UNIT 4

SEQUENCES AND SERIES

Structure

4.1 INTRODUCTION

You have heard or used the word sequence in some context like putting things in a sequence. In this unit, you will study the sequence of real numbers. In Sec. 4.2 we will define what is the meaning of sequence in mathematics in particular real sequence. In the same section, we will also visualise the sequence geometrically. If after a certain value of n terms of a sequence come closer and closer to a particular real number as n increases then we say that sequence converges to that real number. In Sec. 4.3 first we will state some results related to the convergence of the sequence and then we will apply them to test the convergence of the sequence. In Sec. 4.4 we will do similar things for the series of positive terms. If a series has infinite terms of both signs positive and negative then the underlying series is known as a series of arbitrary terms. Some terms related to a series of arbitrary terms are discussed in Sec. 4.5 in brief.

What we have discussed in this unit is summarised in Sec. 4.6. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit, some more questions based on the entire unit are given in Sec. 4.7 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 4.8.

In the next unit, you will study Riemann integration.

Expected Learning Outcomes

After completing this unit, you should be able to:

- define what is a sequence and what is meaning of convergence and divergence of a sequence and how to test the convergence of a sequence;
- define what is a series and what is meaning of convergence and divergence of a series and how to test the convergence of a series; and
- explain the difference between absolute and conditional convergence of a series.

4.2 SEQUENCE IN THE FAMILY OF REAL NUMBERS

In Units 1 and 2 of this course, you have studied function. You know that every function has a domain and range. **Sequence** is also a function having domain as the family of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and range may be any set but, in this course, we will discuss only sequences of real numbers. Hence, the range of every sequence that we are going to consider in this course will be a subset of $\mathbb{R}$, the set of all real numbers. So, throughout the course sequence will mean a sequence of real numbers known as a real sequence. So, let us formally define a real sequence.

Real Sequence: A real sequence is a function whose domain is the set of all natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and range may be any subset of $\mathbb{R}$, the set of all real numbers. In general, a sequence is written as

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots \qquad \ldots (4.1)$$

In short, we write it as $\{a_n\}$ or (a_n) or $< a_n >$. … (4.2)

In (4.1) or (4.2), a_1 is known as the first term of the sequence $< a_n >$. Similarly, $a_2, a_3, \ldots$ are known as the second, and third terms respectively and so on. Therefore, a_n is known as n^{th} the term of the sequence $< a_n >$. A sequence may be represented in any of the following four ways.

First Way of Representing a Sequence: A sequence can be represented by writing the first few terms of the sequence till a definite rule for writing down other terms becomes clear. For example, 1, 4, 9, 16, 25, … is a sequence.

Second Way of Representing a Sequence: A sequence can also be represented by giving a formula for its n^{th} term. For example, $< \frac{n+1}{n} >$.

Third Way of Representing a Sequence (Recursive Relation): A sequence can also be represented by specifying its first few terms and a formula to write the remaining terms in terms of preceding terms. For example,
$a_1 = 1, a_2 = 1, a_{n+1} = a_n + a_{n-1} \quad \forall \ n \geq 2, n \in \mathbb{N}$

Fourth Way of Representing a Sequence: Sometimes sequence is such that it cannot be represented by giving a single formula for its n term. But its terms are generated by two or more relations. For example,

$$a_n = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

If we write this sequence using the first way of representing a sequence then it can be written as $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$

Let us consider six examples of sequences, i.e., Sequence 1 to Sequence 6 given as follows.

Sequence 1: Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = \frac{1}{n}, \; n \in \mathbb{N}$.

The sequence defined by this function can be written as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ … (4.3)

or simply by $< \frac{1}{n} >$. If we take $f(n) = a_n, \; n \in \mathbb{N}$, then this sequence can be written as $a_n = \frac{1}{n}$. … (4.4)

So, the first term = $a_1 = 1$, second term = $a_2 = \frac{1}{2}$, third term = $a_3 = \frac{1}{3}$, and so on.

Sequence 2: Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = \frac{n+1}{n}, \; n \in \mathbb{N}$.

The sequence defined by this function can be written as $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ (4.5)

or simply by $< \frac{n+1}{n} >$. In this sequence, we have $a_n = \frac{n+1}{n}$. … (4.6)

So, the first term = $a_1 = 2$, second term = $a_2 = \frac{3}{2}$, third term = $a_3 = \frac{4}{3}$, and so on.

Sequence 3: Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = 2n - 1, \; n \in \mathbb{N}$.

The sequence defined by this function can be written as $1, 3, 5, 7, \ldots$ … (4.7)
or simply by $< 2n - 1 >$. In this sequence, we have $a_n = 2n - 1$. … (4.8)
So, the first term = $a_1 = 1$, second term = $a_2 = 3$, third term = $a_3 = 5$, and so on.

Sequence 4: Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = -2n + 1, \; n \in \mathbb{N}$.

The sequence defined by this function can be written as $-1, -3, -5, \ldots$ (4.9)
or simply by $< -2n + 1 >$. In this sequence, we have $a_n = -2n + 1$. … (4.10)
So, the first term = $a_1 = -1$, second term = $a_2 = -3$, third term = $a_3 = -5$, and so on.

Sequence 5: Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = (-1)^n, \; n \in \mathbb{N}$.

The sequence defined by this function can be written as $-1, 1, -1, 1, \ldots$ (4.11)
or simply by $< (-1)^n >$. In this sequence, we have $a_n = (-1)^n$. … (4.12)

Clearly the first term = $a_1 = -1$, second term = $a_2 = 1$, third term = $a_3 = -1$, and so on.

Sequence 6: Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = (-1)^n n$, $n \in \mathbb{N}$.

The sequence defined by this function can be written as $-1, 1, -2, 2, \ldots$ (4.13) or simply by $< (-1)^n n >$. In this sequence, we have $a_n = (-1)^n n$. ... (4.14) So, the first term = $a_1 = -1$, second term = $a_2 = 1$, third term = $a_3 = -2$, and so on.

Since sequence is a function, so you can make its graph by taking values of n on horizontal axis and values of the terms of the sequence on vertical axis. Following this rule graphs of the sequences given by (4.3), (4.5), (4.7), (4.9), (4.11) and (4.13) are shown in Fig. 4.1 (a) to (f) respectively. Here values of terms of the sequence are plotted by solid points and by a vertical line from horizontal axis to the corresponding solid point. The vertical lines from horizontal line up to the solid point will help you to understand where the values are going as n increases.

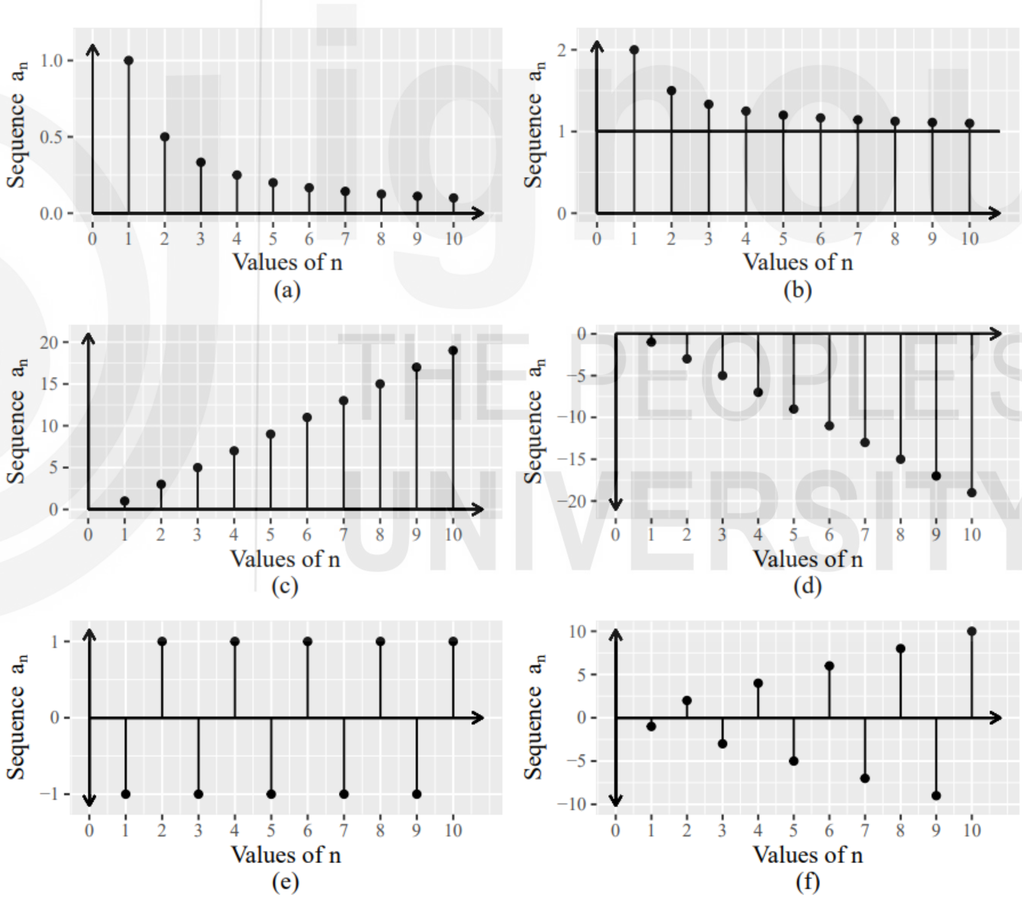

Fig. 4.1: Visualisation of Sequences 1 to 6 (a) 1 (b) 2 (c) 3 (d) 4 (e) 5 (f) 6

Let us comment on the behaviours of the sequences shown in Fig. 4.1 (a) to (f) one at a time.

