** dimension of left kernel of a matrix $n \times m$ = $[n - (\text{Rank of the matrix})]$

i.e $(n-R) \times n$

(2) $x_1^2 - x_2^3 x_3 = 0$

$x_1^2 = x_2^3 x_3$

$x_1^2 x_2^{-3} x_3^{-1} = 1.$

$\Rightarrow \gamma = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} & & \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 & 1 & -3 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 0.$

(why we took its dimension like this) see top of this page

we can choose any left kernel, which satisfies the eqn. Here we have taken this.

$\Rightarrow$ So the parametrize form ~~of solution~~ is:

$$(t_1^0 t_2^1, t_1^1 t_2^1, t_1^{-3} t_2^{-1})$$

•) we can convert fournomial to trinomial like:

$m_1 + m_2 + m_3 + m_4 = 0.$

$$\boxed{\begin{array}{l} z_1 = m_1 + m_2 \\ z_2 = m_3 + m_4 \\ z_1 + z_2 = 0 \end{array}}$$

but it came it cost of introducing 2 more variable z_1 & z_2.

$(m_1, m_2, m_3, m_4, z_1, z_2)$

this was to convince this statement:

(Any polynomial can be parametrized to at most trinomial)*

hum trinomial ko binomial mai nahi ~~convert~~ parametrize karsakte hai.

28th/march/25

(Prev class Reviving) •)

$x^{\gamma_j} = c_j , \quad c_j > 0$

~~Case~~ If u was a solution to this system.

then $(u_1 t^{a_1}, u_2 t^{a_2}, \dots u_n t^{a_n})$ are solutions to the system.

$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \qquad A[\gamma_1\ \gamma_2 \dots \gamma_n] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

∘) Laplacian Matrix:

$$\frac{dx}{dt} = Y \underbrace{I_E\, k}_{A_k} x^Y$$

Graph Laplacian **

* dimentions:

$Y : \mathbb{R}^{n \times m}$

$I_E : \mathbb{R}^{m \times r}$ $\Rightarrow A_k : m \times m$ Square matrix

$k : \mathbb{R}^{r \times m}$

$x^Y : \mathbb{R}^{m \times 1}$

(Piche se deklena esc dim kyu hai)

*) $\dot{x} = Y A_k x^Y$

**

x is a steady state $\iff x^Y \in \ker(Y A_k)$

• If $x^Y \in \ker(A_k)$. (yaha Right kernel seyd not buse)

or

$x^Y \in \ker(Y) \cap \mathrm{Im}(A_k)$

its called complex-balanced equilibrium

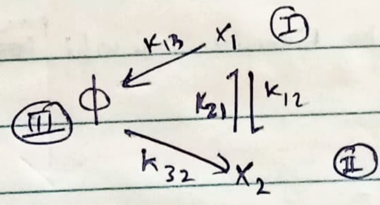
** Imp Now, Let come back to A_k

*) $A_k : \; a_{ij} = \begin{cases} k_{ji} & \text{if } y_j \xrightarrow{k_{ji}} y_i, \; i \neq j \\ -\sum_{i=1}^{m} k_{ji} & \text{if } i = j \end{cases}$

{Note a_{ij} mai ij hai aur k_{ji} ma ji, note the order}

(this is essentially '-' of sum of all column value expect it to make sum of column = 0) *

exercise:



write A_k for it.

$$A_k = \begin{bmatrix} -(k_{12}+k_{13}) & k_{21} & 0 \\ k_{12} & -k_{21} & k_{32} \\ k_{13} & 0 & -k_{32} \end{bmatrix}$$

* (it is '0' if there is No k_{ji}, i.e No Rxn from $y_j \to y_i$)

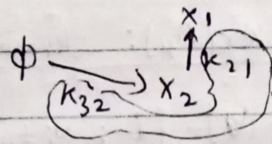
• Kernel of A_κ:

$$Ker(A_\kappa) = \begin{pmatrix} k_{21}k_{32} \\ k_{12}k_{32} + k_{13}k_{32} \\ k_{21}k_{13} \end{pmatrix}$$

This is Right Kernel. Such that A_κ. This = 0

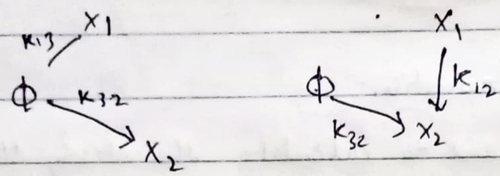
Another way to calculate this,

• Spanning tree rooted at X_1, (no outgoing from X_1)



→ Product of this

• Spanning tree rooted at X_2.



