Summer Project 2024
Report

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1 Summary

The initial description of my summer project was to read given chapters of two books, `Introduction to Cosmology - Barbara Ryden` and the `Introductionn to Elementary Particles - David Griffiths`. But when reading certain chapters of Ryden, I noticed that I don't really have the mathematical background for understanding certain topics like tensors, curvature and special theory of relativity. So following my advisor's recommendation, I had to read certain chapters of `Mathematical Methods for Physicists - George B. Arfken`, and also some parts of `Introduction to Electrodynamics - Griffiths` to make it easier for me to understand concepts. After doing so I have completed the required chapters in cosmology, but for the second book in the reading project, I have only covered two chapters. Thus in this report I am going to include both Cosmology and related Mathematics that I have learned.

2 Problems

As instructed by my advisor, I have to pick a few problems and solve them as a part of my report. I have picked these problems from a combination of books as referenced.

1. 1.1 (dodelson)
2. 1.4
3. 2.5
4. 2.8
5. 2.4 (ryden)
6. 4.5
7. 4.1.2 (arfken)

Problem 1. *Suppose that H scales as temperature squared all the way back until the time when the temperature of the universe was $10^{19} GeV/k_B$ (universe was dominated by radiation all the way upto planktime). Also suppose that dark energy is in the form of a cosmological constant Λ, such that ρ_Λ remains constant throughout the history of the universe. What was $\rho_\Lambda/(3H^2/8\pi G)$, back then? (Source: `Exercise 1.1: Modern Cosmology - Scott Dodelson`)*

Solution:

Given H scales as temperature squared:

$$H(t) \propto T^2$$

$$\frac{H(t)}{H_0} \propto \left(\frac{T}{T_0}\right)^2$$

Given $T = 10^{19} GeV/k_B$, converting into kelvin:

$$T = 10^{19} GeV/k_B = 10^{19} \times \frac{1.6 \times 10^{-10}}{1.38 \times 10^{-23}} K = 1.159 \times 10^{32} K$$

Using $T_0 = 2.7K$ (CMB radiation), we have

$$\left(\frac{T}{T_0}\right) = \frac{1.159 \times 10^{32}}{2.7} = 4.29 \times 10^{31}$$

Thus the value, $\rho_\Lambda/(3H^2/8\pi G)$ can be given as:

$$\frac{\rho_\Lambda}{\left(\frac{3H^2}{8\pi G}\right)} = \frac{\rho_\Lambda}{\rho_{c,0}} \times \left(\frac{H_0}{H}\right)^2$$

Using the above relation between H and T:

$$\frac{\rho_\Lambda}{\left(\frac{3H^2}{8\pi G}\right)} = \frac{\rho_\Lambda}{\rho_{c,0}} \times \left(\frac{T_0}{T}\right)^4$$

Writing $\frac{\rho_\Lambda}{\rho_{c,0}}$ as $\Omega_{\Lambda,0} = 0.7$, (cosmological cosmological constant remains fixed).

$$\frac{\rho_\Lambda}{\left(\frac{3H^2}{8\pi G}\right)} = 0.7 \times \left(\frac{1}{4.29 \times 10^{31}}\right)^4$$

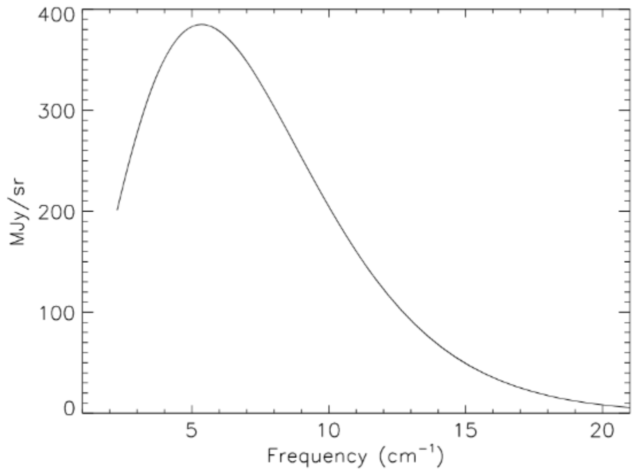
$$\frac{\rho_\Lambda}{\left(\frac{3H^2}{8\pi G}\right)} = 9 \times 10^{-128}$$

Problem 2. *Convert the specific intensity in Eq. (1.9) into an expression for what is plotted in Fig. 1.7, the energy per area, time, frequency and steradian. Show that the peak of a 2.73 K black-body spectrum does lie at $1/\lambda \simeq 5cm^{-1}$. What frequency does this correspond to? (Source:* `Exercise 1.4: Modern Cosmology - Scott Dodelson` *)*

