\text{out}} = a\frac{\rho}{\varepsilon_0}\ \forall x \geq a$
$-a\frac{\rho}{\varepsilon_0}\ \forall x \leq -a$

$\therefore$ outside the beam,
$V_{\text{out}} = -\frac{a\rho x}{\varepsilon_0} + C(\text{const.})\ \forall x \geq a$
$\frac{a\rho x}{\varepsilon_0}$+D(const.) $\forall x \leq -a$



$$V_{\text{in}} = -\frac{\rho x^2}{2\varepsilon_0} + B(\text{const.})$$

$$V_{\text{out}} = -\frac{a\rho x}{\varepsilon_0} + C(\text{const.}) \ \forall x \geq a$$

$$\frac{a\rho x}{\varepsilon_0} + \ \text{D(const.)} \ \forall x \leq -a$$

B.C. $V = 0$ at $x = \pm b$

$$\implies \quad C = D = \frac{a\rho b}{\varepsilon_0}$$

x	V_{out}	E_{out}
$\geq a$	$\frac{a\rho}{\varepsilon_0}(b-x)$	$\frac{a\rho}{\varepsilon_0}$
$\leq -a$	$\frac{a\rho}{\varepsilon_0}(b+x)$	$-\frac{a\rho}{\varepsilon_0}$

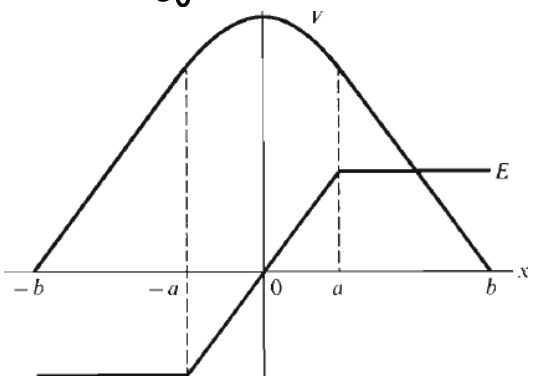
at $x = \pm a$, $V_{in} = V_{out}$

$$\implies \quad -\frac{\rho a^2}{2\varepsilon_0} + B = \frac{a\rho}{\varepsilon_0}(b-a)$$

or, $B = \frac{a\rho}{\varepsilon_0}\left(b - \frac{a}{2}\right)$

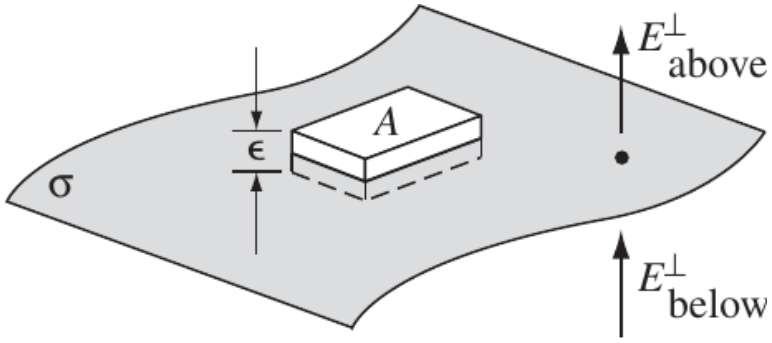
$\therefore V_{\text{in}} = \frac{\rho}{\varepsilon_0}\left[\frac{x^2}{2} + a\left(b - \frac{a}{2}\right)\right]$ and

$E_{\text{in}} = \frac{\rho x}{\varepsilon_0}$



B.C. : electric field undergoes a discontinuity crossing a surface charge σ

draw a thin Gaussian pillbox, extending just barely over the edge in each direction



Gauss's law : $\oint_S \vec{E}.\vec{da} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$

sides of pillbox contribute nothing to flux in the limit as thickness$\rightarrow 0$

$A =$ area of pilbox head

$\therefore E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$

$\Longrightarrow$ normal component of $\vec{E}$ is discontinuous by an amount $\frac{\sigma}{\epsilon_0}$ at any boundary

where there is no surface charge, $\vec{E}$ is continuous

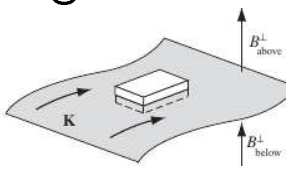
e.g., at the surface of a uniformly charged solid sphere

tangential component of $\vec{E}$, is always continuous : $\vec{E}^{\parallel}_{\text{above}} = \vec{E}^{\parallel}_{\text{below}}$

the ends contribute nothing to flux in the limit as thickness$\rightarrow 0$

magnetic field undergoes a discontinuity at a surface current $\vec{K}$

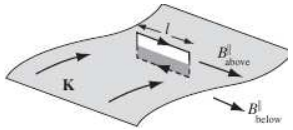
thin Gaussian pillbox, barely over the edge in each direction



sides of pillbox contribute nothing to flux in the limit as thickness$\rightarrow 0$

: $\oint_S \vec{B}.d\vec{a}=0 \implies B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$

consider an Amperian loop running perpendicular to the current



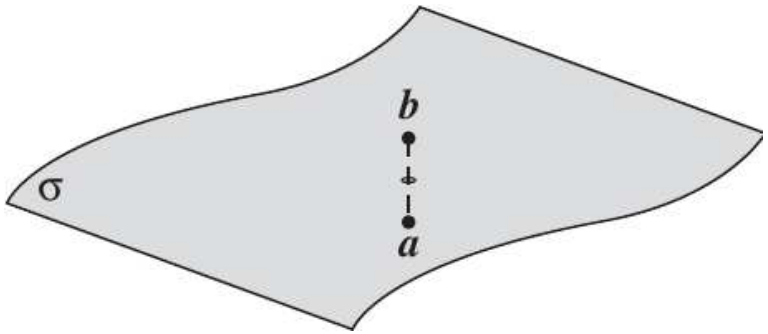
$\oint_S \vec{B}.d\vec{l}=\left(\vec{B}^{\parallel}_{\text{above}}-\vec{B}^{\parallel}_{\text{below}}\right)l=\mu_0 I_{\text{enc}}=\mu_0 Kl$

$\implies$ tangential component of $\vec{B}$ is discontinuous ($\parallel$ to surface but $\perp$ to current), $\vec{B}^{\parallel}_{\text{above}}-\vec{B}^{\parallel}_{\text{below}}=\mu_0 K$

similar Amperian loop running parallel to the current reveals that the parallel component is continuous

In general, B.C. : $\vec{E}_{\text{above}}-\vec{E}_{\text{below}}=\frac{\sigma}{\epsilon_0}\hat{n}$; unit vector $\hat{n}\perp$ surface pointing "upward."
and $\vec{B}^{\parallel}_{\text{above}}-\vec{B}^{\parallel}_{\text{below}}=\mu_0\vec{K}\times\hat{n}$

potential is continuous across any boundary



$$V_{\text{above}} - V_{\text{below}} = -\int_a^b \vec{E}.\vec{dl} \xrightarrow[a\to b]{} 0$$

$$\vec{\nabla} V_{\text{above}} - \vec{\nabla} V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{n}$$

or, $\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$

where $\frac{\partial V}{\partial n} = \vec{\nabla} V.\hat{n}$ normal derivative of V

these b. c. relate fields and potentials just above and just below the surface

derivatives are the limiting values as we approach the surface from either side

similarly, vector potential is continuous across any boundary: $\vec{A}_{\text{above}} = \vec{A}_{\text{below}}$

$\vec{\nabla}.\vec{A} = 0$ guarantees that normal component is continuous; and in $\vec{\nabla} \times \vec{A} = \vec{B}$ in the form $\oint \vec{A}.\vec{dl} = \int \vec{B}.\vec{da} = \Phi \implies$ tangential components are continuous (flux through an Amperian loop of vanishing thickness is zero)

But, $\frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{K}$: discontinuous

Surface Charge and the Force on a Conductor

immediately outside a Conductor, $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n} \implies -\frac{\partial V}{\partial n} = \frac{\sigma}{\epsilon_0}$ or, $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$

In presence of an electric field, surface charge experiences a force

force per unit area, $\vec{f} = \sigma \vec{E}$

electric field is discontinuous at a surface charge, so use the average

In general, $\vec{f} = \sigma \langle \vec{E} \rangle = \frac{1}{2}\sigma \left(\vec{E}_{\text{above}} + \vec{E}_{\text{below}} \right)$

for a conductor, $\vec{f} = \sigma \langle \vec{E} \rangle = \frac{1}{2}\sigma \left(\vec{E}_{\text{above}} + 0 \right) = \frac{1}{2}\sigma . \frac{\sigma}{\epsilon_0}\hat{n} = \frac{\sigma^2}{2\epsilon_0}\hat{n}$

$\implies$ outward electrostatic pressure on surface, drawing conductor into the field, $P = \frac{\epsilon_0}{2} E^2$

method of virtual work : assume a small change in a system and use principle of conservation of energy

imagine : conducting bodies in field are disconnected from power supplies

if electric forces perform mechanical work, they must use electric energy stored in the field

consider a small area a of a conductor pulled into the field by a small distance x

mechanical work performed $= ax.\frac{\text{force}}{\text{area}} =$energy lost by field, ax.energy-density

$\therefore$ force per unit area$=$ energy density

for a gas, energy density$=$ pressure

electric force on a conductor tends to pull it into the field
To visualize electric forces, imagine that electric lines of force are real
and under tension
— arranged in space as if they repelled each other

CAPACITANCE of an isolated conductor

Imagine a conductor situated long distance away from any other body

As charge is added to it, its potential rises
change in potential $\propto$ amount of charge added
and depends on the geometrical configuration of the conductor

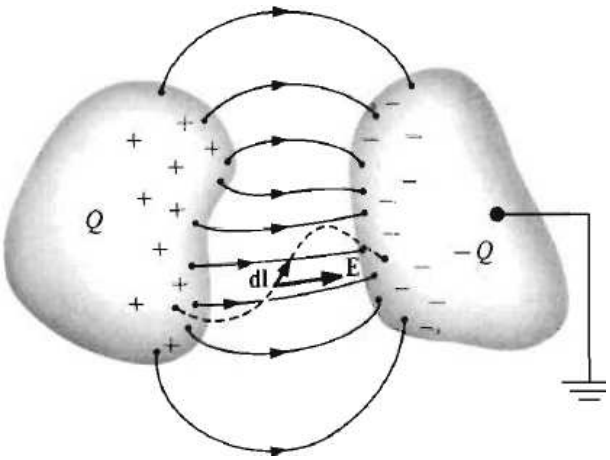
$$\text{capacitance}: \; C = \frac{Q}{V}$$

$$\text{unit}: \; \text{C/V or farad}$$

Ex. : isolated spherical conductor of radius R carrying a total surface charge Q

$$\text{capacitance, } C = \frac{Q}{V} = \frac{Q}{Q/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R$$

Ex. : Two conductors carrying equal and opposite charges



Initially, both are uncharged

assume : one is grounded; potential $= 0$

gradually add small charges to the other until it has a charge Q

charges of opposite sign attracted to grounded conductor until it carries charge $-Q$

from one conductor at $V = 0$, to the other at potential V, along any path

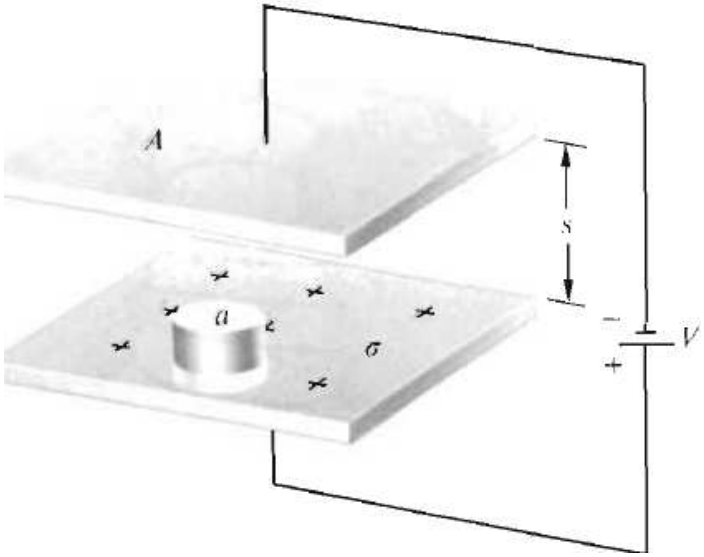
$$\int \vec{E}.\overrightarrow{dl} = V$$

capacitance between the conductors $C = \frac{Q}{V}$

Ex. : PARALLEL-PLATE Capacitor

uniform $\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}$ charge density $\begin{matrix}\sigma\\ -\sigma\end{matrix}$ on $\begin{matrix}\text{top}\\ \text{bottom}\end{matrix}$ surface of $\begin{matrix}\text{lower}\\ \text{upper}\end{matrix}$ plate

spaced s distance away from each other



electric field intensity $\vec{E}$ points upward from lower plate

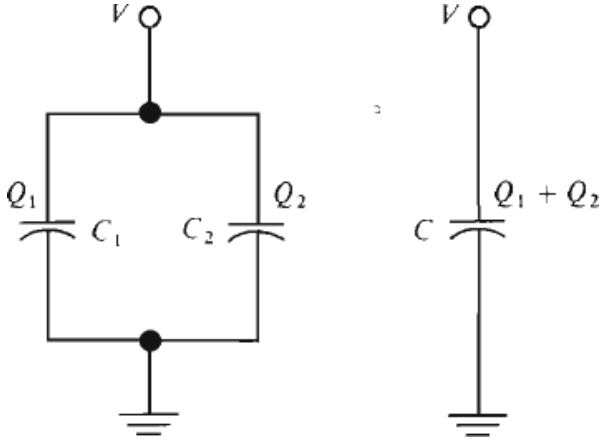
$$E = \frac{\sigma}{\varepsilon_0}$$

$$\therefore\ V = Es = \frac{\sigma s}{\varepsilon_0}$$

$$\therefore\ C = \frac{Q}{V} = \frac{\sigma A}{\frac{\sigma s}{\varepsilon_0}} = \frac{\varepsilon_0 A}{s}$$

For square plates with side= 1 cm, held 1 mm apart, $C = 9 \times 10^{-13}$F

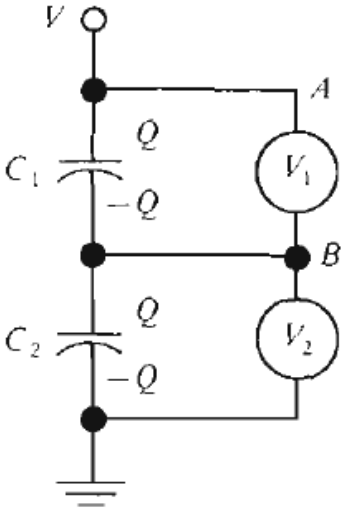
Capacitors connected in PARALLEL

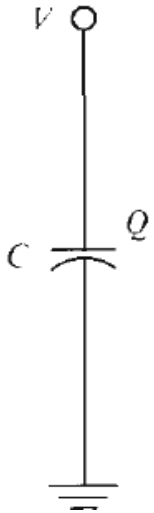


$$Q_1 + Q_2 = C_1 V + C_2 V = CV \Longrightarrow \qquad C = C_1 + C_2$$

$\therefore$ capacitance of capacitors connected in parallel = sum of individual capacitances

Capacitors connected in SERIES





connect points A and B to ground $\Longrightarrow$
voltages $V_1 = V_2 = 0$
capacitor plates are uncharged
now disconnect A and B from ground
apply a charge Q to A
assume : voltmeters draw zero current
No charge flows into or out of B

charge $-Q$ leaves upper plate of C_2 and goes to lower plate of C_1
This leaves a charge $+Q$ on the top plate of C_2

and lower plate of C_1 takes on a charge $-Q$

$V_1 = \frac{Q}{C_1}, \quad V_2 = \frac{Q}{C_2}$ and $V = V_1 + V_2 = \frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2}$

$\Longrightarrow \quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ or, $C = \frac{C_1 C_2}{C_1 + C_2}$; $\qquad \frac{1}{C}$: elastance

capacitors connected in **parallel** : **voltages** are **same**

capacitors connected in **series** : **charges** are **same**

Energy density

energy stored in a parallel-plate capacitor $= \int_0^Q V dq = \int_0^Q \frac{q}{C} dq = \frac{1}{2} \frac{Q^2}{C}$

$= \frac{1}{2} CV^2 = \frac{1}{2} \epsilon_0 \frac{A}{s} V^2 = \frac{1}{2} \epsilon_0 \left(\frac{V}{s}\right)^2 A.s = \left(\frac{1}{2} \epsilon_0 E^2\right) A.s$
where $A.s =$ volume occupied by the field

$\therefore$ stored energy : each point associated with an energy density $\boxed{\frac{1}{2} \epsilon_0 E^2}$ J/m^3

energy associated with a charge distribution, $W = \frac{\varepsilon_0}{2} \int E^2 d\tau = \frac{1}{2} \int V \rho d\tau$

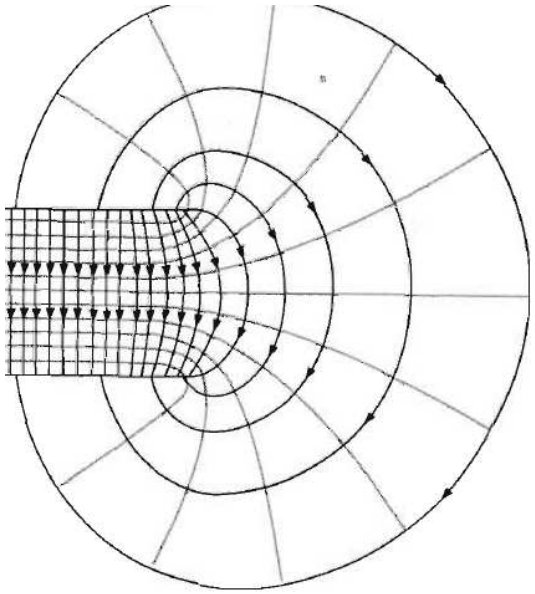
Note: at a point E^2 and $V\rho$ may not be the same

Ex. : Isolated spherical conductor of radius R and carrying a charge Q

$$W = \frac{\varepsilon_0}{2} \int_R^\infty E^2 d\tau = \frac{\varepsilon_0}{2} \int_R^\infty \left(\frac{Q}{4\pi\varepsilon_0 r^2} \right)^2 4\pi\varepsilon_0 r^2 dr = \frac{Q^2}{8\pi\varepsilon_0 R}$$

This energy is $\frac{1}{2}QV$, or $\frac{1}{2}CV^2$, where C is the capacitance

lines of force near edge of a parallel-plate capacitor



lines of force between plates of a capacitor tend to pull them together and bulge out at edges

maximum electric field intensity that can be sustained in air
at normal temperature and pressure $= 3 \times 10^6 \mathrm{V/m}$

$$\text{force} = 40 \ \mathrm{N/m^2}$$

(c) What is the force of attraction between two such wires, 1 cm apart?

(d) If you could somehow remove the stationary positive charges, what would the electrical repulsion force be? How many times greater than the magnetic force is it?

Problem 5.21 Is Ampère's law consistent with the general rule (Eq. 1.46) that divergence-of-curl is always zero? Show that Ampère's law *cannot* be valid, in general, outside magnetostatics. Is there any such "defect" in the other three Maxwell equations?

Problem 5.22 Suppose there *did* exist magnetic monopoles. How would you modify Maxwell's equations and the force law to accommodate them? If you think there are several plausible options, list them, and suggest how you might decide experimentally which one is right.

5.4 ■ MAGNETIC VECTOR POTENTIAL

5.4.1 ■ The Vector Potential

Just as $\nabla \times \mathbf{E} = \mathbf{0}$ permitted us to introduce a scalar potential (V) in electrostatics,

$$\mathbf{E} = -\nabla V,$$

so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a *vector* potential $\mathbf{A}$ in magnetostatics:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.} \tag{5.61}$$

The former is authorized by Theorem 1 (of Sect. 1.6.2), the latter by Theorem 2 (The proof of Theorem 2 is developed in Prob. 5.31). The potential formulation automatically takes care of $\nabla \cdot \mathbf{B} = 0$ (since the divergence of a curl is always zero); there remains Ampère's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{5.62}$$

Now, the electric potential had a built-in ambiguity: you can add to V any function whose gradient is zero (which is to say, any *constant*), without altering the *physical* quantity $\mathbf{E}$. Likewise, you can add to $\mathbf{A}$ any function whose *curl* vanishes (which is to say, the *gradient of any scalar*), with no effect on $\mathbf{B}$. We can exploit this freedom to eliminate the divergence of $\mathbf{A}$:

$$\boxed{\nabla \cdot \mathbf{A} = 0.} \tag{5.63}$$

To prove that this is always possible, suppose that our original potential, $\mathbf{A}_0$, is *not* divergenceless. If we add to it the gradient of λ ($\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$), the new divergence is

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2 \lambda.$$

We can accommodate Eq. 5.63, then, if a function λ can be found that satisfies

$$\nabla^2 \lambda = -\nabla \cdot \mathbf{A}_0.$$

But this is mathematically identical to Poisson's equation (2.24),

$$\nabla^2 V = -\frac{\rho}{\epsilon_0},$$

with $\nabla \cdot \mathbf{A}_0$ in place of ρ/ϵ_0 as the "source." And we *know* how to solve Poisson's equation—that's what electrostatics is all about ("given the charge distribution, find the potential"). In particular, if ρ goes to zero at infinity, the solution is Eq. 2.29:

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\mathscr{r}}\, d\tau',$$

and by the same token, if $\nabla \cdot \mathbf{A}_0$ goes to zero at infinity, then

$$\lambda = \frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{A}_0}{\mathscr{r}}\, d\tau'.$$

If $\nabla \cdot \mathbf{A}_0$ does *not* go to zero at infinity, we'll have to use other means to discover the appropriate λ, just as we get the electric potential by other means when the charge distribution extends to infinity. But the essential point remains: *It is always possible to make the vector potential divergenceless*. To put it the other way around: the definition $\mathbf{B} = \nabla \times \mathbf{A}$ specifies the *curl* of $\mathbf{A}$, but it doesn't say anything about the *divergence*—we are at liberty to pick that as we see fit, and zero is ordinarily the simplest choice.

With this condition on $\mathbf{A}$, Ampère's law (Eq. 5.62) becomes

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.} \tag{5.64}$$

This *again* is nothing but Poisson's equation—or rather, it is *three* Poisson's equations, one for each Cartesian[19] component. Assuming $\mathbf{J}$ goes to zero at infinity, we can read off the solution:

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau'.} \tag{5.65}$$

[19]In Cartesian coordinates, $\nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$, so Eq. 5.64 reduces to $\nabla^2 A_x = -\mu_0 J_x$, $\nabla^2 A_y = -\mu_0 J_y$, and $\nabla^2 A_z = -\mu_0 J_z$. In curvilinear coordinates the unit vectors *themselves* are functions of position, and must be differentiated, so it is *not* the case, for example, that $\nabla^2 A_r = -\mu_0 J_r$. Remember that even if you plan to *evaluate* integrals such as 5.65 using curvilinear coordinates, you must first express $\mathbf{J}$ in terms of its *Cartesian* components (see Sect. 1.4.1).

When charge density $\rho = 0$ in a region
Poisson's eqn becomes $\nabla^2 V = 0 \Leftarrow$ **Laplace's equation**

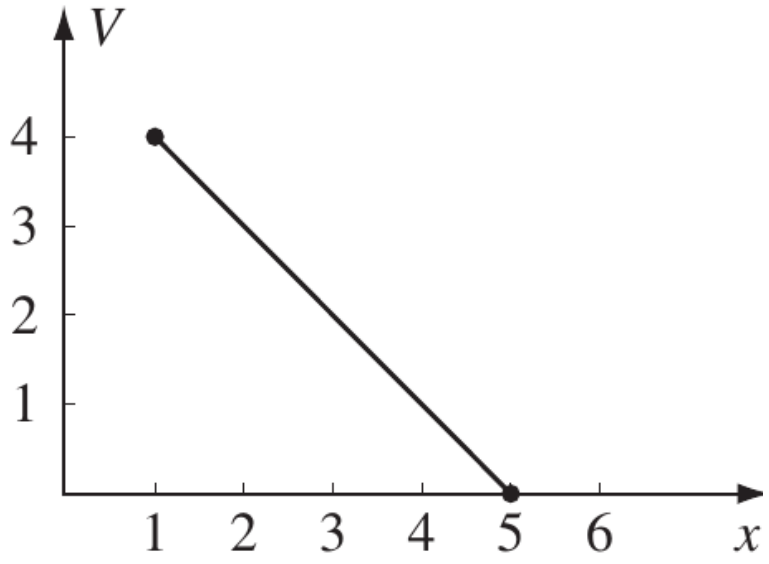
This equation applies not only to electrostatics but also to heat conduction, hydro- and aerodynamics, elasticity, etc.

1-D : $\frac{d^2 V}{dx^2} = 0 \Longrightarrow$ general soln. : $V(x) = mx + b$: straight line

Const. m and b are obtained using B.C.

Ex. : $V = 4$ at $x = 1$
and $V = 0$ at $x = 5$

$\Longrightarrow \quad m = -1$ and $b = 5$
so $V = -x + 5$



two features : trivial here, but more interesting in higher dimensions

1. $V(x)$ is the average of $V(x+a)$ and $V(x-a)\ \forall\, a$

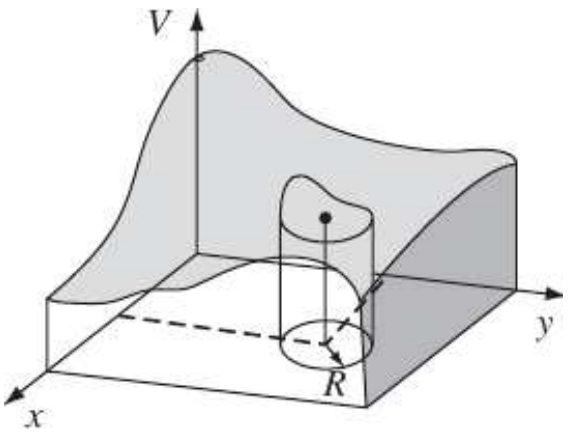
$V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$: Laplace's equation is an averaging instruction
it tells us to assign to point x the average of values to the left and to right of x

2. Laplace's equation tolerates no local maxima or minima
extreme values of V must occur at end points$\Leftarrow$ consequence of (1)

Also, $\frac{d^2 V}{dx^2} = 0$ (not $<$ or $>$ as for max/min)

Laplace's Equation in 2-D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$



1. $V(x, y)$=average of V around the point
   draw a circle of radius R about point (x, y)
   average value of V on the circle= value at center

   $$V(x, y) = \frac{1}{2\pi R} \oint V dl$$

   2. V has no local max/min
   all extrema occur at boundaries
   Laplace's equation picks most featureless function possible, consistent with B.C.
   : no hills, no valleys, smoothest surface

Laplace's Equation in 3-D

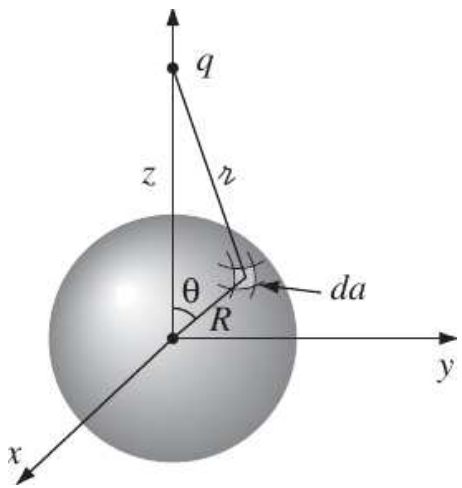
1. $V(\vec{r})$ at a point = average of V over spherical surface of radius R centered at $\vec{r}$

$$V(x,y) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V da$$

2. V has no local max/min - all extrema occur at boundaries

if V had a local maximum at $\vec{r}$
then we can draw a sphere around $\vec{r}$ over which all values of V
— and hence the average — would be less than at $\vec{r}$

Proof : find average potential over a spherical surface of radius R due to single point charge q located outside the sphere



locate q on z-axis

potential at a point on surface $= \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}$

with $\mathscr{r} = \sqrt{z^2 + R^2 - 2zR\cos\theta}$

average over the surface is

$$V_{\text{av}} = \frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\int\left[\frac{1}{\sqrt{z^2+R^2-2zR\cos\theta}}R^2\sin\theta d\theta d\phi\right]$$

$$= \frac{1}{2zR}\frac{q}{4\pi\varepsilon_0}\sqrt{z^2 + R^2 - 2zR\cos\theta}\Big|_0^\pi$$

$$= \frac{1}{2zR}\frac{q}{4\pi\varepsilon_0}\left[(z+R) - (z-R)\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{z}$$

this is precisely potential due to q at center of the sphere!

same goes for any collection of charges outside the sphere $\Leftarrow$ superposition

average potential over the sphere $=$ net potential at the center

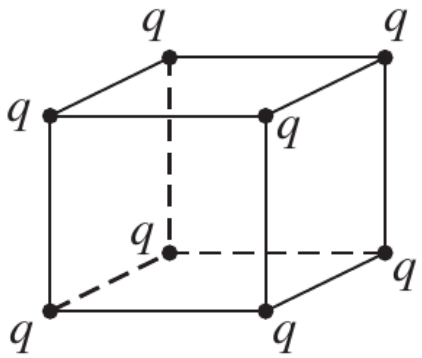
Earnshaw's Theorem:

A charged particle cannot be in a stable equilibrium by electrostatic forces alone

$$\Downarrow$$

local minimum in potential energy

Ex. : cubical arrangement of fixed charges



It seems that a charge at center should remain stable, repelled by charges at vertices

potential energy$= qV$
Laplace's equation allows no local minima for V

What looks like a minimum, is in fact a saddle point

UNIQUENESS THEOREM :

for a given charge density $\rho(x, y, z)$ and for given boundary conditions Poisson's equation has only one solution $V(x, y, z)$

If, by intuition or by analogy, we find a soln. V satisfying both Poisson's equation and given B.C., then it is the correct V

What are appropriate B.C., sufficient to determine V?

1-D : easy; general solution : $V = mx + b$ - two constants require two B.C.