Spanning Tree:

A spanning tree T of a digraph is a graph of G such that:

(i) nodes of T ≡ nodes of G

(ii) It has no cycles.

• T is said to be rooted at a node v, if v is the only node with no outgoing edges.

⇕

There exists a directed path from any other node to v.

• For a spanning Tree,

$K(T) \equiv$ product of labels of edges of T.

(*) Matrix-Tree Theorem:

Consider a strongly connected digraph that is labelled. Then the $Ker(A_\kappa) = \langle J \rangle$

$J_i = \sum k(T)$ T: spanning tree rooted at i.

The only time sir ka mtb 1_g kernel tha is only for monomial parametrization. Everywhere else he meant Right kernel.

- Can extend the matrix-free theorem to connected graphs?
  - → you can see peeche kernel 1D bana hai, kuki graph ko strongly connected liya hai.
  - → for general connected graph, it could be more than 1D which corresponds to the terminal strong linkage classes of the graph.

(not sure my understanding)

⇒ $\boxed{\text{Support of } Ker(A_k): \text{terminal strong linkage classes}}$

(defn) $Support(v) = \{ i : v_i \neq 0 \}$

Linear Elimination:

- If i want to calculate the steady state of a Rxn network i can do it using
  - Groebner basis
  - Manual (using software)
  - Linear elimination

- Hybrid - Histidine kinase

$$X_1 \xrightarrow{k_1} X_2 \xrightarrow{k_2} X_3 \xrightarrow{k_3} X_4$$

$$X_3 + X_5 \xrightarrow{k_4} X_1 + X_6$$

$$X_4 + X_5 \xrightarrow{k_5} X_2 + X_6$$

$$X_6 \xrightarrow{k_6} X_5$$

1st April 25

Continued...

write the ODE's now:

$$\dot{x}_1 = -k_1 x_1 + k_4 x_3 x_5$$

$$\dot{x}_2 = -k_2 x_2 + k_5 x_4 x_5 + k_1 x_1$$

$$\dot{x}_3 = -k_3 x_3 - k_4 x_3 x_5 + k_2 x_2$$

$$\dot{x}_4 = k_3 x_3 - k_5 x_4 x_5.$$

$$\dot{x}_5 = -k_4 x_3 x_5 - k_5 x_4 x_5 + k_6 x_6$$

$$\dot{x}_6 = k_4 x_3 x_5 + k_5 x_4 x_5 - k_6 x_6.$$

We can see the conservation laws here:

$$x_1 + x_2 + x_3 + x_4 = T_1 ,$$

$$x_5 + x_6 = T_2 .$$

(T_1 & T_2 are constant)

Non-Interacting Species:

It is subset of species such that they do not occur together in reactants or product.

<u>ex</u>: for above Rxn networks $\{X_1, X_2, X_3, X_4\}$ is an example of non-~~...~~interacting species.

Now, write OD's for the Non-interacting set in matrix from

$$\begin{pmatrix} -k_1 & 0 & k_4 k_5 & 0 \\ k_1 & -k_2 & 0 & k_5 x_5 \\ 0 & k_2 & -k_3 - k_4 k_5 & 0 \\ 0 & 0 & k_3 & -k_5 x_5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

draw the graph (Laplacian):

$$\begin{tikzcd} X_1 \arrow[r, "k_1"] & X_2 \arrow[r, "k_2"] & X_3 \arrow[r, "k_3"] \arrow[ll, bend left, "k_4 x_5"] & x_4 \arrow[ll, bend right, "k_5 x_5"'] \end{tikzcd}$$

Now,

$$\ker A_k = \langle \rho \rangle = \lambda \rho$$

[ρ is the generator of $\ker A_k$]

ρ_1 = Spanning tree rooted at x_1

$$= k_4 x_5 \, k_2 \, k_5 x_5$$

$$= k_2 k_4 k_5 x_5^2 .$$

Similarly,

ρ_2 = spanning tree rooted at x_2

ρ_3 :

