Population Growth

P. H. Leslie, "On the Use of Matrices in Certain Population Mathematics," *Biometrika* 33 (1945), pp. 183–212.

- The *Leslie model* describes the growth of the female portion of a population, which is assumed to have a maximum lifespan.
- The females are divided into age classes, all of which span an equal number of years.
- Using data about the average birthrates and survival probabilities of each class, the model is then able to determine the growth of the population over time.

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A certain species of German beetle, the Vollmar-Wasserman beetle (or VW beetle, for short), lives for at most 3 years. We divide the female VW beetles into three age classes of 1 year each: youths (0–1 year), juveniles (1–2 years), and adults (2–3 years). The youths do not lay eggs; each juvenile produces an average of four female beetles; and each adult produces an average of three females.

The survival rate for youths is 50% (that is, the probability of a youth's surviving to become a juvenile is 0.5), and the survival rate for juveniles is 25%. Suppose we begin with a population of 100 female VW beetles: 40 youths, 40 juveniles, and 20 adults. Predict the beetle population for each of the next 5 years.

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Solution After 1 year, the number of youths will be the number produced during that year:

$$40 \times 4 + 20 \times 3 = 220$$

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$$40 \times 4 + 20 \times 3 = 220$$

The number of juveniles will simply be the number of youths that have survived:

$$40 \times 0.5 = 20$$

Likewise, the number of adults will be the number of juveniles that have survived:

$$40 \times 0.25 = 10$$

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We can combine these into a single matrix equation

$$\begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 40 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 220 \\ 20 \\ 10 \end{bmatrix}$$

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or $L\mathbf{x}_0 = \mathbf{x}_1$, where $\mathbf{x}_0 = \begin{bmatrix} 40 \\ 40 \\ 20 \end{bmatrix}$ is the initial population distribution vector and $\mathbf{x}_1 = \begin{bmatrix} 220 \\ 20 \\ 10 \end{bmatrix}$ is the distribution after 1 year. We see that the structure of the equation is exactly the same as for Markov chains: $\mathbf{x}_{k+1} = L\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$ (although the interpretation is quite different). It follows that we can iteratively compute successive population distribution vectors. (It also follows that $\mathbf{x}_k = L^k\mathbf{x}_0$ for $k = 0, 1, 2, \ldots$, as for Markov chains, but we will not use this fact here.)

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$$\mathbf{x}_5 = L\mathbf{x}_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 302.5 \\ 227.5 \\ 13.75 \end{bmatrix} = \begin{bmatrix} 951.2 \\ 151.2 \\ 56.88 \end{bmatrix}$$

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The matrix L in Example 3.67 is called a ***Leslie matrix.*** In general, if we have a population with n age classes of equal duration, L will be an $n \times n$ matrix with the following structure:

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}$$

Here, $b_1, b_2, \ldots$ are the *birth parameters* (b_i = the average numbers of females produced by each female in class i) and $s_1, s_2, \ldots$ are the *survival probabilities* (s_i = the probability that a female in class i survives into class $i + 1$).

