# UNIT 8

# BETA AND GAMMA FUNCTIONS

## Structure



## 8.1 INTRODUCTION

You know about factorial function from earlier classes. Factorial function works for non-negative integers. For example, $\underline{|0} = 1$, $\underline{|1} = 1$, $\underline{|2} = 2 \times 1 = 2$, $\underline{|3} = 3 \times 2 \times 1 = 6, \ldots$. In this unit you will study a function known as gamma function which is related to factorial function by the relation gamma(n+1) = factorial(n), where n is non-negative integer, but other than this it also interpolates factorial function for non-integers values. Gamma function and its graphical behaviour are discussed in Sec. 8.2. Another function which is related to the gamma function is beta function which is discussed in Sec. 8.3. Graphical behaviour of the beta function is also discussed in the same section. Some properties of gamma and beta functions are discussed in Sec 8.4.

What we have discussed in this unit is summarised in Sec. 8.5. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit, some more questions based on the entire unit are given in Sec. 8.6 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 8.7.

In the next unit, you will study about change of order of Sigma and integration.

### Expected Learning Outcomes

After completing this unit, you should be able to:

- define gamma and beta functions;
- explain the effect of parameters of gamma and beta functions on their shapes; and
- establish some properties of gamma and beta functions.

## 8.2 GAMMA FUNCTION AND ITS GRAPHICAL BEHAVIOUR

You know that the factorial function is defined as follows.

$$\underline{|n} = n(n-1)(n-2)(n-3)\ldots 3.2.1 \qquad \ldots (8.1)$$

Consider the function $\overline{|n} = \int_0^\infty x^{n-1} e^{-x}\, dx,\ n > 0 \qquad \ldots (8.2)$

which is known as the gamma function.

After replacing n by n + 1 in equation (8.2), we get

$$\overline{|n+1} = \int_0^\infty x^{n} e^{-x}\, dx,\ n+1 > 0 \qquad \ldots (8.3)$$

If you evaluate values of the functions given by (8.1) and (8.3) at $n = 1, 2, 3, 4, 5, \ldots$. You will see that $\overline{|n+1} = \underline{|n}$. If you evaluate the value of the gamma function given in (8.2) or (8.3) manually then it will be time-consuming. You can do it easily in R. In the course MSTL-011 you have studied factorial() and gamma() functions to evaluate values of (8.1) and (8.2) respectively in R. The screenshot of R codes and their outputs in the R console is shown as follows.

```
> factorial(0:5) # to get factorial of the numbers 0, 1, 2, 3, 4 and 5
[1]   1   1   2   6  24 120
> gamma(1:6)  # to get values of gamma function at 1, 2, 3, 4, 5 and 6
[1]   1   1   2   6  24 120
```

Equation (8.1) makes sense only when $n = 1, 2, 3, 4, 5, \ldots$. By convention we also define $\underline{|0} = 1$. But equation (8.2) gives finite values for each real number n > 0. Again using R you can obtain values of the gamma function at any value of n > 0. Values of the Gamma Function at $n = 0.05, 0.15, 0.25, 0.50$ using R are given by running the following code on R console.

```
> gamma(c(0.05, 0.15, 0.25, 0.5))
[1] 19.470085  6.220273  3.625610  1.772454
```
… (8.4)

To see the similarity in the values of the functions (8.1) and (8.3) for $n = 1, 2, 3, 4, 5, \ldots$ graphically, you may refer to Fig. 8.1 (a) and (b). Note that the domain of the factorial function given by (8.1) is $\{0, 1, 2, 3, 4, 5, \ldots\}$ while the domain of the gamma function given by (8.3) is $(0, \infty)$. To see the effect of the domain on a function you may refer to Fig. 1.8 in Unit 1 of this course.

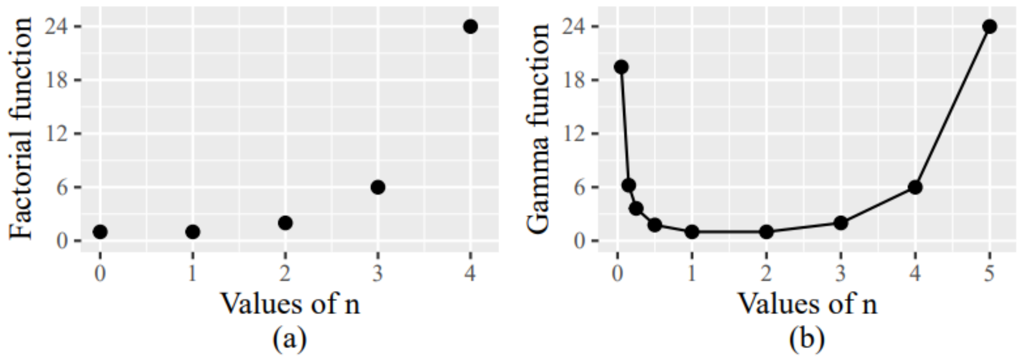


**Fig. 8.1: Visualisation of (a) factorial function (b) gamma function**

**Graphical Behaviour of the Function $y = f(x) = x^n e^{-x}$ and its Integral known as Gamma Function for different values of n**

From the discussion of Sec. 5.2 of Unit 5 of this course you know that single integral $\int_a^b f(x)\,dx$, represents area bounded by two vertical lines $x = a$, $x = b$, one horizontal line x-axis itself and the curve $y = f(x)$, refer (5.1). So, equation (8.3) gives area bounded by two vertical lines $x = a$, $x \to \infty$, one horizontal line x-axis itself and the curve $y = f(x) = x^n e^{-x}$. To understand the effect of n on the shape of the function $y = f(x) = x^n e^{-x}$ and value of the area given by equation (8.3), let us visualise both in Fig. 8.2 (a) to (f) for n = 1, 2, 3, 4, 5, 6 respectively. From the graph of the function $y = f(x) = x^n e^{-x}$ we observe that:

