Indira Gandhi National Open University
School of Sciences

MST-011
Real Analysis, Calculus and Geometry

Block

2

HYPERPLANE AND CALCULUS

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REAL ANALYSIS, CALCULUS AND GEOMETRY

BLOCK 1: Real Analysis

Unit 1: Function of a Single Variable

Unit 2: Types, Continuity and Differentiability of Function of a Single Variable

Unit 3: Set Function and Distance Function

Unit 4: Sequences and Series

Unit 5: Riemann Integration

BLOCK 2: Hyperplane and Calculus

Unit 6: Hyperplane

Unit 7: Convex and Concave Function

Unit 8: Beta and Gamma Functions

Unit 9: Change of Order of Summation and Integration

BLOCK 2 HYPERPLANE AND CALCULUS

In this block, you will study some concepts of geometry and calculus. A review of equations of lines in 2 and 3-dimension in various forms and equations of the plane is discussed. The equation of the hyperplane and the ideas of a convex set and convex function with their properties are also discussed in this block. Two important functions namely: beta and gamma functions are discussed. The last unit of this block will make you familiar with the change of order of double summation and double integration.

This block is divided into four units, namely, Units 6- 9.

Unit 6: This unit first gives an overview of the equation of a line in 2 and 3-dimension, vector and the equation of a plane. Finally, the equation of hyperplane is discussed. A brief overview of solutions to inequations in one and two variables is also provided in this unit.

Unit 7: This unit explains the convex set and convex function through lots of examples with their visualisation. Some properties of the convex set and convex functions are also discussed.

Unit 8: This unit visualises the effect of parameters of beta and gamma functions on their shapes. Some properties of beta and gamma functions are also discussed in this unit.

Unit 9: Change of order of summations in double summation and change of order of integration in double integration will be used in the course MST-012. How you can change the order of two summations as well as double integration is discussed in this unit.

This material has been developed for self-study. However, if you are interested in studying the content in greater depth/more detail, it would be useful for you to attempt related exercises given in the relevant chapters of the books listed at the end of the course introduction. We hope you will enjoy studying this block.

Expected Learning Outcomes

After completing this block, you should be able to:

- write equations of line, plane and hyperplane in different forms;
- write equations of a line and line segment as a set of points;
- explain the idea of the convex set and convex function. You will be also able to state and prove some properties of the convex set and convex function;
- solve problems based on beta and gamma functions; and
- change the order of double summation and integration.

Course Preparation Team

Notations and Symbols

m	:	slope of a line
Γn	:	gamma function of n
$B(m, n)$	:	beta function with parameters m and n
$\int dx$	:	single integral with respect to x
$\iint dx\,dy$	:	double integral first with respect to x and then with respect to y
$\iint dy\,dx$	:	double integral first with respect to y and then with respect to x

UNIT 6

HYPERPLANE

Structure

6.1 Introduction

Expected Learning Outcomes

6.2 Equation of a Line in 2-Dimension

Point Slope Form

Two Point Form

Intercept Form

Slope Intercept Form

General Form

6.3 A Review of Vector Algebra

6.4 Equation of a Line in 3-Dimension

6.5 Equation of a Plane

6.6 Equation of Hyperplane

6.7 Inequation in One and Two Variables

6.8 Summary

6.9 Terminal Questions

6.10 Solutions/Answers

6.1 INTRODUCTION

In earlier classes, you have studied different forms of the equation of a line in 2 and 3-dimension. A brief overview of the equation of a line in 2-dimension is given in Sec. 6.2. To study the equation of a line in 3-dimension you should have an understanding of some basic results of vector algebra which are discussed in Sec. 6.3 and then the equation of a line in 3-dimension is discussed in Sec. 6.4. In the course MST-026 you will study support vector machine (SVM). In SVM sometimes two categories are not linearly separable. So, we go to a higher dimension to find a separating plane. If you work in 3-D then you need a plane which separates two categories and if you work in a higher dimension of more than 3 then we try to identify a hyperplane which separates two categories. These two things plane and hyperplane are discussed in Secs. 6.5 and 6.6 respectively. In some courses of this programme, you will use inequations and so some basic results related to inequation are discussed in Sec. 6.7.

What we have discussed in this unit is summarised in Sec. 6.8. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit some more questions based on the entire unit are given in Sec. 6.9 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 6.10.

In the next unit, you will study about convexity of a set and function.

Expected Learning Outcomes

After completing this unit, you should be able to:

- ❖ obtain the equation of a line in different forms in 2 and 3-dimension;
- ❖ explain the basic results of vector algebra which are required to deal with equations of line, plane and hyperplane;
- ❖ obtain the equation of a plane and hyperplane in different forms; and
- ❖ deal with inequalities in one and two variables.

6.2 EQUATION OF A LINE IN 2-DIMENSION

The slope of a line plays a crucial role in the equation of a straight line in 2-dimension. The slope of a line is directly related to increment. Or we can say that increment plays an important role in the slope of a line. So, before discussing the equation of a line in 2-dimension let us first understand what is increment. And then we will define what is the slope of a line.

Increment: Let an object changes its position from point $A(x_1, y_1)$ to point $B(x_2, y_2)$ then the net changes in the coordinates of point A are known as increments. An increment in the direction of the x-axis is known as **run** and an increment in the direction the of y-axis is known as **rise**. Thus, we have

$$\text{Run} = (\text{x coordinate of point B}) - (\text{x coordinate of point A}) = x_2 - x_1$$

$$\text{Rise} = (\text{y coordinate of point B}) - (\text{y coordinate of point A}) = y_2 - y_1$$

Slope: Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the line l refer to Fig. 6.1. Draw a line from point A and parallel to the x-axis and another line from point B parallel the to y-axis both intersect at point C. So, we have

$$\text{Run} = AC = x_2 - x_1, \quad \text{Rise} = CB = y_2 - y_1 \qquad \ldots (6.1)$$

Consider the ratio $\dfrac{\text{Rise}}{\text{Run}} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{CB}{AC}$ … (6.2)

Similarly, if we consider two points P and Q on the same line l then in ΔPQR, we have

$$\frac{\text{Rise}}{\text{Run}} = \frac{RQ}{PR} \qquad \ldots (6.3)$$

Now, in ΔABC and ΔPQR, we have

$\underline{|1} = \underline{|3}$ [Corresponding angles]

$\underline{|2} = \underline{|4}$ [Each 90^0]

$\therefore\ \Delta ABC \sim \Delta PQR$ [AA similarity]

So, $\dfrac{CB}{AC} = \dfrac{RQ}{PR}$ [Corresponding parts of similar triangles (cpst)]… (6.4)

In view of equations (6.2), (6.3) and (6.4) we see that if we take any two points on a line then the ratio of Rise and Run remains constant. This constant ratio is known as the **slope** of the line.

Also note that if the line l makes an angle θ with a positive direction of the x-axis then $\underline{|1}$ and θ being corresponding angles are equal. So, in right-angled ΔABC, we have

$$\tan\theta = \frac{CB}{AC} = \frac{Rise}{Run} = \frac{y_2 - y_1}{x_2 - x_1} = \text{Slope of the line l} \qquad \ldots (6.5)$$

Also, observe from equation (6.2) that if the coordinates of two points on a line are given then the slope of a line is the ratio of the difference between their y-coordinates to their corresponding x-coordinates.

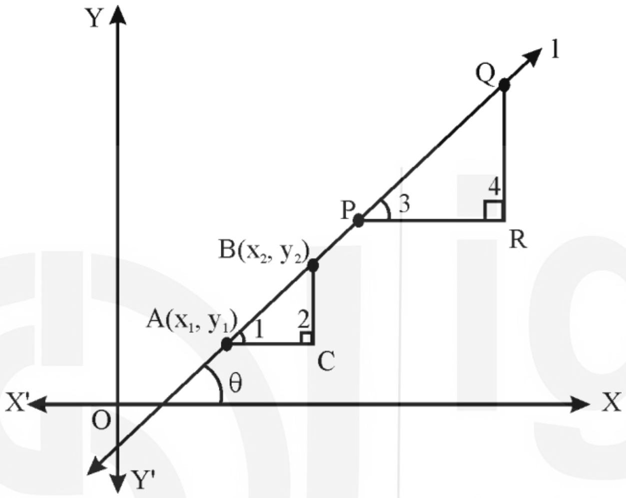

Fig. 6.1: Visualisation of rise, run and concept of slope of a line

Now, we discuss different forms of equation of a line in 2-dimension one at a time.

6.2.1 Point Slope Form

Here we are given two things: a point on the line and slope of the line. Refer Fig. 6.2 (a).

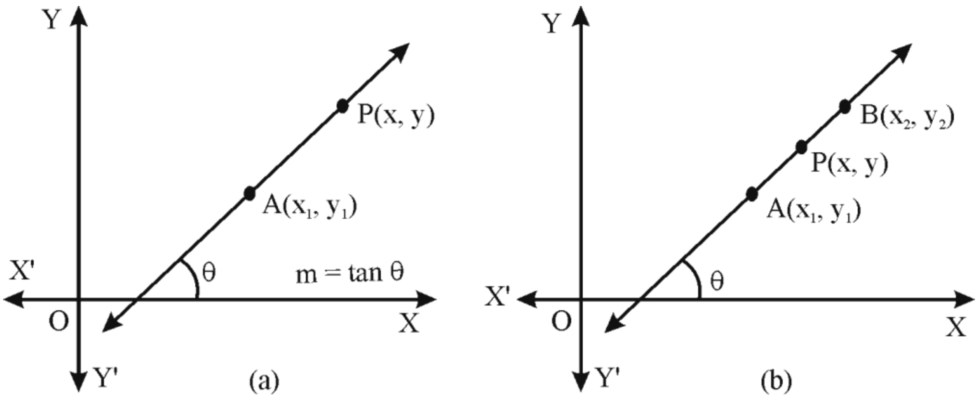

Fig. 6.2: (a) Point slope form (b) Two-point form

Suppose given point is $A(x_1, y_1)$ and slope of the line is m. Let $P(x, y)$ be any point on the line. So, the slope of the line AP is given by $\frac{y - y_1}{x - x_1}$. But the slope of this line is given to us as m. So, equating two slopes we get

$$\frac{y - y_1}{x - x_1} = m$$

$$\Rightarrow y - y_1 = m(x - x_1) \qquad \ldots (6.6)$$

This is the equation of the line in **point-slope form**. Some authors also call it **one-point form**.

Example 1: Find slope of the line in each of the following cases:

(i) passing through the points (2, 3) and (6, 10).

(ii) making an angle of 60^0 with positive direction of x-axis.

Solution: (i) We know that slope of a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. So, in our case it is given by

$$m = \frac{10 - 3}{6 - 2} = \frac{7}{4}$$

(ii) Slope of the line which makes an angle of 60^0 with positive direction of the x-axis is $m = \tan\theta = \tan 60^0 = \sqrt{3}$.

Example 2: Slope of a line is 5/3 and of another line is $-7/9$ interpret both the slopes.

Solution: We know that slope of a line is rise/run. So, slope 5/3 means when a particle moves along this line then it rises 5 units for every run of 3 units. Or we can say that y increases 5 units every time when x increases 3 units. It can also be written as $m = \frac{5}{3} = \frac{5/3}{1}$, so we can say that y increases 5/3 units every time when x increases 1 unit. In the case, slope is $-7/9$ it means y decreases by 7 units every time when x increases by 9 units.

6.2.2 Two Point Form

Here we are given two points on a line and we are interested in the equation of the line passing through these two points refer Fig. 6.2 (b).

Suppose given two points be $A(x_1, y_1)$ and $B(x_2, y_2)$. Let P(x, y) be any point on the line passing through points A and B. Since all the three points A, B and P lie on the same line So, we must have

$$\text{Slope of line AP} = \text{Slope of line AB}$$

$$\Rightarrow \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad [\text{Using } (6.5)]$$

$$\text{or} \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \ldots (6.7)$$

$$\text{or} \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \qquad \ldots (6.8)$$

$$\text{or} \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \lambda \text{ (say)} \qquad \ldots (6.9)$$

$$\Rightarrow x = x_1 + \lambda(x_2 - x_1), \quad y = y_1 + \lambda(y_2 - y_1) \qquad \ldots (6.10)$$

where λ is any real number

Or it can also be written as

$$\Rightarrow x = (1-\lambda)x_1 + \lambda x_2, \quad y = (1-\lambda)y_1 + \lambda y_2, \ -\infty < \lambda < \infty \qquad \ldots (6.11)$$

This line as a set of points can be written as

$$\{(1-\lambda)(x_1, y_1) + \lambda(x_2, y_2) : -\infty < \lambda < \infty\} \qquad \ldots (6.12)$$

Or $\{(1-\lambda)x + \lambda y : -\infty < \lambda < \infty\}$, where $x = (x_1, y_1)$, $y = (x_2, y_2)$ … (6.13)

Equation (6.10) gives us a general point on the line given by (6.8) or (6.9). Note that

(i) when $\lambda = 0$, it gives the point (x_1, y_1).

(ii) when $\lambda = 1$, it gives the point (x_2, y_2).

(iii) when $\lambda = 1/2$, it gives the midpoint of AB, i.e., $\left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}\right)$.

(iv) when $\lambda = 2$, it gives a point 2 units aways from the point A in the same side of position of the point B from the point A.

(v) when $\lambda = -2$, it gives a point 2 units aways from the point A in the opposite side of position of the point B from point A, etc.

(vi) Using (6.9) and on the basis of the discussion done in points (i) to (v), we can say that all the points between points $A(x_1, y_1)$ and $B(x_2, y_2)$ including points A and B (or on the line segment AB) are given by

$$x = x_1 + \lambda(x_2 - x_1), \quad y = y_1 + \lambda(y_2 - y_1), \text{ where } 0 \le \lambda \le 1 \qquad \ldots (6.14)$$

Or you can say that (6.14) represents equation of the line segment AB.