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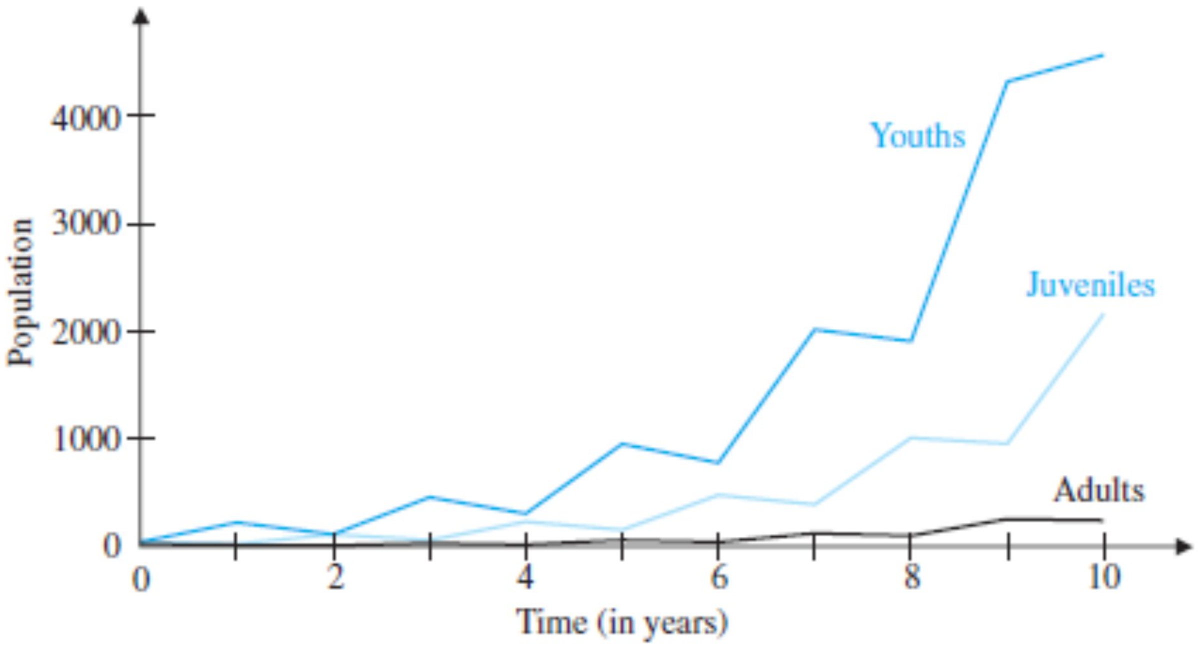

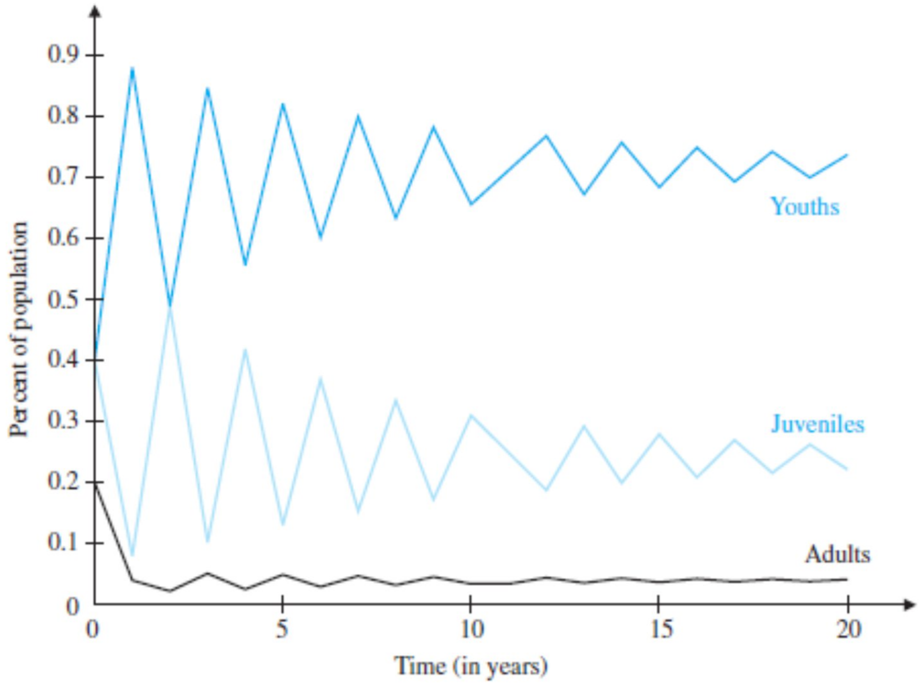

LU factorization and permutation matrix