Introduction to Cosmology

Instructor's Manual

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1
Introduction

The purpose of this instructor's manual is primarily to provide worked solutions for the end-of-chapter problems. In writing up the solutions, I didn't try to mimic the solutions that an A+ student will submit; instead, I tried to go into "verbose" mode, giving general solutions that emphasize the physics behind the problem. I also indulge in occasional asides, and recommendations for further problems you can pose for your students. (The solutions were written in some haste, so errors could lurk – let me know if you find any.)

In the worked solutions, I refer to equations both in the textbook *Introduction to Cosmology (2nd edition)* and in this Instructor's Manual. To help keep things clear, when I refer to an equation in the textbook, I use the format "Eq. 6.66"; when I refer to an equation in this Instructor's Manual, I use the format "equation (6.66)".

In addition to worked solutions, I also give a brief summary of the changes that I made in going from the first edition to the second edition. I also give references for some of the assertions made in the text. (I didn't want to clutter up the text with references, but I assume that some of you will be curious about the source of some of my less-obvious assertions.)

In writing the second edition, I took the opportunity to correct the typographical errors present in the first edition of *Introduction to Cosmology*. However, I am sure I introduced some new errors! If you find any, please let me know at `ryden.1@osu.edu`. Future versions of the Instructor's Manual will contain a list of errata.

2
Fundamental observations

In the second edition of *Introduction to Cosmology*, Section 2.1 contains a more physically realistic analysis of Olbers' paradox. Instead of treating stars as point light sources, as I did in the first edition, I acknowledge that they are spheres of finite size, and use a mean free path analysis to find the resulting surface brightness of the night sky in an infinite Euclidean universe.

In Section 2.2, I clarify the difference between the Copernican principle and the cosmological principle, which I failed to make clear in the first edition.

Section 2.3 contains a bit more historical background on Hubble's law, placing it in the context of the work by Lemaître and others. My default value of the Hubble constant, taken to be $H_0 = 70 \pm 7\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in the first edition, is $H_0 = 68 \pm 2\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in the second edition. (Note that although this value of H_0 is consistent with the WMAP 9-year results and the Planck 2015 results, it is lower than value found by direct measurements in the local universe. For instance, Riess et al. (2016) find $H_0 = 73.24 \pm 1.74\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. In the textbook, I do not bring up this tension; you may want to use it as a topic of discussion with your students.)

In Section 2.4, I clarify the discussion of the different neutrino mass states, placing upper and lower limits on the sum of the neutrino masses. (These limits reappear in Chapter 11, during the discussion of the difference between hot dark matter and cold dark matter.)

Exercises

2.1 *Assume you are a perfect blackbody at a temperature of* $T = 310\,\mathrm{K}$*. What is the rate, in watts, at which you radiate energy? (For the purposes of this problem, you may assume you are spherical.)*

I have a mass $M_{\mathrm{me}} = 70\,\mathrm{kg}$, and my density is comparable to that of water, $\rho_{\rm me} = 1000\,{\rm kg\,m^{-3}}$. If I curl myself into a ball, my radius is

$$R_{\rm me} = \left(\frac{3M_{\rm me}}{4\pi\rho_{\rm me}}\right)^{1/3} = 0.256\,{\rm m} = 3.65 \times 10^{-10}\,{\rm R}_\odot \ . \tag{2.1}$$

My temperature is

$$T_{\rm me} = 310\,{\rm K} = 0.0534\,{\rm T}_\odot \ . \tag{2.2}$$

(The Sun's radius is given Section 2.1 of the text, and the Sun's effective temperature is given in Section 2.4.) Since the luminosity of a spherical blackbody is $L = 4\pi R^2 \sigma_{\rm sb} T^4$, where $\sigma_{\rm sb}$ is the Stefan-Boltzmann constant, my luminosity is

$$\begin{aligned} L_{\rm me} &= L_\odot \left(\frac{R_{\rm me}}{{\rm R}_\odot}\right)^2 \left(\frac{T_{\rm me}}{{\rm T}_\odot}\right)^4 \\ &= 3.838 \times 10^{26}\,{\rm watts}(3.65 \times 10^{-10})^2 (0.534)^4 = 420\,{\rm watts} \ . \end{aligned} \tag{2.3}$$

(Note that the text never explicitly states that $L \propto R^2 T^4$ for a spherical blackbody; your students may need a prompt, depending on their physics background.)

2.2 *Since you are made mostly of water, you are very efficient at absorbing microwave photons. If you were in intergalactic space, how many CMB photons would you absorb per second? (The assumption that you are spherical will be useful.) What is the rate, in watts, at which you would absorb radiative energy from the CMB?*

Since my radius is $R_{\rm me} = 0.256\,{\rm m}$ (from the previous problem), my geometric cross-section is

$$\sigma_{\rm me} = \pi R_{\rm me}^2 = 0.205\,{\rm m}^2 \ . \tag{2.4}$$

The number density of CMB photons is (from Eq. 2.35 of the text),

$$n_\gamma = 4.107 \times 10^8\,{\rm m}^{-3} \ . \tag{2.5}$$

At what rate will I absorb CMB photons? This is easiest to compute if I adopt the fiction that all the CMB photons are moving in the same direction. In that case, during a time interval dt, I would sweep up all the photons in a cylinder of length cdt and cross-sectional area $\sigma_{\rm me}$. That is, the number dN of photons absorbed during time interval dt is

$$dN = cdt\sigma_{\rm me} n_\gamma \ , \tag{2.6}$$

or

$$\frac{dN}{dt} = c\sigma_{\rm me} n_\gamma = 2.52 \times 10^{16}\,{\rm s}^{-1} \ . \tag{2.7}$$

The average energy of a CMB photon is $E_{\rm mean} = 6.344 \times 10^{-4}\,{\rm eV} = 1.016 \times 10^{-22}\,{\rm J}$, from Eq. 2.36 of the text. Thus, the heating rate from absorbing CMB photons will be

$$G_{\rm me} = \frac{dN}{dt} E_{\rm mean} = 2.6 \times 10^{-6}\,{\rm J\,s^{-1}} = 2.6 \times 10^{-6}\,{\rm watts}\ . \qquad (2.8)$$

[A student wise in the ways of the Stefan-Boltzmann law might make the following alternative argument: If I had a temperature equal to that of the CMB, I would be in an equilibrium state, with the rate at which I absorbed energy from the CMB being exactly equal to the rate at which I emitted energy from my surface. Thus, I can state that the rate at which I absorb energy from the CMB is

$$G_{\rm me} = 4\pi R_{\rm me}^2 \sigma_{\rm sb} T_0^4\ , \qquad (2.9)$$

where $T_0 = 2.7255\,{\rm K}$ is the temperature of the CMB. Using $R_{\rm me} = 0.256\,{\rm m}$ and $\sigma_{\rm sb} = 5.670 \times 10^{-8}\,{\rm watts\,m^{-2}\,K^{-4}}$, this works out to

$$G_{\rm me} = 2.58 \times 10^{-6}\,{\rm watts}\ , \qquad (2.10)$$

from which I can work backward to find dN/dt, the rate at which CMB photons are absorbed.]

2.3 *Suppose that intergalactic space pirates toss you out the airlock of your spacecraft without a spacesuit. Combining the results of the two previous questions, at what rate would your temperature change? (Assume your heat capacity is that of pure water, $C = 4200\,{\rm J\,kg^{-1}\,K^{-1}}$.) Would you be most worried about overheating, freezing, or asphyxiating?*

From the previous two problems, I know that since my temperature ($T_{\rm me} \approx 310\,{\rm K}$) is much greater than that of the CMB ($T_0 = 2.7255\,{\rm K}$), I will *lose* energy rather than gain it, at a net rate

$$L_{\rm net} = L_{\rm me} - G_{\rm me} = 420\,{\rm watts} - 0.000003\,{\rm watts} = 420\,{\rm watts}\ . \qquad (2.11)$$

My temperature will then drop at the rate

$$\frac{dT_{\rm me}}{dt} = -\frac{L_{\rm net}}{CM_{\rm me}} = -\frac{420\,{\rm J\,s^{-1}}}{2.94 \times 10^5\,{\rm J\,K^{-1}}} = -1.4 \times 10^{-3}\,{\rm K\,s^{-1}}\ . \qquad (2.12)$$

It takes about 12 minutes for my temperature to drop by one degree; even if I hyperventilate with panic at the prospect of being tossed out the airlock, I will asphyxiate before I freeze.

2.4 *A hypothesis once used to explain the Hubble relation is the "tired light hypothesis." The tired light hypothesis states that the universe is not expanding, but that photons simply lose energy as they move through* *space (by some unexplained means), with the energy loss per unit distance being given by the law*

$$\frac{dE}{dr} = -kE, \tag{2.13}$$

where k *is a constant. Show that this hypothesis gives a distance-redshift relation that is linear in the limit* $z \ll 1$*. What must the value of* k *be in order to yield a Hubble constant of* $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$*?*

If a photon starts with energy E_0 at $r = 0$, then the "tired light" equation

$$\frac{dE}{dr} = -kE \tag{2.14}$$

can be integrated to find the solution

$$E(r) = E_0 e^{-kr} \ . \tag{2.15}$$

Since a photon's energy E is related to wavelength λ by the equation

$$E = hf = \frac{hc}{\lambda} \ , \tag{2.16}$$

the redshift z of the light can be written as

$$z \equiv \frac{\lambda - \lambda_0}{\lambda_0} = \frac{1/E - 1/E_0}{1/E_0} = \frac{E_0 - E}{E} \ . \tag{2.17}$$

Substituting from equation 2.15 above, we find the distance-redshift relation

$$z = \frac{1 - e^{-kr}}{e^{-kr}} = e^{kr} - 1 \ . \tag{2.18}$$

In the limit $kr \ll 1$, we can use the expansion $\exp(kr) \approx 1 + kr$, and thus

$$z \approx kr \ . \tag{2.19}$$

This is the Hubble law, with $k = H_0/c$. To yield $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, the "tired light" parameter must be

$$k = \frac{H_0}{c} = \frac{68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}{3 \times 10^5\,\mathrm{km\,s^{-1}}} = 2.3 \times 10^{-4}\,\mathrm{Mpc^{-1}} \ . \tag{2.20}$$

2.5 *Consider blackbody radiation at a temperature* T*. Show that for an energy threshold* $E_0 \gg kT$*, the fraction of the blackbody photons that have energy* $hf > E_0$ *is*

$$\frac{n(hf > E_0)}{n_\gamma} \approx 0.42 \left(\frac{E_0}{kT}\right)^2 \exp\left(-\frac{E_0}{kT}\right) . \tag{2.21}$$

The cosmic background radiation is currently called the "cosmic microwave background." However, photons with $\lambda < 1\,\mathrm{mm}$ actually lie in the far infrared range of the electromagnetic spectrum. It's time for truth in advertising: what fraction of the photons in today's "cosmic microwave background" are actually far infrared photons?

At photon energies $hf \gg kT$, the number density of photons as a function of frequency is (from Eq. 2.30 of the text),

$$n(f) = \frac{8\pi}{c^3}\frac{f^2}{\exp(hf/kT)-1} \approx \frac{8\pi}{c^3} f^2 \exp(-hf/kT) \ . \tag{2.22}$$

Thus, the number density of photons with energy greater than some threshold energy $E_0 = hf_0 \gg kT$ is

$$n(hf > E_0) \approx \frac{8\pi}{c^3}\int_{f_0}^{\infty} f^2 \exp(-hf/kT) df \ . \tag{2.23}$$

Making the substitution $x = hf/kT$, this becomes

$$n(hf > E_0) \approx 8\pi \left(\frac{kT}{hc}\right)^3 \int_{x_0}^{\infty} x^2 e^{-x} dx \ , \tag{2.24}$$

where $x_0 = hf_0/kT = E_0/kT \gg 1$. The total number density of photons is, from Eqs. 2.31 and 2.32 of the text,

$$n_\gamma = \frac{2.4041}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 = 2.4041(8\pi)\left(\frac{kT}{hc}\right)^3 \ . \tag{2.25}$$

Thus, the fraction of the photons with $hf > E_0 \gg kT$ is

$$F(hf > E_0) = \frac{n(hf > E_0)}{n_\gamma} \approx \frac{1}{2.4041}\int_{x_0}^{\infty} x^2 e^{-x} dx \ . \tag{2.26}$$

Doing the integral, we find that

$$F(hf > E_0) = 0.416 e^{-x_0}[x_0^2 + 2x_0 + 2] \ , \tag{2.27}$$

where $x_0 = E_0/kT$. Since we have already made the approximation $x_0 \gg 1$, equation (2.27) can be adequately approximated as

$$F(hf > E_0) \approx 0.416 x_0^2 e^{-x_0} \approx 0.42 \left(\frac{E_0}{kT}\right)^2 \exp\left(-\frac{E_0}{kT}\right) \ . \tag{2.28}$$

A photon with wavelength $\lambda_0 = 1\,\mathrm{mm}$, at the threshold of the far-infrared range, has an energy $E_0 = hc/\lambda_0 = 1.240 \times 10^{-3}\,\mathrm{eV}$. For the CMB, $kT_0 = 2.349 \times 10^{-4}\,\mathrm{eV}$, yielding $x_0 = E_0/kT_0 = 5.28$, which I declare (slightly rashly) to be much larger than one. The fraction of CMB photons in the far-infrared range, with $hf > E_0$, is then

$$F(hf > 1.24\,\mathrm{meV}) \approx 0.42(5.28)^2 e^{-5.28} \approx 0.06\ . \tag{2.29}$$

2.6 *Show that for an energy threshold $E_0 \ll kT$, the fraction of blackbody photons that have energy $hf < E_0$ is*

$$\frac{n(hf < E_0)}{n_\gamma} \approx 0.21 \left(\frac{E_0}{kT}\right)^2 . \tag{2.30}$$

Microwave (and far infrared) photons with a wavelength $\lambda < 3\,\mathrm{cm}$ are strongly absorbed by H_2O and O_2 molecules. What fraction of the photons in today's cosmic microwave background have $\lambda > 3\,\mathrm{cm}$, and thus are capable of passing through the Earth's atmosphere and being detected on the ground? At photon energies $hf \ll kT$, the number density of photons as a function of frequency is (from Eq. 2.30 of the text),

$$n(f) = \frac{8\pi}{c^3}\frac{f^2}{\exp(hf/kT) - 1} \approx \frac{8\pi}{c^3}\frac{f^2}{hf/kT} \approx \frac{8\pi kT}{hc^3} f\ . \tag{2.31}$$

Thus, the number density of photons with energy less than some threshold energy $E_0 = hf_0 \ll kT$ is

$$\begin{aligned} n(hf < E_0) &\approx \frac{8\pi kT}{hc^3}\int_0^{f_0} f df \\ &\approx \frac{4\pi kT f_0^2}{hc^3} \approx \frac{4\pi kT E_0^2}{(hc)^3}\ . \end{aligned} \tag{2.32}$$

The total number density of photons is, from Eqs. 2.31 and 2.32 of the text,

$$n_\gamma = \frac{2.4041}{\pi^2}\left(\frac{kT}{hc}\right)^3 = 2.4041(8\pi)\left(\frac{kT}{hc}\right)^3 . \tag{2.33}$$

Combining the two above results, we find that the fraction of photons with $hf < E_0 \ll kT$ is

$$F(hf < E_0) = \frac{n(hf < E_0)}{n_\gamma} \approx 0.208\left(\frac{E_0}{kT}\right)^2 . \tag{2.34}$$

A wavelength $\lambda_0 = 3\,\mathrm{cm}$ corresponds to a photon energy $E_0 = hc/\lambda_0 = 4.133\times10^{-5}\,\mathrm{eV}$. For the CMB, $kT_0 = 2.349\times10^{-4}\,\mathrm{eV}$, yielding $E_0/kT_0 = 0.176$, which I declare, somewhat rashly, to be much smaller than one. The fraction of CMB photons with $\lambda > 3\,\mathrm{cm}$ is then

$$F(hf < 41.33\,\mu\mathrm{eV}) \approx 0.2080(0.176)^2 \approx 0.006\ . \tag{2.35}$$

3
Newton versus Einstein

Section 3.1 of the first edition has been expanded into three sections in the second edition, giving a fuller description of Newtonian gravity (Section 3.1 of the second edition), special relativity (Section 3.2), and general relativity (Section 3.3).

Section 3.2 introduces the concepts of an inertial reference frame, the Lorentz transformation, and 4-dimensional spacetime. By talking about the distance between two events in Minkowski space, Section 3.2 provides a gentler introduction to the concept of a "metric."

Exercises

3.1 *What evidence can you provide to support the assertion that the universe is electrically neutral on large scales?*

I pose this question to my students mainly in order to stimulate their thinking. I don't really think it has a "best" answer; I'll just give one interesting point of view on this problem. The gravitational force between two protons separated by a distance r is

$$F_{\rm gr} = -\frac{Gm_p^2}{r^2} \ . \tag{3.1}$$

The electrostatic force between the two protons, each of charge $+q$, is

$$F_{\rm el} = \frac{k_e q^2}{r^2} \ , \tag{3.2}$$

where $k_e = 8.99 \times 10^9\,{\rm N\,m^2\,C^{-2}}$ and $q = 1.602 \times 10^{-19}\,{\rm C}$. This means that the ratio of the electrostatic force to the gravitational force between a pair of protons is

$$\frac{F_{\rm el}}{F_{\rm gr}} = \frac{k_e q^2}{G m_p^2} = 1.24 \times 10^{36} \ . \tag{3.3}$$

In other words, gravity is a really wimpy force.

Now, consider a pair of identical galaxies, which I will approximate as electrically neutral blobs of hydrogen. Each galaxy has N protons, each of charge $+q$, and N electrons, each of charge $-q$. The total mass of each galaxy is $M_{\rm gal} = N m_p$, ignoring the small contribution by the electrons. I now remove a fraction ϵ of all the electrons in each galaxy, so that the net charge of each galaxy is $+\epsilon N q$. The ratio of the electrostatic force to the gravitational force between the pair of galaxies is then

$$\frac{F_{\rm el}}{F_{\rm gr}} = \frac{k_e(\epsilon N q)^2}{G(N m_p)^2} = \epsilon^2 \frac{k_e q^2}{G m_p^2} \ . \tag{3.4}$$

By comparison with equation (3.3) above, giving the ratio of forces for a single pair of protons, we find that the gravitational force between the galaxies exactly balances the electrostatic force when the fraction of electrons removed is

$$\epsilon = \left(\frac{G m_p^2}{k_e q^2}\right)^{1/2} = (1.24 \times 10^{36})^{-1/2} = 9.0 \times 10^{-19} \ . \tag{3.5}$$

(This represents removing one electron from every two micrograms of hydrogen.) Since, for instance, the Milky Way Galaxy and M31 are falling toward each other rather than being repelled from each other, we know that any deficiency (or excess) of electrons in the Local Group must be at a level smaller than one part in $\sim 10^{18}$. [Notice, however, that galaxies aren't made just of protons and electrons. Given the presence of electrically neutral dark matter, for instance, you'd have to increase the electron deficiency (or excess) relative to protons by roughly an order of magnitude, to 1 part in $\sim 10^{17}$, before the electrostatic repulsion between galaxies would overcome the gravitational attraction between them.]

3.2 *Suppose you are a two-dimensional being, living on the surface of a sphere with radius R. An object of width $d\ell \ll R$ is at a distance r from you (remember, all distances are measured on the surface of the sphere). What angular width $d\theta$ will you measure for the object? Explain the behavior of $d\theta$ as $r \to \pi R$.*

On the surface of a sphere, the metric is (from Eq. 3.26 of the text),

$$d\ell^2 = dr^2 + R^2 \sin^2(r/R) d\theta^2 \ . \tag{3.6}$$

The distance between the two points at position (r, θ) and $(r, \theta + d\theta)$ is then

$$d\ell = R \sin(r/R) d\theta \ . \tag{3.7}$$

What does this mean physically, for my two-dimensional self? If I take a rod of length $d\ell$, and position it so that both ends are at a distance r from me, then equation (3.7) above tells me that it will subtend an angle

$$d\theta = \frac{d\ell/R}{\sin(r/R)} \ . \tag{3.8}$$

Note that in the limit $r \ll R$, $\sin(r/R) \approx r/R$, and thus $d\theta \approx d\ell/r$, the same result you find on a plane. However, in the limit $r \to \pi R$, equation (3.8) above solemnly assures me that $d\theta \to \infty$, which is unphysical. In fact, you must remember that equation (3.8) is derived from a metric that assumes that $d\theta \ll 1$. What actually happens when I place an object at the antipodes to me ($r = \pi R$) is that I see it in every direction that I look; on the surface of a sphere, this means that it subtends an angle $d\theta = 2\pi$ as seen from my location at $r = 0$.

3.3 *Suppose you are still a two-dimensional being, living on the same sphere of radius R. Show that if you draw a circle of radius r, the circle's circumference will be*

$$C = 2\pi R \sin(r/R). \tag{3.9}$$

Idealize the Earth as a perfect sphere of radius $R = 6371\,\mathrm{km}$. *If you could measure distances with an error of* $\pm 1\,\mathrm{meter}$, *how large a circle would you have to draw on the Earth's surface to convince yourself that the Earth is spherical rather than flat?*

Suppose that you draw a circle of radius r centered on yourself. Equation (3.7) shows that an arc of the circle that subtends an angle $d\theta$ as seen by you has a physical length

$$d\ell = R \sin(r/R) d\theta \ . \tag{3.10}$$

Integrating from $\theta = 0$ to $\theta = 2\pi$, the circumference C of the circle is computed to be

$$C = R \sin(r/R) \int_0^{2\pi} d\theta = 2\pi R \sin(r/R) \ . \tag{3.11}$$

On a flat plane, the circumference of a circle with radius r is

$$C_{\text{flat}} = 2\pi r \ . \tag{3.12}$$

Thus, the difference in circumference between a circle drawn on a sphere and a circle of the same radius drawn on a plane is

$$\Delta C \equiv C_{\text{flat}} - C = 2\pi R \left[\frac{r}{R} - \sin\left(\frac{r}{R}\right) \right] \ . \tag{3.13}$$

Using the expansion

$$\sin(r/R) \approx \frac{r}{R} - \frac{1}{6}\left(\frac{r}{R}\right)^3 \ , \tag{3.14}$$

the difference in circumference is found to be

$$\Delta C \approx \frac{\pi}{3} R \left(\frac{r}{R}\right)^3 \ . \tag{3.15}$$

If you can measure distances with an accuracy $\delta r = 1\,\text{m}$, the critical distance at which $\Delta C = \delta r$ is then

$$r_{\text{crit}} \approx \left(\frac{3}{\pi}\right)^{1/3} \left(\frac{\delta r}{R}\right)^{1/3} R \ . \tag{3.16}$$

Given $R = 6371\,\text{km}$ and $\delta r / R = 1.570 \times 10^{-7}$, this yields a critical radius

$$r_{\text{crit}} \approx 0.0053 R = 34\,\text{km} \ , \tag{3.17}$$

or about 80% of a marathon.

3.4 *Consider an equilateral triangle, with sides of length L, drawn on a two-dimensional surface of uniform curvature. Can you draw an equilateral triangle of arbitrarily large area A on a surface with $\kappa = +1$ and radius of curvature R? If not, what is the maximum possible value of A? Can you draw an equilateral triangle of arbitrarily large area A on a surface with $\kappa = 0$? If not, what is the maximum possible value of A? Can you draw an equilateral triangle of arbitrarily large area A on a surface with $\kappa = -1$ and radius of curvature R? If not, what is the maximum possible value of A?*

On a positively curved surface ($\kappa = +1$), with total area $4\pi R^2$, you *cannot* drawn a triangle of arbitrarily large area. The largest triangle area A depends on how you define "triangle." When you choose three points on the surface of a sphere, and connect them with great circle arcs, you are actually dividing the surface of the sphere into two regions. The smaller region, called the "inner triangle," has area $A_{\text{in}} \leq 2\pi R^2$. The complementary larger region, called the "outer triangle," has area

$A_{\text{out}} = 4\pi R^2 - A_{\text{in}} \geq 2\pi R^2$. Usually, when mathematicians talk about a triangle on a sphere, they implicitly mean the *inner* triangle; using this definition, the maximum area is $A_{\max} = 2\pi R^2$, which occurs when the three points defining the triangle lie along a single great circle. Sometimes, however, the definition of "triangle" is broadened to include both the inner triangle and the outer triangle. Using this definition, the maximum area is $A_{\max} = 4\pi R^2$, which is the area of the outer triangle in the degenerate case when the three points defining the equilateral triangle coincide.