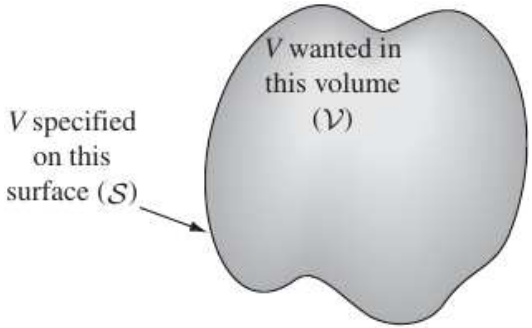
specify value of function at each end
or value of function and its derivative at one end
or value at one end and derivative at the other, and so on

But just value or just derivative at one end $\longrightarrow$ insufficient information
Nor derivatives at both ends $\longrightarrow$ this is either redundant (if two are equal) or inconsistent (if they are not)

uniqueness theorem : proof that a proposed set of B.C. suffice

First uniqueness theorem: solution to Laplace's equation in some volume $\mathcal{V}$ uniquely determined if V is specified on the boundary surface $\mathcal{S}$

Proof : Consider the region shown :



There can be islands inside

boundary could be at infinity, $V = 0$

If $\exists$ 2 solns. $\nabla^2 V_1 = \nabla^2 V_2 = 0$

then $V_3 = V_1 - V_2$ obeys Laplace eqn.

at boundary, $V_1 = V_2 \implies V_3 = 0$

Laplace eqn allows no local maxima or minima

— all extrema occur on the boundaries

max/min of V_3 are both zero

$\therefore V_3 = 0$ everywhere

$\implies V_1 = V_2$ everywhere

In case there are charges enclosed, then V satisfies Poisson's eqn

same argument as above can be applied :

If two solns : $\nabla^2 V_1 = \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho$

difference, $V_3 = V_1 - V_2$ satisfies Laplace's equation and is zero on all boundaries

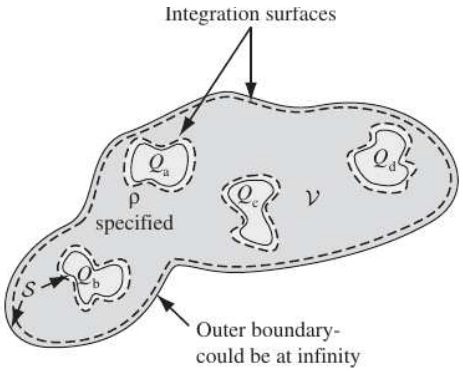
$\therefore V_3 = 0$ everywhere $\implies V_1 = V_2$ everywhere

Corollary: potential in a volume $\mathcal{V}$ uniquely determined if

(a) charge density throughout the region

and (b) value of V on all boundaries, are specified

Second Uniqueness Theorem : In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, electric field is uniquely determined if total charge on each conductor is given
(region as a whole can be bounded by another conductor, or else unbounded)



Proof :
If $\exists$ 2 solns. both obeying
Gauss's law in space between conductors:

$$\vec{\nabla}.\vec{E}_1 = \vec{\nabla}.\vec{E}_2 = \frac{1}{\epsilon_0}\rho$$

for a Gaussian surface
enclosing each conductor

$$\oint_{\substack{\text{ith conducting}\\ \text{surface}}} \vec{E}_1.d\vec{a} = \oint_{\substack{\text{ith conducting}\\ \text{surface}}} \vec{E}_2.d\vec{a} = \frac{1}{\epsilon_0} Q_i$$

for outer boundary

$$\oint_{\substack{\text{outer}\\ \text{boundary}}} \vec{E}_1.d\vec{a} = \oint_{\substack{\text{outer}\\ \text{boundary}}} \vec{E}_2.d\vec{a} = \frac{1}{\epsilon_0} Q_{\text{total}}$$

difference $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$ obeys $\vec{\nabla}.\vec{E}_3 = 0$ in region between conductors and $\oint \vec{E}_3.d\vec{a} = 0$ over each boundary surface

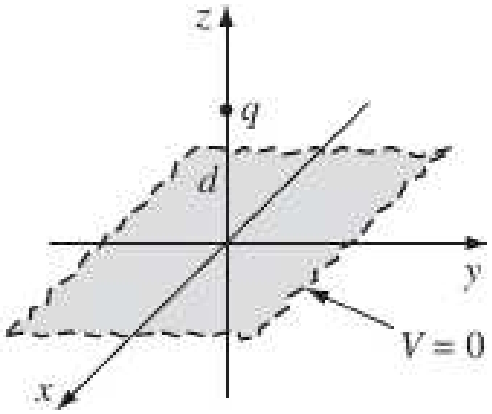
use $\int_{\mathcal{V}} \vec{\nabla}.\left(V_3\vec{E}_3\right) d\tau = \oint_{\mathcal{S}} V_3 \vec{E}_3.d\vec{a} = -\int_{\mathcal{V}} E_3^2\, d\tau = 0$ Last equality comes from V_3 being constant over each surface $\implies E_3 = 0 \implies \vec{E}_1 = \vec{E}_2$

method of images : remarkable application of the uniqueness theorem

conversion of an electric field into another equivalent one that is simpler to calculate

it modifies the field in such a way that, for the region of interest, both the charge distribution and the B.C. are conserved

EX.: Point charge near an infinite conducting plate grounded

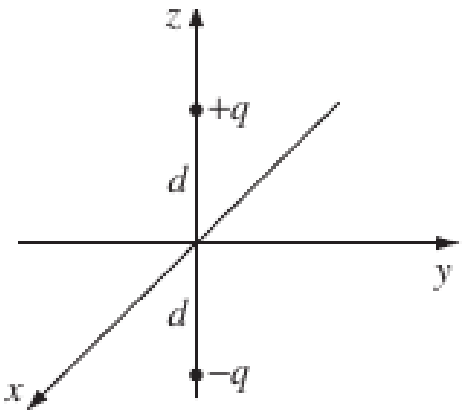


B.C. : $V = 0$
on points where plane is located
and at infinite distances

trick : remove grounded conductor ($V = 0$)
and replace it by a charge $-q$ at distance d
below the plane,
then every point of the plane is equidistant
from q and from $-q$
and will be at zero potential

same boundary conditions as before
$\therefore$ by uniqueness theorem, field to the left
of conducting sheet is unaffected

charge $-q$ is the image of charge q



at a point above $z = 0$,

$V(x, y, z) = \frac{q}{r_+} - \frac{q}{r_-}$;

$r_\pm = \sqrt{x^2 + y^2 + (z \mp d)^2}$

when $x = 0$, $V(\vec{r}) = 0$ and when $x^2 + z^2 \gg d^2$, $V \to 0$

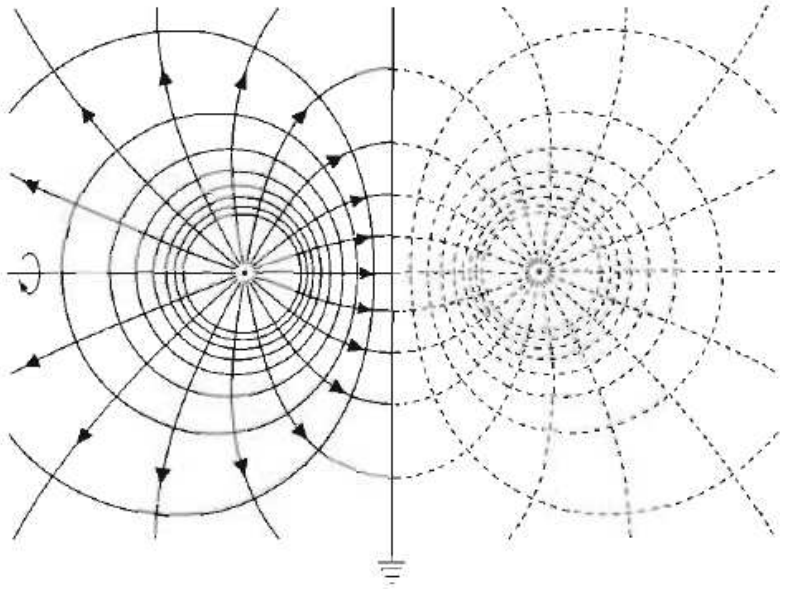
components of the electric field intensity,

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\varepsilon_0}\left[\frac{qx}{r_+^3} - \frac{qx}{r_-^3}\right]$$

$$E_z = -\frac{\partial V}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q(z-d)}{r_+^3} - \frac{q(z+d)}{r_+^3}\right]$$

At surface of the conducting plate, $z = 0$, $r_+ = r_- = r$, $E_z = -\frac{2qd}{4\pi\varepsilon_0 r^3}$; $E_x = 0$

Lines of force Image field
(z-axis shown horizontal)



At surface of conducting plate
$E_z = -\frac{2qd}{4\pi\varepsilon_0 r^3}; \quad E_x = 0;$
induced charge density$= -\varepsilon_0 E_z$

As expected, induced charge is negative and greatest at points on the plane

Equipotential surfaces are generated by rotating the figure about the axis designated by the curved arrow.

lines near the charge cannot be shown because they get too close together
surface charge density is negative, and electric field intensity points to the right at the surface of the conducting plate

total induced charge : $\mathcal{Q} = \int \sigma da = -\int \frac{2qd}{4\pi r^3} r dr d\phi = \frac{2qd}{4\pi} 2\pi \int_d^\infty \frac{dr}{r^2} = -qd\frac{1}{d} = -q$

image charges are located outside the region where we calculate the field

electrostatic force of attraction between charge q and grounded plate$= \frac{q^2}{4\pi\varepsilon_0 (2d)^2}$

$=$ the same as between two charges q and $-q$ separated by a distance $2d$

q cannot tell whether it is in presence of point charge $-q$ or of a grounded plate

force between a charge and a conductor is given correctly by Coulomb force

between the charge and its image charges

With two point charges and no conductor, energy of bringing them together $W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}$

For single point charge and a conductor, energy is half of above, $W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$

Think of energy stored in fields : $W = \frac{\epsilon_0}{2}\int E^2 d\tau$

1st case : both above and below regions contribute

2nd case : only above region contributes

Alternatively, $W = \int_{\infty}^{d} \vec{F}.d\vec{l} = \frac{q^2}{4\pi\varepsilon_0}\int_{\infty}^{d} \frac{1}{4x^2}dx = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$

move q toward the conductor, work done only on q
induced charge is over the conductor, but the whole conductor is at potential zero.

if simultaneously two point charges brought in (with no conductor),
work done on both of them, and the total is twice as great

Solving by separation of variables :

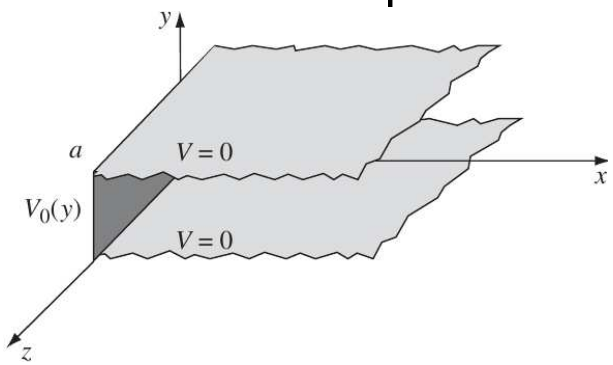
potential, V, or charge density, σ, specified on boundaries of some region

find potential in interior - look for solutions as products of functions
each depending on only one of the coordinates

$V(x, y, z) = X(x)Y(y)Z(z)$, or, $V(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$

Ex. : Two infinite grounded metal plates parallel to the x-z plane

one at $y = 0$, the other at $y = a$



left end, at $x = 0$, is closed off with an infinite strip insulated from plates, and maintained at specific potential $V_0(y)$

Task : Find potential inside this "slot."

$$\text{B.C.} : V = \begin{matrix} 0 \\ V_0(y) \\ \to 0 \end{matrix} \quad \text{when} \quad \begin{matrix} y = 0 \text{ and } a \\ x = 0 \\ x \to \infty \end{matrix}$$

Configuration independent of z

put $V(x, y) = X(x)Y(y)$ in $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \implies Y\frac{d^2 X}{dx^2} = -X\frac{d^2 Y}{dy^2}$

divide by XY, get $\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{\text{f(x) only}} = -\underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{\text{f(y) only}}$: each term must be a constant k^2

or, $\frac{d^2 X}{dx^2} = k^2 X$; $\frac{d^2 Y}{dy^2} = -k^2 Y$; try $X = e^{\alpha x}$ and $Y = e^{\beta y}$, get $\alpha = \pm k$; $\beta = \pm ik$

General solution : $V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

$$V = 0 \text{ at } \begin{array}{l} x \to \infty \implies A = 0 \\ y = 0 \implies D = 0 \end{array} ; \therefore V(x,y) = C'e^{-kx}\sin ky$$

$$V = 0 \text{ at } y = a \implies k = \frac{n\pi}{a}; \therefore V(x,y) = C'e^{-\frac{n\pi}{a}x}\sin\frac{n\pi}{a}y \text{ with } n > 0 \text{ and integer}$$

$\exists$ infinite solutions and a linear combination is also a solution (Fourier series)

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x}\sin\frac{n\pi}{a}y; \qquad V_0(y) = V(0,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a}y$$

completeness and orthogonality :
set of functions $f_n(y)$ is

- complete if any other function $f(y)$ can be expressed as linear combination $f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$
  - functions $\sin(n\pi y/a)$ are complete on the interval $0 \leq y \leq a$
- orthogonal if integral of product of any two different members of the set is zero: $\int f_n(y) f_m(y) dy = 0 \; \forall \; m \neq n$

Fourier basis is complete and orthogonal :

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy = \begin{array}{l} 0 \; \forall \; m \neq n \\ \frac{a}{2} \text{ if } m = n \end{array}$$

$$\therefore \int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy = \sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy = C_n \frac{a}{2}$$

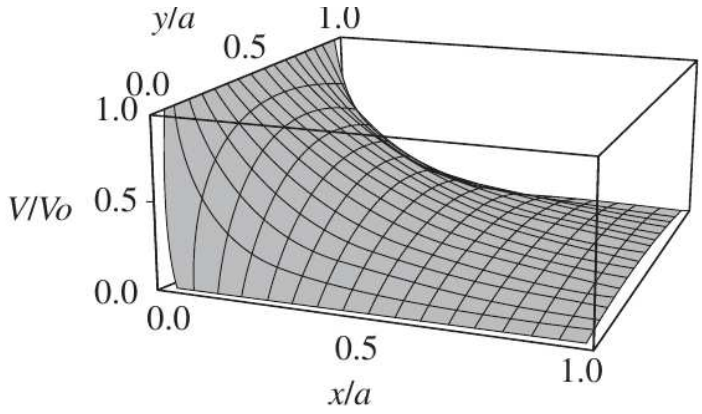
$$\therefore C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

if strip at $x = 0$ is a metal plate with constant potential V_0 insulated from grounded plates at $y = 0$ and a,

then $C_n = \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0}{a}\left(1 - \cos n\pi\right) = \begin{array}{c} 0 \ \forall \ n \,\text{even} \\ \frac{4V_0}{n\pi} \forall \ n \,\text{odd} \end{array}$

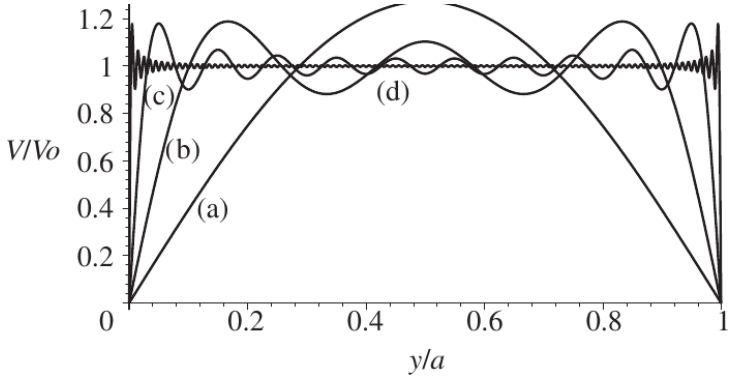
$\therefore V(x, y) = \frac{4V_0}{\pi} \sum_{n>0,\,\text{odd}} \frac{1}{n} e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi y}{a}\right)$

$$\therefore\ V(x,y)=\frac{4V_0}{\pi}\sum_{n>0,\,\mathrm{odd}}\frac{1}{n}e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi y}{a}\right)$$



Fourier sum :

$$V(x,y)=\frac{4V_0}{\pi}\tan^{-1}\left(\frac{\sin\frac{\pi y}{a}}{\sinh\frac{\pi x}{a}}\right)$$



a : $n = 1$; b : n up to 5

c : sum of 10 terms

d: sum of 100 terms

Spherical polar Coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

With azimuthal symmetry : $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$

put $V(r,\theta) = R(r)\Theta(\theta)$, divide by $R\Theta$ and get $\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = 0$

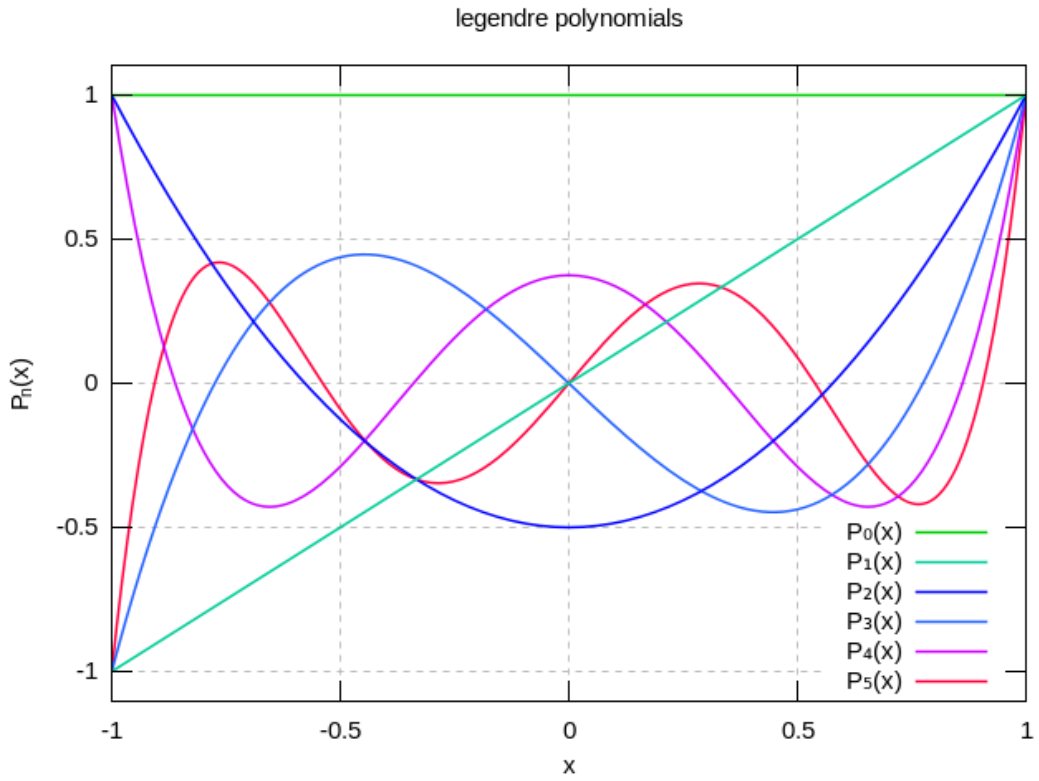
first term depends only on r , and the second only on θ ,
$\therefore$each must be a constant: $\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = l(l+1)$

radial equation, $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$, has general soln. : $R(r) = Ar^l + \frac{B}{r^{l+1}}$

angular equation, $\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta.\Theta$, has general solns. called

Legendre polynomials, $\Theta(\theta) = P_l(\cos\theta)$, a l-th degree polynomial (has l nodes)

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$



$$P_0(x)=1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2-1)$$
$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

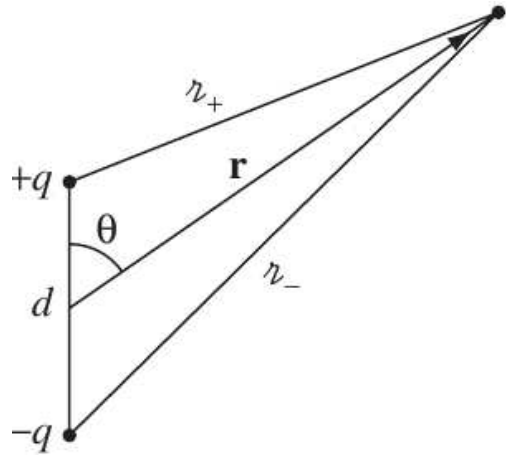
For azimuthal symmetry, general separable solution to Laplace's equation :

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

MULTIPOLE EXPANSION

An electric dipole : two equal and opposite charges ($\pm q$) separated by a distance d

Task : Find approximate potential at points far from the dipole



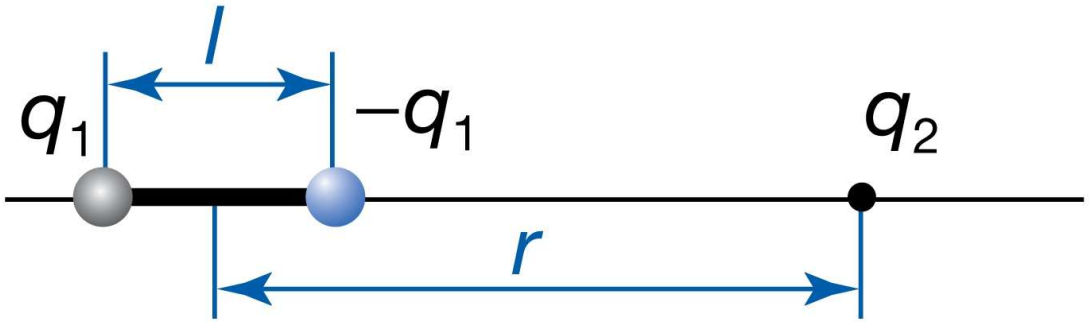
Task : Find approximate potential at points far from the dipole

$$V(r) = \frac{1}{4\pi\epsilon_0}\left(-\frac{q}{\mathscr{r}_-} + \frac{q}{\mathscr{r}_+}\right) \text{ with } \frac{1}{\mathscr{r}_\pm} \approx \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\therefore\ V(r) \approx \frac{1}{4\pi\epsilon_0}\frac{qd}{r^2}\cos\theta$$

The potential of a dipole goes like $1/r^2$ at large r

van der Waals interactions:
point charge - dipole:



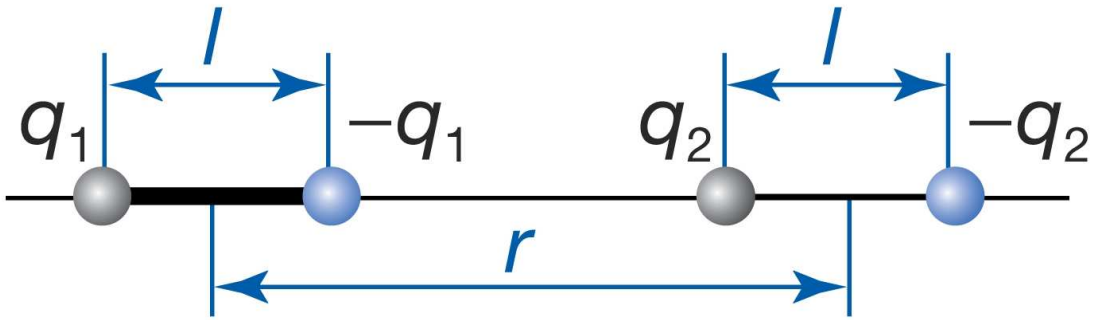
$V = \frac{1}{4\pi\epsilon_0}\left(-\frac{q_1 q_2}{r-\frac{1}{2}l} + \frac{q_1 q_2}{r+\frac{1}{2}l}\right) = \frac{q_1 q_2}{4\pi\epsilon_0 r}\left(-\frac{1}{1-x} + \frac{1}{1+x}\right)$, where $x = \frac{l}{2r}$

point dipole, $l \ll 2r \implies V = \frac{q_1 q_2}{4\pi\epsilon_0 r}\left(-(1 + x + ...) + (1 - x + ...)\right)$

$$= -\frac{2xq_1 q_2}{4\pi\epsilon_0 r} = -\frac{q_1 q_2 l}{4\pi\epsilon_0 r^2} = -\frac{\mu_1 q_2}{4\pi\epsilon_0 r^2}$$

multiplied by $\cos\theta$ when point charge lies at an angle θ to the axis of dipole

dipole-dipole: For a pair of equal and opposite dipoles making a quadrupole potential goes like $1/r^3$

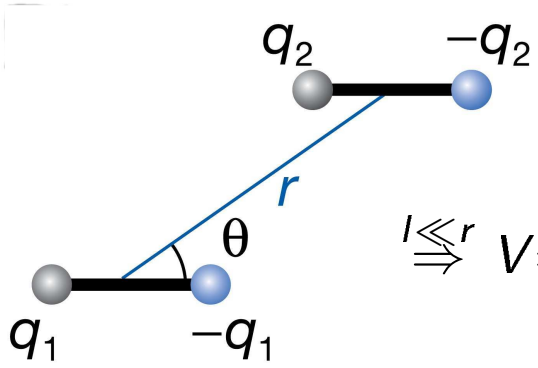


$$V = \frac{1}{4\pi\epsilon_0}\left(-\frac{q_1 q_2}{r+l} + \frac{q_1 q_2}{r} + \frac{q_1 q_2}{r} - \frac{q_1 q_2}{r-l}\right) \overset{x=\frac{l}{r}}{=} -\frac{q_1 q_2}{4\pi\epsilon_0 r}\left(\frac{1}{1+x} - 2 + \frac{1}{1-x}\right)$$

$$\overset{l \ll r}{\Longrightarrow} V = -\frac{2x^2 q_1 q_2}{4\pi\epsilon_0 r} = -\frac{\mu_1 \mu_2}{2\pi\epsilon_0 r^3}$$

$$V = \frac{1}{4\pi\epsilon_0}\left(-\frac{2q_1q_2}{\sqrt{r^2+l^2}} + \frac{2q_1q_2}{r}\right) = \frac{q_1q_2}{4\pi\epsilon_0 r}\left(2 - \frac{2}{\sqrt{1+x^2}}\right)$$

$$(l \ll r) \implies V = \frac{q_1q_2}{4\pi\epsilon_0 r}\left(2 - 2\left\{1 - \frac{1}{2}x^2 + \ldots\right\}\right) = \frac{\mu_1\mu_2}{4\pi\epsilon_0 r^3}$$



$$V = \frac{1}{4\pi\epsilon_0}\left(-\frac{q_1 q_2}{\sqrt{r^2+l^2+2rl\cos\theta}} + \frac{2q_1 q_2}{r} - \frac{q_1 q_2}{\sqrt{r^2+l^2-2rl\cos\theta}}\right)$$

$$\underset{x=\frac{l}{r}}{=} \frac{q_1 q_2}{4\pi\epsilon_0 r}\left(2 - \frac{1}{\sqrt{1+(x^2+2x\cos\theta)}} - \frac{1}{\sqrt{1+(x^2-2x\cos\theta)}}\right)$$

$$\overset{l \ll r}{\Rightarrow} V = \frac{q_1 q_2}{4\pi\epsilon_0 r}\left[2 - \left\{1 - \frac{1}{2}\left(x^2+2x\cos\theta\right) + \frac{3}{8}\left(x^2+2x\cos\theta\right)^2 \ldots\right\} - \left\{1 - \frac{1}{2}\left(x^2-2x\cos\theta\right) + \frac{3}{8}\left(x^2-2x\cos\theta\right)^2 \ldots\right\}\right]$$

ignore terms $\mathcal{O}(x^2)$, $V = \frac{\mu_1 \mu_2 f}{4\pi\epsilon_0 r^3}$; $f = 1 - 3\cos^2\theta$

Generally :

potential at point $\vec{r}$ due to distribn of charges at points $\vec{r_i}$

$$\phi = \sum_i \frac{q_i}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r_i}|} = \sum_i \frac{q_i}{4\pi\epsilon_0}\left[\frac{1}{r} + \vec{\nabla}_{\vec{r_i}}\left(\frac{1}{|\vec{r}-\vec{r_i}|}\right)\Big|_{\vec{r_i}=0} . \vec{r_i} + \cdots\right]$$

$$= \sum_i \frac{q_i}{4\pi\epsilon_0}\left[\frac{1}{r} + \frac{\vec{r}.\vec{r_i}}{r^3} + \cdots\right] \overset{\sum q_i = 0}{=} \frac{1}{4\pi\epsilon_0 r^3}\vec{\mu}.\vec{r} + \cdots$$

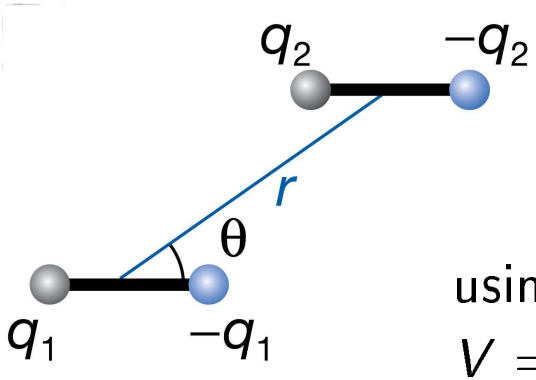
$$\phi_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{\mu}.\vec{r}}{r^3}$$

the electric field at a pt. $\vec{r}$ due to a dipole $\vec{\mu_1}$, is $\vec{\mathcal{E}}_{\text{dip}} = -\vec{\nabla}\phi_{\text{dip}} = -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{\mu_1}.\vec{r}}{r^3}\right)$

or, $\vec{\mathcal{E}}_{\text{dip}} = \frac{\vec{\mu_1}}{4\pi\epsilon_0 r^3} + \frac{1}{4\pi\epsilon_0}\vec{\mu}_1.\vec{r}\vec{\nabla}\left(\frac{1}{r^3}\right) = -\frac{\vec{\mu_1}}{4\pi\epsilon_0 r^3} - \frac{3}{4\pi\epsilon_0}\left(\vec{\mu}_1.\vec{r}\right)\frac{\vec{r}}{r^5}$

The potential energy due to the interaction between the field $\vec{\mathcal{E}}$ of a dipole $\vec{\mu_1}$ and a dipole $\vec{\mu_2}$:

$$V = -\vec{\mathcal{E}}.\vec{\mu_2} = \left[\frac{\vec{\mu_1}}{4\pi\epsilon_0 r^3} - \frac{3}{4\pi\epsilon_0}\left(\vec{\mu}_1.\vec{r}\right)\frac{\vec{r}}{r^5}\right].\vec{\mu_2} = \frac{\vec{\mu_1}.\vec{\mu_2}}{4\pi\epsilon_0 r^3} - \frac{3}{4\pi\epsilon_0}\left(\vec{\mu}_1.\vec{r}\right)\frac{\vec{\mu_2}.\vec{r}}{r^5}$$



using $\vec{\mu_i}.\vec{r} = \mu_i r\cos\theta_i$

$V = \frac{\mu_1\mu_2 f}{4\pi\epsilon_0 r^3}$; $f = 1 - 3\cos^2\theta$

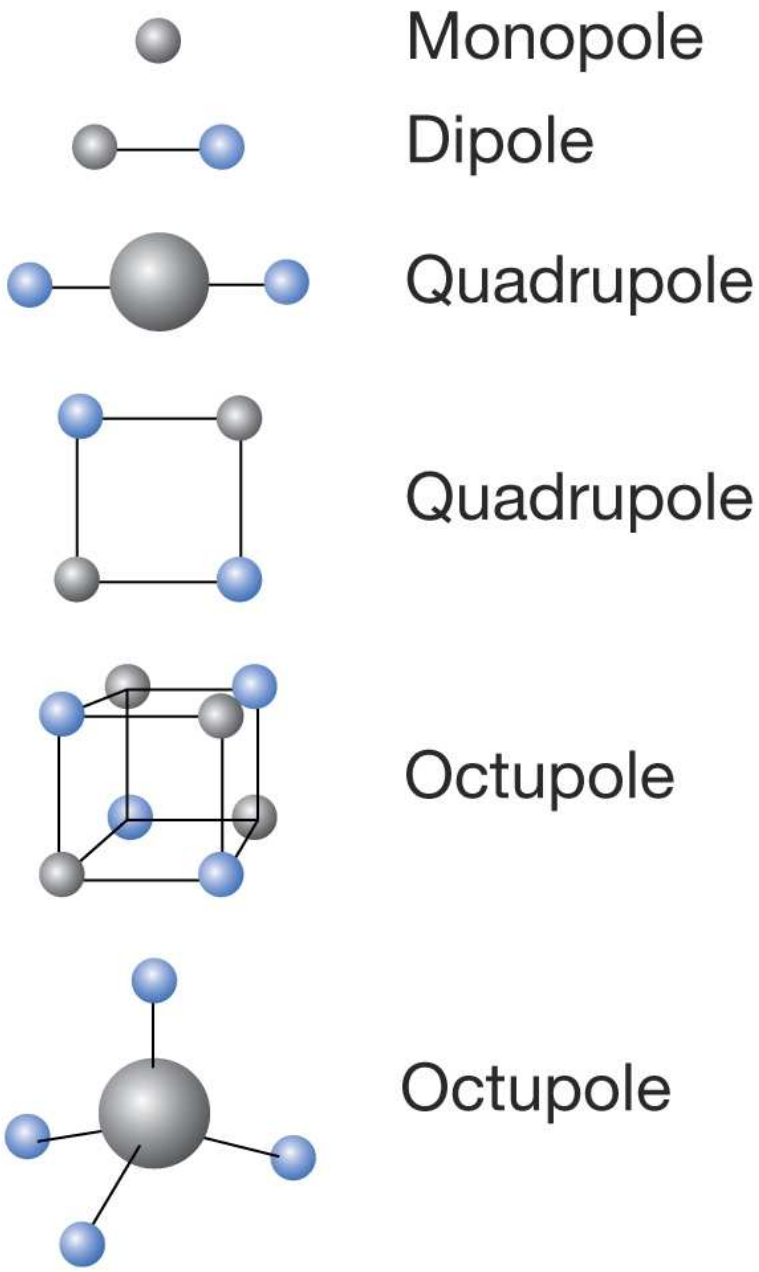
$\vec{\mathcal{E}}_{\text{dip}} = -\frac{\vec{\mu}}{4\pi\epsilon_0 r^3} - \frac{3}{4\pi\epsilon_0}\left(\vec{\mu}.\vec{r}\right)\frac{\vec{r}}{r^5}$

dipole-induced dipole interaction:
The dipole induced due to a field $\vec{\mathcal{E}}$ of a dipole in a neighbouring system:
$\vec{\mu'} = \alpha\vec{E}$, where α is the polarizabilty of the system

interaction, $U = -\vec{\mathcal{E}}.\vec{\mu'} = -\alpha\vec{\mathcal{E}}.\vec{\mathcal{E}} = -\frac{\alpha}{(4\pi\epsilon_0)^2 r^6}\left[\vec{\mu} - 3\left(\vec{\mu}.\vec{r}\right)\frac{\vec{r}}{r^2}\right]^2$

$$= -\frac{\alpha}{(4\pi\epsilon_0)^2 r^6}\left(\mu^2 - 3\mu^2\cos^2\theta - 3\mu^2\cos^2\theta + 9\mu^2\cos^2\theta\right)$$