Comment on the Behaviour of Sequence 1: From Fig. 4.1 (a) you see that terms of the sequence $< \frac{1}{n} >$ are coming closer and closer to horizontal axis as n increases. In fact, terms of this sequence are coming closer and closer to the value 0 $\left[\because \lim_{n\to\infty} a_n = 0\right]$. If $\lim_{n\to\infty} a_n$ exists and equal to a finite number k (say) then we say that sequence $< a_n >$ **converges to finite number k**. Here value of k is 0.

Comment on the Behaviour of Sequence 2: Following similar approach, the sequence $< \frac{n+1}{n} >$ shown in Fig. 4.1 (b) converges to the number 1 $\left[\because \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1+0 = 1\right]$. You can see it in Fig. 4.2 (b) as n increases terms of the sequence are coming closer and closer to the horizontal line which is 1 unit above the horizontal axis.

Comment on the Behaviour of Sequence 3: In Fig. 4.1 (c) you see that terms of the sequence $< 2n-1 >$ are increasing as n increases. In fact, given any real number no matter how large it is there exist terms in this sequence which are even greater than that real number. This is an important point, so let us consider two examples to explain this point.

(i) If you consider the real number 10000 then there exists $m = 5001$ such that $a_n > 10000 \ \forall \ n \ge m = 5001$. $[\because 2n-1 = 10000 \Rightarrow 2n = 10001 \Rightarrow n = 5000.5]$

(ii) If you consider the real number 2^{1000} then there exists $m = 2^{999}+1$ such that

$$a_n > 2^{1000} \ \forall \ n \ge m = 2^{999}+1. \left[\begin{array}{l}\because 2n-1 = 2^{1000} \Rightarrow 2n = 2^{1000}+1 \\ \Rightarrow n = \frac{2^{1000}+1}{2} \Rightarrow n = \frac{2^{1000}}{2}+\frac{1}{2} = 2^{999}+0.5\end{array}\right]$$

If a sequence behaves like this then we say that sequence diverges to infinity. So, sequence $< 2n-1 >$ diverges to ∞. Also, you can note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(2n-1) = \infty$.

Comment on the Behaviour of Sequence 4: In Fig. 4.1 (d) you see that terms of the sequence $< -2n+1 >$ are decreasing as n increases. In fact, given any real number no matter how small it is there exist terms in this sequence which are even less than that real number. Like sequence 3, let us consider two examples to explain this point.

(i) If you consider the real number -10000 then there exists $m = 5001$ such that $a_n < -10000 \ \forall \ n \ge m = 5001$. $\left[\begin{array}{l}\because -2n+1 = -10000 \\ \Rightarrow -2n = -10001 \Rightarrow n = 5000.5\end{array}\right]$

(ii) If you consider the real number -2^{1000} then there exists $m = 2^{999}+1$ such that

$$a_n < -2^{1000} \ \forall \ n \ge m = 2^{999}+1. \left[\begin{array}{l}\because -2n+1 = -2^{1000} \Rightarrow -2n = -2^{1000}-1 \\ \Rightarrow n = \frac{-2^{1000}-1}{-2} \Rightarrow n = \frac{2^{1000}}{2}+\frac{1}{2} = 2^{999}+0.5\end{array}\right]$$

If a sequence behaves like this then we say that sequence diverges to minus infinity. So, sequence $< -2n+1 >$ diverges to $-\infty$. Also, you can note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-2n+1) = -\infty.$$

Comment on the Behaviour of Sequence 5: In Fig. 4.1 (e) you see that even number terms $a_2, a_4, a_6, \ldots$ of the sequence $< (-1)^n >$ converges to 1 while odd number terms $a_1, a_3, a_5, \ldots$ of the sequence $< (-1)^n >$ converges to -1. So, as n increases its behaviour depends on whether n is even or odd. That is

$$a_n = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$$

So,

$$\lim_{n\to\infty} a_n = \begin{cases} \lim\limits_{n\to\infty} 1, & \text{if } n \text{ is even} \\ \lim\limits_{n\to\infty}(-1), & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$$

If a sequence like this converges to more than one finite number then we say that sequence oscillates finitely. So, sequence $< (-1)^n >$ oscillates finitely.

Comment on the Behaviour of Sequence 6: In Fig. 4.1 (f) you see that even number terms $a_2, a_4, a_6, \ldots$ of the sequence $< (-1)^n n >$ are increasing as n increases. In fact, given any real number no matter how large it is there exist even number terms in this sequence which are greater than that real number. For example,

(i) If you consider the real number 10000 then there exists even natural number $m = 2k = 10002$ such that for all even natural numbers n

$$a_n > 10000 \quad \forall\ n \geq m = 2k = 10002. \quad \left[\begin{array}{l} \because \text{If } n \text{ is even then } (-1)^n n = 10000 \\ \Rightarrow n = 10000 \end{array}\right]$$

(ii) If you consider the real number -2^{1000} then there exists odd natural number $m = 2k+1 = 2^{1000}+1$ such that for all odd natural numbers n

$$a_n < -2^{1000} \quad \forall\ n \geq m = 2k+1 = 2^{1000}+1. \quad \left[\begin{array}{l} \because \text{If } n \text{ is odd then } (-1)^n n = -2^{1000} \\ \Rightarrow -n = -2^{1000} \Rightarrow n = 2^{1000} \end{array}\right]$$

If a sequence behaves like this then we say that sequence oscillates infinitely.

Also, note that $a_n = \begin{cases} n, & \text{if } n \text{ is even} \\ -n, & \text{if } n \text{ is odd} \end{cases}$

So,

$$\lim_{n\to\infty} a_n = \begin{cases} \lim\limits_{n\to\infty} n, & \text{if } n \text{ is even} \\ \lim\limits_{n\to\infty}(-n), & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \infty, & \text{if } n \text{ is even} \\ -\infty, & \text{if } n \text{ is odd} \end{cases}$$

With the help of graphical presentation and discussion of convergence, divergence and oscillation of these six sequences it will be easy for you to understand definition of these ideas which are given as follows.

Convergence of a Sequence: A real sequence $<a_n>$ is said to converges a real number 'a' if for a given $\varepsilon > 0$ however small it may be there exists a natural number m depending on ε such that $|a_n - a| < \varepsilon \ \forall \ n \geq m$ … (4.15)

The real number a is called the limit of the sequence $<a_n>$, and we write it as $a_n \to a$ as $n \to \infty$ or $\lim\limits_{n\to\infty} a_n = a$. … (4.16)

Condition (4.15) can be written as:

$$-\varepsilon < a_n - a < \varepsilon \ \forall \ n \geq m$$
$$\Rightarrow a - \varepsilon < a_n < a + \varepsilon \ \forall \ n \geq m$$

This means all the terms from m onward lie inside a strip of width ε about point a refer Fig. 4.2.

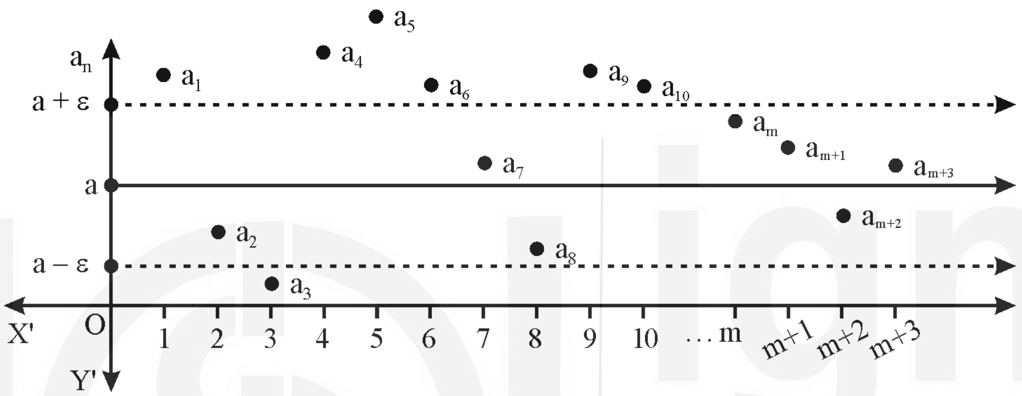

Fig. 4.2: Visualisation of the definition of convergence of a sequence

For example, sequence $<\frac{1}{n}>$ shown in Fig. 4.1 (a) converges to 0. Let us consider two values of $\varepsilon > 0$ so that you can understand well what this definition says.

- Suppose $\varepsilon = 0.1 > 0$ then any positive integer ≥ 11 can be taken as value of m. So,

$$|a_n - 0| < 0.1 \ \forall \ n \geq 11 \text{ or } \left|\frac{1}{n} - 0\right| < 0.1 \ \forall \ n \geq 11 \ [\because \text{ here } a = 0]$$

$$\text{or } \left|\frac{1}{n}\right| < 0.1 \ \forall \ n \geq 11 \left[\begin{array}{l} \because 11 > 10 \Rightarrow \frac{1}{11} < \frac{1}{10} \text{ refer (6.89)} \\ \Rightarrow \frac{1}{11} < 0.1 \\ \text{So, } \frac{1}{n} < 0.1 \ \forall \ n \geq 11 \Rightarrow \left|\frac{1}{n}\right| < 0.1 \ \forall \ n \geq 11 \end{array}\right]$$

- Suppose $\varepsilon = 0.01 > 0$ then any positive integer ≥ 101 can be taken as value of m. So, $|a_n - 0| < 0.01 \ \forall \ n \geq 101$ or $\left|\frac{1}{n} - 0\right| < 0.01 \ \forall \ n \geq 101$ $[\because \text{ here } a = 0]$

$$\text{or } \left|\frac{1}{n}\right| < 0.01 \ \forall \ n \geq 101 \left[\begin{array}{l} \because 101 > 100 \Rightarrow \frac{1}{101} < \frac{1}{100} \Rightarrow \frac{1}{101} < 0.01 \\ \text{So, } \frac{1}{n} < 0.01 \ \forall \ n \geq 101 \Rightarrow \left|\frac{1}{n}\right| < 0.01 \ \forall \ n \geq 101 \end{array}\right]$$

Divergence of a Sequence: A real sequence $<a_n>$ is said to diverges to ∞ if given any positive real number K no matter how large it is there exists a positive integer m such that $a_n > K \ \forall \ n \geq m,$ and we write it as

$$a_n \to \infty \text{ as } n \to \infty \text{ or } \lim_{n\to\infty} a_n = \infty. \qquad \ldots (4.17)$$

A real sequence $<a_n>$ is said to diverges to $-\infty$ if given any positive real number K no matter how large it is there exists a positive integer m such that $a_n < -K \ \forall \ n \geq m,$ and we write it as

$$a_n \to -\infty \text{ as } n \to \infty \text{ or } \lim_{n\to\infty} a_n = -\infty. \qquad \ldots (4.18)$$

For example, sequence $<2n-1>$ shown in Fig. 4.1 (c) diverges to ∞. Let us consider two values of the positive real number K so that you can understand well what this definition says.