For reference Eq. (1.9) is given by:

$$I_\nu = \frac{4\pi\hbar}{c^2} \frac{\nu^3}{e^{2\pi\hbar\nu/k_B T} - 1}$$

And Fig. 1.7 is:



Solution:

By differentiating equation by ν we get

$$\frac{dI_\nu}{d\nu} = \frac{4\pi\hbar}{c^2}\frac{d}{d\nu}\left(\frac{\nu^3}{e^{2\pi\hbar\nu/k_BT}-1}\right)$$

$$\frac{dI_\nu}{d\nu} = \frac{4\pi\hbar}{c^2}\left(\frac{3\nu^2(e^{2\pi\hbar\nu/k_BT}-1)-\nu^3(e^{2\pi\hbar\nu/k_BT}-1)(\frac{2\pi\hbar}{k_BT})}{(e^{2\pi\hbar\nu/k_BT}-1)^2}\right)$$

$$\frac{dI_\nu}{d\nu} = \frac{4\pi\hbar}{c^2}\left(\frac{3\nu^2-\nu^3(\frac{2\pi\hbar}{k_BT})}{e^{2\pi\hbar\nu/k_BT}-1}\right)$$

$$\frac{dI_\nu}{d\nu} = \frac{4\pi\hbar\nu^2}{k_BTc^2}\left(\frac{3k_BT-\nu(2\pi\hbar)}{e^{2\pi\hbar\nu/k_BT}-1}\right)$$

Thus when equating it to zero to find the peak

$$\frac{dI_\nu}{d\nu} = 0 = 3k_BT-\nu(2\pi\hbar)$$

This gives

$$v_{max} = \frac{3k_BT}{2\pi\hbar}$$

When $T = 2.73K$, we get

$$v_{max} = \frac{3(1.38\times10^{-23})(2.73)}{6.6\times10^{-34}}$$

$$v_{max} = 1.68\times10^{11}Hz$$

Using $\lambda = \frac{c}{\nu}$

$$\frac{1}{\lambda_{max}} = \frac{1.68\times10^{11}}{3\times10^{8}}m^{-1}$$

$$\frac{1}{\lambda_{max}} = 5.59\times10^{2}m^{-1}$$

$$\frac{1}{\lambda_{max}} \approx 5cm^{-1}$$

Now in the given diagram the units used is MJy/sr which is Mega Jansky per Steridian, whose dimention is similar to RHS

$$1Jy = 10^{-26} SI units$$

Thus

$$Intensity[MJy/sr] = 10^{20} I_\nu(SI units)$$

Which is the same units used in the graph.

Problem 3. *At early times, the cosmological constant can be neglected. Using this approximation, integrate Eq. (1.3) in a Euclidean universe to obtain a(t). Using $T(t) = T_0/a(t)$, determine the times when the cosmic temperature was 0.1 MeV and 1/4 eV. (Source:* ***`Exercise 2.5: Modern Cosmology - Scott Dodelson`*** *)*

For reference: Eq. (1.3) is the friedmann equation:

$$H^2(t) = \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_{cr} - \rho(t_0)}{a(t)^2}\right]$$

Solution:

We know that the hubble rate is defined as:

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}$$

Using this time can be written as:

$$dt = \frac{1}{a}\frac{da}{H(t)}$$

When value of a goes from 0 to a.

$$t = \int_0^a \frac{1}{a}\frac{da}{H(t)}$$

Since the we ignore curvature and the cosmological constant we have:

$$\left(\frac{H(t)}{H_0}\right)^2 = \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4}$$

Using this in the above equation we get:

$$t = \frac{1}{H_0}\int_0^a \frac{1}{a}\frac{da}{\sqrt{\frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4}}}$$

$$H_0 t = \frac{1}{\sqrt{\Omega_{r,0}}}\int_0^a \frac{a.da}{\sqrt{\frac{\Omega_{m,0}a}{\Omega_{r,0}} + 1}}$$

We can define $a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$

$$H_0 t = \frac{1}{\sqrt{\Omega_{r,0}}} \int_0^a \frac{a.da}{\sqrt{\frac{a}{a_{rm}} + 1}}$$

We can use taylor expansion to integrate this, and ignore higher terms:

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2} \right]$$

Since the given temperatures we can estimate $a << a_{rm}$, when radiation dominates.