On a plane ($\kappa = 0$) with no boundary, an equilateral triangle can have an arbitrarily large area. On a plane, the area of an equilateral triangle with sides of length L is $A = \sqrt{3}L^2/4$. Thus, as $L \to \infty$, $A \to \infty$.

On a negatively curved surface ($\kappa = -1$), you can draw an equilateral triangle of arbitrarily long *perimeter*, but not of arbitrarily large *area.* To see why this is true, consider the relation between the area A of a triangle on a surface of uniform negative curvature and the angles α, β, and γ at its vertices (see Eq. 3.27 of the text):

$$\alpha + \beta + \gamma = \pi - A/R^2 \ . \tag{3.18}$$

For an equilateral triangle, $\alpha = \beta = \gamma$. However, the smallest angles you can have are $\alpha = \beta = \gamma = 0$; this is the limit that you approach as L, the length of the sides of the triangle, approaches infinity. In the limit that $L \to \infty$ and $\alpha \to 0$, the area of the triangle, from equation (3.18) above, approaches

$$A_{\max} = \pi R^2 \ . \tag{3.19}$$

Thus, if you want triangles of arbitrarily large area, you should stick to flat space.

3.5 *By making the substitutions $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$, demonstrate that Equations 3.29 and 3.30 represent the same metric.*

I admit this is not the most exciting exercise in the book. However, it is important to emphasize that two metrics that look different can represent the same space.

Here goes: if

$$x = r\sin\theta\cos\phi \ , \tag{3.20}$$

then

$$dx = dr\sin\theta\cos\phi + d\theta r\cos\theta\cos\phi - d\phi r\sin\theta\sin\phi \tag{3.21}$$

and

$$\begin{aligned} dx^2 &= dr^2(\sin^2\theta\cos^2\phi) + d\theta^2(r^2\cos^2\theta\cos^2\phi) \\ &\quad + d\phi^2(r^2\sin^2\theta\sin^2\phi) + drd\theta(2r\sin\theta\cos\theta\cos^2\phi) \\ &\quad - drd\phi(2r\sin^2\theta\sin\phi\cos\phi) - d\theta d\phi(2r^2\sin\theta\cos\theta\sin\phi\cos\phi) \ . \end{aligned} \tag{3.22}$$

Similarly, if

$$y = r\sin\theta\sin\phi \ , \tag{3.23}$$

then

$$dy = dr\sin\theta\sin\phi + d\theta r\cos\theta\sin\phi + d\phi r\sin\theta\cos\phi \tag{3.24}$$

and

$$\begin{aligned} dy^2 &= dr^2(\sin^2\theta\sin^2\phi) + d\theta^2(r^2\cos^2\theta\sin^2\phi) \\ &\quad + d\phi^2(r^2\sin^2\theta\cos^2\phi) + drd\theta(2r\sin\theta\cos\theta\sin^2\phi) \\ &\quad + drd\phi(2r\sin^2\theta\sin\phi\cos\phi) + d\theta d\phi(2r^2\sin\theta\cos\theta\sin\phi\cos\phi) \ . \end{aligned} \tag{3.25}$$

Adding together equations (3.22) and (3.25), and taking advantage of the identity $\cos^2\phi + \sin^2\phi = 1$, I find

$$\begin{aligned} dx^2 + dy^2 &= dr^2\sin^2\theta + d\theta^2 r^2\cos^2\theta \\ &\quad + d\phi^2 r^2\sin^2\theta + drd\theta(2r\sin\theta\cos\theta) \ . \end{aligned} \tag{3.26}$$

Next, starting with

$$z = r\cos\theta \ , \tag{3.27}$$

I find that

$$dz = dr\cos\theta - d\theta r\sin\theta \tag{3.28}$$

and

$$dz^2 = dr^2\cos^2\theta + d\theta^2 r^2\sin^2\theta - drd\theta(2r\sin\theta\cos\theta) \ . \tag{3.29}$$

Finally, adding together equations (3.26) and (3.29), and taking advantage of the identity $\cos^2\theta + \sin^2\theta = 1$, I find

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= dr^2 + d\theta^2 r^2 + d\phi^2 r^2\sin^2\theta \\ &= dr^2 + r^2[d\theta^2 + \sin^2\theta d\phi^2] \ . \end{aligned} \tag{3.30}$$

Quod erat demonstrandum, and all that.

4
Cosmic dynamics

In the introductory section of this chapter, I offer geometric arguments, based on observations of distant galaxies, why the radius of curvature of the universe cannot be much smaller than the Hubble radius.

Section 4.1 (Einstein's field equation) is new to the second edition. Without touching on the numerical details of general relativity (Christoffel symbols – just say no!), I discuss qualitatively some of the properties of the field equation.

Exercises

4.1 *Suppose the energy density of the cosmological constant is equal to the present critical density $\varepsilon_\Lambda = \varepsilon_{c,0} = 4870\,\mathrm{MeV\,m^{-3}}$. What is the total energy of the cosmological constant within a sphere $1\,\mathrm{AU}$ in radius? What is the rest energy of the Sun ($E_\odot = M_\odot c^2$)? Comparing these two numbers, do you expect the cosmological constant to have a significant effect on the motion of planets within the solar system?*

Given that $1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$, the total energy of Λ in a sphere of radius $1\,\mathrm{AU}$ is

$$\begin{aligned} E_\Lambda &= (4870\,\mathrm{MeV\,m^{-3}})\left[\frac{4\pi}{3}(1.496 \times 10^{11}\,\mathrm{m})^3\right] \\ &= 6.830 \times 10^{37}\,\mathrm{MeV} = 1.094 \times 10^{25}\,\mathrm{J}\ . \end{aligned} \tag{4.1}$$

Given that $1\,\mathrm{M_\odot} = 1.989 \times 10^{30}\,\mathrm{kg}$, the rest energy of the Sun is

$$E_\odot = M_\odot c^2 = (1.989 \times 10^{30}\,\mathrm{kg})(2.998 \times 10^{8}\,\mathrm{m\,s^{-1}})^2 = 1.788 \times 10^{47}\,\mathrm{J}\ . \tag{4.2}$$

Thus, the energy of the cosmological constant represents a fraction $f = E_\Lambda/E_\odot = 6.21\times10^{-23}$ of the Sun's rest energy. The gravitational effects of the cosmological constant provide a minuscule perturbation to the dynamics of the solar system. To put it another way, $E_\Lambda/L_\odot \approx 0.03\,\mathrm{s}$; the energy of Λ in a sphere of radius 1 AU is equivalent to the energy lost by the Sun in one-tenth of the blink of an eye (given $t_{\mathrm{blink}} \sim 0.3\,\mathrm{s}$). To put it yet another way, $E_\Lambda/c^2 \approx 1.2 \times 10^8\,\mathrm{kg}$; the energy of Λ in a sphere of radius 1 AU is comparable to the mass of a meteoroid just 40 meters across.

4.2 *Consider Einstein's static universe, in which the attractive force of the matter density ρ is exactly balanced by the repulsive force of the cosmological constant, $\Lambda = 4\pi G\rho$. Suppose that some of the matter is converted into radiation (by stars, for instance). Will the universe start to expand or contract? Explain your answer.*

The matter density ρ provides an energy density $\varepsilon_m = \rho c^2$ with no pressure. The cosmological constant provides an energy density

$$\varepsilon_\Lambda = \frac{c^2}{8\pi G}\Lambda = \frac{1}{2}\rho c^2 \tag{4.3}$$

and a pressure

$$P_\Lambda = -\varepsilon_\Lambda = -\frac{1}{2}\rho c^2 \ . \tag{4.4}$$

Now I take a tiny fraction f of the matter and convert it to radiation. The energy density of matter is now reduced to

$$\varepsilon_m = (1-f)\rho c^2 \ . \tag{4.5}$$

The energy density of radiation is

$$\varepsilon_r = f\rho c^2 \tag{4.6}$$

and the pressure of the radiation is

$$P_r = \frac{1}{3}\varepsilon_r = \frac{f}{3}\rho c^2 \ . \tag{4.7}$$

The acceleration equation for a universe containing matter, radiation, and Λ is (from Eq. 4.49 of the text)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}[\varepsilon_m + \varepsilon_r + \varepsilon_\Lambda + 3P_r + 3P_\Lambda] \ . \tag{4.8}$$

Substituting in the appropriate energy densities and pressures for a slightly perturbed static universe, I find

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3c^2}\rho c\left[(1-f) + \frac{1}{2} + f + f - \frac{3}{2}\right] \\ &= -\frac{4\pi}{3}G\rho f \ .\end{aligned} \tag{4.9}$$

The positive pressure of the photons will cause the universe to contract.

[A possible additional question: Given an energy density $\varepsilon_m = \rho c^2$ in matter and an energy density ε_r in radiation, what value for Λ would be required to make the universe static? The answer is that you would need

$$\varepsilon_\Lambda = \varepsilon_r + \varepsilon_m/2 \ , \tag{4.10}$$

corresponding to

$$\Lambda = 8\pi G \left(\frac{\varepsilon_r}{c^2} + \frac{\rho}{2} \right) \ . \tag{4.11}$$

Thus, for any value of ε_m and ε_r, there is a value of Λ that produces a static universe.]

4.3 *If $\rho = 2.7 \times 10^{-27}\,\mathrm{kg\,m^{-3}}$, what is the radius of curvature R_0 of Einstein's static universe? How long would it take a photon to circumnavigate such a universe?*

From Eq. 4.73 of the text, the radius of curvature of Einstein's static universe is

$$R_0 = \frac{c}{(\pi G \rho)^{1/2}} \ . \tag{4.12}$$

For a density $\rho = 2.7 \times 10^{-27}\,\mathrm{kg\,m^{-3}}$, comparable to the mass density of the actual universe, the radius of curvature is

$$\begin{aligned} R_0 &= \frac{2.998 \times 10^8\,\mathrm{m\,s^{-1}}}{2(5.660 \times 10^{-37}\,\mathrm{s^{-2}})^{1/2}} \\ &= 1.99 \times 10^{26}\,\mathrm{m} = 6460\,\mathrm{Mpc} \ . \end{aligned} \tag{4.13}$$

This implies a circumference $C_0 = 2\pi R_0 = 40{,}600\,\mathrm{Mpc} = 1.32 \times 10^{11}\,\mathrm{ly}$. Thus, it would take 132 Gyr for a photon to circumnavigate the universe.

4.4 *Suppose that the universe were full of regulation baseballs, each of mass $m_{bb} = 0.145\,\mathrm{kg}$ and radius $r_{bb} = 0.0369\,\mathrm{m}$. If the baseballs were distributed uniformly throughout the universe, what number density of baseballs would be required to make the density equal to the critical density? (Assume nonrelativistic baseballs.) Given this density of baseballs, how far would you be able to see, on average, before your line of sight intersected a baseball? In fact, we can see galaxies at a distance $\sim c/H_0 \sim 4000\,\mathrm{Mpc}$; does the transparency of the universe on this length scale place useful limits on the number density of intergalactic baseballs? (Note to readers outside North America or Japan: feel free to substitute regulation cricket balls, with $m_{cr} = 0.160\,\mathrm{kg}$ and $r_{cr} = 0.0360\,\mathrm{m}$.)*

The number density of baseballs required to provide a critical density $\rho_{c,0}$ is

$$n_{bb,0} = \frac{\rho_{c,0}}{m_{bb}} = \frac{8.7 \times 10^{-27}\,\mathrm{kg\,m^{-3}}}{0.145\,\mathrm{kg}} = 6.0 \times 10^{-26}\,\mathrm{m^{-3}} \;, \tag{4.14}$$

or about 200 million baseballs per cubic astronomical unit. The average distance you could see before your line of sight intersected a baseball would then be (compare to Eq. 2.2 of the textbook)

$$\begin{aligned} \lambda_{bb} = \frac{1}{n_{bb}\pi r_{bb}^2} &= 3.90 \times 10^{27}\,\mathrm{m} \\ &= 29c/H_0 \;. \end{aligned} \tag{4.15}$$

As I mention in Section 4.2 of the text, the density of the universe lies within 0.5% of the critical density $\rho_{c,0}$. Even if nearly all that density were in the form of intergalactic baseballs, you would still have $\lambda_{bb} \approx 29c/H_0 \gg c/H_0$, and galaxies at a distance $\sim c/H_0$ would be readily visible. Thus, I would argue that the transparency of the universe on length scales $\sim c/H_0$ doesn't place a particularly useful upper limit on the number density of baseballs. [Skipping slightly ahead in the text, a tighter upper limit is placed by the fact that a regulation baseball must be made of baryonic matter; thus, we require that $\rho_{bb,0} \leq \Omega_{\mathrm{bary},0}\rho_{c,0} = 0.048\rho_{c,0}$. This leads to fewer than 10 million baseballs per cubic AU.]

If you want to generalize from baseballs (or cricket balls), you can have your students show that for opaque spheres of mass m and radius r contributing a density equal to the critical density $\rho_{c,0}$, the criterion $\lambda > c/H_0$ required for transparency translates to the requirement

$$\frac{m}{\pi r^2} > \frac{3H_0 c}{8\pi G} = 1.2\,\mathrm{kg\,m^{-2}} \;. \tag{4.16}$$

4.5 *The principle of wave-particle duality tells us that a particle with momentum p has an associated de Broglie wavelength of $\lambda = h/p$; this wavelength increases as $\lambda \propto a$ as the universe expands. The total energy density of a gas of particles can be written as $\varepsilon = nE$, where n is the number density of particles, and E is the energy per particle. For simplicity, let's assume that all the gas particles have the same mass m and momentum p. The energy per particle is then simply*

$$E = (m^2c^4 + p^2c^2)^{1/2} = (m^2c^4 + h^2c^2/\lambda^2)^{1/2} \;. \tag{4.17}$$

Compute the equation-of-state parameter w for this gas as a function of the scale factor a. Show that $w = \frac{1}{3}$ in the highly relativistic limit

($a \to 0$, $p \to \infty$) and that $w = 0$ in the highly nonrelativistic limit ($a \to \infty$, $p \to 0$).

The de Broglie wavelength increases as the universe expands, as $\lambda \equiv h/p = \lambda_0 a(t)$. Thus, the momentum of the particle decreases as $p = p_0/a(t)$. The number density of particles decreases as $n = n_0/a(t)^3$, assuming conservation of particle number. Thus, the dependence of the energy density on scale factor is, from equation (4.17) above,

$$\varepsilon(t) = n_0 a(t)^{-3} mc^2 (1 + x_0^2 a(t)^{-2})^{1/2} \ , \tag{4.18}$$

where $x_0 \equiv p_0/mc$ is a dimensionless momentum. From the continuity equation,

$$\dot{\varepsilon} = -3\frac{\dot{a}}{a}(\varepsilon + P) = -3\frac{\dot{a}}{a}\varepsilon\left(1 + \frac{P}{\varepsilon}\right) \ , \tag{4.19}$$

we can write

$$w = \frac{P}{\varepsilon} = -\frac{\dot{\varepsilon}/\varepsilon + 3\dot{a}/a}{3\dot{a}/a} \ . \tag{4.20}$$

Taking the logarithm of equation (4.18) above, we find

$$\ln\varepsilon = \ln n_0 - 3\ln a + \ln(mc^2) + \frac{1}{2}\ln(1 + x_0^2 a^{-2}) \ . \tag{4.21}$$

Taking the time derivative of each side yields

$$\frac{\dot{\varepsilon}}{\varepsilon} = -3\frac{\dot{a}}{a} - \frac{x_0^2 a^{-2}}{1 + x_0^2 a^{-2}}\frac{\dot{a}}{a} \ . \tag{4.22}$$

Substituting this back into the continuity equation, in the form of equation (4.20), we find that

$$w = \frac{1}{3}\frac{x_0^2 a^{-2}}{1 + x_0^2 a^{-2}} \ . \tag{4.23}$$

When $a \gg x_0$, corresponding to the nonrelativistic case $p = p_0/a \ll mc$, we find that

$$w \approx \frac{1}{3}\frac{x_0^2}{a^2} \approx \frac{1}{3}\frac{p^2}{m^2c^2} \ll \frac{1}{3} \ , \tag{4.24}$$

which goes to $w = 0$ in the limit $a \to \infty$. When $a \ll x_0$, corresponding to the relativistic case $p = p_0/a \gg mc$,

$$w \approx \frac{1}{3}\left[1 - \frac{a^2}{x_0^2}\right] \approx \frac{1}{3}\left[1 - \frac{m^2c^2}{p^2}\right] \ , \tag{4.25}$$

which goes to $w = 1/3$ in the limit $a \to 0$.

5
Model Universes

Chapter 5 in the second edition (Model Universes) is a slightly compressed combination of Chapters 5 and 6 in the first edition.

The “Benchmark Model,” discussed in detail in Section 5.5, has updated parameters. (I forget why I first adopted the term “Benchmark Model” rather than the other commonly used terms, “Concordance Model” or “Consensus Model”; perhaps I subconsciously wanted to indicate that there is still ample room for discord and dissent in cosmology.)

Exercises

5.1 *A light source in a flat, single-component universe has a redshift z when observed at a time t_0. Show that the observed redshift z changes at a rate*

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3(1+w)/2}. \tag{5.1}$$

For what values of w does the observed redshift increase with time?

Prefatory comment: This was the problem that triggered the most emails from readers of the first edition, and is likely to be a problem that your students find challenging. To show how the correct answer is derived, I will work it in detail, emphasizing the physics of the problem.

A photon is emitted at a time t_e, when the scale factor was $a(t_e)$; it is observed at a later time t_0, when the scale factor is $a(t_0)$. The redshift of the photon is

$$z = \frac{a(t_0)}{a(t_e)} - 1 \ . \tag{5.2}$$

A second photon is emitted at a time $t_e + dt_e$, when the scale factor is

$$a(t_e + dt_e) \approx a(t_e) + \left.\frac{da}{dt}\right|_{t_e} dt_e \approx a(t_e)\left[1 + H(t_e)dt_e\right] \ . \tag{5.3}$$

It is observed at a later time $t_0 + dt_0$, when the scale factor is

$$a(t_0 + dt_0) \approx a(t_0) + \left.\frac{da}{dt}\right|_{t_0} dt_0 \approx a(t_0)\left[1 + H(t_0)dt_0\right] \ . \tag{5.4}$$

The redshift of the second photon is

$$z + dz = \frac{a(t_0 + dt_0)}{a(t_e + dt_e)} - 1 \tag{5.5}$$

Substituting from equations (5.3) and (5.4), we can rewrite this redshift as

$$z + dz \approx \frac{a(t_0)}{a(t_e)} \frac{1 + H_0 dt_0}{1 + H(t_e)dt_e} - 1 \tag{5.6}$$

$$\approx \frac{a(t_0)}{a(t_e)} \left[1 + H_0 dt_0 - H(t_e)dt_e\right] - 1. \tag{5.7}$$

Substituting from equation (5.2) above, we can write the difference in redshift of the two photons as

$$dz \approx (1 + z)[H_0 dt_0 - H(t_e)dt_e] \ , \tag{5.8}$$

which leads to

$$\frac{dz}{dt_0} = H_0(1 + z)\left[1 - \frac{H(t_e)}{H_0}\frac{dt_e}{dt_0}\right] \ . \tag{5.9}$$

Note that the above equation is true for any universe described by a Robertson-Walker metric. (The most common error in solving this problem is omitting the second term in the square brackets of equation (5.9). As Heraclitus nearly said, you cannot observe the same photon twice. If you observe two photons separated by a time interval dt_0, in general they were emitted separated by a time interval $dt_e \neq 0$.)

In a flat, single-component universe, the Hubble parameter is (compare to Eq. 5.41 of the text)

$$H(t) = \frac{2}{3 + 3w} t^{-1} \ , \tag{5.10}$$

where $w \neq -1$ is the equation-of-state parameter. In a universe with $H(t) \propto t^{-1}$, equation (5.9) can be rewritten as

$$\frac{dz}{dt_0} = H_0(1 + z)\left[1 - \frac{t_0}{t_e}\frac{dt_e}{dt_0}\right] \ . \tag{5.11}$$

In a flat, single-component universe, the relation between t_e and t_0, expressed in terms of the observed z, is (see Eq. 5.48 of the text)

$$\frac{t_0}{t_e} = (1+z)^{(3+3w)/2} \ . \tag{5.12}$$

And what is the relation between dt_e and dt_0? If two photons are emitted in the same direction, separated by a time interval dt_e, the proper distance between them is initially cdt_e. However, by the time they are observed, the expansion of the universe has stretched the distance between them to an interval $cdt_e(1+z)$, and they will thus be observed separated by a longer time interval $dt_0 = dt_e(1+z)$. Therefore,

$$\frac{dt_e}{dt_0} = (1+z)^{-1} \ . \tag{5.13}$$

Finally, substituting equations (5.12) and (5.13) into equation (5.11), we find the desired result

$$\frac{dz}{dt_0} = H_0(1+z)[1-(1+z)^{(3+3w)/2-1}] \tag{5.14}$$

$$= H_0(1+z) - H_0(1+z)^{3(1+w)/2} \ . \tag{5.15}$$

The observed value of z increases with time if

$$(1+z)^{(1+3w)/2} > 1 \ . \tag{5.16}$$

For $z > 0$, this requires $(1+3w)/2 > 0$, or $w < -1/3$. In a universe with nothing but dark energy, the redshift of a distant light source increases with time as the observer and light source accelerate away from each other.

5.2 *Suppose you are in a flat, matter-only universe that has a Hubble constant $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. You observe a galaxy with $z = 1$. How long will you have to keep observing the galaxy to see its redshift change by one part in 10^6? [Hint: use the result from the previous problem.]*

A flat, matter-only universe has $w = 0$, and thus (from the previous problem)

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3/2} \ . \tag{5.17}$$

For a $z = 1$ galaxy, this means

$$\frac{dz}{dt_0} = H_0(2-2^{3/2}) = -0.828 H_0 \tag{5.18}$$

and thus the time dt_0 required to see the redshift change by an amount dz is

$$dt_0 = -1.207 H_0^{-1} dz \ . \tag{5.19}$$

For $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, the Hubble time is $H_0^{-1} = 14.38\,\mathrm{Gyr}$. The time required to see the redshift change by one part in 10^6 is

$$dt_0 = 1.207 \times 10^{-6}(14.38 \times 10^9\,\mathrm{yr}) = 17{,}400\,\mathrm{yr} \ . \tag{5.20}$$

Not exactly a weekend project.

5.3 *In a positively curved universe containing only matter ($\Omega_0 > 1$, $\kappa = +1$), show that the present age of the universe is given by the formula*

$$H_0 t_0 = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \cos^{-1}\left(\frac{2 - \Omega_0}{\Omega_0}\right) - \frac{1}{\Omega_0 - 1}. \tag{5.21}$$

Assuming $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, plot t_0 as a function of Ω_0 in the range $1 \le \Omega_0 \le 3$.