Also, if λ is allowed to take any real value then it will represent equation of line AB and it is known as the parametric equation of the line. So, the parametric equation of the line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$x = x_1 + \lambda(x_2 - x_1), \quad y = y_1 + \lambda(y_2 - y_1), \text{ where } -\infty < \lambda < \infty \qquad \ldots (6.15)$$

Like (6.12) and (6.13), the line segment AB as a set of points can be written as

$$\{(1-\lambda)(x_1, y_1) + \lambda(x_2, y_2) : 0 \le \lambda \le 1\} \qquad \ldots (6.16)$$

Or $\{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}$, where $x = (x_1, y_1)$, $y = (x_2, y_2)$ … (6.17)

Example 3: Find equation of a line passing through the points (4, 5) and $(-3, 7)$. Also find the coordinates of a point on this line which is 100 units away from the point (4, 5) in the direction opposite to the point $(-3, 7)$.

Solution: Using two-point form equation of a line passing through the given points is given by

$$\frac{x-4}{-3-4} = \frac{y-5}{7-5} \Rightarrow \frac{x-4}{-7} = \frac{y-5}{2} \Rightarrow 2x + 7y - 43 = 0$$

Now, coordinates of a point on this line which is 100 units away from the point (4, 5) in the direction opposite to the point $(-3, 7)$ is given by solving following for x and y

$$\frac{x-4}{-7}=\frac{y-5}{2}=-100 \Rightarrow x=704,\ y=-195$$

So, required point is $(704, -195)$.

6.2.3 Intercept Form

Let us first define what are x and y-intercepts. If a line which is neither horizontal nor vertical intersects x-axis at a point having coordinates (a, 0) and y-axis at a point having coordinates (0, b) then a and b are known as **x and y-intercepts** respectively.

Here we are interested in the equation of a line which has a and b as x and y-intercepts respectively. Refer Fig. 6.3 (a). By definition of intercepts this line passes through the points A(a, 0) and B(0, b). Therefore, using two-point form equation of such a line is given by

$$y-0=\frac{b-0}{0-a}(x-a) \Rightarrow -ay=bx-ab \Rightarrow bx+ay=ab \Rightarrow \frac{x}{a}+\frac{y}{b}=1 \quad \ldots (6.18)$$

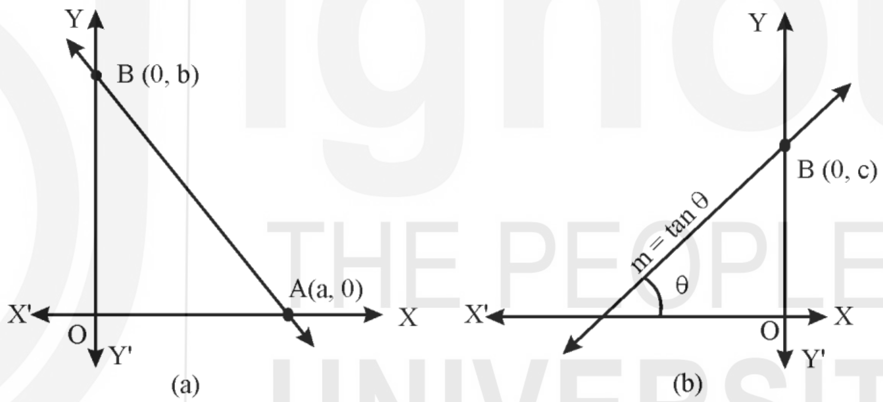

Fig. 6.3: (a) Intercept form (b) Slope intercept form

Example 4: If a line makes -3 and 4 as x and y intercepts respectively. Find equation of the line.

Solution: Using intercept form equation of the required line is given by

$$\frac{x}{-3}+\frac{y}{4}=1 \Rightarrow 4x-3y+12=0$$

6.2.4 Slope Intercept Form

Here we are given two things: (i) slope of the line (ii) y-intercept made by the line. We are interested in the equation of such a line. Refer Fig. 6.3 (b).

Let slope of the line be m and c be the y-intercept made by the line. So, coordinates of the point where line intercepts y-axis are (0, c). Now, equation of the line using slope point form is given by

$$\Rightarrow y-c=m(x-0) \Rightarrow y=mx+c \quad \ldots (6.19)$$

If c = 0 then line will pass through origin and equation (6.19) reduces to

$$y=mx \quad \ldots (6.20)$$

So, remember equation (6.20) represents line passing through origin and having slope m.

Example 5: If slope of a line is -5 and y-intercept is 2. Find equation of the line.

Solution: Using slope intercept form equation of the required line is given by $y = mx + c \Rightarrow y = -5x + 2$

or $5x + y - 2 = 0$

6.2.5 General Form

All the forms discussed in sub-Secs. 6.2.1 to 6.2.4 after simplification reduces to the form

$$ax + by + c = 0 \qquad \dots (6.21)$$

where not both a and b are zero because if both a = 0 and b = 0 then equation (6.21) will not represent a straight line

The form given in equation (6.21) is known as equation of a line in general form. So, remember graph of any equation of the form given in (6.21) will represent a straight line and equation of any straight line can be represented in the form of equation (6.21).

One more thing to note: you are aware that any two separate points lead to a unique line. But equation (6.21) seems that it has three unknown parameters a, b and c so we need three conditions to obtain equation of a line. Actually, equation (6.21) has only two parameters A and B explained as follows

$$Ax + By = 1 \left[\because ax + by = -c \Rightarrow \frac{a}{-c}x + \frac{b}{-c}y = 1 \Rightarrow Ax + By = 1, A = \frac{a}{-c}, B = \frac{b}{-c}\right]$$

Example 6: If equation of a line is $5x + 3y + 7 = 0,$ then find:

(i) slope of the line and y-intercept.

(ii) intercepts of both x and y.

Solution: Given equation of the line is $5x + 3y + 7 = 0 \qquad \dots (6.22)$

(i) Equation (6.22) can be written as $y = -\frac{5}{3}x - \frac{7}{3} \qquad \dots (6.23)$

Comparing equation (6.23) with equation (6.19), we get $m = -\frac{5}{3}, c = -\frac{7}{3}.$

So, slope of the given line is $m = -\frac{5}{3},$ and y-intercept is $c = -\frac{7}{3}.$

Another way of obtaining slope of a line is $= -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{5}{3}$

But keep in mind that before using this formula make sure that terms of x and y both should be on the same side of equality sign.

(ii) Equation (6.22) can be written as $-\frac{5}{7}x - \frac{3}{7}y = 1$

or $\dfrac{x}{-7/5} + \dfrac{y}{-7/3} = 1$ … (6.24)

Comparing equation (6.24) with equation (6.18), we get $a = -\dfrac{7}{5}, b = -\dfrac{7}{3}$.

So, x-intercept is $a = -\dfrac{7}{5}$, and y-intercept is $b = -\dfrac{7}{3}$.

Also, remember following results related to a line.

Perpendicular Distance of a Point from a Line: Refer Fig. 6.4 (a). If d is the perpendicular distance of a point $A(x_1, y_1)$ from the line ax + by + c = 0 then

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad \text{… (6.25)}$$

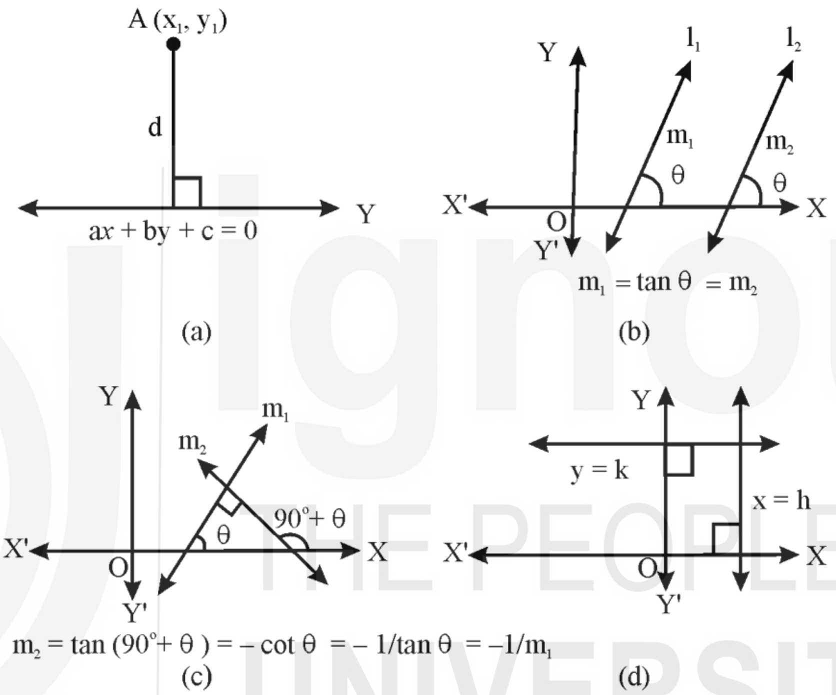

Fig. 6.4: Visualisation of (a) ⊥ distance of a point A from a line (b) Slopes of parallel lines (c) Slopes of ⊥ lines (d) Equation of lines parallel to axes

Slope of a Line Parallel to a Given Line: Refer Fig. 6.4 (b). If m_1 is the slope of a given line l_1, and line l_2 is parallel to line l_1, then the slope m_2 of line l_2 is equal to slope of line l_1. That is $m_1 = m_2$ … (6.26)

Slope of a Line Perpendicular to a Given Line: Refer Fig. 6.4 (c). If m_1 is the slope of a given line l_1, and m_2 be the slope of a line l_2 which is perpendicular to given line l_1, then $m_1 m_2 = -1$. … (6.27)

Except in the special cases slope of horizontal and vertical line. Slope of a horizontal line is zero $\left[\because \tan 0^0 = 0\right]$ and slope of vertical line is not defined. $\left[\because \tan 90^0 = \infty\right]$

Equation of a line Parallel to x-axis or y-axis: Refer Fig. 6.4 (d). For any particular line parallel to x-axis coordinates of each point satisfy the following conditions:

(i) x-coordinates varies from point to point, but

(ii) y-coordinate is a fixed constant.

So, equation of any particular line parallel to x-axis is given by y = k. … (6.28)

Similarly,

equation of any particular line parallel to y-axis is given by x = h. … (6.29)

To obtain value of h or k we need only one condition. Given condition may be: a point is given from which line passes or distance from respective axis is given, and to which side of the respective axis.

Now, you can try the following Self-Assessment Question.

SAQ 1

(a) Find the equation of a line passing through the point (2, 3) and having slope 7/8.

(b) Find the perpendicular distance of the point $(2, -7)$ from the line $3x + 4y + 9 = 0$.

(c) Find the slope of a line which is parallel to the line $5x + 6y + 8 = 0$.

6.3 A REVIEW OF VECTOR ALGEBRA

When we work in higher dimension the use of vector notations makes things simple to write and understand. That is why we are recalling some results of vector algebra from earlier classes before discussing equation of a line and plane in 3-dimension and hyperplane in n-dimension. So, in this section we will review some notations and basic concepts of vector algebra which you have studied in earlier classes. We will use these notations and concepts in the next three sections of this unit and in the courses MST-024 and MST-026. Let us start this journey with definition of a vector.

Vector: A vector is a quantity that has both magnitude and direction. If a vector has initial point A and terminal point B then it is denoted by $\overrightarrow{AB}$. Magnitude of the vector $\overrightarrow{AB}$ is the length of the line segment AB and direction is from point A to point B.

Unit Vector: A vector having magnitude 1 (unity) is called unit vector. A unit vector is denoted by putting a cap on it. Unit vectors along x-axis, y-axis and z-axis are denoted by $\hat{i}, \hat{j}, k$ respectively.

Position Vector of a Point: If O is the origin and A is any point then position vector of the point A is denoted by $\overrightarrow{OA}$ and is a vector having magnitude equal to the length OA and direction is from origin O to the point A. Refer Fig. 6.5 (a). If coordinates of point A are (a_1, b_1, c_1) then you know that a_1, b_1, c_1 are the distances of point A from origin along x-axis, y-axis, z-axis respectively. So, position vector of point A can be written as

$$\overrightarrow{OA} = a_1\hat{i} + b_1\hat{j} + c_1 k \quad \text{… (6.30)}$$

and magnitude of vector $\overrightarrow{OA}$ is denoted by $\left|\overrightarrow{OA}\right|$ and given by

$$\left|\overrightarrow{OA}\right| = \sqrt{a_1^2 + b_1^2 + c_1^2} \qquad \dots (6.31)$$

where $a_1,\ b_1,\ c_1$ are known as components of the vector $\overrightarrow{OA}$

Keep in mind that throughout this course we will denote the position vector of a general point P(x, y, z) by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. … (6.32)

Remark 1: In the courses MST-018, MST-024 and MST-026 we will denote the vector given by (6.30) in any of the following three ways:

(i) Like a point in 3-dimension as (a_1, b_1, c_1) … (6.33)

(ii) As a column matrix $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ … (6.34)

(iii) Or as a transpose of a row matrix $\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}^T$ … (6.35)

These notations will also be extended to n-dimension in the exactly similar way. So, keep this structure of different notations for the same thing in your mind.

Vector Joining Two Points: If A and B are two points and O is the origin then the vector joining points A and B is denoted by $\overrightarrow{AB}$ and defined as

$$\overrightarrow{AB} = (\text{Position vector of point B}) - (\text{Position vector of point A})$$

$$= \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a} \qquad \dots (6.36)$$

Refer Fig. 6.5 (b).