- Suppose $K = 10000$ then any positive integer ≥ 5001 can be taken as value of m. So,

$$a_n > 10000 \ \forall \ n \geq m = 5001 \text{ or } 2n-1 > 10000 \ \forall \ n \geq m = 5001$$

$$\begin{bmatrix} \because n > 5001 \Rightarrow 2n > 10002 \\ \Rightarrow 2n-1 > 10001 \Rightarrow 2n-1 > 10000 \Rightarrow a_n > 10000 \end{bmatrix}$$

- Suppose $K = 2^{100000}$ then any positive integer ≥ $2^{99999}+1$ can be taken as value of m. So,

$$a_n > 2^{100000} \ \forall \ n \geq m = 2^{99999}+1 \text{ or } 2n-1 > 2^{100000} \ \forall \ n \geq m = 2^{99999}+1$$

$$\begin{bmatrix} \because n > 2^{99999}+1 \Rightarrow 2n > 2\times 2^{99999}+2 \Rightarrow 2n-1 > 2^{100000}+1 \\ \Rightarrow 2n-1 > 2^{100000} \Rightarrow a_n > 2^{100000} \end{bmatrix}$$

Following similar steps as in the case of divergence to ∞. You can consider some values of K and obtain corresponding value of m in the case $a_n \to -\infty$.

Oscillatory Sequence: If a real sequence $<a_n>$ neither converges to a finite real number and nor diverges to ∞ or $-\infty$, then we say that sequence oscillates. As discussed earlier oscillatory sequence may oscillate finitely or infinitely. … (4.19)

For example, sequence $<(-1)^n>$ shown in Fig. 4.1 (e) oscillates finitely because it converges to two finite real numbers 1 and – 1. The sequence $<(-1)^n n>$ shown in Fig. 4.1 (f) oscillates infinitely because it diverges to ∞ and $-\infty$.

Let us now do some examples to explain the idea of convergence and divergence of a sequence.

Example 1: Discuss the convergence of the sequence $<\frac{n+1}{n}>$ using definition.

Solution: Here $a_n = \frac{n+1}{n}$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(\frac{n}{n}+\frac{1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 + \lim_{n\to\infty}\left(\frac{1}{n}\right) = 1+0 = 1$$

Now, using definition we prove that it converges to 1. Let $\varepsilon > 0$ be given then

$$|a_n - 1| < \varepsilon \Rightarrow \left|\frac{n+1}{n} - 1\right| < \varepsilon \Rightarrow \left|1 + \frac{1}{n} - 1\right| < \varepsilon \Rightarrow \left|\frac{1}{n}\right| < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon \left[\because n > 0, \text{ so } \left|\frac{1}{n}\right| = \frac{1}{n}\right]$$

$$\Rightarrow n > \frac{1}{\varepsilon} \left[\because a > 0,\ b > 0 \text{ and } a < b \Rightarrow \frac{1}{a} > \frac{1}{b} \text{ refer (6.89)}\right]$$

If we take positive integer $m > \frac{1}{\varepsilon}$, then we have $|a_n - 1| < \varepsilon \ \forall\ n \geq m$

Hence, by definition sequence $< \frac{n+1}{n} >$ converges to 1.

Example 2: Discuss the convergence of the sequence $< \frac{5n+4}{2n+3} >$ using definition.

Solution: Here $a_n = \frac{5n+4}{2n+3}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n+4}{2n+3} = \lim_{n\to\infty}\left(\frac{5+\frac{4}{n}}{2+\frac{3}{n}}\right) = \left(\frac{5+0}{2+0}\right) = \frac{5}{2}$

Now, using definition we prove that it converges to 5/2. Let $\varepsilon > 0$ be given then

$$\left|a_n - \frac{5}{2}\right| < \varepsilon \Rightarrow \left|\frac{5n+4}{2n+3} - \frac{5}{2}\right| < \varepsilon \Rightarrow \left|\frac{10n+8-10n-15}{2(2n+3)}\right| < \varepsilon \Rightarrow \left|\frac{-7}{2(2n+3)}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{7}{2(2n+3)}\right| < \varepsilon \Rightarrow \frac{7}{2(2n+3)} < \varepsilon \left[\begin{array}{l}\because n > 0, \text{ so } \dfrac{7}{2(2n+3)} > 0 \\ \Rightarrow \left|\dfrac{7}{2(2n+3)}\right| = \dfrac{7}{2(2n+3)}\end{array}\right]$$

Now, $\frac{7}{2(2n+3)} < \varepsilon \Rightarrow \frac{7}{2n+3} < 2\varepsilon \Rightarrow 7 < 4\varepsilon n + 6\varepsilon$

$$\Rightarrow 7 - 6\varepsilon < 4\varepsilon n \Rightarrow \frac{7-6\varepsilon}{4\varepsilon} < n \text{ or } n > \frac{7-6\varepsilon}{4\varepsilon}$$

If we take positive integer $m > \frac{7-6\varepsilon}{4\varepsilon}$, then we have $\left|a_n - \frac{5}{2}\right| < \varepsilon \ \forall\ n \geq m$

Hence, by definition sequence $< \frac{5n+4}{2n+3} >$ converges to $\frac{5}{2}$.

Remark 1: For better understanding of the concept, you should try experiments like: In particular, if $\varepsilon = 0.01$, then $m > \frac{7-0.06}{0.04} = \frac{6.94}{0.04} = 173.5$.

So, in this case we may take m = 174 or any value higher than 174 but not less than 173. **In other words, it means only first 173 terms of the given sequence are at a distance of 0.01 or higher from 5/2. All other terms from 174 onward are at a distance less than 0.01 from 5/2**.

Example 3: Show that the sequence $< 3^n >$ diverges using definition.

Solution: Here $a_n = 3^n$. Let K be any positive real number no matter how large it is. We claim that there exists positive integer m such that $a_n > K \ \ \forall\ n \geq m$

Now, $a_n > K \Rightarrow 3^n > K \Rightarrow n\log 3 > \log K \Rightarrow n > \dfrac{\log K}{\log 3}$

If we take positive integer $m > \dfrac{\log K}{\log 3}$, then we have $a_n > K \ \forall \ n \geq m$.

Hence, by definition sequence $< 3^n >$ diverges to ∞.

Remark 2: For better understanding of the concept, you should try experiments like: In particular, if $K = 10000$, then $m > \dfrac{\log 10000}{\log 3} \simeq 8.3836$. So, in this case we may take m = 9.

Now, you can try the following Self-Assessment Question.

SAQ 1

Write n^{th} term of the sequence $9, \dfrac{11}{4}, \dfrac{13}{9}, \dfrac{15}{16}, \dfrac{17}{25}, \ldots$

4.3 SOME RESULTS RELATED TO CONVERGENCE AND DIVERGENCE OF SEQUENCE AND THEIR USE

Before stating some results, which help in checking whether the given sequence converges or diverges or oscillates, let us explain some terms which will be required in the results which we are going to discuss here.

Bounded above Sequence: A sequence $< a_n >$ is said to be bounded above if there exists a real number K such that

$$a_n \leq K \ \ \forall \ n \in \mathbb{N} \qquad \ldots (4.20)$$

For example, consider the sequence $< \dfrac{1}{2^n} >$, this sequence is bounded above since $a_n \leq \dfrac{1}{2} \ \ \forall \ n \in \mathbb{N} \left[\because \ a_1 = \dfrac{1}{2} \leq \dfrac{1}{2}, \ a_2 = \dfrac{1}{4} \leq \dfrac{1}{2}, \ a_3 = \dfrac{1}{8} \leq \dfrac{1}{2}, \text{ and so on} \right]$

Bounded below Sequence: A sequence $< a_n >$ is said to be bounded below if there exists a real number k such that

$$a_n \geq k \ \ \forall \ n \in \mathbb{N} \qquad \ldots (4.21)$$

For example, consider the sequence $< \dfrac{1}{2^n} >$, this sequence is bounded below since $a_n \geq 0 \ \ \forall \ n \in \mathbb{N} \left[\because \ a_1 = \dfrac{1}{2} \geq 0, \ a_2 = \dfrac{1}{4} \geq 0, \ a_3 = \dfrac{1}{8} \geq 0, \text{ and so on} \right]$

Bounded Sequence: A sequence $< a_n >$ is said to be bounded if it is both bounded above and bounded below. So, if a sequence is bounded then there will exist two real numbers k and K such that

$$k \leq a_n \leq K \ \ \forall \ n \in \mathbb{N} \qquad \ldots (4.22)$$

For example, consider the sequence $< \dfrac{1}{2^n} >$, this sequence is bounded since it is both bounded above and bounded below. Here k = 0 and $K = \dfrac{1}{2}$, i.e.,

$0 \le a_n \le \frac{1}{2} \ \forall \ n \in \mathbb{N} \left[\because a_1 = \frac{1}{2} \text{ and } 0 \le \frac{1}{2} \le \frac{1}{2}, a_2 = \frac{1}{4} \text{ and } 0 \le \frac{1}{4} \le \frac{1}{2}, \text{ and so on} \right]$

Least Upper Bound (lub) of a Sequence: Least upper bound of a sequence can be defined in two ways. Let us call them definition 1 and definition 2 given as follows.