A positively curved, matter-only universe has a current age t_0 given by the relation (from Eq. 5.89 of the text)

$$H_0 t_0 = \Omega_0^{-1/2} \int_0^1 \frac{a^{1/2} da}{\left[1 - \frac{\Omega_0 - 1}{\Omega_0} a\right]^{1/2}} \ . \tag{5.22}$$

Being a grizzled elder of science, I will not resort to MATLAB (or a similar symbolic manipulation package) at this point, but will introduce the new variable of integration

$$x \equiv \left[1 - \frac{\Omega_0 - 1}{\Omega_0} a\right]^{1/2} \ . \tag{5.23}$$

With this new variable, equation (5.22) above can be rewritten as

$$H_0 t_0 = 2\Omega_0(\Omega_0 - 1)^{-3/2} \int_{\Omega_0^{-1}}^1 (1 - x^2)^{1/2} dx \ . \tag{5.24}$$

I then take down my well-thumbed table of indefinite integrals from the shelf, and find that

$$\int (1 - x^2)^{1/2} dx = \frac{1}{2}\left[x(1 - x^2)^{1/2} + \sin^{-1} x\right] \ . \tag{5.25}$$

Using this relation in equation (5.24) I find that

$$H_0 t_0 = \Omega_0(\Omega_0 - 1)^{-3/2}\left[\frac{\pi}{2} - \frac{1}{\Omega_0^{1/2}}\left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2} - \sin^{-1}\frac{1}{\Omega_0^{1/2}}\right] \ , \tag{5.26}$$

which simplifies to

$$H_0 t_0 = -\frac{1}{\Omega_0 - 1} + \Omega_0(\Omega_0 - 1)^{-3/2}\left[\frac{\pi}{2} - \sin^{-1}\frac{1}{\Omega_0^{1/2}}\right] . \tag{5.27}$$

Unfortunately, this isn't the form that was requested. I can, however, make use of the inverse trig identity

$$\frac{\pi}{2} - \sin^{-1} u = \cos^{-1} u \tag{5.28}$$

to simplify equation (5.27) to

$$H_0 t_0 = -\frac{1}{\Omega_0 - 1} + \Omega_0(\Omega_0 - 1)^{-3/2}\cos^{-1}\frac{1}{\Omega_0^{1/2}} . \tag{5.29}$$

Unfortunately, this *still* isn't the form that was requested. I can, however, dig once more into the big bag of inverse trig identities to pull out

$$\cos^{-1} u = \frac{1}{2}\cos^{-1}(2u^2 - 1) . \tag{5.30}$$

With $u = \Omega_0^{-1/2}$, this means

$$\cos^{-1}\frac{1}{\Omega_0^{1/2}} = \frac{1}{2}\cos^{-1}\left(\frac{2 - \Omega_0}{\Omega_0}\right) . \tag{5.31}$$

Substituting this back into equation (5.29), we get the sought-for result

$$H_0 t_0 = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}\cos^{-1}\left(\frac{2 - \Omega_0}{\Omega_0}\right) - \frac{1}{\Omega_0 - 1} . \tag{5.32}$$

The requested plot is shown in figure 5.1

5.4 *In a negatively curved universe containing only matter ($\Omega_0 < 1$, $\kappa = -1$), show that the present age of the universe is given by the formula*

$$H_0 t_0 = \frac{1}{1 - \Omega_0} - \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}}\cosh^{-1}\left(\frac{2 - \Omega_0}{\Omega_0}\right) . \tag{5.33}$$

Assuming $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, plot t_0 as a function of Ω_0 in the range $0 \le \Omega_0 \le 1$.

A negatively curved, matter-only universe has a current age t_0 given by the relation (from Eq. 5.89 of the text)

$$H_0 t_0 = \int_0^1 \frac{da}{[\Omega_0/a + (1 - \Omega_0)]^{1/2}} \tag{5.34}$$

$$= \Omega_0^{-1/2}\frac{a^{1/2}da}{[1 - \frac{1-\Omega_0}{\Omega_0}a]^{1/2}} . \tag{5.35}$$

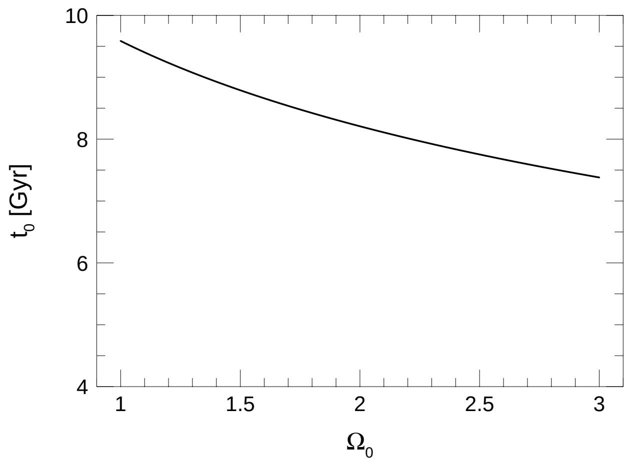

Figure 5.1 Age of universe versus Ω_0 for a positively curved, matter-only universe.

Taking a hint from the previous problem, I adopt a new variable of integration

$$y \equiv \left[1 - \frac{\Omega_0 - 1}{\Omega_0} a\right]^{1/2} \ . \tag{5.36}$$

With this new variable, equation (5.35) is rewritten as

$$H_0 t_0 = 2\Omega_0 (1 - \Omega_0)^{-3/2} \int_1^{\Omega_0^{-1/2}} (y^2 - 1)^{1/2} dy \ . \tag{5.37}$$

Taking down the crumbling papyrus containing the table of indefinite integrals, I find that

$$\int (y^2 - 1) dy = \frac{1}{2} \left[y(y^2 - 1)^{1/2} - \ln(y + (y^2 - 1)^{1/2}) \right] \tag{5.38}$$

$$= \frac{1}{2} \left[y(y^2 - 1)^{1/2} - \cosh^{-1} y \right] \ . \tag{5.39}$$

Using this relation in equation (5.37), I find that

$$H_0 t_0 = \Omega_0 (1 - \Omega_0)^{-3/2} \left[\frac{1}{\Omega_0} \left(\frac{1 - \Omega_0}{\Omega_0} \right)^{1/2} - \cosh^{-1} \frac{1}{\Omega_0^{1/2}} \right] \ , \tag{5.40}$$

which simplifies to

$$H_0 t_0 = \frac{1}{1 - \Omega_0} - \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} \cosh^{-1} \frac{1}{\Omega_0^{1/2}} \ . \tag{5.41}$$

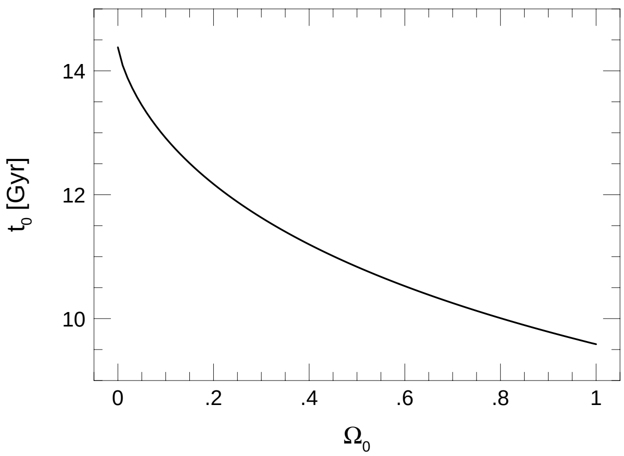

Figure 5.2 Age of universe versus Ω_0 for a negatively curved, matter-only universe.

Unfortunately, this isn't the form that was requested. However, the inverse cosh function follows the same relation as the inverse cos function [see equation (5.30) above]:

$$\cosh^{-1} u = \frac{1}{2}\cosh^{-1}(2u^2 - 1) \ . \tag{5.42}$$

With $u = \Omega_0^{-1/2}$, this means

$$\cosh^{-1}\frac{1}{\Omega_0^{1/2}} = \frac{1}{2}\cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right) \ . \tag{5.43}$$

Substituting this back into equation (5.41), we get the sought-for result

$$H_0 t_0 = \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right) \ . \tag{5.44}$$

The requested plot is shown in figure 5.2

5.5 *One speculation in cosmology is that the dark energy may take the form of "phantom energy" with an equation-of-state parameter $w < -1$. Suppose that the universe is spatially flat and contains matter with a density parameter $\Omega_{m,0}$, and phantom energy with a density parameter $\Omega_{p,0} = 1 - \Omega_{m,0}$ and equation-of-state parameter $w_p < -1$. At what scale factor a_{mp} are the energy density of phantom energy and matter equal? Write down the Friedmann equation for this universe in the limit that $a \gg a_{mp}$. Integrate the Friedmann equation to show that the scale factor a goes to infinity at a finite cosmic time $t_{\rm rip}$, given by the* *relation*

$$H_0(t_{\rm rip} - t_0) \approx \frac{2}{3|1+w_p|}(1-\Omega_{m,0})^{-1/2}. \tag{5.45}$$

This fate for the universe is called the "Big Rip." Current observations of our own universe are consistent with $H_0 = 68\,{\rm km\,s^{-1}\,Mpc^{-1}}$, $\Omega_{m,0} = 0.3$, and $w_p = -1.1$. If these numbers are correct, how long do we have remaining until the "Big Rip"?

Tne density of matter goes as

$$\varepsilon_m = \varepsilon_{m,0} a^{-3} \ . \tag{5.46}$$

The density of phantom energy goes as

$$\varepsilon_p = \varepsilon_{p,0} a^{-3-3w_p} \ . \tag{5.47}$$

(Thus, for $w < -1$, the energy density *increases* as the universe expands.) The scale factor a_{mp} of equality between matter and phantom energy is given by

$$\frac{\varepsilon_{m,0}}{\varepsilon_{p,0}} \frac{a_{mp}^{-3}}{a_{mp}^{-3-3w_p}} = \frac{\Omega_{m,0}}{1-\Omega_{m,0}}(a_{mp})^{3w_p} = 1 \ . \tag{5.48}$$

This leads to

$$a_{mp} = \left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/(3w_p)} \ . \tag{5.49}$$

The Friedmann equation for this universe is

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{1-\Omega_{m,0}}{a^{3+3w_p}} \ . \tag{5.50}$$

In the limit $a \gg a_{mp}$, the phantom energy term dominates on the right hand side, and the Friedmann equation reduces to

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2 \frac{1-\Omega_{m,0}}{a^{3+3w_p}} \tag{5.51}$$

or

$$\frac{da}{dt} = H_0(1-\Omega_{m,0})^{1/2} a^{-(1+3w_p)/2} \ . \tag{5.52}$$

Doing the integral,

$$H_0 \int_{t_0}^{t_{\rm rip}} dt = (1-\Omega_{m,0})^{-1/2} \int_1^\infty a^{(1+3w_p)/2} da \ , \tag{5.53}$$

we find that

$$H_0(t_{\rm rip} - t_0) = (1 - \Omega_{m,0})^{-1/2} \frac{2}{3|1 + w_p|} \tag{5.54}$$

when $w_p < -1$. For $H_0 = 14.38\,{\rm Gyr}$, $\Omega_{m,0} = 0.3$, and $w_p = -1.1$, the time until the Big Rip is

$$t_{\rm rip} - t_0 = 14.38\,{\rm Gyr} \frac{1}{\sqrt{0.7}} \frac{2}{0.3} = 115\,{\rm Gyr}\ . \tag{5.55}$$

The Big Rip will occur many Hubble times in the future, unless w_p is much less than -1. (For $\Omega_{m,0} \approx 0.3$, having $t_{\rm rip} - t_0 \approx 1/H_0$ requires $w_p \approx -1.8$.)

5.6 *Suppose you wanted to "pull an Einstein," and create a static universe ($\dot{a} = 0$, $\ddot{a} = 0$) in which the gravitational attraction of matter is exactly balanced by the gravitational repulsion of dark energy with equation-of-state parameter $-1/3 < w_q < -1$ and energy density ε_q. What is the necessary matter density (ε_m) required to produce a static universe, expressed in terms of ε_q and w_q? Will the curvature of this static universe be negative or positive? What will be its radius of curvature, expressed in terms of ε_q and w_q?*

Having a static universe containing dark energy and matter requires $\ddot{a} = 0$. From the acceleration equation (see Eq. 4.53 of the text), this implies

$$-\frac{4\pi G}{3c^2}(\varepsilon_m + \varepsilon_q + 3P_q) = 0\ , \tag{5.56}$$

or, since $P_q = w_q \varepsilon_q$,

$$\varepsilon_m + (1 + 3w_q)\varepsilon_q = 0\ . \tag{5.57}$$

Thus, $\ddot{a} = 0$ requires

$$\varepsilon_m = (-3w_q - 1)\varepsilon_q\ . \tag{5.58}$$

Since $\varepsilon_m > 0$ and $w_q < -1/3$, we require $\varepsilon_q > 0$ for a static universe.

Having a static universe also requires $\dot{a} = 0$. From the Friedmann equation (see Eq. 4.51 of the text), this implies

$$\frac{8\pi G}{3c^2}(\varepsilon_m + \varepsilon_q) - \frac{\kappa c^2}{R^2} = 0\ , \tag{5.59}$$

where R is the radius of curvature of the universe. Since $\varepsilon_m + \varepsilon_q > 0$, this implies $\kappa = +1$, and a *positively curved* universe. In terms of ε_q

and w_q, the radius of curvature is

$$R = \frac{c^2}{(8\pi G|w_q|\varepsilon_q)^{1/2}} \ . \tag{5.60}$$

5.7 *Consider a positively curved universe containing only matter (the "Big Crunch" model discussed in Section 5.4.1). At some time $t_0 > t_{\rm Crunch}/2$, during the contraction phase of this universe, an astronomer named Elbbuh Niwde discovers that nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance. He then measures H_0 and Ω_0, finding $H_0 < 0$ and $\Omega_0 > 1$. Given H_0 and Ω_0, how long a time will elapse between Dr. Niwde's observations at $t = t_0$ and the final Big Crunch at $t = t_{\rm crunch}$? What is the highest amplitude blueshift that Dr. Niwde is able to observe? What is the lookback time to an object with this blueshift?*

Since the contraction phase of the Big Crunch universe is the time reversal of the expansion, the time elapsed from Dr. Niwde's observation until $t_{\rm crunch}$ is equal to the time elapsed from $t = 0$ until the universe has a Hubble parameter equal in magnitude to that observed by Dr. Niwde (but opposite from sign). If the scale factor at the time of Dr. Niwde's observation is $a(t_0) = 1$, we can use the relation given in Eq. 5.89 of the text to find

$$|H_0|\,(t_{\rm crunch} - t_0) = \int_0^1 \frac{da}{[\Omega_0/a - (\Omega_0 - 1)]^{1/2}} \ . \tag{5.61}$$

Lifting the answer given in problem 5.3 above, we can write this as

$$|H_0|\,(t_{\rm crunch} - t_0) = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \cos^{-1}\left(\frac{2 - \Omega_0}{\Omega_0}\right) - \frac{1}{\Omega_0 - 1} \ . \tag{5.62}$$

The maximum blueshift that Dr. Niwde is able to observe corresponds to the moment of maximum expansion, when $a = a_{\rm max} > 1$. Eq. 5.88 of the text tells us that at maximum expansion of a matter-only Big Crunch universe,

$$a_{\rm max} = \frac{\Omega_0}{\Omega_0 - 1} \ . \tag{5.63}$$

Thus, the highest amplitude blueshift that Dr. Niwde could theoretically observe is

$$z = \frac{1}{a_{\rm max}} - 1 = \frac{\Omega_0 - 1}{\Omega_0} - 1 = -\frac{1}{\Omega_0} \ . \tag{5.64}$$

The lookback time to an object with $z = -1/\Omega_0$ is $t_0 - t_{\rm crunch}/2$. From

Eq. 5.92 of the text, the age of the universe at the time of maximum expansion is

$$t_{\rm crunch}/2 = \frac{\pi}{2}\,|H_0|^{-1}\,\frac{\Omega_0}{(\Omega_0-1)^{3/2}}\ . \tag{5.65}$$

Using equation (5.62) above to tell us the time from t_0 to the Big Crunch, we find that

$$\begin{aligned} t_0 - t_{\rm crunch}/2 &= t_{\rm crunch}/2 - (t_{\rm crunch} - t_0) \qquad (5.66)\\ &= |H_0|^{-1}\left(\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\left[\pi - \cos^{-1}(\frac{2-\Omega_0}{\Omega_0})\right] + \frac{1}{\Omega_0 - 1}\right)\ . \end{aligned}$$

Using the identity $\pi - \cos^{-1} u = \cos^{-1}(-u)$, we can rewrite the lookback time to the moment of maximum expansion in the simpler form

$$t_0 - t_{\rm crunch}/2 = |H_0|^{-1}\left[\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\cos^{-1}(\frac{\Omega_0-2}{\Omega_0}) + \frac{1}{\Omega_0-1}\right]\ . \tag{5.67}$$

(For instance, if Dr. Niwde measures $\Omega_0 = 2$, the lookback time to the most blueshifted galaxies (those with $z = -0.5$) is $\sim 2.57/|H_0|$,[1] the time remaining until the Big Crunch is $\sim 0.57/|H_0|$, and the time elapsed since the Big Bang is $\sim 5.71/|H_0|$.)

5.8 *Consider an expanding, positively curved universe containing only a cosmological constant ($\Omega_0 = \Omega_{\Lambda,0} > 1$). Show that such a universe underwent a "Big Bounce" at a scale factor*

$$a_{\rm bounce} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2}, \tag{5.68}$$

and that the scale factor as a function of time is

$$a(t) = a_{\rm bounce}\cosh[\sqrt{\Omega_0}H_0(t - t_{\rm bounce})], \tag{5.69}$$

where $t_{\rm bounce}$ is the time at which the Big Bounce occurred. What is the time $t_0 - t_{\rm bounce}$ that has elapsed since the Big Bounce, expressed as a function of H_0 and Ω_0?

If $\Omega_0 = \Omega_{\Lambda,0} > 1$, then the Friedmann equation is (see Eq. 5.81 of the text)

$$\frac{1}{H_0^2}\left(\frac{\dot a}{a}\right)^2 = \Omega_0 - \frac{\Omega_0 - 1}{a^2}\ . \tag{5.70}$$

[1] Perhaps I should say ~ 2.57 Elbbuh times.

At the time of the Big Bounce, $\dot{a} = 0$ and thus

$$a^2_{\rm bounce} = \frac{\Omega_0 - 1}{\Omega_0} \ , \tag{5.71}$$

or

$$a_{\rm bounce} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2} \ . \tag{5.72}$$

The Friedmann equation can be rewritten in terms of $a_{\rm bounce}$ as

$$\frac{1}{H_0^2}\left(\frac{da}{dt}\right)^2 = \Omega_0(a^2 - a^2_{\rm bounce}) \ , \tag{5.73}$$

or

$$\frac{1}{a_{\rm bounce}}\frac{da}{dt} = \sqrt{\Omega_0}H_0\left(a^2/a^2_{\rm bounce} - 1\right)^{1/2} \ . \tag{5.74}$$

Using the variable $x \equiv a/a_{\rm bounce}$, this can be written as the integral equation

$$\sqrt{\Omega_0}H_0\int_{t_{\rm bounce}}^{t} dt = \int_1^{a/a_{\rm bounce}} \frac{dx}{(x^2-1)^{1/2}} \ . \tag{5.75}$$

Hauling the moth-eaten table of integrals from the bottom drawer of my roll-top desk, I find that

$$\int \frac{dx}{(x^2-1)^{1/2}} = \ln(x + \sqrt{x^2-1}) = \cosh^{-1} x \ . \tag{5.76}$$

Thus, equation (5.75) can be integrated to find

$$\sqrt{\Omega_0}H_0(t - t_{\rm bounce}) = \cosh^{-1}(a/a_{\rm bounce}) \ , \tag{5.77}$$

or

$$a(t) = a_{\rm bounce}\cosh[\sqrt{\Omega_0}H_0(t - t_{\rm bounce})] \ . \tag{5.78}$$

At a time t_0 where $a(t_0) = 1$, the time elapsed since the Big Bounce is, from equation (5.77) above,

$$t_0 - t_{\rm bounce} = \frac{1}{\sqrt{\Omega_0}H_0}\cosh^{-1}(1/a_{\rm bounce}) \ . \tag{5.79}$$

Using the relation for $a_{\rm bounce}$ from equation (5.72) above,

$$t_0 - t_{\rm bounce} = \frac{1}{\sqrt{\Omega_0}H_0}\cosh^{-1}\left(\sqrt{\frac{\Omega_0}{\Omega_0 - 1}}\right) \ . \tag{5.80}$$

5.9 *A universe is spatially flat, and contains both matter and a cosmological constant. For what value of $\Omega_{m,0}$ is t_0 exactly equal to H_0^{-1}?*

If a universe has $\Omega_{m,0}$ in matter and $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$ in a cosmological constant, then the age of the universe, from Eq. 5.104 of the text, is

$$t_0 = H_0^{-1} \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\frac{\sqrt{1-\Omega_{m,0}}+1}{\sqrt{\Omega_{m,0}}}\right] \ . \tag{5.81}$$

Writing a snippet of computer code to step through values of $\Omega_{m,0}$, I find that

$$\ln\left[\frac{\sqrt{1-\Omega_{m,0}}+1}{\sqrt{\Omega_{m,0}}}\right] = \frac{3\sqrt{1-\Omega_{m,0}}}{2} \ , \tag{5.82}$$

and thus $t_0 = H_0^{-1}$, when $\Omega_{m,0} = 0.263$.

5.10 *In the Benchmark Model, what is the total mass of all the matter within our horizon? What is the total energy of all the photons within our horizon? How many baryons are within the horizon?*

In the Benchmark Model, the horizon distance today is (from Eq. 5.115 of the text)

$$d_{\rm hor}(t_0) = 3.20c/H_0 = 14{,}000\,{\rm Mpc} \ . \tag{5.83}$$

The proper volume inside the horizon today is then

$$V_{\rm hor} = \frac{4\pi}{3} d_{\rm hor}^3 = 137(c/H_0)^3 = 1.15 \times 10^{13}\,{\rm Mpc}^3 \ . \tag{5.84}$$

The mass density of matter today, assuming the Benchmark Model, is

$$\rho_{m,0} = \Omega_{m,0}\rho_{c,0} = 0.31(1.28\times10^{11}\,{\rm M}_\odot\,{\rm Mpc}^{-3}) = 3.97\times10^{10}\,{\rm M}_\odot\,{\rm Mpc}^{-3} \ . \tag{5.85}$$

The total mass of matter within the horizon is thus

$$M_{\rm hor} = \rho_{m,0} V_{\rm hor} = 4.6 \times 10^{23}\,{\rm M}_\odot \ . \tag{5.86}$$

(In Section 12.4 of the text, I quote this number as $4.3 \times 10^{23}\,{\rm M}_\odot$. Somewhere, I have been slightly careless with rounding.)

The energy density of CMB photons today is (from Eq. 2.34 of the text)

$$\varepsilon_\gamma = 4.175 \times 10^{-14}\,{\rm J\,m}^{-3} \left(\frac{3.086 \times 10^{22}\,{\rm m}}{1\,{\rm Mpc}}\right)^3 \tag{5.87}$$

$$= 1.227 \times 10^{54}\,{\rm J\,Mpc}^{-3} \ . \tag{5.88}$$

The total energy of CMB photons within the horizon is thus

$$E_{\rm hor} = \varepsilon_\gamma V_{\rm hor} = 1.41 \times 10^{67}\,{\rm J}\ . \tag{5.89}$$

Adding in all the starlight (direct and reprocessed by dust) that has accumulated over the history of the universe would boost this number by $\sim 10\%$, to $E_{\rm hor} \sim 1.6 \times 10^{67}\,{\rm J}$.

The number density of baryons today, making the approximation that all baryons are protons, is

$$\begin{aligned} n_{\rm bary} &= \frac{\Omega_{\rm bary,0}\varepsilon_{c,0}}{m_p c^2} \\ &= \frac{0.048(4870\,{\rm MeV\,m^{-3}})}{938.27\,{\rm MeV}} = 0.249\,{\rm m^{-3}} = 7.32 \times 10^{66}\,{\rm Mpc^{-3}}\ . \end{aligned} \tag{5.90}$$

The total number of baryons within the horizon is thus

$$N_{\rm hor} = n_{\rm bary} V_{\rm hor} = 8.4 \times 10^{79}\ . \tag{5.91}$$

6

Measuring cosmological parameters

Chapter 6 in the second edition corresponds to Chapter 7 in the first edition.

The basic discussion of standard candles and standard yardsticks is unchanged from the first edition. After some consideration, I decided to keep a short discussion of the deceleration parameter q_0, partly because of its historical importance, and partly because it is still a useful way of parameterizing $\ddot{a}$ at the present moment.