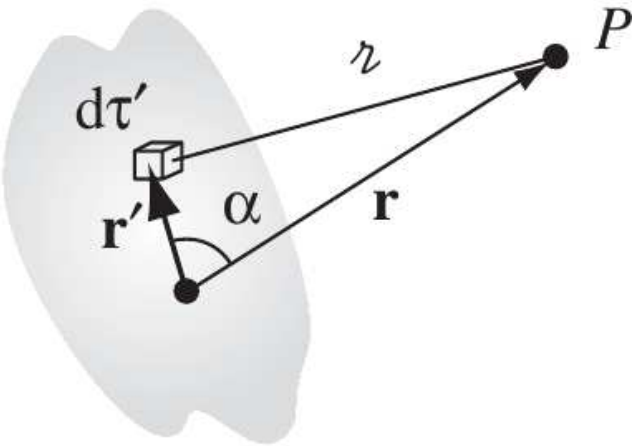
$$= -\frac{\alpha}{(4\pi\epsilon_0)^2}\left(1 + 3\cos^2\theta\right)\frac{\mu^2}{r^6}$$



potential for back-to-back quadrupoles (an octopole), potential goes like $1/r^4$

.

systematic expansion for the potential of any localized charge distribution :



$$\mathscr{r}^2 = \sqrt{r^2 + r'^2 - 2rr'\cos\alpha} = r^2\left[1+\left(\frac{r'}{r}\right)^2\right] - 2r^2\left(\frac{r'}{r}\right)\cos\alpha$$

$$\text{or, } \mathscr{r} = r\sqrt{1+\epsilon}\ ; \quad \epsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right)$$

potential at $\vec{r} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}|} d\tau'$ $\qquad \frac{1}{\mathscr{r}} = \frac{(1+\epsilon)^{-\frac{1}{2}}}{r} = \frac{1}{r}\left(1 - \frac{\epsilon}{2} + \frac{3\epsilon^2}{8} - \frac{5\epsilon^3}{16} + \cdots\right)$

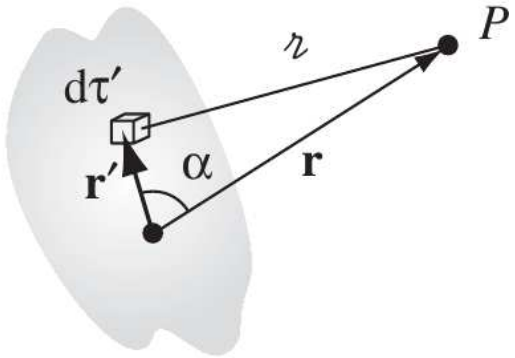
$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5\epsilon^3}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2 \frac{3\cos^2\alpha - 1}{2} - \left(\frac{r'}{r}\right)^3 \frac{5\cos^3\alpha - 3\cos\alpha}{2} + \cdots\right] = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$\therefore\ V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\alpha)\rho(\vec{r}')d\tau'$: multipole expansion of V in powers of $\frac{1}{r}$

$\therefore V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\alpha)\rho(\vec{r'})d\tau'$ dominated at large r by the monopole term:

$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$ where $Q = \int \rho d\tau =$ total charge



if $Q = 0$, dominant term is the dipole

$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\rho(\vec{r'})d\tau'$

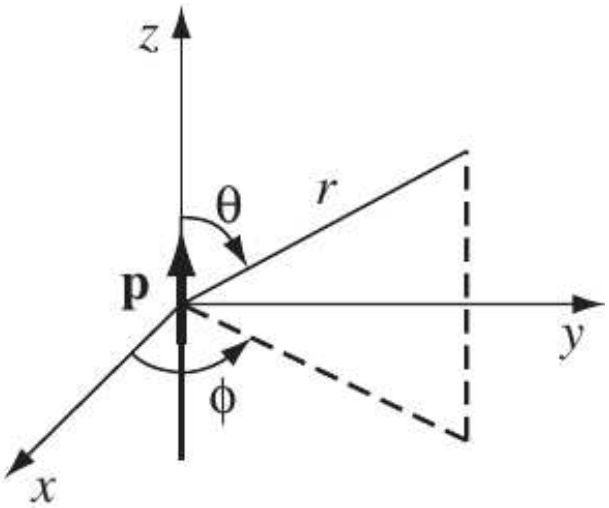
use $r'\cos\alpha = \hat{r}.\vec{r'}$, get $V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}.\underbrace{\int \vec{r'}\rho(\vec{r'})d\tau'}_{\text{dipole moment}}$

dipole moment , $\vec{p} = \int \vec{r'}\rho(\vec{r'})d\tau'$ $\quad V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}.\hat{r}}{r^2}$

dipole moment is determined by geometry (size, shape, and density) of charge distribution

dipole moment of a collection of point charges, $\vec{p} = \sum_{i=1}^{n} q_i \vec{r_i'}$

Electric Field of a Dipole

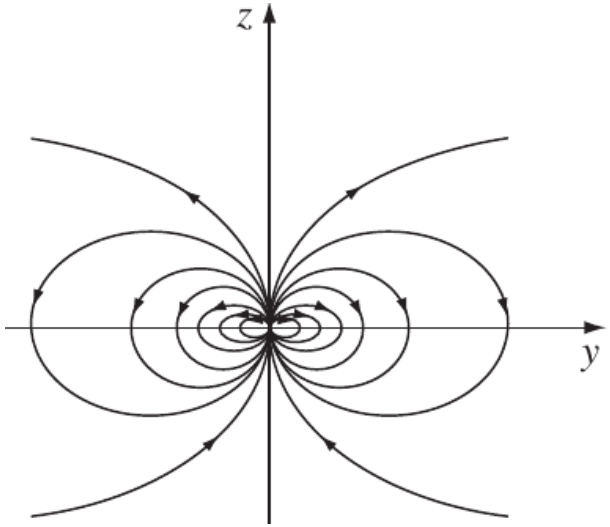


$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}.\hat{r}}{r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

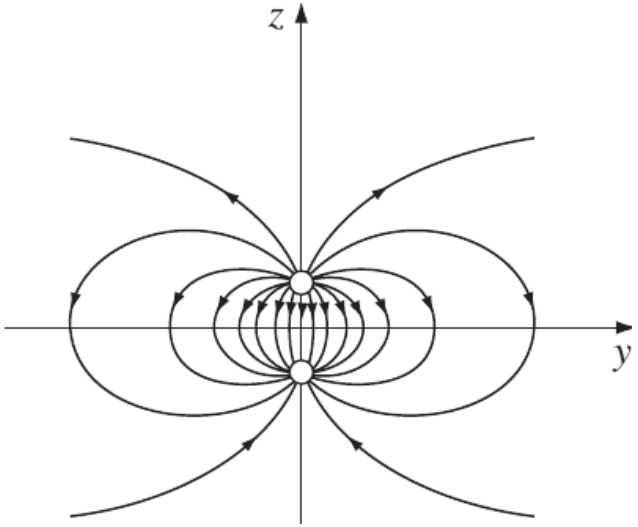
$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3};\ E = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

$$\therefore\ \vec{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$



pure dipole

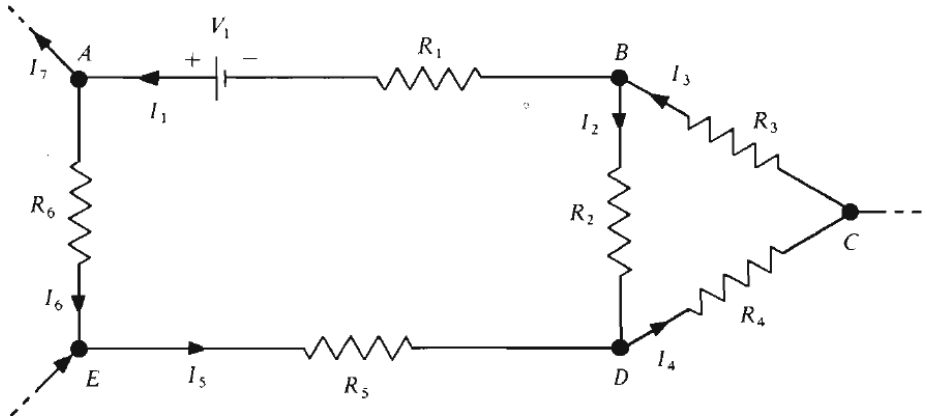


physical dipole

Resistors connected in series and parallel : left as exercise

Kirchoff's law

consider a general circuit composed of resistors and voltage sources



Junctions, e.g., A, B, C, D, E are called nodes

connections between nodes, e.g., AB, BD, DC .. ., are called branches

closed circuits, e.g., as ABDE, or ABCDE, or BCD, are called meshes

Kirchoff's current law :

at any given node, the sum of the incoming currents is equal to the sum of the outgoing currents

e.g., at A, $I_1 = I_6 + I_7$

Kirchoff's voltage law :

voltage rise, or voltage drop, around any given mesh is zero

e.g., the voltage rise from A to B, to D, to E, and back to A, is clearly zero

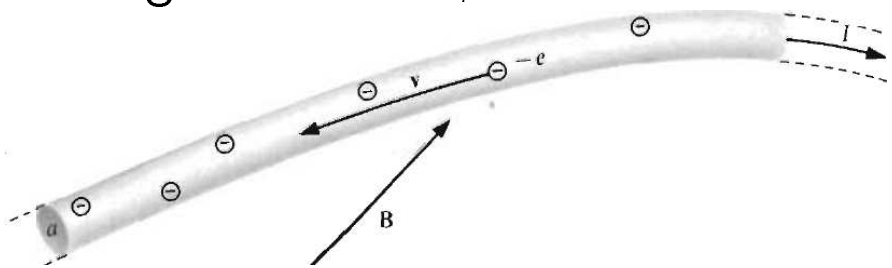
$$-V_1 + I_1 R_1 - I_2 R_2 + I_5 R_5 + I_6 R_6 = 0$$

Force on a wire carrying a current in a magnetic field

$$\vec{F}_{\text{mag}} = \int \vec{v} \times \vec{B} dq = \int \vec{v} \times \vec{B} \lambda dl = \int \vec{I} \times \vec{B} dl = \int I.d\vec{l} \times \vec{B} \overset{\text{I const}}{=} I \int d\vec{l} \times \vec{B}$$

Consider a wire of cross-sectional area a carrying a current I
n conduction electrons per unit volume, each carrying charge $-e$ at velocity $\vec{v}$

in a magnetic field $\vec{B}$,



for length dl containing
$n.a.dl$ conduction electrons
$d\vec{F} = n.a.dl \left(-e\vec{v} \times \vec{B}\right)$

magnetic force on one electron $= -e\vec{v} \times \vec{B}$

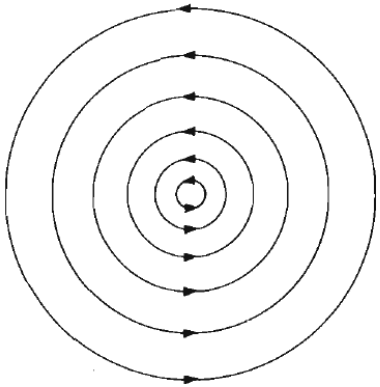
charge passing through a given cross-section per second
=charge on carriers contained in a length v of the wire

$\vec{I} = -nae\vec{v}$ then $d\vec{F} = Id\vec{l} \times \vec{B}$ $\therefore \vec{F}_{\text{mag}} \overset{\text{I const}}{=} I \int d\vec{l} \times \vec{B}$

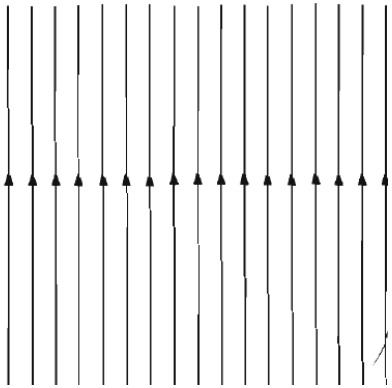
For a straight length l of wire carrying a current I in a uniform magnetic field $\vec{B}$

$$\vec{F} = I.\vec{l} \times \vec{B}$$

Magnetic field near a current-carrying wire situated in uniform magnetic field



field due to current $\perp$ paper

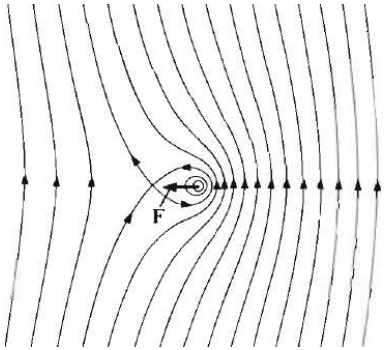


uniform field $\parallel$ paper

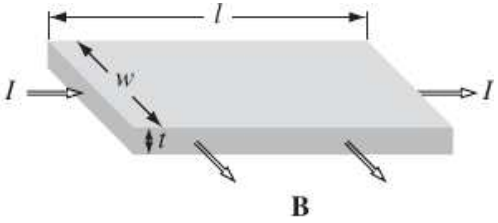
Superposition of fields and resulting magnetic force $\vec{F}$

ex. : current 10 amp; uniform field $\vec{B} = 2 \times 10^{-4}$tesla

point where lines of $\vec{B}$ are broken is at 10 mm from center of wire



Hall effect (1879) :
production of a potential difference (Hall voltage) across an electrical conductor transverse to an electric current in the conductor
and to an applied magnetic field $\perp$ current



current I flows to the right through rectangular bar of conducting material in presence of uniform magnetic field $\vec{B}$ pointing out of the page

If the moving charges are positive, in which direction are they deflected by the magnetic field?

$\vec{F} = q\vec{v} \times \vec{B}$: they are deflected down

bottom plate acquires a positive charge $\Longrightarrow$
potential difference between the top and bottom plates

equilibrium : $qvB = qE$, or, $E = vB$
for thickness d, $V = Ed = vBd$

If - ve charges flow to the left, they also deflected down :
bottom plate acquires a negative charge
potential difference still same, but this time top plate
at higher potential.

DIELECTRICS: no free charges that can move through material under influence of an electric field

all electrons bound to specific atoms/molecules

only motion : minute displacement of $+$ve and $-$ve charges in opposite directions

polarized dielectric : charge displacement - stretching and rotating

induced dipole moments produce their own field, adding to field of external charges

dipole field and external field can be comparable in magnitude

applied field orients molecules with permanent dipole moments

$\Longrightarrow$ torque that tends to align them with the field
but collisions due to thermal agitation tend to destroy the alignment

equilibrium polarization $\Longrightarrow$ on average, a net alignment

induced dipole moment $\vec{\mu}_{\text{ind}} = \alpha \vec{E}$; const. α : polarizability

Atomic Polarizabilities ($\frac{\alpha}{4\pi\epsilon_0}$, in units of 10^{-30} m^3)

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

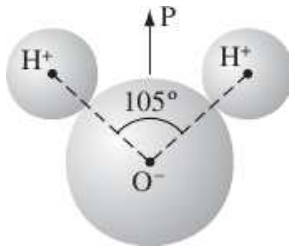
molecules polarize more readily in some directions than in others

for CO_2, $\vec{p} = \alpha_\perp \vec{E}_\perp + \alpha_\parallel \vec{E}_\parallel$ $\quad \alpha_\parallel = 4.05 \times 10^{-30}\ \text{m}^3 (\vec{E} \parallel \text{axis of molecule})$

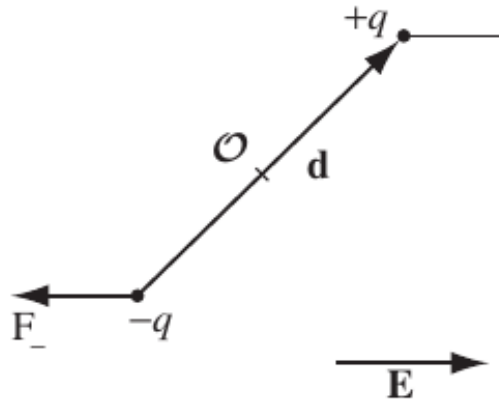
$\alpha_\perp = 1.8 \times 10^{-30}\ \text{m}^3 (\vec{E} \perp \text{axis of molecule})$

Generally : $\vec{\mu}_{\text{ind}} = \underline{\boldsymbol{\alpha}} \vec{E}$, where $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \Leftarrow$ Polarizability tensor

Alignment of Polar Molecules :



For molecules with permanent dipole moments and uniform applied field, e.g., H_2O



force on positive end, $\vec{F}_+ = q\vec{E}$

exactly cancels force on negative end $\vec{F}_- = -q\vec{E}$

$\exists$ torque: $\vec{N} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$

$= (\vec{d}/2) \times (q\vec{E}) + (-\vec{d}/2) \times (-q\vec{E})$

$= q\vec{d} \times \vec{E} = \vec{\mu} \times \vec{E}$

dipole $\vec{\mu} = q\vec{d}$ in a uniform field $\vec{E}$ experiences a torque $\vec{N} = \vec{\mu} \times \vec{E}$

polar molecule free to rotate swings around until $\vec{\mu}$ is in the direction of applied field

nonuniform field : $\vec{F}_+$ does not balance $\vec{F}_-$

$\Longrightarrow$ $\exists$ net force on dipole in addition to torque : not serious issue for small molecules

$$\vec{F} = \vec{F}_+ + \vec{F}_- = q\left(\vec{E}_+ - \vec{E}_-\right) = q\Delta\vec{E} = q(\vec{d}.\vec{\nabla})\vec{E} = \left(\vec{\mu}.\vec{\nabla}\right)\vec{E}, \text{ Note : } \Delta\vec{E} = \sum_i \left(\vec{\nabla} E_{x_i}\right).\vec{d}$$

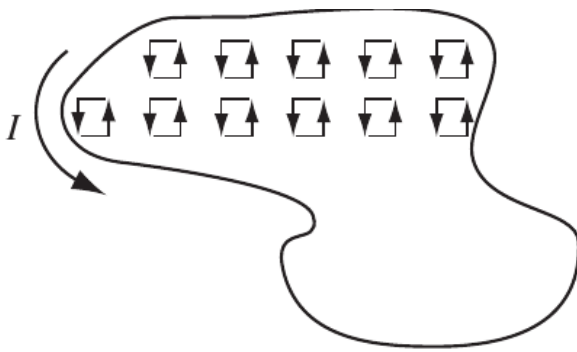
In the present context, $\left(\vec{\mu}.\vec{\nabla}\right)\vec{E} = \vec{\nabla}\left(\vec{\mu}.\vec{E}\right)$: $\vec{\mu}$ is not differentiated

For a 'perfect' dipole of infinitesimal length
torque about center of dipole in a nonuniform field, $\vec{N} = \vec{\mu} \times \vec{E}$;

about any other point $\vec{N} = \vec{\mu} \times \vec{E} + \vec{r} \times \vec{F}$

Magnetic Torques and Forces

magnetic dipole experiences a torque in a magnetic field



any current loop could be built up from infinitesimal rectangular loops, with all 'internal' sides canceling

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consider a rectangular current loop in a uniform field $\vec{B}$

Center loop at origin, and tilt it an angle θ from z-axis towards y-axis

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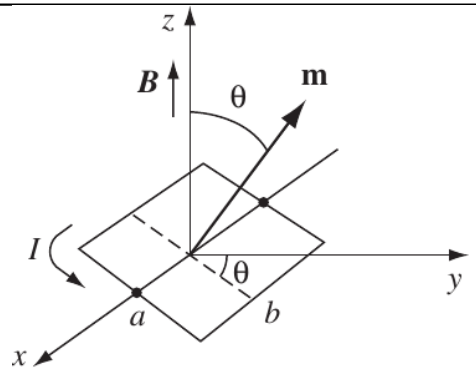
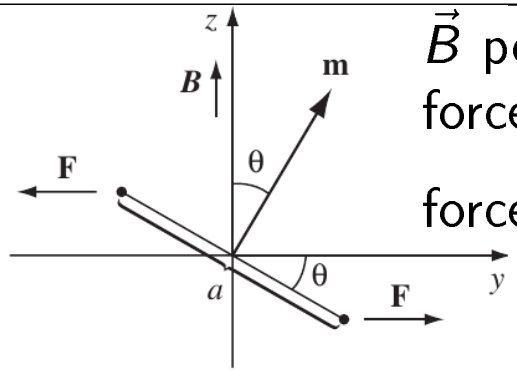




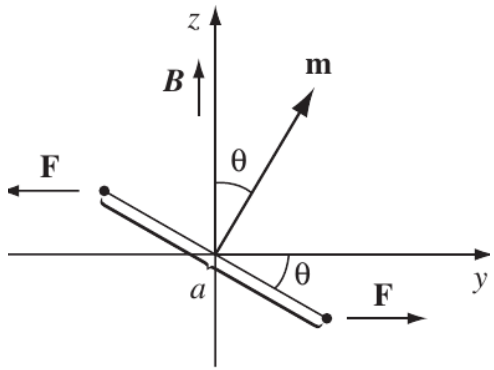
$\vec{B}$ points in z direction

forces on two sloping sides cancel

forces on 'horizontal' sides are equal and opposite

net force on loop$= 0$

torque: $\vec{N} = aF \sin\theta . \hat{x}$



magnitude of force on each segment, $F = IbB$

$\vec{N} = IabB \sin\theta.\hat{x} = mB \sin\theta.\hat{x}$, or, $\vec{N} = \vec{m} \times \vec{B}$

where $m = Iab$ = magnetic dipole moment of loop

$\implies$ torque on localized current distribution

torque (about the center) for dipole of infinitesimal size $\implies$ accounts for paramagnetism

$\vec{N} = \vec{m} \times \vec{B}$: Eq. is identical in form to the electrical analog

torque is in a direction as to line the dipole $\parallel$ field

Since every electron constitutes a magnetic dipole, paramagnetism should be a universal phenomenon

Pauli exclusion principle locks electrons together in pairs with opposing spins and this effectively neutralizes the torque on the combination

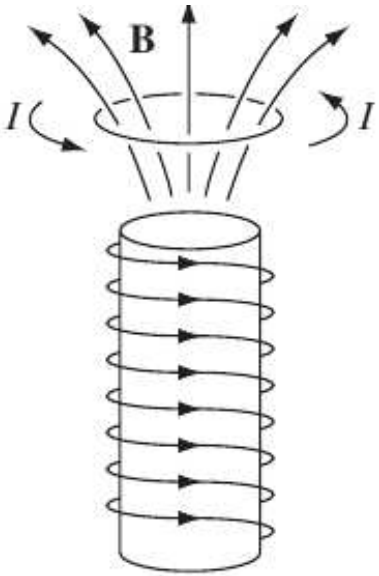
$\therefore$ paramagnetism mostly occurs in atoms/molecules with odd number of electrons, where unpaired spin is subject to magnetic torque

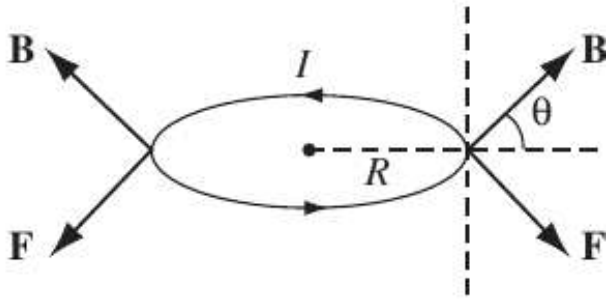
alignment is far from complete, since random thermal collisions destroy the order

In a uniform field, net force on a current loop$= 0$

$$\vec{F} = I \oint d\vec{l} \times \vec{B} = I \left(\oint d\vec{l} \right) \times \vec{B} = 0$$

In a nonuniform field this is no longer the case
e.g., consider circular wire ring of radius R, carrying current I , suspended above a short solenoid in the 'fringing' region





$\vec{B}$ has a radial component, and there is a net downward force on the loop,

$$F = I \oint d\vec{l} \times \vec{B} = I \int B.dl.\sin(\tfrac{\pi}{2} - \theta) = 2\pi IRB\cos\theta$$

(B constant, direction alone varies)

For an infinitesimal loop, with dipole moment $\vec{m}$, in a field $\vec{B}$, $\vec{F} = \vec{\nabla}(\vec{m}.\vec{B})$

ELECTRIC POLARIZATION $\vec{P} = N\langle\vec{\mu}\rangle$, where $N = \#$molecules per unit volume

BOUND Charge densities : displacements on molecular scale by polarization

action of a field : $+$ve and $-$ve charges separate by an average distance

$\Longrightarrow$ net volume and surface charge densities

$\rho_b = -\vec{\nabla}.\vec{P}; \quad \sigma_b = \vec{P}.\hat{n}; \quad \hat{n}$: normal to surface, pointing outward

potential (hence field) of polarized object same as that produced by ρ_b and σ_b

Instead of integrating contributions of all infinitesimal dipoles

find bound charges, and calculate fields they produce

other charges are free charges : e.g., conduction electrons in a conductor

and electrons injected with high-energy electron beam

MAGNETIC Materials :tiny currents of electrons orbit around nuclei with spin angular momentum

macroscopic scale : current loops are like small magnetic dipoles
MAGNETIZATION, net alignment of magnetic dipoles on an atomic scale -
$\vec{M} \equiv$ magnetic dipole moment per unit volume

Ordinarily, they cancel each other out because of random orientation of atoms
When magnetic field is applied, net alignment of magnetic dipoles occurs

Unlike electric polarization, which is almost always in the same direction as $\vec{E}$
in some materials **spins** $\parallel \vec{B}$ (**paramagnets**); some **opposite to** $\vec{B}$ **(diamagnets**)

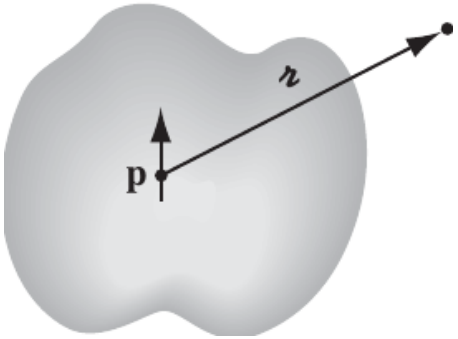
few substances (**ferromagnets**) retain magnetization even after external field is removed
—for these, magnetization is not determined by the whole magnetic 'history'

Permanent magnets made of iron : familiar but theoretically most complicated
diamagnetism and paramagnetism are extremely weak : It takes delicate experiment and a powerful magnet to detect them

when a sample is in a region of nonuniform field : paramagnet attracted into field
diamagnet repelled away

actual forces are very weak : force on comparable sample of iron $10^4 - 10^5$ times greater

electro : Field produced due to polarization :



For a single dipole $\phi_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{\mu}.\vec{\mathscr{r}}}{\mathscr{r}^3}$

With net polarization $\vec{P}(\vec{r}')$, $\vec{\mu} = \vec{P}(\vec{r}')d\tau'$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}').\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int \vec{P}(\vec{r}').\vec{\nabla}_{\vec{r}'}\left(\frac{1}{\mathscr{r}}\right)d\tau' \quad [\because \vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\int \vec{\nabla}_{\vec{r}'}.\left(\frac{\vec{P}}{\mathscr{r}}\right)d\tau' - \int \frac{1}{\mathscr{r}}\vec{\nabla}_{\vec{r}'}.\vec{P}d\tau'\right]$$

$$= \frac{1}{4\pi\epsilon_0}\oint \frac{\vec{P}}{\mathscr{r}}.d\vec{a}' - \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\vec{\nabla}_{\vec{r}'}.\vec{P}d\tau'$$

1st term $\implies$ potential of a surface charge : $\sigma_b \equiv \vec{P}.\hat{n}$

2nd term $\implies$ potential of a volume charge : $\rho_b \equiv -\vec{\nabla}.\vec{P}$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma_b}{\mathscr{r}}.da' + \frac{1}{4\pi\epsilon_0}\int \frac{\rho_b}{\mathscr{r}}d\tau'$$

$Q = \int (\rho_f + \rho_b)\, d\tau \implies \vec{\nabla}.\vec{E} = \frac{\rho_f + \rho_b}{\varepsilon_0}$: GAUSS'S LAW

electrostatics : scalar potential due to polarization :
$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma_b}{\mathscr{r}}.da' + \frac{1}{4\pi\epsilon_0}\int \frac{\rho_b}{\mathscr{r}} d\tau'$

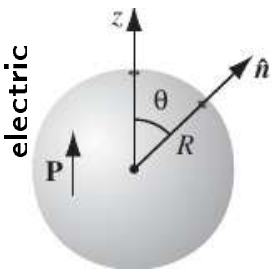
similarly, vector potential due to magnetisation : $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\int \frac{\vec{K}_b}{r_s} da' + \int \frac{\vec{J}_b}{r_s} d\tau'\right]$

Bound Currents : $\vec{J}_b = \vec{\nabla} \times \vec{M}$; $\quad \vec{K}_b = \vec{M} \times \hat{n}$ and $\vec{\nabla}.\vec{J}_b = 0$

potential (and field) of a magnetized object is same as produced by a volume current $\vec{J}_b = \vec{\nabla} \times \vec{M}$ throughout the material, plus a surface current $\vec{K}_b = \vec{M} \times \hat{n}$ on the boundary

Instead of integrating the contributions of infinitesimal dipoles, determine bound currents, and then the field they produce
in the same way as we calculate field of any volume and surface currents

Ex. : field produced by uniformly polarized/magnetized sphere of radius R



choose z-axis as direction of polarization

uniform $\vec{P} \Longrightarrow \rho_b = 0$

$\sigma_b = \vec{P}.\hat{n} = P\cos\theta$

$V(r,\theta) =$

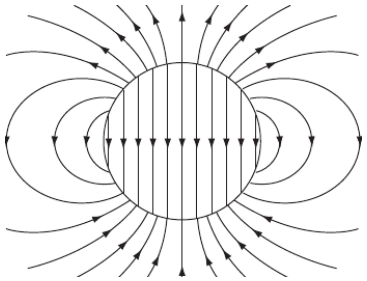
$$\frac{P\cos\theta.r}{3\epsilon_0} = \frac{Pz}{3\epsilon_0} \forall r \leq R$$

$$\frac{P\cos\theta.R^3}{3\epsilon_0.r^2} = \frac{PzR^3}{3\epsilon_0 r^3} \forall r \geq R$$