Definition 1: Suppose sequence $< a_n >$ is bounded above then there will exists a real number u such that

$a_n \le u \ \forall \ n \in \mathbb{N}$

The real number u is known as **upper bound** of the sequence $< a_n >$. The upper bound u is said to be the least upper bound of the sequence $< a_n >$ if whenever we have any other upper bound u_1 then $u \le u_1$. That is no real number less than u can be an upper bound of the sequence $< a_n >$. … (4.23)

For example, $u = \frac{1}{2}$ is the least upper bound of the sequence $< \frac{1}{2^n} >$, since if we have any other upper bound u_1 then $\frac{1}{2} \le u_1$. e.g., $u_1 = 0.6$ is an upper bound of the sequence $< \frac{1}{2^n} >$, clearly $u = 0.5 < 0.6 = u_1$.

Definition 2: Let $< a_n >$ be a real and bounded above sequence then a real number u is said to be least upper bound (lub) of the sequence $< a_n >$ if

(i) $a_n \le u \ \forall \ n \in \mathbb{N}$ and

(ii) given any $\varepsilon > 0$, no matter how small it is, there exists at least one $n \in \mathbb{N}$ such that $a_n > u - \varepsilon$. … (4.24)

For example, $u = \frac{1}{2}$ is the least upper bound of the sequence $< \frac{1}{2^n} >$, since if we take any value of $\varepsilon > 0$, then there exists at least one value 1 of n such that

$a_1 = \frac{1}{2} > \frac{1}{2} - \varepsilon$

Remark 3: (i) In particular, if $\varepsilon = 0.4 > 0$, then there are only three values 1, 2 and 3 of n such that

$a_1 = \frac{1}{2} = 0.5 > 0.1 = \frac{1}{2} - 0.4 = \frac{1}{2} - \varepsilon, \ a_2 = \frac{1}{4} = 0.25 > 0.1 = \frac{1}{2} - 0.4 = \frac{1}{2} - \varepsilon,$

$a_3 = \frac{1}{8} = 0.125 > 0.1 = \frac{1}{2} - 0.4 = \frac{1}{2} - \varepsilon.$

But $a_4 = \frac{1}{16} = 0.0625 < 0.1 = \frac{1}{2} - 0.4 = \frac{1}{2} - \varepsilon$

(ii) If $\varepsilon = 0.3 > 0$, then there are only two values 1 and 2 of n such that

$a_1 = \frac{1}{2} = 0.5 > 0.2 = \frac{1}{2} - 0.3 = \frac{1}{2} - \varepsilon, \ a_2 = \frac{1}{4} = 0.25 > 0.2 = \frac{1}{2} - 0.3 = \frac{1}{2} - \varepsilon.$

But $a_3 = \frac{1}{8} = 0.125 < 0.2 = \frac{1}{2} - 0.3 = \frac{1}{2} - \varepsilon.$

(iii) If $\varepsilon = 0.2 > 0$, then there is only one value 1 of n such that

$$a_1 = \frac{1}{2} = 0.5 > 0.3 = \frac{1}{2} - 0.2 = \frac{1}{2} - \varepsilon.$$

But $a_2 = \frac{1}{4} = 0.25 < 0.3 = \frac{1}{2} - 0.2 = \frac{1}{2} - \varepsilon.$

If you take any value of $\varepsilon > 0,$ no matter how small it is then you will always find that $a_1 = \frac{1}{2} > \frac{1}{2} - \varepsilon.$ Hence, $\frac{1}{2}$ is the lub of the sequence $< \frac{1}{2^n} >$.

Greatest Lower Bound (glb) of a Sequence: Greatest lower bound of a sequence can be defined in two ways. Let us call them definition 1 and definition 2 given as follows.

Definition 1: Suppose sequence $< a_n >$ is bounded below then there will exists a real number g such that

$a_n \geq g \;\; \forall \; n \in \mathbb{N}.$

A real number g which satisfy this condition is known as **lower bound** of the sequence $< a_n >$. The lower bound g is said to be the greatest lower bound of the sequence $< a_n >$ if whenever we have any other lower bound g_1 then $g_1 \leq g.$ That is no real number > g can be a lower bound of the sequence $< a_n >$. … (4.25)

For example, $g = 0$ is the greatest lower bound of the sequence $< \frac{1}{2^n} >,$ since if we have any other lower bound g_1 then $g_1 \leq g.$ e.g., $g_1 = -0.1$ is a lower bound of the sequence $< \frac{1}{2^n} >,$ clearly $g_1 = -0.1 < 0 = g.$

Definition 2: Let $< a_n >$ be a real and bounded below sequence then a real number g is said to be greatest lower bound (glb) of the sequence $< a_n >$ if

(i) $a_n \geq g \;\; \forall \; n \in \mathbb{N}$ and

(ii) given any $\varepsilon > 0,$ no matter how small it is, there exists at least one $n \in \mathbb{N}$ such that $a_n < g + \varepsilon.$ … (4.26)

For example, $g = 0$ is the greatest lower bound of the sequence $< \frac{1}{2^n} >,$ since if we take any value of $\varepsilon > 0,$ then there exists at least one value of n such that $a_n < 0 + \varepsilon.$

Remark 4: (i) In particular, if $\varepsilon = 0.1 > 0,$ then except first three terms all other terms are ≤ 0 + 0.1 = g + $\varepsilon.$ That is

$$a_4 = \frac{1}{16} = 0.0625 < 0.1 = 0 + 0.1 = g + \varepsilon,$$

$$a_5 = \frac{1}{32} = 0.03125 < 0.1 = 0 + 0.1 = g + \varepsilon, \text{ and so on.}$$

But $a_3 = \frac{1}{8} = 0.125 > 0.1 = 0 + 0.1 = g + \varepsilon$

(ii) If $\varepsilon = 0.2 > 0,$ then except first two terms all other terms are ≤ 0 + 0.1 = g + $\varepsilon.$ That is

$$a_3 = \frac{1}{8} = 0.125 < 0.2 = 0 + 0.2 = g + \varepsilon,$$

$$a_4 = \frac{1}{16} = 0.0625 < 0.2 = 0 + 0.2 = g + \varepsilon, \text{ and so on.}$$

But $a_2 = \frac{1}{4} = 0.25 > 0.2 = 0 + 0.2 = g + \varepsilon.$

(iii) If $\varepsilon = 0.3 > 0,$ then except first term all other terms are ≤ 0 + 0.1 = g + ε. That is

$$a_2 = \frac{1}{4} = 0.25 < 0.3 = 0 + 0.3 = g + \varepsilon,$$

$$a_3 = \frac{1}{8} = 0.125 < 0.3 = 0 + 0.3 = g + \varepsilon, \text{ and so on.}$$

But $a_1 = \frac{1}{2} = 0.5 > 0.3 = 0 + 0.3 = g + \varepsilon.$

For the sequence $< \frac{1}{2^n} >,$ if you will take any value of $\varepsilon > 0,$ which is greater than 0.5 then all terms of the given sequence will be < $g + \varepsilon.$ But if you take any value of $\varepsilon > 0,$ no matter how small it is then you will always find at least one value of n such that $a_n < g + \varepsilon.$ Hence, 0 is the glb of the sequence $< \frac{1}{2^n} >.$

Monotonic Sequence

Before defining monotonic sequence, we have to define monotonically increasing and decreasing sequences.

Monotonically Increasing Sequence: A real sequence $< a_n >$ is said to be monotonically increasing sequence if $a_n \le a_{n+1} \ \forall \ n \in \mathbb{N}$

i.e., $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \ \ldots$ … (4.27)

For example, consider the sequence $< 2^n >,$ this sequence is monotonically increasing since

$$a_n \le a_{n+1} \ \forall \ n \in \mathbb{N} \left[\because \forall \ n \in \mathbb{N} \Rightarrow n < n+1 \Rightarrow 2^n < 2^{n+1} \Rightarrow a_n < a_{n+1} \Rightarrow a_n \le a_{n+1}\right]$$

Monotonically Decreasing Sequence: A real sequence $< a_n >$ is said to be monotonically decreasing sequence if $a_n \ge a_{n+1} \ \forall \ n \in \mathbb{N}$

i.e., $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5 \ \ldots$ … (4.28)

For example, consider the sequence $< \frac{1}{2^n} >,$ this sequence is monotonically decreasing since

$$a_n \ge a_{n+1} \ \forall \ n \in \mathbb{N} \left[\begin{array}{l} \because \forall \ n \in \mathbb{N} \Rightarrow n < n+1 \Rightarrow 2^n < 2^{n+1} \Rightarrow \frac{1}{2^n} > \frac{1}{2^{n+1}} \text{ refer (6.89)} \\ \Rightarrow a_n > a_{n+1} \Rightarrow a_n \ge a_{n+1} \end{array}\right]$$

Monotonic Sequence: A real sequence $< a_n >$ is said to be monotonic if either it is monotonically increasing or monotonically decreasing sequence.

… (4.29)

Cauchy Sequence: A sequence $< a_n >$ is said to be a Cauchy sequence if given $\varepsilon > 0,$ no matter how small it is there exists a positive integer n_0 such that $|a_n - a_m| < \varepsilon \ \forall$ m, $n \geq n_0$ … (4.30)

Now, let us state some results without proof as proofs are beyond the scope of this course.

