Review

$n=4$

$\vec{a} = (a_0, a_1, a_2, a_3) \to R^4$

Obj: Obtain a DFT of $\vec{a}$

4^{th} primitive root of unity.

$+1, -1, i, -i$

$\omega = i$

DFT: $R^4 \to \mathbb{C}^4$

$$DFT_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

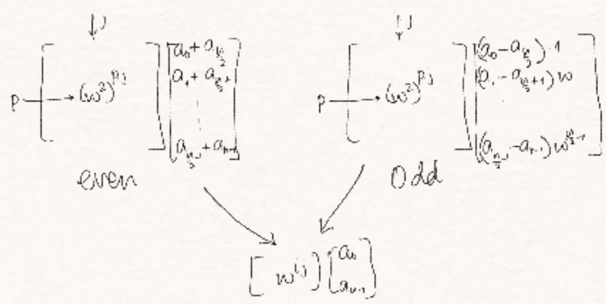
$$DFT_4(\vec{a}) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 + a_2 + a_3 \\ a_0 - a_2 + i(a_1 - a_3) \\ a_0 + a_2 - (a_1 + a_3) \\ a_0 - a_2 - i(a_1 - a_3) \end{bmatrix} = \begin{matrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{matrix}$$

$$\begin{bmatrix} b_0 \\ b_2 \end{bmatrix} = \begin{bmatrix} DFT_2 \end{bmatrix} \begin{bmatrix} a_0 + a_2 \\ a_1 + a_3 \end{bmatrix} \qquad \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} = \begin{bmatrix} DFT_2 \end{bmatrix} \begin{bmatrix} (a_0 - a_2) \cdot 1 \\ (a_1 - a_3) \cdot i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (a_0 - a_2) + i(a_1 - a_3) \\ (a_0 - a_2) - i(a_1 - a_3) \end{bmatrix}$$

$$= \begin{bmatrix} a_0 + a_2 + a_1 + a_3 \\ a_0 + a_2 - (a_1 + a_3) \end{bmatrix}$$



Inverse DFT?

$(M^{-1})_{i,j} = \frac{w^{-i\cdot j}}{n}$

$= \frac{w^{n-ij}}{n}$

$\vec{b} = (b_0, b_1, b_2, b_3)$.

$$iDFT_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} b_0+b_1+b_2+b_3 \\ b_0-b_2-i(b_1-b_3) \\ b_0+b_2-(b_1+b_3) \\ b_0-b_2+i(b_1-b_3) \end{bmatrix}$$

$\frac{w^{-1}}{4} = \quad i^{-1} = i^3$

$\frac{w^{4-2}}{4} = w^2$

$w^{4-6} = w^{-2} = w^{4-2}$

$= w^2$.

$$\begin{bmatrix} a_0+a_1+a_2+a_3 \\ a_0-a_2+i(a_1-a_3) \\ a_0+a_2-(a_1+a_3) \\ a_0-a_2-i(a_1-a_3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4a_0 \\ 4a_1 \\ 4a_2 \\ 4a_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\left.\begin{aligned} f(z) &= \sum_{i=0}^{n-1} c_i \cdot z^i \\ g(z) &= \sum_{i=0}^{n-1} e_i \cdot z^i \end{aligned}\right\} \quad \underbrace{f\cdot g}_{h}(z) = \sum_{i,j=0}^{n-1} c_i \cdot e_j \cdot z^{i+j}$$

$\deg(f), \deg(g) \leq n-1$

$\Rightarrow \deg(h) \leq 2n-2$.

$w \leftarrow n^{th}$ prim root of unity

DFT gives us $\left\{\begin{matrix} f(w^0), \dots, f(w^{n-1}) \\ g(w^0), \dots, g(w^{n-1}) \end{matrix}\right\}$ $h(w^i) = f(w^i).g(w^i)$

$h(w^0), \dots, h(w^{n-1})$

w is no longer n^{th} prim root but $2n^{th}$ primitive root.