ρ_4 :

Now, use the conservation laws to get the steady states;

$$T_1 = x_1 + x_2 + x_3 + x_4$$

$$= \lambda (\rho_1 + \rho_2 + \rho_3 + \rho_4)$$

$$\Rightarrow \lambda = \frac{T}{\rho_1 + \rho_2 + \rho_3 + \rho_4}$$

ex - 2

$$X_1 + X_3 \xrightarrow{k_1} X_2 \;,\quad X_2 + X_4 \xrightarrow{k_2} X_1$$

$$\phi \underset{k_4}{\overset{k_3}{\rightleftharpoons}} X_1 \;,\quad \phi \xrightarrow{k_5} X_3 \quad \phi \xrightarrow{k_6} X_4 .$$

- One possible non-interacting set is $\{X_1, X_2\}$.
- Now write ODEs for it.

$$\dot{x}_1 = -k_1 x_1 x_3 + k_2 x_2 x_4 + k_3 - k_4 x_1$$

$$\dot{x}_2 = k_1 x_1 x_3 - k_2 x_2 x_4$$

- write it in Graph Laplacian : If it is not Graph Laplacian make it Graph laplacian.

$$\begin{pmatrix} -k_1 x_3 - k_4 & k_2 x_4 & k_3 \\ k_1 x_3 & -k_2 x_4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Is this a graph laplacian , No ,
we can add an extra row here,

$$\begin{pmatrix} -k_1 x_3 - k_4 & k_2 x_4 & k_3 \\ k_1 x_3 & -k_2 x_4 & 0 \\ k_4 & 0 & -k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we got this row as, at steady state
$\dot{x}_2 = k_1 x_1 x_3 - k_2 x_2 x_4 = 0$
Putting this in $\dot{x}_1 = 0$ we get.

$$\dot{x}_1 = -\underbrace{(k_1 x_1 x_3 - k_2 x_2 x_4)}_{=0} + k_3 - k_4 x_1 = 0$$

$$\Rightarrow k_3 - k_4 x_1 = 0$$

(Here we ~~have~~ got one more steady stat eqn we just used this & added it in the row)

- Now draw the graph.

$$\begin{tikzcd} X_1 \arrow[r, shift left, "k_1 x_3"] \arrow[d, shift left, "k_4"] & X_2 \arrow[l, shift left, "k_2 x_4"] \\ \phi \arrow[u, shift left, "k_3"] & \end{tikzcd}$$

8^{th} April 25

- we will study injectivity* today.
- *) A network is said to Multistationarity, if there exists $K \in \mathbb{R}^{r}_{>0}$, $x \neq y \in \mathbb{R}^{n}_{>0}$, $x-y \in S$, such that
$$f_K(x) = f_K(y) = 0.$$

f_K is injective $\Rightarrow$ No Multistationarity.

$$\boxed{f_K(x) = N \operatorname{diag}(K)\, x^B}$$

(x^B = monomial matrix)*
B is matrix of exponents

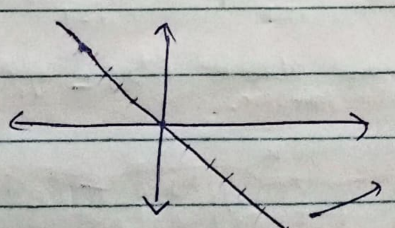
*) f is not injective $\Rightarrow$ $\exists\, K$, $x \neq y$ s.t.
$$f_K(x) = f_K(y)$$

$$\Rightarrow N \operatorname{diag}(K)\, x^B = N \operatorname{diag}(K)\, y^B.$$
$$N \operatorname{diag}(K)[x^B - y^B] = 0.$$

$$\Rightarrow \operatorname{diag}(K)[x^B - y^B] \in \ker N$$
$$\Updownarrow$$
$$\exists\, v \in \ker(N) \quad \text{s.t. } \sigma(v) = \sigma(x^B - y^B)$$

*{ when σ is the 'sign pattern'
Jaise: $\sigma\begin{pmatrix}1\\-2\end{pmatrix} = \begin{bmatrix}+\\-\end{bmatrix}$ }