Result 1: If a sequence $< a_n >$ converges then it will converge to a unique limit. … (4.31)

Result 2: A monotonic sequence $< a_n >$ is convergent if and only if it is bounded. … (4.32)

Result 3: A sequence $< a_n >$ which is both monotonically increasing and bounded above converges to its lub. … (4.33)

Result 4: A sequence $< a_n >$ which is both monotonically decreasing and bounded below converges to its glb. … (4.34)

Result 5: A sequence $< a_n >$ which is monotonically increasing but not bounded above diverges to ∞. … (4.35)

Result 6: A sequence $< a_n >$ which is monotonically decreasing but not bounded below diverges to $-\infty$. … (4.36)

Remark 5: In view of results 2 to 6 you can say that a monotonic sequence either converges or diverges. … (4.37)

Result 7: Every convergent sequence is bounded. … (4.38)

Result 8: If a sequence $< a_n >$ converges to a and $a_n \geq 0 \ \forall \ n \in \mathbb{N},$ then $a \geq 0.$ … (4.39)

Result 9: If sequences $< a_n >$ and $< b_n >$ are such that $a_n \leq b_n \ \forall \ n \in \mathbb{N}$ and $a_n \to a$ and $b_n \to b$ then $a \leq b.$ … (4.40)

Result 10: **Squeeze Principle**: If sequences $< a_n >, < b_n >$ and $< c_n >$ are such that $a_n \leq b_n \leq c_n \ \forall \ n \geq n_0$ for some positive integer n_0 and $a_n \to a$ and $c_n \to a$ then $b_n \to a.$ … (4.41)

Result 11: **Cauchy's First Theorem on Limit**: If sequence $< a_n >$ converges to 'a' then the sequence $< b_n >$ also converges to 'a' where $b_n = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \ \forall \ n \in \mathbb{N},$ i.e., b_n is arithmetic mean of the first n terms of the sequence $< a_n >$. … (4.42)

Result 12: **Cauchy's Second Theorem on Limit**: If all the terms of the sequence $< a_n >$ are positive then $\lim\limits_{n\to\infty}(a_n)^{1/n} = \lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}$ provided $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}$ exists as finite or infinite. … (4.43)

Result 13: A real sequence $< a_n >$ is convergent if and only if it is a Cauchy sequence. … (4.44)

Result 14: **Algebra of Limits**: If sequences $< a_n >$ and $< b_n >$ converge to a and b respectively. i.e., $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b,$ then following hold:

- $\lim_{n\to\infty}(a_n + b_n) = a + b$ … (4.45)
- $\lim_{n\to\infty}(a_n - b_n) = a - b$ … (4.46)
- $\lim_{n\to\infty} a_n b_n = ab$ … (4.47)
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b},\ b \neq 0$ … (4.48)
- $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a},\ a \neq 0$ … (4.49)
- $\lim_{n\to\infty} |a_n| = |a|$ … (4.50)
- $\lim_{n\to\infty} \alpha a_n = \alpha a,\ \alpha$ is any real number … (4.51)

Now, let us apply these results in solving problems on convergence of sequence.

Example 4: Discuss the convergence of the sequence $< \frac{1}{n} >$ which is shown in Fig. 4.1 (a) and given by (4.3) or (4.4).

Solution: Here $a_n = \frac{1}{n}$, so $a_{n+1} = \frac{1}{n+1}$.

Now $n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1} \Rightarrow a_n > a_{n+1} \Rightarrow < a_n >$ is decreasing. … (4.52)

Another way of showing the sequence $< a_n >$ as a decreasing sequence is:

$$a_n - a_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} > 0 \quad [\because n > 0 \Rightarrow n(n+1) > 0]$$

$\Rightarrow a_n - a_{n+1} > 0 \Rightarrow a_n > a_{n+1}$. Hence, $< a_n >$ is decreasing.

You can prove it decreasing using either way you like.

Also, $a_n = \frac{1}{n} > 0\ \forall\ n \in \mathbb{N} \Rightarrow$ Sequence $< a_n >$ is bounded below. … (4.53)

By (4.52) and (4.53) the given sequence $< a_n >$ is both bounded below as well as decreasing. Hence, using (4.34) given sequence $< \frac{1}{n} >$ converges to its glb where $\text{glb} = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$ Hence, $< \frac{1}{n} > \to 0.$

Example 5: Discuss the convergence of the sequence $< \frac{n}{n+1} >$.

Solution: Here $a_n = \frac{n}{n+1}$, so $a_{n+1} = \frac{n+1}{n+2}$.

We claim that given sequence $< a_n >$ is an increasing sequence.

$$a_n - a_{n+1} = \frac{n}{n+1} - \frac{n+1}{n+2} = \frac{n^2 + 2n - (n^2 + 2n + 1)}{(n+1)(n+2)}$$

$$= \frac{-1}{(n+1)(n+2)} < 0 \ [\because n > 0 \Rightarrow (n+1)(n+2) > 0]$$

$\Rightarrow a_n - a_{n+1} < 0 \Rightarrow a_n < a_{n+1}$. Hence, $< a_n >$ is increasing. … (4.54)

Also, $n < n+1 \Rightarrow \frac{n}{n+1} < 1 \ \forall \ n \in \mathbb{N} \Rightarrow a_n < 1 \ \ \forall \ n \in \mathbb{N}$

$\Rightarrow$ Sequence $< a_n >$ is bounded above. … (4.55)

By (4.54) and (4.55) the given sequence $< a_n >$ is both bounded above as well as increasing. Hence, using (4.33) given sequence $< \frac{n}{n+1} >$ converges to its lub where $\text{lub} = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$. Hence, $< \frac{n}{n+1} > \to 1$.

Example 6: Discuss the convergence of the sequence $< n^{1/n} >$.

Solution: Let $a_n = n$, so $a_{n+1} = n+1$

Now, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 + 0 = 1$ … (4.56)

By Cauchy's second theorem on limit, we have

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \qquad [\text{Using } (4.56)]$$

Hence, the sequence $< n^{1/n} >$ converges to 1. … (4.57)

First 35 terms of this sequence are shown in Table 4.1 and have been visualised in Fig. 4.3 (a) by 35 vertical lines. From Table 4.1 note that first three terms are increasing as n increases but after that continuously terms are decreasing as n increases. So, this trend verifies the same thing that we have shown in Example 6 that $a_n \to 1$ as $n \to \infty$.

Example 7: Discuss the convergence of the sequence

$$< \frac{1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}}{n} >.$$

Solution: Let $a_n = n^{1/n}$, then using (4.57) it converges to 1.

By Cauchy's first theorem on limit, we have

$$\lim_{n\to\infty} \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} = \lim_{n\to\infty} a_n = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}}{n} = 1.$$

Hence, the given sequence $< \frac{1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}}{n} >$ converges to 1.

First 35 terms of this sequence are shown in Table 4.2 and have been visualised in Fig. 4.3 (b) by 35 vertical lines. From Table 4.2 note that first six terms are increasing as n increases but after that continuously terms are decreasing as n increases. So, this trend verifies the same thing that we have shown in Example 7 that $a_n \to 1$ as $n \to \infty$.

Table 4.1: First 35 terms of the sequence discussed in Example 6. Values of these terms are obtained using R software

Value of n	n^{th} term of the sequence	Value of n	n^{th} term of the sequence	Value of n	n^{th} term of the sequence
1	1.000000	13	1.218114	25	1.137411
2	1.414214	14	1.207442	26	1.133501
3	1.442250	15	1.197860	27	1.129831
4	1.414214	16	1.189207	28	1.126378
5	1.379730	17	1.181352	29	1.123124
6	1.348006	18	1.174187	30	1.120050
7	1.320469	19	1.167623	31	1.117142
8	1.296840	20	1.161586	32	1.114387
9	1.276518	21	1.156013	33	1.111772
10	1.258925	22	1.150851	34	1.109286
11	1.243575	23	1.146055	35	1.106920
12	1.230076	24	1.141586		

Table 4.2: First 35 terms of the sequence discussed in Example 7. Values of these terms are obtained using R software

Value of n	n^{th} term of the sequence	Value of n	n^{th} term of the sequence	Value of n	n^{th} term of the sequence
1	1.000000	13	1.295610	25	1.234164
2	1.207107	14	1.289312	26	1.230293
3	1.285488	15	1.283215	27	1.226572
4	1.317669	16	1.277340	28	1.222993
5	1.330081	17	1.271694	29	1.219550
6	1.333069	18	1.266277	30	1.216233
7	1.331269	19	1.261084	31	1.213037
8	1.326965	20	1.256109	32	1.209954
9	1.321360	21	1.251343	33	1.206978
10	1.315116	22	1.246775	34	1.204105
11	1.308613	23	1.242396	35	1.201328
12	1.302068	24	1.238196		

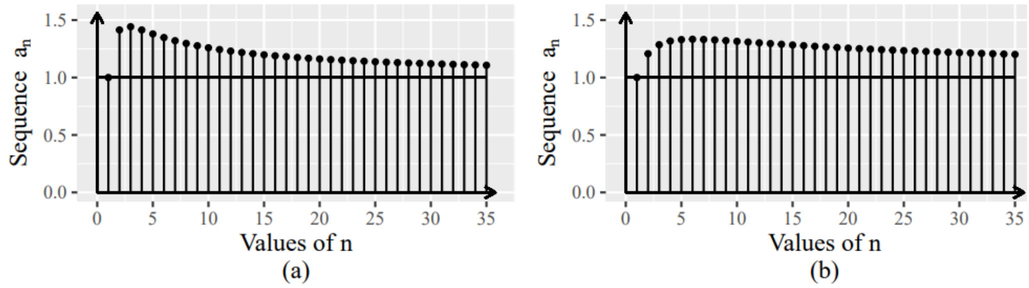

Fig. 4.3: Visualisation of successive terms of the sequence discussed in Examples (a) 6 (b) 7

Example 8: Prove that the sequence $< \dfrac{1}{n} >$ is a Cauchy sequence.