$$t \approx \frac{a^2}{2H_0\sqrt{\Omega_{r,0}}}$$

Using the temperature equation given in the question:

$$T(t) = \frac{T_0}{a(t)}$$

$$a(t) = \frac{T_0}{T(t)}$$

$$a(t) = \frac{k_B T_0}{k_B T(t)}$$

Taking $k_B T_0 = 2.35 \times 10^{-4} eV$ (CMB), and using
(i) $T(t) = 0.1MeV = 10^5 eV$

$$a(t) = 2.35 \times 10^{-9}$$

$$t = \frac{a^2}{2H_0\sqrt{\Omega_{r,0}}}$$

Now taking $\Omega_{r,0} = 9 \times 10^{-5}$, $H_0^{-1} = 14.4Gyr$

$$t = 4.19 \times 10^{-6} yr$$

$$t = 132s$$

(ii) $T(t) = 0.25eV$

$$a(t) = 9.4 \times 10^{-4}$$

$$t = \frac{a^2}{2H_0\sqrt{\Omega_{r,0}}}$$

$$t = 3.88 \times 10^5 yr$$

Problem 4. *How is the energy density of a gas of photons with a black-body spectrum related to the specific intensity of the radiation? That is, what is the relation between ρ_γ and I_ν defined in Eq. (1.9) (Source:* `Exercise 2.8: Modern Cosmology - Scott Dodelson` *)*

For reference Eq. (1.9) is given by:

$$I_\nu = \frac{4\pi\hbar}{c^2} \frac{\nu^3}{e^{2\pi\hbar\nu/k_B T} - 1}$$

Solution:

WKT Energy density from frequency ν to $\nu + d\nu$, is given by:

$$\epsilon(\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/k_B T} - 1}$$

$$\epsilon(\nu)d\nu = \frac{16\pi^2\hbar}{c^3}\frac{\nu^3 d\nu}{e^{2\pi\hbar\nu/k_B T} - 1}$$

$$\epsilon(\nu)d\nu = \frac{4\pi}{c}I_\nu d\nu$$

Writing $E_{mean} = pc = h\nu$ for radiation, we have

$$\frac{2\pi}{c}d\nu = dp\frac{1}{\hbar}$$

$$\epsilon(\nu)d\nu = \frac{2}{\hbar}I_\nu dp$$

By integrating over all frequencies we have

$$\epsilon = \frac{2}{\hbar}\int_0^\infty I_\nu dp$$

Problem 5. *A hypothesis once used to explain the Hubble relation is the "tired light hypothesis." The tired light hypothesis states that the universe is not expanding, but that photons simply lose energy as they move through space (by some unexplained means), with the energy loss per unit distance being given by the law*

where k is a constant.

$$\frac{dE}{dr} = -kE$$

Show that this hypothesis gives a distance– redshift relation that is linear in the limit $z << 1$. What must the value of k be in order to yield a Hubble constant of $H_0 = 68kms^{-1}Mpc^{-1}$? (Source: `Exercise 2.4: Introduction to Cosmology - Barbara Ryden` *)*

Solution:

Given equation,

$$\frac{dE}{dr} = -kE$$

can be written as

$$E(r) = E_0 e^{-kr}$$

For light from stars we can write energy in terms of wave length as $\lambda = \frac{hc}{E}$

We can write redshift in terms of this wavelength as:

$$z = \frac{\lambda_e - \lambda_0}{\lambda_0} = \frac{\frac{hc}{E} - \frac{hc}{E_0}}{\frac{hc}{E_0}}$$

$$z = e^{kr} - 1$$

Comparing this with the Hubble's equation:

$$z = \frac{H_0}{c}r$$

We have:

$$\frac{H_0}{c}r = e^{kr} - 1$$

By using taylor expansion:

$$\frac{H_0}{c}r \approx kr$$

$$k = \frac{H_0}{c} = 2.3 \times 10^{-4} Mpc^{-1}$$

Problem 6. *The principle of wave-particle duality tells us that a particle with momentum p has an associated de Broglie wavelength of $\lambda = h/p$; this wavelength increases as $\lambda \propto a$ as the universe expands. The total energy density of a gas of particles can be written as $\epsilon = nE$, where n is the number density of particles, and E is the energy per particle. For simplicity, let's assume that all the gas particles have the same mass m and momentum p. The energy per particle is then simply*

$$E = (m^2c^4 + p^2c^2)^{1/2} = (m^2c^4 + h^2c^2/\lambda^2)^{1/2}$$

Compute the equation-of-state parameter w for this gas as a function of the scale factor a. Show that $w = 1/3$ in the highly relativistic limi ($a \to 0, p \to \infty$) and that $w = 0$ in the highly nonrelativistic limit ($a \to \infty, p \to 0$). (Source: `Exercise 4.5: Introduction to Cosmology - Barbara Ryden`)