- **To the Left Side of Origin**
  - the graph goes downward when n is odd refer to Fig. 8.2 (a), (c) and (e). It happens so due to the reason that $e^{-x}$ being exponential function is always positive and $x^n$ will be negative when x is negative and n is odd. So, the product of a positive number and a negative number is negative. So, $x^n e^{-x}$ is negative to the left side of the origin. … (8.5)
  - the graph goes upward when n is even refer to Fig. 8.2 (b), (d) and (f). It happens so due to the reason that $e^{-x}$ being exponential function is always positive and $x^n$ will be positive when x is negative and n is even. So, the product of two positive numbers is positive. So, $x^n e^{-x}$ is positive to the left side of the origin.
- **To the Right Side of Origin**
  - the graph initially increases and attains its maximum value after attaining maximum value it starts decreasing and goes on decreasing and ultimately it tends to zero as x tends to infinity. You may refer to any of the Fig. 8.2 (a) to (f). It happens so due to the reason that initially the function $x^n$ maintains growth over the function $e^{-x}$ but after a certain value of x, the function $e^{-x}$ starts dominating the function $x^n$ and ultimately the value of the product $x^n e^{-x}$ is dictated by the exponential function $e^{-x}$ which tends to zero as x tends to infinity.
  - the area under the curve $y = f(x) = x^n e^{-x}$ increases as n increases but the important point is, it remains finite for all finite values of n. The actual value of the area under the curve is given by $\overline{|n+1}$. For example, Table

8.1 shows areas of shaded regions in Fig. 8.2 (a) to (f) which is given as follows.

**Table 8.1: Actual values of areas of shaded regions in Fig. 8.2 (a) to (f)**

| Value of n | Figure number | Shaded area in terms of gamma function | Shaded area in terms of factorial function | Actual shaded area |
|---|---|---|---|---|
| 1 | 8.2 (a) | $\overline{\vert 2}$ | $\underline{\vert 1}$ | 1 |
| 2 | 8.2 (b) | $\overline{\vert 3}$ | $\underline{\vert 2}$ | 2 |
| 3 | 8.2 (c) | $\overline{\vert 4}$ | $\underline{\vert 3}$ | 6 |
| 4 | 8.2 (d) | $\overline{\vert 5}$ | $\underline{\vert 4}$ | 24 |
| 5 | 8.2 (e) | $\overline{\vert 6}$ | $\underline{\vert 5}$ | 120 |
| 6 | 8.2 (f) | $\overline{\vert 7}$ | $\underline{\vert 6}$ | 720 |

- The shape of the function $y = f(x) = x^n e^{-x}$ to the right of zero looks like normal density which is discussed in Unit 12 of the course MST-12.

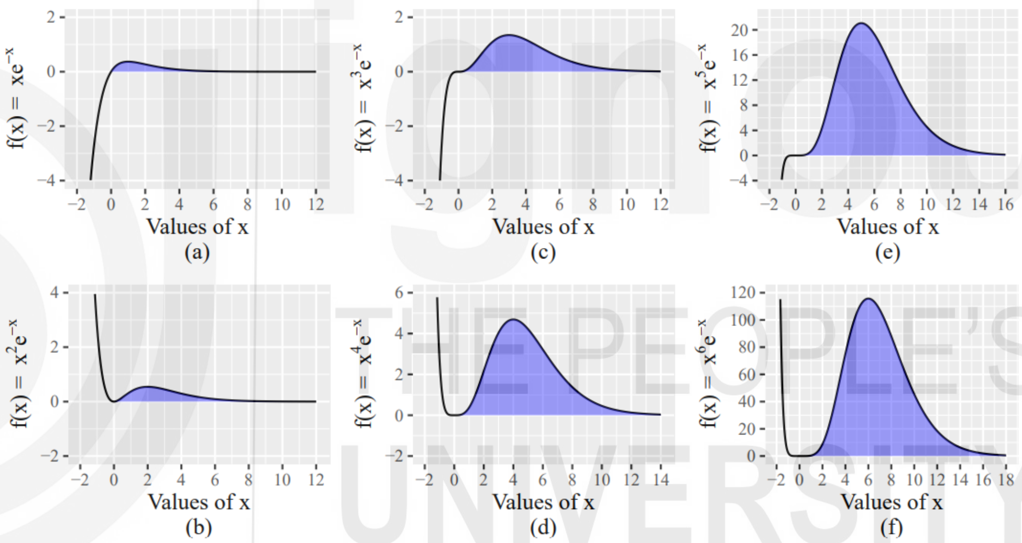


**Fig. 8.2: Visualisation of the function $y = f(x) = x^{n-1} e^{-x}$ by curve and**

**$\overline{|n} = \int_0^\infty x^{n-1} e^{-x} dx$ by shaded region for different values of n (a) n = 1 (b) n = 2 (c) n = 3 (d) n = 4 (e) n = 5 (f) n = 6**

So far in this section we have compared gamma function with factorial function and we have also seen effect of n on the shape of the function $y = f(x) = x^n e^{-x}$ which is integrand in the gamma function. Let us now formally define gamma function as follows.

**Definition of Gamma Function**: A function $\overline{|}\ : (0, \infty) \to (0, \infty)$ defined by

$\overline{|n} = \int_0^\infty x^{n-1} e^{-x} dx,\ n \in (0, \infty)$ is called gamma function.

Graph of gamma function for n > 0 is shown in Fig. 8.3 given as follows. Note that:

- It is a convex function since if you join any two points on it then the graph of the function will lie below the chord joining two points. … (8.6)

- Global minimum is at the point 1.461632145. It cannot be obtained graphically. To obtain it using calculus is beyond the scope of this course. You can verify it in R. Global minimum value at the point 1.461632145 is 0.8856032. … (8.7)

In the next section you will study similar analysis about beta function.

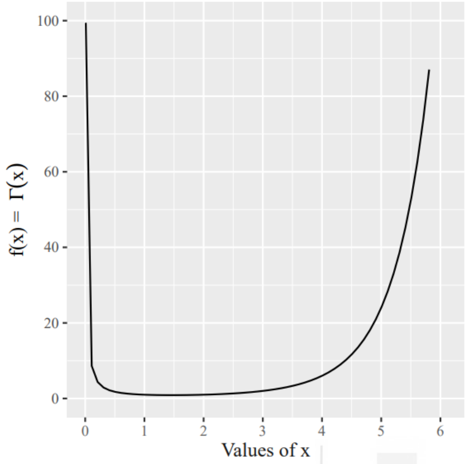


**Fig. 8.3: Visualisation of gamma function**

Now, you can try the following Self-Assessment Question.

**SAQ 1**

Does $\left\lceil 8 \right. = \left\lfloor 8 \right.$?

## 8.3 BETA FUNCTION AND ITS GRAPHICAL BEHAVIOUR

In Sec. 8.2 you have studied about graphical behaviour of:

- gamma function $\left\lceil n \right. = \int_0^\infty x^{n-1} e^{-x} dx,\ n \in (0, \infty)$;

- integrand of gamma function $y = f(x) = x^n e^{-x}$

for different values of n. We have also seen effect of n on the shape of the function $y = f(x) = x^n e^{-x}$ which is integrand in the gamma function.