Solution: Here $a_n = \dfrac{1}{n}$. Let $\varepsilon > 0$ be given. Let n > m then

$$|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|-\left(\frac{1}{m} - \frac{1}{n}\right)\right|$$

$$= \left|\frac{1}{m} - \frac{1}{n}\right| = \frac{1}{m} - \frac{1}{n} \quad \left[\begin{array}{l} \because n > m \Rightarrow \dfrac{1}{n} < \dfrac{1}{m} \Rightarrow \dfrac{1}{m} - \dfrac{1}{n} > 0 \\ \Rightarrow \left|\dfrac{1}{m} - \dfrac{1}{n}\right| = \dfrac{1}{m} - \dfrac{1}{n} \text{ as if } x \geq 0 \text{ then } |x| = x \end{array}\right]$$

$$\Rightarrow |a_n - a_m| = \frac{1}{m} - \frac{1}{n} < \frac{1}{m} \quad \left[\because n > 0 \Rightarrow \frac{1}{n} > 0\right]$$

So, $|a_n - a_m| < \varepsilon \Rightarrow \dfrac{1}{m} < \varepsilon \Rightarrow m > \dfrac{1}{\varepsilon}$

If we take a positive integer $n_0 > \dfrac{1}{\varepsilon}$ then

$\Rightarrow |a_n - a_m| < \varepsilon \quad \forall\ m, n \geq n_0$

Hence, the given sequence $< \dfrac{1}{n} >$ is a Cauchy sequence.

Remark 6: In particular, if $\varepsilon = 0.1 > 0$, then $n_0 > \dfrac{1}{\varepsilon} \Rightarrow n_0 > \dfrac{1}{0.1} \Rightarrow n_0 > 10$. It means for $\varepsilon = 0.1 > 0$, difference between any two terms after tenth term is less than 0.1. if $\varepsilon = 0.01 > 0$, then $n_0 > \dfrac{1}{\varepsilon} \Rightarrow n_0 > \dfrac{1}{0.01} \Rightarrow n_0 > 100$. It means for $\varepsilon = 0.01 > 0$, difference between any two terms after 100^{th} term is less than 0.01.

Example 9: Prove that the sequence $< \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \ldots + \dfrac{1}{(2n)^2} >$ converges to 0.

Solution: Here $a_n = \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \ldots + \dfrac{1}{(2n)^2}$, then

$$a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}$$

$$< \frac{1}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \ldots + \frac{1}{n^2} \quad \left[\because \frac{1}{(n+1)^2} < \frac{1}{n^2}, \text{ etc.}\right]$$

$$= \frac{n+1}{n^2} \qquad \left[\because \underbrace{a + a + a + \ldots + a}_{(n+1)\text{ times}} = (n+1)a, \text{ here } a = \frac{1}{n^2}\right]$$

Also,

$$a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}$$

$$> \frac{1}{(2n)^2} + \frac{1}{(2n)^2} + \frac{1}{(2n)^2} + \ldots + \frac{1}{(2n)^2} \quad \left[\because \frac{1}{n^2} > \frac{1}{(2n)^2}, \text{ etc.}\right]$$

$$= \frac{n+1}{(2n)^2} \quad \left[\because \underbrace{a + a + a + \ldots + a}_{(n+1)\text{ times}} = (n+1)a, \text{ here } a = \frac{1}{(2n)^2}\right]$$

Thus, we have

$$\frac{n+1}{(2n)^2} < a_n < \frac{n+1}{n^2} \quad \ldots (4.58)$$

But $\lim\limits_{n\to\infty} \frac{n+1}{(2n)^2} = \lim\limits_{n\to\infty} \frac{n+1}{4n^2} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n}}{4n} = 0$ and $\lim\limits_{n\to\infty} \frac{n+1}{n^2} = \lim\limits_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n^2}\right) = 0$ … (4.59)

Taking limit in (4.58), we have

$$\lim_{n\to\infty} \frac{n+1}{(2n)^2} < \lim_{n\to\infty} a_n < \lim_{n\to\infty} \frac{n+1}{n^2}$$

$$\Rightarrow 0 < \lim_{n\to\infty} a_n < 0 \quad [\text{Using } (4.59)]$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0 \quad [\text{Using Squeeze principle refer } (4.41)]$$

$$\therefore \lim_{n\to\infty} a_n = 0 \Rightarrow \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2} \to 0 \text{ as } n \to \infty$$

Hence, given sequence $< \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2} >$ converges to 0.

Now, you can try the following Self-Assessment Question.

SAQ 2

If n^{th} term of a sequence $< a_n >$ is $< \frac{1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}}{n} >$, then find a_3 and a_{n+1}.

4.4 INFINITE SERIES

In Sec. 4.3 you have studied about sequence and discussed their converges. In this section you will study about infinite series of **positive terms**.

You know that a sequence $< a_n >$ can also be written as $a_1, a_2, a_3, a_4, \ldots$

If the terms of this sequence are added instead of separated by commas, then resulting expression will be $a_1 + a_2 + a_3 + a_4 + \ldots$ … (4.60)

In mathematics the expression written in (4.60) can be written using summation notation as follows

$$\sum_{n=1}^{\infty} a_n \quad \ldots (4.61)$$

Both expressions given by (4.60) and (4.61) are known as **infinite series**.

You know that if you add some finite real numbers then result is a finite real number. For example, (i) $5+10+15+20=50$ which is a finite number.

(ii) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{16+8+4+2+1}{16}=\frac{31}{16}$ which is a finite number. What about if you add infinite real numbers. Answer of this question is: it may be infinite or finite depends on the numbers in our series. Throughout this section we will trying to answer this question through different examples. But before going to discuss examples we have to define some terms which are necessary for further discussion.

Sequence of Partial Sum: If we have an infinite series $\sum_{n=1}^{\infty} a_n$ then the sequence $s_1, s_2, s_3, s_4, \ldots, s_n, \ldots$, where

$s_1=a_1,\ s_2=a_1+a_2,\ s_3=a_1+a_2+a_3,\ s_4=a_1+a_2+a_3+a_4, \ldots,$

$s_n=a_1+a_2+a_3+\ldots+a_n, \ldots$

is known as sequence of partial sum of the given infinite series $\sum_{n=1}^{\infty} a_n$.

Convergence and Divergence of Infinite Series: If the sum of infinite terms

$$a_1+a_2+a_3+\ldots+a_n+\ldots \qquad \ldots (4.62)$$

is a real number then we say that infinite series $\sum_{n=1}^{\infty} a_n$ converges, otherwise diverges. But we will not obtain sum of all terms to check its convergence. We will test convergence of series using some results. One easy way of testing whether given series converges or not is discussed as follows.

Note that $a_1+a_2+a_3+\ldots+a_n+\ldots=\lim_{n\to\infty}(a_1+a_2+a_3+\ldots+a_n)=\lim_{n\to\infty} s_n$ … (4.63)

From (4.63), we note that convergence or divergence of infinite series $\sum_{n=1}^{\infty} a_n$ depends on convergence or divergence of the sequence $<s_n>$ of partial sum. So, one of the testing procedures to test convergence of a given series is via testing convergence of the sequence $<s_n>$ of partial sum.

Now, let us state some results without proof as proofs are beyond the scope of this course which will be used to test whether a given series converges or diverges.

Result 15: If an infinite series $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n=0$. … (4.64)

Result 16: **Geometric Series**: Consider the infinite geometric series

$$\sum_{n=1}^{\infty} r^{n-1}=1+r+r^2+r^3+\ldots$$

It converges if $0<r<1$ and diverges if $r\geq 1$. … (4.65)

Result 17: **p-series**: An infinite series of the form

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots \qquad \ldots (4.66)$$

is known as p-series. It converges if $p>1$ and diverges if $p\leq 1$. … (4.67)

Result 18: **Comparison Test**: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two positive terms series and:

- $a_n \le k\, b_n \;\; \forall \; n \in \mathbb{N}$

then (i) convergence of $\sum_{n=1}^{\infty} b_n$ implies $\sum_{n=1}^{\infty} a_n$ converges. … (4.68)

(ii) divergence of $\sum_{n=1}^{\infty} a_n$ implies $\sum_{n=1}^{\infty} b_n$ diverges. … (4.69)

- $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, where l is a non-zero finite real number … (4.70)

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges or diverges together. … (4.71)

if l is 0 and $\sum_{n=1}^{\infty} b_n$ converges implies $\sum_{n=1}^{\infty} a_n$ converges. … (4.72)

if l is ∞ and $\sum_{n=1}^{\infty} b_n$ diverges implies $\sum_{n=1}^{\infty} a_n$ diverges. … (4.73)

Result 19: **D' Alembert's Ratio Test**: If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = l$, then

(i) $\sum_{n=1}^{\infty} a_n$ converges if l > 1 … (4.74)

(ii) $\sum_{n=1}^{\infty} a_n$ diverges if l < 1. … (4.75)

(iii) Test fails if l = 1 … (4.76)

(iv) $\sum_{n=1}^{\infty} a_n$ converges if $l = \infty$. … (4.77)

Result 20: **Cauchy's Root Test**: If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and

$\lim_{n\to\infty} (a_n)^{1/n} = l$, then

(i) $\sum_{n=1}^{\infty} a_n$ converges if l < 1 … (4.78)

(ii) $\sum_{n=1}^{\infty} a_n$ diverges if l > 1. … (4.79)

(iii) Test fails if l = 1 … (4.80)

(iv) $\sum_{n=1}^{\infty} a_n$ diverges if $l = \infty$. … (4.81)

Result 21: Convergence or divergence of an infinite series $\sum_{n=1}^{\infty} a_n$ remains unaltered on omission of some finite terms. … (4.82)

Result 22: If a series $\sum_{n=1}^{\infty} a_n$ converges to a then for a real number c series $\sum_{n=1}^{\infty} c\,a_n$ also converges and converges to ca. … (4.83)

Result 23: If a series $\sum_{n=1}^{\infty} a_n$ diverges then for a non-zero real number c series $\sum_{n=1}^{\infty} c\,a_n$ also diverges. … (4.84)

Result 24: If series $\sum_{n=1}^{\infty} a_n$ converges to a and series $\sum_{n=1}^{\infty} b_n$ converges to b then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ also converges and converges to a + b. … (4.85)

Result 25: If series $\sum_{n=1}^{\infty} a_n$ converges to a and series $\sum_{n=1}^{\infty} b_n$ converges to b then the series $\sum_{n=1}^{\infty} (a_n - b_n)$ also converges and converges to a – b. … (4.86)

Now, let us test convergence of some series using these results.