The data in Figure 6.5 are from the Union 2.1 compilation of the Supernova Cosmology Project. The results of Figure 6.6 are derived from the Joint Lightcurve Analysis (JLA) dataset.

Exercises

6.1 *Suppose that a polar bear's foot has a luminosity of $L = 10\,\mathrm{watts}$. What is the bolometric absolute magnitude of the bear's foot? What is the bolometric apparent magnitude of the foot at a luminosity distance of $d_L = 0.5\,\mathrm{km}$? If a bolometer can detect the bear's foot at a maximum luminosity distance of $d_L = 0.5\,\mathrm{km}$, what is the maximum luminosity distance at which it could detect the Sun? What is the maximum luminosity distance at which it could detect a supernova with $L = 4 \times 10^9\,\mathrm{L}_\odot$?*

The luminosity of the bear's foot is

$$L_{\mathrm{bf}} = 10\,\mathrm{W}\left(\frac{1\,\mathrm{L}_\odot}{3.828 \times 10^{26}\,\mathrm{W}}\right) = 2.612 \times 10^{-26}\,\mathrm{L}_\odot \ . \tag{6.1}$$

Its bolometric absolute magnitude is then (from Eq. 6.46 of the text)

$$M_{\mathrm{bf}} = -2.5\log_{10}\left(\frac{2.612 \times 10^{-26}\,\mathrm{L}_\odot}{78.7\,\mathrm{L}_\odot}\right) = 68.70 \ . \tag{6.2}$$

Suppose that the bear's foot is at a luminosity distance

$$d_L = 500\,\mathrm{m}\left(\frac{1\,\mathrm{Mpc}}{3.086\times 10^{22}\,\mathrm{m}}\right) = 1.620\times 10^{-20}\,\mathrm{Mpc}\ . \tag{6.3}$$

The bolometric apparent magnitude of the bear's foot is then (from Eq. 6.28 of the text)

$$m_{\mathrm{bf}} = M_{\mathrm{bf}} + 5\log_{10}\left(\frac{d_L}{1\,\mathrm{Mpc}}\right) + 25 = -5.26\ . \tag{6.4}$$

(Remember that in the apparent magnitude system, negative magnitudes correspond to a high flux.)

The Sun has a bolometric absolute magnitude $M_\odot = 4.74$. It has a bolometric apparent magnitude equal to $m_{\mathrm{bf}} = -5.26$ at a distance given by the relation (from Eq. 6.47 of the text)

$$\log_{10}\left(\frac{d_L}{10\,\mathrm{pc}}\right) = 0.2(m_{\mathrm{bf}} - M_\odot)\ , \tag{6.5}$$

leading to

$$d_L = 10\,\mathrm{pc}\left[10^{0.2(-5.26-4.74)}\right] = 0.10\,\mathrm{pc}\ . \tag{6.6}$$

Thus, a bolometer that can measure the heat from a polar bear's feet at a distance of half a kilometer can measure the heat from the Sun at a distance $\sim 20{,}000\,\mathrm{AU}$.

The bolometric absolute magnitude of the supernova is

$$M_{\mathrm{sn}} = -2.5\log_{10}\left(\frac{4\times 10^9\,\mathrm{L}_\odot}{78.7\,\mathrm{L}_\odot}\right) = -19.27\ . \tag{6.7}$$

It has a bolometric apparent magnitude equal to $m_{\mathrm{bf}} = -5.26$ at a luminosity distance

$$d_L = 10\,\mathrm{pc}\left[10^{0.2(-5.26+19.27)}\right] = 6300\,\mathrm{pc}\ . \tag{6.8}$$

Thus, a bolometer that can measure the heat from a polar bear's feet at a distance of half a kilometer can measure the heat from a supernova at a distance more than 3/4 of the distance to the galactic center.

6.2 *Suppose that a polar bear's foot has a diameter of $\ell = 0.16\,\mathrm{m}$. What is the angular size $\delta\theta$ of the foot at an angular-diameter distance of $d_A = 0.5\,\mathrm{km}$? In the Benchmark Model, what is the minimum possible angular size of the polar bear's foot?*

At an angular-diameter distance $d_A = 500\,\text{m}$, the bear's foot has an angular size

$$\delta\theta = \frac{\ell}{d_A} = \frac{0.16\,\text{m}}{500\,\text{m}} \tag{6.9}$$

$$= 3.2 \times 10^{-4}\,\text{radians} = 66\,\text{arcsec}\ , \tag{6.10}$$

barely at the limit of angular resolution of the human eye.

In the Benchmark Model, angular diameter is minimized, and angular-diameter distance is maximized, at a redshift $z_c = 1.6$, where $d_A(\text{max}) = 1770\,\text{Mpc} = 5.46 \times 10^{25}\,\text{m}$. A polar bear at a redshift $z = 1.6$ will have feet with an angular size

$$\delta\theta = \frac{\ell}{d_A(\text{max})} = \frac{0.16\,\text{m}}{5.46 \times 10^{25}\,\text{m}} \tag{6.11}$$

$$= 2.9 \times 10^{-27}\,\text{radians} = 6.0 \times 10^{-16}\,\text{microarcsec}\ . \tag{6.12}$$

Although observing a polar bear from an angular-diameter distance of 1770 Mpc is certainly safer than viewing it from an angular-diameter distance of half a kilometer,[1] its angular size at $d_A = 1770\,\text{Mpc}$ is inconveniently small.

6.3 *Suppose that you are in a spatially flat universe containing a single component with a unique equation-of-state parameter w. What are the current proper distance $d_P(t_0)$, the luminosity distance d_L and the angular-diameter distance d_A as a function of z and w? At what redshift will d_A have a maximum value? What will this maximum value be, in units of the Hubble distance?*

In a flat, single-component universe, the proper distance to a light source with redshift z is (from Eq. 5.50 of the text)

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1 - (1+z)^{-(1+3w)/2}\right] \tag{6.13}$$

[1] A polar bear can sprint at $v \approx 40\,\text{km}\,\text{hr}^{-1}$; if it can keep up this speed, it can finish a 500-meter dash in about 45 seconds.

if $w \neq -1/3$, and

$$d_p(t_0) = \frac{c}{H_0} \ln(1+z) \tag{6.14}$$

if $w = -1/3$. Since the universe is flat, the luminosity distance is

$$d_L = (1+z) d_p(t_0) \ , \tag{6.15}$$

corresponding to

$$d_L = \frac{c}{H_0} \frac{2}{1+3w} \left[(1+z) - (1+z)^{(1-3w)/2} \right] \tag{6.16}$$

if $w \neq -1/3$, and

$$d_L = \frac{c}{H_0} (1+z) \ln(1+z) \tag{6.17}$$

if $w = -1/3$. Since the universe is flat, the angular-diameter distance is

$$d_A = \frac{d_p(t_0)}{1+z} \ , \tag{6.18}$$

corresponding to

$$d_A = \frac{c}{H_0} \frac{2}{1+3w} \left[(1+z)^{-1} - (1+z)^{-(3+3w)/2} \right] \tag{6.19}$$

if $w \neq -1/3$, and

$$d_A = \frac{c}{H_0} \frac{\ln(1+z)}{1+z} \tag{6.20}$$

if $w = -1/3$. Taking the derivative of equations (6.19) and (6.20) above, it is found that d_A has an extremum (a maximum, in this case), at

$$1 + z_c = \left(\frac{3+3w}{2} \right)^{2/(1+3w)} \ . \tag{6.21}$$

if $w > -1$ and $w \neq -1/3$. When $w = -1/3$, then

$$1 + z_c = \mathrm{e} \approx 2.718 \ . \tag{6.22}$$

Substituting this critical redshift back into equations (6.19) and (6.20), we find that the maximum angular-diameter distance is

$$d_A(\mathrm{max}) = \frac{c}{H_0} \left(\frac{2}{3+3w} \right)^{(3+3w)/(1+3w)} \tag{6.23}$$

if $w > -1$ and $w \neq -1/3$. When $w = -1/3$, then

$$d_A(\mathrm{max}) = \frac{1}{\mathrm{e}} \frac{c}{H_0} \approx 0.368 \frac{c}{H_0} \ . \tag{6.24}$$

6.4 *Verify that Equation 6.51 is correct in the limit of small z. (You will probably want to use the relation $\log_{10}(1+x) \approx 0.4343\ln(1+x) \approx 0.4343x$ in the limit $|x| \ll 1$.)*

The distance modulus is related to the luminosity distance by the relation (from Eq. 6.49 of the text)

$$m - M = 5\log_{10}\left(\frac{d_L}{1\,\mathrm{Mpc}}\right) + 25\ . \tag{6.25}$$

In the limit $z \ll 1$, the luminosity distance can be approximated as (compare to Eq. 6.30 of the text)

$$d_L \approx \frac{c}{H_0} z \left(1 + \frac{1-q_0}{2} z\right)\ . \tag{6.26}$$

Substituting into equation (6.25) above, we have

$$m - M \approx 5\log_{10}\left(\frac{c/H_0}{1\,\mathrm{Mpc}}\right) + 5\log_{10} z + 5\log_{10}\left(1 + \frac{1-q_0}{2} z\right) + 25\ . \tag{6.27}$$

Since

$$\frac{c}{H_0} = 4409\,\mathrm{Mpc}\left(\frac{H_0}{68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right)^{-1}\ , \tag{6.28}$$

then

$$5\log_{10}\left(\frac{c/H_0}{1\,\mathrm{Mpc}}\right) = 18.221 - 5\log_{10}\left(\frac{H_0}{68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right)\ . \tag{6.29}$$

Since

$$\log_{10}(1+x) \approx 0.4343\ln(1+x) \approx 0.4343x \tag{6.30}$$

when $x \ll 1$, we can write

$$5\log_{10}\left(1 + \frac{1-q_0}{2} z\right) \approx 2.1715\left(\frac{1-q_0}{2} z\right) \approx 1.086(1-q_0)z \tag{6.31}$$

when $(1-q_0)z \ll 2$. Substituting the results of equation (6.29) and (6.31) into equation (6.27) above, I reach the result

$$m - M \approx 43.221 - 5\log_{10}\left(\frac{H_0}{68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right) + 5\log_{10} z + 1.086(1-q_0)z\ . \tag{6.32}$$

Comparing to Eq. 6.51 of the text, I find that I made a rounding error somewhere that resulted in my rounding up to 43.23 magnitudes rather than down to 43.22 magnitudes.

6.5 *The surface brightness Σ of an astronomical object is its observed flux divided by its observed angular area; thus, $\Sigma \propto f/(\delta\theta)^2$. For a class of objects that are both standard candles and standard yardsticks, what is Σ as a function of redshift? Would observing the surface brightness of this class of objects be a useful way of determining the value of the deceleration parameter q_0? Why or why not?*

For an object of known luminosity L at a redshift z, the observed flux is (Eq. 6.27 of the text)

$$f = \frac{L}{4\pi S_\kappa(r)^2(1+z)^2} \ . \tag{6.33}$$

For an object of known physical diameter ℓ at a redshift z, the observed angular size is (Eq. 6.34 of the text)

$$\delta\theta = \frac{\ell(1+z)}{S_\kappa(r)} \ . \tag{6.34}$$

The two above relations hold true for any expanding (or contracting) universe described by a Robertson-Walker metric. In such a universe, the surface brightness of an object at redshift z is

$$\Sigma \propto \frac{f}{(\delta\theta)^2} \propto \frac{L}{4\pi S_\kappa(r)^2(1+z)^2} \cdot \frac{S_\kappa(r)^2}{\ell^2(1+z)^2} \tag{6.35}$$

$$\propto \frac{L}{\ell^2}(1+z)^{-4} \ . \tag{6.36}$$

Note that this result is independent of the spatial curvature of the universe, and is independent of the acceleration or deceleration of the universe. As such, it is *not* a useful way of determining the deceleration constant q_0 (nor is it useful for determining the curvature constant κ).

Observing whether $\Sigma \propto (1+z)^{-4}$ is called the "Tolman surface brightness test." It was proposed by Richard Tolman in 1930 as a test of the expansion of the universe that was independent of pesky complicating factors like curvature and cosmic acceleration. In an expanding universe, observed surface brightness depends purely on the growth of the scale factor between the time of emission and observation: $a(t_e)^4/a(t_0)^4 = (1+z)^{-4}$.

6.6 *You observe a quasar at a redshift $z = 5.0$, and determine that the observed flux of light from the quasar varies on a timescale $\delta t_0 = 3\,\text{days}$. If the observed variation in flux is due to a variation in the intrinsic luminosity of the quasar, what was the variation timescale δt_e at the time the light was emitted? For the light from the quasar to vary on a timescale δt_e, the bulk of the light must come from a region of physical* *size $R \leq R_{\max} = c(\delta t_e)$. What is $R_{\max}$ for the observed quasar? What is the angular size of $R_{\max}$ in the Benchmark Model?*

Observed variations on a timescale $\delta t_0 = 3\,\text{days}$ imply that the variation at the time of emission was

$$\delta t_e = \frac{\delta t_0}{1+z} = \frac{3\,\text{days}}{6} = 0.5\,\text{day} = 43{,}200\,\text{s} \ . \tag{6.37}$$

The maximum size of the region emitting the light is then

$$R_{\max} = c(\delta t_e) = 2.998 \times 10^8\,\text{m}\,\text{s}^{-1}(43{,}200\,\text{s}) \tag{6.38}$$

$$= 1.295 \times 10^{13}\,\text{m} = 86.6\,\text{AU} = 4.20 \times 10^{-10}\,\text{Mpc} \ . \tag{6.39}$$

This is comparable in size to the orbit of the dwarf planet Haumea. From Fig. 6.4 of the text, I judge the angular-diameter distance to an object with $z = 5$ to be

$$d_A = 0.30c/H_0 = 1310\,\text{Mpc} \tag{6.40}$$

in the Benchmark Model. The angular size subtended by $R_{\max}$ is then

$$\delta\theta = \frac{R_{\max}}{d_A} = \frac{4.20 \times 10^{-10}\,\text{Mpc}}{1310\,\text{Mpc}} \tag{6.41}$$

$$= 3.2 \times 10^{-13}\,\text{radian} = 0.066\,\text{microarcsec} \ . \tag{6.42}$$

Although this angle is gargantuan compared to the minimum angular size of a polar bear's foot (see problem 6.2 above), it is still "challenging" to observe, to use a favorite euphemism of astronomers.

6.7 *Derive the relation $A_p(t_0) = 4\pi S_\kappa(r)^2$, as given in Equation 6.24, starting from the Robertson–Walker metric of Equation 6.22.*

The Robertson–Walker metric states that

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + S_\kappa(r)^2 d\Omega^2] \ , \tag{6.43}$$

where

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 \ . \tag{6.44}$$

Now, let's fix the time at $t = t_0$, when $a(t) = a(t_0) = 1$. At this fixed time, the spatial metric is then

$$d\ell^2 = dr^2 + S_\kappa(r)^2[d\theta^2 + \sin^2\theta d\phi^2] \ . \tag{6.45}$$

Next, consider a sphere of proper radius r centered on the origin. The proper distance between two points on the sphere, at positions (r, θ, ϕ) and $(r, \theta + d\theta, \phi + d\phi)$, is then

$$d\ell^2 = S_\kappa(r)^2[d\theta^2 + \sin^2\theta d\phi^2] \ . \tag{6.46}$$

This metric looks familiar. Compare it to the metric on the surface of a sphere of radius R (Eq. 3.26 of the text):

$$d\ell^2 = (dr')^2 + R^2 \sin^2(r'/R)(d\theta')^2 \ , \tag{6.47}$$

where r' is the distance from the north pole, measured along a great circle, and θ' is the longitude, measured from a prime meridian. If we chose the new variable $\psi' \equiv r'/R$, which runs from 0 at the north pole to π at the south pole, the metric on a sphere can be rewritten as

$$d\ell^2 = R^2[(d\psi')^2 + \sin^2\psi'(d\theta')^2] \ . \tag{6.48}$$

Comparison of equation (6.46) with equation (6.48) reveals that the metric of a sphere of proper radius r in a 4-d universe described by a Robertson–Walker metric is *identical* to that of a 2-d spherical surface with radius $R = S_\kappa(r)$. Thus, since the 2-d spherical surface has an area $A = 4\pi R^2$, the sphere of proper radius r has an area

$$A_p(t_0) = 4\pi S_\kappa(r)^2 \ . \tag{6.49}$$

6.8 *A spatially flat universe contains a single component with equation-of-state parameter w. In this universe, standard candles of luminosity L are distributed homogeneously in space. The number density of the standard candles is n_0 at $t = t_0$, and the standard candles are neither created nor destroyed. Show that the observed flux from a single standard candle at redshift z is*

$$f(z) = \frac{L(1+3w)^2}{16\pi(c/H_0)^2}\frac{1}{(1+z)^2}\left[1-(1+z)^{-(1+3w)/2}\right]^{-2} \tag{6.50}$$

when $w \neq -\frac{1}{3}$. What is the corresponding relation when $w = -\frac{1}{3}$? Show that the observed intensity (that is, the power per unit area per steradian of sky) from standard candles with redshifts in the range $z \to z + dz$ is

$$dJ(z) = \frac{n_0 L(c/H_0)}{4\pi}(1+z)^{-(7+3w)/2}dz. \tag{6.51}$$

What will be the total intensity J of all standard candles integrated over all redshifts? Explain why the night sky is of finite brightness even in universes with $w \leq -\frac{1}{3}$, which have an infinite horizon distance.

The flux from a single standard candle of luminosity L is

$$f = \frac{L}{4\pi d_L^2} \ . \tag{6.52}$$

In a flat universe, the luminosity distance to a light source with measured redshift z is (Eq. 6.29 of the text)

$$d_L = (1+z)d_p(t_0) \ . \tag{6.53}$$

In a flat, single-component universe, the proper distance to the light source at the time of observation is (Eq. 5.50 of the text)

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1-(1+z)^{-(1+3w)/2}\right] , \tag{6.54}$$

when $w \neq -1/3$. Combining the three equations above, the flux of the standard candle is found to be

$$f = \frac{L}{16\pi}\frac{(1+3w)^2}{(c/H_0)^2}(1+z)^{-2}\left[1-(1+z)^{-(1+3w)/2}\right]^{-2} \tag{6.55}$$

when $w \neq -1/3$. In the special case that $w = -1/3$,

$$d_p(t_0) = \frac{c}{H_0}\ln(1+z) \ , \tag{6.56}$$

and the flux of the standard candle is thus

$$f = \frac{L}{4\pi}\frac{1}{(c/H_0)^2}\left[(1+z)\ln(1+z)\right]^{-2} \ . \qquad \left[w = -\frac{1}{3}\right] \tag{6.57}$$

A standard candle with observed redshift z is currently at a proper distance (see equation (6.54) above)

$$r \equiv d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1-(1+z)^{-(1+3w)/2}\right] \ . \tag{6.58}$$

Thus, standard candles with redshifts in the range $z \to z+dz$ are currently at proper distances in the range $r \to r+dr$, where

$$dr = \frac{dr}{dz}dz = \frac{c}{H_0}(1+z)^{-(3+3w)/2}dz \ . \tag{6.59}$$

Thus, the total number of standard candles in the redshift range $z \to z+dz$ is

$$\begin{aligned} dN &= n_0 \cdot 4\pi r^2 \frac{dr}{dz}dz & (6.60) \\ &= 4\pi n_0 d_p(t_0)^2 \cdot \frac{c}{H_0}(1+z)^{-(3+3w)/2}dz \ , & (6.61) \end{aligned}$$

where n_0 is the current number density of standard candles. The intensity (flux per steradian) of standard candles in the redshift range

$z \to z + dz$ is then

$$dJ = \frac{dN \cdot f}{4\pi} = n_0 d_p(t_0)^2 \frac{c}{H_0}(1+z)^{-(3+3w)/2} \cdot \frac{L}{4\pi} \frac{1}{d_p(t_0)^2(1+z)^2}$$

$$= \frac{n_0 L(c/H_0)}{4\pi}(1+z)^{-(7+3w)/2} \; . \tag{6.62}$$

Integrated over all redshifts, the total intensity provided by the standard candles is

$$J = \frac{n_0 L(c/H_0)}{4\pi} \int_0^\infty u^{-(7+3w)/2} du \tag{6.63}$$

$$= \frac{n_0 L(c/H_0)}{2\pi(5+3w)} \tag{6.64}$$

when $w > -5/3$.

The horizon distance in a flat, single-component universe is

$$d_{\rm hor} = \int_0^\infty \frac{dr}{dz} dz \propto \int_0^\infty (1+z)^{-(3+3w)/2} dz \tag{6.65}$$

$$\propto (1+z)^{-(1+3w)/2} \Big|_0^\infty \; . \tag{6.66}$$

This diverges in the limit $z \to \infty$ if $w < -1/3$. However, calculating the intensity J requires introducing an additional factor of $(1+z)^{-2}$ from the flux for each standard candle. Thus, the integral for J is

$$J \propto \int_0^\infty \frac{dr}{dz}(1+z)^{-2} dz \propto \int_0^\infty (1+z)^{-(7+3w)/2} dz \tag{6.67}$$

$$\propto (1+z)^{-(5+3w)/2} \Big|_0^\infty \; . \tag{6.68}$$

This diverges in the limit $z \to \infty$ if $w < -5/3$. Thus, if $-5/3 < w < -1/3$, the night sky has finite intensity despite the fact that the horizon distance is infinite (and thus an infinite number of immortal standard candles are within reach of our telescope). This is because the expansion of the universe redshifts the standard candles to a low enough flux that their integrated intensity is still finite.

6.9 *In the Benchmark Model, at what scale factor a did $\ddot{a} = 0$? [This represents the moment when expansion switched from slowing down ($\ddot{a} < 0$) to speeding up ($\ddot{a} > 0$).] Is this scale factor larger or smaller than the scale factor $a_{m\Lambda}$ at which the energy density of matter equaled the energy density of the cosmological constant?*

In the Benchmark Model, the switch from deceleration to positive acceleration occurred long after the radiation-dominated era. Thus, I can approximate the universe as containing only matter ($\Omega_{m,0} = 0.31$)

and a cosmological constant ($\Omega_{\Lambda,0} = 0.69$). The acceleration equation is then

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}[\varepsilon_m + \varepsilon_\Lambda + 3P_\Lambda] = -\frac{4\pi G}{3c^2}[\varepsilon_m - 2\varepsilon_\Lambda] \ . \tag{6.69}$$

Thus, $\ddot{a} = 0$ when $\varepsilon_m = 2\varepsilon_\Lambda$ (not at the time of matter-Λ equality, when $\varepsilon_m = \varepsilon_\Lambda$). Since $\varepsilon_m = \Omega_{m,0}\varepsilon_{c,0}a^{-3}$ and $\varepsilon_\Lambda = \Omega_{\Lambda,0}\varepsilon_{c,0}$, we find that the "switchover" scale factor $a_{\rm sw}$ at which $\ddot{a} = 0$ is given by the relation

$$\Omega_{m,0}\varepsilon_{c,0}a_{\rm sw}^{-3} = \Omega_{\Lambda,0}\varepsilon_{c,0} \ , \tag{6.70}$$

or

$$a_{\rm sw} = \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3} = 0.608 \ , \tag{6.71}$$

corresponding to a redshift $z_{\rm sw} = 1/a_{\rm sw} - 1 = 0.645$. Since

$$a_{\rm sw} = \frac{1}{2^{1/3}}a_{m\Lambda} = 0.794a_{m\Lambda} \ , \tag{6.72}$$

the universe starts to accelerate ($\ddot{a} > 0$) before the time of matter-Λ equality at $a_{m\Lambda} = 0.766$, $z_{m\Lambda} = 0.306$.

7
Dark matter

Chapter 7 in the second edition corresponds to Chapter 8 in the first edition.

Section 7.1 (Visible Matter) now contains a discussion of the initial mass function of stars. In part, this information is included to emphasize the difficulty of deriving the mass density of stars from their luminosity density. However, it will also turn out to be useful in Chapter 12, when discussing reionization and star formation.