In this section we will do similar study for beta function and its integrand. Let us start with the definition of beta function.

**Definition of Beta Function**: The integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0, n > 0$ is known as beta function and is denoted by B(m, n).

i.e., $$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0, n > 0 \qquad \ldots (8.8)$$

Let us first evaluate value of this function manually for some particular values of m and n as follows.

$$B(1, 1) = \int_0^1 x^{1-1}(1-x)^{1-1} dx = \int_0^1 1 dx \quad \left[\because x^0 = 1,\ (1-x)^0 = 1.\ \text{In fact, if a is any finite number other than zero then } a^0 = 1\right]$$

$$= [x]_0^1 = 1 - 0 = 1 \qquad \ldots (8.9)$$

$$B(2,\ 1) = \int_0^1 x^{2-1}(1-x)^{1-1}\,dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}(1-0) = \frac{1}{2} \qquad \ldots (8.10)$$

$$B(1,\ 2) = \int_0^1 x^{1-1}(1-x)^{2-1}\,dx = \int_0^1 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = \left(1 - \frac{1}{2} - 0 + 0\right) = \frac{1}{2} \qquad \ldots (8.11)$$

$$B(2,\ 2) = \int_0^1 x^{2-1}(1-x)^{2-1}\,dx = \int_0^1 x(1-x)\,dx = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3} - 0 + 0\right) = \frac{1}{6} \qquad \ldots (8.12)$$

Like this you can obtain values of beta function B(m, n) manually for different values of m and n. You can also calculate these values in R using beta() function. Screenshot of R codes and their outputs in R console is shown as follows.

```
> beta(1,1)    # m = 1, n = 1
[1] 1
> beta(2,1)    # m = 2, n = 1
[1] 0.5
> beta(1,2)    # m = 1, n = 2
[1] 0.5
> beta(2,2)    # m = 2, n = 2
[1] 0.1666667
```

… (8.13)

**Remark 1**: Note that values of B(2, 1) and B(1, 2) are equal. This did not happen by chance. In Sec. 8.4 you will prove that B(m, n) = B(n, m) for all values of m and n where m > 0, n > 0. This is known as **symmetric property of beta function**.

You can also obtain these values in a single command using c() function. Screenshot of R code and its output in R console is shown as follows.

```
> beta(c(1,2,1,2),c(1,1,2,2))
[1] 1.0000000 0.5000000 0.5000000 0.1666667
```

… (8.14)

So far we have obtained values of B(m, n) at positive integer values of m and n. But this function gives finite value for any values of m and n which are greater than zero. You can see it using R. Screenshot of R codes and their outputs in R console is shown as follows.

```
> beta(1/2, 1/2)
[1] 3.141593
> beta(1/2, 3/2)
[1] 1.570796
> beta(11/3, 7/6)
[1] 0.1987252
> beta(1/10, 1/3)
[1] 12.4657
```

Now, like gamma function let us analyse graphical behaviour of beta function and its integrand $y = f(x) = x^{m-1}(1-x)^{n-1}$ for different values of m and n. You may refer to Fig. 8.4 (a) to (j). In beta function, range of integral is 0 to 1 so it will represent area of the region bounded by two vertical lines x = 0, x = 1 one horizontal line x-axis itself and the function $y = f(x) = x^{m-1}(1-x)^{n-1}$. This area is shaded by light blue colour in the Fig. 8.4 (a) to (j).

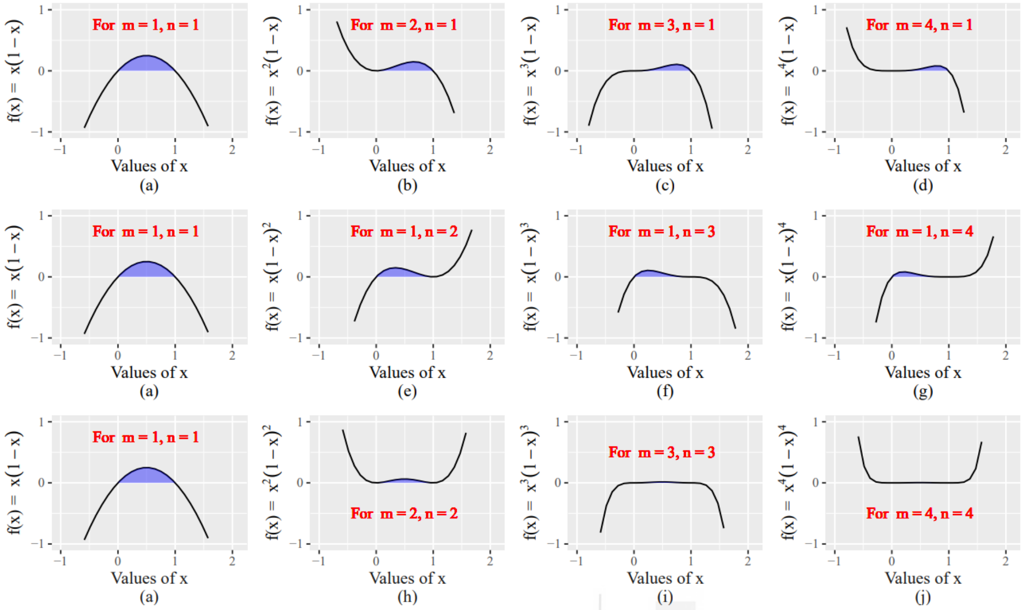


**Fig. 8.4: Visualisation of the function $y = f(x) = x^{m-1}(1-x)^{n-1}$ by curve and $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ by shaded region for different values of m and n (a) m = 1, n = 1 (b) m = 2, n = 1 (c) m = 3, n = 1 (d) m = 4, n = 1 (e) m = 1, n = 2 (f) m = 1, n = 3 (g) m = 1, n = 4 (h) m = 2, n = 2 (i) m = 3, n = 3 (j) m = 4, n = 4**

### Graphical Behaviour of the Function $y = f(x) = x^n e^{-x}$ and its Integral known as Beta Function for different values of m and n

Before studying graphical behaviour of the function $y = f(x) = x^{m-1}(1-x)^{n-1}$ and its integral first of all we make a remark.