** what is sign pattern of a subspace or a set
$$\sigma(C) = \{\sigma(v) \mid v \in C\}$$



sign pattern of this line is
$$\begin{pmatrix}-\\+\end{pmatrix}\begin{pmatrix}+\\-\end{pmatrix} \text{ also union } \begin{Bmatrix}0\\0\end{Bmatrix}$$

* Imp

• f_k is injective wrt S for all K $\iff$ $\forall K, \forall x, y$ s.t $x-y \in S$
(INJ) $f_k(x) \neq f_k(y)$

$\Updownarrow$

(LIN)
The linear map where matrix $N \operatorname{diag}(\mu) B^T \operatorname{diag}(\lambda)$ is injective on S $\forall$ positive vector μ, λ.

$\Updownarrow$

~~(NOT LIN)~~ $\forall z \in S$ where

$$N (\operatorname{diag}(\mu)) B^T \operatorname{diag}(\lambda) z \neq 0 \quad \forall \mu, \lambda > 0$$

(NOT LIN)

$\exists z \in S$, $\mu, \lambda > 0$ such that

$$N \operatorname{diag}(\mu) B^T \operatorname{diag}(\lambda) z = 0$$

Construct x, y s.t $x - y = z \in S$.
→ we can take x, y as

$$y_i = \begin{cases} 1 & \text{if } z_i = 0 \\ \dfrac{z_i}{e^{\lambda z_i} - 1} & \text{if } z_i \neq 0 \end{cases}$$

$$x_i = y_i e^{\lambda z_i}$$

{ we can verify it ki x, y ko esa lene se condition satisfy hori i.e $x - y = z \in S$ just by putting values yaa }

Our aim is to relate $\sigma(x^B - y^B)$ $\sigma(\ker N)$

$$(x^B - y^B)_j \equiv \prod_{i=1}^{n} x_i^{b_{ij}} - \prod_{i=1}^{n} y_i^{b_{ij}}$$

we have from above ($x_i = y_i e^{\lambda z_i}$ & $y_i = \frac{z_i}{e^{\lambda z_i} - 1}$)
so putting x_i value from here.

$$:) \quad = \prod_{i=1}^{n} y_i^{b_{ij}} \left[\prod_{i=1}^{n} (e^{\lambda z_i})^{b_{ij}} - 1 \right]$$

$$= \prod_{i=1}^{n} y_i^{b_{ij}} \left[e^{\sum b_{ij} \lambda_i z_i} - 1 \right]$$

$$\sigma(x^B - y^B) \equiv \sigma(B^T \operatorname{dig}(\lambda) z)$$
$$\equiv \sigma(\operatorname{diag}(\mu) B^T \operatorname{diag}(\lambda) z)$$

11^{th}/April/25

∂) Revision!

$$f_k = N \operatorname{diag}(k) x^B$$

f_k is not injective for all k

$$\Updownarrow$$

$\exists k$, $\exists x, y$ s.t

$$N \operatorname{diag}(k) x^B = N \operatorname{diag}(k) y^B$$

$$N \operatorname{diag}(k)[x^B - y^B] = 0$$

$$\operatorname{diag}(k)[x^B - y^B] \in \ker N$$

$\Updownarrow$ (?) Sochne ke bola kya ye hoga

$$\exists v \in \ker N, \quad \text{s.t} \quad \sigma(v) = \sigma(x^B - y^B)$$

∂) (INJ) f_k is injective wrt S $\forall k$.

(LIN) The linear map $N \operatorname{diag}(\gamma) B^T \operatorname{dig}(\lambda)$ is injective on S $\forall \gamma$, $\forall \lambda$

$\Updownarrow$ Actually ye soln hai, eska forward toh agle page mai prove karaya hai, backward ~~Adg~~ mai hai.

(NOT LIN) $\exists z \in S$ s.t $x - y \in S$ and

$$N \operatorname{diag}(\gamma) B^T \operatorname{diag}(\lambda) z = 0$$

W.T.S $\exists x, y, k$ s.t $x - y \in S$ and

$$\operatorname{diag}(k)(x^B - y^B) \in \ker N.$$

So, first construct x, y s.t $x - y \equiv z \in S$.