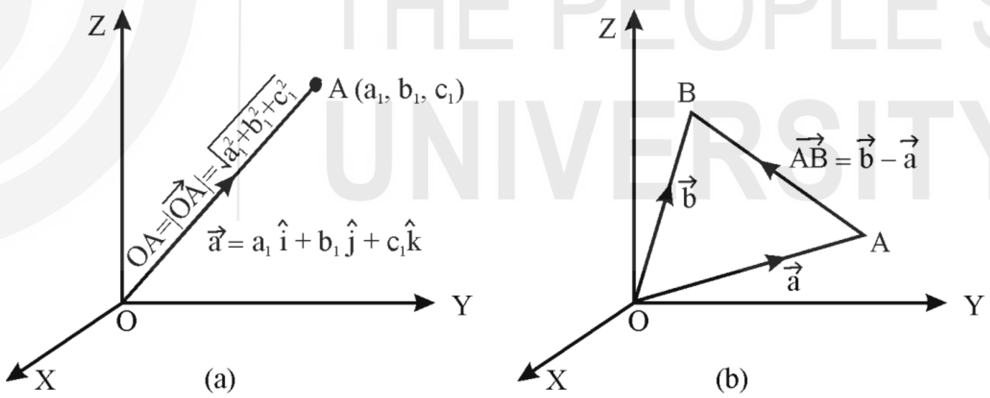

Fig. 6.5: (a) Position vector of a point (b) Vector joining two points

Addition of Vectors: Let $\vec{a}$ and $\vec{b}$ are two vectors refer Fig. 6.6 (a). You know from the knowledge of earlier classes that a vector remains the same if it is shifted parallel to itself anywhere in its plane. Addition of two vectors is based on this property. Let us shift vector $\vec{b}$ parallel to itself such that initial point of $\vec{b}$ coincides with terminal point of $\vec{a}$, refer Fig. 6.6 (b). Join initial point of vector $\vec{a}$ to the terminal point of $\vec{b}$ then the vector thus obtained is known as $\vec{a} + \vec{b}$ refer Fig. 6.6 (c). In particular, using this concept of addition of two vectors in any triangle ABC (say), we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \qquad \dots (6.37)$$

This is known as **triangle law of addition** refer Fig. 6.6 (d).

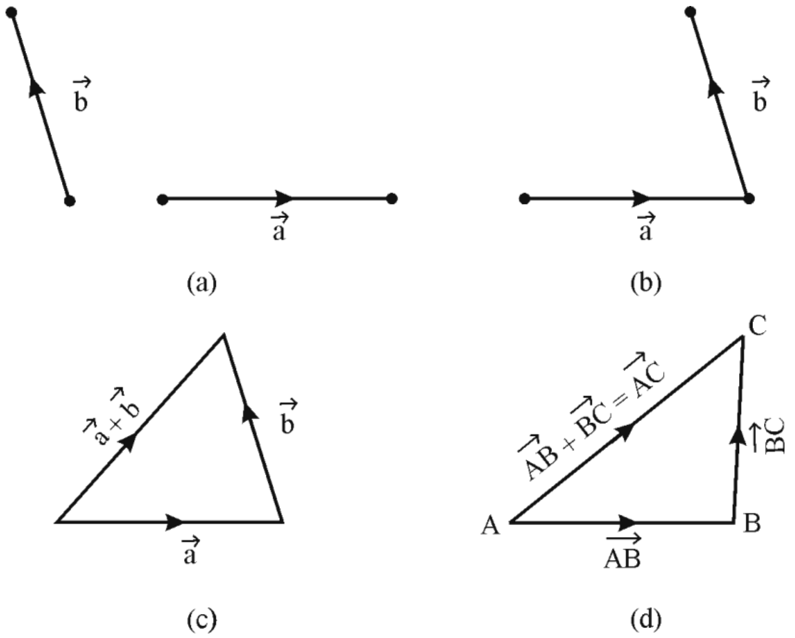

Fig. 6.6: Visualisation of (a) vectors $\vec{a}$ and $\vec{b}$ (b) shifting of vector $\vec{b}$ (c) addition of $\vec{a}$ and $\vec{b}$ (d) triangle law of addition of two vectors

Dot Product of two Vectors: Dot product of two vectors $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as follows.

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \qquad \dots (6.38)$$

where θ is the angle between vectors $\vec{a}$ and $\vec{b}$ refer Fig. 6.7 (a).

Some important observations from the definition of dot product are listed as follows.

- If $\vec{a}\perp\vec{b}$ then $\theta = 90^0$, and so $\cos\theta = \cos 90^0 = 0$.

  Hence, if $\vec{a}\perp\vec{b}$ then $\vec{a}.\vec{b} = 0$. … (6.39)

  In particular $\hat{i}.\hat{j} = 0 = \hat{j}.\hat{i},\ \hat{j}.k = 0 = k.\hat{j},\ \ k.\hat{i} = 0 = \hat{i}.k$. … (6.40)

- If $\vec{a}.\vec{b} = 0$, then either $|\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\vec{a}\perp\vec{b}$. So, if $\vec{a}$ and $\vec{b}$ are non-zero vectors and $\vec{a}.\vec{b} = 0$, then $\vec{a}\perp\vec{b}$. … (6.41)

- If $\vec{a} = \vec{b}$, then $\vec{a}.\vec{a} = |\vec{a}||\vec{a}|\cos 0 = |\vec{a}|^2$ $[\because \cos 0 = 1]$.

  So, $\vec{a}.\vec{a} = |\vec{a}|^2$. … (6.42)

  In particular $\hat{i}.\hat{i} = 1,\ \hat{j}.\hat{j} = 1,\ k.k = 1,$ … (6.43)

- So, if $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1 k,\ \vec{b} = a_2\hat{i} + b_2\hat{j} + c_2 k$, then using (6.40) and (6.43), we get

  $$\vec{a}.\vec{b} = a_1 a_2 + b_1 b_2 + c_1 c_2 \qquad \dots (6.44)$$

- $|\vec{a}.\vec{b}| \le |\vec{a}||\vec{b}|$ or $|\vec{a}.\vec{b}| \le \|\vec{a}\|\|\vec{b}\|$ known as **Cauchy Schwartz Inequality**…(6.45)

  where in higher dimension $|\vec{a}|$ is denoted by $\|\vec{a}\|$ and read as norm of $\vec{a}$

  Its proof is very simple given as follows.

Result holds trivially if either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, because then both sides will become 0 and so equality will hold. So, consider the case that neither $\vec{a}=\vec{0}$ nor $\vec{b}=\vec{0}$. So, when $\vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{0}$, equation (6.38) can be written as

$$\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\cos\theta \Rightarrow \left|\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right|=|\cos\theta| \Rightarrow \frac{|\vec{a}.\vec{b}|}{|\vec{a}||\vec{b}|}\leq 1 \ [\because -1\leq\cos\theta\leq 1\Rightarrow|\cos\theta|\leq 1]$$

$$\Rightarrow |\vec{a}.\vec{b}|\leq|\vec{a}||\vec{b}| \text{ or } |\vec{a}.\vec{b}|\leq\|\vec{a}\|\|\vec{b}\|$$

Some particular cases of Cauchy Schwartz Inequality are given as follows:

In 2-dimension $|a_1b_1+a_2b_2|\leq\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}$, … (6.45a)

where $\vec{a}=a_1\hat{i}+a_2\hat{j}, \vec{b}=b_1\hat{i}+b_2\hat{j}$ or $\vec{a}=(a_1, a_2), \vec{b}=(b_1, b_2)$

In 3-dimension $|a_1b_1+a_2b_2+a_3b_3|\leq\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}$ … (6.45b)

where $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3k, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3k$ or $\vec{a}=(a_1, a_2, a_3), \vec{b}=(b_1, b_2, b_3)$

In n-dimension, we have

$|a_1b_1+a_2b_2+\ldots+a_nb_n|\leq\sqrt{a_1^2+a_2^2+\ldots+a_n^2}\sqrt{b_1^2+b_2^2+\ldots+b_n^2}$ … (6.45c)

or $|a_1b_1+a_2b_2+\ldots+a_nb_n|\leq\|(a_1, a_2, \ldots, a_n)\|\|(b_1, b_2, \ldots, b_n)\|$ … (6.45d)

or $|a_1b_1+a_2b_2+\ldots+a_nb_n|\leq\|\vec{a}\|\|\vec{b}\|$ … (6.45e)

where

$\vec{a}=a_1e_1+a_2e_2+\ldots+a_ne_n, \vec{b}=b_1e_1+b_2e_2+\ldots+b_ne_n$ or $\vec{a}=(a_1, a_2, \ldots, a_n)$,

$\vec{b}=(b_1, b_2, \ldots, b_n)$

- $|\vec{a}+\vec{b}|\leq|\vec{a}|+|\vec{b}|$ known as **Triangle Inequality** … (6.46)

If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then equality will hold. So, consider the case that neither $\vec{a}=\vec{0}$ nor $\vec{b}=\vec{0}$. Using equation (6.42), we have

$$|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=|\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2 \ [\because \vec{a}.\vec{b}=\vec{b}.\vec{a}]$$

$$\leq|\vec{a}|^2+2|\vec{a}.\vec{b}|+|\vec{b}|^2 \ [\because \vec{a}.\vec{b}\leq|\vec{a}.\vec{b}|]$$

$$\leq|\vec{a}|^2+2|\vec{a}||\vec{b}|+|\vec{b}|^2 \ [\because \text{Using Cauchy Schwartz Inequality}]$$

$$=(|\vec{a}|+|\vec{b}|)^2$$

$$\Rightarrow|\vec{a}+\vec{b}|^2\leq(|\vec{a}|+|\vec{b}|)^2$$

$$\Rightarrow|\vec{a}+\vec{b}|\leq|\vec{a}|+|\vec{b}| \ [\because \text{if } a>0, b>0 \text{ then } a^2\leq b^2\Rightarrow a\leq b]$$

Projection of a Vector on another Vector: Let $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$ be the position vectors of points A and B respectively. Draw perpendicular from point B to OA which intersects OA at point C then the length $|\overrightarrow{OC}|$ is known as scalar projection of vector $\vec{b}$ on vector $\vec{a}$ and $\overrightarrow{OC}$ is known as vector projection of vector $\vec{b}$ on vector $\vec{a}$ refer Fig. 6.7 (b).

So, scalar projection is given by

$$OC = OB\cos\theta = \left(\left|\vec{b}\right|\cos\theta\right)\left(\frac{1}{\left|\vec{a}\right|}\right)\left|\vec{a}\right| = \frac{1}{\left|\vec{a}\right|}\left(\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta\right) = \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|} \quad \text{[Using (6.38)]}$$

So, scalar projection of $\vec{b}$ on $\vec{a} = OC = \dfrac{\vec{a}.\vec{b}}{\left|\vec{a}\right|}$ … (6.47)

and vector projection of $\vec{b}$ on $\vec{a} = \overrightarrow{OC} = \dfrac{\vec{a}.\vec{b}}{\left|\vec{a}\right|}a$ … (6.48)

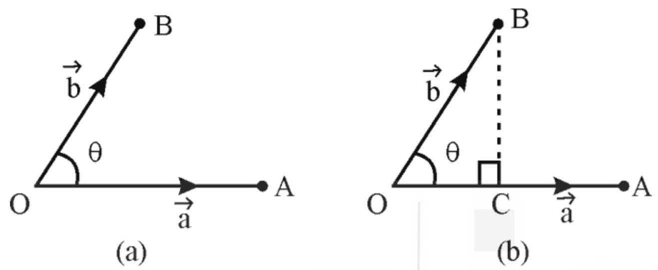

Fig. 6.7: Visualisation of (a) vectors $\vec{a}$ and $\vec{b}$ and angle θ between them (b) projection of $\vec{b}$ on $\vec{a}$

Example 7: Find the scalar projection of the vector $\vec{a} = 2\hat{i} + \hat{j} - 3k$ on the vector $\vec{b} = 2\hat{i} + 3\hat{j} - 6k$.

Solution: Using (6.31) and (6.44), we have

$$\left|\vec{a}\right| = \sqrt{(2)^2 + (1)^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14},\ \left|\vec{b}\right| = \sqrt{(2)^2 + (3)^2 + (-6)^2} = \sqrt{49} = 7$$

$$\vec{a}.\vec{b} = (2)(2) + (1)(3) + (-3)(-6) = 4 + 3 + 18 = 25$$

Using (6.47) scalar projection of $\vec{a}$ on $\vec{b}$ is given by $\dfrac{\vec{a}.\vec{b}}{\left|\vec{b}\right|} = \dfrac{25}{7}$

Now, you can try the following Self-Assessment Question.

SAQ 2

Find the vector projection of the vector $\vec{a} = 2\hat{i} + \hat{j} - 3k$ on the vector $\vec{b} = 2\hat{i} + 3\hat{j} - 6k$.

6.4 EQUATION OF A LINE IN 3-DIMENSION

To understand equation of a line in 3-dimension you should be familiar with two things:

(i) basic concepts of vector algebra discussed in Sec. 6.3, and

(ii) direction cosines and direction ratios of a line.

You have already gone through a review of basic concepts of vector algebra in the previous section. So, let us discuss what are direction cosines and direction ratios of a line.