Example 10: Test the convergence of the series $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n}}$.

Solution: Here $a_n = \sqrt{\frac{n+1}{n}}$.

Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = \sqrt{1+0} = 1 \neq 0$

Hence, the given series $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n}}$ does not converges. [Using (4.64)]

Example 11: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$.

Solution: Here given series is $\sum_{n=1}^{\infty} \frac{1}{n^n}$. After omitting first two terms of the given series then it reduces to $\sum_{n=3}^{\infty} \frac{1}{n^n}$.

So, $\sum_{n=3}^{\infty} \frac{1}{n^n} = \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \frac{1}{6^6} + \ldots < \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} + \ldots \quad \left[\begin{array}{l} \because 3 < 4 \Rightarrow 3^4 < 4^4 \\ \Rightarrow \frac{1}{4^4} < \frac{1}{3^4}, \text{ etc.} \end{array}\right]$

But $\frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} + \ldots$ is an infinite geometric series with common ratio $\frac{1}{3} < 1$. Therefore it converges. [refer (4.65)]

But then using comparison test given series converges. [refer (4.68)]

Example 12: Test the convergence of the series $\frac{1}{5.9} + \frac{1}{6.11} + \frac{1}{7.13} + \frac{1}{8.15} + \ldots$

Solution: Here

$$a_n = \frac{1}{(n^{th} \text{ term of } 5, 6, 7, 8, \ldots)(n^{th} \text{ term of } 9, 11, 13, 15, \ldots)}$$

$$= \frac{1}{(n+4)(2n+7)} \quad \left[\begin{array}{l} \because n^{th} \text{ term of } 5, 6, 7, 8, \ldots \text{ which is an AP is} \\ a+(n-1)d = 5+(n-1)(1) = n+4 \\ \text{Similarly, } n^{th} \text{ term of } 9, 11, 13, 15, \ldots \\ \text{which is also an AP is } 9+(n-1)(2) = 2n+7 \end{array}\right]$$

$$= \frac{1}{n^2(1+4/n)(2+7/n)}$$

Let $b_n = \frac{1}{n^2}$

Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2(1+4/n)(2+7/n)}}{1/n^2} = \lim_{n\to\infty} \frac{1}{(1+4/n)(2+7/n)} = \frac{1}{(1+0)(2+0)} = \frac{1}{2}$$

which is non-zero finite number. Thus, by comparison test series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges or diverges together. But series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test as p = 2 > 1. [refer (4.67)]. Hence, given series converges.

Example 13: Test the convergence of the series $\frac{1}{5} + \frac{1}{2.5^2} + \frac{1}{3.5^3} + \frac{1}{4.5^4} + \ldots$

Solution: Here

$$\sum_{n=1}^{\infty} a_n = \frac{1}{5} + \frac{1}{2.5^2} + \frac{1}{3.5^3} + \frac{1}{4.5^4} + \ldots = \frac{1}{1.5} + \frac{1}{2.5^2} + \frac{1}{3.5^3} + \frac{1}{4.5^4} + \ldots$$

$$a_n = \frac{1}{(n^{th} \text{ term of } 1, 2, 3, 4, \ldots)(n^{th} \text{ term of } 5, 5^2, 5^3, 5^4, \ldots)}$$

$$= \frac{1}{n.5^n} \quad \left[\begin{array}{l} \because n^{th} \text{ term of } 1, 2, 3, 4, \ldots \text{ which is an AP is} \\ a+(n-1)d = 1+(n-1)(1) = n \\ \text{Similarly, } n^{th} \text{ term of } 5, 5^2, 5^3, 5^4, \ldots \text{ which is a GP is} \\ ar^{n-1} = 5(5)^{n-1} = 5^n \end{array}\right]$$

$$\therefore a_{n+1} = \frac{1}{(n+1)5^{n+1}}$$

Now,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{n5^n}}{\frac{1}{(n+1)5^{n+1}}} = \lim_{n\to\infty} \frac{5(n+1)}{n} = \lim_{n\to\infty} 5\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} 5\left(1+\frac{1}{n}\right) = 5(1+0) = 5$$

Since 5 > 1, so by D Alembert Ratio test given series converges. [refer (4.74)]

Example 14: Test the convergence of the series $\frac{x}{2.3} + \frac{x^2}{5.8} + \frac{x^3}{8.13} + \frac{x^4}{11.18} + \ldots$ where $x > 0$.

Solution: Here $a_n = \frac{n^{th} \text{ term of } x, x^2, x^3, x^4, \ldots}{(n^{th} \text{ term of } 2, 5, 8, 11, \ldots)(n^{th} \text{ term of } 3, 8, 13, 18, \ldots)}$

$$= \frac{x^n}{(3n-1)(5n-2)} \quad \left[\begin{array}{l} \because n^{th} \text{ term of } x, x^2, x^3, x^4, \ldots \text{ which is a GP is} \\ ar^{n-1} = x(x)^{n-1} = x^n \\ n^{th} \text{ term of } 2, 5, 8, 11, \ldots \text{ which is an AP is} \\ a+(n-1)d = 2+(n-1)(3) = 3n-1 \\ \text{Similarly, } n^{th} \text{ term of } 3, 8, 13, 18, \ldots \text{ which is also} \\ \text{an AP is } a+(n-1)d = 3+(n-1)(5) = 5n-2 \end{array}\right]$$

$$\therefore a_{n+1} = \frac{x^{n+1}}{(3(n+1)-1)(5(n+1)-2)} = \frac{x^{n+1}}{(3n+2)(5n+3)}$$

Now,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\dfrac{x^n}{(3n-1)(5n-2)}}{\dfrac{x^{n+1}}{(3n+2)(5n+3)}} = \lim_{n\to\infty} \frac{(3n+2)(5n+3)}{(3n-1)(5n-2)} \frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{\left(3+\dfrac{2}{n}\right)\left(5+\dfrac{3}{n}\right)}{\left(3-\dfrac{1}{n}\right)\left(5-\dfrac{2}{n}\right)} \frac{1}{x} = \frac{(3+0)(5+0)}{(3-0)(5-0)} \frac{1}{x} = \frac{1}{x}$$

So, by D Alembert Ratio test given series converges if $\frac{1}{x} > 1 \Rightarrow x < 1$, diverges if $\frac{1}{x} < 1 \Rightarrow x > 1$ and test fails if $\frac{1}{x} = 1 \Rightarrow x = 1$. [refer (4.74) to (4.76)]

Now, when x = 1, then

$$a_n = \frac{x^n}{(3n-1)(5n-2)} = \frac{1}{(3n-1)(5n-2)} = \frac{1}{n^2\left(3-\dfrac{1}{n}\right)\left(5-\dfrac{2}{n}\right)}$$

Let $b_n = \frac{1}{n^2}$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1}{n^2\left(3-\dfrac{1}{n}\right)\left(5-\dfrac{2}{n}\right)}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\left(3-\dfrac{1}{n}\right)\left(5-\dfrac{2}{n}\right)} = \frac{1}{(3-0)(5-0)} = \frac{1}{15}$$

which is non-zero finite number. Thus, by comparison test series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges or diverges together [refer (4.71)]. But series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test as p = 2 > 1. [refer (4.67)]. Hence, given series converges.

Now, you can try the following Self-Assessment Question.

SAQ 3

Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$.

4.5 ABSOLUTE AND CONDITIONAL CONVERGENCE

In Sec. 4.4 you have studied about convergence or divergence of positive term series. But there may be series having infinite terms of both signs positive and negative such series are known as **series of arbitrary terms**. If we have a series $\sum_{n=1}^{\infty} u_n$ of arbitrary terms then we say that it **converges absolutely** if series $\sum_{n=1}^{\infty} |u_n|$ converges. … (4.87)

We say that series $\sum_{n=1}^{\infty} u_n$ of arbitrary terms **converges conditionally** if series $\sum_{n=1}^{\infty} u_n$ converges but $\sum_{n=1}^{\infty} |u_n|$ diverges. … (4.88)

We will not discuss idea of absolute and conditional convergence in detail. We shall discuss only one example which is based on Leibnitz's test. So, first we have to state Leibnitz's test. But Leibnitz's test is used to check the convergence behaviour of alternating term series only, therefore, first we define what we mean by alternating series.

Alternating Series: A series having terms alternatively positive and negative is known as alternating series.

For example, series $\sum_{n=1}^{\infty} (-1)^{n-1} n = 1-2+3-4+5-6+7-8+\ldots$ is an alternating series.

Now, we state Leibnitz's test.

Leibnitz's Test: Alternating series
$\sum_{n=1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + u_7 - u_8 + \ldots$ converges if

(i) $u_{n+1} \le u_n \ \forall \ n \in \mathbb{N}$ and (ii) $\lim_{n\to\infty} u_n = 0$, where $u_i > 0$ for all i. … (4.89)

Example 15: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

Solution: Here given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$,

where $v_n = \frac{1}{n}$

$\therefore \ v_{n+1} = \frac{1}{n+1}$

Since $n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1} \Rightarrow v_n > v_{n+1}$. Also, $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Thus, both conditions of Leibnitz's test satisfied and hence given alternating series converges [refer (4.89)]. But series $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test as p = 1. [refer (4.67)]

Hence, given series converges conditionally not absolutely.