Toh ye put karne ke baad we got

$$(x^B - y^B)_j = \prod_{i=1}^{n} y_i^{b_{ij}} [e^{\sum b_{ij} \lambda_i z_i} - 1]$$

$\Rightarrow$ Sign of $(x^B - y^B)$ is same as sign of $B^T \operatorname{diag}(\lambda) z$

$$\Rightarrow \sigma(x^B - y^B) = \sigma(B^T \operatorname{diag}(\lambda) z)$$

* (Agar global injectivity dekni hai toh just check kernel (N) has only the zero vector

~~If we want to check injectivity wrt a subspace S:~~

(*) what does it mean to be injective on subspace S.

i.e. only check for this eq.^{n} not the 'ω' eq.^{n} following it

$$\boxed{N \,\mathrm{diag}(y)\, B^T \mathrm{diag}(\lambda)\, z = 0}$$

$$\omega z = 0$$

where $\mathrm{row\,Span}(\omega) = S^{\perp}$.

Only has the solution $z = 0$.

We can write it as,

$$\begin{bmatrix} N \,\mathrm{diag}(\gamma)\, B^T \mathrm{diag}(\lambda) \\ W \end{bmatrix} z = 0$$

Now we remove the redundant rows of matrix (Sayd N), (Sayd linearly dependent ko hata diya)
to get N'

we can do this only if this is true.

** If $\mathrm{rank}(N) = \dim(S) = s$

i.e.

choose $N' \in \mathbb{R}^{s \times n}$ s.t. $\ker(N') = \ker(N)$ **

**

So, $M_{\gamma,\lambda} \equiv \begin{bmatrix} N' \mathrm{diag}(\gamma)\, B^T \mathrm{diag}(\lambda) \\ W \end{bmatrix} z = 0$

has $z = 0$

iff

$\det(M_{\gamma,\lambda}) \neq 0$ for every γ, λ

we will see that:

- ω will come out to be the conservation laws matrix

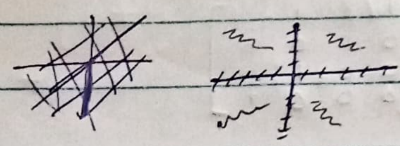
Global injectivity:

To check for Global injectivity, we just check ~~vector~~ for

$$N \,\mathrm{diag}(y)\, B^T \mathrm{diag}(\lambda)\, z = 0$$

If kernel of this matrix contains only zero vector (i.e. $= 0$) then it ~~[illegible]~~, is globally injective

*) Orthant: 9 orthant in $\mathbb{R}^2$ $= \{+,0,-\}^n$ $= 3^n$ orthant



★ Gale Duals:

$A_{d\times n}$ $\quad d \le n$

$J \subseteq [1,n]$

A_J : matrix with indicies in J.

$$\overset{n}{d\begin{pmatrix} & & \end{pmatrix}}_{A} \overset{n\times n-d}{\begin{pmatrix} & & \end{pmatrix}}_{B} = 0$$

Columns of B generate $\ker(A)$

A, B are Gale Duals.

15^{th}/April/25

•) We will do an example for what we studied in the last class:

→ Calcium transport Network:

$$X_1 \xrightarrow{k_1} \phi \quad , \quad X_1 + X_2 \xrightarrow{k_2} 2X_1$$

$$X_1 + X_3 \xrightarrow{k_3} X_4 \xrightarrow{k_4} X_2 + X_3 .$$

: Let's write down N for it

$$N = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{bmatrix} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

We see x_3 & x_4 row are linearly dependent (they are multiple), so

$$N' = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Now, lets write down the exponent matrix B,

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The left kernel of N

$$W = (0 \; 0 \; 1 \; 1)$$

Now, the matrix $M_{\mu,\lambda}$,

$$M_{\mu,\lambda} = \begin{pmatrix} N \operatorname{diag}(\mu) B^T \operatorname{diag}(\lambda) \\ W \end{pmatrix}$$

$$M_{\mu,\lambda} = \begin{pmatrix} (-\mu_1+\mu_2+\mu_3)\lambda_1 & \mu_2\lambda_2 & -\mu_3\lambda_3 & 0 \\ -\mu_2\lambda_1 & -\mu_2\lambda_2 & 0 & \mu_4\lambda_4 \\ -\mu_3\lambda_1 & 0 & -\mu_3\lambda_3 & \mu_4\lambda_4 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

notebook ke piche calculation kr hai kinematic analysis wala page me dekho

$$\det(M_{\mu,\lambda}) = \lambda_1\lambda_2\lambda_3\mu_1\mu_2(\mu_3+\mu_4) \neq 0.$$