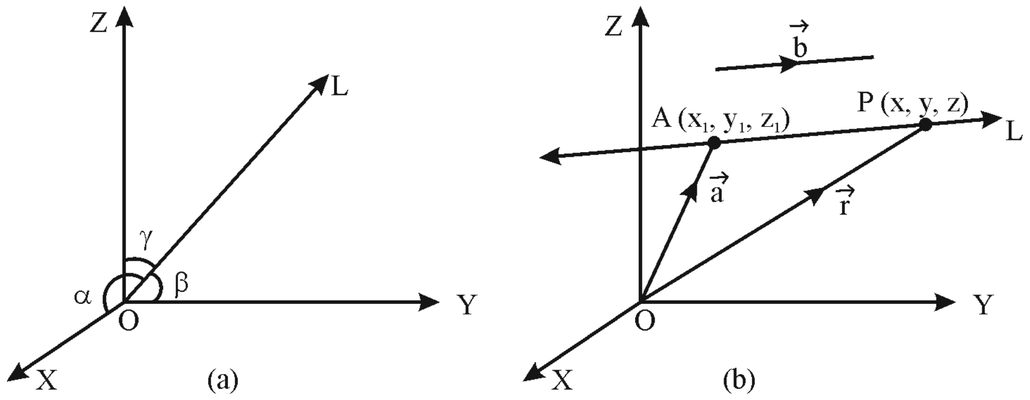

Fig. 6.8: Visualisation of (a) angles of a line with coordinate axes (b) equation of a line passing through a point and parallel to a given vector

Direction Cosines: If a line L makes angles α, β, γ with positive direction of x-axis, y-axis, z-axis respectively refer to Fig. 6.8 (a), then $\cos\alpha, \cos\beta, \cos\gamma$ are known as direction cosines of the line L. For example, x-axis is also a line which makes angles of $0^0, 90^0, 90^0$ with x-axis, y-axis, z-axis respectively, so direction cosines of x-axis are $\cos 0^0, \cos 90^0, \cos 90^0$ or $1, 0, 0.$ Direction cosines of a line are generally denoted by l, m, n and written as $< l, m, n >.$ So, direction cosines of x-axis can be written as $< 1, 0, 0 >.$ Similarly, direction cosines of y-axis and z-axis are $< 0, 1, 0 >$ and $< 0, 0, 1 >$ respectively. If $< l, m, n >$ are the direction cosines of a line then they always satisfy the condition $l^2 + m^2 + n^2 = 1$ … (6.49)

Direction Ratios: Direction ratios are nothing but simply they are proportional to direction cosines. For example, $< \frac{2}{7}, \frac{3}{7}, \frac{6}{7} >$ are direction cosines of a line because $l^2 + m^2 + n^2 = \left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 = \frac{4 + 9 + 36}{49} = \frac{49}{49} = 1.$ If you multiply these direction cosines by 7 then they reduce to $< 2, 3, 6 >$ so these are direction ratios of the same line. Instead of 7 if you multiply by 70 then they reduce to $< 20, 30, 60 >$ these are also the direction ratios of the same line. Thus, direction ratios of a line are not unique. In fact, there are infinite number of ways to write direction ratios of the same line. You have seen that direction cosines of a line are generally denoted by l, m, n and written as $< l, m, n >,$ but direction ratios of a line are generally denoted by a, b, c and written as $< a, b, c >.$

From earlier classes you also know that direction ratios of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are given by $< x_2 - x_1, y_2 - y_1, z_2 - z_1 >.$ … (6.50)

Remark 2: If direction ratios of a line are $< a, b, c >,$ then the vector $a\hat{i} + b\hat{j} + ck$ will be parallel to the same line. … (6.51)

Now, we obtain equation of a line in the following two situations.

- Equation of a line passing through a point and parallel to a given line
- Equation of a line passing through two points

Let us discuss these taken one at a time.

Equation of a Line Passing Through a Point and Parallel to a Given Line

Let line L passes through a point A(x_1, y_1, z_1) and parallel to a vector $\vec{b} = a\hat{i} + b\hat{j} + c k$, refer Fig. 6.8 (b). Let P(x, y, z) be any point on line L then $\overrightarrow{AP}$ lies on the line and so is parallel to the vector $\vec{b}$. Therefore,

$$\overrightarrow{AP} = \lambda\, \vec{b} \qquad \left[\because \text{If } \vec{a} \parallel \vec{b} \text{ then } \vec{a} = \lambda \vec{b} \text{ for some real number } \lambda\right]$$

$$\Rightarrow \vec{r} - \vec{a} = \lambda \vec{b} \qquad \left[\text{Using } (6.36) \text{ in LHS}\right] \qquad \ldots (6.52)$$

$$\text{or} \quad \vec{r} = \vec{a} + \lambda \vec{b} \qquad \ldots (6.53)$$

This is known as **vector equation** of a line passing through a point having position vector $\vec{a}$ and parallel to the vector $\vec{b}$.

Putting values of $\vec{r}$, $\vec{a}$, $\vec{b}$ in component form in (6.52), we get

$$\Rightarrow x\hat{i} + y\hat{j} + z k - \left(x_1\hat{i} + y_1\hat{j} + z_1 k\right) = \lambda\left(a\hat{i} + b\hat{j} + c k\right)$$

$$\Rightarrow (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)k = \lambda a\hat{i} + \lambda b\hat{j} + \lambda c k$$

Equating coefficients of $\hat{i}$, $\hat{j}$, k on both sides, we get

$$(x - x_1) = \lambda a,\ (y - y_1) = \lambda b,\ (z - z_1) = \lambda c \Rightarrow \frac{x - x_1}{a} = \lambda,\ \frac{y - y_1}{b} = \lambda,\ \frac{z - z_1}{c} = \lambda$$

$$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda \qquad \ldots (6.54)$$

$$\text{or} \quad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \ldots (6.55)$$

This is known as **cartesian equation** of a line which passes through a point A(x_1, y_1, z_1) and is parallel to a line having direction ratios $< a, b, c >$.

Any point on line (6.54) is given by

$$\Rightarrow x = x_1 + \lambda a,\ \ y = y_1 + \lambda b,\ \ z = z_1 + \lambda c \qquad \ldots (6.56)$$

where λ is any real number

Equation (6.56) gives us a general point on the line given by (6.54) or (6.55).

Equation of a Line Passing Through Two Points

Let line L passes through two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) refer Fig. 6.9. Let P(x, y, z) be any point on line L then $\overrightarrow{AP}$ lies on the line. Also, the vector $\overrightarrow{AB}$ lies on the same line and hence we can say that $\overrightarrow{AP} \parallel \overrightarrow{AB}$. Therefore,

$$\overrightarrow{AP} = \lambda \overrightarrow{AB} \qquad \left[\because \text{If } \vec{a} \parallel \vec{b} \text{ then } \vec{a} = \lambda \vec{b} \text{ for some real number } \lambda\right]$$

$$\Rightarrow \vec{r} - \vec{a} = \lambda\left(\vec{b} - \vec{a}\right) \qquad \left[\text{Using equation } (6.36) \text{ in LHS and RHS}\right] \qquad \ldots (6.57)$$

$$\Rightarrow \vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right) \qquad \ldots (6.58)$$

This is known as **vector equation** of a line passing through two points having position vectors $\vec{a}$ and $\vec{b}$.

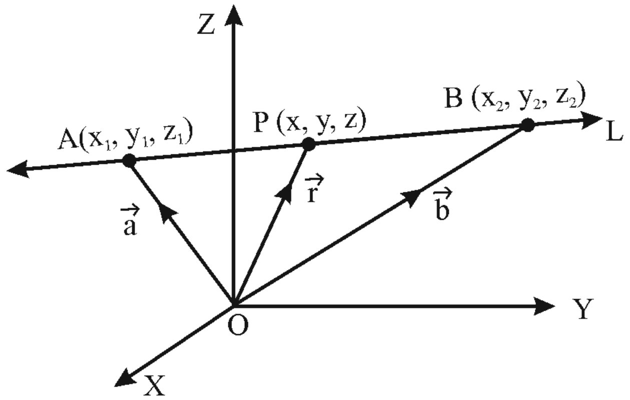

Fig. 6.9: Visualisation of equation of a line passing through two points

Putting values of $\vec{r},\ \vec{a},\ \vec{b}$ in component form in equation (6.57), we get

$$\Rightarrow x\hat{i}+y\hat{j}+zk-\left(x_1\hat{i}+y_1\hat{j}+z_1k\right)=\lambda\left(\left(x_2\hat{i}+y_2\hat{j}+z_2k\right)-\left(x_1\hat{i}+y_1\hat{j}+z_1k\right)\right)$$

$$\Rightarrow (x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)k=\lambda(x_2-x_1)\hat{i}+\lambda(y_2-y_1)\hat{j}+\lambda(z_2-z_1)k$$

Equating coefficients of $\hat{i},\ \hat{j},\ k$ on both sides, we get

$$(x-x_1)=\lambda(x_2-x_1),\quad (y-y_1)=\lambda(y_2-y_1),\quad (z-z_1)=\lambda(z_2-z_1)$$

$$\Rightarrow \frac{x-x_1}{x_2-x_1}=\lambda,\quad \frac{y-y_1}{y_2-y_1}=\lambda,\quad \frac{z-z_1}{z_2-z_1}=\lambda$$

or $$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=\lambda \qquad \ldots (6.59)$$

or $$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \qquad \ldots (6.60)$$

This is known as **cartesian equation** of a line which passes through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Any point on line (6.59) is given by

$$\Rightarrow x=x_1+\lambda(x_2-x_1),\quad y=y_1+\lambda(y_2-y_1),\quad z=z_1+\lambda(z_2-z_1) \qquad \ldots (6.61)$$

where λ is any real number

Like equation (6.14) and in view of equation (6.59) we can say that all the points between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ including points A and B (or on the line segment AB) are given by

$$x=x_1+\lambda(x_2-x_1),\ y=y_1+\lambda(y_2-y_1),\ z=z_1+\lambda(z_2-z_1) \qquad \ldots (6.62)$$

where $0\le\lambda\le 1$

Or $x=(1-\lambda)x_1+\lambda x_2,\ y=(1-\lambda)y_1+\lambda y_2,\ z=(1-\lambda)z_1+\lambda z_2,$ … (6.63)

where $0\le\lambda\le 1$

This line as a set of points can be written as

$$\left\{\left((1-\lambda)x_1+\lambda x_2,\ (1-\lambda)y_1+\lambda y_2,\ (1-\lambda)z_1+\lambda z_2\right):-\infty<\lambda<\infty\right\}$$

$$\left\{(1-\lambda)(x_1, y_1, z_1)+\lambda(x_2, y_2, z_2):-\infty<\lambda<\infty\right\} \qquad \ldots (6.64)$$

Or $\left\{(1-\lambda)x+\lambda y:-\infty<\lambda<\infty\right\}$, where $x=(x_1, y_1, z_1),\ y=(x_2, y_2, z_2)$ … (6.65)

Like (6.16) and (6.17), the line segment AB as a set of points can be written as

$$\{(1-\lambda)(x_1, y_1, z_1)+\lambda(x_2, y_2, z_2): 0\le\lambda\le 1\} \quad \dots (6.66)$$

Or $\{(1-\lambda)x+\lambda y: 0\le\lambda\le 1\}$, where $x=(x_1, y_1, z_1)$, $y=(x_2, y_2, z_2)$ … (6.67)

Example 8: If a line passes through two points $A(2, 5, -4)$ and $B(1, 6, 4)$, then find:

(i) Direction ratios of the line passing through the points A and B, and also direction cosines of this line.

(ii) Find the cartesian equation of the line passing through points A and B.

Solution: Given points are $A(2, 5, -4)$ and $B(1, 6, 4)$.

(i) Direction ratios of the line passing through the points $A(2, 5, -4)$ and $B(1, 6, 4)$ are given by

$$< x_2-x_1, y_2-y_1, z_2-z_1 >=< 1-2, 6-5, 4-(-4) >=< -1, 1, 8 >$$

Direction cosine of the same line are given by

$$< \frac{-1}{\sqrt{(-1)^2+(1)^2+(8)^2}}, \frac{1}{\sqrt{(-1)^2+(1)^2+(8)^2}}, \frac{8}{\sqrt{(-1)^2+(1)^2+(8)^2}} >$$

$$=< \frac{-1}{\sqrt{66}}, \frac{1}{\sqrt{66}}, \frac{8}{\sqrt{66}} >$$

(ii) Cartesian equation of the line passing through the points $A(2, 5, -4)$ and $B(1, 6, 4)$ using (6.60) is given by

$$\frac{x-2}{1-2}=\frac{y-5}{6-5}=\frac{z-(-4)}{4-(-4)}$$

or $\dfrac{x-2}{-1}=\dfrac{y-5}{1}=\dfrac{z+4}{8}$

Now, you can try the following Self-Assessment Question.

SAQ 3

If a line passes through two points $A(2, 5, -4)$ and $B(1, 6, 4)$, then find the vector equation of the line.

6.5 EQUATION OF A PLANE

If you are working in two dimension and want to divide the cartesian plane in two parts then it can be done by drawing a line. If you are working in three dimension and want to divide the 3-dimensional space in two parts then it can be done by drawing a plane. If you are working in more than three-dimension n (say) and want to divide the n dimensional space in to two parts then it can be done by drawing a hyperplane. In this section we will discuss equation of a plane and in the next section we will discuss equation of a hyperplane. Now, a natural question that may arise in your mind is why we want to divide plane or space in two parts? You will understand its importance when you will study support vector machine (SVM) algorithm and binary classification problem in the course MST-026: Introduction to Machine Learning.

Here, we will obtain equation of a plane in the following two situations.

- When normal to the plane and perpendicular distance of the plane from the origin are given (known as normal form)
- When normal to the plane and a point on the plane are given

Let us discuss these taken one at a time.

- **When Normal to the Plane and Perpendicular Distance of the Plane from the Origin are Given (Normal Form)**

Let us first define what is normal to the plane. A vector which is perpendicular to the plane is known as normal to the plane. A unit vector in the direction of the normal is known as unit normal vector.

Let π be the given plane and d be the perpendicular distance of the plane from origin. Refer Fig. 6.10 (a). Draw perpendicular from origin to the plane and let it intersects the plane π at the point N then $\overrightarrow{ON}$ will be normal to the plane. Let n be the unit normal vector along the normal $\overrightarrow{ON}$ so $\overrightarrow{ON} = dn$. Let P be any point on the plane and $\vec{r}$ be the position vector of P then $\overrightarrow{NP}$ will lie on the plane therefore

$\overrightarrow{NP} \perp \overrightarrow{ON}$ $\quad\left[\because\ \overrightarrow{ON}\ \text{is normal to the plane}\right]$

$\overrightarrow{NP}.\overrightarrow{ON} = 0$ $\quad\left[\because\ \text{Dot product of two perpendicular vectors is zero}\right]$

$\Rightarrow \left(\vec{r} - dn\right).\left(dn\right) = 0$ $\quad\left[\text{Using equation (6.36) for } \overrightarrow{NP}\right]$

$\Rightarrow \left(\vec{r} - dn\right).n = 0$ $\quad\left[\because\ d \neq 0 \text{ so dividing by d on both sides}\right]$

$\Rightarrow \vec{r}.n - d = 0$ $\quad\left[\because\ n.n = 1\right]$

or $\vec{r}.n = d$ … (6.68)

This is known as **vector equation** of the plane which is at a perpendicular distance of d unit from the origin and have n as unit normal vector to the plane.