**Remark 2**: In Table 8.2 we shall use three properties of beta and gamma functions which will be discussed in the next Sec. 8.4. So, for the time being assume that they hold. These properties are: (1) $\overline{|1} = 1$ (2) $\overline{|n} = \underline{|n-1}$

(3) $B(m, n) = \dfrac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$.

Now, from the Fig. 8.4 (a) to (j) we note the following points about the graphical behaviour of the function $y = f(x) = x^{m-1}(1-x)^{n-1}$ and its integral which is known as beta function and is denoted by B(m, n).

- When m is odd then curve is left downward, refer to Fig. 8.4 (a), (c), (e), (f). (g) and (i). … (8.15)
- When m is even then curve is left upward, refer to Fig. 8.4 (b), (d), (h) and (j). … (8.16)
- When n is odd then curve is right downward, refer to Fig. 8.4 (a) to (d), (f) and (i). … (8.17)
- When n is even then curve is right upward, refer to Fig. 8.4 (e), (g), (h) and (j). … (8.18)
- Area under the curve decreases as we keep one of m and n fix and increase other. This point is explained as follows. … (8.19)

For example: (1) In Fig. 8.4 (a) to (d) n = 1 is fix and m increases so area under the curve decreases. To see numerical values of area, refer to column 6 and row 1 to row 4 of the Table 8.2.

(2) In Fig. 8.4 (a), (e) to (g) m = 1 is fix and n increases so area under the curve decreases. To see numerical values of area, refer to column 6 and row 1, row 5 to row 8 of the Table 8.2.

- Area under the curve decreases as both m and n increase. … (8.20)

For example, in Fig. 8.4 (a), (h) to (j) both m and n increase so area under the curve decreases. To see numerical values of area, refer to column 6 and row 1, row 8 to row 10 of the Table 8.2.

**Table 8.2: Actual values of areas of shaded regions in Fig. 8.4 (a) to (j)**

| **Value of m** | **Value of n** | **Figure number** | **Shaded area in terms of beta function** | **Shaded area in terms of gamma function using property 3** | **Actual shaded area** | |
|---|---|---|---|---|---|---|
| 1 | 1 | 8.4 (a) | B(1, 1) | $\frac{\overline{\lvert 1}\,\overline{\lvert 1}}{\overline{\lvert 1+1}} = \frac{\underline{\lvert 0}\,\underline{\lvert 0}}{\underline{\lvert 1}} = 1$ | 1 | Decreasing ↓ |
| 2 | 1 | 8.4 (b) | B(2, 1) | $\frac{\overline{\lvert 2}\,\overline{\lvert 1}}{\overline{\lvert 2+1}} = \frac{\underline{\lvert 1}\,\underline{\lvert 0}}{\underline{\lvert 2}} = \frac{1}{2}$ | 0.5 | |
| 3 | 1 | 8.4 (c) | B(3, 1) | $\frac{\overline{\lvert 3}\,\overline{\lvert 1}}{\overline{\lvert 3+1}} = \frac{\underline{\lvert 2}\,\underline{\lvert 0}}{\underline{\lvert 3}} = \frac{1}{3}$ | $0.\bar{3}$ | |
| 4 | 1 | 8.4 (d) | B(4, 1) | $\frac{\overline{\lvert 4}\,\overline{\lvert 1}}{\overline{\lvert 4+1}} = \frac{\underline{\lvert 3}\,\underline{\lvert 0}}{\underline{\lvert 4}} = \frac{1}{4}$ | 0.25 | |
| 1 | 2 | 8.4 (e) | B(1, 2) | $\frac{\overline{\lvert 1}\,\overline{\lvert 2}}{\overline{\lvert 1+2}} = \frac{\underline{\lvert 0}\,\underline{\lvert 1}}{\underline{\lvert 2}} = \frac{1}{2}$ | 0.5 | Decreasing ↓ |
| 1 | 3 | 8.4 (f) | B(1, 3) | $\frac{\overline{\lvert 1}\,\overline{\lvert 3}}{\overline{\lvert 1+3}} = \frac{\underline{\lvert 0}\,\underline{\lvert 2}}{\underline{\lvert 3}} = \frac{1}{3}$ | $0.\bar{3}$ | |
| 1 | 4 | 8.4 (g) | B(1, 4) | $\frac{\overline{\lvert 1}\,\overline{\lvert 4}}{\overline{\lvert 1+4}} = \frac{\underline{\lvert 0}\,\underline{\lvert 3}}{\underline{\lvert 4}} = \frac{1}{4}$ | 0.25 | |
| 2 | 2 | 8.4 (h) | B(2, 2) | $\frac{\overline{\lvert 2}\,\overline{\lvert 2}}{\overline{\lvert 2+2}} = \frac{\underline{\lvert 1}\,\underline{\lvert 1}}{\underline{\lvert 3}} = \frac{1}{6}$ | $0.1\bar{6}$ | Decreasing ↓ |
| 3 | 3 | 8.4 (i) | B(3, 3) | $\frac{\overline{\lvert 3}\,\overline{\lvert 3}}{\overline{\lvert 3+3}} = \frac{\underline{\lvert 2}\,\underline{\lvert 2}}{\underline{\lvert 5}} = \frac{1}{30}$ | $0.0\bar{3}$ | |
| 4 | 4 | 8.4 (j) | B(4, 4) | $\frac{\overline{\lvert 4}\,\overline{\lvert 4}}{\overline{\lvert 4+4}} = \frac{\underline{\lvert 3}\,\underline{\lvert 3}}{\underline{\lvert 7}} = \frac{1}{140}$ | $\simeq 0.007$ | |

In the next section you will study some properties of gamma and beta functions.

Now, you can try the following Self-Assessment Question.

**SAQ 2**

Without calculating values of B(4, 2) and B(5, 2) give reason whether B(4, 2) > B(5, 2) is true or false?

## 8.4 PROPERTIES OF GAMMA AND BETA FUNCTIONS

Properties of gamma function help us to find value of the gamma function $\overline{|n}$ at various values of n with some known values of it. First, we will list some properties of gamma function which are required in different courses of this programme and then we will discuss their proofs.