The luminosity density $\Psi_V = 1.1 \times 10^8 L_{\odot,V}\,\mathrm{Mpc}^{-3}$ quoted in Equation 7.1 is drawn from Table 10 of Blanton et al. (2003).

On page 126, I quote a mass-to-light ratio of $M/L_V \approx 0.3\mathrm{M}_\odot/\mathrm{L}_{\odot,V}$ for star-forming galaxies and $M/L_V \approx 8\mathrm{M}_\odot/\mathrm{L}_{\odot,V}$ for quiescent galaxies; here, I am using the color – M/L relations of Inlo & Portinari (2013).

For the relative sizes of the disks of the Milky Way and M31, quoted in Section 7.2, I draw on Yin et al. (2009). For the Milky Way Galaxy, I adopt a circular speed $v = 235\,\mathrm{km\,s^{-1}}$ and a distance $R = 8.2\,\mathrm{kpc}$ to the galactic center, following the review article of Bland-Hawthorn & Gerhard (2016).

At the end of Section 7.2, I emphasize the uncertainty in the extent of our galaxy's dark halo; although numerical simulations seem to be generally consistent with a total mass $M \sim 1.5 \times 10^{12}\,\mathrm{M}_\odot$ for our galaxy (implying $M/L_V \sim 75\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$), this is a subject that I should probably revisit in the third edition.

Section 7.4 (Gravitational Lensing) now emphasizes the strict upper limit placed on the contribution of MACHOs to the dark halo of our galaxy.

Exercises

7.1 *Suppose it were suggested that black holes of mass* $10^{-8}\,\mathrm{M}_\odot$ *made up all the dark matter in the halo of our galaxy. How far away would you expect the nearest such black hole to be? How frequently would you* *expect such a black hole to pass within 1 AU of the Sun? (An order-of-magnitude estimate is sufficient.)*

Suppose it were suggested that MACHOs of mass $10^{-3}\,\mathrm{M}_\odot$ *(about the mass of Jupiter) made up all the dark matter in the halo of our galaxy. How far away would you expect the nearest MACHO to be? How frequently would such a MACHO pass within 1 AU of the Sun? (Again, an order-of-magnitude estimate will suffice.)*

Since I used the magic phrase "order-of-magnitude estimate," I have liberated myself to use a series of approximations in this calculation. First, I approximate the dark halo as being spherically symmetric, with a mass (from Eq. 7.12 of the text)

$$M(r) = \frac{v^2 r}{G} \ , \tag{7.1}$$

where v is the orbital velocity of a test mass on a circular orbit. The density is then

$$\rho(r) = \frac{1}{4\pi r^2}\frac{dM}{dr} = \frac{v^2}{4\pi G r^2} \ , \tag{7.2}$$

where I have made the further approximation that v is constant at our location. If the dark halo is made entirely of black holes with mass $m_{\mathrm{bh}} = 10^{-8}\,\mathrm{M}_\odot = 1.53\,\mathrm{M}_{\mathrm{pluto}}$, then the number density of black holes is

$$n_{\mathrm{bh}} = \frac{\rho}{m_{\mathrm{bh}}} = \frac{v^2}{4\pi G m_{\mathrm{bh}} r^2} \ . \tag{7.3}$$

At a distance $r = 8.2\,\mathrm{kpc}$ from the galactic center, this translates to a number density

$$n_{\mathrm{bh}} = 5.17 \times 10^{-44}\,\mathrm{m}^{-3} \left(\frac{v}{235\,\mathrm{km\,s^{-1}}}\right)^2 \left(\frac{m_{\mathrm{bh}}}{10^{-8}\,\mathrm{M}_\odot}\right)^{-1} \ . \tag{7.4}$$

If the black holes are randomly strewn about, the nearest to the solar system is likely to be at a distance

$$\begin{aligned} d_{\mathrm{bh}} \sim n_{\mathrm{bh}}^{-1/3} &= 2.7 \times 10^{14}\,\mathrm{m} \left(\frac{v}{235\,\mathrm{km\,s^{-1}}}\right)^{-2/3} \left(\frac{m_{\mathrm{bh}}}{10^{-8}\,\mathrm{M}_\odot}\right)^{1/3} \\ &= 1800\,\mathrm{AU} \left(\frac{v}{235\,\mathrm{km\,s^{-1}}}\right)^{-2/3} \left(\frac{m_{\mathrm{bh}}}{10^{-8}\,\mathrm{M}_\odot}\right)^{1/3} \ . \end{aligned} \tag{7.5}$$

To see how often such a black hole comes within a distance ℓ of the Sun, consider what happens as the Sun moves along its orbit about the galactic center with an orbital speed $v \approx 235\,\mathrm{km\,s^{-1}}$. During a time t, a circle of radius ℓ, centered on the Sun and oriented perpendicular to the Sun's orbital motion, sweeps out a cylinder with volume

$$V = \pi \ell^2 \cdot vt \ . \tag{7.6}$$

The typical time between black hole encounters is the time it takes for this volume to contain one black hole; that is, the time for which

$$n_{\rm bh} V = n_{\rm bh} \pi \ell^2 vt = 1 \ . \tag{7.7}$$

The time t between encounters is thus

$$t = \frac{1}{n_{\rm bh} \pi \ell^2 v} = \frac{4 G m_{\rm bh} r^2}{\ell^2 v^3} \ , \tag{7.8}$$

substituting for $n_{\rm bh}$ from equation (7.3) above. At a distance $r = 8.2\,{\rm kpc}$ from the galactic center, this becomes

$$t = 37\,{\rm Myr} \left(\frac{v}{235\,{\rm km\,s^{-3}}} \right)^{-3} \left(\frac{m_{\rm bh}}{10^{-8}\,{\rm M_\odot}} \right) \left(\frac{\ell}{1\,{\rm AU}} \right)^{-2} \ . \tag{7.9}$$

Thus, if the dark halo is made of black holes comparable in mass to a dwarf planet, the nearest black hole is probably $\sim 2000\,{\rm AU}$ away, and we find a black hole within 1 AU of the Sun every $\sim 40\,{\rm Myr}$.

If the dark halo is made of MACHOs with $m_{\rm macho} = 10^{-3}\,{\rm M_\odot} = 1.05\,{\rm M_{jupiter}}$, then we can substitute into equation (7.5) above to find that the nearest MACHO is probably at a distance

$$d_{\rm macho} \sim n_{\rm macho}^{-1/3} = 83{,}000\,{\rm AU} \left(\frac{v}{235\,{\rm km\,s^{-1}}} \right)^{-2/3} \left(\frac{m_{\rm macho}}{10^{-3}\,{\rm M_\odot}} \right)^{1/3} \ . \tag{7.10}$$

We can then substitute into equation (7.9) above to find that the time between close encounters with a MACHO is

$$t = 3700\,{\rm Gyr} \left(\frac{v}{235\,{\rm km\,s^{-1}}} \right)^{-3} \left(\frac{m_{\rm macho}}{10^{-3}\,{\rm M_\odot}} \right) \left(\frac{\ell}{1\,{\rm AU}} \right)^{-2} \ . \tag{7.11}$$

Thus, if the dark halo is made of MACHOs comparable in mass to Jupiter, the nearest MACHO is probably $\sim 0.4\,{\rm pc}$ away (about a third the distance to Proxima Centauri), and there is one chance in 800 that a MACHO has passed within 1 AU of the Sun during the existence of the solar system.

7.2 *The Draco galaxy is a dwarf galaxy within the Local Group. Its luminosity is $L = (1.8 \pm 0.8) \times 10^5\,{\rm L_\odot}$ and half its total luminosity is contained within a sphere of radius $r_h = 120 \pm 12\,{\rm pc}$. The red giant stars in the Draco galaxy are bright enough to have their line-of-sight* *velocities measured. The measured velocity dispersion of the red giant stars in the Draco galaxy is $\sigma_r = 10.5 \pm 2.2\,\mathrm{km\,s^{-1}}$. What is the mass of the Draco galaxy? What is its mass-to-light ratio? Describe the possible sources of error in your mass estimate of this galaxy.*

Note: this problem is suitable for students who have a little background in propagation of uncertainty. For instance, I assume that the 21% uncertainty in σ_r corresponds to a 42% uncertainty in σ_r^2. Since the 10% uncertainty in r_h should be uncorrelated with the uncertainty in σ_r, the estimated mass $M \propto \sigma_r^2/r_h$ has a 43% uncertainty, since uncorrelated random errors add in quadrature.

If I assume that the velocity dispersion of the red giants in the Draco galaxy is isotropic, then their mean square velocity is

$$\langle v^2 \rangle = 3\sigma_r^2 \ . \tag{7.12}$$

where $\sigma_r = (1.05 \pm 0.22) \times 10^4\,\mathrm{m\,s^{-1}}$ is the line-of-sight velocity dispersion of the red giants. The mean square velocity is then

$$\langle v^2 \rangle = 3[(1.05 \pm 0.22) \times 10^4\,\mathrm{m\,s^{-1}}]^2 = (3.31 \pm 0.69) \times 10^8\,\mathrm{m^2\,s^{-2}} \ . \tag{7.13}$$

The half-mass radius of the Draco galaxy is $r_h = 120 \pm 12\,\mathrm{pc} = (3.70 \pm 0.37) \times 10^{18}\,\mathrm{m}$. Using the virial equation as shown in Eq. 7.37 of the text, I find

$$\begin{aligned} M_{\rm draco} &= \frac{\langle v^2 \rangle r_h}{\alpha G} = \frac{[(3.31 \pm 0.69) \times 10^8\,\mathrm{m^2\,s^{-2}}][(3.70 \pm 0.37) \times 10^{18}\,\mathrm{m}]}{(0.45)(6.67 \times 10^{-11}\,\mathrm{m^2\,s^{-2}\,kg^{-1}})} \\ &= (4.1 \pm 1.8) \times 10^{37}\,\mathrm{kg} = (2.05 \pm 0.88) \times 10^7\,\mathrm{M_\odot} \ . \end{aligned} \tag{7.14}$$

Combined with a luminosity $L_{\rm draco} = (1.8 \pm 0.8) \times 10^5\,\mathrm{L_\odot}$, this implies a mass-to-light ratio

$$\left(\frac{M}{L}\right)_{\rm draco} = \frac{(2.05 \pm 0.88) \times 10^7\,\mathrm{M_\odot}}{(1.8 \pm 0.8) \times 10^5\,\mathrm{L_\odot}} = 110 \pm 70\,\mathrm{M_\odot/\,L_\odot} \ . \tag{7.15}$$

In estimating the uncertainty of the mass-to-light ratio, I assumed that the uncertainty in M and L were uncorrelated. This is not strictly true: the uncertainties in r_h and L are correlated, since (for instance) overestimating the distance to Draco will increase your estimated value of both r_h and L. However, the uncertainty in M is contributed mainly by σ_r, which doesn't give a hoot about the distance to Draco.

In addition to the uncertainty in the measured values of σ_r and r_h, there are additional sources of error in the computed virial mass of Draco. These include all the assumptions that go into the virial theorem: that the Draco galaxy is self-gravitating (with no tidal effects from neighboring galaxies) and that it is in a steady state (not expanding or contracting). These also include the additional assumptions that go into the virial mass estimate of equation (7.14): that the velocity dispersion is isotropic, that the half-light radius r_h is equal to the half-mass radius, and that the numerical factor $\alpha = 0.45$ is the same for dwarf galaxies as for clusters of galaxies. (I tell my students that the virial theorem is like a chainsaw: it's a powerful tool, but you shouldn't wield it blindly.)

7.3 *A light ray just grazes the surface of the Earth ($M = 6.0 \times 10^{24}$ kg, $R = 6.4 \times 10^6$ m). Through what angle α is the light ray bent by gravitational lensing? (Ignore the refractive effects of the Earth's atmosphere.) Repeat your calculation for a white dwarf ($M = 2.0 \times 10^{30}$ kg, $R = 1.5 \times 10^7$ m) and for a neutron star ($M = 3.0 \times 10^{30}$ kg, $R = 1.2 \times 10^4$ m).*

From Eqs. 7.43 and 7.44 of the text, when a light ray just grazes an object of mass M and radius R, it is deflected through an angle

$$\alpha = \frac{4GM}{c^2 R} = 1.751\,\text{arcsec}\left(\frac{M}{1\,\mathrm{M}_\odot}\right)\left(\frac{R}{1\,\mathrm{R}_\odot}\right)^{-1} . \tag{7.16}$$

For the Earth, which has mass $M_\oplus = 5.972 \times 10^{24}\,\text{kg} = 3.00 \times 10^{-6}\,\mathrm{M}_\odot$ and radius $R_\oplus = 6371\,\text{km} = 9.16 \times 10^{-3}\,\mathrm{R}_\odot$, the deflection angle is

$$\alpha_\oplus = 1.751\,\text{arcsec}\frac{3.00 \times 10^{-6}}{9.16 \times 10^{-3}} = 5.7 \times 10^{-4}\,\text{arcsec} . \tag{7.17}$$

For a white dwarf with mass $M_{\text{wd}} = 2.0 \times 10^{30}\,\text{kg} = 1.006\,\mathrm{M}_\odot$ and radius $R_{\text{wd}} = 15{,}000\,\text{km} = 0.0216\,\mathrm{R}_\odot$, the deflection angle is

$$\alpha_{\text{wd}} = 1.751\,\text{arcsec}\frac{1.006}{0.0216} = 82\,\text{arcsec} = 1.36\,\text{arcmin} . \tag{7.18}$$

For a neutron star with mass $M_{\text{ns}} = 3.0 \times 10^{30}\,\text{kg} = 1.508\,\mathrm{M}_\odot$ and radius $R_{\text{ns}} = 12\,\text{km} = 1.725 \times 10^{-5}\,\mathrm{R}_\odot$, the deflection is

$$\alpha_{\text{ns}} = 1.751\,\text{arcsec}\frac{1.508}{1.725 \times 10^{-5}} = 153{,}000\,\text{arcsec} = 43^\circ . \tag{7.19}$$

7.4 *If the halo of our galaxy is spherically symmetric, what is the mass density $\rho(r)$ within the halo? If the universe contains a cosmological constant with density parameter $\Omega_{\Lambda,0} = 0.7$, would you expect it to significantly affect the dynamics of our galaxy's halo? Explain why or why not.*

From equation(7.2) above, the mass density of a spherically symmetric halo is

$$\rho(r) = \frac{v^2}{4\pi G r^2} \ , \tag{7.20}$$

if we make the approximation that the orbital speed v is constant with radius. Taking $v = 235\,\mathrm{km\,s^{-1}}$, this leads to a mass density

$$\rho(r) = 1.03 \times 10^{-21}\,\mathrm{kg\,m^{-3}} \left(\frac{r}{8.2\,\mathrm{kpc}}\right)^{-2} \tag{7.21}$$

$$= 1.2 \times 10^5 \rho_{c,0} \left(\frac{r}{8.2\,\mathrm{kpc}}\right)^{-2} \ . \tag{7.22}$$

Since the equivalent mass density of the cosmological constant is $\rho_\Lambda = \varepsilon_\Lambda/c^2 = 0.7\rho_{c,0}$, the dark energy density is equal to just $f \sim 6 \times 10^{-6}$ of the dark matter density locally, at $r \approx 8.2\,\mathrm{kpc}$. Even if you go to a distance $r \sim 100\,\mathrm{kpc}$ from the galactic center, the halo density is still $\rho \sim 800\rho_{c,0}$, and the dark energy density is equal to $f \sim 10^{-3}$ of the dark matter density. Thus, even when you go to the fringes of the halo, the dark energy messes with the dynamical time at the part-per-thousand level, which I would argue is not dynamically significant.

7.5 *In the previous chapter, we noted that galaxies in rich clusters are poor standard candles because they tend to grow brighter as they merge with other galaxies. Let's estimate the galaxy merger rate in the Coma cluster to see whether it's truly significant. The Coma cluster contains* $N \approx 1000$ *galaxies within its half-mass radius of* $r_h \approx 1.5\,\mathrm{Mpc}$*. What is the mean number density of galaxies within the half-mass radius? Suppose that the typical cross-sectional area of a galaxy is* $\Sigma \approx 10^{-3}\,\mathrm{Mpc}^2$*. How far will a galaxy in the Coma cluster travel, on average, before it collides with another galaxy? The velocity dispersion of the Coma cluster is* $\sigma \approx 880\,\mathrm{km\,s^{-1}}$*. What is the average time between collisions for a galaxy in the Coma cluster? Is this time greater than or less than the Hubble time?*

The mean number density of galaxies inside the half-mass radius of the Coma cluster is

$$\overline{n} = \frac{3N}{4\pi r_h^3} \approx \frac{3(1000)}{4\pi(1.5\,\mathrm{Mpc})^3} \approx 71\,\mathrm{Mpc}^{-3} \ . \tag{7.23}$$

For purposes of this problem, I will approximate the galaxies in Coma as identical spheres. A cross-sectional area of $\Sigma_{\rm gal} \sim 10^{-3}\,\mathrm{Mpc}$ then corresponds to a radius $r_{\rm gal} = (\Sigma/\pi)^{1/2} \sim 18\,\mathrm{kpc}$. I will define a "collision" as occurring when the centers of two galaxies come within a distance $2r_{\rm gal}$ of each other, and the two spherical galaxies overlap. Thus, as a galaxy moves through a distance d, it sweeps out a cylinder of radius $2r_{\rm gal}$, cross-sectional area $\pi(2r_{\rm gal})^2 = 4\Sigma_{\rm gal}$, and length d. If the center of any other galaxy is inside that cylinder as the first galaxy sweeps by, then there will be a collision. The volume of the cylinder is

$$V = 4\Sigma d \tag{7.24}$$

and the expected number of galaxies whose center is inside the cylinder is

$$N_{\rm cyl} = \overline{n}V = 4\overline{n}\Sigma d \ . \tag{7.25}$$

The typical distance $d_{\rm coll}$ that a galaxy travels before a collision is the value of d for which $N_{\rm cyl} = 1$. That is,

$$d_{\rm coll} = \frac{1}{4\overline{n}\Sigma} \approx 3.5\,{\rm Mpc} \ . \tag{7.26}$$

Thus, as a galaxy moves through a distance comparable to the half-mass radius, $r_h \approx 1.5\,{\rm Mpc}$, it's nearly a toss-up as to whether it collides with another galaxy or not. The velocity dispersion of the Coma cluster is

$$\begin{aligned} \sigma &\approx 880\,{\rm km\,s^{-1}} \left(\frac{1\,{\rm Mpc}}{3.086\times 10^{19}\,{\rm km}}\right)\left(\frac{3.156\times 10^{16}\,{\rm s}}{1\,{\rm Gyr}}\right) \\ &\approx 0.90\,{\rm Mpc\,Gyr^{-1}} \ . \end{aligned} \tag{7.27}$$

(One of the more useful coincidences in cosmology is that $1\,{\rm km\,s^{-1}} \sim 1\,{\rm kpc\,Gyr^{-1}}$.) Abandoning all factors of order unity, I estimate that the typical time between collisions for any given galaxy is

$$t_{\rm coll} \sim \frac{d_{\rm coll}}{\sigma} \sim 4\,{\rm Gyr} \ . \tag{7.28}$$

This is shorter than the Hubble time unless the neglected factor of order unity turns out to be a factor of order π. In any case, the fact that $t_{\rm coll}$ is not very much larger than H_0^{-1} means that collisions are not rare in rich clusters like the Coma cluster.

7.6 *Fusion reactions in the Sun's core produce* 2×10^{38} *neutrinos per second. If these solar neutrinos radiate isotropically away from the Sun, about how many solar neutrinos are inside your body at any given time? Is this larger or smaller than the number of neutrinos from the cosmic neutrino background that are inside your body at the same moment?*

The rate at which the Sun produces neutrinos is $R_{\rm sn} = 2\times 10^{38}\,{\rm s^{-1}}$; given the tiny cross-section of neutrinos for interaction, we can assume that none of the neutrinos are absorbed before they reach a distance $r = 1\,\mathrm{AU}$ from the Sun. Now, consider a thin spherical shell of radius r and thickness dr centered on the Sun; if $dr \ll r$, the volume of the shell is

$$dV = 4\pi r^2 dr \ . \tag{7.29}$$

Solar neutrinos have an energy very much larger than their rest energy,[1] so they travel essentially at the speed of light. Thus, the time it takes a solar neutrino to pass through the shell is

$$dt = \frac{dr}{c} \ . \tag{7.30}$$

The rate at which neutrinos enter this spherical shell is $R_{\rm sn}$, given my assumption that neutrinos are never absorbed. The number of solar neutrinos inside the shell at any instant is therefore

$$dN_{\rm sn} = R_{\rm sn} dt = \frac{R_{\rm sn}}{c} dr \tag{7.31}$$

and the number density of solar neutrinos inside the shell is

$$n_{\rm sn} = \frac{dN_{\rm sn}}{dV} = \frac{R_{\rm sn}}{4\pi r^2 c} \tag{7.32}$$

$$= 2.4 \times 10^6\,\mathrm{m}^{-3} \left(\frac{r}{1\,\mathrm{AU}}\right)^{-2} \ . \tag{7.33}$$

My mass is $m = 70\,\mathrm{kg}$ and my density is $\rho = 1000\,\mathrm{kg\,m^{-3}}$. This means I have a volume

$$V = m/\rho = 0.070\,\mathrm{m}^3 \ . \tag{7.34}$$

Thus, the number of solar neutrinos inside my body at any instant is

$$N_{\rm sn} = n_{\rm sn} V = 1.7 \times 10^5 \left(\frac{m}{70\,\mathrm{kg}}\right) \left(\frac{r}{1\,\mathrm{AU}}\right)^{-2} \ . \tag{7.35}$$

The number density of cosmic neutrinos is (from Eq. 7.49 of the text)

$$n_{\rm cn} = 3.36 \times 10^8\,\mathrm{m}^{-3} \ . \tag{7.36}$$

Therefore, the number of cosmic neutrinos inside my body is

$$N_{\rm cn} = n_{\rm cn} V = 2.35 \times 10^7 \left(\frac{m}{70\,\mathrm{kg}}\right) \ . \tag{7.37}$$

[1] The average energy of a solar neutrino is $\sim 0.5\,\mathrm{MeV}$, with the great majority having $E > 0.1\,\mathrm{MeV}$.

I conclude that the number of solar neutrinos in my body is smaller than the number of cosmic neutrinos, by a ratio

$$\frac{N_{\rm sn}}{N_{\rm cn}} = \frac{R_{\rm sn}}{n_{\rm cn}} \frac{1}{4\pi r^2 c} \approx \frac{1}{140} \left(\frac{r}{1\,{\rm AU}} \right)^{-2} . \tag{7.38}$$

(However, the solar neutrinos mostly have $E_{\rm sn} > 10^5\,{\rm eV}$, while the average energy of a cosmic neutrino is $m_\nu c^2 < 0.1\,{\rm eV}$.) For the number of solar neutrinos to exceed the number of cosmic neutrinos, I'd have to move within 0.08 AU of the Sun.

8

The cosmic microwave background

Chapter 8 in the second edition corresponds to Chapter 9 in the first edition.

I have rewritten Section 8.3 (The Physics of Recombination) to correct errors that I made in the first edition. The new version now clarifies the difference between kinetic equilibrium (which permits the distribution of particles to be written as a Bose-Einstein or Fermi-Dirac distribution) and chemical equilibrium (which permits the chemical potential terms to vanish from the Saha equation).

The discussion, in Section 8.5, of acoustic oscillations has been rewritten for clarity and to tie in with a discussion of baryon acoustic oscillations in Section 11.6.

The results in Figure 8.7 are drawn from the *Planck 2015* data release. The baryon-to-photon ratio $\eta = (6.10 \pm 0.06) \times 10^{-10}$ quoted in Equation 8.67 is the *Planck 2015* TT + lowP + lensing number.