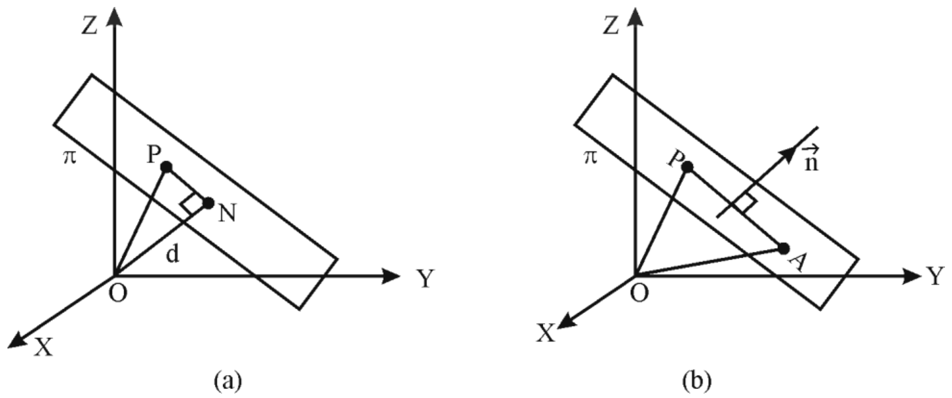

Fig. 6.10: Visualisation of equation of a plane in (a) normal form (b) passing through a point and normal to the plane is given

If $n = l\hat{i} + m\hat{j} + nk$ then putting values of $\vec{r}$ and n in equation (6.68), we get

$$\Rightarrow \left(x\hat{i} + y\hat{j} + zk\right).\left(l\hat{i} + m\hat{j} + nk\right) = d$$

$$\Rightarrow lx + my + nz = d \qquad \dots (6.69)$$

This is known as **cartesian equation** of the plane which is at a perpendicular distance of d unit from the origin and direction cosines of the line normal to the plane are $< l, m, n >$.

Remark 3: It is important to note that direction cosines of the normal to the plane are coefficients of x, y, z in equation (6.69). It did not happen by chance. Remember this important fact it is true for every equation of a plane.

Normal to the Plane and a Point on the Plane are Given

Let π be the given plane and $A(x_1, y_1, z_1)$ be the point on the plane. Let $\vec{n}$ be normal to the plane. Let $\vec{r}$ be the position vector of any point P on the plane and $\vec{a}$ be the position vector of the point A. Both the points A and P lie on the plane so normal vector $\vec{n}$ will be perpendicular to the vector $\overrightarrow{AP}$.

$\therefore \overrightarrow{AP} \perp \vec{n}$ $\quad \left[\because \vec{n} \text{ is normal to the plane}\right]$

$\overrightarrow{AP}.\vec{n} = 0$ $\quad \left[\because \text{ if } \vec{a} \perp \vec{b} \text{ then } \vec{a}.\vec{b} = 0\right]$

$$\Rightarrow \left(\vec{r} - \vec{a}\right).\vec{n} = 0 \quad \left[\because \overrightarrow{AP} = \vec{r} - \vec{a}\right] \quad .. (6.70)$$

This is known as **vector equation** of the plane passing through a point having position vector $\vec{a}$ and $\vec{n}$ is the normal to the plane.

If $\vec{n} = a\hat{i} + b\hat{j} + ck$ then putting values of $\vec{r}$ and $\vec{a}$ in equation (6.70), we get

$$\Rightarrow \left(\left(x\hat{i} + y\hat{j} + zk\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1k\right)\right).\left(a\hat{i} + b\hat{j} + ck\right) = 0$$

$$\Rightarrow \left((x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)k\right).\left(a\hat{i} + b\hat{j} + ck\right) = 0$$

$$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \ldots (6.71)$$

This is known as **cartesian equation** of the plane passing through a point $A(x_1, y_1, z_1)$ and having direction ratios of the normal to the plane as $< a, b, c >$.

Note that after simplification equation (6.71) can be written as

$$ax + by + cx + d = 0, \text{ where } d = -(ax_1 + by_1 + cz_1) \quad \ldots (6.72)$$

Example 9: Find the vector equation of a plane which passes through the point A(2, 5, 4) and perpendicular to the line BC where B(1, 4, 6) and C(7, 2, 10).

Solution: Direction ratios of line BC are
$< x_2 - x_1, y_2 - y_1, z_2 - z_1 > = < 7 - 1, 2 - 4, 10 - 6 > = < 6, -2, 4 > = < 3, -1, 2 >$

Since plane is perpendicular to the line BC so normal to the plane will be parallel to the line BC and so normal vector to the plane is

$$\vec{n} = 3\hat{i} - \hat{j} + 2k$$

Also, position vector of point A is $\vec{a} = 2\hat{i} + 5\hat{j} + 4k$

Now, vector equation of the plane passing through a point having position vector $\vec{a}$ and $\vec{n}$ as the normal to the plane is given by

$$\left(\vec{r} - \vec{a}\right).\vec{n} = 0 \qquad \left[\text{Using } (6.70)\right]$$

$$\Rightarrow \left(\left(x\hat{i}+y\hat{j}+zk\right)-\left(2\hat{i}+5\hat{j}+4k\right)\right).\left(3\hat{i}-\hat{j}+2k\right)=0 \qquad \ldots (6.73)$$

Which is required vector equation of the plane.

Now, you can try the following Self-Assessment Question.

SAQ 4

Find the cartesian form of the plane given in Example 9.

6.6 EQUATION OF HYPERPLANE

You can geometrically visualise:

- a line, For example, real line.
- two perpendicular lines. For example, x-axis and y-axis, and
- three mutually perpendicular lines. For example, x-axis, y-axis and z-axis.

But you cannot geometrically visualise four or more mutually perpendicular lines. But theory of three mutually perpendicular axes can be extended to any dimension n (say). For example, direction cosines of x-axis, y-axis and z-axis in 3-dimension are $<1, 0, 0>, <0, 1, 0>, <0, 0, 1>$ respectively. This idea of direction cosines says that we can write direction cosines of 4, 5, 6, 7, … mutually perpendicular lines. For example, direction cosines of four lines $<1, 0, 0, 0>, <0, 1, 0, 0>, <0, 0, 1, 0>, <0, 0, 0, 1>$ are such that they are mutually perpendicular which can be seen by taking their dot product as follows:

$$(1, 0, 0, 0).(0, 1, 0, 0)=(1)(0)+(0)(1)+(0)(0)+(0)(0)=0$$
$$(1, 0, 0, 0).(0, 0, 1, 0)=(1)(0)+(0)(0)+(0)(1)+(0)(0)=0$$
$$(1, 0, 0, 0).(0, 0, 0, 1)=(1)(0)+(0)(0)+(0)(0)+(0)(1)=0$$
$$(0, 1, 0, 0).(0, 0, 1, 0)=(0)(0)+(1)(0)+(0)(1)+(0)(0)=0$$
$$(0, 1, 0, 0).(0, 0, 0, 1)=(0)(0)+(1)(0)+(0)(0)+(0)(1)=0$$
$$(0, 0, 1, 0).(0, 0, 0, 1)=(0)(0)+(0)(0)+(1)(0)+(0)(1)=0$$

So, if we denote unit vectors along these four directions by e_1, e_2, e_3, e_4 respectively then because their dot products are zero so using equation (6.41) we can say that there exists four directions which are mutually perpendicular. So, in general, if we denote the unit vectors $e_1, e_2, e_3, e_4, \ldots, e_n$ along the lines having direction cosines given as follows:

$$<\underbrace{1, 0, 0, 0, \ldots, 0}_{\text{n positions}}>, <\underbrace{0, 1, 0, 0, \ldots, 0}_{\text{n positions}}>, <\underbrace{0, 0, 1, 0, \ldots, 0}_{\text{n positions}}>, \ldots, <\underbrace{0, 0, 0, 0, \ldots, 1}_{\text{n positions}}>,$$

then we can say that there exist n directions which are mutually perpendicular.

Now, you have gotten the idea why theoretically there exists four or more mutually perpendicular directions. In 2-dimensional geometry the geometric entity of one less dimension, i.e., 2 – 1 = 1 dimension is known as straight line which we have discussed in Sec. 6.2. In 3-dimensional geometry the geometric entity of one less dimension, i.e., 3 – 1 = 2 dimension is known as plane which we have discussed in Sec. 6.5. In n-dimensional geometry the geometric entity of one less dimension, i.e., n – 1 dimension is known as **hyperplane** which we will discuss in this section.

Let us recall general equation of a line given by (6.21) which is

$$ax + by + c = 0$$

General equation of the plane given by (6.72) is

$$ax + by + cx + d = 0$$

With the notations a, b, c, d and x, y, z it will be difficult to write equation of a plane in n dimension. So, to write equation of a plane in n dimension which is called hyperplane we have to change these notations as follows.

Equation (6.21) of a line in 2-dimension can be written as

$$\omega_1 x_1 + \omega_2 x_2 + \omega_0 = 0,\ \omega_i \in \mathbb{R},\ i = 0, 1, 2 \qquad \dots (6.74)$$

where not all $\omega_i = 0,\ i = 1, 2$

Equation (6.72) of a plane in 3-dimension can be written as

$$\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_0 = 0,\ \omega_i \in \mathbb{R},\ i = 0, 1, 2, 3 \qquad \dots (6.75)$$

where not all $\omega_i = 0,\ i = 1, 2, 3$

Similarly, equation of a plane in n-dimension called hyperplane can be written as

$$\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \dots + \omega_n x_n + \omega_0 = 0,\ \omega_i \in \mathbb{R},\ i = 0, 1, 2, 3, \dots, n \qquad \dots (6.76)$$

where not all $\omega_i = 0,\ i = 1, 2, 3, \dots, n$

After extending theory of 3-dimension and in view of Remark 3 we can say that direction ratios of the normal to the hyperplane given by (6.76) are $< \omega_1, \omega_2, \omega_3, \dots, \omega_n >$.

In view of the notations mentioned in equations (6.34) and (6.35) under Remark 1 and using rule of matrix multiplication, equations (6.74), (6.75) and (6.76) all can be written as

$$\omega^T x + b = 0, \qquad \dots (6.77)$$

where

for equation (6.74), we have

$$\omega^T = [\omega_1 \quad \omega_2],\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \quad x_2]^T,\ b \in \mathbb{R} \qquad \dots (6.78)$$

for equation (6.75), we have

$$\omega^T = [\omega_1 \quad \omega_2 \quad \omega_3],\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1 \quad x_2 \quad x_3]^T,\ b \in \mathbb{R} \qquad \dots (6.79)$$

for equation (6.76), we have

$$\omega^T = [\omega_1 \quad \omega_2 \quad \omega_3 \quad \dots \quad \omega_n],\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n]^T,\ b \in \mathbb{R} \quad \dots (6.80)$$

Remark 4: All the results of the plane discussed in Sec. 6.5 can be extended to n-dimensional hyperplane. For example, if $\omega_0 = 0$ then the point (0, 0, 0, …, 0) will satisfy hyperplane given by (6.76) and so it will pass through the origin. Also, equation of the hyperplane in vector form which passes through a point having position vector $\vec{a}$ and $\vec{n}$ is the normal to the hyperplane is given by

$$(\vec{r} - \vec{a}).\vec{n} = 0 \quad \dots (6.81)$$

where

$$\vec{r} = x_1 e_1 + x_2 e_2 + x_3 e_3 + \dots + x_n e_n,\ \vec{a} = a_1 e_1 + a_2 e_2 + a_3 e_3 + \dots + a_n e_n, \quad \dots (6.82)$$

$$\vec{n} = n_1 e_1 + n_2 e_2 + n_3 e_3 + \dots + n_n e_n \quad \dots (6.83)$$

$$\text{where } \vec{e}_i = (0, 0, \dots, 0, \underset{i^{th}\text{ position}}{1}, 0, 0, \dots, 0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ position} \quad \dots (6.84)$$

Example 10: Find the equation of the hyperplane in $\mathbb{R}^5$ in cartesian form (or scalar form) which passes through the point $A(2, 1, -1, 3, -2)$ and perpendicular to the direction having direction ratios $< 3, 1, 2, 5, 6 >$.

Solution: We know that equation of a hyperplane having normal vector ω is given by

$$\omega^T x + b = 0$$

Here $\omega^T = [3 \quad 1 \quad 2 \quad 5 \quad 6]$, $x = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T$, so we have

$$[3 \quad 1 \quad 2 \quad 5 \quad 6] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + b = 0 \Rightarrow 3x_1 + x_2 + 2x_3 + 5x_4 + 6x_5 + b = 0$$

But it passes through the point $A(2, 1, -1, 3, -2)$, we have

$$3(2) + 1 + 2(-1) + 5(3) + 6(-2) + b = 0 \quad \Rightarrow 6 + 1 - 2 + 15 - 12 + b = 0 \quad \Rightarrow b = -8$$

Putting value of b, we get

$$3x_1 + x_2 + 2x_3 + 5x_4 + 6x_5 - 8 = 0$$

6.7 INEQUATION IN ONE AND TWO VARIABLES

You know that when two expressions are equated to each other than resulting relation is called equation in mathematics. For example, $4x + 5 = 7 - x$ is an equation in one variable x and $3x + y = 4 + 3y - 2x$ is also an equation but it is in two variables x and y. Now, in place of equality sign $(=)$ if we have any of the four $\leq, <, \geq,$ or $>$ inequality signs then resulting relation is called an **inequation**.

For example, (i) $2x < 5$ (ii) $2x - (3 - x) \leq 8$ (iii) $3x + 2 > 8 - x$ (iv) $3x \geq 2 + x$ all are examples of inequations.

Like equations you can also solve inequations for x. But there are some rules which you should keep in mind while solving inequations. If a, b, c, d are real numbers then following are some rules which you should be familiar before start working with inequalities.