$\vec{E}(\vec{r}) = -\vec{\nabla}V =$

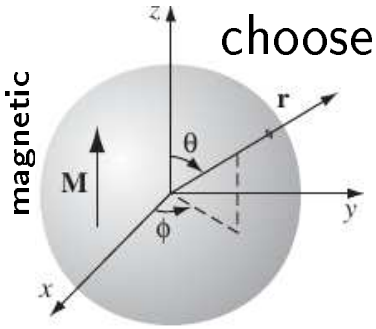
$$-\frac{P}{3\epsilon_0}\hat{z} = -\frac{\vec{P}}{3\epsilon_0} \forall r \leq R$$

$$\frac{P}{3\epsilon_0 r^4}\left[2z\hat{r} + \sqrt{x^2+y^2}\hat{\theta}\right] \forall r \geq R$$



outside potential identical to that of perfect dipole at origin

$\phi_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{\mu}.\hat{r}}{r^2}$ where $\vec{\mu} = \frac{4}{3}\pi R^3 \vec{P}$



choose z-axis along direction of $\vec{M}$

$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$

$\vec{K}_b = \vec{M} \times \hat{n} = M\sin\theta\hat{\phi}$

rotating spherical shell, of uniform surface charge σ, corresponds to a surface current density

$$\vec{K} = \sigma\vec{v} = \sigma\omega R\sin\theta\hat{\phi}$$

$\therefore$ field identical to that of a spinning spherical shell, with $\sigma R\vec{\omega} \equiv \vec{M}$

inside sphere, $\vec{B} = \frac{2}{3}\mu_0\vec{M}$, outside : same as that of perfect dipole, $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$

internal field is uniform

Physical Interpretation of Bound Charges

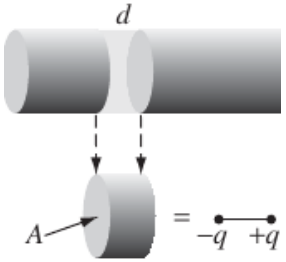
Imagine a long string of dipoles

head of one dipole effectively cancels the tail of its neighbor

but at the ends there are two charges left over:

plus at the right end and minus at the left

examine a 'tube' of dielectric parallel to $\vec{P}$



dipole moment of tiny chunk $= P(Ad) = qd$, where

$A =$ cross-sectional area, $d =$ length

$\therefore q = PA \implies \sigma_b = \frac{q}{A} = P$

For an oblique cut, charge is still same

$A = A_{\text{end}} \implies \sigma_b = \frac{q}{A_{\text{end}}}$

$= P\cos\theta = \vec{P}.\hat{n}$

polarization $\implies$ bound charge $\sigma_b = \vec{P}.\hat{n}$ spread over surface of the material

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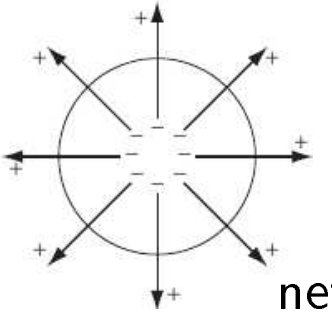
when polarization is nonuniform



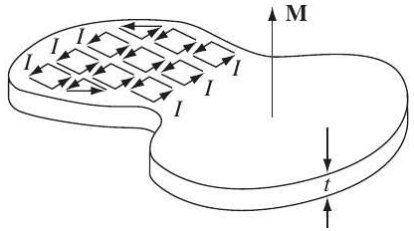
bound charge accumulates within the material as well as on surface

a diverging $\vec{P}$ results in a pileup of negative charge

net bound charge, $\int_{\mathcal{V}} \rho_b d\tau = \oint_{\mathcal{S}} \vec{P}.d\vec{a} = -\int_{\mathcal{V}} \vec{\nabla}.\vec{P} d\tau \implies \rho_b = -\vec{\nabla}.\vec{P}$

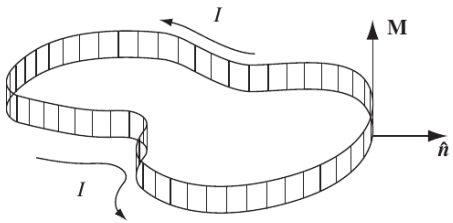
Physical Interpretation of Bound Currents

consider a thin slab of uniformly magnetized material, with dipoles represented by tiny current loops



all internal currents cancel

equivalent to single ribbon of current I around the boundary



In terms of magnetization M, its dipole moment $m = M \times \text{area} \times \text{thickness} = Mad$

In terms of current I, $m = Ia$, $\quad \therefore I = Md$
so surface current $K_b = I/d = M$; direction indicated by $\vec{K}_b = \vec{M} \times \hat{n}$
a peculiar current : each charge moves only in tiny little loop within single atom
net effect : macroscopic current flowing over the surface
it is "bound" current : every charge is attached to particular atom

When magnetization is nonuniform, internal currents no longer cancel
actual microscopic magnetic field inside matter fluctuates wildly from point to point and instant to instant

magnetic field in matter - macroscopic : average over regions large enough to contain many atoms

ELECTRIC DISPLACEMENT $\vec{D}$

use : $\rho_b = -\vec{\nabla}.\vec{P}$ and $\vec{\nabla}.\vec{E} = \frac{\rho_f + \rho_b}{\varepsilon_0}$

$\vec{\nabla}.\vec{E} = \frac{\rho_f - \vec{\nabla}.\vec{P}}{\varepsilon_0}$, or, $\vec{\nabla}.\left(\varepsilon_0 \vec{E} + \vec{P}\right) = \rho_f$

electric displacement $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \quad \therefore \vec{\nabla}.\vec{D} = \rho_f$

integral form, Gauss's law : $\int\limits_S \vec{D}.\overrightarrow{da} = \int \rho_f \, d\tau = Q_{f_{enc}}$, total free charge enclosed

flux of electric displacement $\vec{D}$ through a closed surface = free charge enclosed

$\vec{\nabla}.\vec{D}$ as well as $\int\limits_S \vec{D}.\overrightarrow{da}$ are both unaffected by the bound charges

AUXILIARY FIELD $\vec{H}$

Ampère's Law in Magnetized Materials

field due to magnetization of a medium is the field produced by bound currents

everything else other than bound currents — call it the free current

free current flows through wires imbedded in magnetized substance or, if it is a conductor, through the material itself

total current can be written as $\vec{J} = \vec{J}_f + \vec{J}_b$

free current involves actual transport of charge
bound current is there because of magnetization

Ampère's law : $\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \vec{\nabla}\times\vec{M}$

or, $\vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \vec{J}_f$

write : $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$; Ampère's law : $\vec{\nabla}\times\vec{H} = \vec{J}_f$

or, in integral form, $\oint \vec{H}.d\vec{l} = I_{f_{\text{enc}}}$, total free current passing through Amperian loop

electro- : $\vec{D}$	magneto- : $\vec{H}$
Gauss's law in terms of free charge alone	Ampère's law in terms of free current

B.C. in terms of $\vec{D}$

$\int_S \vec{D}.\overrightarrow{da} = Q_{f_{\text{enc}}} \implies D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$

and $\vec{D} = \epsilon_0 \vec{E} + \vec{P} \implies \vec{D}^{\parallel}_{\text{above}} - \vec{D}^{\parallel}_{\text{below}} = \vec{P}^{\parallel}_{\text{above}} - \vec{P}^{\parallel}_{\text{below}}$

Deceptive Parallel : Eqn $\vec{\nabla}.\vec{D} = \rho_f$ looks like Gauss's law

false; there is no "Coulomb's law" for D: $\vec{D}(\vec{r}) \neq \frac{1}{4\pi\varepsilon_0} \int \frac{\hat{\mathscr{r}} \rho_f(\vec{r'}) d\tau'}{\mathscr{r}^2}$

divergence alone is insufficient to determine a vector field

need to know curl also - for electrostatic fields curl $= 0$

$\vec{\nabla} \times \vec{D} = \epsilon_0 (\vec{\nabla} \times \vec{E}) + (\vec{\nabla} \times \vec{P}) = \vec{\nabla} \times \vec{P}$

ELECTRIC SUSCEPTIBILITY, χ_e

in linear and isotropic dielectrics molecular charge separation $\propto \vec{E}$ and $\parallel \vec{E}$

Most dielectrics are homogeneous, linear and isotropic

dipole moment per unit volume $= \vec{P} = N\vec{\mu} = \chi_e \varepsilon_0 \vec{E}$

$$\vec{\mu}_{\text{ind}} = \epsilon_0 \underline{\boldsymbol{\chi}} \vec{E}, \text{ where } \underline{\boldsymbol{\chi}} = \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \Leftarrow \text{Susceptibility tensor}$$

electric susceptibility of dielectric : χ_e - dimensionless const.

RELATIVE PERMITTIVITY ε_r (dielectric constant)

For a linear and isotropic dielectric : $\vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$

$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$: dimensionless const$>$ 1, formerly called dielectric constant

typically $2 < \varepsilon_r < 7$

In vacuum, $\chi_e = 0$, $\varepsilon_r = 1$; ε is called the permittivity

Type	Frequency (hertz) 100	10^6	10^{10}
Bakelite	5.50	4.45	3.55
Butyl rubber	2.43	2.40	2.38
Fused silica	3.78	3.78	3.78
Lucite	3.20	2.63	2.57
Neoprene	6.70	6.26	4.0
Polystyrene	2.56	2.56	2.54
Steatite	6.55	6.53	6.51
Styrofoam	1.03	1.03	1.03
Teflon	2.1	2.1	2.08
Water	81.	78.2	34.

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

some non-linear dielectrics have ε_r as high as 10^5

Dielectrics : Relation between free charge densities and bound charge densities

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \varepsilon_r \vec{E} \implies \vec{P} = (\varepsilon_r - 1)\,\varepsilon_0 \vec{E} = \frac{\varepsilon_r - 1}{\varepsilon_r} \vec{D}$

Taking divergence of both sides and using $\rho_b = -\vec{\nabla}.\vec{P}$ and $\vec{\nabla}.\vec{D} = \rho_f$

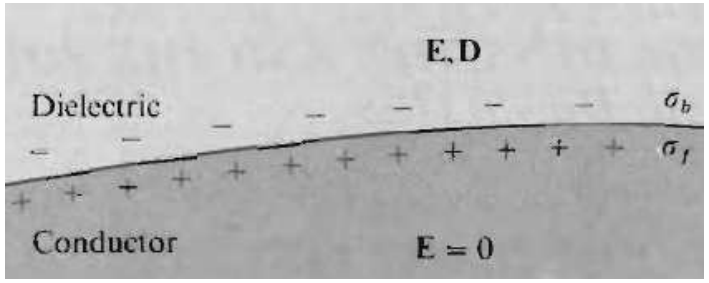
$\rho_b = -\frac{\varepsilon_r - 1}{\varepsilon_r} \rho_f$

ρ_b and ρ_f have opposite signs, so total charge density$< \rho_f$

$\rho_f + \rho_b = \frac{\rho_f}{\varepsilon_r}$

surface charge densities σ_b and σ_f

At interface between a dielectric and a conductor, there is bound surface charge density σ_b on the dielectric



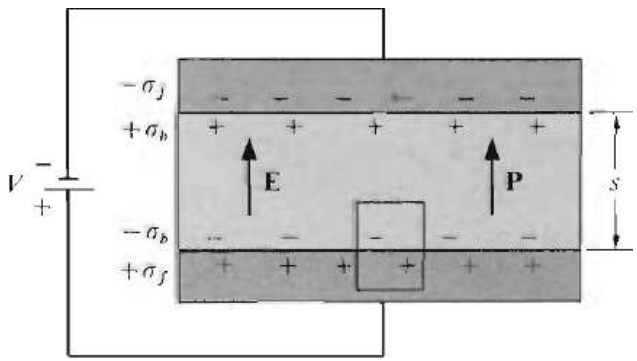
and free surface charge density σ_f on conductor

steady-state conditions : zero electric field inside conductor

inside dielectric, Gauss's law : $\varepsilon_0 \vec{E} = \sigma_f + \sigma_b$, $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \sigma_f$

$\therefore \sigma_f + \sigma_b = \frac{\sigma_f}{\varepsilon_r}$

DIELECTRIC-INSULATED parallel-plate capacitor



assume s small compared to linear extent of plates

neglect fringing field at edges

Then $\vec{P}$ is uniform, $\vec{\nabla}.\vec{P} = 0$

and $\rho_b = 0$ consistent with $\rho_f = 0$

surface-charge densities $\begin{matrix}\sigma_f \\ -\sigma_f\end{matrix}$ on $\begin{matrix}\text{lower} \\ \text{upper}\end{matrix}$ plate $\implies$ uniform field directed upward

polarization $\implies$ bound surface charge density $\begin{matrix}-\sigma_b \\ \sigma_b\end{matrix}$ on $\begin{matrix}\text{lower} \\ \text{upper}\end{matrix}$ surface

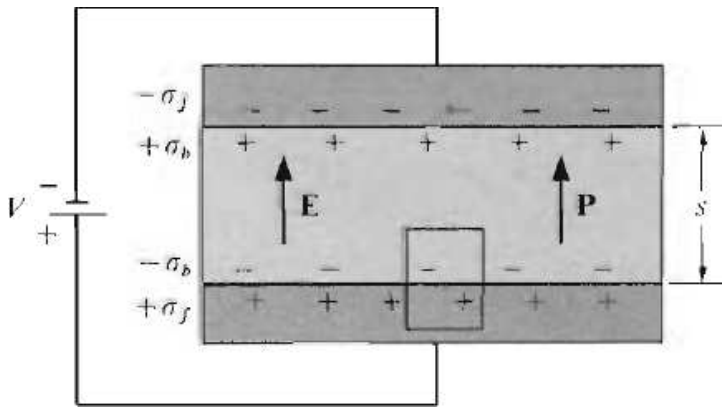
bound charges produce uniform electric field directed downward that cancels part of the field of charges situated on plates

$\because$ net field between plates $= \frac{V}{s}$, free charge densities σ_f and $-\sigma_f$ on plates must be larger than when dielectric is absent

$\therefore$ presence of dielectric increases charges on plates for a given V, and hence increases capacitance

Task : calculate capacitance

apply Gauss's law for displacement $\vec{D}$ to an imaginary cylinder



only flux of $\vec{D}$ through Gaussian surface is through the top, and $D = \sigma_f$

Then $E = \frac{\sigma_f}{\varepsilon_0 \varepsilon_r}$

potential difference between plates $= \frac{\sigma_f s}{\varepsilon_0 \varepsilon_r}$

capacitance $C = \frac{\sigma_f A}{\frac{\sigma_f s}{\varepsilon_0 \varepsilon_r}} = \varepsilon_0 \varepsilon_r \frac{A}{s}$;

$A =$area of plate

$\therefore$ capacitance is increased by factor of ε_r in presence of dielectric

measurement of capacitance with and without a dielectric provides a convenient method for obtaining ε_r

POISSON'S AND LAPLACE'S EQUATIONS

In homogeneous, linear, and isotropic dielectrics, $\vec{D} \propto \vec{E}$ and ε_r is independent of the coordinates

$\vec{\nabla}.\vec{E} = \frac{\rho_f + \rho_b}{\varepsilon_0} = \frac{\rho_f}{\varepsilon_r \varepsilon_0}$ and $\vec{E} = -\vec{\nabla} V$

$\therefore \nabla^2 V = -\frac{\rho_f + \rho_b}{\varepsilon_0} = -\frac{\rho_f}{\varepsilon_r \varepsilon_0}$: Poisson's equation for dielectrics

expressions involving ε_r are valid only in linear and isotropic dielectrics

Laplace's equation : $\nabla^2 V = 0$ when both ρ_b and ρ_f are zero

Polarizabilities

induced dipole moment, $\mu^* = \underline{\underline{\alpha}}\vec{\mathcal{E}} + \frac{1}{2}\underline{\underline{\beta}}\vec{\mathcal{E}}^2 + \cdots$

polarizability, written as a number : $\alpha = \frac{1}{3}\mathrm{Tr}(\underline{\underline{\alpha}})$

$\underline{\underline{\beta}}$: first hyperpolarizability.

Experimentally, $\mu^* = \alpha\vec{\mathcal{E}} + \frac{1}{2}\beta\vec{\mathcal{E}}^2 + \cdots$

α : units $(\mathrm{C\ m})^2\,\mathrm{J}^{-1}$

polarizability volume, $\alpha' = \frac{\alpha}{4\pi\epsilon_0}$ has units of volume, $\mathcal{O}(10^{-30})\mathrm{m}^3 \equiv 1\text{Å}^3$

	μ/D	$\alpha'/10^{-30}\mathrm{m}^3$
CCl_4	0	10.5
H_2	0	0.819
H_2O	1.85	1.48
HCl	1.08	2.63
HI	0.42	5.45

For an isotropic fluid $P = 0$ in absence of applied field

because molecules adopt ceaselessly changing random orientations

due to thermal motion, so $\langle \mu \rangle = 0$

In presence of weak electric field, orientations of molecular dipoles fluctuate

task: find $\langle \mu_z \rangle$ in presence of a field:

probability dp for a dipole orientation $\theta \to \theta + d\theta$,

$$dp = \frac{e^{-\frac{E(\theta)}{k_B T}} \sin\theta d\theta}{\int_0^\pi e^{-\frac{E(\theta)}{k_B T}} \sin\theta d\theta}, \qquad E(\theta) = -\mu\mathcal{E}\cos\theta$$

$$\therefore \langle \mu_z \rangle = \int \mu \cos\theta dp = \frac{\mu \int_0^\pi e^{x\cos\theta} \cos\theta \sin\theta d\theta}{\int_0^\pi e^{x\cos\theta} \sin\theta d\theta} \qquad x = \frac{\mu\mathcal{E}}{k_B T}$$

write $y = \cos\theta$, then $dy = -\sin\theta d\theta$ and

$$\langle \mu_z \rangle = \frac{\mu \int_{-1}^{1} y e^{xy} dy}{\int_{-1}^{1} e^{xy} dy} = \mu L(x) \qquad L(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{1}{x}$$

$L(x)$ is called Langevin function. For $x \ll 1$, $L(x) = \frac{1}{3}x + \cdots$ $\qquad \therefore \langle \mu_z \rangle = \frac{\mu^2 \mathcal{E}}{3k_B T}$

Frequency dependence : When applied field changes direction slowly, permanent $\vec{\mu}$ has time to reorient
—the whole molecule rotates into a new direction—and follows the field

when field frequency is high, molecule cannot change direction fast enough to follow change in direction of applied field
and permanent $\vec{\mu}$ makes no contribution to polarization of sample

orientation polarization, arising from the permanent $\vec{\mu}$, is lost at such high frequencies

molecule takes about 1 ps to turn through about 1 radian in a fluid
loss of contribution of orientation polarization to total polarization occurs when measurements are made at frequencies greater than about 10^{11} Hz (MW)

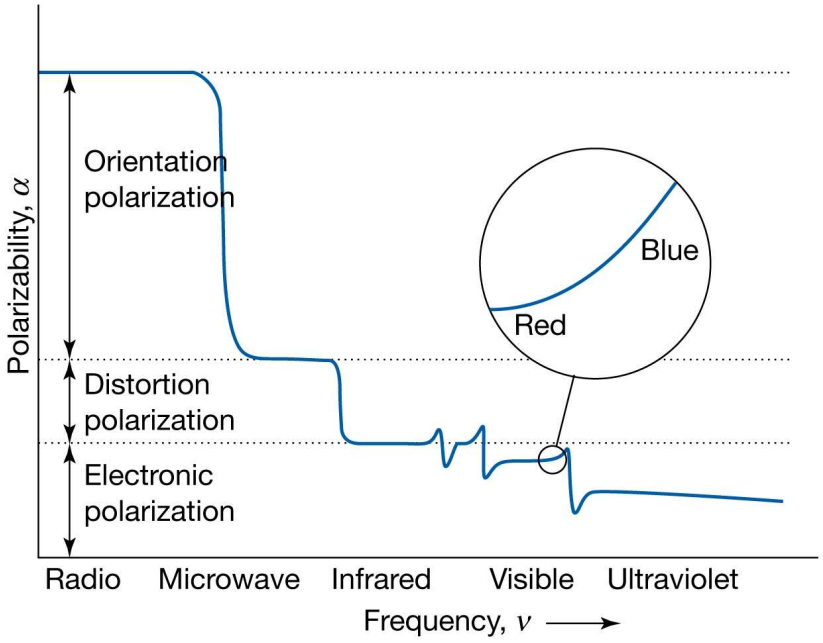
next contribution to polarization lost as the frequency is raised is distortion polarization, that arises from the distortion of positions of nuclei by applied field

molecule is bent and stretched by applied field, and $\vec{\mu}$ changes accordingly

time taken for molecule to bend is approximately the inverse of the molecular vibrational frequency, so distortion polarization disappears when frequency of the radiation is increased through IR

disappearance of polarization occurs in stages: each successive stage occurs as the incident frequency rises above frequency of a particular mode of vibration

- At even higher frequencies, in visible region, only electrons are mobile enough to respond to rapidly changing direction of applied field
- Polarization that remains is due entirely to the distortion of electron distribution, and the surviving contribution to molecular polarizability is called electronic polarizability



Molar polarization

For two charges Q_1 and Q_2 separated by a distance r in vacuum, Coulomb potential energy of interaction $= \frac{Q_1 . Q_2}{4\pi\epsilon . r}$

$$\text{relative permittivity, } \epsilon_r = \frac{\epsilon}{\epsilon_0}$$

ϵ_r has a significant effect on strength of interactions between ions in solution ($\epsilon_{H_2O} = 78$)

$$\text{Debye equation: } \frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{\rho P_m}{M}$$

where ρ =density, M = molar mass, P_m = molar polarization, $\frac{N_A}{3\epsilon_0}\left(\alpha + \frac{\mu^2}{3k_B T}\right)$

Without contribution from permanent μ,

$$\text{Clausius–Mossotti equation: } \frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{\rho N_A \alpha}{3M\epsilon_0}$$

relative permittivity of chlorobenzene $= 5.71$ at 20^oC and 5.62 at 25^oC

Assume constant density $= 1.11$ g cm^{-3}; task: estimate α' and μ

$P_M = \frac{N_A}{3\epsilon_0}\left(\alpha + \frac{\mu^2}{3k_B T}\right)$ and $\frac{\epsilon-1}{\epsilon+2} = \frac{\rho P_M}{M}$; or, $P_M = \frac{M}{\rho}\frac{\epsilon-1}{\epsilon+2}$

$M = 12 \times 6 + 35.45 + 5 = 112.45$g/mol; and $\frac{M}{\rho} = \frac{112.45}{1.11} = 101.351$ cm^{3}mol^{-1}

T	P_M
293	$101.35 \times \frac{5.71-1}{5.71+2} = 61.915$
298	$101.35 \times \frac{5.62-1}{5.62+2} = 61.449$

$\therefore$slope for P_M vs $\frac{1}{T}$ plot $= \frac{61.915-61.449}{\frac{1}{293}-\frac{1}{298}} = \frac{0.466}{5.727\times10^{-5}} = 8136.895$ cm^{3}mol^{-1}K

$\therefore \frac{N_A\alpha}{3\epsilon_0} = 61.915 - 8136.895 \times \frac{1}{293} = 34.144$

$\therefore \alpha = 34.144 \times \frac{3\epsilon_0}{N_A}$ and $\alpha' = \frac{N_A\alpha}{4\pi\epsilon_0} = \frac{3\times34.144\times7}{4\times22} \approx 8.15$cm^3/mol

Also, $\frac{N_A}{3\epsilon_0}.\frac{\mu^2}{3k_B} = 8136.895 \times 10^{-6}$m^{3}mol^{-1}K

$$\therefore \mu \overset{\text{SI units}}{=} \sqrt{\frac{3\times8136.895\times10^{-6}\times8.8542\times10^{-12}\times3\times1.381\times10^{-23}}{6.023\times10^{23}}} = 385.581\times10^{-32}\text{Cm}$$

$$= \frac{385.581\times10^{-32}}{3.336\times10^{-30}} = 1.16 \text{ D}$$

Derivation of Debye equation :

1 moleculae occupies the vol. $\frac{4}{3}\pi R^3$, $\therefore$density of atoms $N = \frac{1}{\frac{4}{3}\pi R^3} = \frac{\rho N_A}{M}$
macroscopic field $\vec{E} = \vec{E}_{\text{self}} + \vec{E}_{\text{else}}$, where $\vec{E}_{\text{self}} =$ average field over the sphere due to the molecule itself
$\vec{\mu}_{\text{ind}} = \alpha \vec{E}_{\text{else}} \implies \vec{P}_{\text{ind}} = N\alpha \vec{E}_{\text{else}}$ and $\vec{E}_{\text{self}} = -\frac{1}{4\pi\epsilon_0} \cdot \frac{\vec{\mu}_{\text{ind}}}{R^3}$
$\therefore \vec{E} = -\frac{1}{4\pi\epsilon_0} \cdot \frac{\alpha}{R^3} \vec{E}_{\text{else}} + \vec{E}_{\text{else}} = \left(1 - \frac{1}{4\pi\epsilon_0} \cdot \frac{\alpha}{R^3}\right) \vec{E}_{\text{else}} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right) \vec{E}_{\text{else}}$

$\vec{P} = \frac{N\alpha}{1 - \frac{N\alpha}{3\epsilon_0}} \vec{E} = \epsilon_0 \chi_e \vec{E}$, or, $\chi_e = \frac{\frac{N\alpha}{\epsilon_0}}{1 - \frac{N\alpha}{3\epsilon_0}}$

or, $\chi_e - \frac{N\alpha}{3\epsilon_0}\chi_e = \frac{N\alpha}{\epsilon_0} \implies \frac{N\alpha}{\epsilon_0}\left(1 + \frac{\chi_e}{3}\right) = \chi_e$, or, $\alpha = \frac{\epsilon_0}{N} \cdot \frac{\chi_e}{1 + \frac{\chi_e}{3}} = \frac{3\epsilon_0}{N} \cdot \frac{\chi_e}{3 + \chi_e}$

using $\chi_e = \epsilon_r - 1$, $\quad \alpha = \frac{3\epsilon_0}{N} \cdot \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$ or, $\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{\alpha}{3} \cdot \frac{N}{\epsilon_0} = \frac{\rho N_A \alpha}{3M\epsilon_0}$

when contrbn. from permanent dipoles is added, $\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{\rho N_A}{3M\epsilon_0}\left(\alpha + \frac{\mu^2}{3k_B T}\right) = \frac{\rho P_M}{M}$

refractive index:

Maxwell's theory of electromagnetic radiation :
refractive index at specified wavelength (visible or uv)

$$n_r = \sqrt{\epsilon_r}, \text{ where } n_r = \frac{c(\text{vacuum})}{c'(\text{medium})}$$

molar polarization, P_M, and molecular polarizability, α, can be measured at frequencies typical of visible light (about $\sim 10^{15}$ Hz) by measuring refractive index and using the Clausius–Mossotti equation

$$\frac{n_r^2-1}{n_r^2+2} = \frac{\rho N_A \alpha}{3M\epsilon_0} \text{ : Lorenz-Lorenz equation}$$

Magnetic properties - LINEAR AND NONLINEAR MEDIA

Magnetic Susceptibility and Permeability

for most substances magnetization $\propto$ field, provided field is not too strong

linear media : $\vec{M} = \chi_m \vec{H}$

χ_m: magnetic susceptibility - dimensionless quantity

$+$ve for paramagnets and $-$ve for diamagnets

Typical values are around 10^{-5};

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.7×10^{-4}	Oxygen (O_2)	1.7×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.2×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.0×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.7×10^{-4}
Carbon Dioxide	-1.1×10^{-8}	Liquid Oxygen ($-200°$ C)	3.9×10^{-3}
Hydrogen (H_2)	-2.1×10^{-9}	Gadolinium	4.8×10^{-1}

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) = \mu_0 \left(1 + \chi\right) \vec{H} = \mu \vec{H}$$

$\boxed{\vec{B} = \mu \vec{H}}$ permeability $\mu = \mu_0 \left(1 + \chi\right)$

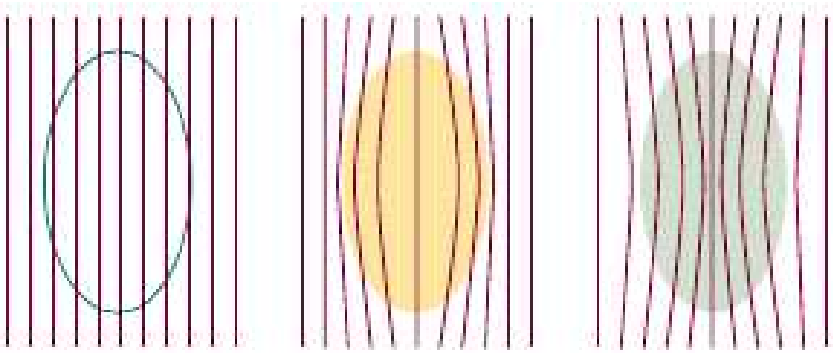
μ_0 is permeability of free space

magnetization contributes to density of lines of force in the material

$\chi > 0$: paramagnetic - tend to move into magnetic field
and density of lines of force in them is greater than in vacuum

ions/molecules with unpaired electrons, e.g. radicals and many d-metal complexes

vacuum diamagnetic paramagnetic



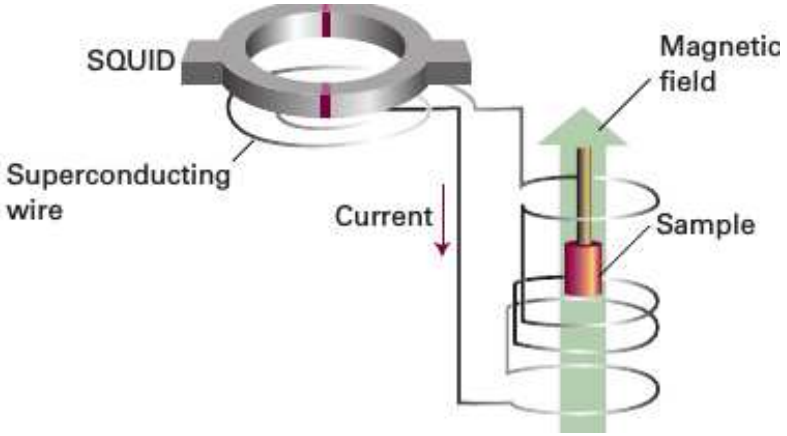
$\chi < 0$: diamagnetic - tend to move out of magnetic field;
density of lines of force within is lower than in vacuum

- no unpaired electrons

Measurement of susceptibility : **Gouy balance**
sensitive balance from which sample hangs in narrow cylinder
and lies between poles of a magnet
paramagnetic : drawn into field; apparent weight greater than when field is off
diamagnetic : expelled from field and weighs less when field is on
balance normally calibrated against sample of known susceptibility

modern version : superconducting quantum interference device (SQUID) : takes advantage of quantization of magnetic flux and property of current loops in superconductors that include a weakly conducting link through which electrons must tunnel

current flowing in loop in a magnetic field depends on magnetic flux



sample moved upwards in small increments and potential difference across SQUID measured

SQUIDs are sensitive enough to measure fields as low as 5×10^{-14} T with a few days of averaged measurements

generally, the dimensionless susceptibility is called the volume susceptibility χ_V

χ_m is used for molar susceptibility, $\chi_m = \frac{M}{\rho}\chi_V$
where M = mol. wt. and ρ = density

	$\chi_V/10^{-4}$	$\chi_m/10^{-10}$ m^{3}mol^{-1}
$H_2O(l)$	-9.02	-1.63
NaCl(s)	-16	-3.8
Cu(s)	-9.7	-0.69
$CuSO_4.5H_2O(s)$	$+167$	$+183$

permanent magnetic moment arises from unpaired electron spins

magnetic moment $\propto$ spin angular momentum

$$m = g_e\sqrt{s(s+1)}\mu_B; \quad \mu_B = \frac{e\hbar}{2m_e} = 9.274 \times 10^{-24}\,\mathrm{JT}^{-1}; \quad g_e = 2.0023$$

multiple electrons: addition of spin angular momenta

χ and $\mathcal{M}$ depend on temperature because orientations of electron spins fluctuate whether molecules are in fluid phases or trapped in solids :
some orientations have lower energy than others
magnetization depends on randomizing influence of thermal motion
contribution to susceptibility due to thermal averaging of permanent magnetic moments in presence of applied magnetic field $\propto \frac{m^2}{3k_B T}$

spin contribution, $\chi_m = \frac{N_A g_e^2 \mu_0 \mu_B^2 S(S+1)}{3k_B T} > 0$: paramagnetic

Curie law: $\boxed{\chi_m = \frac{C}{T}}$ $\quad C = 6.3001 \times 10^{-6} S(S+1)\ \mathrm{m^3 mol^{-1} K^{-1}}$

χ_m decreases with increasing T, due to thermalizing of spin orientations

Consider a complex salt with three unpaired electrons per complex cation at 298 K and molar volume 61.7 cm^3 mol^{-1}.

$$\chi_m = 6.3001 \times 10^{-6} \frac{\frac{3}{2}\left(\frac{3}{2}+1\right)}{298} = 7.93 \times 10^{-3} m^3 \text{ mol}^{-1}$$

$$\chi = \frac{\chi_m}{V_m} = \frac{7.93 \times 10^{-3}}{6.17 \times 10^{-5}} = 1.29 \times 10^{-3}$$