Exercises

8.1 *The purpose of this problem is to determine how changing the value of the baryon-to-photon ratio, η, affects the recombination temperature in the early universe. Plot the fractional ionization X as a function of temperature, in the range $3000\,\mathrm{K} < T < 4500\,\mathrm{K}$; first make the plot assuming $\eta = 4 \times 10^{-10}$, then assuming $\eta = 8 \times 10^{-10}$. How much does this change in η affect the computed value of the recombination temperature $T_{\rm rec}$, if we define $T_{\rm rec}$ as the temperature at which $X = \frac{1}{2}$?*

The accompanying plot (figure 8.1) shows X as a function of T for a baryon-to-photon ratio $\eta = 4 \times 10^{-10}$ (solid line) and for $\eta = 8 \times 10^{-10}$ (dotted line). For a higher baryon-to-photon ratio, we see that recombination occurs as a higher temperature. Using $\eta = 4 \times 10^{-10}$, I find a recombination temperature $T_{\rm rec} = 3721\,\mathrm{K}$. Using $\eta = 8 =$

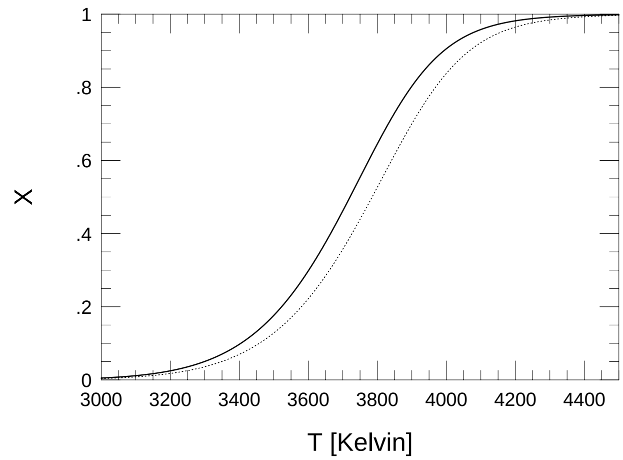

Figure 8.1 Fractional ionization of hydrogen as a function of temperature. The solid line assumes $\eta = 4\times10^{-10}$; the dotted line assumes $\eta = 8\times10^{-10}$.

10^{-10}, I find $T_{\rm rec} = 3785\,{\rm K}$. Thus, doubling the baryon-to-photon ratio increases the recombination temperature by just 1.7% in this range of η. (This is the basis of my statement in the text that "the exact value of η doesn't strongly affect the value of $T_{\rm rec}$.")

8.2 *Assuming a baryon-to-photon ratio $\eta = 6.1\times10^{-10}$, at what temperature T will there be one ionizing photon, with $hf > Q = 13.6\,{\rm eV}$, per baryon? [Hint: the result of exercise 2.5 will be useful.] Is the temperature you calculate greater than or less than $T_{\rm rec} = 3760\,{\rm K}$?*

When $\eta = 6.1 \times 10^{-10}$, there are $\eta^{-1} \approx 1.6 \times 10^9$ photons for every baryon. By asking the temperature at which there is one photon with $hf > Q$ for every baryon, we are asking the temperature at which

$$F(hf > Q) = \eta \ , \tag{8.1}$$

where F is the fraction of photons with energy $hf > Q = 13.6\,{\rm eV}$. I will start by assuming that this occurs at a temperature T_c such that $kT_c \ll Q$, or $T_c \ll Q/k = 160{,}000\,{\rm K}$. (This is an assumption that I can check at the end of the problem.) With the assumption that $kT_c \ll Q$, I can adopt the approximation of equation (2.28) in exercise 2.5 above:

$$F(hf > Q) \approx 0.416 \left(\frac{Q}{kT}\right)^2 \exp\left(-\frac{Q}{kT}\right) \ . \tag{8.2}$$

Numerically solving the equation

$$\left(\frac{Q}{kT_c}\right)^2 \exp\left(-\frac{Q}{kT_c}\right) = \frac{\eta}{0.416} = 1.47 \times 10^{-9} \tag{8.3}$$

for T_c, I find $kT_c = 0.0385Q = 0.524\,\text{eV}$, which justifies my assumption above that $kT_c \ll Q$. This result corresponds to $T_c = 6080\,\text{K}$, about 60% higher than the recombination temperature $T_{\text{rec}} = 3760\,\text{K}$. [Because of the exponential dependence on temperature in equation (8.3), by the time the temperature drops to T_{rec}, there is only one ionizing photon left for every 1.4×10^6 baryons.]

8.3 *Imagine that at the time of recombination, the baryonic portion of the universe consisted entirely of ^{4}He (that is, helium with two protons and two neutrons in its nucleus). The ionization energy of helium (that is, the energy required to convert neutral He to He$^+$) is $Q_{\text{He}} = 24.6\,\text{eV}$. At what temperature would the fractional ionization of the helium be $X = \frac{1}{2}$? Assume that $\eta = 6 \times 10^{-10}$ and that the number density of He^{++} is negligibly small. [The relevant statistical weight factor for the ionization of helium is $g_{\text{He}}/(g_e g_{\text{He}^+}) = \frac{1}{4}$.]*

Let $n(\text{He}^+)$ be the number density of singly ionized helium, $n(\text{He}^0)$ be the number density of neutral helium atoms, and $n(\text{He}) = n(\text{He}^+) + n(\text{He}^0)$ be the total number density of helium nuclei. (I am assuming that the temperature is sufficiently low that the amount of doubly ionized helium is negligible.) With the proper statistical weights, the Saha equation for the balance between He and He$^+$ is (compare to Eq. 8.29 of the text):

$$\frac{n(\text{He}^0)}{n(\text{He}^+)n_e} = \frac{1}{4}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q_{\text{He}}}{kT}\right) . \tag{8.4}$$

The fractional ionization of helium is

$$X = \frac{n(\text{He}^+)}{n(\text{He})} , \tag{8.5}$$

so we may write

$$n(\text{He}^+) = Xn(\text{He}) \tag{8.6}$$

and

$$n(\text{He}^0) = (1 - X)n(\text{He}) . \tag{8.7}$$

The Saha equation can then be rewritten as

$$\frac{1 - X}{X} = \frac{n_e}{4}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q_{\text{He}}}{kT}\right) . \tag{8.8}$$

In the real universe, the recombination of He^+ to He happens while hydrogen is still ionized, implying that the free electron density n_e in the real universe will still remain high even when the helium fractional ionization X goes to zero. However, for the purposes of this problem, I have created my own fictional universe in which the baryonic component is pure helium. In this helium-filled universe,

$$n_e = n(\mathrm{He}^+) = Xn(\mathrm{He}) \ , \tag{8.9}$$

and thus

$$\frac{1-X}{X^2} = \frac{n(\mathrm{He})}{4}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2}\exp\left(\frac{Q_{\mathrm{He}}}{kT}\right) \ . \tag{8.10}$$

There are 4 baryons in a ^{4}He nucleus, so

$$n(\mathrm{He}) = \frac{1}{4}n_{\mathrm{bary}} = \frac{\eta}{4}n_\gamma = \frac{\eta}{4}0.2436\left(\frac{kT}{\hbar c}\right)^3 \ . \tag{8.11}$$

Substituting this result back into equation (8.10) above yields the result:

$$\frac{1-X}{X^2} = 0.240\eta\left(\frac{kT}{m_e c^2}\right)^{3/2}\exp\left(\frac{Q_{\mathrm{He}}}{kT}\right) \ . \tag{8.12}$$

This result for a helium-only universe looks similar to the equivalent relation for a hydrogen-only universe (see Eq. 8.34 of the text). The only changes are a different numerical factor out front, and the use of a higher ionization energy, $Q_{\mathrm{He}} \approx 1.8Q_{\mathrm{H}}$. Solving equation (8.12) for the temperature at which $X = 1/2$, I find $T_{\mathrm{rec}} = 6500\,\mathrm{K}$, or

$$kT_{\mathrm{rec}} = 0.560\,\mathrm{eV} = \frac{Q_{\mathrm{He}}}{44} \ . \tag{8.13}$$

If I had done this problem in an order-of-magnitude way, I would have said, "The ionization energy of He is 80% greater than that of H; thus, its recombination temperature will be 80% greater as well, or $T_{\mathrm{rec}} \approx 1.8(3760\,\mathrm{K}) \approx 6800\,\mathrm{K}$."

8.4 *What is the proper distance d_p to the surface of last scattering? What is the luminosity distance d_L to the surface of last scattering? Assume that the Benchmark Model is correct, and that the redshift of the last scattering surface is $z_{\mathrm{ls}} = 1090$.*

The current proper distance to the surface of last scattering is (see Eq. 5.33 of the text)

$$d_p = c\int_{t_{\mathrm{ls}}}^{t_0}\frac{dt}{a(t)} \ . \tag{8.14}$$

This can be rewritten as

$$d_p = d_{\mathrm{hor}} - c\int_0^{t_{\mathrm{ls}}} \frac{dt}{a(t)} \ , \tag{8.15}$$

where

$$d_{\mathrm{hor}} = c\int_0^{t_0} \frac{dt}{a(t)} = 3.20\frac{c}{H_0} \tag{8.16}$$

is the horizon distance in the Benchmark Model. Since last scattering took place early in the history of the universe, the second term on the right hand side of equation (8.15) represents a correction term small compared to the horizon distance. To find its numerical value, we note that last scattering took place in the early universe, when only radiation and matter contributed significantly. Thus, we can use Eq. 5.109, which tells us that in a radiation + matter universe,

$$dt = H_0^{-1}\Omega_{r,0}^{-1/2}\frac{a\,da}{(1 + a/a_{\mathrm{rm}})^{1/2}} \ , \tag{8.17}$$

where a_{rm} is the scale factor of radiation-matter equality. Using this relation, we find

$$c\int_0^{t_{\mathrm{ls}}} \frac{dt}{a(t)} = \frac{c}{H_0}\Omega_{r,0}^{-1/2}\int_0^{a_{\mathrm{ls}}} \frac{da}{(1 + a/a_{rm})^{1/2}} \tag{8.18}$$

$$= \frac{c}{H_0}\Omega_{r,0}^{-1/2} 2a_{rm}\left[(1 + a_{\mathrm{ls}}/a_{rm})^{1/2} - 1\right] \ . \tag{8.19}$$

Using the Benchmark values $\Omega_{r,0} = 9.0 \times 10^{-5}$, $a_{rm} = 2.9 \times 10^{-4}$, and $a_{\mathrm{ls}} = 1/(1 + z_{\mathrm{ls}}) = 9.17 \times 10^{-4}$, this becomes

$$c\int_0^{t_{\mathrm{ls}}} \frac{dt}{a(t)} = 0.064\frac{c}{H_0} \ , \tag{8.20}$$

equal to 2% of the current horizon distance. Thus, the proper distance to the last scattering surface is

$$d_p = 0.98 d_{\mathrm{hor}} = 3.14\frac{c}{H_0} = 13{,}700\,\mathrm{Mpc} \tag{8.21}$$

in the Benchmark Model. The luminosity distance to the last scattering surface, since the Benchmark Model is spatially flat, takes the simple form

$$d_L = (1 + z_{\mathrm{ls}})d_p = (1091)(13{,}700\,\mathrm{Mpc}) = 1.50 \times 10^7\,\mathrm{Mpc} \ . \tag{8.22}$$

The angular-diameter distance, incidentally, is

$$d_A = \frac{d_p}{1 + z_{\mathrm{ls}}} = \frac{13{,}700\,\mathrm{Mpc}}{1091} = 12.6\,\mathrm{Mpc} \ , \tag{8.23}$$

just 2% smaller than the approximation used in the text (which ignores the 2% correction factor that I laboriously calculated above).

9

Nucleosynthesis and the early universe

Chapter 9 in the second edition corresponds to Chapter 10 in the first edition.

In Section 9.2, for clarity, I make a more explicit contrast between deuterium synthesis in the early universe and the (far more laborious) synthesis of deuterium as the first step in the p-p chain within the Sun today.

In Section 9.4, the deuterium-to-hydrogen ratio $D/H \approx 1.6 \times 10^{-5}$ that I quote for the local interstellar gas is the value found by Linsky et al. (2006) within the Local Bubble. The value $D/H = (2.53 \pm 0.04) \times 10^{-5}$ that I quote for intergalactic gas is from Cooke et al. (2014).

Exercises

9.1 *Suppose the neutron decay time were $\tau_n = 88\,\mathrm{s}$ instead of $\tau_n = 880\,\mathrm{s}$, with all other physical parameters unchanged. Estimate $Y_{\rm max}$, the maximum possible mass fraction in ^4He, assuming that all available neutrons are incorporated into ^4He nuclei.*

If the neutron decay time were $\tau_n = 88\,\mathrm{s}$, this would still be large compared to the time of neutron freezeout, $t_{\rm freeze} \approx 1\,\mathrm{s}$. Thus, the neutron-to-proton ratio at $t_{\rm freeze}$ will still be, from Eq. 9.17 of the text,

$$f = \frac{n_n}{n_p} = \exp\left(-\frac{Q_n}{kT_{\rm freeze}}\right) = \exp\left(-\frac{1.29}{0.8}\right) = 0.199\ , \qquad (9.1)$$

or one neutron for every 5.015 protons. However, a neutron decay time of 88 seconds is short compared to the time of deuterium synthesis, $t_{\rm nuc} \approx 200\,\mathrm{s} \approx 2.3\tau_n$. Thus, the neutron-to-proton ratio at $t_{\rm freeze}$ will be significantly lowered (compare to Eq. 9.32 of the text):

$$\begin{aligned} f = \frac{n_n}{n_p} &= \frac{\exp(-200/88)}{5.015 + [1 - \exp(-200/88)]} \\ &= \frac{0.103}{6.015 - 0.103} = 0.0174 \ , \end{aligned} \tag{9.2}$$

or 2 neutrons for every 114.8 protons. If 2 neutrons team up with 2 protons to make a ^{4}He nucleus, that leaves 112.8 protons left over, and the mass fraction of ^{4}He will then be

$$Y_{\max} = \frac{4}{116.8} = 0.034 \ . \tag{9.3}$$

9.2 *Suppose the difference in rest energy of the neutron and proton were $Q_n = (m_n - m_p)c^2 = 0.129\,\mathrm{MeV}$ instead of $Q_n = 1.29\,\mathrm{MeV}$, with all other physical parameters unchanged. Estimate $Y_{\max}$, the maximum possible mass fraction in ^{4}He, assuming that all available neutrons are incorporated into ^{4}He nuclei.*

If the difference in rest energy of the neutron and proton were $Q_n = 0.129\,\mathrm{MeV}$, this would be relatively small compared to the energy of neutron freezeout, $kT_{\rm freeze} = 0.8\,\mathrm{MeV}$. Thus, the neutron-to-proton ratio at $t_{\rm freeze}$ would be fairly close to one (compare to Eq. 9.17 of the text):

$$f = \frac{n_n}{n_p} = \exp\left(-\frac{0.129}{0.8}\right) = 0.851 \ , \tag{9.4}$$

or one neutron for every 1.175 protons. A subtlety of this problem is that $Q_n = 0.129\,\mathrm{MeV}$ is less than the rest energy of an electron, $m_e c^2 = 0.511\,\mathrm{MeV}$. Thus, the spontaneous decay of the neutron,

$$n \rightarrow p + e^- + \overline{\nu}_e \ , \tag{9.5}$$

will be energetically impossible. This means that after neutrinos have decoupled ($t > 1\,\mathrm{s}$), neutrons will be able to decay only by positron capture (compare to Eq. 9.10 of the text):

$$n + e^+ \rightarrow p + \overline{\nu}_e \ . \tag{9.6}$$

At temperatures $kT \ll m_e c^2$ (or times $t \gg 4\,\mathrm{s}$), positrons will have disappeared (through annihilation with electrons) and neutrons will not decay. Assuming negligible decay by positron capture, the neutron-to-proton ratio will remain at $f = 0.851$, representing 2 neutrons for every 2.35 protons. Thus, with complete fusion to ^{4}He, there will be one ^{4}He nucleus for every 0.35 leftover protons. The mass fraction of ^{4}He will then be

$$Y_{\rm max} = \frac{4}{4.35} = 0.92 \ . \tag{9.7}$$

[If your students permit the neutrons to decay with a lifetime $\tau_n = 880\,{\rm s}$ (forgetting about energy conservation), they will find $Y_{\rm max} = 0.73$.]

9.3 *The total luminosity of the stars in our galaxy is $L \approx 3 \times 10^{10}\,{\rm L}_\odot$. Suppose that the luminosity of our galaxy has been constant for the past* 10 Gyr*. How much energy has our galaxy emitted in the form of starlight during that time? Most stars are powered by the fusion of H into ^{4}He, with the release of* 28.4 MeV *for every helium nucleus formed. How many helium nuclei have been created within stars in our galaxy over the course of the past* 10 Gyr*, assuming that the fusion of H into ^{4}He is the only significant energy source? If the baryonic mass of our galaxy is $M \approx 10^{11}\,{\rm M}_\odot$, by what amount has the helium fraction Y of our galaxy been increased over its primordial value $Y_4 = 0.24$?*

The luminosity of our galaxy is

$$L_{\rm gal} \approx 3 \times 10^{10}\,{\rm L}_\odot \left(\frac{3.828 \times 10^{26}\,{\rm J\,s^{-1}}}{1\,{\rm L}_\odot} \right) \approx 1.15 \times 10^{37}\,{\rm J\,s^{-1}} \ . \tag{9.8}$$

Suppose that this luminosity has been constant for a time

$$t_{\rm gal} \approx 10\,{\rm Gyr} \left(\frac{3.156 \times 10^{16}\,{\rm s}}{1\,{\rm Gyr}} \right) \approx 3.16 \times 10^{17}\,{\rm s} \ . \tag{9.9}$$

The total amount of radiative energy emitted by all the stars in our galaxy is then

$$E_{\rm gal} = L_{\rm gal} t_{\rm gal} \approx (1.15 \times 10^{37}\,{\rm J\,s^{-1}})(3.16 \times 10^{17}\,{\rm s}) \approx 3.6 \times 10^{54}\,{\rm J} \ , \tag{9.10}$$

or $E_{\rm gal} \approx 2.3 \times 10^{67}\,{\rm MeV}$. The energy released in creating one helium nucleus is $E_{\rm He} = 28.4\,{\rm MeV}$. The number of helium nuclei created in the production of $E_{\rm gal}$ is

$$N_{\rm He} = \frac{E_{\rm gal}}{E_{\rm He}} \approx \frac{2.3 \times 10^{67}\,{\rm MeV}}{28.4\,{\rm MeV}} \approx 8.0 \times 10^{65} \ . \tag{9.11}$$

Since the mass of an individual helium nucleus is $m_{\rm He} = 6.645 \times 10^{-27}\,{\rm kg}$, this represents a mass

$$M_{\rm He} = N_{\rm He} m_{\rm He} \approx (8.0 \times 10^{65})(6.645 \times 10^{-27}\,{\rm kg}) \approx 5.3 \times 10^{39}\,{\rm kg} \ , \tag{9.12}$$

or $M_{\rm He} \approx 2.7 \times 10^9\,{\rm M}_\odot$. Given a total baryon mass $M_{\rm bary} \approx 10^{11}\,{\rm M}_\odot$, this means that fusion in stars has increased the helium mass fraction by an amount

$$\Delta Y \approx \frac{M_{\rm He}}{M_{\rm bary}} \approx \frac{2.7 \times 10^9\,{\rm M}_\odot}{10^{11}\,{\rm M}_\odot} \approx 0.027\ . \tag{9.13}$$

This represents an increase of $\sim 11\%$ over the primordial helium mass fraction $Y_p = 0.24$.

9.4 *In Section 9.2, it is asserted that the maximum possible value of the primordial helium fraction is*

$$Y_{\rm max} = \frac{2f}{1+f}, \tag{9.14}$$

where $f = n_n/n_p \leq 1$ is the neutron-to-proton ratio at the time of nucleosynthesis. Prove that this assertion is true. If the neutron-to-proton ratio is $f \equiv n_n/n_p$, then the number density of neutrons is $n_n = f n_p$, and the total number density of baryons is $n_{\rm bary} = n_n + n_p = (1+f)n_p$. Suppose that a region of space contains N baryons in total, with a mass $M_{\rm bary} \approx m_p N$. (In this problem, I can get reasonable accuracy by assuming that all baryons have the same mass, regardless of whether they are protons or neutrons, or whether they are free or bound.) The number of protons will then be

$$N_p = \frac{1}{1+f} N \tag{9.15}$$

and the number of neutrons will be

$$N_n = N - N_p = \frac{f}{1+f} N\ . \tag{9.16}$$

If $f < 1$, then the production of ^{4}He will be limited by the neutron scarcity. The maximum number of helium nuclei will be half the number of neutrons,

$$N_{\rm He} = \frac{f}{2(1+f)} N\ , \tag{9.17}$$

and the maximum mass of helium will be

$$M_{\rm He} = 4 m_p N_{\rm He} = \frac{2f}{1+f} m_p N = \frac{2f}{1+f} M_{\rm bary}\ . \tag{9.18}$$

The maximum possible mass fraction of helium is then

$$Y_{\rm max} = \frac{M_{\rm He}}{M_{\rm bary}} = \frac{2f}{1+f}\ . \tag{9.19}$$

9.5 *The typical energy of a neutrino in the cosmic neutrino background, as pointed out in Chapter 5, is $E_\nu \sim kT_\nu \sim 5 \times 10^{-4}\,\mathrm{eV}$. What is the approximate interaction cross-section σ_w for one of these cosmic neutrinos? Suppose you had a large lump of ^{56}Fe (with density $\rho = 7900\,\mathrm{kg\,m^{-3}}$). What is the number density of protons, neutrons, and electrons within the lump of iron? How far, on average, would a cosmic neutrino travel through the iron before interacting with a proton, neutron, or electron? (Assume that the cross-section for interaction is simply σ_w, regardless of the type of particle the neutrino interacts with.)*

In Eq. 9.15 of the text, I give a typical weak cross-section

$$\sigma_w \sim 10^{-47}\,\mathrm{m}^2 \left(\frac{kT}{1\,\mathrm{MeV}}\right)^2 . \tag{9.20}$$

If I daringly extrapolate to a temperature $kT \sim 5 \times 10^{-10}\,\mathrm{MeV}$, this implies a cross-section

$$\sigma_w \sim 10^{-47}\,\mathrm{m}^2 (5 \times 10^{-10})^2 \sim 2 \times 10^{-66}\,\mathrm{m}^2 \sim 10^4 \ell_P^2 . \tag{9.21}$$

Given the uncertainty in the cross-section at these low energies, I will scale my answers to a cross-section $10^{-66}\,\mathrm{m}^2$.

A single atom of ^{56}Fe has 26 protons, 30 neutrons, and 26 electrons (82 particles in total), and a mass $m = 9.288 \times 10^{-26}\,\mathrm{kg}$. The number density of iron atoms, given $\rho = 7900\,\mathrm{kg\,m^{-3}}$, is

$$n_{\mathrm{Fe}} = \frac{\rho}{m} = \frac{7900\,\mathrm{kg\,m^{-3}}}{9.288 \times 10^{-26}\,\mathrm{kg}} = 8.51 \times 10^{28}\,\mathrm{m^{-3}} . \tag{9.22}$$

Thus, the number density of the different particles is

$$n_p = n_e = 26 n_{\mathrm{Fe}} = 2.21 \times 10^{30}\,\mathrm{m^{-3}} \tag{9.23}$$

for protons and electrons, and

$$n_n = 30 n_{\mathrm{Fe}} = 2.55 \times 10^{30}\,\mathrm{m^{-3}} \tag{9.24}$$

for neutrons. The total number density of neutrons, protons, and electrons is then

$$n_{\mathrm{part}} = 82 n_{\mathrm{Fe}} = 6.97 \times 10^{30}\,\mathrm{m^{-3}} . \tag{9.25}$$

The average distance that a neutrino can travel before interacting with a particle is then (compare to Eq. 2.2 of the text)

$$\lambda = \frac{1}{n_{\mathrm{part}} \sigma_w} , \tag{9.26}$$

where σ_w is the assumed cross-section for interaction between a proton, neutron, or electron with a neutrino. Scaling to my assumed weak cross-section, $\sigma_w = 10^{-66}\,\mathrm{m}^2$,

$$\lambda = 1.4 \times 10^{35}\,\mathrm{m} \left(\frac{\sigma_w}{10^{-66}\,\mathrm{m}^2} \right)^{-1} \approx 10^9 \frac{c}{H_0} \left(\frac{\sigma_w}{10^{-66}\,\mathrm{m}^2} \right)^{-1} . \qquad (9.27)$$

A mean free path equal to a billion times the Hubble distance poses obvious problems for building a cosmic neutrino detector.