**Property 1**: If $n > 0$ then recurrence relation for gamma function is $\overline{|n+1} = n\overline{|n}$. … (8.21p1)

**Property 2**: Prove that $\overline{|1} = 1$. … (8.21p2)

**Property 3**: If n is a positive integer, then prove that $\overline{|n} = \underline{|n-1}$. … (8.21p3)

**Property 4**: For n > 0, a > 0, prove that $\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\overline{|n}}{a^n}$ … (8.21p4)

**Property 5**: **Relation between beta and gamma functions**: For $m > 0,\ n > 0$, prove that $B(m, n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$. … (8.21p5)

**Remark 3**: In the proof of this property, we need idea of double integral which is discussed in the next unit, i.e., Unit 9. So, we will prove it in the next unit. So, right now assume that this result holds.

**Property 6**: For $p > -1,\ q > -1$, prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\overline{\left|\frac{p+1}{2}\right.}\,\overline{\left|\frac{q+1}{2}\right.}}{2\overline{\left|\frac{p+q+2}{2}\right.}}$.
… (8.21p6)

**Property 7**: Prove that $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$. … (8.21p7)

**Property 8**: Prove that beta function is symmetric in m and n, i.e., prove that B(m, n) = B(n, m). … (8.21p8)

**Property 9**: For m > 0, n > 0, prove that $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ … (8.21p9.1)

Hence, prove that $B(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$. … (8.21p9.2)

and $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$. … (8.21p9.3)

**Property 10**: Prove that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. … (8.21p10)

**Property 11**: Prove that $\int_{-\infty}^0 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. … (8.21p11)

**Property 12**: Prove that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$. … (8.21p12)

**Property 13**: Prove that

(i) $B(p, q) = B(p+1, q) + B(p, q+1)$ ... (8.21p13.1)

(ii) $\dfrac{B(p, q+1)}{q} = \dfrac{B(p+1, q)}{p} = \dfrac{B(p, q)}{p+q}$ ... (8.21p13.2)

**Property 14**: **Duplication Formula**: For m > 0 prove that

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m). \quad \ldots (8.21p14)$$

**Property 15**: Prove that $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi.$ … (8.21p15)

**Property 16**: Prove that $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)},\ n \notin \mathbb{Z}$ … (8.21p16)

Let us now prove these properties one at a time except property 16 whose proof is beyond the scope of this course.

**Proof of Property 1**: By definition of gamma function, we have

$$\Gamma(n+1) = \int_0^\infty x^{n+1-1} e^{-x} dx,\ n \in (0, \infty)$$

$$= \int_0^\infty x^n e^{-x} dx$$

Integrating by parts keeping $x^n$ as the first function and $e^{-x}$ as the second function, we get

$$\Gamma(n+1) = \left[x^n \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty n x^{n-1} \frac{e^{-x}}{-1} dx$$

$$= -(0-0) + n\int_0^\infty x^{n-1} e^{-x} dx \quad \left[\begin{array}{l}\because \text{ as } x \to \infty \text{ then } x^n \to \infty \text{ and } e^{-x} \to 0 \text{ but} \\ x^n e^{-x} \to 0 \text{ since } e^{-x} \text{ goes faster to zero} \\ \text{than } x^n \text{ goes to } \infty\end{array}\right]$$

$$= n\Gamma(n) \quad \left[\because \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx\right]$$

Hence, $\Gamma(n+1) = n\Gamma(n)$ … (8.22)

**Proof of Property 2**: We know that

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx,\ n \in (0, \infty)$$

Replacing n by 1, we get

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x} dx = \int_0^\infty e^{-x} dx \quad \left[\because x^0 = 1\right]$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = -(0-1) = 1 \quad \left[\because e^{-x} \to 0 \text{ as } x \to \infty\right]$$

Hence, $\Gamma(1) = 1.$ … (8.23)



**Proof of Property 3**: We know that (refer 8.22 and 8.23)

$\Gamma(n+1) = n\Gamma(n)$ … (8.24)

$\Gamma(1) = 1$ … (8.25)

Replacing n by n – 1 in (8.24), we get

$\Gamma(n) = (n-1)\Gamma(n-1)$ … (8.26)

Replacing n by n – 1 in (8.26), we get

$\Gamma(n-1) = (n-2)\Gamma(n-2)$ … (8.27)

Using (8.27) in RHS of (8.26), we get

$\Gamma(n) = (n-1)(n-2)\Gamma(n-2)$ … (8.28)

Replacing n by n – 1 in (8.27), we get

$\Gamma(n-2) = (n-3)\Gamma(n-3)$ … (8.29)

Using (8.29) in RHS of (8.28), we get

$\Gamma(n) = (n-1)(n-2)(n-3)\,\Gamma(n-3)$ … (8.30)

Continuing in this way after n – 4 more steps, we get

$$\Gamma(n) = (n-1)(n-2)(n-3)\ldots3.2.1\,\Gamma(1)$$
$$= (n-1)(n-2)(n-3)\ldots3.2.1 \quad \left[\because \Gamma(1)=1 \text{ using } (8.25)\right]$$
$$= \underline{|n-1}$$

**Proof of Property 4**: Let $I = \int_0^\infty x^{n-1} e^{-ax} dx,\ n \in (0, \infty)$ … (8.31)

Let us put ax = y so that (8.31) reduces to the form of gamma function

Differentiating, we get a dx = dy

Also, when x = 0, then y = 0 and

when $x \to \infty \Rightarrow y \to \infty \quad [\because a > 0]$

$\therefore$ (8.31) becomes

$$I = \int_0^\infty \left(\frac{y}{a}\right)^{n-1} e^{-y} \frac{1}{a} dy = \frac{1}{a^n}\int_0^\infty y^{n-1} e^{-y} dy = \frac{1}{a^n}\Gamma(n) \quad \left[\because \int_0^\infty x^{n-1} e^{-x} dx = \Gamma(n)\right]$$

Hence, $\int_0^\infty x^{n-1} e^{-ax} dx = \dfrac{\Gamma(n)}{a^n}$. … (8.32)

**Proof of Property 6**: We know that $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0, n > 0$

… (8.33)

Having a look at the required relation, we have to put $x = \sin^2\theta$

Differentiating, we get $dx = 2\sin\theta\cos\theta\, d\theta$

Also, when x = 0, then $\theta = 0$ and

when $x = 1 \Rightarrow \theta = \dfrac{\pi}{2}$

$\therefore$ (8.33) becomes

$$B(m,n)=\int_0^{\pi/2}\sin^{2m-2}\theta(1-\sin^2\theta)^{n-1}2\sin\theta\cos\theta\,d\theta=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

So, we have

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{1}{2}B(m,n)$$

$$=\frac{1}{2}\frac{\Gamma m\,\Gamma n}{\Gamma m+n}\qquad[\text{Using }(8.21\text{p}5)]\qquad\ldots(8.34)$$