Orbital angular momentum also contributes to paramagnetism

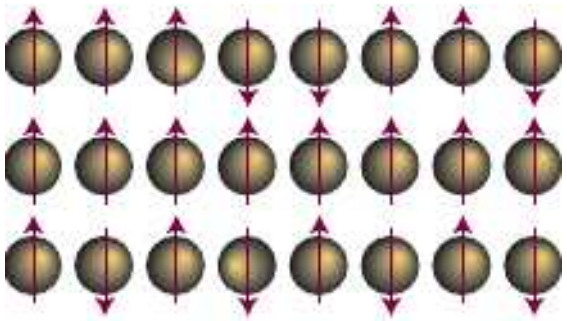
ferromagnetism : nonlinear medium

At low temperatures, some paramagnetic solids make a phase transition to a state in which large domains of spins cooperatively align in parallel orientations

• has nonzero magnetization in absence of applied field;alignment 'frozen in' interaction between nearby dipoles such that each dipole points in the same direction as its neighbour; correlation so strong as to align virtually 100% of the unpaired electron spins

antiferromagnetism: exchange interactions lead to alternating spin orientations: spins locked into a low-magnetization arrangement

zero magnetization in absence of applied field - spin magnetic moments cancel



paramagnetic

ferromagnetic

antiferromagnetic

ferromagnetic transition occurs at Curie temperature

antiferromagnetic transition occurs at the Néel temperature

Nature of cooperative behaviour depends on details of band structure
Magnetic moments can also be induced in molecules
circulation of electronic currents induced by an applied field gives rise to a magnetic field which usually opposes the applied field : diamagnetic
induced electron currents occur within orbitals of molecule that are occupied in ground state

temperature-independent paramagnetism (TIP)

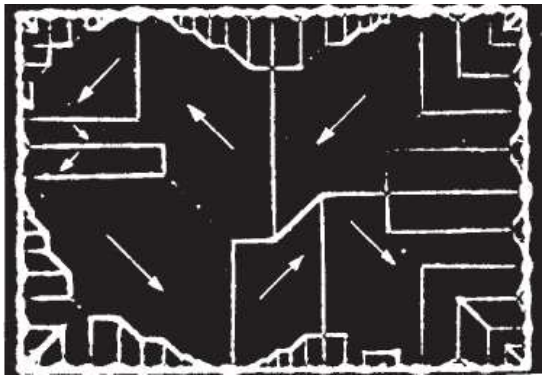
For some molecules with no unpaired electrons, induced electron currents flow in opposite direction using unoccupied orbitals close to HOMO in energy
This orbital paramagnetism can be distinguished from spin paramagnetism by its temperature-independence

All molecules have a diamagnetic component to their susceptibility

but it is dominated by spin paramagnetism if electrons are unpaired

In few cases (where there are low-lying excited states) TIP is strong enough to make molecules paramagnetic even though their electrons are paired

why isn't every wrench and nail a powerful magnet?
answer : alignment occurs in relatively small patches $\Rightarrow$ domains
Each domain contains billions of dipoles, all lined up

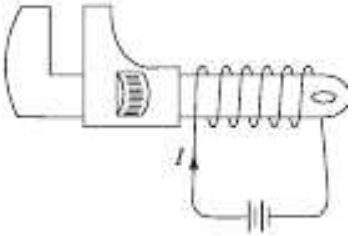


domains visible
under microscope,
using suitable
etching techniques
randomly oriented

wrench has enormous number of domains
their magnetic fields cancel, so wrench is not magnetized
orientation of domains is not completely random; within a given crystal, there may be some preferential alignment along the crystal axes. But there will be just as many domains pointing one way as the other, so there is still no large-scale magnetization

How do we produce a permanent magnet?

simple way :wrap coil of wire around object to be magnetized

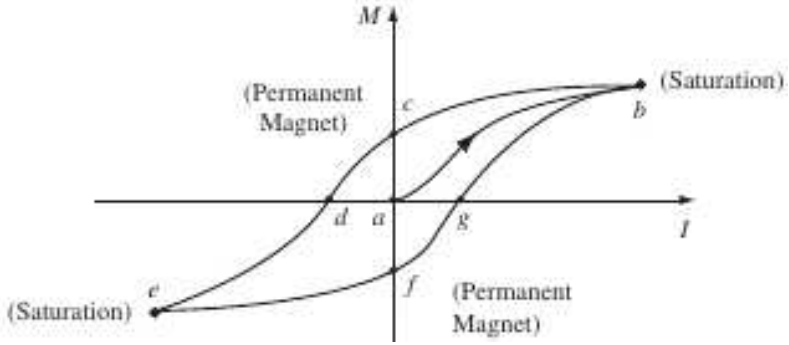


Run current I through the coil

As we increase current, field increases, domain boundaries move, magnetization grows until saturation point, with all dipoles aligned

further increase in current has no effect on $\vec{M}$

Hysteresis loop



magnetization depends not only on the applied field, but also on its previous magnetic 'history'

current is zero at a, c, and f , yet the magnetization is different for each

it is customary to draw hysteresis loops as plots of B against H

If coil $\approx$ long solenoid, with n turns/ length, then

$$H = nI$$

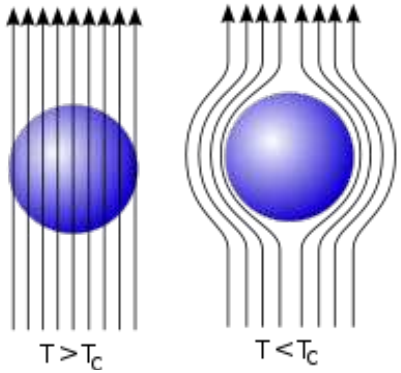
$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

in practice $M \gg H$, $\therefore \vec{B} \propto \vec{M}$

field without the iron$= \mu_0 \vec{H}$

Some superconductors, classed as Type I, show abrupt loss of superconductivity when applied magnetic field > critical value $\mathcal{H}_C$ characteristic of material

$$\mathcal{H}_C = \mathcal{H}_0 \left(1 - \frac{T^2}{T_C^2}\right)$$



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Type I superconductors are completely diamagnetic below $\mathcal{H}_C$: magnetic field does not penetrate into material

This complete exclusion of magnetic field is called **Meissner effect**, demonstrated by levitation of a superconductor above a magnet

Type II superconductors including HTSCs show a gradual loss of superconductivity and diamagnetism with increasing magnetic field

Electrodynamics

Direct currents in electric circuits : Conduction of electricity

Current density vector $\vec{J} \propto$ force per unit charge, or, $\vec{J} = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right)$

force that drives the current — chemical, gravitational, etc. or,
$\vec{J} = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right)$

velocity of charges small $\implies \left| \vec{v} \times \vec{B} \right| \ll E$

$\implies$ Ohm's law : $\vec{J} = \sigma \vec{E}$, where σ is called conductivity

units of conductivity : amperes/Vm, or Siemens/m

even insulators conduct slightly, though conductivity of a metal is much greater

metals can be regarded as perfect conductors, with $\sigma = \infty$
for insulators $\sigma \approx 0$

in plasmas, the magnetic contribution to $\vec{f}$ can be significant
Note : $\vec{E} = 0$ inside a conductor for stationary charges ($\vec{J} = 0$)

for perfect conductors, $\vec{E} = \frac{\vec{J}}{\sigma} = 0$ even if current is flowing
metals are good conductors and field required to drive current is negligible

We treat connecting wires in electric circuits as equipotentials

Resistors, by contrast, are made from poorly conducting materials

Conductor	Conductivity σ (siemens per meter)
Aluminum	3.54×10^7
Brass (65.8 Cu, 34.2 Zn)	1.59×10^7
Chromium	3.8×10^7
Copper	5.80×10^7
Gold	4.50×10^7
Graphite	1.0×10^5
Magnetic iron	1.0×10^7
Mumetal (75 Ni, 2 Cr, 5 Cu, 18 Fe)	0.16×10^7
Nickel	1.3×10^7
Sea water	≈ 5
Silver	6.15×10^7
Tin	0.870×10^7
Zinc	1.86×10^7

reciprocal of σ : resistivity: $\rho = 1/\sigma$
Table : ρ in Ωm

Material	Resistivity	Material	Resistivity
Conductors:		*Semiconductors:*	
Silver	1.59×10^{-8}	Sea water	0.2
Copper	1.68×10^{-8}	Germanium	0.46
Gold	2.21×10^{-8}	Diamond	2.7
Aluminum	2.65×10^{-8}	Silicon	2500
Iron	9.61×10^{-8}	*Insulators:*	
Mercury	9.61×10^{-7}	Water (pure)	8.3×10^{3}
Nichrome	1.08×10^{-6}	Glass	$10^{9} - 10^{14}$
Manganese	1.44×10^{-6}	Rubber	$10^{13} - 10^{15}$
Graphite	1.6×10^{-5}	Teflon	$10^{22} - 10^{24}$

drift velocity $\vec{v}$ of conduction electrons

$\vec{J} = ne\vec{v}$

Cu : 1 conduction electron per atom : 8.5×10^{28} atoms/m^3

for current= 1 A in wire of cross-section= 1m^2, $J = 10^6$S/m

$v = \frac{10^6}{8.5 \times 10^{28} \times 1.6 \times 10^{-19}} \approx 7.4 \times 10^{-5}$m/s= 0.26m/Hr

Conservation of charge

$\Longrightarrow$ eq. of continuity : $\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$ - law of (local) conservation of charge

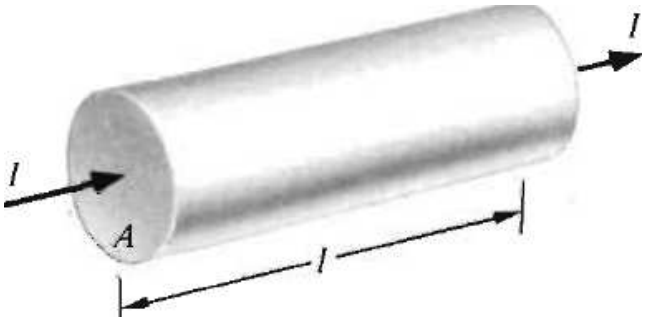
Charge density in a homogeneous conductor :

steady-state $\Longrightarrow$ $\frac{\partial \rho}{\partial t} = 0 \Longrightarrow \vec{\nabla}.\vec{J} = \vec{\nabla}.\sigma\vec{E} = 0$

for homogeneous conductor with σ independent of coordinates, $\vec{\nabla}.\vec{E} = \frac{\vec{\nabla}.\vec{J}}{\sigma} = 0$

$\therefore$ under steady-state conditions for homogeneous conductor carrying a current net charge density inside $= 0$ - external field is non-zero.

unbalanced charge resides on surface
- still true when charges allowed to move



element of conductor of length l and cross-section A

potential between ends $= V$

$$J = \frac{I}{A} = \sigma E = \sigma \frac{V}{l}$$

$$\therefore\ I = \frac{\sigma V A}{l} = \frac{V}{\frac{l}{\sigma A}} = \frac{V}{R}$$

where resistance $R = \frac{l}{\sigma A}$
in Ω, ohms=V/amp

Ohm's law : $V = IR$

resistor satisfying the law is called ohmic or linear

collision theory based derivation :

Consider velocity, $\vec{v_0}$, of an electron at t= 0

time since last collision= t; $\langle t \rangle \sim \frac{\lambda}{v_{\text{th}}} = \tau$, where $\begin{array}{c} \lambda = \text{mean free path} \\ v_{\text{th}} = \text{thermal velocity} \end{array}$

accn.= $-\frac{eE}{m}$; $\therefore \vec{v} = -\frac{e\vec{E}}{m}t + \vec{v_0}$

$\langle \vec{v} \rangle = -\frac{e\vec{E}}{m}\langle t \rangle = -\frac{e\vec{E}\tau}{m} = \mu\vec{E}$, where μ is called mobility

$\vec{J} = -ne\langle \vec{v} \rangle = \frac{ne^2\tau}{m}.\vec{E}$

$\vec{J} = \sigma\vec{E} \implies \sigma = \frac{ne^2\tau}{m} = \frac{n\lambda e^2}{2mv_{\text{th}}}$

work done by electrical force is converted into heat in resistor

work done per unit charge $= V$ and charge flowing per unit time $= I$

$\therefore$ power delivered, $P = VI = I^2R \Longleftarrow$ Joule heating law

I in amp, R in Ω, P in watts (J/s)

Electromotive Force (electromotance)

forces involved in driving current around a circuit: $\vec{f} = \vec{f_s} + \vec{E}$
source, $\vec{f_s}$: confined to one portion of the loop, say, battery,
electrostatic force, $\vec{E}$: smoothens flow and communicates influence of the source to distant parts of circuit

physical agency for $\vec{f_s}$

battery: chemical force

piezoelectric crystal : mechanical pressure converted into electrical impulse

thermocouple : temperature gradient

photoelectric cell : light

Van de Graaff generator : electrons loaded onto a conveyer belt and swept along

net effect is determined by Electromotive Force $\mathcal{E} = \oint \vec{f}.d\vec{l} = \oint \vec{f_s}.d\vec{l}$

not a force — integral of a force per unit charge : electromotance

Within ideal source of emf (resistanceless battery), net force on charges$= 0$

then $\vec{f} = \vec{f_s} + \vec{E} \implies \vec{E} = -\vec{f_s}$

potential difference between the terminals (say, a and b)

$$V = -\int_a^b \vec{E}.d\vec{l} = -\int_a^b \vec{f_s}.d\vec{l} = \oint \vec{f_s}.d\vec{l} = \mathcal{E}$$

integral extended to entire loop because $\vec{f_s} = 0$ outside the source

function of battery : establish and maintain a voltage difference equal to emf
e.g., a 6 V battery holds the positive terminal 6 V above the negative terminal
resulting electrostatic field drives current around rest of the circuit
- inside the battery f_s drives current in a direction opposite to $\vec{E}$

Motional emf - arise when we move a wire through a magnetic field

If Φ =flux of $\vec{B}$ through a loop $= \int \vec{B}.d\vec{a}$

emf generated in the loop $\mathcal{E} = -\frac{d\Phi}{dt}$: flux rule for motional emf

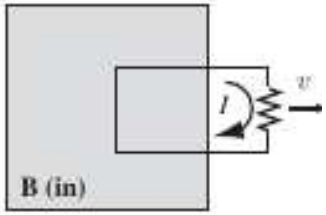
If a circuit or part of it, moves in a magnetic field
then one must take into account the $\vec{v} \times \vec{B}$ term
conduction electrons in moving conductors feel an electric field $\vec{v} \times \vec{B}$
and $\oint \vec{E}.d\vec{l} \neq 0$

FARADAY INDUCTION LAW : $\oint \vec{E}.d\vec{l} = -\frac{d\Phi}{dt}$
electromotance (emf) induced in a circuit =
$-$rate of change of magnetic flux linking the circuit
where magnetic flux $\Phi =$ area swept by wire per unit time$\times B$

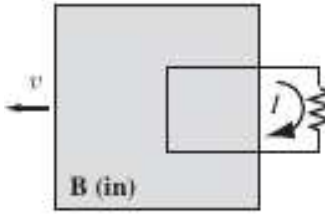
$\boxed{\text{changing magnetic field induces electric field}}$

path of integration need not lie in conducting material
Whenever magnetic flux through a loop changes, emf $\mathcal{E} = -\frac{d\Phi}{dt}$ appears

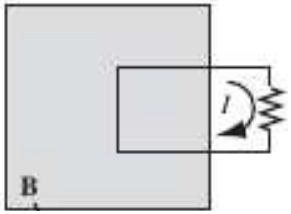
1831 : Faraday reported on a series of experiments



pull a loop of wire to the right through a magnetic field . Current flows in the loop



move magnet to the left, holding the loop still; current flows in the loop



With both the loop and magnet at rest, change field-strength; again, current flows in the loop

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

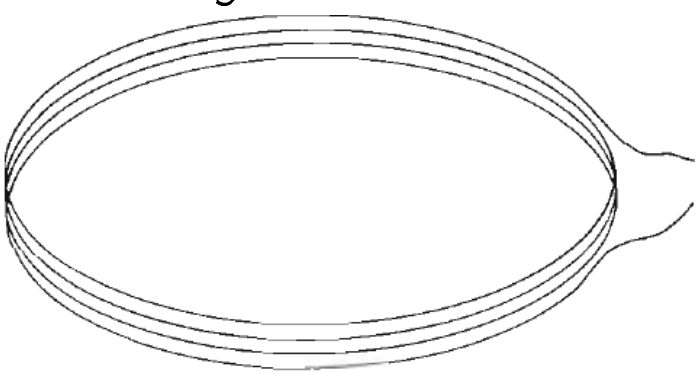
If no sources in a circuit
current = induced electromotance/ resistance ($V = IR$, Ohm's law)

for a rigid coil and a time-dependent $\vec{B}$

add the fluxes through each turn

sum of these fluxes is called the flux linkage, Λ

using $\Phi = \int_S \vec{B}.d\vec{a}$ for a coil of N turns with all turns close together



$\Lambda = N\Phi$

$\therefore \oint \vec{E}.d\vec{l} = -\frac{d\Lambda}{dt}$

flux linkage is expressed in weber-turns

LENZ'S LAW

If flux linkage Λ $\begin{matrix}\text{increases} \\ \text{decreases}\end{matrix}$, then $\begin{matrix}\frac{d\Lambda}{dt} > 0 \\ \frac{d\Lambda}{dt} < 0\end{matrix}$

and electromotance $\begin{matrix}< 0 : \text{negative direction} \\ > 0 : \text{positive direction}\end{matrix}$ $\left(\oint \vec{E}.d\vec{l} = -\frac{d\Lambda}{dt}\right)$

direction of induced current is such that
it produces magnetic field that opposes change in flux

If Λ increases, induced current produces an opposing flux

if Λ decreases, induced current produces an aiding flux : Lenz's law

Nature abhors a change in flux

currents induced by changing magnetic fields in conductors other than wires are called eddy currents

FARADAY INDUCTION LAW IN DIFFERENTIAL FORM

Using Stokes's theorem, $\oint \vec{E}.d\vec{l} = \int\limits_S \left(\vec{\nabla} \times \vec{E}\right).d\vec{a} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int\limits_S \vec{B}.d\vec{a}$

$$\text{or, } \int\limits_S \left(\vec{\nabla} \times \vec{E}\right).d\vec{a} = -\int\limits_S \frac{\partial \vec{B}}{\partial t}.d\vec{a}$$

$\therefore \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: another of Maxwell equations - general law for stationary media

- a differential equation that relates space derivatives of $\vec{E}$ and time rate of change of $\vec{B}$ at a point

Both $\vec{E}$ and $\vec{B}$ are measured in same reference frame

The equation does not give the value of $\vec{E}$, unless it can be integrated

Electric field intensity $\vec{E}$ in terms of ϕ and $\vec{A}$

$\because \vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{\nabla} \times \vec{E} = -\frac{\partial(\vec{\nabla}\times\vec{A})}{\partial t} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$

or, $\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0; \quad \therefore \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

for steady currents, $\vec{A}$ =const. and $\vec{E} = -\vec{\nabla}\phi$

electric field intensity can arise both from accumulations of charge, through $\vec{\nabla}\phi$ and from changing magnetic fields, through the $\frac{\partial \vec{A}}{\partial t}$ term

All three quantities, $\vec{E}$, $\vec{A}$, and ϕ, are measured in same reference frame

Induced Electric Field

Faraday's law generalizes electrostatic rule $\vec{\nabla} \times \vec{E} = 0$ to time-dependent régime

If $\vec{E}$ is a pure Faraday field (due exclusively to a changing $\vec{B}$, with $\rho = 0$),

then $\vec{\nabla}.\vec{E} = 0; \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

mathematically analogous to magnetostatics, $\vec{\nabla}.\vec{B} = 0; \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$\Longrightarrow$ Faraday-induced electric fields are determined by $-\frac{\partial \vec{B}}{\partial t}$

in exactly same way as magnetostatic fields are determined by $\mu_0 \vec{J}$

The analog to Biot-Savart $\vec{B} = \frac{\mu_0}{4\pi} I \oint \frac{\vec{dl} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ is

$$\vec{E} = -\frac{1}{4\pi} \int \frac{\left(\frac{\partial \vec{B}}{\partial t}\right) \times \vec{\mathscr{r}}}{\mathscr{r}^2} d\tau = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\vec{B} \times \vec{\mathscr{r}}}{\mathscr{r}^2} d\tau$$

and if symmetry permits, tricks analogous to those used for Ampère's law in integral form $\oint_C \vec{B}.d\vec{l} = \mu_0 I_{\text{enc}}$ can be used for Faraday's law in integral form: $\oint_C \vec{E}.d\vec{l} = -\frac{d\Phi}{dt}$

rate of change of (magnetic) flux through Amperian loop

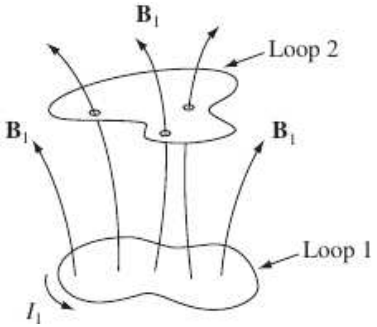
plays the role formerly assigned to $\mu_0 I_{\text{enc}}$

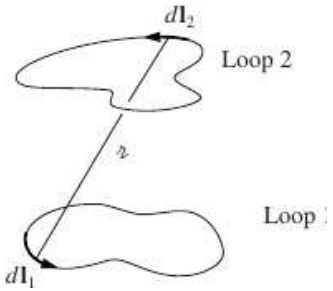
Inductance

Consider two loops of wire, at rest

run steady current I_1 around loop 1

$\Longrightarrow$ produces magnetic field $\vec{B}_1$





Some of the field lines pass through loop 2

flux of $\vec{B}_1$ through 2, $\Phi_2 = \int \vec{B}_1 . d\vec{a}_2 \propto I_1$ (Biot-Savart)$= M_{21} I_1$

$M_{21}$is known as the mutual inductance of the two loops

Stokes' theorem $\implies \Phi_2 = \int \vec{B}_1 . d\vec{a}_2 = \int \left(\vec{\nabla} \times \vec{A}_1\right) . d\vec{a}_2 = \oint \vec{A}_1 . d\vec{l}_2$

$= \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\vec{l}_1}{\mathscr{r}}\right) . d\vec{l}_2 \implies M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 . d\vec{l}_2}{\mathscr{r}}$: Neumann formula

1. $M_{21}$is purely geometrical quantity, having to do with sizes, shapes, and relative positions of two loops
2. integral in Neumann formula unchanged if we switch roles of loops 1 and 2; it follows that $M_{21} = M_{12}$

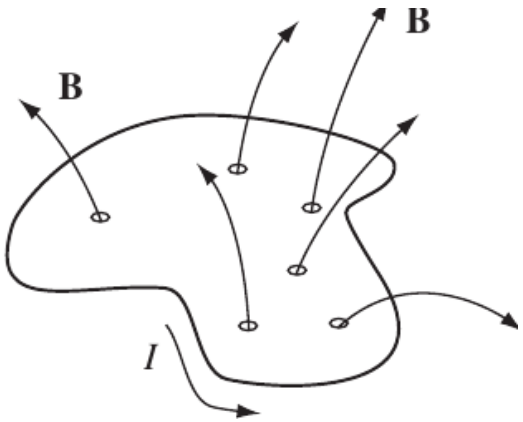
Whatever the shapes and positions of the loops
flux through 2 when we run a current I around 1
$\equiv$ flux through 1 when we send the same current I around 2

We may as well drop the subscripts and call them both M

vary current in loop 1

flux through loop 2 will vary accordingly
Faraday's law says this changing flux will induce emf in loop 2: $\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$

Every time we change the current in loop 1,
induced current flows in loop 2— even though there are no wires connecting them!



changing current not only induces emf in nearby loops, it also induces an emf in source loop itself

the field (and flux) $\propto I \implies$ flux $= LI$
$L \Rightarrow$ (self)-inductance of the loop

As with M, it depends on geometry (size and shape) of the loop

Inductance is measured in henries (H)$\equiv$ Vs/A

If the current changes, the emf induced in the loop is $\mathcal{E} = -L\frac{dI}{dt}$

Inductance (like capacitance) is an intrinsically positive quantity

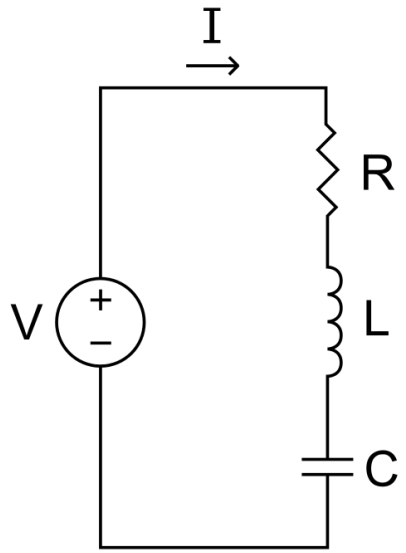
Lenz's law : emf is in such a direction as to oppose any change in current

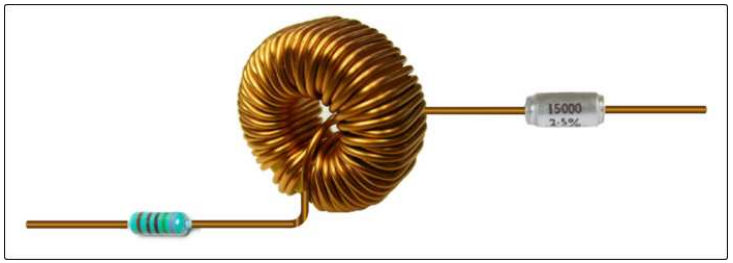
$\therefore$ it is called a back emf

Whenever we alter current in a wire, we go against this back emf

Inductance plays the same role in electric circuits as mass in mechanical systems :
The greater L is, the harder it is to change the current
just as the larger the mass, the harder it is to change an object's velocity

R L C circuit





natural frequency $\omega_0 = \frac{1}{\sqrt{LC}}$

$$V(t) = V_R + V_L + V_C = RI(t) + L\frac{dI(t)}{dt} + \frac{1}{C}\int_0^t I(\tau)d\tau$$

or, $\frac{d^2I(t)}{dt^2} + 2\alpha\frac{dI(t)}{dt} + \omega_0^2 I(t) = 0; \quad \alpha = \frac{R}{2L}$

α and ω_0 are both in units of angular frequency

α is called neper frequency, or attenuation - measure of how fast the transient response of the circuit will die away after the stimulus has been removed

Electrodynamics Before Maxwell

(i) $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ (Gauss)

(ii) $\vec{\nabla}.\vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$(Faraday)

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere)

inconsistency : divergence of curl is always zero
but $\vec{\nabla}.\left(\vec{\nabla} \times \vec{B}\right) = \mu_0 \vec{\nabla}.\vec{J} \neq 0$ generally

For steady currents, $\vec{\nabla}.\vec{J} = 0$
but beyond magnetostatics Ampère's law cannot be right

continuity equation : $\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial\left(\varepsilon_0 \vec{\nabla}.\vec{E}\right)}{\partial t} = -\vec{\nabla}.\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$

$\therefore$ generally, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

when $\vec{E}$ is time independent, we have $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$$\boxed{\text{A changing electric field induces a magnetic field}}$$

Maxwell called the extra term displacement current: $\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

(i) $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ (Gauss)

(ii) $\vec{\nabla}.\vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$(Faraday)

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$(Ampere)

given $\rho\left(\vec{r},\, t\right)$ and $\vec{J}\left(\vec{r},\, t\right)$, what are the fields $\vec{E}\left(\vec{r},\, t\right)$ and $\vec{B}\left(\vec{r},\, t\right)$?

(iii) $\Longrightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial\left(\vec{\nabla} \times \vec{A}\right)}{\partial t} \Longrightarrow \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

Gauss's law : $\vec{\nabla}.\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon_0}$

or, $\nabla^2 V + \frac{\partial\left(\vec{\nabla}.\vec{A}\right)}{\partial t} = -\frac{\rho}{\varepsilon_0} \cdots\cdots$ (A) replaces Poisson's equation

Ampère/Maxwell law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

or, $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 \vec{J} - \mu_0 \varepsilon_0 \frac{\partial\left(\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t}\right)}{\partial t} = \mu_0 \vec{J} - \mu_0 \varepsilon_0 \vec{\nabla}\left(\frac{\partial V}{\partial t}\right) - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$

use : $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2 \vec{A}$

$\left(\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J} \cdots\cdots$ (B)

Eqns. (A) and (B) contain all the information in Maxwell's equations

Gauge Transformations

Eqns. (A) and (B) are ugly
However, we have reduced six variables problem ($\vec{E}$ and $\vec{B}$)
to four variables problem : V and $\vec{A}$

Eqns. $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$ do not uniquely define potentials
we impose conditions on V and $\vec{A}$, as long as nothing happens to $\vec{E}$ and $\vec{B}$
- this is the gauge freedom

consider two sets of potentials
(V, $\vec{A}$) and ($V' = V + \beta$, $\vec{A}' = \vec{A} + \vec{\alpha}$) which correspond to same $\vec{E}$ and $\vec{B}$

$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}' \implies \vec{\nabla} \times \vec{\alpha} = 0 \implies \vec{\alpha} = \vec{\nabla}\lambda$, where λ is a scalar

$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} \implies \vec{\nabla}\beta + \frac{\partial \vec{\alpha}}{\partial t} = 0$

or, $\vec{\nabla}\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0 \implies \beta + \frac{\partial \lambda}{\partial t} = k(t)$, independent of position

we can absorb k in λ.

simple possibility :

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

For any scalar function $\lambda(\vec{r}, t)$, we can add $\vec{\nabla}\lambda$ to $\vec{A}$
provided we simultaneously subtract $\frac{\partial \lambda}{\partial t}$ from V
This will not affect $\vec{E}$ and $\vec{B}$

Such changes in V and A are called gauge transformations

They are exploited to adjust divergence of $\vec{A}$ and to simplify the eqns.

Coulomb Gauge : $\vec{\nabla}.\vec{A} = 0$;
i.e., a given $\vec{A}$ is replaced by a new $\vec{A}$ s.t. $\vec{\nabla}.(\vec{A}+\vec{\nabla}\lambda)=0$

or, $\vec{\nabla}\lambda \equiv -\vec{A}+$arbitrary const. vector

absorb the constant vector in $\vec{A}$, and $\vec{\nabla}V \rightarrow \vec{\nabla}V - \vec{\nabla}\left(\frac{\partial\lambda}{\partial t}\right) = \vec{\nabla}V + \frac{\partial\vec{A}}{\partial t}$

$$\therefore \vec{\nabla}.\vec{E} = -\vec{\nabla}.\left(\vec{\nabla}V + \frac{\partial\vec{A}}{\partial t}\right) = -\nabla^2 V$$

or, $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$: Poisson eqn.

advantage : scalar potential is simple to calculate
disadvantage :

Eqn. (B) : $\left(\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$ becomes

$\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \mu_0\varepsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right)$: $\vec{A}$ is difficult to calculate

Lorenz gauge : $\vec{\nabla}.\vec{A} = -\mu_0\varepsilon_0 \frac{\partial V}{\partial t}$; i.e., $\vec{\nabla}.\vec{E} = \vec{\nabla}.\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right)$

or, $\nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$

Eqn. (B) $\implies \nabla^2 \vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$

Lorenz gauge treats V and $\vec{A}$ on an equal footing : the same differential operator

$\nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} \equiv \Box^2$ (d'Alembertian) occurs in both equations:

$$\Box^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\Box^2 \vec{A} = -\mu_0 \vec{J}$$

$\Box^2 f = 0$: 4-D version of Poisson's eqn.