- If $a < b$, then $a + c < b + c$. For example, $2 < 5 \Rightarrow 2 + 3 < 5 + 3$.
- If $a < b$, then $a - c < b - c$. For example, $2 < 5 \Rightarrow 2 - 3 < 5 - 3$.
- If $c > 0$ and $a < b$, then $ac < bc$. … (6.85)
  For example, $2 < 5 \Rightarrow 2(3) < 5(3)$, i.e., $6 < 15$
- If $c < 0$ and $a < b$, then $ac > bc$ … (6.86)
  For example, $2 < 5 \Rightarrow 2(-3) > 5(-3)$, i.e., $-6 > -15$.
- If $a > 0$, then
  $ax < b \Rightarrow x < \frac{b}{a}$, $ax \leq b \Rightarrow x \leq \frac{b}{a}$, $ax > b \Rightarrow x > \frac{b}{a}$, $ax \geq b \Rightarrow x \geq \frac{b}{a}$ … (6.87)
  For example, $6 \times 4 < 30 \Rightarrow 4 < \frac{30}{6}$, i.e., $4 < 5$.
- If $a < 0$, then
  $ax < b \Rightarrow x > \frac{b}{a}$, $ax \leq b \Rightarrow x \geq \frac{b}{a}$, $ax > b \Rightarrow x < \frac{b}{a}$, $ax \geq b \Rightarrow x \leq \frac{b}{a}$ … (6.88)
  For example, $(-6)4 < 30 \Rightarrow 4 > \frac{30}{-6}$, i.e., $4 > -5$.
- If $a > 0 \Rightarrow \frac{1}{a} > 0$. For example, $5 > 0 \Rightarrow \frac{1}{5} = 0.2 > 0$.
- If $a < 0 \Rightarrow \frac{1}{a} < 0$. For example, $-5 < 0 \Rightarrow \frac{1}{-5} = -0.2 < 0$.
- If $a > 0, b > 0$ and $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$. … (6.89)

  For example, $5 > 0, 2 > 0$. So, $5 > 2 \Rightarrow \frac{1}{5} < \frac{1}{2}$, i.e., $0.2 < 0.5$.
- If $a < 0, b < 0$ and $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$. … (6.90)

  For example, $-5 < 0, -2 < 0$. So, $-2 > -5 \Rightarrow \frac{1}{-2} < \frac{1}{-5}$, i.e., $-0.5 < -0.2$.
- If $a < b, c < d$ then $a + c < b + d$. … (6.91)
  For example, $2 < 5$, and $6 < 7 \Rightarrow 2 + 6 < 5 + 7$, i.e., $8 < 12$.

Further, you have already studied modulus function in Unit 1 of this course. In some courses you will also need to deal with both modulus function and its appearance in inequalities. If a is a real number, then following are some rules which you should keep in mind while working with inequalities and modulus function simultaneously.

- $|x| = a \Leftrightarrow x = \pm a.$ For example, $|x| = 2 \Leftrightarrow x = \pm 2.$ … (6.92)
- $|x| \le a \Leftrightarrow -a \le x \le a.$ For example, $|x| \le 2 \Leftrightarrow -2 \le x \le 2.$ … (6.93)
- $|x| \ge a \Leftrightarrow x \ge a$ or $x \le -a$ For example, $|x| \ge 2 \Leftrightarrow x \ge 2$ or $x \le -2$ … (6.94)
- $|x| < a \Leftrightarrow -a < x < a.$ For example, $|x| < 2 \Leftrightarrow -2 < x < 2.$ … (6.95)
- $|x| > a \Leftrightarrow x > a$ or $x < -a.$ For example, $|x| > 2 \Leftrightarrow x > 2$ or $x < -2.$ … (6.96)

Some inequalities when square of variable is involved are given as follows:

- If $a > 0$ then $x^2 \le a \Leftrightarrow -\sqrt{a} \le x \le \sqrt{a}.$ … (6.97)
  For example, $|x| \le 2 \Leftrightarrow -\sqrt{2} \le x \le \sqrt{2}.$
- If $a > 0$ then $x^2 \ge a \Leftrightarrow x \ge \sqrt{a}$ or $x \le -\sqrt{a}$ … (6.98)
  For example, $x^2 \ge 2 \Leftrightarrow x \ge \sqrt{2}$ or $x \le -\sqrt{2}$

Let us now do some examples to apply these inequalities.

Example 11: Solve for x: $3x \le 15$

Solution: $3x \le 15 \Rightarrow x \le \dfrac{15}{3} \Rightarrow x \le 5.$ $\quad$ $\left[\text{Using } (6.87)\right]$

So, required solution set is $(-\infty, 5].$

Example 12: Solve for x: $-3x \le 15$

Solution: $-3x \le 15 \Rightarrow x \ge \dfrac{15}{-3}$ $\begin{bmatrix}\text{Using } (6.88) \\ \text{as } -3 < 0, \text{ So, inequality sign has reversed}\end{bmatrix}$

$\Rightarrow x \ge -5.$ So, required solution set is $[-5, \infty).$

Example 13: Solve for x: $8 < -2(3x - 2) \le 20$

Solution: $8 < -2(3x - 2) \le 20$

$$\Rightarrow \frac{8}{-2} > 3x - 2 \ge \frac{20}{-2} \quad \left[\because -2 < 0, \text{ So, both inequalities signs have reversed}\right]$$

$$\Rightarrow -4 > 3x - 2 \ge -10 \Rightarrow -4 + 2 > 3x \ge -10 + 2 \Rightarrow -2 > 3x \ge -8$$

$$\Rightarrow \frac{-2}{3} > x \ge \frac{-8}{3} \qquad \left[\text{Using } (6.87)\right]$$

So, required solution set is $\left[-\dfrac{8}{3}, -\dfrac{2}{3}\right).$

Example 14: Write solution set of: $3x + 2y \le 6.$ Also, show its solution graphically with shaded region.

Solution: $3x + 2y \le 6$ … (6.99)

Solution set of the inequation (6.99) is given by $\{(x, y) \in \mathbb{R}^2 : 3x + 2y \le 6\}$.

Using (6.87), inequation (6.99) can be written as

$$\frac{x}{2} + \frac{y}{3} \le 1 \qquad \ldots (6.100)$$

Equation corresponding to inequation (6.100) is

$$\frac{x}{2} + \frac{y}{3} = 1 \qquad \ldots (6.101)$$

which is equation of a line in intercept form having 2 and 3 as its x and y intercepts respectively, refer equation (6.18). So, graph of the equation (6.101) not inequation (6.100) is given in Fig. 6.11 (a).

Now, to show solution of inequation (6.100) graphically with shaded region, we have to check on which side (above or below) of line given by (6.100) its solution lies. This can be decided by putting coordinates of any point in (6.100) which is not on the line given by (6.101). If a line does not pass through the origin in that case to save calculation work, we put coordinates of origin in the inequation. So, putting x = 0, y = 0 in inequation (6.99), we get $0 + 0 \le 6$ which is true. Hence, solution set of inequation (6.99) or (6.100) contains origin. So, graphical solution of inequation (6.99) is all the region below the line given by (6.101) including the line itself and have been shaded refer to Fig. 6.11 (c).

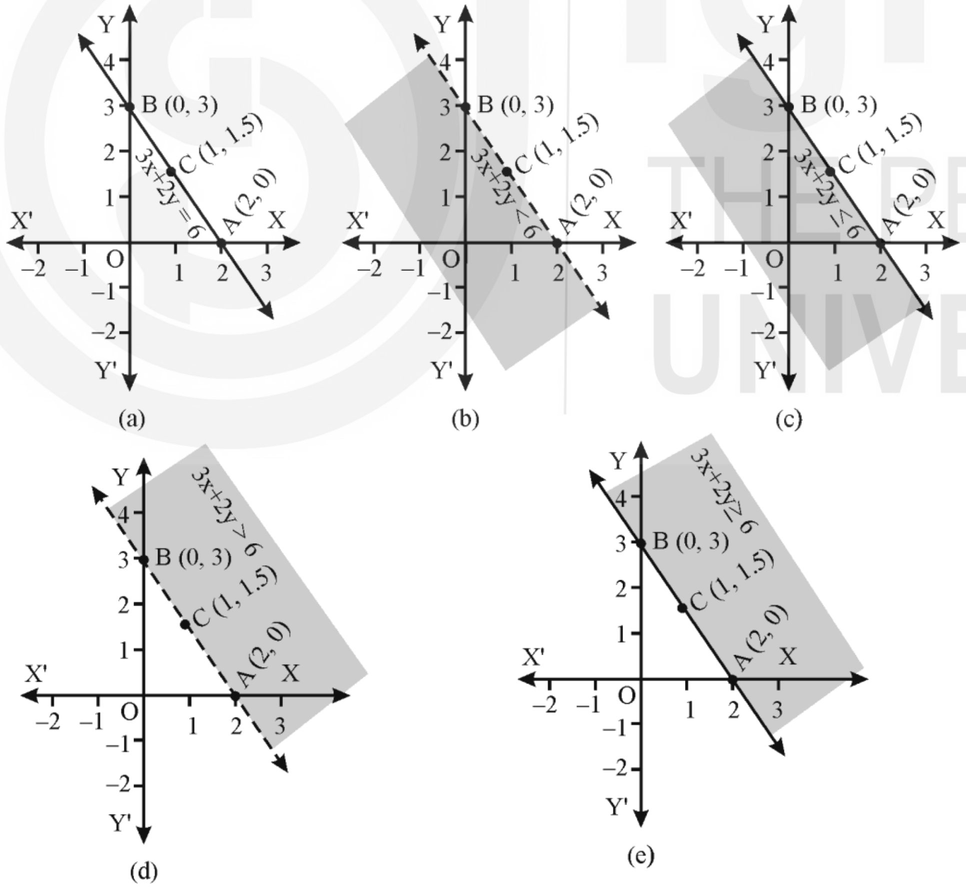

Fig. 6.11: Visualisation of solution of (a) line given by (6.101) (b) the inequation 3x + 2y < 6 by shaded region (c) (6.99) or (6.100) by shaded region (d) 3x + 2y > 6 by shaded region (e) 3x + 2y ≥ 6 by shaded region

Remark 5: Line given by (6.101) divides the plane in two parts known as **half planes**. One half plane is the shaded region and the other half plane is unshaded region. So, all inequations of the types $ax+b\geq c$, $ax+b>c$, $ax+b\leq c$, $ax+b<c$, where $a, b, c\in\mathbb{R}$ represents half planes. If we have $\leq$ or $\geq$ signs then straight line which divides xy-plane into two halves is included in the shaded region refer Fig. 6.11 (c) and (e). While in the case we have $<$ or $>$ signs then straight line which divides xy-plane into two halves is included in the unshaded region not in shaded region refer Fig. 6.11 (b) and (d). Keep this in mind regarding notion of half planes.

Now, you can try the following Self-Assessment Question.

SAQ 5

(a) Solve for x: $5-3x\geq 17$

(b) Solve the given inequations graphically and represent the solution set by shaded region: $3x+2y\geq 6$, $x+3y\leq 9$, $x-y\leq 1$.

6.8 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- **Increment**: Let an object changes it position from the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$ then the net changes in the coordinates of point A are known as increments. Increment in the direction of x-axis is known as **run** and increment in the direction of y-axis is known as **rise**.
- **Slope**: If we take any two points on a line then the ratio of Rise and Run remains constant. This constant ratio is known as **slope** of the line.
- Equation of the line in **point slope form** is: $y-y_1=m(x-x_1)$
- Equation of the line in **two point form** is: $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$
- Line as a set of points $\{(1-\lambda)(x_1, y_1)+\lambda(x_2, y_2): -\infty<\lambda<\infty\}$
- Line segment as a set of points $\{(1-\lambda)(x_1, y_1)+\lambda(x_2, y_2): 0\leq\lambda\leq 1\}$
- Equation of the line in **intercept form** is: $\frac{x}{a}+\frac{y}{b}=1$
- Equation of the line in **slope intercept form** is: $y=mx+c$
- Equation of the line **through origin** is: $y=mx$
- Equation of the line in **general form** is: $ax+by+c=0$ or $\omega_1x_1+\omega_2x_2+\omega_0=0$
- **Perpendicular Distance of a Point from a Line**: $d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$
- Slopes of two **parallel lines are equal**.
- **Product** of slopes of two **perpendicular lines is** -1.
- Equation of any particular line **parallel to x-axis** is given by y = k.
- Equation of any particular line **parallel to y-axis** is given by x = h.

- **Position vector of a point** $A(a_1, b_1, c_1)$ is written as $\overrightarrow{OA} = a_1\hat{i} + b_1\hat{j} + c_1 k$
- **Magnitude** of vector $\overrightarrow{OA}$ is $\left|\overrightarrow{OA}\right| = \sqrt{a_1^2 + b_1^2 + c_1^2}$
- **Vector Joining Two Points**: If A and B are two points and O is the origin then

$\overrightarrow{AB} =$ Position vector of point B $-$ Position vector of point A $= \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

- **Triangle law of addition** is $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- **Dot Product of two Vectors** $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as: $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- If $\vec{a} \perp \vec{b}$ then $\vec{a}.\vec{b} = 0$.
- Scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$
- Vector projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|} a$
- If $< l, m, n >$ are the **direction cosines** of a line then $l^2 + m^2 + n^2 = 1$.
- **Direction Ratios** are proportional to direction cosines.
- Equation of a line **passing through a point** having position vector $\vec{a}$ and **parallel to a vector** $\vec{b}$ is

  $\vec{r} - \vec{a} = \lambda\vec{b}$ or $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$
- **Equation of a line passing through two points** having position vectors $\vec{a}$ and $\vec{b}$ **is** $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$ or $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$
- **Equation of a plane in normal form is** $\vec{r}.n = d$ or $lx + my + nz = d$
- **Equation of a plane when normal to the plane and a point on the plane are given is** $\left(\vec{r} - \vec{a}\right).\vec{n} = 0$ or $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
- Equation of **hyperplane** is $\omega^T x + b = 0,$ where

$$\omega^T = [\omega_1 \quad \omega_2 \quad \omega_3 \quad \ldots \quad \omega_n],\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_n]^T,\ b \in \mathbb{R}$$

6.9 TERMINAL QUESTIONS

1. Find the slope of a line which is perpendicular to the line $2x - 3y + 7 = 0$.

2. Write the direction ratios of the normal to the hyperplane:
   $2x_1 - 3x_2 + 7x_3 + 5x_4 + 4x_5 - 9x_6 + 10 = 0$.
   Also, write direction cosines of the normal to this hyperplane.