As per the need of the required form, we have to replace 2m – 1 by p and 2n – 1 by q in (8.34). After doing so we have

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{1}{2}\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}\qquad\left[\begin{array}{l}\because 2m-1=p,\ 2n-1=q\\ \Rightarrow m=\dfrac{p+1}{2}\ \text{and}\ n=\dfrac{q+1}{2}\\ \text{Also, } m>0, n>0\Rightarrow p>-1, q>-1\end{array}\right]$$

$$\text{Hence, }\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}},\ p>-1,\ q>-1.\qquad\ldots(8.35)$$

**Proof of Property 7**: We know that

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{1}{2}\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}},\ p>-1,\ q>-1\qquad\ldots(8.36)$$

Putting p = 0, q = 0 in (8.36), we get

$$\int_0^{\pi/2}\sin^0\theta\cos^0\theta\,d\theta=\frac{1}{2}\frac{\Gamma\frac{0+1}{2}\,\Gamma\frac{0+1}{2}}{\Gamma\frac{0+0+2}{2}}\Rightarrow\int_0^{\pi/2}1\,d\theta=\frac{1}{2}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{2}{2}}\Rightarrow[\theta]_0^{\pi/2}=\frac{1}{2}\frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1}$$

$$\Rightarrow\frac{\pi}{2}-0=\frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2\quad[\because\Gamma 1=1\text{ using }(8.23)]$$

$$\Rightarrow\left(\Gamma\frac{1}{2}\right)^2=\pi\Rightarrow\Gamma\frac{1}{2}=\sqrt{\pi}\qquad\ldots(8.37)$$

**Proof of Property 8**: We know that

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m>0, n>0\qquad\ldots(8.38)$$

Putting $x=1-y$ in (8.38),

Differentiating, we get $dx=-dy$

Also, when x = 0, then $y=1$ and

when $x=1\Rightarrow y=0$

$\therefore$ (8.38) becomes

$$B(m, n)=\int_1^0 (1-y)^{m-1}(1-(1-y))^{n-1}(-dy),\ m>0, n>0$$

$$=-\int_1^0 (1-y)^{m-1} y^{n-1} dy$$

$$=\int_0^1 y^{n-1}(1-y)^{m-1} dy \qquad \left[\because \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx\right]$$

$$=B(n, m)$$

Hence, B(m, n) = B(n, m) … (8.39)

**Proof of Property 9**: We know that

$$B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m>0, n>0 \qquad \dots (8.40)$$

Putting $x=\frac{t}{1+t}$ in (8.40), … (8.41)

Differentiating, we get $dx=\frac{(1+t).dt-t(0+dt)}{(1+t)^2} \Rightarrow dx=\frac{1}{(1+t)^2}dt$

Also, (8.41) implies $t=\frac{x}{1-x}$ and so when x = 0, then $t=0$ and when $x \to 1 \Rightarrow t \to \infty$

Further, $1-x=1-\frac{t}{1+t}=\frac{1+t-t}{1+t}=\frac{1}{1+t}$, i.e., $1-x=\frac{1}{1+t}$

$\therefore$ (8.40) becomes

$$B(m, n)=\int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(\frac{1}{1+t}\right)^{n-1}\frac{1}{(1+t)^2}dt,\ m>0, n>0$$

$$=\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt,\ m>0, n>0$$

$$=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \left[\because \int_a^b f(u)\,du=\int_a^b f(v)\,dv\right]$$

Hence, $B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx,\ m>0, n>0$ … (8.42)

So, using (8.42), we get $B(n, m)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx,\ m>0, n>0$ … (8.43)

But B(m, n) = B(n, m) [using (8.39)]

Hence, $B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx,\ m>0, n>0$ … (8.44)

**Proof of Property 10**: Let $I=\int_0^\infty e^{-x^2}dx$ … (8.45)

Putting $x=\sqrt{t}$ in (8.45),

Differentiating, we get $dx = \frac{1}{2\sqrt{t}}dt$

Also, when x = 0, then $t = 0$ and

when $x \to \infty \Rightarrow t \to \infty$

$\therefore$ (8.45) becomes

$$I = \int_0^\infty e^{-t}\frac{1}{2\sqrt{t}}dt = \frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/2}\,dt = \frac{1}{2}\int_0^\infty e^{-t}\,t^{1/2-1}\,dt = \frac{1}{2}\left\lceil\frac{1}{2}\right. \quad \left[\because \lceil n = \int_0^\infty x^{n-1}e^{-x}dx,\ n > 0\right]$$

$$= \frac{1}{2}\sqrt{\pi} \qquad \left[\because \left\lceil\frac{1}{2}\right. = \sqrt{\pi} \quad \text{refer } (8.37)\right]$$

Hence, $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$ … (8.46)

**Proof of Property 11**: Let $I = \int_{-\infty}^{0} e^{-x^2}dx$ … (8.47)

Putting $x = -t$ in (8.47),

Differentiating, we get $dx = -dt$

Also, when x = 0, then $t = 0$ and

when $x \to -\infty \Rightarrow t \to \infty$

$\therefore$ (8.47) becomes

$$I = \int_\infty^0 e^{-t^2}(-dt) = -\int_\infty^0 e^{-t^2}\,dt = \int_0^\infty e^{-t^2}\,dt \quad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \frac{1}{2}\sqrt{\pi} \qquad \left[\because \int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi} \quad \text{refer } (8.46)\right]$$

Hence, $\int_{-\infty}^{0} e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$ … (8.48)

**Proof of Property 12**:

$$\text{Let } I = \int_{-\infty}^{\infty} e^{-x^2}dx = \int_{-\infty}^{0} e^{-x^2}dx + \int_0^\infty e^{-x^2}dx \quad \left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$$

$$= \frac{1}{2}\sqrt{\pi} + \frac{1}{2}\sqrt{\pi} \qquad [\text{Using } (8.46) \text{ and } (8.48)]$$

$$= \sqrt{\pi}$$

Hence, $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ … (8.49)