Maxwell's equations in free space :

(i) $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ (Gauss)

(ii) $\vec{\nabla}.\vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$(Faraday)

(iv) $\vec{\nabla} \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$ (Ampere)

in matter : replace μ_0, ε_0 by μ, ε

(i) $\vec{\nabla}.\vec{D} = \rho_f$

(ii) $\vec{\nabla}.\vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J_f}$

Electromagnetic Waves
The Wave Equation

A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity

In the presence of absorption, the wave will diminish in size as it moves

dispersive medium : , different frequencies travel at different speeds

in 2-D or 3-D, as the wave spreads out, its amplitude decreases

and standing waves don't propagate at all

Ex. : wave of fixed shape traveling in z direction

$$f(z,t) = f(z - vt, 0) = g(z - vt)$$

wave form shifts by an amount vt; v is the velocity

waves (with different shapes)

$f_1(z,t) = Ae^{-b(z-vt)^2}$; $f_2(z,t) = A\sin[b(z-vt]$; $f_3(z,t) = \frac{A}{b(z-vt)^2+1}$

These are not waves : $f_4(z,t) = Ae^{-b(z^2-vt)}$; $f_5(z,t) = A\sin(bz)\cos(bvt)$

classical wave eqn. : $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

exercise : Show that if $f(z,t) = g(z-vt)$
then $\frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$, where $u = z - vt$

another class of solutions : $f(z,t) = h(z+vt)$

general solution : sum of a wave to the right and a wave to the left :

$f(z,t) = g(z-vt) + h(z+vt)$

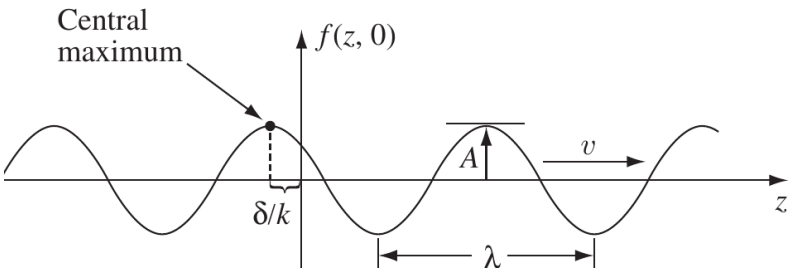
wave equation is linear

exercise: refresh memory -

wavelength, phase, phase constant, wave number, wave length, period, frequency, angular frequency

sinusoidal soln. :

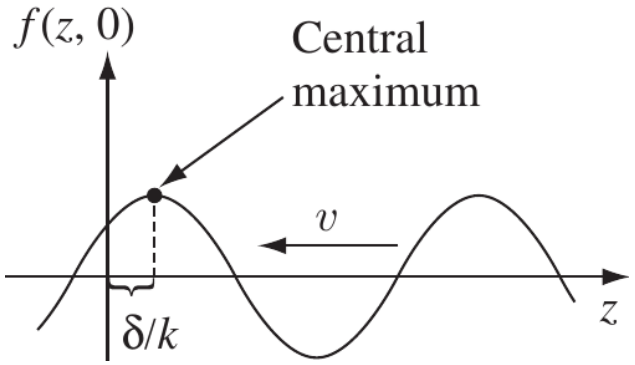
$$f(z,t) = A\cos[k(z - vt) + \delta]$$
$$= A\cos(kz - \omega t + \delta)$$



traveling backwards : $t \to -t$

$A\cos(kz + \omega t - \delta) = A\cos(-kz - \omega t + \delta)$

switch sign of k to produce a wave with same amplitude, phase constant, frequency, and wavelength, traveling in opposite direction



$\delta/k =$ distance by which wave is "delayed"

Complex notation

complex wave function : $\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$; $\tilde{A} = Ae^{i\delta}$: complex amplitude

actual wave function : the real part of $\tilde{f}$: $f(z,t) = Re[\tilde{f}(z,t)]$

Linear combinations of sinusoidal waves

any wave can be expressed as a linear combination of sinusoidal ones:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}dk$$

Boundary Conditions: Reflection and Transmission

reflecting surface : incident wave $\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)} (z < 0)$ coming in from left

reflected wave : $\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}, (z < 0)$ traveling back

transmitted wave : $\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)} (z > 0)$

incident wave $\tilde{f}_I$ is a sinusoidal oscillation that extends all the way back to $z = -\infty$
same goes for $\tilde{f}_R$ and $\tilde{f}_T$ (the latter extends to $z = -\infty$)

All parts of system oscillate at same frequency ω

incident wave of finite extent : linear combination of true sinusoidal functions
with range of frequencies and wavelengths

For a single incident frequency the waves extend to infinity

For a sinusoidal incident wave, after encountering the reflecting surface

$$\tilde{f}(z,t) = \begin{cases} \tilde{A_I} e^{i(k_1 z - \omega t)} + \tilde{A_R} e^{i(-k_1 z - \omega t)}, \ \forall z < 0 \\ \tilde{A_T} e^{i(k_2 z - \omega t)}, \ \forall z > 0 \end{cases}$$

B.C. : $\tilde{f}(0-,t) = \tilde{f}(0+,t)$ and $\left.\frac{\partial \tilde{f}}{\partial z}\right|_{0-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0+}$

$$\implies \tilde{A_I} + \tilde{A_R} = \tilde{A_T} \text{ and } k_1 \left(\tilde{A_I} - \tilde{A_R}\right) = k_2 \tilde{A_T}$$

$$\implies \tilde{A_R} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right) \tilde{A_I} \text{ and } \tilde{A_T} = \left(\frac{2k_1}{k_1 + k_2}\right) \tilde{A_I}$$

or, using $\omega = kv$, $\tilde{A_R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) \tilde{A_I}$; and $\tilde{A_T} = \left(\frac{2v_2}{v_2 + v_1}\right) \tilde{A_I}$

or, $A_R e^{i\delta_R} = \frac{v_2 - v_1}{v_2 + v_1} A_I e^{i\delta_I}$ and $A_T e^{i\delta_T} = \frac{2v_2}{v_2 + v_1} A_I e^{i\delta_I}$

if $\delta_R = \delta_T = \delta_I$, then $A_R = \frac{v_2 - v_1}{v_2 + v_1} A_I$ and $A_T = \frac{2v_2}{v_2 + v_1} A_I$

if $v_2 < v_1$, reflected wave is out of phase by 180°, $\delta_R + \pi = \delta_T = \delta_I$,
then $\because \cos(-k_1 z - \omega t + \delta_I - \pi) = -\cos(-k_1 z - \omega t + \delta_I)$

$A_R = \frac{v_1 - v_2}{v_2 + v_1} A_I$ and $A_T = \frac{2v_2}{v_2 + v_1} A_I$

completely opaque surface : no transmission—all of it reflects back $A_R = A_I$ and $A_T = 0$

waves that travel down on shaking a string :
transverse - displacement $\perp$ direction of propagation

elastic string : compression waves -
longitudinal, displacement $\parallel$ direction of propagation

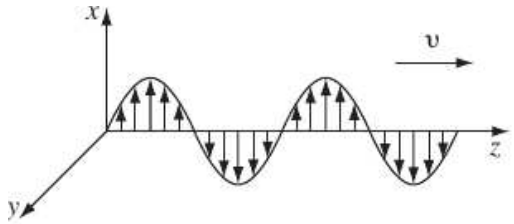
Sound waves : longitudinal

electromagnetic waves : transverse

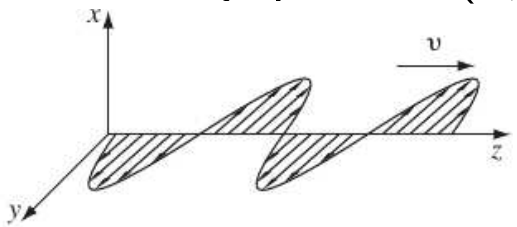
Polarization : two dimensions perpendicular to any given line of propagation
Accordingly, transverse waves occur in two independent states of polarization

$\tilde{f}_x(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{x}$ and $\tilde{f}_y(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{y}$

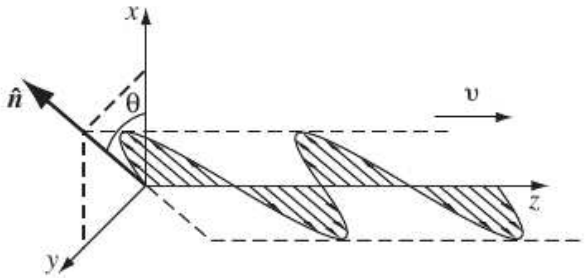
or along any other direction in the x-y plane :$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n};\ \hat{n}.\hat{z} = 0$



(a) Vertical polarization



(b) Horizontal polarization



(c) Polarization vector

generally,
$\tilde{f}(z,t) = \tilde{A}\cos\theta e^{i(kz-\omega t)}\hat{x} + \tilde{A}\sin\theta e^{i(kz-\omega t)}\hat{y}$

Electromagnetic waves in vacuum

In regions of space where there is no charge or current, Maxwell's equations :

(i) $\vec{\nabla}.\vec{E} = 0$ (iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(ii) $\vec{\nabla}.\vec{B} = 0$ (iv) $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

- set of coupled, first-order, partial differential equations for $\vec{E}$ and $\vec{B}$

can be decoupled by applying curl to (iii) and (iv):

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{E}\right) - \nabla^2 \vec{E} = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial\left(\vec{\nabla} \times \vec{B}\right)}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{B}\right) - \nabla^2 \vec{B} = \vec{\nabla} \times \left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = \mu_0 \varepsilon_0 \frac{\partial\left(\vec{\nabla} \times \vec{E}\right)}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

using (i) and (ii), $\begin{array}{l} \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \end{array}$ - separate second order eqns for $\vec{E}$ and $\vec{B}$

$\therefore$ In vacuum, each Cartesian component of $\vec{E}$ and $\vec{B}$ satisfies the 3-D wave eqn.

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

Maxwell's eqns $\Longrightarrow$ empty space speed of em waves$= \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 2.98 \times 10^{10}$m/s

Monochromatic Plane Waves : waves of single frequency traveling in one direction
- no x or y dependence
$\because$ fields are uniform over every plane $\perp$ direction of propagation

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(kz-\omega t)};\ \tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(kz-\omega t)}; \quad \omega = ck$$

and components of $\tilde{\vec{E}}_0$ and $\tilde{\vec{B}}_0$ are constants

$\because \vec{\nabla}.\vec{E} = 0$ and $\vec{\nabla}.\vec{B} = 0$,

$$\therefore \tilde{E}_{0_z} \frac{\partial}{\partial z}\left[e^{i(kz-\omega t)}\right] = 0 = \tilde{B}_{0_z} \frac{\partial}{\partial z}\left[e^{i(kz-\omega t)}\right] \implies \tilde{E}_{0_z} = \tilde{B}_{0_z} = 0$$

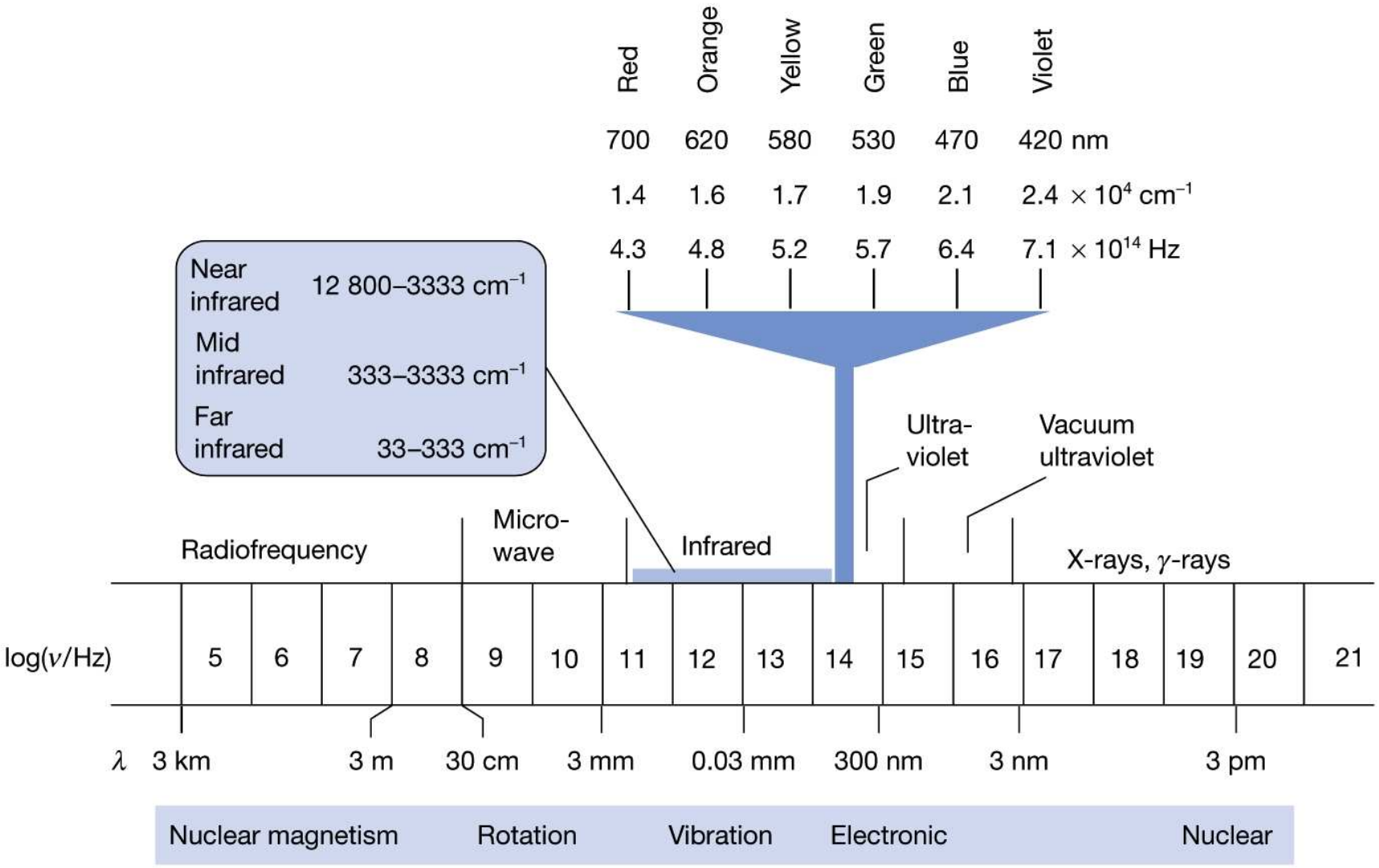
i. e., electromagnetic waves are transverse
electric and magnetic fields are perpendicular to the direction of propagation.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t},\ \text{or,}\ \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_x & \tilde{E}_y & \tilde{E}_z \end{vmatrix} = ik\left(-\hat{x}\tilde{E}_{0_y} + \hat{y}\tilde{E}_{0_x}\right)e^{i(kz-\omega t)} = i\omega\left(\hat{x}\tilde{B}_{0_x} + \hat{y}\tilde{B}_{0_y}\right)e^{i(kz-\omega t)}$$

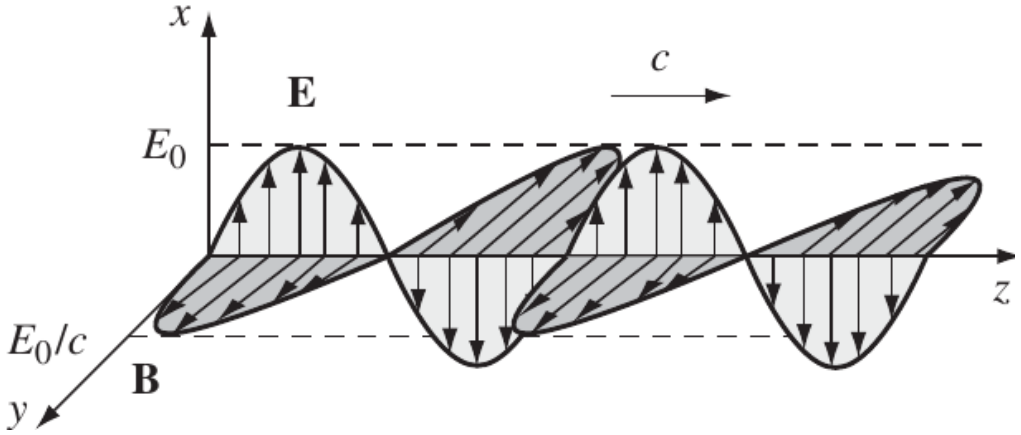
gives $-k\tilde{E}_{0_y} = \omega\tilde{B}_{0_x}$ and $k\tilde{E}_{0_x} = \omega\tilde{B}_{0_y}$

or, $\tilde{\vec{B}}_0 = \frac{k}{\omega}\left(\hat{z}\times\tilde{\vec{E}}_0\right)$; $\vec{E}$ and $\vec{B}$ are in phase and mutually perpendicular

their (real) amplitudes are related : $B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0$


Red
Orange
Yellow
Green
Blue
Violet
700 620 580 530 470 420 nm
1.4 1.6 1.7 1.9 2.1 2.4 × 10⁴ cm⁻¹
4.3 4.8 5.2 5.7 6.4 7.1 × 10¹⁴ Hz
Near infrared 12 800–3333 cm⁻¹
Mid infrared 333–3333 cm⁻¹
Far infrared 33–333 cm⁻¹
Ultra-violet
Vacuum ultraviolet
Radiofrequency
Micro-wave
Infrared
X-rays, γ-rays
log(ν/Hz)
5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
λ 3 km 3 m 30 cm 3 mm 0.03 mm 300 nm 3 nm 3 pm
Nuclear magnetism
Rotation
Vibration
Electronic
Nuclear

If $\vec{E}$ points in the x-direction, then $\vec{B}$ points in the y-direction



$$\tilde{\vec{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}\hat{x}$$

$$\tilde{\vec{B}}(z,t) = \frac{1}{c}\tilde{E}_0 e^{i(kz-\omega t)}\hat{y}$$

or, $\vec{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\,\hat{x}, \quad \vec{B}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{y}$

generalized :

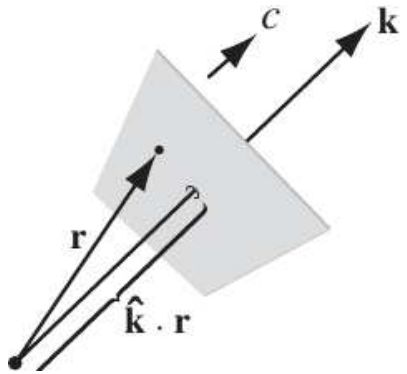
$$\tilde{\vec{E}}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}.\vec{r}-\omega t)}\hat{n}$$

$$\tilde{\vec{B}}(\vec{r},t) = \frac{1}{c}\tilde{E}_0 e^{i(\vec{k}.\vec{r}-\omega t)}\left(\hat{k}\times\hat{n}\right) = \frac{1}{c}\hat{k}\times\tilde{\vec{E}} \quad ; \quad \hat{k}.\hat{n} = 0$$

actual (real) electric and magnetic fields in a monochromatic plane wave
with propagation vector $\vec{k}$ and polarization $\hat{n}$

$$\vec{E}(\vec{r},t) = E_0\cos(\vec{k}.\vec{r}-\omega t+\delta)\hat{n}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}E_0\cos(\vec{k}.\vec{r}-\omega t+\delta)\left(\hat{k}\times\hat{n}\right)$$



Electromagnetic waves in matter

Inside matter, but in regions where there is no free charge or free current, Maxwell's equations become

(i) $\vec{\nabla}.\vec{D} = 0$ (iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(ii) $\vec{\nabla}.\vec{B} = 0$ (iv) $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

If the medium is linear, $\vec{D} = \varepsilon \vec{E}$; $\vec{H} = \frac{1}{\mu}\vec{B}$

and homogeneous

(i) $\vec{\nabla}.\vec{E} = 0$ (iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(ii) $\vec{\nabla}.\vec{B} = 0$ (iv) $\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$

these differ from the vacuum analogs in the replacement of $\mu_0\varepsilon_0$ by $\mu\varepsilon$

electromagnetic waves propagate through linear homogeneous medium at speed $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$

$n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \Leftarrow$ index of refraction

For most materials, $\mu \approx \mu_0$, $\therefore n \approx \sqrt{\epsilon_r}$

Classical wave polarization : $\vec{E} = A\hat{n}e^{i\left(\vec{k}.\vec{r}-\omega t\right)}$

$\hat{n}$ is polarization vector, $\perp \hat{k}$, unit vector in direction of propagation

If $\vec{E}$ vibrates along $\begin{matrix} x \\ y \end{matrix}$ axis then light is linearly $\begin{matrix} x \\ y \end{matrix}$ polarized , $\hat{n} = \begin{matrix} \hat{n}_x \parallel \hat{x} \\ \hat{n}_y \parallel \hat{y} \end{matrix}$

generally, $\hat{n} = a_1\hat{n}_x + a_2 e^{i\delta}\hat{n}_y; \quad a_1^2 + a_2^2 = 1$ and $|\hat{n}|^2 = 1$

corresponds to linear superposition of 2 waves of equal frequency and same wave vector with amplitudes A_1 and A_2 , polarized along x and y directions, respectively, with definite phase difference δ :

$$\vec{E} = A_1\hat{n}_x e^{i\left(\vec{k}.\vec{r}-\omega t\right)} + A_2\hat{n}_y e^{i\left(\vec{k}.\vec{r}-\omega t\right)} = A\left(a_1\hat{n}_x + a_2 e^{i\delta}\hat{n}_y\right)e^{i\left(\vec{k}.\vec{r}-\omega t\right)}$$

where $a_i = \frac{A_i}{A}$ and $A = \sqrt{A_1^2 + A_2^2}$

define β, s.t., $a_1 = \cos\beta$; $a_2 = \sin\beta$, then $\hat{n} = \cos\beta\hat{n}_x + e^{i\delta}\sin\beta\hat{n}_y$

waves of equal amplitude oscillating in phase $(a_1 = a_2;\ \delta = 0)$: $\hat{n} = \frac{1}{\sqrt{2}}(\hat{n}_x + \hat{n}_y)$

for $a_1 = a_2; \delta = 90^o$: $\hat{n} \sim \hat{n}_x \pm i\hat{n}_y$: left or right handed circular polarization

for $a_1 \neq a_2; \delta \neq 0$; ellptical polarization

light is completely polarized
if its properties are specified in terms of single polarization vector $\hat{n}$

Optical rotatory dispersion (ORD):

variation in optical rotation with change in wavelength of light

- is used to find absolute configuration of metal complexes

When plane-polarized white light is passed through a cylinder of sucrose solution, spiral rainbow is observed perpendicular to it

When light passes through a polarizer, extent of rotation depends on wavelength

Short wavelengths are rotated more than longer wavelengths, per unit distance

Variation of color with distance through the tube is observed

variation caused by two different phenomena :
first accounts in most cases for majority of variation in rotation and should not strictly be termed rotatory dispersion

optical activity is actually **circular birefringence**

optically active substance transmits right circularly polarized light with different velocity from left circularly polarized light

$$\text{rotation (rad cm}^{-1}) = \frac{2\pi l}{\lambda}\left(n_L - n_R\right)$$

In addition to pseudodispersion which depends on material thickness,

true rotatory dispersion depends on variation of indices of refraction with wavelength for right and left circularly polarized light

CD (Circular dichroism) : circularly polarized light is 'chiral',
it interacts differently with chiral molecules

Two types of circularly polarized light are absorbed to different extents

Equal amounts of left and right circularly polarized light of a selected wavelength are alternately radiated into a (chiral) sample.
One of the two polarizations is absorbed more than other

wavelength-dependent difference of absorption is measured, yielding CD spectrum

Due to interaction with molecule, electric field vector of light traces out an elliptical path after passing through sample

$$\Delta A = A_L - A_R \overset{\text{Beer's}}{\underset{\text{law}}{=}} (\epsilon_L - \epsilon_R)\, c.l$$

$$\text{CD} = \epsilon_L - \epsilon_R = \frac{A_L - A_R}{c.l}$$

with c as molar concentration, $\epsilon_L - \epsilon_R =$ molar CD
- intrinsic property and function of wavelength and temperature.

ellipticity of polarization, $\tan\theta = \frac{E_R - E_L}{E_R + E_L}$

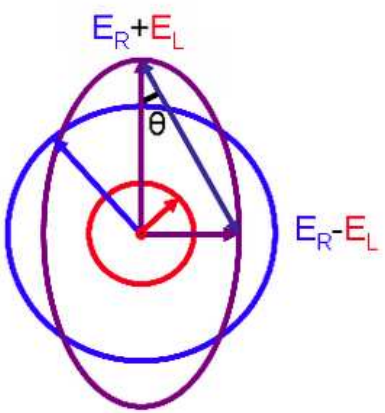
($\theta = 0 \implies$ linear polarization;
$\theta = \frac{\pi}{4} \implies$ circular polarization)

$E \propto \sqrt{I}$, $\therefore \tan\theta = \frac{\sqrt{I_R} - \sqrt{I_L}}{\sqrt{I_R} + \sqrt{I_L}}$

for small θ,
using $I = I_0 e^{-A \ln 10}$

$$\theta = \frac{e^{-\Delta A \frac{\ln 10}{2}} - 1}{e^{-\Delta A \frac{\ln 10}{2}} + 1} \overset{\Delta A \ll 1}{=} \frac{1}{2}\Delta A \frac{\ln 10}{2}$$

or, $\theta = \Delta A \left(\frac{\ln 10}{4}\right)\left(\frac{180}{\pi}\right)$

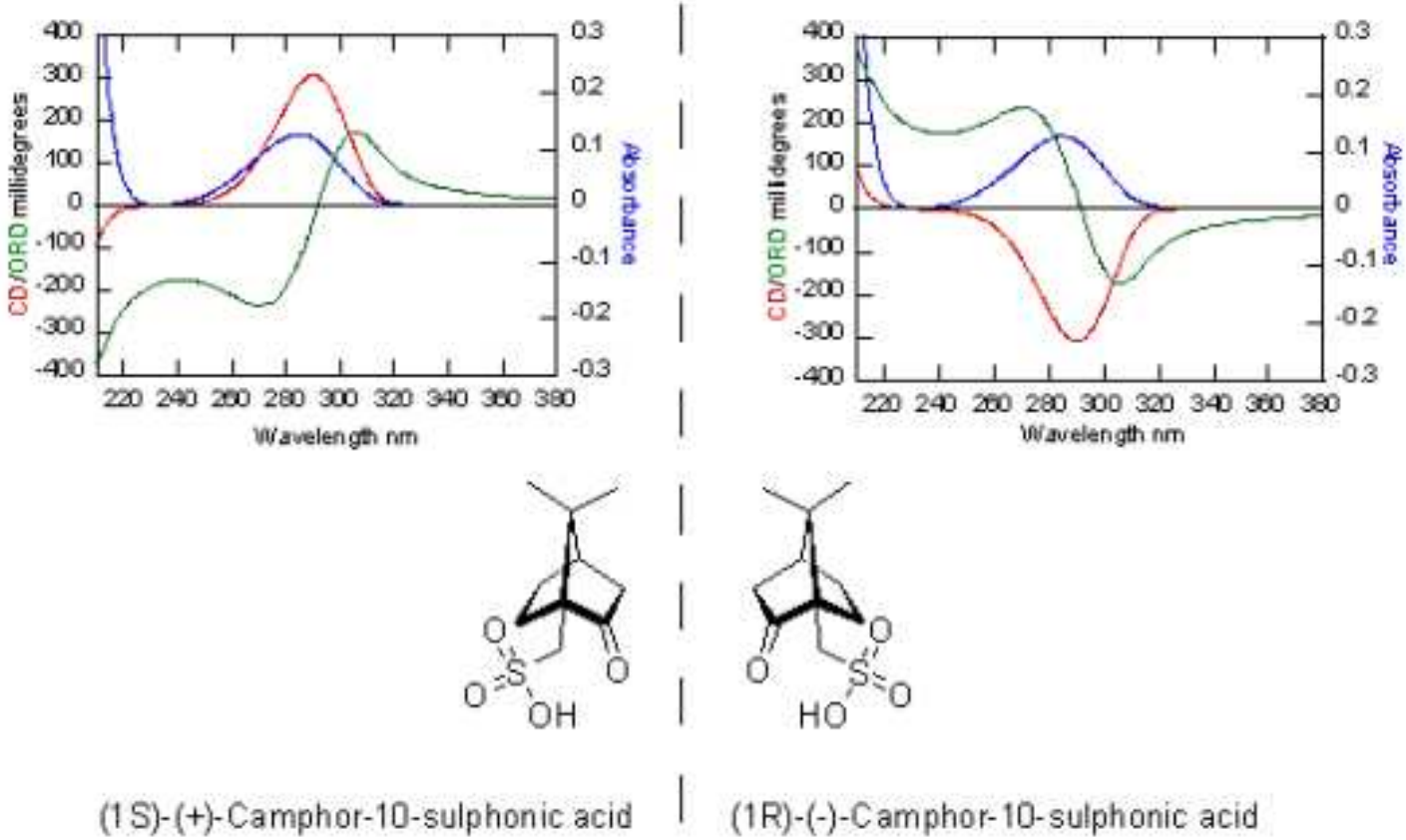


molar ellipticity: $[\theta] = \frac{100\theta}{c.l}$

$$\therefore [\theta] = 100\Delta\epsilon \left(\frac{\ln 10}{4}\right)\left(\frac{180}{\pi}\right)$$
$$= 3298.2\Delta\epsilon$$

units: deg·cm^2/dmol

CD, ORD and Absorbance spectra of R and S forms of camphor sulphonic acid

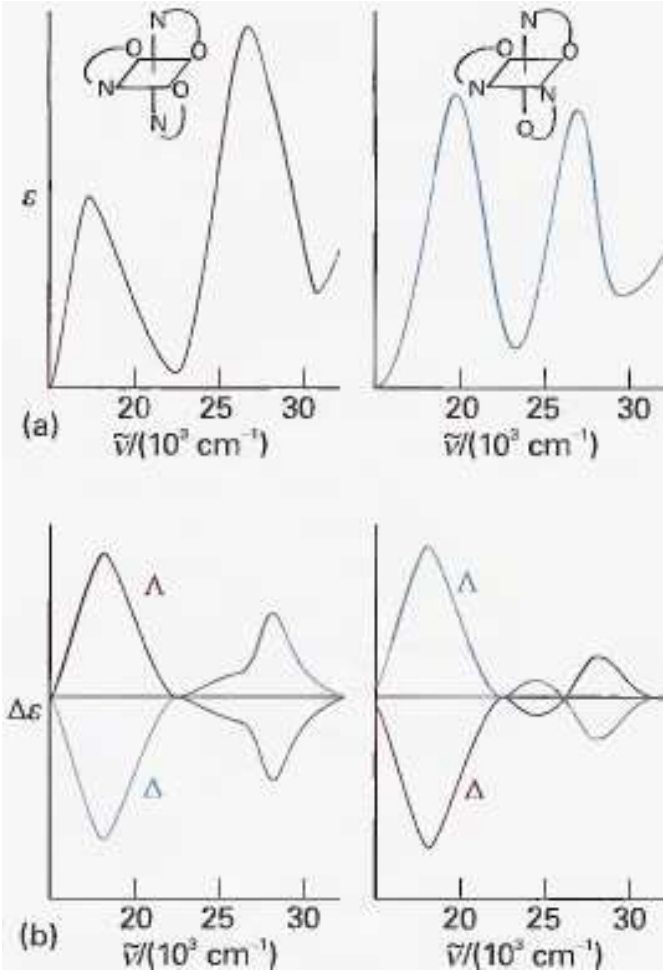


(a) absorption spectra of two isomers of $[Co(ala)_3]$

left- and right-handed forms of isomers give identical absorption spectra

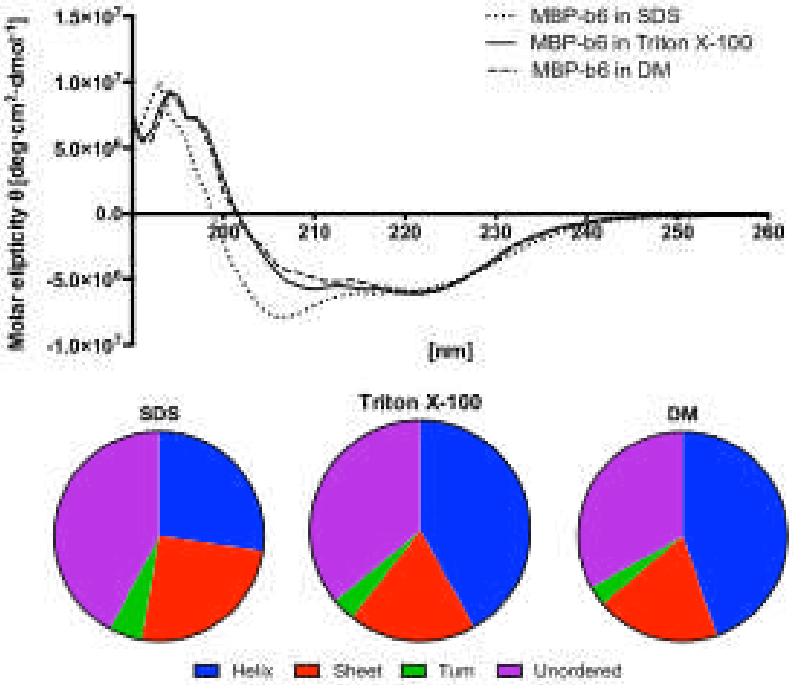
(b) corresponding CD spectra

However, CD spectra are distinctly different, and absolute configurations (Λ and Δ) are assigned by comparison with CD spectra of complex of known absolute configuration



CD is very sensitive to the secondary structure of polypeptides and proteins.