10

Inflation and the very early universe

Chapter 10 in the second edition corresponds to Chapter 11 in the first edition.

In the first edition, I baldly asserted that $N \sim 60$ is the minimum number of e-foldings of inflation required to flatten the universe. In the second edition, I work through the derivation of that number, in Section 10.4.

In the first edition, I used $N = 100$ e-foldings of inflation as my default model (for no better reason than my devoted fervor for the decimal system). In the second edition, I use $N = 65$ e-foldings, with the explanation that this is a minimal inflationary model, with just enough exponential growth to agree with the observed flatness of today's universe and the observed isotropy of the cosmic microwave background.

In the second edition, I also corrected an error found in the first edition, where I erroneously implied that the particle horizon distance grows $\propto t$ in the post-inflationary era. It actually grows $\propto a(t)$; that is, $\propto t^{1/2}$ when radiation dominates.

Exercises

10.1 *What upper limit is placed on $\Omega(t_P)$ by the requirement that the universe not end in a Big Crunch between the Planck time, $t_P \approx 5\times10^{-44}\,$s, and the start of the inflationary epoch at t_i? Compute the maximum permissible value of $\Omega(t_P)$, first assuming $t_i \approx 10^{-36}\,$s, then assuming $t_i \approx 10^{-26}\,$s. (Hint: prior to inflation, the Friedmann equation will be dominated by the radiation term and the curvature term.)*

In the pre-inflationary era, the Friedmann equation contained only a radiation term and a curvature term. Thus, the general Friedmann equation (see, for instance, Eq. 5.81 of the text) reduces to

$$\frac{H^2}{H_0^2} = \frac{\Omega_0}{a^4} - \frac{\Omega_0 - 1}{a^2} \ . \tag{10.1}$$

For the sake of convenience, I will take t_0, the time when $a(t_0) = 1$, to be the Planck time, t_P. At this time, the density parameter in radiation was $\Omega_0 = \Omega(t_P)$, and the Hubble constant was

$$H_0 = H(t_P) \approx \frac{1}{2} t_P^{-1} \ . \tag{10.2}$$

(Here, I am assuming that the universe was sufficiently close to flat at the Planck time that the radiation-only relation, $H = (1/2)t^{-1}$, is a good approximation. I can check this assumption later on.)

When $\Omega_0 > 1$, the time of maximum expansion, when $H = 0$, corresponds to an scale factor, from equation (10.1) above,

$$\frac{\Omega_0}{a_{\rm max}^4} = \frac{\Omega_0 - 1}{a_{\rm max}^2} \ , \tag{10.3}$$

or

$$a_{\rm max} = \left(\frac{\Omega_0}{\Omega_0 - 1} \right)^{1/2} \ . \tag{10.4}$$

Since the contraction phase of a Big Crunch universe is an exact reversal of the expansion, the time of maximum expansion is $t_{\rm crunch}/2$, where $t_{\rm crunch}$ is the time of the Big Crunch.

The Friedmann equation of the pre-inflationary universe, as given in equation (10.1), can be rewritten as

$$\frac{1}{H_0} \frac{da}{dt} = \frac{1}{a} \left[\Omega_0 - (1 - \Omega_0) a^2 \right]^{1/2} \ . \tag{10.5}$$

Integrating from the Planck time to the time of maximum expansion, this becomes

$$\Omega_0^{-1/2} \int_1^{a_{\rm max}} \frac{a da}{[1 - a^2/a_{\rm max}^2]^{1/2}} = H_0 \int_{t_P}^{t_{\rm crunch}/2} dt \ . \tag{10.6}$$

The left side of the above equation is neatly integrable; the entire left-hand side comes to $1/(\Omega_0 - 1)$. On the right-hand side, I will assume that $t_{\rm crunch} \gg 2t_P$. This leads to

$$\frac{1}{\Omega_0 - 1} = \frac{1}{2} H_0 t_{\rm crunch} \ , \tag{10.7}$$

or, using the relations $H_0 = H(t_P) \approx (1/2)t_P^{-1}$ and $\Omega_0 = \Omega(t_P)$,

$$t_{\rm crunch} \approx \frac{4}{\Omega(t_P) - 1} t_P \ . \tag{10.8}$$

The requirement that $t_{\rm crunch} > t_i$, where t_i is the time when inflation starts, corresponds to the requirement that

$$\Omega(t_P) - 1 < 4\frac{t_P}{t_i} \ . \tag{10.9}$$

If $t_i = 10^{-36}\,{\rm s} = 2 \times 10^7 t_P$, this implies the requirement that

$$\Omega(t_P) - 1 < 2 \times 10^{-7} \ . \tag{10.10}$$

If $t_i = 10^{-26}\,{\rm s} = 1.9 \times 10^{17} t_P$, this implies the requirement that

$$\Omega(t_P) - 1 < 2 \times 10^{-17} \ . \tag{10.11}$$

In a positively curved universe, delaying the inflationary epoch thus leads to a recurrence of the flatness problem. Although a coincidence at the level of 2 parts in 10^{17} isn't as bad as a coincidence at the level of 2 parts in 10^{62} (as discussed in Section 10.1), it is still unsettling.

10.2 *Current observational limits on the density of magnetic monopoles tell us that their density parameter is currently $\Omega_{M,0} < 10^{-6}$. If monopoles formed at the GUT time, with one monopole per horizon of mass $m_M = m_{\rm GUT}$, how many e-foldings of inflation would be required to drive the current density of monopoles below the bound $\Omega_{M,0} < 10^{-6}$? Assume that inflation took place immediately after the formation of monopoles.*

A monopole has mass

$$m_M = m_{\rm GUT} = \frac{E_{\rm GUT}}{c^2} \ . \tag{10.12}$$

Given a GUT energy $E_{\rm GUT} = 10^{12}\,{\rm TeV} = 1.6 \times 10^5\,{\rm J}$, this yields a monopole mass

$$m_M = \frac{1.6 \times 10^5\,{\rm J}}{(3 \times 10^8\,{\rm m\,s^{-1}})^2} = 1.8 \times 10^{-12}\,{\rm kg} \ . \tag{10.13}$$

If the density parameter in monopoles is $\Omega_{M,0}$, the number density of monopoles today is

$$n_M(t_0) = \frac{\Omega_{M,0}\rho_{c,0}}{m_M} = 4.9 \times 10^{-21}\,{\rm m^{-3}} \left(\frac{\Omega_{M,0}}{10^{-6}}\right) \ . \tag{10.14}$$

If there were one monopole per horizon volume at the GUT time, then the number density of monopole at the time they formed was (from Eq. 10.14 of the text)

$$n_M(t_{\rm GUT}) \sim \frac{1}{(2ct_{\rm GUT})^3} \sim 10^{82}\,{\rm m}^{-3}\ . \tag{10.15}$$

How did we get from the plethora of monopoles at $t_{\rm GUT}$ to the paucity of monopoles today? Suppose that exponential inflation started at a time $t_i \approx t_{\rm GUT} \approx 10^{-36}\,{\rm s}$, just after the formation of monopoles, and continued until $t_f = (N+1)t_i$. After N e-foldings of inflation, volumes increase by a factor e^{3N}, and the number density of monopoles is driven down to

$$n_M(t_f) = e^{3N} n_M(t_{\rm GUT}) \sim e^{-3N} \cdot 10^{82}\,{\rm m}^{-3}\ . \tag{10.16}$$

Extrapolating backward from today, when $a(t_0) = 1$, the scale factor at the end of inflation, when $t_f = (N+1)\cdot 10^{-36}\,{\rm s}$, was

$$a(t_f) \approx 2 \times 10^{-28}(N+1)^{1/2}\ . \tag{10.17}$$

(Compare to Eq. 10.30 of the text.) With the number density of monopoles today given by equation (10.14), the number density of monopoles at the end of inflation was

$$n_M(t_f) = \frac{n_M(t_0)}{a(t_f)^3} = 6.1 \times 10^{62}\,{\rm m}^{-3} \left(\frac{\Omega_{M,0}}{10^{-6}}\right)(N+1)^{3/2}\ . \tag{10.18}$$

Comparing equation (10.16), which gives the number density of monopoles expected after inflation, to equation (10.18), which gives the constraint from the monopole density today, we find that

$$e^{-3N} \cdot 10^{82}\,{\rm m}^{-3} = 6.1 \times 10^{62}\,{\rm m}^{-3} \left(\frac{\Omega_{M,0}}{10^{-6}}\right)(N+1)^{3/2}\ , \tag{10.19}$$

or

$$e^{-2N}(N+1) = 1.5 \times 10^{-13} \left(\frac{\Omega_{M,0}}{10^{-6}}\right)^{2/3}\ . \tag{10.20}$$

In the end-of-chapter problem (lazily cut-and-pasted from the first edition of the text), I quote the limit $\Omega_{M,0} < 10^{-6}$. Solving equation (10.20) numerically, this implies a minimum of $N = 16.2$ e-foldings of inflation. However, in the new, improved second edition of the text, I quote the new, improved limit $\Omega_{M,0} < 5 \times 10^{-16}$, which implies a minimum of $N = 23.5$ e-foldings.

[Remember that a minimum of $N \approx 60$ e-foldings are required to make the radius of curvature R_0 greater than the Hubble distance

c/H_0 today. This amount of inflation will dilute the number density of monopoles so that there is less than one per Hubble volume today. One monopole per Hubble volume corresponds to $\Omega_{M,0} \sim 10^{-64}$.]

10.3 *It has been speculated that the present-day acceleration of the universe is due to the existence of a false vacuum, which will eventually decay. Suppose that the energy density of the false vacuum is* $\varepsilon_\Lambda = 0.69\varepsilon_{c,0} = 3360\,\mathrm{MeV\,m^{-3}}$*, and that the current energy density of matter is* $\varepsilon_{m,0} = 0.31\varepsilon_{c,0} = 1510\,\mathrm{MeV\,m^{-3}}$*. What will be the value of the Hubble parameter once the false vacuum becomes strongly dominant? Suppose that the false vacuum is fated to decay instantaneously to radiation at a time* $t_f = 50t_0$*. (Assume, for simplicity, that the radiation takes the form of blackbody photons.) To what temperature will the universe be reheated at* $t = t_f$*? What will the energy density of matter be at* $t = t_f$*? At what time will the universe again be dominated by matter?*

A universe containing both Λ and matter has a scale factor $a(t)$ given by the equation (see Eq. 5.101 of the text)

$$H_0 t = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1 + (a/a_{m\Lambda})^3}\right] , \tag{10.21}$$

where the scale factor of matter-lambda equality is

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/2} = \left(\frac{0.31}{0.69}\right)^{1/3} = 0.766 . \tag{10.22}$$

In the limit that Λ is strongly dominant and $a \gg a_{m\Lambda}$, the scale factor reduces to (Eq. 5.103 of the text)

$$a(t) \approx a_{m\Lambda} \exp(\sqrt{\Omega_{\Lambda,0}} H_0 t) . \tag{10.23}$$

In this limit, the Hubble constant is

$$H_\Lambda = \frac{\dot{a}}{a} \approx \sqrt{\Omega_{\Lambda,0}} \cdot H_0 = \sqrt{0.69} \cdot 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = 56.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}} , \tag{10.24}$$

corresponding to a Hubble time

$$H_\Lambda^{-1} = \frac{H_0^{-1}}{\sqrt{\Omega_{\Lambda,0}}} = \frac{14.38\,\mathrm{Gyr}}{\sqrt{0.69}} = 17.31\,\mathrm{Gyr} . \tag{10.25}$$

Suppose that the cosmological constant is provided by a false vacuum that decays at a time

$$t_f = 50t_0 = 47.75 H_0^{-1} = 690\,\mathrm{Gyr} . \tag{10.26}$$

(Here I am using the relation $t_0 = 0.955 H_0^{-1}$ for the Benchmark Model.) The scale factor at that time will be, from equation (10.23) above,

$$a(t_f) = a_{m\Lambda} \exp(\sqrt{\Omega_{\Lambda,0}} H_0 t_f) \tag{10.27}$$

$$= 0.766 \exp(\sqrt{0.69} \cdot 47.75) = 1.29 \times 10^{17} \ . \tag{10.28}$$

If the false vacuum, with energy $\varepsilon_\Lambda = 3360\,\mathrm{MeV\,m^{-3}}$, or $\varepsilon_\Lambda = 5.38 \times 10^{-10}\,\mathrm{J\,m^{-3}}$, is converted entirely to blackbody radiation, then the temperature T_f of the radiation is given by the relation

$$\varepsilon_\Lambda = \alpha T_f^4 \ , \tag{10.29}$$

or

$$T_f = \left(\frac{\varepsilon_\Lambda}{\alpha}\right)^{1/4} = \left(\frac{5.38 \times 10^{-10}\,\mathrm{J\,m^{-3}}}{7.566 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-1}}}\right)^{1/4} = 29.0\,\mathrm{K} \ . \tag{10.30}$$

At the time of reheating, then, the universe will be a relatively balmy 29 Kelvin, with an energy density in blackbody radiation

$$\varepsilon_r(t_f) = \varepsilon_\Lambda = 3360\,\mathrm{MeV\,m^{-3}} \ . \tag{10.31}$$

However, the energy density in matter will have been diluted to

$$\varepsilon_m(t_f) = \frac{\varepsilon_{m,0}}{a(t_f)^3} = \frac{1510\,\mathrm{MeV\,m^{-3}}}{(1.29 \times 10^{17})^3} = 7.03 \times 10^{-49}\,\mathrm{MeV\,m^{-3}} \ . \tag{10.32}$$

Thus, at the time of reheating, the universe will be very strongly radiation-dominated, with $\Omega_m(t_f) = 2.09 \times 10^{-52}$ and $\Omega_r(t_f) = 1 - \Omega_m(t_f) \approx 1$. The matter will not start to dominate again until a scale factor given by the relation

$$\Omega_m(t_f) \left(\frac{a}{a(t_f)}\right)^{-3} = \Omega_r(t_f) \left(\frac{a}{a(t_f)}\right)^{-4} \ , \tag{10.33}$$

or

$$a_{rm} = a(t_f) \frac{\Omega_r(t_f)}{\Omega_m(t_f)} = 1.29 \times 10^{17} \frac{1}{2.09 \times 10^{-52}} = 6.17 \times 10^{68} \ . \tag{10.34}$$

At this time, the temperature of the radiation created at the time of reheating will have dropped to a downright chilly $4.7 \times 10^{-68}\,\mathrm{K}$.

After reheating, the Friedmann equation of the universe will be

$$H(t)^2 = \frac{8\pi G}{3c^2} [\varepsilon_r(t) + \varepsilon_m(t)] \ . \tag{10.35}$$

Since at the time of reheating, $\varepsilon_r(t_f) = \varepsilon_\Lambda$, the Friedmann equation can be rewritten as

$$H(t)^2 = H_\Lambda^2 \left[\Omega_r(t_f) \frac{a(t_f)^4}{a^4} + \Omega_m(t_f) \frac{a(t_f)^3}{a^3} \right] \ , \tag{10.36}$$

where

$$H_\Lambda^2 = \frac{8\pi G}{3c^2} \varepsilon_\Lambda = (17.31\,\mathrm{Gyr})^{-2} \ . \tag{10.37}$$

In Section 5.4.4 of the text, I give the Friedmann equation for the radiation + matter stage of the early universe:

$$H(t)^2 = H_0^2 \left[\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \right] \ . \tag{10.38}$$

Comparing this equation with equation (10.36) above, I find that I can loot and pillage the results of Section 5.4.4 by making the substitutions $H_0 \to H_\Lambda$, $\Omega_{r,0} \to \Omega_r(t_f) a(t_f)^4$, and $\Omega_{m,0} \to \Omega_m(t_f) a(t_f)^3$. In particular, Eq. 5.113 of the text gives the time of radiation-matter equality in the early universe:

$$t_{rm} = \frac{4}{3} \left(1 - \frac{1}{\sqrt{2}} \right) \frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2} H_0^{-1} \ . \tag{10.39}$$

In the far future universe, this translates to the equation

$$t_{rm} = \frac{4}{3} \left(1 - \frac{1}{\sqrt{2}} \right) \frac{\Omega_r(t_f)^{3/2}}{\Omega_m(t_f)^2} H_\Lambda^{-1} \tag{10.40}$$

$$= 0.391 \frac{1}{(2.09 \times 10^{-52})^2} 17.31\,\mathrm{Gyr} = 1.5 \times 10^{104}\,\mathrm{Gyr} \tag{10.41}$$

Now that I have caused your students' calculators to explode, my job here is done.

11
Structure formation: Gravitational instability

Chapter 11 in the second edition corresponds to Chapter 12 in the first edition.

In the course of revising Chapter 11, I slipped in the phrases "cosmic web" and "ΛCDM," not explicitly used in the first edition, but useful to know if you are going to spend any time chatting with cosmologists.

Section 11.5 (Hot versus Cold) was extensively rewritten to clarify the distinction between "hot" and "cold" in the context of structure formation.

For structure formation aficionados, I point out that the normalization $\delta M/M = 1$ at $M = 10^{14}\,\mathrm{M}_\odot$ corresponds to $\sigma_8 = 0.80$. (For non-aficionados, σ_8 is the rms mass fluctuation in a a sphere of comoving radius $r = 8h^{-1}\,\mathrm{Mpc}$, where $h \equiv H_0/100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.)[1]

Section 11.6 (Baryon Acoustic Oscillations) is completely new, taking advantage of the measurement of the "BAO bump" since the first edition was published.

Exercises

11.1 *Consider a spatially flat, matter-dominated universe ($\Omega = \Omega_m = 1$) that is contracting with time. What is the functional form of $\delta(t)$ in such a universe?*

In the general case, the linear perturbation equation for the growth of structure is (Eq. 11.49 of the text)

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0 \; . \tag{11.1}$$

[1] While writing *Introduction to Cosmology*, I swore a mighty oath not to use the parameter $h = H_0/100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in the text. I hope that letting it slip into the Instructor's Manual and onto the back cover of the second edition doesn't constitute a violation of my oath.

In a flat, matter-dominated universe, $\Omega_m = 1$. If that universe is *contracting*, then

$$H = \frac{\dot{a}}{a} = -\frac{2}{3t} < 0 \ . \tag{11.2}$$

The growth equation in a contracting, flat, matter-dominated universe is then

$$\ddot{\delta} - \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0 \ . \tag{11.3}$$

Because of the change in sign of H, the term involving $\dot{\delta}$ is no longer a "Hubble friction" term, slowing the growth of δ, but a "Hubble boost" term, speeding the growth of δ. If I try a powerlaw solution, of the form $\delta = Dt^m$, I find that

$$m(m-1)t^{m-2} - \frac{4}{3t}mt^{m-1} - \frac{2}{3t^2}t^m = 0 \ , \tag{11.4}$$

implying that m must be a solution of the quadratic equation

$$m^2 - \frac{7}{3}m - \frac{2}{3} = 0 \ . \tag{11.5}$$

The two solutions are

$$m = \frac{7 \pm \sqrt{73}}{6} \ , \tag{11.6}$$

leading to a solution for δ that has the form

$$\delta = D_1 t^{2.59} + D_2 t^{-0.26} \ . \tag{11.7}$$

Thus, the growing mode grows more rapidly than the $\delta \propto t^{0.67}$ dependence in an expanding universe, and the decaying mode decays less rapidly than the $\delta \propto t^{-1}$ dependence in an expanding universe.

11.2 *Consider an empty, negatively curved, expanding universe, as described in Section 5.2. If a dynamically insignificant amount of matter ($\Omega_m \ll 1$) is present in such a universe, how do density fluctuations in the matter evolve with time? That is, what is the functional form of $\delta(t)$?*

The general form of the perturbation equation for the growth of structure is (Eq. 11.49 of the text)

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0 \ . \tag{11.8}$$

In a (nearly) empty universe, with $\Omega = \Omega_m \ll 1$, the scale factor grows (nearly) linearly with time (Eq. 5.28)

$$a(t) \approx t/t_0 \ , \tag{11.9}$$

and thus

$$H(t) \equiv \frac{\dot{a}}{a} \approx \frac{1}{t} \ . \tag{11.10}$$

The perturbation equation for the growth of δ then reduces to

$$\ddot{\delta} + \frac{2}{t}\dot{\delta} = 0 \tag{11.11}$$

in the limit $\Omega_m \to 0$. If I try solutions of the form $\delta = Ct^n$, I find that

$$n(n-1)t^{n-2} + 2nt^{n-2} = 0 \ , \tag{11.12}$$

leading to $n = 0$ and $n = -1$. Thus, in a nearly empty universe,

$$\delta = C_1 + C_2 t^{-1} \ . \tag{11.13}$$

There is no growing mode; in a nearly empty universe, the amplitude of fluctuations in the matter density approaches a constant value, while the mean matter density falls as $\rho \propto a(t)^{-3} \propto t^{-3}$.

11.3 *A volume containing a photon-baryon fluid is adiabatically expanded or compressed. The energy density of the fluid is $\varepsilon = \varepsilon_\gamma + \varepsilon_{\rm bary}$, and the pressure is $P = P_\gamma = \varepsilon_\gamma/3$. What is $dP/d\varepsilon$ for the photon-baryon fluid? What is the sound speed, c_s? In Equation 11.26, how large an error did we make in our estimate of λ_J(before) by ignoring the effect of the baryons on the sound speed of the photon-baryon fluid?*

A box full of photon-baryon fluid undergoes homogeneous, isotropic expansion (or contraction) described by a scale factor $a(t)$. The energy density of the fluid is then

$$\varepsilon(a) = \varepsilon_\gamma(a) + \varepsilon_{\rm bary}(a) = \varepsilon_{\gamma,0} a^{-4} + \varepsilon_{\rm bary} a^{-3} \ , \tag{11.14}$$

and the ratio of the rest energy of the baryons to the energy of the photons is

$$f(a) \equiv \frac{\varepsilon_{\rm bary}}{\varepsilon_\gamma} = \frac{\varepsilon_{\rm bary,0}}{\varepsilon_{\gamma,0}} a \ . \tag{11.15}$$

The pressure of the photon-baryon fluid is

$$P = \frac{1}{3}\varepsilon_\gamma = \frac{1}{3}\varepsilon_{\gamma,0} a^{-4} \ . \tag{11.16}$$

As the box expands (or contracts),

$$\frac{d\varepsilon}{da} = -4\varepsilon_{\gamma,0} a^{-5} - 3\varepsilon_{\rm bary,0} a^{-4} \tag{11.17}$$

and

$$\frac{dP}{da} = -\frac{4}{3}\varepsilon_{\gamma,0} a^{-5} \ . \tag{11.18}$$

Thus, combining the two above results,

$$\frac{dP}{d\varepsilon} = \frac{-(4/3)\varepsilon_{\gamma,0}}{-4\varepsilon_{\gamma,0} - 3\varepsilon_{\text{bary},0}a} \tag{11.19}$$

$$= \frac{1}{3}\frac{1}{1+0.75f(a)} \ . \tag{11.20}$$

The sound speed of the photon-baryon fluid is then

$$c_s = c\left(\frac{dP}{d\varepsilon}\right)^{1/2} = \frac{c}{\sqrt{3}}\frac{1}{[1+0.75f(a)]^{1/2}} \ . \tag{11.21}$$

Thus, as long as $f(a)$, the ratio of the baryon rest energy to the photon energy, remains much smaller than one, the sound speed will be close to the sound speed $c/\sqrt{3}$ for a pure photon gas. At the time of decoupling, the ratio f is (compare to Eqs. 8.59 and 8.60 of the text)

$$f = \frac{\varepsilon_{\text{bary}}(t_{\text{dec}})}{\varepsilon_\gamma(t_{\text{dec}})} = \frac{\Omega_{\text{bary},0}}{\Omega_{\gamma,0}(1+z_{\text{dec}})} = 0.82 \ . \tag{11.22}$$

The sound speed of the photon-baryon fluid immediately before decoupling was then

$$c_s = \frac{c}{\sqrt{3}}\frac{1}{[1+0.75\cdot 0.82]^{1/2}} = 0.79\frac{c}{\sqrt{3}} = 0.46c \ . \tag{11.23}$$

Thus, since $\Lambda_J \propto c_s$, I overestimated Λ_J(before) by a factor $1/0.79 = 1.27$ by ignoring the baryonic contribution to the photon-baryon fluid.