$\Rightarrow$ (LIN) holds $\Rightarrow$ f_k is injective for all k (Kuki LIN $\Rightarrow$ INJ)

★ Back to Gale duals:

A matrix $A_{d\times n}$, $d \le n$.

$$\Big(\;\; \Big)_{A_{d\times n}} \Big(\;\; \Big)_{B_{n\times n-d}} = 0$$

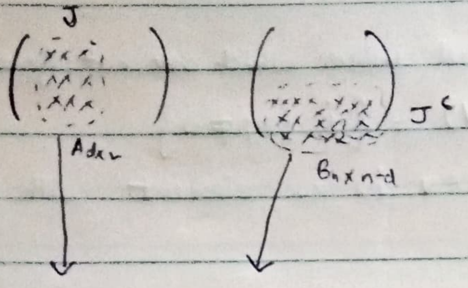
Columns of B generate ker (A)

rank (B) = n - d.

A and B are said to be Gale duals of each other

$J \subseteq \{1, 2, \ldots n\}$, $|J| = d$

A_J is the submatrix with indices in J.



J^c : J ke alawa
Jo bhi indices baki hai
~~[illegible]~~
To B mai se J^c rows
leke we get,
$\Downarrow$
B_{J^c}

So I have A_J B_{J^c}

So, we have a theorem relating it.

Cayley: Let A, B be as before. There exist
$\mu \neq 0$ s.t $\forall J$,
$|J| = d$, $J \subseteq \{1, 2, \ldots n\}$

$$\det(A_J) = \mu (-1)^{\sigma(J)} \det(B_{J^c})$$

$$\sigma(J) = \sum_{j\in J} j$$

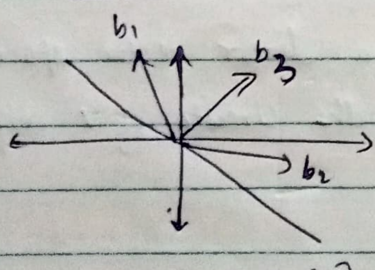
Imp

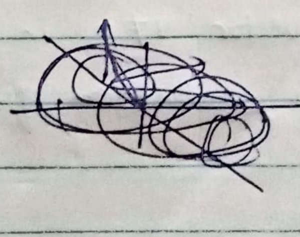
One more imp Observation:

There exist a positive vector in the kernel of A
iff
All vector $b_1\, b_2 \ldots b_n$ lie in an open half space.

($b_1, b_2 \ldots b_n$ are rows of B)

i.e imp $\boxed{\exists\ v \in \ker(A) \cap \mathbb{R}^n_{>0} \iff \text{All vectors } b_1, b_2 \ldots b_n \text{ lie in an open half space.}}$





$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \underset{>0}{\begin{bmatrix} v \end{bmatrix}}$$

? khali explain karne ke liye likha tha.

(Circuit is a minimal support vector)

* **Support of a vector:** all indices which are not zero.

$$Supp(v) = \{i \mid v_i \neq 0\}.$$

A circuit $c \in V$ is an element with minimal Support.

13th April '21

- $Supp(v) = \{i \mid v_i \neq 0\}$

  $v = \{2, 3, 0, 5, 0, 7\}$

  $Supp(v) = \{1, 2, 4, 6\}$

  (Note: ye indices ka set hai na ki us index pe)

Minimal Support vector: (Circuits)

(2, 3, 0, 0, 4) ye minimal supp vector nahi hosakta because, i can find a vector like (1, 0, 0, 0, 7) whose supp vector is contained in it (Not sure).

- $v = (2, 3, 0, -7, 8)$

  $\sigma(v) = (+, +, 0, -, +)$

- A Circuit C is said to be Conformal with a vector v if, whenever $C_i \neq 0 \Rightarrow \sigma(C_i) = \sigma(v_i)$.

  for ex- $C = (1, 0, 0, 0, -7)$ $\quad v = (2, 3, 0, 0, -4)$.

  C is conformal with v here.