**Proof of Property 13**: (i)

$$\text{RHS} = B(p+1,\ q) + B(p,\ q+1) = \frac{\lceil p+1 \lceil q}{\lceil p+q+1} + \frac{\lceil p \lceil q+1}{\lceil p+q+1} \quad [\text{Using } (8.21\text{p}5)]$$

$$= \frac{p\lceil p \lceil q}{(p+q)\lceil p+q} + \frac{\lceil p\, q \lceil q}{(p+q)\lceil p+q} \qquad [\text{Using } (8.22)]$$

$$= \frac{(p+q)\Gamma p\,\Gamma q}{(p+q)\Gamma(p+q)} = \frac{\Gamma p\,\Gamma q}{\Gamma(p+q)} = B(p,\,q) = \text{LHS} \qquad [\text{Using } (8.21\text{p}5)]$$

(ii) $$\frac{B(p,\,q+1)}{q} = \frac{\Gamma p\,\Gamma(q+1)}{q\,\Gamma(p+q+1)} \qquad [\text{Using } (8.21\text{p}5)]$$

$$= \frac{\Gamma p\, q\,\Gamma q}{q\,(p+q)\Gamma(p+q)} \qquad [\text{Using } (8.22)]$$

$$= \frac{\Gamma p\,\Gamma q}{(p+q)\Gamma(p+q)} = \frac{B(p,\,q)}{p+q} \qquad \ldots (8.50)$$

Also,

$$\frac{B(p+1,\,q)}{p} = \frac{\Gamma(p+1)\Gamma q}{p\,\Gamma(p+q+1)} \qquad [\text{Using } (8.21\text{p}5)]$$

$$= \frac{p\,\Gamma p\,\Gamma q}{p\,(p+q)\Gamma(p+q)} \qquad [\text{Using } (8.22)]$$

$$= \frac{\Gamma p\,\Gamma q}{(p+q)\Gamma(p+q)} = \frac{B(p,\,q)}{p+q} \qquad \ldots (8.51)$$

From (8.50) and (8.51), we get $\frac{B(p,\,q+1)}{q} = \frac{B(p+1,\,q)}{p} = \frac{B(p,\,q)}{p+q}$ … (8.52)

**Proof of Property 14**: **Proof of Duplication Formula**: We know that

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)},\ \ p>-1,\ q>-1 \qquad \ldots (8.53)$$

Putting p = q in (8.53), we get

$$\int_0^{\pi/2} \sin^p\theta\cos^p\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{2\,\Gamma\left(\frac{p+p+2}{2}\right)} \Rightarrow \frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{2\,\Gamma(p+1)} = \int_0^{\pi/2}\left(\frac{2\sin\theta\cos\theta}{2}\right)^p d\theta$$

$$\Rightarrow \frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{2\,\Gamma(p+1)} = \frac{1}{2^p}\int_0^{\pi/2}(\sin 2\theta)^p\,d\theta \qquad \ldots (8.54)$$

Putting $2\theta = t$ in (8.54),

Differentiating, we get $2\,d\theta = dt$

Also, when $\theta = 0$, then $t = 0$ and

when $\theta = \frac{\pi}{2} \Rightarrow t \to \pi$

$\therefore$ (8.54) becomes

$$\frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{2\,\Gamma(p+1)} = \frac{1}{2^p}\int_0^{\pi}(\sin t)^p\,\frac{1}{2}\,dt = \frac{1}{2}\frac{1}{2^p}\int_0^{\pi}\sin^p t\,dt$$

$$= \frac{1}{2}\frac{1}{2^p}\, 2\int_0^{\pi/2} \sin^p t\,dt \quad \left[\begin{array}{l} \because \text{ If } f(2a-x)=f(x), \text{ then } \displaystyle\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \\ \text{Here, } a=\pi/2,\ f(t)=\sin^p t, \text{ and} \\ f(\pi-t)=\sin^p(\pi-t)=\sin^p t = f(t) \end{array}\right]$$

$$= \frac{1}{2^p}\int_0^{\pi/2} \sin^p t \cos^0 t\,dt$$

$$= \frac{1}{2^p}\,\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{p+0+2}{2}\right)} \qquad [\text{Using } (8.53)]$$

$$\Rightarrow \frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{\Gamma(p+1)} = \frac{1}{2^p}\,\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+0+2}{2}\right)}$$

$$\Rightarrow \frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)}{\Gamma(p+1)} = \frac{1}{2^p}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{p+2}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\ [\text{Using } (8.37)]\right] \qquad \ldots (8.55)$$

Putting $\frac{p+1}{2} = m$ in (8.55), we get

$$\Rightarrow \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{2m+1}{2}\right)} \qquad \left[\because \frac{p+1}{2} = m \Rightarrow p+1 = 2m \Rightarrow \frac{p+2}{2} = \frac{2m+1}{2}\right]$$

$$\Rightarrow \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m) \qquad \ldots (8.56)$$

Hence, proved.

**Proof of Property 15**: We know that (refer (8.56))

$$\Rightarrow \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m) \qquad \ldots (8.57)$$

Putting $m = \frac{1}{4}$ in (8.57), we get

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{1/2-1}}\,\Gamma\left(2\left(\frac{1}{4}\right)\right) \Rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}}{2^{-1/2}}\,\Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\sqrt{\pi}\sqrt{\pi} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\ [\text{Using } (8.37)]\right]$$

$$= \sqrt{2}\,\pi$$

Hence, $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$ $\qquad \ldots (8.58)$

**Proof of Property 16**: Beyond the scope of this course.

**Example 1:** Evaluate $\int_0^\infty \frac{x^3(1+x^4)}{(1+x)^{12}}\,dt$ using properties of gamma and beta functions.

**Solution:**

$$\int_0^\infty \frac{x^3(1+x^4)}{(1+x)^{12}}\,dx = \int_0^\infty \frac{x^3}{(1+x)^{12}}\,dx + \int_0^\infty \frac{x^7}{(1+x)^{12}}\,dx = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+8}}\,dx + \int_0^\infty \frac{x^{8-1}}{(1+x)^{8+4}}\,dx$$

$$= B(4,\ 8) + B(8,\ 4) \qquad \left[\because B(m,\ n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx\right]$$

$$= 2B(4,\ 8) \qquad \left[\because B(m,\ n) = B(n,\ m)\right]$$

$$= 2\frac{\Gamma 4\,\Gamma 8}{\Gamma(4+8)} \qquad \left[\because B(m,\ n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right]$$

$$= 2\frac{\lfloor 3\,\lfloor 7}{\lfloor 11} = 2\frac{6\,\lfloor 7}{11.10.9.8.\lfloor 7} = \frac{1}{660}$$

Now, you can try the following Self-Assessment Question.

---

**SAQ 3**

Evaluate $\int_0^\infty x^4 e^{-2x}\,dx$ using properties of beta and gamma function.