4.6 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- **Sequence** is a function having domain as the family of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and range may be any set but in this course, we have discussed only sequences of real numbers.
- **Real Sequence**: A real sequence is a function whose domain is the set of all natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and range may be any subset of $\mathbb{R}$, the set of all real numbers.
- **Ways of Representing a Sequence**: Here we have discussed four ways of representing a sequence.
  - **First way**: A sequence can be represented by writing the first few terms till a definite rule for writing down other term becomes clear. For example, 1, 4, 9, 16, 25, … is a sequence written in this way.
  - **Second Way**: A sequence can also be represented by giving a formula for its n^{th} term. For example, $< \frac{n+1}{n} >$.
  - **Third Way (Recursive Relation)**: A sequence can also be represented by specifying its first few terms and a formula to write remaining terms in terms of the preceding terms. For example, $a_1 = 1, a_2 = 1, a_{n+1} = a_n + a_{n-1} \ \forall \ n \geq 2, n \in \mathbb{N}$
  - **Fourth Way**: Sometimes sequence is such that it cannot be represented by giving a single formula for its n term. But its terms are generated by two or more relations. For example, $a_n = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$
- **Convergence of a Sequence**: A real sequence $< a_n >$ is said to converges a real number 'a' if for a given $\varepsilon > 0$ however small it may be there exists a natural number m depending on ε such that $|a_n - a| < \varepsilon \ \forall \ n \geq m$.
- **Divergence of a Sequence**: A real sequence $< a_n >$ is said to diverges to ∞ if given any positive real number K no matter how large it is there exists a positive integer m such that $a_n > K \ \forall \ n \geq m,$ and we write it as $a_n \to \infty$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = \infty$.
- **Oscillatory Sequence**: If a real sequence $< a_n >$ neither converges to a finite real number and nor diverges to ∞ or $-\infty$, then we say that sequence oscillates. An oscillatory sequence may **oscillate finitely or infinitely**.
- **Bounded above Sequence**: A sequence $< a_n >$ is said to be bounded above if there exists a real number K such that $a_n \leq K \ \forall \ n \in \mathbb{N}$
- **Bounded below Sequence**: A sequence $< a_n >$ is said to be bounded

below if there exists a real number k such that $a_n \geq k \quad \forall\, n \in \mathbb{N}$

- **Bounded Sequence**: A sequence $< a_n >$ is said to be bounded if it is both bounded above and bounded below.
- **Least Upper Bound**: Let $< a_n >$ be a real and bounded above sequence then a real number u is said to be least upper bound (lub) of the sequence $< a_n >$ if

  (i) $a_n \leq u \quad \forall\, n \in \mathbb{N}$ and

  (ii) given any $\varepsilon > 0,$ no matter how small it is, there exists at least one $n \in \mathbb{N}$ such that $a_n > u - \varepsilon.$

- **Greatest Lower Bound**: Let $< a_n >$ be a real and bounded below sequence then a real number g is said to be greatest lower bound (glb) of the sequence $< a_n >$ if

  (i) $a_n \geq g \quad \forall\, n \in \mathbb{N}$ and

  (ii) given any $\varepsilon > 0,$ no matter how small it is, there exists at least one $n \in \mathbb{N}$ such that $a_n < g + \varepsilon.$

- **Monotonically Increasing Sequence**: A real sequence $< a_n >$ is said to be monotonically increasing sequence if $a_n \leq a_{n+1} \quad \forall\, n \in \mathbb{N}$
- **Monotonically Decreasing Sequence**: A real sequence $< a_n >$ is said to be monotonically decreasing sequence if $a_n \geq a_{n+1} \quad \forall\, n \in \mathbb{N}$
- **Monotonic Sequence**: A real sequence $< a_n >$ is said to be monotonic if either it is monotonically increasing or monotonically decreasing sequence.
- **Cauchy Sequence**: A sequence $< a_n >$ is said to be a Cauchy sequence if given $\varepsilon > 0,$ no matter how small it is there exists a positive integer n_0 such that $|a_n - a_m| < \varepsilon \quad \forall\, m, n \geq n_0$
- The expression $a_1 + a_2 + a_3 + a_4 + \ldots$ or $\sum_{n=1}^{\infty} a_n$ is known as **infinite series**.
- **Sequence of Partial Sum**: If we have an infinite series $\sum_{n=1}^{\infty} a_n$ then the sequence $s_1, s_2, s_3, s_4, \ldots, s_n, \ldots,$ where

  $s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3,\ s_4 = a_1 + a_2 + a_3 + a_4, \ldots,$
  $s_n = a_1 + a_2 + a_3 + \ldots + a_n, \ldots$

  is known as sequence of partial sum of the given infinite series $\sum_{n=1}^{\infty} a_n.$

- **Convergence and Divergence of Infinite Series**: If the sum of infinite terms $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is a real number, then we say that infinite series $\sum_{n=1}^{\infty} a_n$ converges, otherwise diverges.
- If a series have infinite terms of both signs positive and negative such series are known as **series of arbitrary terms**.
- If we have a series $\sum_{n=1}^{\infty} u_n$ of arbitrary terms then we say that it **converges**

absolutely if series $\sum_{n=1}^{\infty}|u_n|$ converges.

- We say that series $\sum_{n=1}^{\infty}u_n$ of arbitrary terms **converges conditionally** if series $\sum_{n=1}^{\infty}u_n$ converges but $\sum_{n=1}^{\infty}|u_n|$ diverges.

- **Alternating Series**: A series having terms alternatively positive and negative is known as alternating series.

- We have also mentioned some standard results related to test of convergence of sequence and series without proof. We have also learnt to apply these tests with the help of many examples.

4.7 TERMINAL QUESTIONS

1. Test the convergence of the series
$$\frac{3}{4}+\frac{7}{3}+\frac{3}{4^2}+\frac{7}{3^2}+\frac{3}{4^3}+\frac{7}{3^3}+\frac{3}{4^4}+\frac{7}{3^4}+\ldots$$

2. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$.

4.8 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. In numerator, we have an AP, so n^{th} term in numerator is given by
$$a+(n-1)d=9+(n-1)(2)=9+2n-2=2n+7$$
In denominator it is obvious that n^{th} term is n^2. Hence, n^{th} term of the given sequence is $\frac{2n+7}{n^2}$.

2. If $a_n=\frac{1+2^{1/2}+3^{1/3}+\ldots+n^{1/n}}{n}$, then $a_3=\frac{1+2^{1/2}+3^{1/3}}{3}$ and
$$a_{n+1}=\frac{1+2^{1/2}+3^{1/3}+\ldots+n^{1/n}+(n+1)^{1/(n+1)}}{n+1}.$$

3. Here $a_n=\left(\frac{n}{n+2}\right)^{n^2}$, then
$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left(\frac{n}{n+2}\right)^n=\lim_{n\to\infty}\left(\frac{1}{1+\frac{2}{n}}\right)^n=\lim_{n\to\infty}\frac{(1)^n}{\left(1+\frac{2}{n}\right)^n}=\lim_{n\to\infty}\frac{1}{\left(\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right)^2}$$
$$=\frac{1}{\left(\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right)^2}=\frac{1}{(e)^2}\left[\because \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e\right]$$

But $\frac{1}{e^2} < 1$ $\left[\begin{array}{l} \because \text{ Refer (1.32), } e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots \text{ which is } > 1 \\ \Rightarrow e^2 > 1 \Rightarrow \frac{1}{e^2} < 1 \end{array}\right]$

Hence, by Cauchy's root test given series converges. [refer (4.78)]

Terminal Questions

1. Here $\sum_{n=1}^{\infty} a_n = \frac{3}{4} + \frac{7}{3} + \frac{3}{4^2} + \frac{7}{3^2} + \frac{3}{4^3} + \frac{7}{3^3} + \frac{3}{4^4} + \frac{7}{3^4} + \ldots$

$$= \left(\frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \frac{3}{4^4} + \ldots\right) + \left(\frac{7}{3} + \frac{7}{3^2} + \frac{7}{3^3} + \frac{7}{3^4} + \ldots\right)$$

$$= 3\left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \ldots\right) + 7\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots\right)$$

$$= \sum_{n=1}^{\infty} 3b_n + \sum_{n=1}^{\infty} 7c_n$$

Series $\sum_{n=1}^{\infty} b_n = \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \ldots\right)$, $\sum_{n=1}^{\infty} c_n = \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots\right)$ both being geometric series with common ratio $\frac{1}{4} < 1$ and $\frac{1}{3} < 1$ respectively converges. [refer (4.65)]. Further, we know that if series $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} c\, b_n$ also converges for finite real number c. [refer (4.83)]

Thus, series $\sum_{n=1}^{\infty} 3b_n$ and $\sum_{n=1}^{\infty} 7c_n$ both converges. We also know that if series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converges then their sum $\sum_{n=1}^{\infty} (u_n + v_n)$ converges. [refer (4.85)].

Hence, given series converges.

2. Here given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$. Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$,

where $v_n = \frac{1}{n^2}$

$\therefore\ v_{n+1} = \frac{1}{(n+1)^2}$

Since $n^2 < (n+1)^2 \Rightarrow \frac{1}{n^2} > \frac{1}{(n+1)^2} \Rightarrow v_n > v_{n+1}\ \ \forall\ n \in \mathbb{N}$

Also, $\lim_{n \to \infty} v_n = \lim_{n \to \infty} \frac{1}{n^2} = 0.$

Thus, both conditions of Leibnitz's test satisfied and hence given alternating series converges [refer (4.89)]. Also, series

$$\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$ converges by p-test as p = 2>1. [refer (4.67)]

Hence, given series is absolutely convergent also.