6.10 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. (a) We know that equation of a line passing through a point A(x_1, y_1) and having slope m is given by

$$y - y_1 = m(x - x_1)$$

$\therefore$ required equation is given by

$$y - 3 = \frac{7}{8}(x-2) \Rightarrow 8y - 24 = 7x - 14 \Rightarrow 7x - 8y + 10 = 0$$

(b) If d is the perpendicular distance of the point $A(2, -7)$ from the line $3x + 4y + 9 = 0$, then

$$d = \frac{|3(2)+4(-7)+9|}{\sqrt{(3)^2+(4)^2}} = \frac{|6-28+9|}{\sqrt{25}} = \frac{|-13|}{5} = \frac{13}{5}$$

(c) Slope of the line $5x + 6y + 8 = 0$ is $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\dfrac{5}{6}$. We know that slopes of two parallel lines are equal. So, slope of any line parallel to the given line is $-5/6$.

2. In Example 7 we have already obtained $|\vec{b}| = 7$, $\vec{a}.\vec{b} = 25$.

Also, unit vector along $\vec{b}$ is given by $b = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{2\hat{i}+3\hat{j}-6k}{7}$.

So, using (6.48) vector projection of $\vec{a}$ on $\vec{b}$ is given by

$$\frac{\vec{a}.\vec{b}}{|\vec{b}|}b = \frac{25}{7}\left(\frac{2\hat{i}+3\hat{j}-6k}{7}\right) = \frac{25}{49}\left(2\hat{i}+3\hat{j}-6k\right)$$

3. Using equation (6.58) required vector equation of the line is given by

$$\vec{r} = \vec{a} + \lambda\left(\vec{b}-\vec{a}\right) = 2\hat{i}+5\hat{j}-4k+\lambda\left(\left(\hat{i}+6\hat{j}+4k\right)-\left(2\hat{i}+5\hat{j}-4k\right)\right)$$

$$= 2\hat{i}+5\hat{j}-4k+\lambda\left(-\hat{i}+\hat{j}+8k\right), \text{ where } \lambda \in \mathbb{R}$$

4. Using (6.71) required equation of the plane is given by

$$3(x-2)-(y-5)+2(z-4)=0$$
$$3x - y + 2z - 9 = 0$$

5. (a) $5 - 3x \ge 17 \Rightarrow -3x \ge 17 - 5 \Rightarrow -3x \ge 12$

$$\Rightarrow x \le \frac{12}{-3} \quad [\because -3 < 0, \text{ So, inequality sign has changed}]$$

$$\Rightarrow x \le -4$$

So, required solution set is $(-\infty, -4]$.

(b) Given inequations are

$3x + 2y \geq 6$ ... (6.102) $x + 3y \leq 9$ ... (6.103) $x - y \leq 1$ ...(6.104)

Equations corresponding to inequations (6.102) to (6.104) are

$3x + 2y = 6$ ... (6.105) $x + 3y = 9$ ... (6.106) $x - y = 1$ ...(6.107)

Intercept forms of equations (6.105) to (6.107) are given by

$\frac{x}{2} + \frac{y}{3} = 1$ ... (6.108) $\frac{x}{9} + \frac{y}{3} = 1$ ... (6.109) $\frac{x}{1} + \frac{y}{-1} = 1$ ...(6.110)

So, graphs of the equations (6.108) to (6.110) not inequations (6.102) to (6.104) are given in Fig. 6.12 (a).

Like Example 14 to show solution of inequations (6.102) to (6.104) graphically with shaded region, we have to check for each inequation on which side (above or below) of line its solution lies. None of the three lines passes through origin so like Example 14 we put coordinates of origin in each inequation given by (6.102) to (6.104).

From inequation (6.102), we get $0 + 0 \geq 6$ which is false, so solution set of inequation (6.102) does not contain origin. Similarly, from inequations (6.103) and (6.104), we get $0 + 0 \leq 9$ and $0 - 0 \leq 1$ both are true, so solution sets of inequations (6.103) and (6.104) contain origin. Hence, the solution set which is common to all the three inequations (6.102) to (6.104) is shown by shaded region in Fig. 6.12 (b).

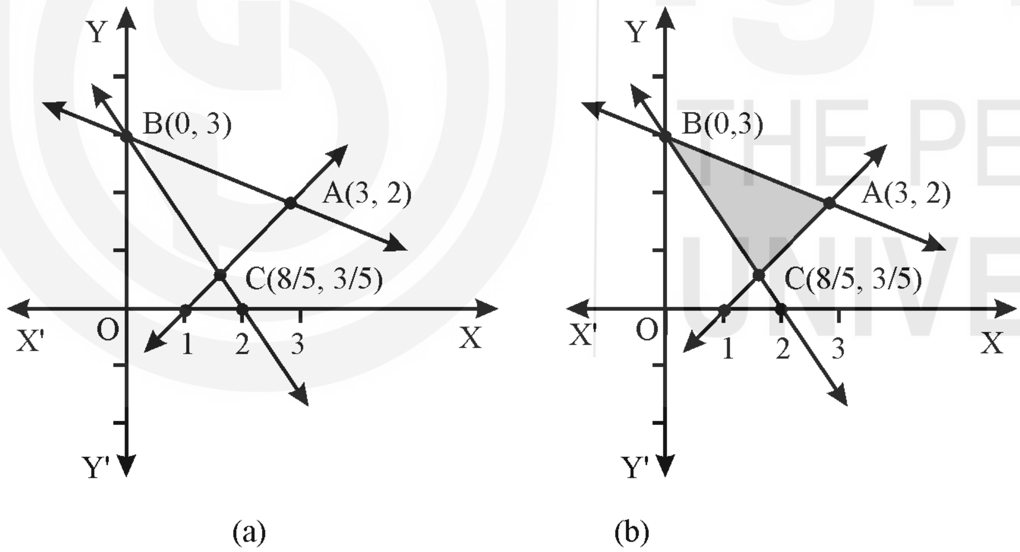

Fig. 6.12: Visualisation of (a) three lines given by (6.108) to (6.110) (b) solution of the inequations (6.102) to (6.104) by shaded region

Terminal Questions

1. Slope of the line $2x - 3y + 7 = 0.$ is $m_1 = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{2}{-3} = \frac{2}{3}$. If m_2 be the slope of a line perpendicular to the given line then we know that $m_1 m_2 = -1 \Rightarrow (2/3) m_2 = -1 \Rightarrow m_2 = -3/2.$

2. We know that coefficients of $x_1, x_2, x_3, x_4, x_5, x_6$ in the equation of the hyperplane are the direction ratios of the normal to the hyperplane.

Hence, direction ratios of the normal to the given hyperplane are: $<2, -3, 7, 5, 4, -9>$.

We know that if $<a_1, a_2, a_3, \cdots, a_n>$ are the direction ratios of the normal to a hyperplane then direction cosines of the normal are given by

$$<\frac{a_1}{\sqrt{a_1^2+a_2^2+\cdots+a_n^2}}, \frac{a_2}{\sqrt{a_1^2+a_2^2+\cdots+a_n^2}}, \cdots, \frac{a_n}{\sqrt{a_1^2+a_2^2+\cdots+a_n^2}}>$$

Hence, required direction cosines are given by

$$<\frac{2}{2\sqrt{46}}, \frac{-3}{2\sqrt{46}}, \frac{7}{2\sqrt{46}}, \frac{5}{2\sqrt{46}}, \frac{4}{2\sqrt{46}}, \frac{-9}{2\sqrt{46}}>$$

$$\text{or } <\frac{1}{\sqrt{46}}, \frac{-3}{2\sqrt{46}}, \frac{7}{2\sqrt{46}}, \frac{5}{2\sqrt{46}}, \frac{2}{\sqrt{46}}, \frac{-9}{2\sqrt{46}}>$$

Since

$$\sqrt{2^2+(-3)^2+7^2+5^2+4^2+(-9)^2} = \sqrt{4+9+49+25+16+81}$$
$$= \sqrt{184} = \sqrt{2\times 2\times 46} = 2\sqrt{46}$$

UNIT 7

CONVEX AND CONCAVE FUNCTION

Structure

7.1 Introduction

Expected Learning Outcomes

7.2 Linear, Affine, Conic and Convex Combinations

7.3 Affine and Convex Sets

7.4 Properties of Convex Set

7.5 Definition of Convex and Concave Functions

7.6 Epigraph and Properties of Convex Function

7.7 Summary

7.8 Terminal Questions

7.9 Solutions/Answers

7.1 INTRODUCTION

The need for ideas of convex set and convex function is required in optimisation problems discussed in courses MST-022, MST-026 and MSTE-011. In this unit, you will study some basic results related to the convex set and convex function. To define a convex set, we will use the idea of linear combination and convex combination of two points in the underlying set. The convex combination is a particular case of affine combination. So, linear, affine and convex combinations are discussed in Sec. 7.2. Conic combination is also defined in Sec. 7.2. Definition of affine and convex sets and their simple examples are discussed in Sec. 7.3. After understanding what is a convex set, some properties of convex set are discussed in Sec. 7.4. The domain of a convex function is a convex set and you have studied about convex set in Secs. 7.3 and 7.4. So, the convex function is defined in Sec. 7.5. Epigraph of a function and some properties of the convex function are discussed in Sec. 7.6. But the idea of the concave function is also related to the convex function so the concave function is also discussed in the same section.

What we have discussed in this unit is summarised in Sec. 7.7. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit, more questions based on the entire unit are given in Sec. 7.8 under the heading Terminal Questions.

Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 7.9.

In the next unit, you will study gamma and beta functions.

Expected Learning Outcomes

After completing this unit, you should be able to:

- ❖ define what are linear, affine, conic and convex combinations;
- ❖ explain what we mean by affine and convex sets and can provide a lot of examples of each;
- ❖ list and prove some properties of the convex set;
- ❖ define convex and concave functions; and
- ❖ list some properties of a convex function and derive some of them.

7.2 LINEAR, AFFINE, CONIC AND CONVEX COMBINATIONS

As mentioned in Sec. 7.1 in courses MST-022, MST-026 and MSTE-011 you will deal with the optimum solution to a given problem. Suppose you are interested in obtaining the optimum value of the function $f(x_1, x_2, x_3, \ldots, x_n)$ known as the objective function. The variables $x_1, x_2, x_3, \ldots, x_n$ are known as decision variables. If there is no restriction on decision variables then you have to obtain the optimum value of the objective function on the entire $\mathbb{R}^n$. But generally, there are restrictions on decision variables in terms of available man-hours, machine hours, money, storage capacity, etc. Restrictions on decision variables reduce $\mathbb{R}^n$ to some subset C of $\mathbb{R}^n$, where all restrictions agree. If this common reduced subset C of $\mathbb{R}^n$ is a convex set as well as objective function and all the constraints in an optimisation problem are linear functions of decision variables then the optimisation problem is known as a **linear programming problem**. You have learnt the graphical method to solve linear programming problems in earlier classes. You will study some more methods to solve LPP in the course MSTE-011. An optimisation problem where either objective function or constraints on decision variables or both are non-linear function(s) of decision variables, then it is known as a **non-linear programming problem**. In both linear and non-linear programming problems convex set and convex function play an important role. So, this unit is devoted to discuss both the convex set and convex function. But the idea of a convex set is based on the idea of a convex combination. So, in this section, we will discuss different types of combinations of points in a set given as follows.

- Linear Combination
- Affine Combination
- Conic Combination, and
- Convex Combination

Let us discuss these one at a time.

Linear Combination

Before defining linear combination first, we have to clarify what are coefficients (scalars) and points (vectors) in the world of convex analysis (linear algebra). Here we are discussing these concepts from a data science or machine learning point of view. So, in an expression of the form

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k,$

we say that $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are **coefficients** or **scalars** and they may be any real numbers. While $x_1, x_2, x_3, \ldots, x_k$ are known as **points** or **vectors** in $\mathbb{R}^n$, where $n = 1, 2, 3, \ldots$

For example, in the expression: $3(4, 2) + 8(-2, 7)$, coefficients or scalars are $\alpha_1 = 3$, $\alpha_2 = 8$, while points or vectors are $x_1 = (4, 2)$, $x_2 = (-2, 7)$. In the expression $2(4, 1, 2) - 7(8, 9, 5)$ or $2(4, 1, 2) + (-7)(8, 9, 5)$, coefficients or scalars are $\alpha_1 = 2$, $\alpha_2 = -7$, while points or vectors are $x_1 = (4, 1, 2)$, $x_2 = (8, 9, 5)$. In the first expression points or vectors are elements of $\mathbb{R}^2$ and in the second expression points or vectors are elements of $\mathbb{R}^3$.

Remark 1: From here onwards we will call α_i's as scalars instead of coefficients and continue this terminology in the courses MST-022 and MST-026 while x_i's will be called points in this unit and we will call them vectors in the courses MST-022 and MST-026 due to standard practice that is followed in most of the books on these topics.