$f(z) = \sum_{i=0}^{2n-1} c_i \cdot z^i \quad | \quad c_i = 0 \;\forall\; i \geq n$

$g(z) = \sum_{i=0}^{2n-1} e_i \cdot z^i \quad | \quad e_i = 0 \;\forall\; i \geq n$

$\xrightarrow{DFT_{2n}}$

$f(w^0)$	$g(w^0)$
$\vdots$	$\vdots$
$f(w^{2n-1})$	$g(w^{2n-1})$

$h(w^0), \dots, h(w^{2n-1}) \longleftarrow h(z) = \sum_{i=0}^{2n-1} c'_i \cdot z^i$

$A = \sum_{i \geq 0} a_i \cdot p^i$ | $f(z) = \sum_{i \geq 0} a_i z^i$

$B = \sum_{i \geq 0} b_i \cdot p^i$ | $g(z) = \sum_{i \geq 0} b_i \cdot z^i$

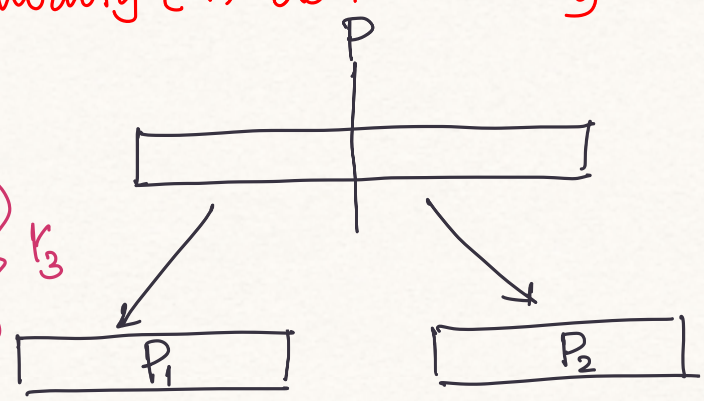
$\left.\begin{matrix} f(z) \\ g(z) \end{matrix}\right\} \to h(z) \xrightarrow{\text{Substitute } z \text{ for } p}$ and we get $A \cdot B$

"Fast Multipoint evaluations"

Counting inversions

Given a sequence of elements $a_1, \dots, a_n$, we want to count the # of pairs $i<j$ s.t $a_i > a_j$.

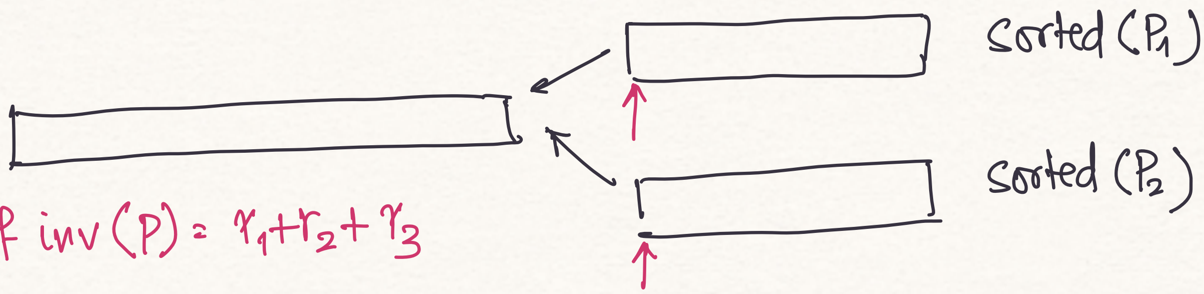
Qn: Can we modify (or) add to merge sort to achieve this?



Are there pairs s.t $i \in P_1$ and $j \in P_2$ s.t $a_i > a_j$? $\Big\}\ r_3$

Suppose, we know # of inversions in $\underbrace{P_1}_{r_1}$, and in $\underbrace{P_2}_{r_2}$

Can we get # of inversions in P?



# of inv(P) = $r_1 + r_2 + r_3$

→ Case-1: Element from Sorted(P_1) is minimal. Continue merging

Case-2: Element from Sorted(P_2) is smaller. $r_3 \leftarrow r_3 +$ remaining elements in sorted(P_i)

$$\operatorname{Det}\begin{bmatrix} a_{11} & & a_{1n} \\ & a_{ij} & \\ a_{n1} & & a_{nn} \end{bmatrix}_{n\times n} = \sum_{\sigma \in S_n} \left\{ \left[(-1)^{\#\mathrm{inv}(\sigma)}\right] \prod_{i=1}^{n} a_{i\sigma(i)} \right\}$$

$$\operatorname{Det}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}\cdot a_{21} \qquad \begin{matrix} 1 \to 1 \\ 2 \to 2 \end{matrix} \qquad \underbrace{\begin{matrix} 1 \to 2 \\ 2 \to 1 \end{matrix}}_{\sigma'}$$

$$= \sum_{\sigma \in S_2} (-1)^{\#\mathrm{inv}(\sigma)} \cdot a_{1\sigma(1)} \cdot a_{2\sigma(2)} = a_{11}a_{22} + \underbrace{(-1)^{\#\mathrm{inv}(\sigma')}}_{-1} \cdot a_{12}a_{21}$$

[Mahajan-Vinay 97] Computing Determinants faster.