---

## 8.5 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- **Factorial function** is defined as $\lfloor n = n(n-1)(n-2)(n-3)\ldots 3.2.1.$ In R it can be obtained by using the built-in function factorial(n).
- **Gamma function** is defined as $\Gamma n = \int_0^\infty x^{n-1}e^{-x}\,dx,\ n > 0.$ In R it can be obtained by using the built-in function gamma(n).
- **Beta function** is defined as $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m > 0, n > 0$ In R it can be obtained by using the built-in function beta(m, n).
- **Some properties of gamma and beta** functions are:
  - $\Gamma 1 = 1$
  - $\Gamma(n+1) = n\Gamma n$
  - $\Gamma n = \lfloor n-1$
  - $\int_0^\infty x^{n-1}e^{-ax}\,dx = \frac{\Gamma n}{a^n}$
  - $\Gamma\frac{1}{2} = \sqrt{\pi}$
  - $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$
  - $\int_{-\infty}^0 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$
  - $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$
  - $\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \sqrt{2}\,\pi$
  - $B(m,\ n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

- $B(n,m)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

- $B(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

- $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{\left\lceil\frac{p+1}{2}\right.\left\lceil\frac{q+1}{2}\right.}{2\left\lceil\frac{p+q+2}{2}\right.}$

- $B(p,q)=B(p+1,q)+B(p,q+1)$

- $\frac{B(p,q+1)}{q}=\frac{B(p+1,q)}{p}=\frac{B(p,q)}{p+q}$

- B(m, n) = B(n, m) (This is known as symmetric property of beta function)

- $B(m,n)=\frac{\lceil m\lceil n}{\lceil m+n}$ (This is known as the relation between gamma and beta functions)

- $\lceil m\left\lceil m+\frac{1}{2}\right.=\frac{\sqrt{\pi}}{2^{2m-1}}\lceil 2m$ (This is known as duplication formula)

- $\lceil n\lceil 1-n=\frac{\pi}{\sin(n\pi)},\ n\notin\mathbb{Z}$ (This is known as **Euler Reflection formula**)

## 8.6 TERMINAL QUESTIONS

1. Express the given integral in beta function:
$\int_0^b x^{p-1}(b-x)^{q-1}dx,\ p>0,\ q>0.$

2. Express the given integral in beta function:
$\int_0^1 x^{p-1}(1-x^a)^{q-1}dx,\ p>0,\ q>0.$

3. Evaluate using properties of beta and gamma functions: $\int_0^\infty \frac{1}{1+x^4}dx.$

## 8.7 SOLUTIONS/ANSWERS

### Self-Assessment Questions (SAQs)

1. Answer is no, since $\lceil n=\lfloor n-1$, so $\lceil 8=\lfloor 7$.

2. We know that area under the curve of beta function decreases as we keep one of m and n fix and increase other. Here value of n is fix and equal to 2 but value m varies. Hence, B(4, 2) < B(5, 2) is false because 5 > 4 so value of B(4, 2) will be greater than value of B(5, 2).

3. We know that $\int_0^\infty x^{n-1}e^{-ax}dx=\frac{\lceil n}{a^n}$

In our case n = 5, a = 2, so

$$\int_0^\infty x^4 e^{-2x}\,dx = \int_0^\infty x^{5-1} e^{-2x}\,dx = \frac{\overline{|5}}{2^5} = \frac{\underline{|4}}{32} = \frac{24}{32} = \frac{3}{4}$$

## Terminal Questions

1. Let $I = \int_0^b x^{p-1}(b-x)^{q-1}\,dx,\ p>0,\ q>0$ … (8.59)

Putting x = bt

Differentiating, we get $dx = b\,dt$

Also, when $x = 0$, then $t = 0$ and

when $x = b \Rightarrow t = 1$

$\therefore$ (8.59) becomes

$$I = \int_0^1 (bt)^{p-1}(b-bt)^{q-1}\,b\,dt = \int_0^1 b^{p-1}t^{p-1}b^{q-1}(1-t)^{q-1}\,b\,dt$$

$$= b^{p+q-1}\int_0^1 t^{p-1}(1-t)^{q-1}\,dt,\ p>0,\ q>0$$

$$= b^{p+q-1}B(p,\ q) \quad \left[\because B(m,\ n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m>0,\ n>0\right]$$

2. Let $I = \int_0^1 x^{p-1}(1-x^a)^{q-1}\,dx,\ p>0,\ q>0$ … (8.60)

Putting $x^a = t \Rightarrow x = t^{1/a}$

Differentiating, we get $dx = \frac{1}{a}t^{a-1}\,dt$

Also, when $x = 0$, then $t = 0$ and

when $x = 1 \Rightarrow t = 1$

$\therefore$ (8.60) becomes

$$I = \int_0^1 \left(t^{1/a}\right)^{p-1}(1-t)^{q-1}\frac{1}{a}t^{1/a-1}\,dt = \frac{1}{a}\int_0^1 t^{\frac{1}{a}(p-1+1)-1}\ (1-t)^{q-1}\,dt$$

$$= \frac{1}{a}\int_0^1 t^{\frac{p}{a}-1}\ (1-t)^{q-1}\,dt,\ p>0,\ q>0$$

$$= \frac{1}{a}B\left(\frac{p}{a},\ q\right) \quad \left[\because B(m,\ n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m>0,\ n>0\right]$$

3. Let $I = \int_0^\infty \frac{1}{1+x^4}\,dx$ … (8.61)

Putting $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$

Differentiating, we get $dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta$

Also, when $x = 0$, then $\theta = 0$ and when $x \to \infty \Rightarrow \theta \to \frac{\pi}{2}$

$\therefore$ (8.61) becomes

$$I = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta} \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta \, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sec^2\theta} \frac{1}{\sqrt{\tan\theta}} \sec^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}} d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}} d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{-1/2+1}{2}\right)\Gamma\left(\frac{1/2+1}{2}\right)}{2\Gamma\left(\frac{-1/2+1/2+2}{2}\right)} \left[\because \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}, \; p > -1, q > -1\right]$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) \left[\because \Gamma(1) = 1\right]$$

$$= \frac{1}{4} \frac{\pi}{\sin(\pi/4)} \qquad \left[\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}, \; n \notin \mathbb{Z}\right]$$

$$= \frac{\sqrt{2}\,\pi}{4} \qquad \left[\because \sin(\pi/4) = \frac{1}{\sqrt{2}}\right]$$