* **Rockafeller's theorem:**

  Any vector $v \neq 0$ in a linear vector space V is a positive linear combination of the circuits of V conformal with v.

This is the procedure for finding circuits of a vector space

Let $C \in \mathbb{R}^{d \times n}$ be of Rank d. Let V be its rowspan. for a subset $I \subseteq \{1,2,\ldots n\}$, Let C_I denote the submatrix of C consisting of the columns of C with indices in I. for $J \subseteq \{1,2,\ldots n\}$ of cardinality d-1 such that the columns of C_J are linearly independent, define

circuits: $$(\gamma_J)_k = \begin{cases} (-1)^{\mu(k,J)} \det(C_{J \cup \{k\}}) & \text{if } k \notin J \\ 0 & \text{if } k \in J \end{cases}$$

$k = 1, 2, \ldots n$

$\mu(k,J)$ are No. of indices in J that are strictly less than k.

ex:- $C = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \end{pmatrix}$ $d = 2$, $n = 4$, $|J| = 1$

$$\begin{array}{llcccc} & & k=1 & k=2 & k=3 & k=4 \\ J=\{1\} & \gamma_{\{1\}} = & (0 & -2 & -4 & -3) \\ J=\{2\} & \gamma_{\{2\}} = & (2 & 0 & 0 & -1) \\ J=\{3\} & \gamma_{\{3\}} & (4 & 0 & 0 & -2) \\ J=\{4\} & \gamma_{\{4\}} & (3 & 1 & 2 & 0) \end{array}$$

→ circuit 1
→ circuit 2
→ " 3
→ " 4

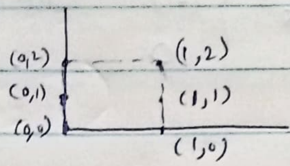
Imp So if someone ask for circuits of a vector space. first calculate the basis of the vectors space. Stack each Basis vector as a row to form a Matrix, (in above example it was C). Then follow the procedure as above.

Brouwer degree :

- To apply Brouwer degree we need two Assumption : A and B.
- ~~[illegible]~~ we calculate Jacobian of its determinant.
- Given a polynomial, we plot its exponent as coordinate & get a Newton polytope.

$$k_1 k_2 k_3 k_6 + (k_1 + k_2) k_4 k_5 k_6 x_5^2 + k_2\left(\frac{k_1}{k_3} - 1\right) x_4 x_5^2 +$$
$$k_3 x_4 x_5 + k_4 x_5 + k_6 x_4 .$$

Newton polytope



(Mtb hum enme kuch esa dikhane wale hai; Given a polynomial, the sign of entire polynomial can match the sign of a single term in the polynomial. Mtb we create Newton polytope of the polynomial, the the sign of polynomial can be determined by the sign of vertex in the newton polytope (Sayd use phase bula raha tha).

-) Bas enme second point se samjhna hai, usse upar to agle class mai Padhaega.

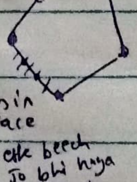
22nd | April | 25

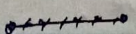
-) $$f(x) = \sum_v \lambda_v x^v$$

Given a face τ of the newton polytope.
Let f_τ be the restriction of f to the monomials supported on the face τ.

for any $y^* \in \mathbb{R}^n_{\geq 0}$, $\exists\ x^* \in \mathbb{R}^n_{>0}$ s.t,

$$\sigma(f(x^*)) = \sigma(f_\tau(y^*))$$





This in a face
(note eke beech mai jo bhi haga we take that also in that (note) like

If our polytope is like this, then the phase is like shown.
(Toh we have to take all polynomial inside the face restriction & say sign of all these polynomial can be equal to the entire polynomial for some value)

recall:

(~~means~~ there exist a conservation law with all coefficient positive)
w: left kernel of stoichiometric matrix

↑

(A) network is conservative
(B) No critical Siphons

*) Siphon: A set $T \subseteq S$ (set of all species) is a siphon if $\forall x_i \in T$, and for all reaction having x_i as a product, the reaction has x_i as a reactant.

