Dynamic Programming (Contd.)

All pairs shortest paths (APSP)

Want: $dist(u,v)$ $\forall\, u,v \in V$.

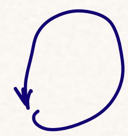
BFS $\to O(m+n)$

Dijkstra's $\hookrightarrow O(m \log n)$

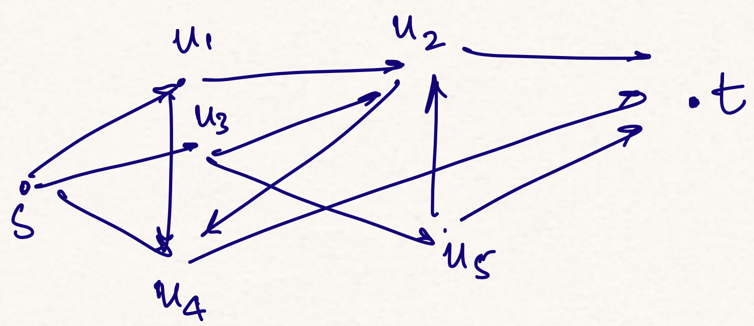
Find "shortest distance" from a fixed node to all other vertices.

- Unweighted graph: $O(n(m+n))$
- Non-negative wts on edges: $O(mn\log n)$

General graphs (without negative cycles).



→ Single source shortest paths:



$$\min_{(v,t)\in E} \{ dist(s,v) + w(v,t) \}$$

$u_2 \to t$

$u_4 \to t$

$u_5 \to t$

$v \in \{u_2, u_4, u_5\}$.

Claim: This could lead to "infinite loops".

$$dist(s,t) = \min_{v} \{ dist(s,v) + w(v,t) \}.$$

$$\hookrightarrow dist(s,v) = \min_{\substack{v' \\ (v',v)\in E}} \{ dist(s,v') + w(v',v) \}.$$

v' could be t.
$\hookrightarrow dist(s,t)$

One of the sub-computations could be "dist(s,t)" itself.

dist(s,t,ℓ) : length of shortest path between s and t with at most ℓ edges.

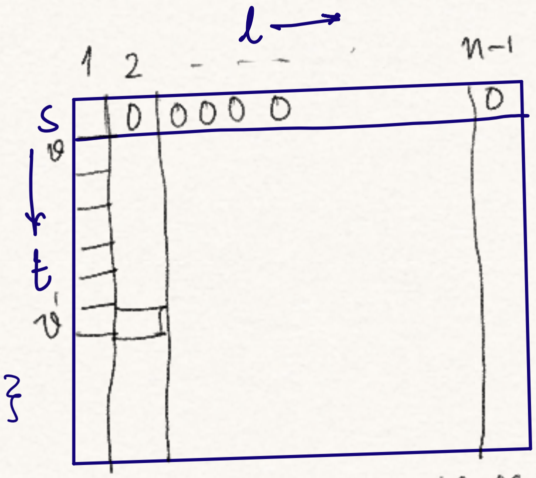
$$\text{dist}(s,t,\ell) = \min_{\substack{v \\ (v,t)\in E}} \{ \text{dist}(s,v,\ell-1) + w(v,t) \}$$

Claim: We are interested in dist$(s,t,n-1)$ $\forall t \in V$? (Sufficient).

↳ Suppose not. If there are more than n edges then by PHP, some ~~edge~~ node repeats / there is a cycle.
Since we are working with graphs w/ no neg. cycles, we can get a shorter path by eliminating that cycle.

↳ Sub problem structure

dist(s, t, ℓ)



$$\text{dist}(s,t,\ell) = \min_{\substack{v \\ (v,t)\in E}} \{ \text{dist}(s,v,\ell-1) + w(v,t) \}$$

$$\text{dist}(s,v,1) = \begin{cases} \infty & \text{if } (s,v) \notin E \\ wt(s,v) & \text{if } (s,v)\in E \end{cases}$$

Base cases:

dist$(s,s,\ell) = 0$ $\forall \ell$

$$\text{dist}(s,v,1) = \begin{cases} wt(s,v) & \text{if } (s,v)\in E \\ \infty & \text{o/w} \end{cases}$$

$$\text{dist}(s,v',2) = \min_{\substack{v \\ (v,v')\in E}} \{ \text{dist}(s,v,1) + wt(v,v') \}$$

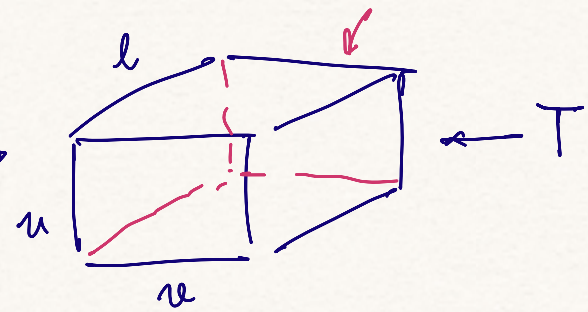
$O(m\cdot n)$ ← Single source

↓

$O(m\cdot n^2)$ ← All pairs.

↳ $O(n^3 \operatorname{polylog} n)$

↳ dist(u, v, l) →



↳ Output: $\{dist(u,v,n-1) \mid u,v \in V\}$.

V^3 many entries in

For a vertex v, $\underline{d_v \cdot (n-1)}$ look ups.

$$\sum_{v\in V} d_v \cdot (n-1) = (n-1)\cdot \sum_{v\in V} d_v = 2m(n-1)$$

Let vertices be renamed $\{1,2,\ldots,n\}$. (arbitrarily).

dist(u, v, r) = length of Shortest path between u and w that uses vertices $\leq r$

$u \to 1 \to 2 \to 3 \to w$

dist(u, v, 5)

dist(u, v, 4)

dist(u, v, 2)

↳ Can we do divide and conquer?

dist(u, v, l) → dist(u, x, i) dist(x, v, l−i)

dist(u, x, $\frac{l}{2}$) dist(x, v, $\frac{l}{2}$)

dist(u, v) → dist(u, x), dist(x, v).