Solution:

We can write Energy in terms of energy density as:

$$\epsilon V = (m^2c^4 + h^2c^2/\lambda^2)^{1/2}$$

$$\epsilon = \frac{1}{V}(m^2c^4 + h^2c^2/\lambda^2)^{1/2}$$

We know that volume scales cubically and, λ scales linearly:

$$\epsilon = \frac{1}{Va^3(t)}\left(m^2c^4 + \frac{h^2c^2}{a^2(t)\lambda^2}\right)^{1/2}$$

Taking log both sides

$$\epsilon = \frac{1}{V_0a^3(t)}\left(m^2c^4 + \frac{p^2c^2}{a^2(t)}\right)^{1/2}$$

$$ln(\epsilon) = -lnV_0 - 3ln(a(t)) + \frac{1}{2}ln\left(m^2c^4 + \frac{p^2c^2}{a^2(t)}\right)$$

Differentiate this equation by time:

$$\frac{\dot{\epsilon}}{\epsilon} = -3\frac{\dot{a}}{a} + \frac{1}{2}\frac{-2\frac{p^2c^2}{a(t)^3}\dot{a}}{\left(m^2c^4 + \frac{p^2c^2}{a^2(t)}\right)}$$

$$\frac{\dot{\epsilon}}{\epsilon} + 3\frac{\dot{a}}{a} = -\frac{p^2}{(a(t)^2m^2c^2 + p^2)}\frac{\dot{a}}{a}$$

$$\frac{\frac{\dot{\epsilon}}{\epsilon} + 3\frac{\dot{a}}{a}}{\frac{3\dot{a}}{a}} = -\frac{p^2}{3\left(a(t)^2m^2c^2 + p^2\right)}$$

Now using fluid equation and equation of state:

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0$$

$$\dot{\epsilon} = -3\frac{\dot{a}}{a}(1+w)\epsilon$$

$$\frac{\dot{\epsilon}}{\epsilon} = -3\frac{\dot{a}}{a}(1+w)$$

$$\frac{\frac{\dot{\epsilon}}{\epsilon}}{\frac{3\dot{a}}{a}} = -(1+w)$$

$$-\left(\frac{\frac{\dot{\epsilon}}{\epsilon} + \frac{3\dot{a}}{a}}{\frac{3\dot{a}}{a}}\right) = w$$

Thus

$$w = \frac{p^2}{3\left(a(t)^2m^2c^2 + p^2\right)}$$

When $p \to \infty$

$$w = \frac{1}{3\left(a(t)^2\frac{m^2c^2}{p^2} + 1\right)}$$

$$w = \frac{1}{3}$$

When $p \to 0$

$$w = 0$$

Problem 7. *The components of tensor A are equal to the corresponding components of tensor B in one particular coordinate system denoted, by the superscript 0; that is,*

$$A^0_{ij} = B^0_{ij}$$

Show that tensor A is equal to tensor B, $A_{ij} = B_{ij}$, in all coordinate systems (Source: ***`Exercise 4.1.2: Mathematical Methods for Physicists - George B. Arfken`*** *)*

Solution:

Let us define a coordinate transformation from the coordinate system x^0_{ij} to say $x_{\alpha\beta}$.

Thus for any given vector A^0_{ij} can be written in terms of the new coordinate system as:

$$A_{\alpha\beta} = \frac{dx_\alpha}{dx^0_i}\frac{dx_\beta}{dx^0_j}A^0_{ij}$$

Since, $A^0_{ij} = B^0_{ij}$

$$A_{\alpha\beta} = \frac{dx_\alpha}{dx^0_i}\frac{dx_\beta}{dx^0_j}A^0_{ij}$$

$$A_{\alpha\beta} = \frac{dx_\alpha}{dx^0_i}\frac{dx_\beta}{dx^0_j}B^0_{ij}$$

Since the same transformation equation also applies to B^0_{ij} $(A^0 and B^0)$ are defined in the same coordinate system:

$$A_{\alpha\beta} = B_{\alpha\beta}$$

Therefore if components of tensor A and B are equal under one coordinate system then they are also equal under any coordinate system as long as we can define a transformation property between those coordinate systems.