CD spectra in uv region (UV-CD) of MBP-cytochrome b6 fusion protein in different detergent solutions:



In DM, as well as in Triton X-100 solution, the prtein recovered its structure.
Spectra from SDS solution shows decreased ellipticity in the range between 200–210 nm, which indicates incomplete secondary structure recovery

Lower panel: Content of secondary structures predicted

Poynting's Theorem

work necessary to assemble a static charge distrilbution $= \frac{\epsilon_0}{2} \int E^2 d\tau$

Likewise, work required to get currents going $= \frac{1}{2\mu_0} \int B^2 d\tau$

total energy stored in electromagnetic fields per unit volume $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$

Suppose at time t, a charge and current configuration produces fields $\vec{E}$ and $\vec{B}$
At $t + dt$, the charges move around a bit

Q.: How much work, dW , is done by the electromagnetic forces acting on these charges, in the interval dt?

work done on charge q is $\vec{F}.d\vec{l} = q\left(\vec{E} + \vec{v} \times \vec{B}\right).\vec{v}dt = q\vec{E}.\vec{v}dt$

use $q = \rho d\tau$ and $\vec{J} = \rho\vec{v} \Longrightarrow \frac{dW}{dt} = \int\limits_{\mathcal{V}} \vec{E}.\vec{J}d\tau$ in volume $\mathcal{V}$

$\therefore$ work done per unit time, per unit volume$= \vec{E}.\vec{J} = \frac{1}{\mu_0}\vec{E}.\vec{\nabla} \times \vec{B} - \varepsilon_0\vec{E}.\frac{\partial\vec{E}}{\partial t}$

$$= \frac{1}{\mu_0}\left[-\vec{\nabla}.\left(\vec{E}\times\vec{B}\right) + \vec{B}.\vec{\nabla}\times\vec{E}\right] - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t} = \frac{1}{\mu_0}\left[-\vec{\nabla}.\left(\vec{E}\times\vec{B}\right) - \vec{B}.\frac{\partial\vec{B}}{\partial t}\right] - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t}$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\vec{\nabla}.\left(\vec{E}\times\vec{B}\right)$$

$$\therefore \frac{dW}{dt} = \int\limits_{\mathcal{V}} \vec{E}.\vec{J}d\tau = -\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\mathcal{V}}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\int\limits_{\mathcal{V}}\vec{\nabla}.\left(\vec{E}\times\vec{B}\right)d\tau$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\mathcal{V}}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint\limits_{\mathcal{S}}\left(\vec{E}\times\vec{B}\right).d\vec{a}$$: Poynting's work-energy theorem

$$\frac{dW}{dt} = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}}\left(\vec{E}\times\vec{B}\right).d\vec{a}$$

1st term : rate of change in energy stored in the fields, $\frac{\partial}{\partial t}\int_{\mathcal{V}} u d\tau$

2nd term : rate at which energy transported out of $\mathcal{V}$, across its boundary surface

Poynting's theorem : work done on the charges by the electromagnetic force = decrease in energy remaining in fields– energy that flowed out through the surface

Poynting vector : energy per unit time, per unit area, transported by the fields

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$\vec{S}.d\vec{a}$ =energy per unit time crossing $d\vec{a}$: energy flux ($\vec{S}$ = energy flux density)

Energy in Electromagnetic Waves

total energy stored per unit volume, $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$

$\because B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2$; $\quad \therefore$ electric and magnetic contributions are equal

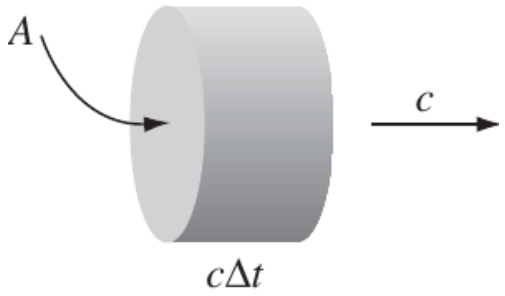
For monochromatic plane waves, $u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$

energy flux density (energy per unit area, per unit time) transported by the fields
$=$ Poynting vector : $\frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\hat{z} = cu\hat{z}$

$\vec{S} =$ energy density, $u \times$ velocity

in a time Δt, length $c\Delta t$ passes through area A
carrying energy $uAc\Delta t$

$\therefore$ energy transported per unit time, per unit area $= uc$



Momentum in Electromagnetic Waves

momentum density stored in the fields, $\vec{g} = \frac{1}{c^2}\vec{S}$

For monochromatic plane waves, $\vec{g} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\hat{z} = \frac{1}{c}u\hat{z}$

For light, wavelength is short and the period brief
- any macroscopic measurement will encompass many cycles

$\therefore$ we want is the average value
average of cosine-squared over a complete cycle is $\frac{1}{2}$

$$\therefore \langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2$$

$$\langle \vec{S} \rangle = \frac{1}{2} c\epsilon_0 E_0^2 \hat{z}$$

$$\langle \vec{g} \rangle = \frac{1}{2c}\epsilon_0 E_0^2 \hat{z}$$

avverage power per unit area transported or intensity, $I = \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2$

When light falls (at normal incidence) on a perfect absorber, it delivers its momentum to the surface

In a time Δt, momentum transfer, $\Delta\vec{p} = \langle\vec{g}\rangle Ac\Delta t$
$\therefore$ radiation pressure (average force per unit area), $P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}$

On a perfect reflector pressure is twice as great, because the momentum switches direction, instead of simply being absorbed

if electric field drives charges in the x-direction,
magnetic field then exerts on them a force $q(\vec{v} \times \vec{B})$ in the z-direction
net force on all charges in the surface produces the pressure

Frequency Dependence of Permittivity

$n \approxeq \sqrt{\epsilon_r}$ is a function of wavelength
A prism or a raindrop bends blue light more sharply than red
and spreads white light out into a rainbow of colors : $\Leftarrow$ dispersion

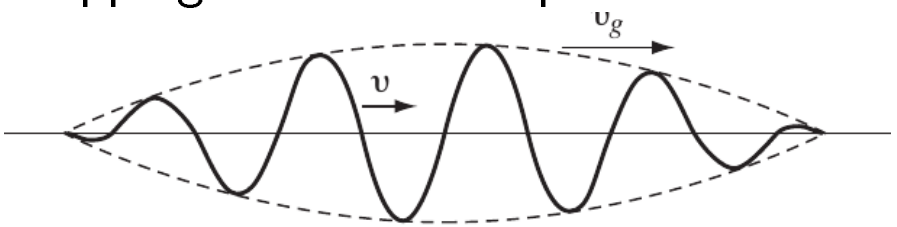
whenever speed of wave depends on its frequency, supporting medium : $\Leftarrow$ dispersive

in a dispersive medium, a wave form that incorporates a range of frequencies will change shape as it propagates

sharply peaked wave typically flattens out
and whereas each sinusoidal component travels
at ordinary wave (or phase) velocity, $v = \frac{\omega}{k}$

packet as a whole ('envelope') moves at group velocity, $v_g = \frac{d\omega}{dk}$

dropping a rock into a pond :



disturbance as a whole spreads out in a circle, moving at speed v_g

ripples that go to make it travel twice as fast, $v = 2v_g$

They appear at the back end of the packet, growing as they move forward to the center, then shrinking again and fading away at the front

energy carried by a wave packet in a dispersive medium does not travel at the phase velocity

in some cases $v > c$

Lorentz Force Law in Potential Form

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = q\left[-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)\right];$$

use : $\vec{\nabla}\left(\vec{v}.\vec{A}\right) = \vec{v} \times \left(\vec{\nabla} \times \vec{A}\right) + \left(\vec{v}.\vec{\nabla}\right)\vec{A}$

$$\frac{d\vec{p}}{dt} = -q\left[\frac{\partial \vec{A}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{A} + \vec{\nabla}\left(\phi - \vec{v}.\vec{A}\right)\right]$$

$\left[\frac{\partial \vec{A}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{A}\right] \Leftarrow$ convective derivative of $\vec{A}$, $\frac{d\vec{A}}{dt}$

$$\therefore \frac{d\left(\vec{p} + q\vec{A}\right)}{dt} = -\vec{\nabla}\left[q\left(\phi - \vec{v}.\vec{A}\right)\right]$$

reminiscent of formula $\frac{d\vec{p}}{dt} = -\vec{\nabla}U$

canonical momentum, $\vec{p}_{\text{can}} = \vec{p} + q\vec{A}$;

$U_{\text{vel}} = q\left(\phi - \vec{v}.\vec{A}\right)$

U_{vel} is a velocity dependent potential

alternatively : $\left(\vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)\right)_x$

$$= v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$= v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} + v_x\frac{\partial A_x}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z} - v_x\frac{\partial A_x}{\partial x}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}\right)$$

$$\therefore \left(\vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)\right)_x = \frac{\partial\left(\vec{v}.\vec{A}\right)}{\partial x} - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

$$F_x = q\left\{-\frac{\partial}{\partial x}\left(\phi - \vec{v}.\vec{A}\right) - \frac{d}{dt}\frac{\partial\left(\vec{v}.\vec{A}\right)}{\partial v_x}\right\}$$

$\because \phi$ is independent of velocity,

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x};\ U = q\left(\phi - \vec{v}.\vec{A}\right)$$

similarly, rate of change of the particle's energy:

$\frac{d(T+qV)}{dt} = \frac{\partial}{\partial t}\left[q\left(\phi - \vec{v}.\vec{A}\right)\right]$, where $T = \frac{1}{2}mv^2$= K.E. and qV = P.E.

same quantity U_{vel} = appears on the right side of both equations

Thus, $\vec{A}$ is a kind of "potential momentum" per unit charge
just as ϕ is potential energy per unit charge

Ex. : consider for a uniform magnetic field, $\vec{A} = -\frac{1}{2}\left(\vec{r} \times \vec{B}\right)$

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{A} = -\frac{1}{2}\left[\frac{\partial\left(\vec{r}\times\vec{B}\right)}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\left(\vec{r} \times \vec{B}\right)\right]$$

1st term $= 0$ [$\because$ no explicit time-dependent term in $\vec{r} \times \vec{B}$

$$\therefore\ -2\frac{d\vec{A}}{dt} = \left(\vec{v}.\vec{\nabla}\right)\left(\vec{r} \times \vec{B}\right)$$

$$= \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\left[\left(yB_z - zB_y\right)\hat{x} + \left(zB_x - xB_z\right)\hat{y} + \left(xB_y - yB_x\right)\hat{z}\right]$$

$$= v_x\left(-B_z\hat{y} + B_y\hat{z}\right) + v_y\left(B_z\hat{x} - B_x\hat{z}\right) + v_z\left(-B_y\hat{x} + B_x\hat{y}\right)$$

$$= \left(v_yB_z - v_zB_y\right)\hat{x} + \cdots\ = \vec{v} \times \vec{B}$$

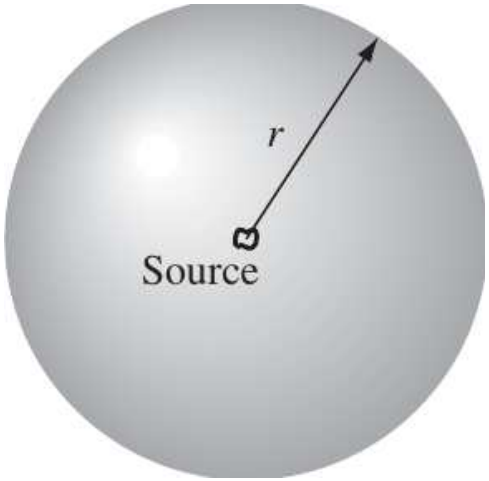
Lagrangian for a charged particle in an electromagnetic field : $L = T - q\left(\phi - \vec{v}.\vec{A}\right)$

Radiation

When charges accelerate, their fields can transport energy irreversibly out to infinity $\Leftarrow$ radiation

assume : source is localized near the origin

we wish to calculate the energy it is radiating at time t_0



Imagine a large sphere of radius r

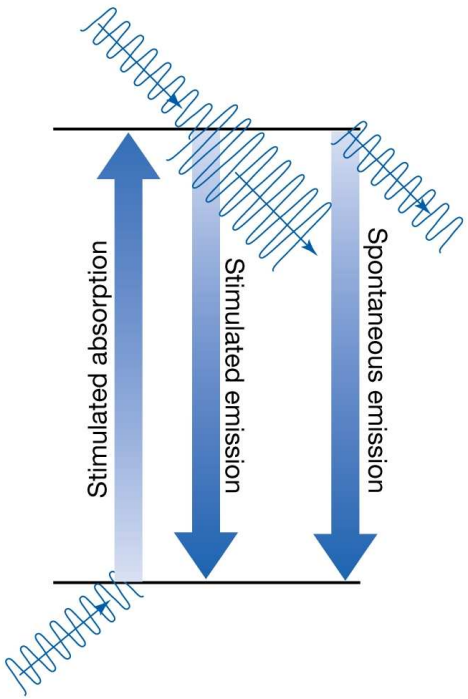
power passing through its surface is the integral of Poynting vector:

$$P(r,t) = \oint \vec{S}.d\vec{a} = \frac{1}{\mu_0} \oint \left(\vec{E} \times \vec{B}\right).d\vec{a}$$

this energy actually left the source at the earlier time $t_0 = t - r/c$

$\therefore$ power radiated $= P_{\text{rad}}(t_0) = \lim_{r\to\infty} P(r, t_0 + r/c)$

Einstein A and B coefficients:



Absorption rate$= B\rho$
where $\rho(T) =$ energy density of radiation
rate of stimulated emission $= B'\rho$
rate of spontaneous emission$= A$

At thermal equilibrium:
$NB\rho = N'(A + B'\rho)$;
$N\,(N')$ =population of ground (upper) state.

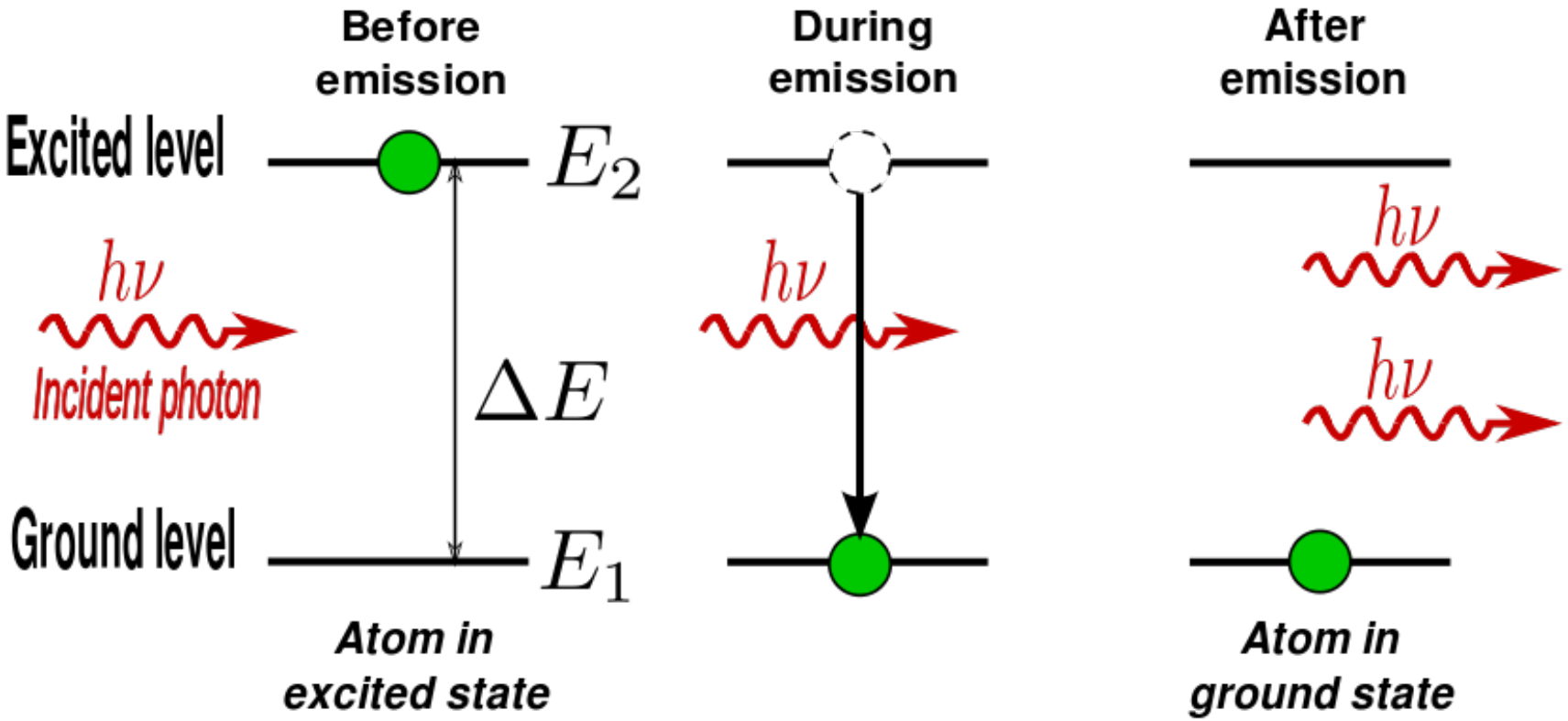
$$\implies \rho = \frac{N'A}{NB - N'B'} = \frac{\frac{A}{B}}{\frac{N}{N'} - \frac{B'}{B}} = \frac{\frac{A}{B}}{e^{\frac{h\nu}{k_B T}} - \frac{B'}{B}}$$

Net rate of absorption $W = NB\rho - N'B'\rho$

For small energy differences, $A \ll B' \approx B \implies W = (N - N')\,B\rho$

Laser action: light amplification by stimulated emission of radiation - General principles

laser action : positive-feedback - the more photons of appropriate frequency present, more photons of that frequency will be stimulated to form



$$E_2 - E_1 = \Delta E = h\nu$$

Requirements of laser action: population inversion :

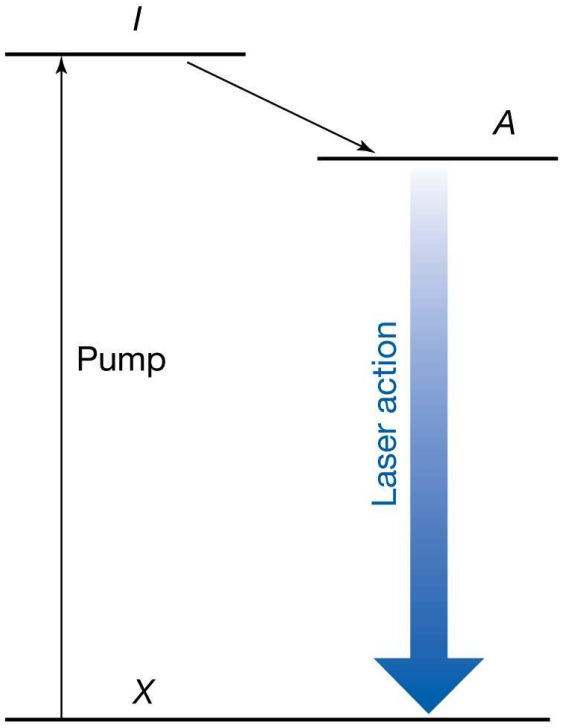
1. metastable excited state with long enough lifetime to participate in stimulated emission

2. greater population in metastable state than in lower state where the transition terminates

Because at thermal equilibrium the opposite is true, it is necessary to achieve a population inversion in which there are more molecules in the upper state than in the lower

Three-level laser

I consists of many states
I→A rapid
A→X slow

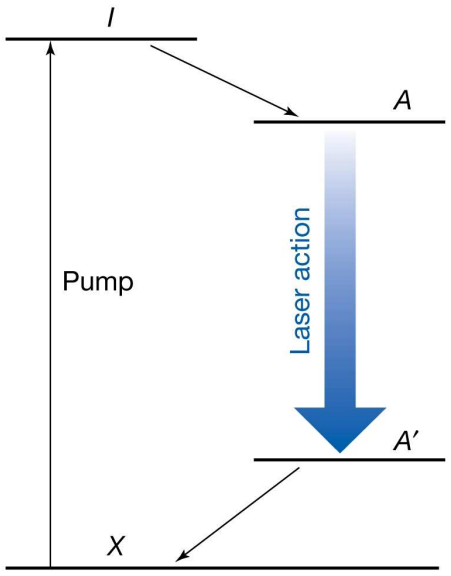


disadvantage: need to pump many molecules from X to I

Four-level laser

I→A rapid
A→A′
A′→X rapid

any population in A is a population inversion for (A, A′) system



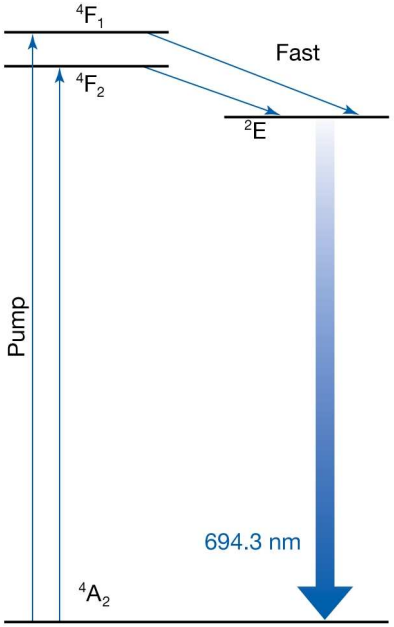
Transitions in a ruby laser

Ruby: Al_2O_3 doped with Cr^{3+} ions

Typical: 2 J pulses for 10 ns

Power: 0.2 GW per pulse

$\lambda = 694$nm



Neodynium laser : Nd:YAG; 1064 nm

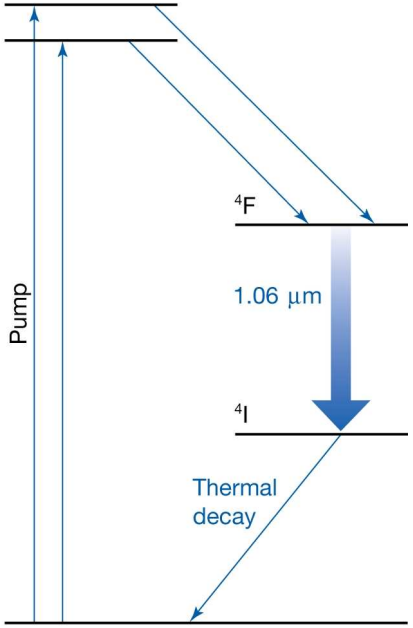
Frequency doubling : 532 nm;

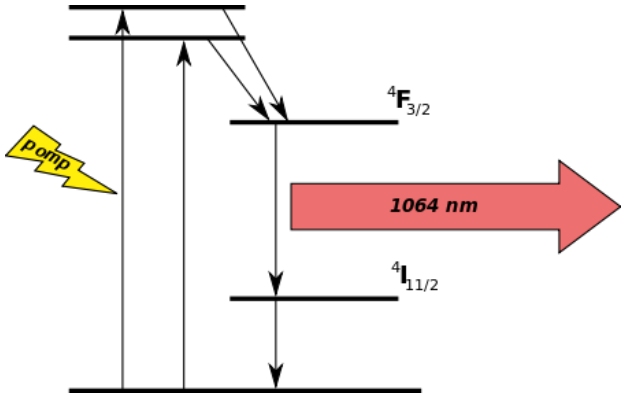
Tripling: 355 nm

Nd^{3+} at low concentrations in $Y_3Al_5O_{12}$-

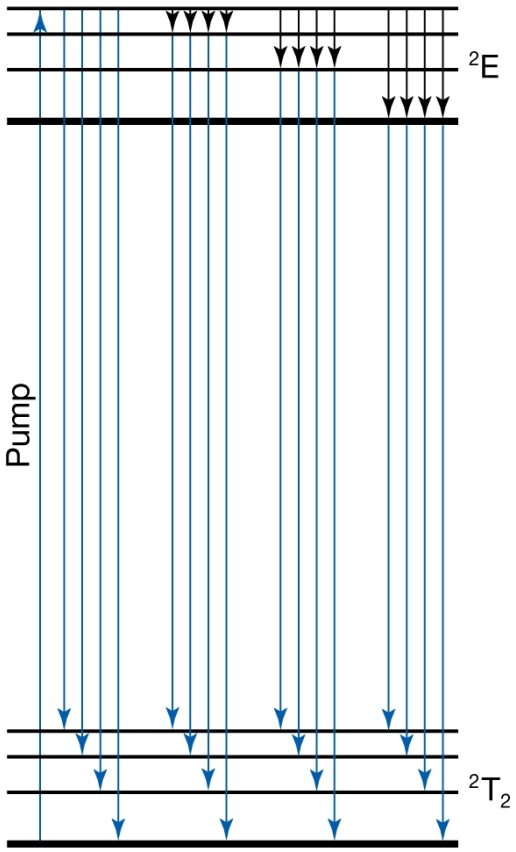
(replaces a small fraction $\approx 1\%$ of Y ions)

used as lasing medium for solid-state lasers

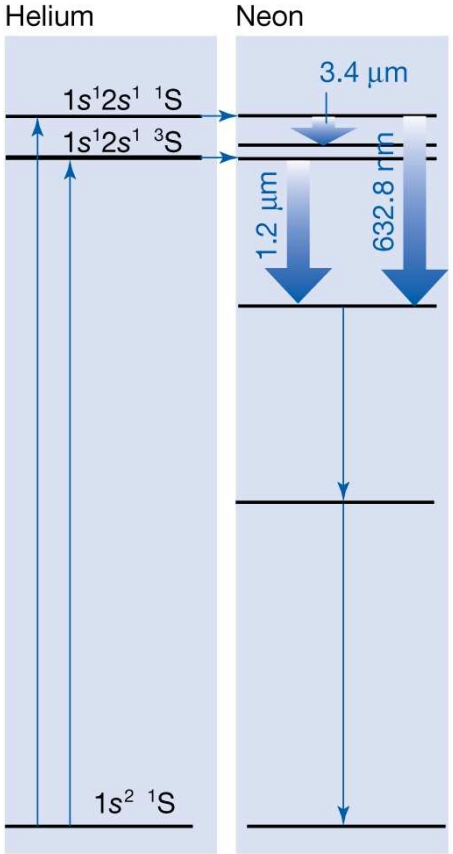




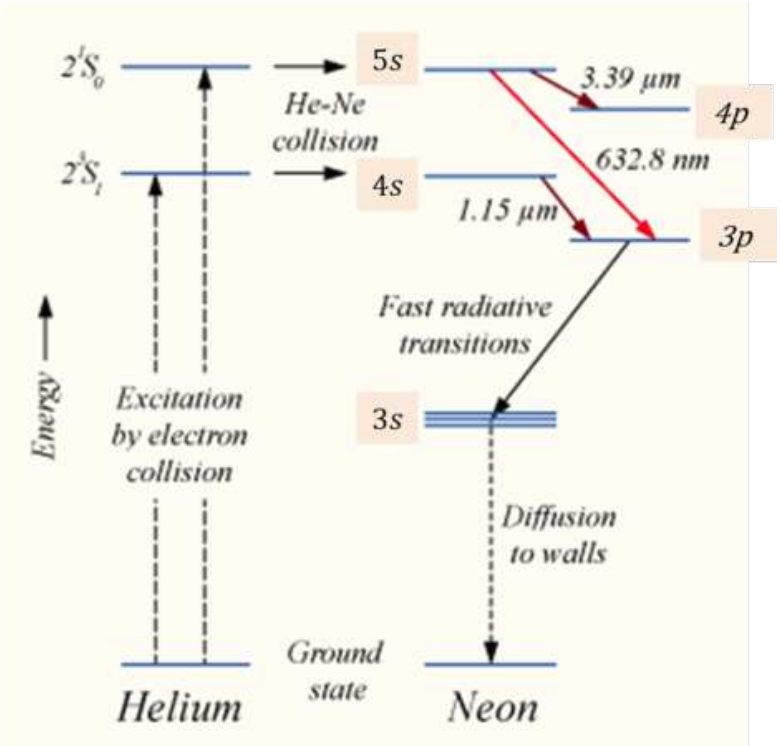
Titanium sapphire laser
A vibronic laser
20mJ – 1 J ; 20 – 100fs
700–1000 nm: tunable



He:Ne :: 5:1



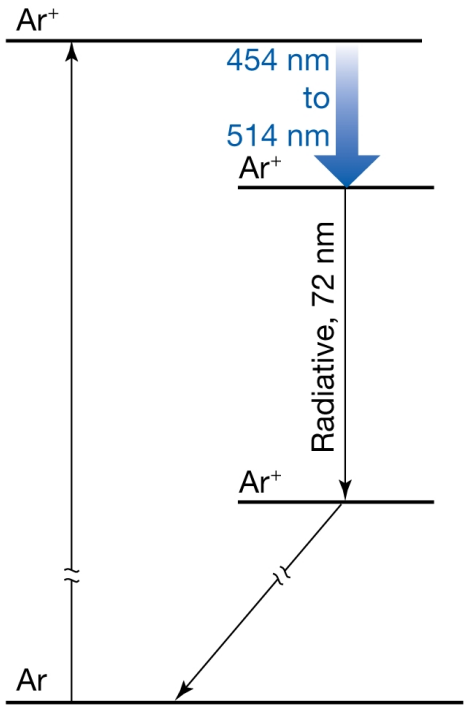
633 nm



Argon-ion laser:

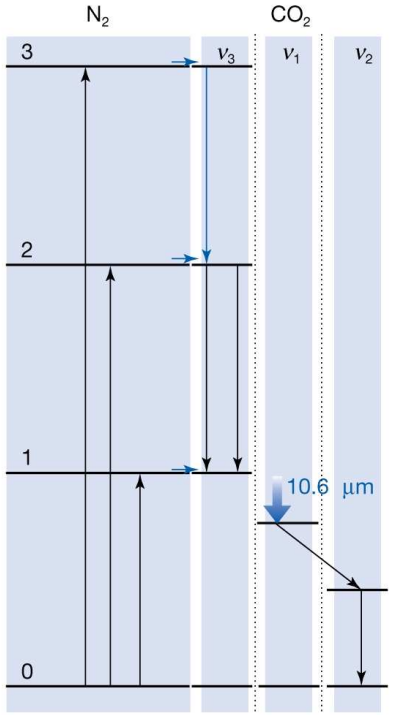
$$\text{Ar (1 Torr)} \xrightarrow[\textit{discharge}]{\textit{electric}} \text{Ar}^{+*} + \text{Ar}^{2+*}$$

excited ions undergo laser transition to lower states (strong : 488nm and 514 nm) then revert to ground state : 72 nm uv and are neutralised by electrodes



Kr ion laser: (most intense 647 nm)

CO_2 laser :9.2-10.8μm
strongest emission at 10.6μm

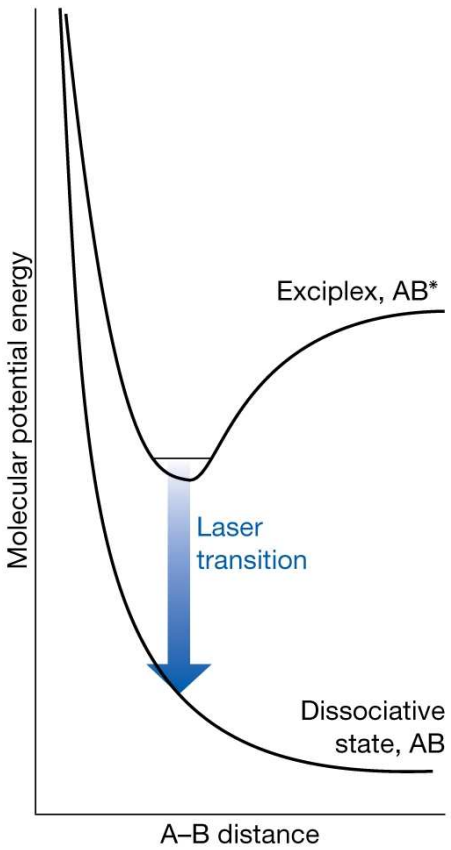


vibrationally excited N_2 has excess energy corresponding to vibrational excitation of antisymmetric stretch of CO_2 transition : $v_3 = 1$ to $v_1 = 1$

exciplex: combination of atoms that survives only in an excited state
and dissociates as excitation energy is discarded

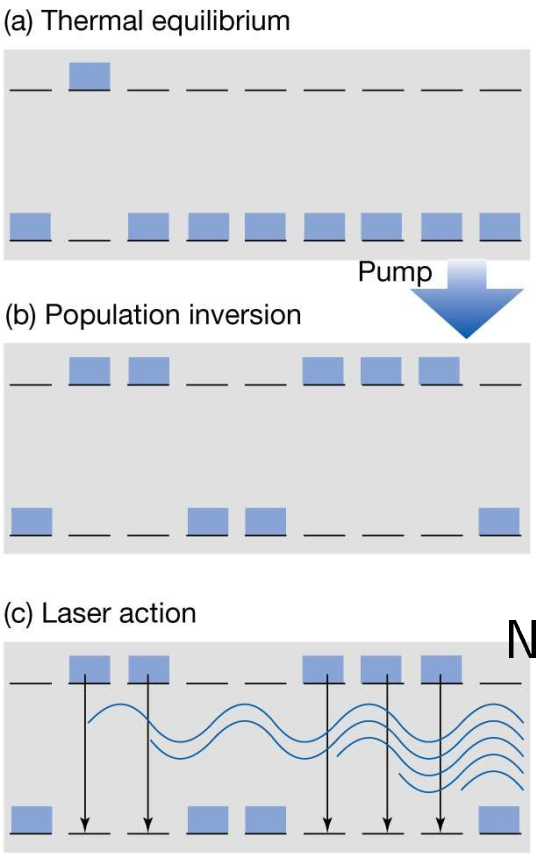
exciplex can be formed in a mixture of Xe, Cl_2, and Ne (acts as buffer). An electric discharge through the mixture produces excited Cl atoms, which attach to Xe atoms to give exciplex XeCl*.