11.4 *Suppose that the stars in a disk galaxy have a constant orbital speed v out to the edge of its spherical dark halo, at a distance R_{halo} from the galaxy's center. If a bound structure, such as a galaxy, forms by gravitational collapse of an initially small density perturbation, the minimum time for collapse is $t_{\min} \approx t_{\text{dyn}} \approx 1/\sqrt{G\overline{\rho}}$. Show that $t_{\min} \approx R_{\text{halo}}/v$ for a disk galaxy. What is $t_{\min}$ for our own galaxy? What is the maximum possible redshift at which you would expect to see galaxies comparable in v and R_{halo} to our own galaxy? (Assume the Benchmark Model is correct.)*

If a galaxy has constant orbital speed v, the mass enclosed within a radius r is (Eq. 7.12 of the text)

$$M(r) = \frac{v^2 r}{G} \ . \tag{11.24}$$

The mean density inside the cutoff radius R_{halo} is then

$$\overline{\rho} = \frac{3}{4\pi R_{\text{halo}}^3}M(R_{\text{halo}}) = \frac{3v^2}{4\pi G R_{\text{halo}}^2} \ . \tag{11.25}$$

This leads to a minimum time for collapse

$$t_{\rm min} \approx (G\overline{\rho})^{-1/2} \approx \left(\frac{4\pi}{3}\right)^{1/2} \frac{R_{\rm halo}}{v} \approx 2.0 \frac{R_{\rm halo}}{v} \ . \tag{11.26}$$

Scaling to $R_{\rm halo} = 100\,{\rm kpc} = 3.09\times 10^{21}\,{\rm m}$ and $v = 235\,{\rm km\,s^{-1}} = 2.35\times 10^5\,{\rm m\,s^{-1}}$, the minimum time for our galaxy to form by gravitational collapse is

$$t_{\rm min} \approx 2.7\times 10^{16}\,{\rm s} \left(\frac{R_{\rm halo}}{100\,{\rm kpc}}\right)\left(\frac{v}{235\,{\rm km\,s^{-1}}}\right)^{-1} \tag{11.27}$$

$$\approx 0.85\,{\rm Gyr} \left(\frac{R_{\rm halo}}{100\,{\rm kpc}}\right)\left(\frac{v}{235\,{\rm km\,s^{-1}}}\right)^{-1} \ . \tag{11.28}$$

A time $t \sim 1\,{\rm Gyr}$ corresponds to the matter-dominated epoch of the Benchmark Model, when the relation between scale factor and time was (Eq. 5.102 of the text)

$$a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}} H_0 t\right)^{2/3} \ , \tag{11.29}$$

leading to a redshift-time relation

$$z = \frac{1}{a(t)} - 1 \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}} H_0 t\right)^{-2/3} - 1. \tag{11.30}$$

Using $t_{\rm min}$ from equation (11.28) above, and adopting the parameters of the Benchmark Model, I find that the maximum possible redshift at which a galaxy like our own could form is

$$z_{\rm max} \approx 7.4 \left(\frac{R_{\rm halo}}{100\,{\rm kpc}}\right)^{-2/3}\left(\frac{v}{235\,{\rm km\,s^{-1}}}\right)^{2/3} - 1 \ . \tag{11.31}$$

11.5 *Within the Coma cluster, as discussed in Section 7.3, galaxies have a root mean square velocity of* $\langle v^2\rangle^{1/2} \approx 1520\,{\rm km\,s^{-1}}$ *relative to the center of mass of the cluster; the half-mass radius of the Coma cluster is* $r_h \approx 1.5\,{\rm Mpc}$*. Using arguments similar to those of the previous problem, compute the minimum time* $t_{\rm min}$ *required for the Coma cluster to form by gravitational collapse.*

From the virial relation given by Eq. 7.31 of the text, the mass of the Coma cluster is

$$M = \frac{\langle v^2\rangle r_h}{\alpha G} \ , \tag{11.32}$$

where $\alpha \approx 0.45$. The mean density inside the half-mass radius is then

$$\overline{\rho}_h = \frac{3}{4\pi r_h^3}\frac{M}{2} = \frac{3\langle v^2\rangle}{8\pi\alpha G r_h^2} \ . \tag{11.33}$$

This implies that the minimum time required to form the central half of the Coma cluster is

$$t_{\rm min} \approx (G\overline{\rho}_h)^{-1/2} \approx \left(\frac{8\pi\alpha}{3}\right)^{1/2} \frac{r_h}{\langle v^2\rangle^{1/2}} \approx 1.9\frac{r_h}{\langle v^2\rangle^{1/2}} \ , \tag{11.34}$$

assuming $\alpha = 0.45$. With $r_h = 1.5\,{\rm Mpc} = 4.63\times10^{22}\,{\rm m}$ and $\langle v^2\rangle^{1/2} = 1520\,{\rm km\,s^{-1}} = 1.52\times10^6\,{\rm m\,s^{-1}}$, this becomes

$$t_{\rm min} \approx 5.8\times10^{16}\,{\rm s} \approx 1.8\,{\rm Gyr} \ . \tag{11.35}$$

(Remember, this is the time for the inner half of the Coma cluster's mass to collapse; the outer half will come piling in later.) Although I didn't ask for it, $t_{\rm min} \approx 1.8\,{\rm Gyr}$ corresponds to $z_{\rm max} \approx 3.5$.

11.6 *Suppose that the density fluctuations $\delta(\vec r)$ in the early universe constitute a Gaussian field with a power spectrum $P(k)$ that equals zero above some maximum wavenumber $k_{\rm max}$. This maximum wavenumber corresponds to a minimum length scale $r_{\rm min} = 2\pi/k_{\rm max}$ and a minimum mass scale $M_{\rm min} = (4\pi/3)\rho_m r_{\rm min}^3$. Show that for $M < M_{\rm min}$, the mean square mass fluctuation, $\langle(\delta M/M)^2\rangle$, is equal to σ_δ^2 for the density field.*

If the power spectrum has $P(k) = 0$ for $k > k_{\rm max}$, this means there is no power on comoving scales $r < r_{\rm min} = 2\pi/k_{\rm max}$. The standard deviation in the density field $\delta(\vec r)$ is, from Eq. 11.67 of the text,

$$\sigma_\delta^2 = \frac{V}{2\pi}\int_0^\infty P(k)k^2 dk \ . \tag{11.36}$$

However, with no power at $k > k_{\rm max}$, we can truncate the range of integration to

$$\sigma_\delta^2 = \frac{V}{2\pi}\int_0^{k_{\rm max}} P(k)k^2 dk \ . \tag{11.37}$$

Now consider the mean square mass fluctuation in spheres of comoving radius r and mass

$$M = \frac{4\pi}{3} r^3 \rho_{m,0} \ . \tag{11.38}$$

From Eq. 11.70 of the text, the mean square mass fluctuation is

$$\langle\left(\frac{\delta M}{M}\right)^2\rangle = \frac{9V}{2\pi^2 r^2}\int_0^\infty P(k) j_1(kr)^2 dk \ , \tag{11.39}$$

where $j_1(x) = (\sin x - x\cos x)/x^2$. However, with no power at $k > k_{\rm max}$, we can truncate the range of integration to

$$\langle\left(\frac{\delta M}{M}\right)^2\rangle = \frac{9V}{2\pi^2 r^2}\int_0^{k_{\rm max}} P(k)j_1(kr)^2 dk \ . \tag{11.40}$$

Now consider the case where the radius of the sphere is $r < r_{\rm min} = 2\pi/k_{\rm max}$. In that case, the entire integral of equation (11.40) has $kr < kr_{\rm min} \leq k_{\rm max}r_{\rm min}$. Since $k_{\rm max}r_{\rm min} = 2\pi$, this means that $kr < 2\pi$ over the entire integral in equation (11.40). With kr limited to relatively small values, it is useful to do a series expansion of $j_1(kr)$ to find

$$j_1(kr) = \frac{1}{k^2r^2}(\sin kr - kr\cos kr) \approx \frac{1}{3}kr(1 - \frac{1}{10}k^2r^2) \ , \tag{11.41}$$

and thus

$$j_1(kr)^2 \approx \frac{1}{9}k^2r^2(1 - \frac{1}{5}k^2r^2) \ . \tag{11.42}$$

By plugging this expansion for j_1 into equation (11.40) above, I find

$$\langle\left(\frac{\delta M}{M}\right)^2\rangle \approx \frac{V}{2\pi^2}\left[\int_0^{k_{\rm max}} P(k)k^2 dk - \frac{r^2}{5}\int_0^{k_{\rm max}} P(k)k^4 dk\right] \ . \tag{11.43}$$

Comparing to equation (11.37) above, I can write this result as

$$\langle\left(\frac{\delta M}{M}\right)^2\rangle \approx \sigma_\delta^2\left[1 - \frac{r^2}{5r_c^2}\right] \ , \tag{11.44}$$

where

$$r_c^2 = \frac{\int_0^{k_{\rm max}} P(k)k^2 dk}{\int_0^{k_{\rm max}} P(k)k^4 dk} \ . \tag{11.45}$$

Thus, $\langle(\delta M/M)^2\rangle \to \sigma_\delta^2$ in the limit $r \to 0$.

Notice that if $P(k) \propto k^n$, with $n > -3$, then

$$r_c^2 = \frac{\frac{1}{3+n}k_{\rm max}^{3+n}}{\frac{1}{5+n}k_{\rm max}^{5+n}} = \frac{5+n}{3+n}\frac{r_{\rm min}^2}{4\pi^2} \ . \tag{11.46}$$

From equation (11.44) above, the limit for which the approximation $\langle(\delta M/M)^2\rangle = \sigma_\delta^2$ holds good is

$$r \ll 5r_c^2 = \frac{5(5+n)}{4\pi^2(3+n)}r_{\rm min}^2 \ , \tag{11.47}$$

or

$$M \ll \left[\frac{5(5+n)}{4\pi^2(3+n)}\right]^{3/2} M_{\rm min} \ . \tag{11.48}$$

For a truncated Harrison-Zeldovich spectrum ($n = 1$), this is equivalent to

$$M \ll 0.1 M_{\rm min} \ , \tag{11.49}$$

stricter than the $M < M_{\rm min}$ criterion that I wrote in the problem.

12

Structure formation: Baryons and photons

Chapter 12 is entirely new for the second edition; it was written in response to a (moderate) clamor by readers for more coverage of structure formation.

In the introduction to this section, the solar neighborhood density $\rho_{\rm sn} \approx 0.095\,{\rm M}_\odot\,{\rm pc}^{-3}$ is taken from Holmberg & Flynn (2000).

Figure 12.1, showing the baryonic census of the universe, is borrowed from Figure 1.2 of *Interstellar & Intergalactic Medium* by Barbara Ryden & Richard Pogge, available from Google Books for the low, low price of $9.99. [End of commercial interruption.]

In Section 12.2, the value $\tau = 0.066 \pm 0.016$ that I use for the optical depth of reionized gas is the *Planck* TT + lowP + lensing value from Ade et al. (2016). The more recent analysis of Aghanim et al. (2016) gives a smaller value $\tau = 0.055 \pm 0.009$. (Calculating the resulting shift in the time of reionization is left as an exercise for the reader.)

The statement that a $30\,{\rm M}_\odot$ star produces $\sim 10^{63}$ ionizing photons in its main sequence lifetime is based on the work of Sternberg, Hoffman, & Pauldrach (2003). Generally, their results show that every O star with initial mass $M \geq 30\,{\rm M}_\odot$ produces $\sim 3 \times 10^{62}$ ionizing photons per solar mass over its main sequence lifetime.

Exercises

12.1 *For the Schechter luminosity function of galaxies (Equation 12.21), find the number density of galaxies more luminous than L, as a function of L^*, Φ^*, and α. In the limit $L \to 0$, show why $\alpha = -1$ leads to problems, mathematically speaking. What is a plausible physical solution to this mathematical problem? [Hint: an acquaintance with incomplete gamma functions will be useful.]*

For a Schechter luminosity function,

$$\Phi(L)dL = \Phi^* \left(\frac{L}{L^*}\right)^\alpha \exp\left(-\frac{L}{L^*}\right)\frac{dL}{L^*} \ , \qquad (12.1)$$

the number density of galaxies more luminous than some value L is

$$n(> L) = \int_L^\infty \Phi(L')dL' \ , \qquad (12.2)$$

or, using the variable of integration $t \equiv L'/L^*$,

$$n(> L) = \Phi^* \int_{L/L^*}^\infty t^\alpha e^{-t} dt \ . \qquad (12.3)$$

Since the incomplete gamma function is defined as

$$\Gamma(\beta, x) \equiv \int_x^\infty t^{\beta-1} e^{-t} dt \ , \qquad (12.4)$$

we may write

$$n(> L) = \Phi^* \Gamma(\alpha + 1, L/L^*) \ . \qquad (12.5)$$

For bright galaxies ($L > L_*$), this can be expanded as

$$n(> L) \approx \Phi^* \left(\frac{L}{L^*}\right)^\alpha \exp\left(-\frac{L}{L^*}\right) [1 + \alpha(L^*/L)] \ . \qquad (12.6)$$

However, problems can arise in the limit $L \to 0$. The total number density of galaxies, integrating over the entire Schechter function, is

$$n_{\rm tot} = \Phi^* \int_0^\infty t^\alpha e^{-t} dt \ . \qquad (12.7)$$

When $\alpha > -1$, this is finite, and can be written as

$$n_{\rm tot} = \Phi^* \Gamma(\alpha + 1) \ , \qquad (12.8)$$

where $\Gamma(x)$ is the standard gamma function. However, when $\alpha \leq -1$, the integral in equation (12.7) diverges at the origin.

Don't panic. The infinite number density of galaxies implied by a Schechter function with $\alpha \leq -1$ should not alarm us. Galaxies cannot have an arbitrarily small luminosity. For example, suppose that the slope of the Schechter function is $\alpha = -1$ down to some minimum luminosity $L_{\rm min}$.[1] The total number density of galaxies will then be

$$n_{\rm tot} = \Phi^* \int_{L_{\rm min}/L^*}^\infty \frac{e^{-t} dt}{t} \sim \Phi^* \ln(L^*/L_{\rm min}) \ , \qquad (12.9)$$

[1] It is unlikely that α is perfectly constant all the way down to $L_{\rm min}$; the luminosity function of dwarf galaxies is poorly constrained. Bear with me, however, for the sake of a simple numerical argument.

assuming $L_{\rm min} \ll L^*$. For our minimum luminosity, let's take the example of a pathetic dwarf galaxy with a single M main sequence star in its dark halo, with $L_{\rm min} \sim 10^{-5}\,{\rm L}_\odot$. Given $L^* \approx 2\times10^{10}\,{\rm L}_\odot \sim 2\times10^{15} L_{\rm min}$, this yields a total number density of galaxies $n_{\rm tot} \sim 35\Phi^* \neq \infty$.

12.2 *For the Schechter luminosity function of galaxies, find the total luminosity density Ψ as a function of L^*, Φ^*, and α. What is the numerical value of the luminosity density Ψ_V in the V band, given $L_V^* = 2 \times 10^{10} L_{\odot,V}$, $\Phi^* = 0.005\,{\rm Mpc}^{-3}$, and $\alpha = -1$?*

The total luminosity density Ψ is given by the relation

$$\Psi = \int_0^\infty L'\Phi(L')dL' \ . \tag{12.10}$$

For the Schechter luminosity function, this becomes

$$\Psi = \Phi^* L^* \int_0^\infty t^{\alpha+1} e^{-t} dt \ . \tag{12.11}$$

If $\alpha > -2$, this is finite, and can be written in terms of the gamma function as

$$\Psi = \Phi^* L^* \Gamma(\alpha + 2) \ . \tag{12.12}$$

In the V band, we take $\Phi^* = 5 \times 10^{-3}\,{\rm Mpc}^{-3}$, $L_V^* = 2 \times 10^{10} L_{\odot,V}$, and $\alpha = -1$. This leads to $\Gamma(\alpha + 2) = \Gamma(1) = 1$, and thus

$$\Psi_V = \Phi^* L_V^* = 10^8 L_{\odot,V}\,{\rm Mpc}^{-3} \ . \tag{12.13}$$

12.3 *On a mass scale $M = 10^{17}\,{\rm M}_\odot$ the root mean square mass fluctuation is $\delta M/M = 0.12$ today (see Figure 11.5). Do you expect to see any gravitationally collapsed structures with a mass $M = 10^{17}\,{\rm M}_\odot$ in the directly visible universe today? Explain why or why not.*

For this problem, I can redo the "density lottery" argument used in Section 12.4. If the mass inside the last scattering surface is, from Eq. 12.38 of the text,

$$M \approx 4.3 \times 10^{23}\,{\rm M}_\odot \ , \tag{12.14}$$

then I can divide the directly visible universe into $\sim 4.3 \times 10^6$ different regions, each of mass $10^{17}\,{\rm M}_\odot$. The densest of these regions will have won a density lottery with a probability $P \approx 1/4.3\times10^6 \approx 2.3\times10^{-7}$ of drawing the winning ticket. In a Gaussian distribution, this corresponds to a 5.1σ deviation. Today, a 1σ fluctuation corresponds to $\delta M/M = 0.12$ on this mass scale. Thus, the densest $10^{17}\,{\rm M}_\odot$ region inside the last scattering surface most likely has an overdensity $\sim 5.1(0.12) \sim 0.6$, and has not yet begun its collapse.

To put it another way, since a 1σ overdensity has $\delta M/M = 0.12$ today, for a region to have collapsed by now, it must have a $> 8.33\sigma$ overdensity. In a Gaussian distribution, this corresponds to a probability $P \sim 4 \times 10^{-17}$; given that we have drawn only $\sim 4.3 \times 10^6$ tickets, that gives us one chance in 6 billion of holding a winning ticket, and seeing a collapsed structure with $M = 10^{17}\,\mathrm{M}_\odot$ in the directly visible universe.

12.4 *The universe will end in a Big Rip if the dark energy takes the form of phantom energy with $w < -1$ (see exercise 5.5). Since energy density has the dependence $\varepsilon \propto a^{-3(1+w)}$, the energy density ε_p of phantom energy increases as the universe expands; when ε_p/c^2 becomes larger than the mass density of a bound object, the object will be ripped apart. Suppose that the universe contains both matter and phantom energy with equation-of-state parameter $w_p = -1.1$. If the density parameters of the two components are $\Omega_{m,0} = 0.3$ and $\Omega_{p,0} = 0.7$, at what scale factor $a_{\rm gal}$ will a galaxy comparable to our Milky Way Galaxy be ripped apart? At what scale factor $a_\star$ will a star comparable to our Sun be ripped apart? If $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, how many years before the Big Rip will a galaxy be ripped apart? How many years before the Big Rip will a sunlike star be ripped apart? [Hint: the Big Rip is defined as the time $t_{\rm rip}$ when $a \to \infty$. The result of exercise 5.5 will be useful.]*

The mass equivalent of the phantom energy density is

$$\rho_p(a) = \rho_{p,0} a^{-(3+3w_p)} \ , \tag{12.15}$$

where the current density is $\rho_{p,0} = \Omega_{p,0}\rho_{c,0}$. In a universe with $\Omega_{p,0} = 0.7$ and $\rho_{c,0} = 8.7\times10^{-27}\,\mathrm{kg\,m^{-3}}$, this implies $\rho_{p,0} = 6.09\times10^{-27}\,\mathrm{kg\,m^{-3}}$. With $w_p = -1.1$, this leads to

$$\rho_p(a) = (6.09 \times 10^{-27}\,\mathrm{kg\,m^{-3}}) a^{0.3} \ . \tag{12.16}$$

The mean density of our galaxy (see Problem 11.4 above) can be written as

$$\overline{\rho}_{\rm gal} = \frac{3v^2}{4\pi G R_{\rm halo}^2} = 2.07 \times 10^{-23}\,\mathrm{kg\,m^{-3}} \left(\frac{R_{\rm halo}}{100\,\mathrm{kpc}}\right)^{-2} \ , \tag{12.17}$$

assuming a constant orbital speed $v = 235\,\mathrm{km\,s^{-1}}$. Thus, the galaxy will be ripped apart at a scale factor

$$a_{\rm gal} = \left(\frac{\overline{\rho}_{\rm gal}}{\rho_{p,0}}\right)^{-1/(3+3w_p)} \approx 6 \times 10^{11} \left(\frac{R_{\rm halo}}{100\,\mathrm{kpc}}\right)^{-6.67} \ . \tag{12.18}$$

The mean density of the Sun is $\overline{\rho}_\odot = 1400\,\mathrm{kg\,m^{-3}}$. Thus, a sunlike star will be ripped apart at a scale factor

$$a_\star = \left(\frac{\overline{\rho}_\odot}{\rho_{p,0}}\right)^{-1/(3+3w_p)} \approx 7 \times 10^{97} \ . \tag{12.19}$$

In the future, when phantom energy totally rules, the Friedmann equation will be

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_{p,0} a^{-(3+3w_p)} \ . \tag{12.20}$$

This can be rewritten as

$$a^{(1-3w_p)/2} \frac{da}{dt} = H_0 \sqrt{\Omega_{p,0}} \ . \tag{12.21}$$

Integrating up to the time of the Big Rip ($t = t_{\mathrm{rip}}$, $a = \infty$), we find

$$H_0 \sqrt{\Omega_{p,0}} \int_t^{t_{\mathrm{rip}}} dt = \int_a^\infty a^{(1-3w_p)/2} da \ , \tag{12.22}$$

or

$$t_{\mathrm{rip}} - t = \frac{H_0^{-1}}{\sqrt{\Omega_{p,0}}} \frac{2}{|3+3w_p|} a^{(3+3w_p)/2} \ . \tag{12.23}$$

This relation gives the time remaining until the Big Rip for any given scale factor. An object with mean density $\overline{\rho}$ will be ripped apart, as we have seen above, at a scale factor

$$a = \left(\frac{\overline{\rho}}{\Omega_{p,0}\rho_{c,0}}\right)^{-1/(3+3w_p)} \ . \tag{12.24}$$

Thus, substituting into equation (12.23) above, we find that it will be ripped apart when the time remaining until The Big Rip is

$$t_{\mathrm{rip}} - t = \frac{H_0^{-1}}{\sqrt{\Omega_{p,0}}} \frac{2}{|3+3w_p|} \left(\frac{\overline{\rho}}{\Omega_{p,0}\rho_{c,0}}\right)^{-1/2} \tag{12.25}$$

$$= H_0^{-1} \frac{2}{|3+3w_p|} \left(\frac{\rho_{c,0}}{\overline{\rho}}\right)^{1/2} \ . \tag{12.26}$$

For $w = -1.1$, $H_0^{-1} = 14.38\,\mathrm{Gyr}$, this becomes

$$t_{\mathrm{rip}} - t = 9.59 \times 10^{10}\,\mathrm{yr} \left(\frac{\rho_{c,0}}{\overline{\rho}}\right)^{1/2} \ . \tag{12.27}$$

For a galaxy like the Milky Way,

$$\overline{\rho}_{\rm gal}/\rho_{c,0} = 2390 \left(\frac{R_{\rm halo}}{100\,{\rm kpc}} \right)^{-2} \ , \tag{12.28}$$

and thus

$$t_{\rm rip} - t_{\rm gal} = 2.0 \times 10^9\,{\rm yr} \left(\frac{R_{\rm halo}}{100\,{\rm kpc}} \right) \ . \tag{12.29}$$

For a star like the Sun, $\overline{\rho}_\star = 1.6 \times 10^{29} \rho_{c,0}$, and thus

$$t_{\rm rip} - t_\star = 2.4 \times 10^{-4}\,{\rm yr} \ , \tag{12.30}$$

or about two hours.