ex:

$$X + Y \to Z$$
$$X \to 2X$$
$$Z + Y \to X$$

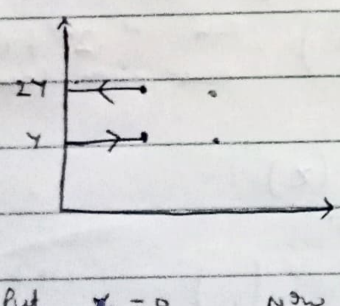
$\{x, y, z\} \equiv S$:

$\{x\}$ → NOT SIPHON
$\{y\}$ → SIPHON (If y never occurs in the product, it is trivially true)
$\{z\}$ → NOT SIPHON
$\{x,y\}$ → SIPHON
$\{y,z\}$ → SIPHON
$\{z,x\}$ → SIPHON

•) A set $T \subseteq S$ is critical if $z_i = 0$ iff $i \in T$, $(Z+S) \cap \mathbb{R}^n_{>0} \neq \phi$.

ex: ~~$X+Y \to Z$~~
~~$X \to 2X$~~

$$Y \to X + Y$$
$$X + 2Y \to 2Y$$



(ese hi likhna hai)

$\{x\}$: critical (not simon)
$\{Y\}$: Not critical (Simon)
$\{x, Y\}$: Not critical (Simon)

• $\{x\}$: Put $x = 0$, Now using the Rxn vector can it enter the positive orthant? Yes we can do it here.
So $\{x\}$ is critical.

• $\{Y\}$ Put $y = 0$, usme mai ab X axis mai kahi bhi hu, but using the Rxn vect → aur ← jo hai, I can't enter the positive orthant, usme mai x-axis mai hi ghoomta rahunga.
∴ $\{Y\}$ is not critical.

(W: row reduced form of the left kernel)

$\{x, y\}$: Put $x=0$ and $y=0$, puts I am at origin and I can't enter the positive orthant using the Rxn vectors. $\therefore$ $\{x,y\}$ is not critical.

So now coming back to brouwer degree;...

i)

$$X_1 \xrightarrow{k_1} X_2$$
$$2X_2 \xrightarrow{k_2} 2X_1$$
$$X_1 + X_2 \xrightarrow{k_3} 2X_2$$

$$N = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

Lets find the ω matrix (row reduced form of Left Kernel or the conservation matrix).

$$\omega = \begin{pmatrix} 1 & 1 \end{pmatrix}$$ So conservation law $= X_1 + X_2 = T$

Now, the Rate eqⁿs

$$\dot{x} = -k_1 x_1 + 2k_2 x_2^2 - k_3 x_1 x_2$$

$$\dot{y} = k_1 x_1 - 2k_2 x_2^2 + k_3 x_1 x_2$$

Now,

$$F_{k,T} = \begin{pmatrix} x_1 + x_2 - T \\ k_1 x_1 - 2k_2 x_2^2 + k_3 x_1 x_2 \end{pmatrix}$$

$\downarrow$

ye kaise banaya. Ideally $F_{k,T}$ mai saare Rate eqⁿ dalte hai, but using our conservation law we can replace the rate equation by the conservation law. Here we had $\omega = (1 \; 1)$ so we replace eqⁿ wrt the first non-zero index in ω. Here it was $\dot{x}$.

Suppose $\omega = \begin{pmatrix} 1 & 1 & 0 & 0 & 7 \\ 0 & 1 & 3 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, Toh 1st, 2nd & 4th row replace Rate conservation law se.

Now

$$M_k(x) = J_{F_{k,T}}(x)$$ i.e Jacobian

$$= \begin{pmatrix} 1 & 1 \\ k_1 + k_3 x_2 & -4k_2 x_2 + k_3 x_1 \end{pmatrix}$$ (ye Jacobian calculate karna hai)

So now

$$\det(M_k(x)) = -k_1 - k_3 x_2 - 4k_2 x_2 + k_3 x_1$$

(ye hai $\dot{x}, \dot{y}$ ye)

$\uparrow$

So, for Steady State to be:

Uniqueness : $(-1)^s \det(M_k(x)) > 0 \ \underline{\forall} \ x^* \in V(f_{k_0}(x))$

Multistationary : $(-1)^s \det(M_k(x)) < 0$ for $\underline{some}$ $x^* \in V(f_{k_0}(x))$

$\downarrow$

(mtb ~~agar~~ always positive hai toh uniqueness aur agar negative hosakta hai for some value then multistationary.)