The exciplex survives for about 10 ns
laser action at 308 nm (uv). As soon as XeCl* has discarded photon, atoms separate because ground state potential energy curve is dissociative



KrF* exciplex laser : 249 nm.

lasers:
cavity and mode characterists



$$n \times \frac{1}{2}\lambda = L$$

Integral number of half wavelengths fit in cavity
Not all wavelengths that can be sustained are amplified

resonant modes : Modes contributing to laser action

Coherence: Laser radiation is coherent

electromagnetic waves are all in step

Spatial coherence: waves in step across the cross-section of the beam emerging from cavity

Temporal coherence: waves remain in step along the beam

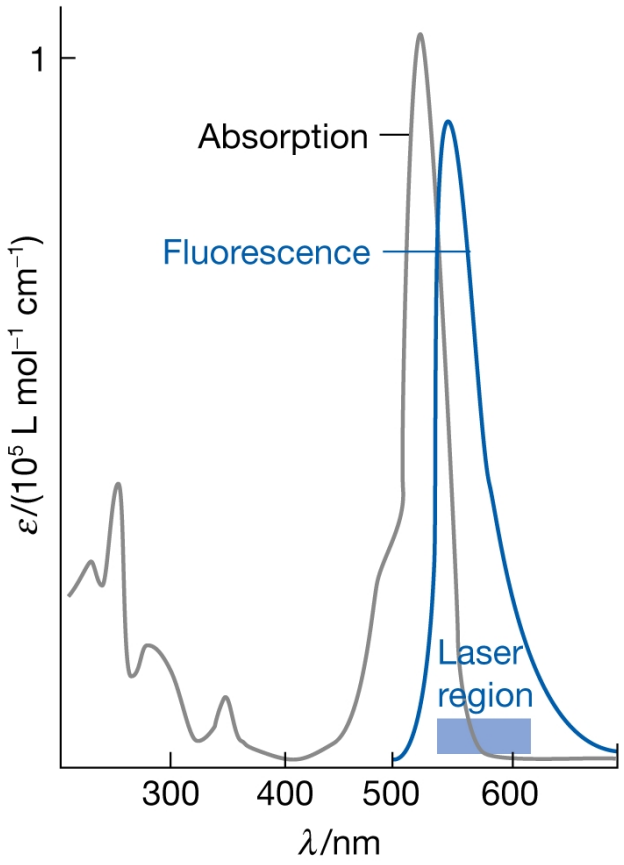
Coherence length: $l_C = \frac{\lambda^2}{2\Delta\lambda}$; $\Delta\lambda =$spread in wavelngth

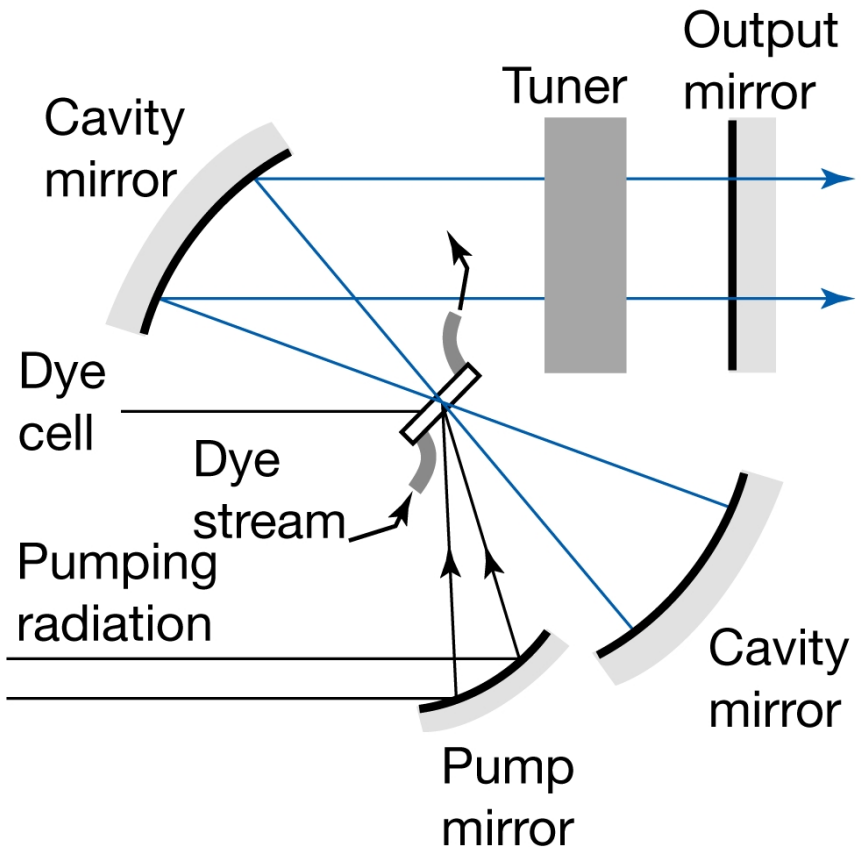
If perfectly monochrmatic: $\Delta\lambda = 0 \implies l_C = \infty$

typical light bulb: $l_C \approx 400$ nm;

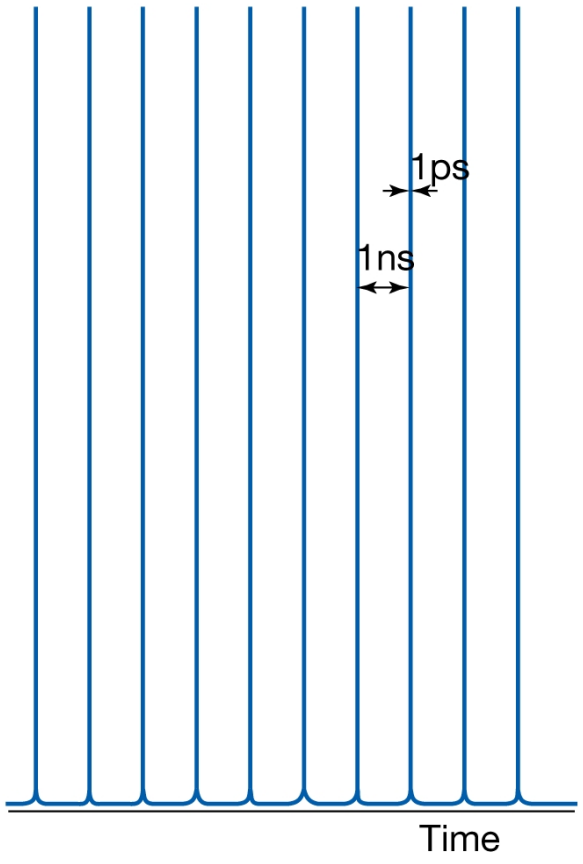
He-Ne laser with $\Delta\lambda = 2$ nm has $l_C \approx 10$ cm

Dye lasers:
Rhodamine 6G (in methanol)





Pulse radiation from lasers:



Using techniques called Q-switching and mode locking, we can get pulse radiation from a laser

Relating the power and energy of a laser

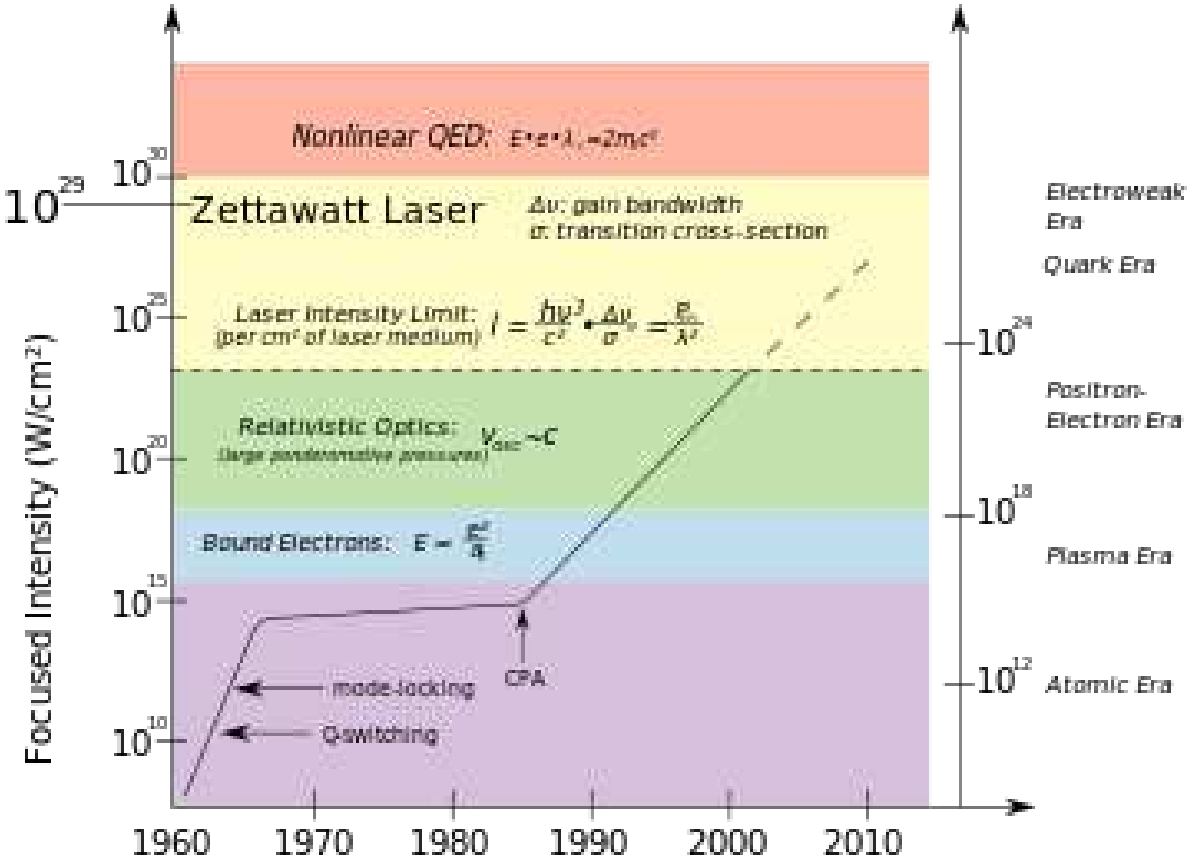
Ex. : laser rated at 0.10 J can generate radiation in 3.0 ns rectangular pulses at pulse repetition rate of 10 Hz

peak power output, $P_{\text{peak}} = \frac{\text{energy released in an interval}}{\text{duration of interval}} = \frac{0.1\ J}{3\times10^{-9}\,s} = 3.3 \times 10^7$ J/s

10 Hz$\equiv$ 1 cycle/0.1s
average power output $P_{\text{average}} = \frac{\text{total energy relased over long period}}{\text{duration of period}} = \frac{0.1\ J}{\frac{1}{10}\,s} = 1$W

In 1 s, there are only 10 cycles of 3.0 ns each
$\therefore$ emission happens only for 30 ns and this is why $P_{\text{peak}} \gg P_{\text{average}}$

history of maximum laser pulse intensity : 40 years.



Characteristics of laser radiation and their chemical applications

Characteristic	Advantage	Application
High power	Multiphoton process	Nonlinear spectroscopy Saturation spectroscopy
	Low detector noise	Improved sensitivity
	High scattering intensity	Raman spectroscopy
Monochromatic	High resolution State selection	Spectroscopy Isotope separation Photochemically precise State-to-state reaction dynamics
Collimated beam	Long path lengths Forward-scattering observable	Sensitivity Nonlinear Raman spectroscopy
Coherent	Interference between separate beams	CARS
Pulsed	Precise timing of excitation	Fast reactions Relaxation Energy transfer

Precision-specified transitions

• monochromatic character of laser radiation allows us to excite specific states with very high precision

• One consequence of state-specificity for photochemistry is that illumination of a sample may be photochemically precise and hence efficient in stimulating a reaction, because its frequency can be tuned exactly to an absorption

• specific excitation of a particular excited state of a molecule greatly enhances rate of a reaction even at low temperatures

• reaction-rate generally increases by raising temperature because energies of various modes of motion are enhanced; this increases energy in all modes, even those that do not contribute reaction rate. With a laser we excite kinetically significant mode : rate enhancement achieved most efficiently

• example : reaction $BCl_3 + C_6H_6 \rightarrow C_6H_5 - BCl_2 + HCl$ normally proceeds above 600°C in presence of catalyst; exposure to CO_2 laser radiation results in formation of products at room temperature without a catalyst

• commercial potential : heat-sensitive compounds, such as pharmaceuticals, may be made at lower temperatures

• related application : state-to-state reaction dynamics, in which a specific state of a reactant is excited and we monitor not only the rate at which it forms products but also states in which they are produced

Alternating voltages and currents (AC)
Most electric and magnetic devices use alternating or, at least, fluctuating currents

two major reasons related to power technology and transmission and processing of information :

1. With AC, electric power supplied at a given voltage can be made available at almost any other convenient voltage by means of transformers
This makes electric power adaptable to a broad variety of uses

Power supplied by a DC source can also be changed from one voltage to another
But this is done by first switching the current periodically, to obtain AC, then feeding this to a transformer, and then rectifying and filtering the output $\Leftarrow$ costly and inefficient

2. AC can transmit information
e.g., A microphone transforms the information contained in a spoken word into a complex fluctuating current
This current modulates a radio-frequency current that is fed to an antenna
Antenna launches an electromagnetic wave, and so forth

$V = V_0 \cos \omega t$; where $V_0 =$ peak voltage, $\omega = 2\pi f$,circular frequency, in radians/s
$f =$frequency in Hz

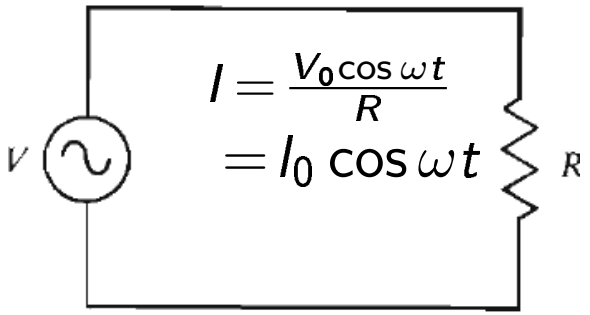
phase of V arbitrarily set to 0 at $t = 0$

If such a voltage is applied to a circuit, then branch currents are of the form

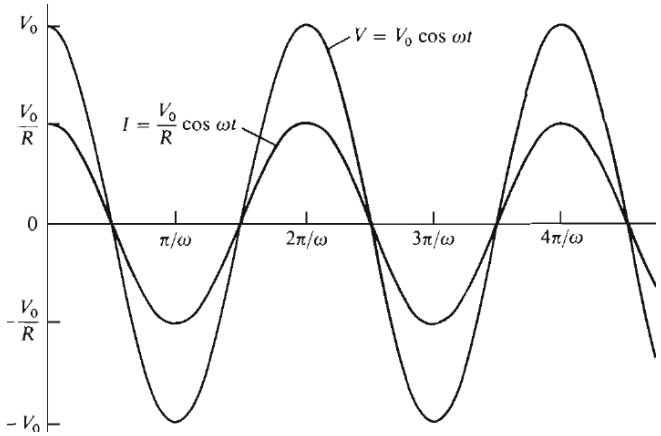
$I = I_0 + \cos(\omega t + \phi)$

R M S OR EFFECTIVE VALUES

AC voltage $V_0 \cos\omega t$ applied across a resistor R



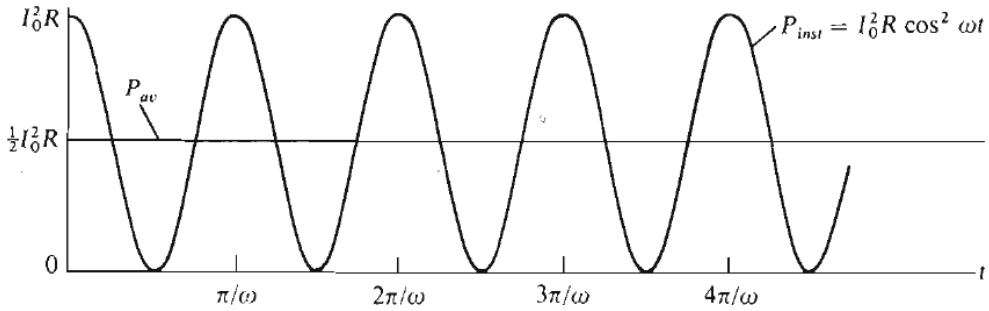
$$I = \frac{V_0 \cos\omega t}{R} = I_0 \cos\omega t$$



current in phase with voltage, and peak current $I_0 = \frac{V_0}{R}$

instantaneous power dissipated in resistor

$$P_{\text{inst}} = VI = \frac{V_0^2}{R}\cos^2\omega t = I_0^2 R \cos^2\omega t$$

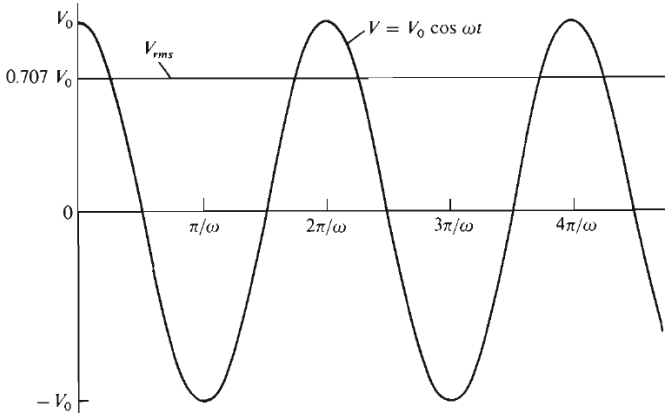


over one cycle, average power dissipation

$$P_{\text{av}} = \frac{V_0^2}{R}\cos^2\omega t = \frac{1}{2}\frac{V_0^2}{R}\frac{\int_0^{\frac{2\pi}{\omega}} \cos^2\omega t\, dt}{\frac{2\pi}{\omega}} = \frac{1}{2} I_0^2 R$$

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}; \quad I_{\text{rms}} = \frac{I_0}{\sqrt{2}}; \; P_{\text{av}} = \frac{V_{\text{rms}}^2}{R} = I_{\text{rms}}^2 R$$

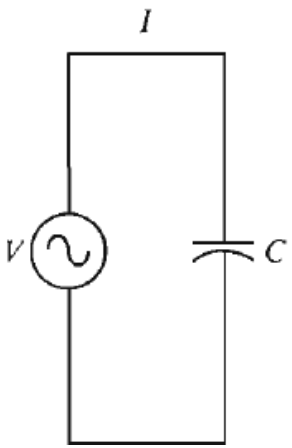
peak values are $\sqrt{2}$ times larger than rms as long as V and I are sinusoidal functions of the time

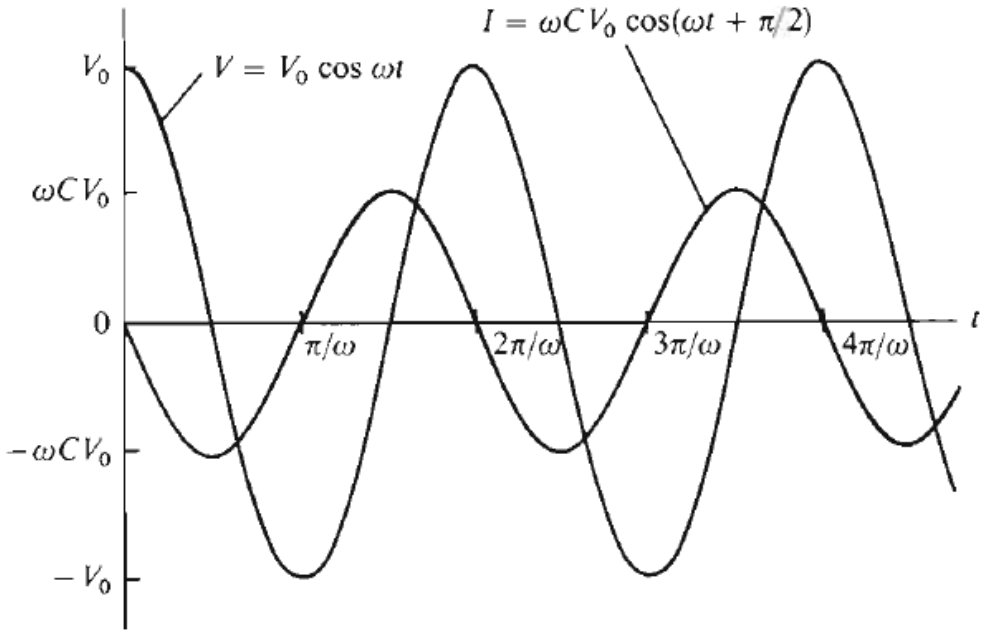


AC IN CAPACITORS AND INDUCTORS

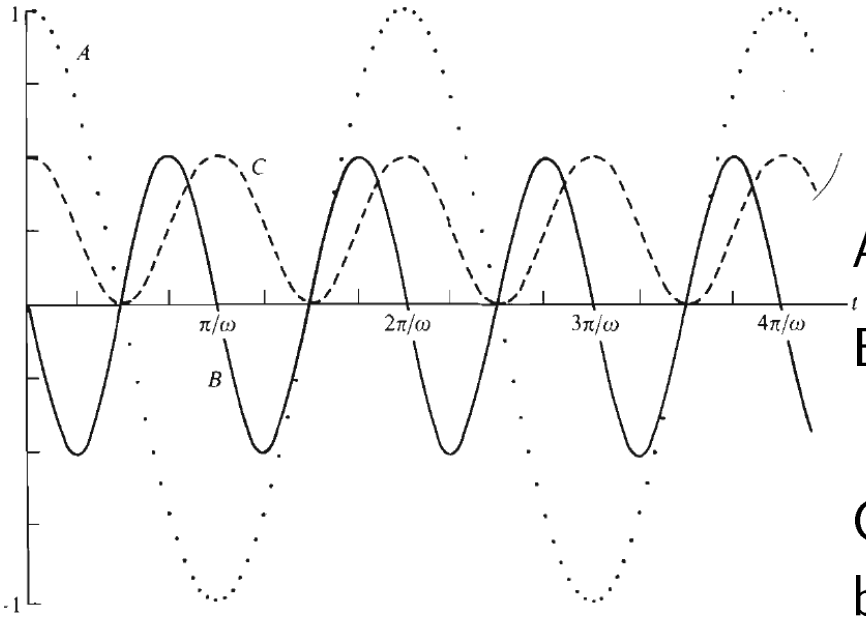
$\frac{Q}{C} = V = V_0 \cos \omega t; \quad I = \frac{dQ}{dt} = C\frac{dV}{dt} = -\omega C V_0 \sin \omega t = \omega C V_0 \cos\left(\omega t + \frac{\pi}{2}\right)$

Ex. : for $\frac{1}{\omega C} = 2\Omega$





Power transfer between a source and a capacitor, for $C = \omega = V_0 = 1$



A : applied voltage$= V_0 \cos \omega t$

B: power absorbed by capacitor

$$= VI = -\omega C V_0^2 \sin \omega t \cos \omega t$$

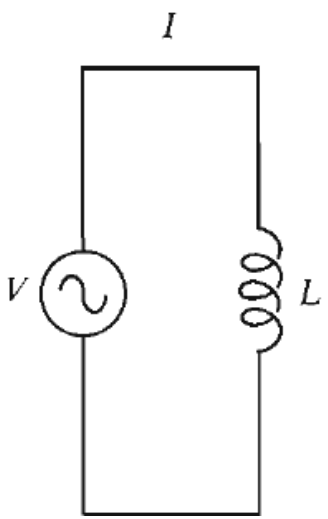
C : energy stored $= \frac{1}{2} C V_0^2 \cos \omega t$ oscillates between 0 and $\frac{1}{2} C V_0^2$

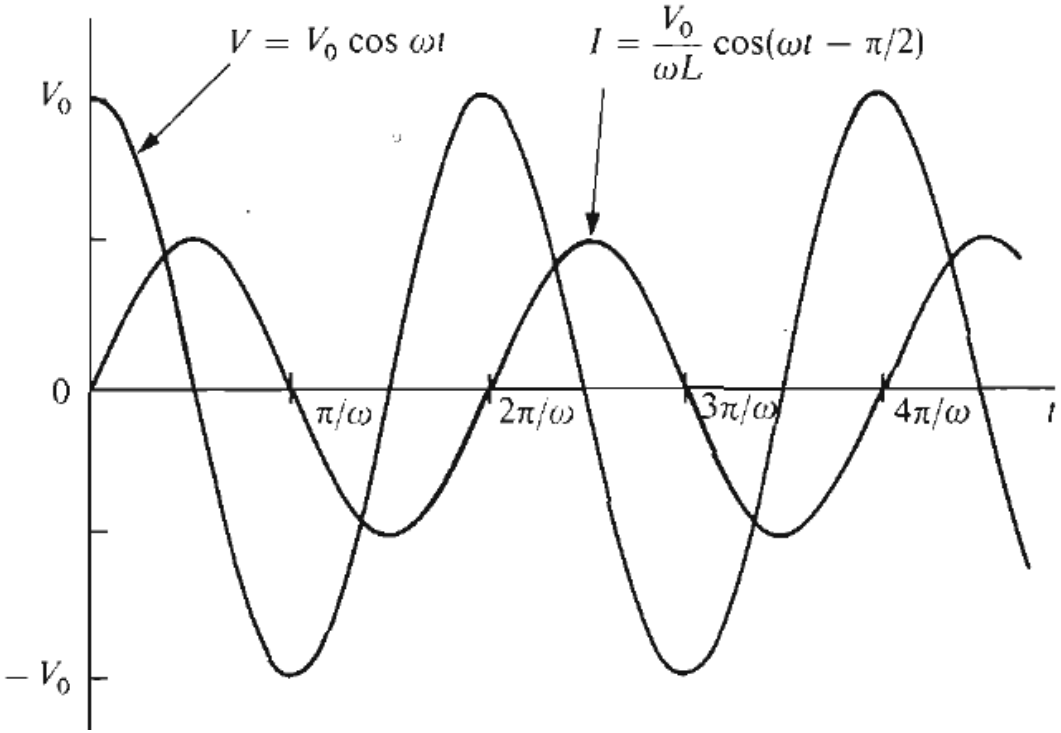
current $\propto$ frequency and $\propto C$ - leads voltage by $\frac{\pi}{2}$

current is said to be in quadrature with the voltage - This expression is used whenever phase difference between two quantities is $\frac{\pi}{2}$, leading or lagging

ideal inductor L with zero resistance connected to a source V

Ex. : for $\omega L = 2\Omega$



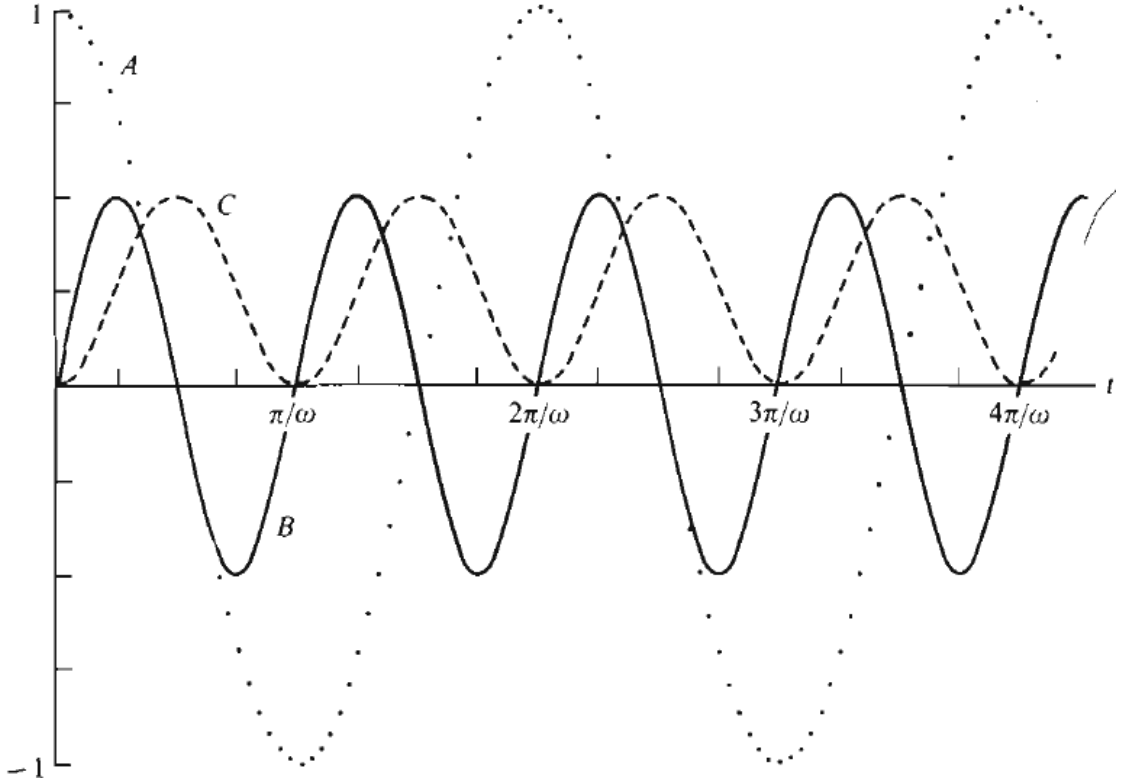


$$L\frac{dI}{dt} = V = V_0 \cos \omega t \Longrightarrow I = \frac{V_0}{\omega L} \sin \omega t = \frac{V_0}{\omega L} \cos\left(\omega t - \frac{\pi}{2}\right)$$

current $\propto$ frequency^{-1} and to L^{-1} ; current is said in quadrature with voltage - It lags the voltage by $\frac{\pi}{2}$

av. energy dissipation$= 0$; $\quad VI = \frac{V_0^2}{\omega L} \sin \omega t \cos \omega t$

Power transfer between a source and an inductor for $L = \omega = V_0 = 1$



A : applied voltage$= V_0 \cos \omega t$

B: power absorbed by inductor $= VI = \frac{V_0^2}{\omega L} \sin \omega t \cos \omega t$

C : energy stored $= \frac{1}{2} L \left(\frac{V_0}{\omega L}\right)^2 \sin^2 \omega t$ - oscillates between 0 and $\frac{1}{2} L \left(\frac{V_0}{\omega L}\right)^2 = \frac{1}{2} L I_0^2$