Now, we define linear combination. A point x is a **linear combination** of points $x_1, x_2, x_3, \ldots, x_k$ if and only if

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k, \quad \alpha_i \in \mathbb{R}, \; i = 1, 2, 3, \ldots, k \qquad \ldots (7.1)$$

For example, a point $x = (3, 7)$ is a linear combination of points $x_1 = (1, 1)$, $x_2 = (1, -1)$ as there exist real numbers $\alpha_1 = 5$, $\alpha_2 = -2$, such that

$$(3, 7) = 5(1, 1) + (-2)(1, -1) \quad \left[\begin{array}{l} \because \text{ if } (3, 7) = \alpha_1(1, 1) + \alpha_2(1, -1) \\ \Rightarrow (3, 7) = (\alpha_1 + \alpha_2, \alpha_1 - \alpha_2) \Rightarrow \alpha_1 + \alpha_2 = 3, \\ \alpha_1 - \alpha_2 = 7. \text{ After solving, we get } \alpha_1 = 5, \alpha_2 = -2 \end{array}\right]$$

Affine Combination

In a linear combination there was no restriction on values of α_i's. In linear combination α_i's. may be any real numbers. But in an affine combination, there is a restriction on their sum defined as follows.

A point x is an **affine combination** of points $x_1, x_2, x_3, \ldots, x_k$ if and only if

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k, \; \alpha_i \in \mathbb{R}, \; i = 1, 2, 3, \ldots, k \text{ and } \sum_{i=1}^{k} \alpha_i = 1 \ldots (7.2)$$

For example, point $x = (1, 5)$ is an affine combination of points $x_1 = (1, 1)$, $x_2 = (1, -1)$ as there exist real numbers $\alpha_1 = 3$, $\alpha_2 = -2$, such that

$$(1, 5) = 3(1, 1) + (-2)(1, -1) \quad \left[\begin{array}{l} \because 3(1, 1) + (-2)(1, -1) = (3, 3) + (-2, 2) \\ \qquad = (3 - 2, 3 + 2) = (1, 5) \end{array}\right]$$

Conic Combination

In affine combination there was a restriction on sum of all α_i's. But in conic combination there is no restriction on their sum but there are restrictions on individual α_i's defined as follows.

A point x is a **conic combination** of points $x_1, x_2, x_3, \ldots, x_k$ if and only if

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k, \quad \alpha_i \in \mathbb{R}, \text{ and } \alpha_i > 0, \quad i = 1, 2, 3, \ldots, k \quad \ldots (7.3)$$

For example, point $x = (7, -1)$ is a conic combination of points $x_1 = (1, 1)$, $x_2 = (1, -1)$ as there exist real numbers $\alpha_1 = 3$, $\alpha_2 = 4$, such that $\alpha_1 > 0$, $\alpha_2 > 0$, and

$$(7, -1) = 3(1, 1) + 4(1, -1) \quad \left[\begin{aligned} \because 3(1, 1) + 4(1, -1) &= (3, 3) + (4, -4) \\ &= (3+4, 3-4) = (7, -1) \end{aligned}\right]$$

Convex Combination

In affine combination there was a restriction only on sum of all α_i's. In conic combination there was a restriction only on individual α_i's. But in convex combination there are restrictions on both individual α_i's as well as on sum of all α_i's defined as follows.

A point x is a **convex combination** of points $x_1, x_2, x_3, \ldots, x_k$ if and only if

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k, \; \alpha_i \in \mathbb{R}, \; \alpha_i \geq 0, \; i = 1, 2, 3, \ldots, k \text{ and } \sum_{i=1}^{k} \alpha_i = 1 \quad \ldots (7.4)$$

For example, point $x = \left(1, -\frac{1}{2}\right)$ is a convex combination of points $x_1 = (1, 1)$, $x_2 = (1, -1)$ as there exist real numbers $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{3}{4}$, such that $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$ and

$$\left(1, -\frac{1}{2}\right) = \frac{1}{4}(1, 1) + \frac{3}{4}(1, -1) \quad \left[\begin{aligned} \because \frac{1}{4}(1, 1) + \frac{3}{4}(1, -1) &= \left(\frac{1}{4}, \frac{1}{4}\right) + \left(\frac{3}{4}, \frac{-3}{4}\right) \\ &= \left(\frac{1}{4} + \frac{3}{4}, \frac{1}{4} + \frac{-3}{4}\right) = \left(1, -\frac{1}{2}\right) \end{aligned}\right]$$

Now, you can try the following Self-Assessment Question.

SAQ 1

In each part identify whether given combination is affine or conic or convex.

(a) $-2(7, 5) + 3(0, -1)$ (b) $3(7, 5) + 2(0, -1)$ (c) $0.3(7, 5) + 0.7(0, -1)$

7.3 AFFINE AND CONVEX SETS

This section is divided into two subsections. In the first subsection we will discuss affine set and in the second subsection we will discuss convex set.

7.3.1 Affine Set

In Sec. 7.2 you have understood what we mean by an affine combination which will help you in understanding an affine set. So, let us define an affine set.

A set S is said to be an **affine set** if affine combination of any two members of it is in the set S. That is for all $x, y \in S$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = 1$ then $\alpha_1 x + \alpha_2 y \in S.$ Let us try to understand this definition geometrically. This definition says that:

$$\alpha_1 x + \alpha_2 y \in S, \ \forall \ x, y \in S \text{ and } \alpha_1, \alpha_2 \in \mathbb{R} \text{ with } \alpha_1 + \alpha_2 = 1 \qquad \dots (7.5)$$

$$\Rightarrow (1-\alpha_2)x + \alpha_2 y \in S, \ \forall \ x, y \in S, \ \alpha_2 \in \mathbb{R} \ [\because \alpha_1 + \alpha_2 = 1 \Rightarrow \alpha_1 = 1 - \alpha_2]$$

Or taking $\alpha_2 = \alpha,$ we get

$$(1-\alpha)x + \alpha y \in S, \ \forall \ x, y \in S, \ \alpha \in \mathbb{R} \qquad \dots (7.6)$$

On comparing equation (7.6) with equation (6.13) of the previous unit, we see that (7.6) represents all points of a straight line passing through the points x and y. But as per the requirements mentioned in (7.6) all these points belong to the set S. In other words, whole line lies inside the set S. So, **geometrically** a set S is said to be an **affine set** if we take any two points in the set S then the whole line joining the two points should lie in the set S.

After getting geometrical interpretation of an affine set, you can easily name some sets which are affine sets. Think geometrically what are the sets which contain whole line joining any two points on them. Recall that some such sets you have studied in the previous unit which contain whole line joining any two points in that set. Obviously, now some of the names which are coming in your mind includes: (i) a line in 2 or higher dimension (ii) plane in 3-dimension (iii) hyperplane in more than three dimension (iv) $\mathbb{R}^n$ itself for each $n = 1, 2, 3, \dots,$ etc. Two more sets which always fall in this category are empty set and the singleton set. Why empty set is an affine set? It is affine set because there are no two points in it to give a counter example to fail the definition. So, requirements to become affine set automatically satisfied. Why singleton set is an affine set? Let us explain it. Suppose S = {x}. Let $u, v \in S,$ then u = x and v = x you know why we have taken both u and v as x because S has only one element x. So, there is no other option for u and v. Now, take any $\alpha_1, \alpha_2 \in \mathbb{R},$ such that $\alpha_1 + \alpha_2 = 1$ then

$$\begin{aligned} \alpha_1 u + \alpha_2 v &= \alpha_1 u + (1-\alpha_1)v \quad [\because \alpha_1 + \alpha_2 = 1 \Rightarrow \alpha_2 = 1 - \alpha_1] \\ &= \alpha_1 x + (1-\alpha_1)x \quad [\because u = x \text{ and } v = x] \\ &= (\alpha_1 + 1 - \alpha_1)x = x \in S \end{aligned}$$

Hence, $\alpha_1 u + \alpha_2 v \in S, \ \forall \ u, v \in S$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = 1$

Thus, S is an affine set and therefore,

every singleton set is an affine set. … (7.7)

Now, let us do some examples related to affine set.

Example 1: Give an example of a set which is not an affine set.

Solution: Consider the half plane

$$3x + 2y \le 6 \qquad \dots (7.8)$$

discussed in Example 14 in the previous unit. You may refer Remark 5 mentioned in the previous unit for the notion of half plane. We shall prove that given half plane is not an affine set using two ways (i) graphically so that you can understand it intuitively (ii) using definition.

Graphically: Let us extend Fig. 6.11 (b) and added information is shown in Fig. 7.1 (a). Note that points $A(0, 1)$ and $B(1, 1)$ lie in the shaded region which represent the set S where $S = \{(x, y) \in \mathbb{R}^2 : 3x + 2y \le 6\}$. Now, the line passing through the points A and B contains the point C(3, 1) but the point C(3, 1) does not lie in the set S. So, set S is not an affine set because we know that a set is said to be an affine set if it contains whole line passing through any two points of it. This completes geometric proof. Remember there is nothing special in the points $A(0, 1)$ and $B(1, 1)$ you can take any other two points in S such that the line joining them does not wholly lie inside the set S.

Using Definition: Let S be as defined in graphical method. We have to show that S is not an affine set using definition. To do so we have to give a counter example where definition of affine set fails. So, let us consider two points $x = (-1, 0),\ y = (1, 0)$ obviously both belong to the half plane given by (7.8) $[\because 3(-1) + 2(0) \le 6 \text{ true and } 3(1) + 2(0) \le 6 \text{ true}]$. Now consider

$\alpha_1 = -1,\ \alpha_2 = 2,$ so $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 + \alpha_2 = 1$ but

$$\alpha_1 x + \alpha_2 y = -1(-1, 0) + 2(1, 0)$$
$$= (1, 0) + (2, 0) = (3, 0) \notin S \quad [\because 3(3) + 2(0) \le 6 \text{ which is False}]$$

$\Rightarrow$ S is not an affine set. To have a look on this argument graphically you may refer Fig. 7.1 (b).

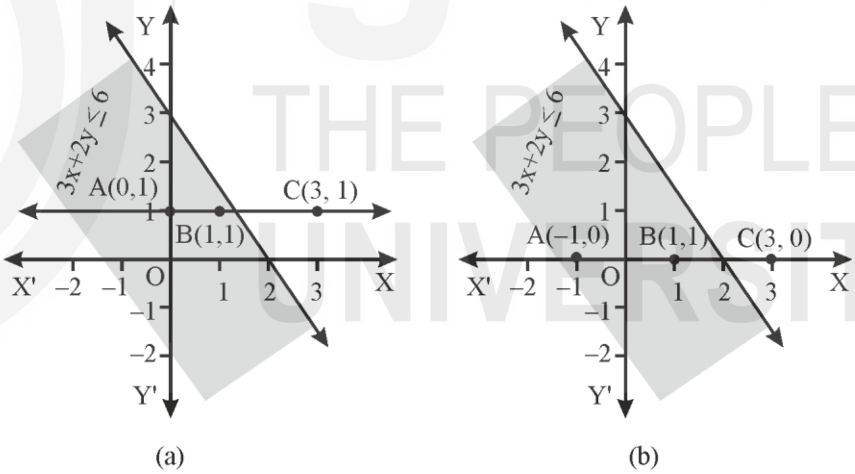

Fig. 7.1: Visualisation of (a) a half plane 3x + 2y ≤ 6 as an example of a set which is not an affine set (b) counter example given in the proof by definition

Example 2: Prove that hyperplane is an affine set.

Solution: We know that equation of a hyperplane is given by (you may refer equation (6.77) in Unit 6)

$$\omega^T x + b = 0 \quad \dots (7.9)$$

Let u, v lie on the hyperplane given by equation (7.9). Required to prove for any $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = 1 \Rightarrow \alpha_1 u + \alpha_2 v$ lies on the hyperplane.

i.e., required to prove $\omega^T (\alpha_1 u + \alpha_2 v) + b = 0$

Since u and v lie on the hyperplane (7.9), therefore

$$\omega^T u + b = 0 \text{ or } \omega^T u = -b \quad \dots (7.10)$$

$\omega^T v + b = 0$ or $\omega^T v = -b$ … (7.11)

Let $y = \alpha_1 u + \alpha_2 v$ then

$$\omega^T y = \omega^T\left(\alpha_1 u + \alpha_2 v\right) = \alpha_1 \omega^T u + \alpha_2 \omega^T v$$

$$= \alpha_1(-b) + \alpha_2(-b) \qquad \left[\text{Using } (7.10) \text{ and } (7.11)\right]$$

$$= (\alpha_1 + \alpha_2)(-b)$$

$$= -b \qquad [\because\ \alpha_1 + \alpha_2 = 1]$$

$$\Rightarrow \omega^T\left(\alpha_1 u + \alpha_2 v\right) = -b$$

$$\Rightarrow \omega^T\left(\alpha_1 u + \alpha_2 v\right) + b = 0$$

$\Rightarrow \alpha_1 u + \alpha_2 v$ lies on the hyperplane (7.9). Hence, hyperplane is an affine set.

7.3.2 Convex Set

In Sec. 7.2 you have understood what we mean by a convex combination which will help you in understanding a convex set. So, let us define a convex set.

A set S is said to be a **convex set** if convex combination of any two members of it is in the set S. That is if for all $x, y \in S$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \geq 0,\ \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$ then $\alpha_1 x + \alpha_2 y \in S.$ Let us try to understand this definition geometrically. This definition says that:

$\alpha_1 x + \alpha_2 y \in S, \forall x, y \in S$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1$… (7.12)

$\Rightarrow (1 - \alpha_2)x + \alpha_2 y \in S, \forall\ x, y \in S, \alpha_2 \in \mathbb{R}, \alpha_2 \in [0, 1]\ [\because\ \alpha_1 + \alpha_2 = 1 \Rightarrow \alpha_1 = 1 - \alpha_2]$

Or taking $\alpha_2 = \alpha,$ we get

$(1 - \alpha)x + \alpha y \in S,\ \ \forall\ x, y \in S,\ \alpha \in \mathbb{R},\ \alpha \in [0, 1]$ … (7.13)

On comparing equation (7.13) with equation (6.13) of the previous unit, we see that (7.13) represents all points on the line segment joining points x and y. But as per the requirements mentioned in (7.13) all these points belong to the set S. In other words, whole line segment joining points x and y lies inside the set S. So, **geometrically** a set S is said to be a **convex set** if we take any two points in the set S then the whole line segment joining points x and y lies inside the set S.

After getting geometrical interpretation of a convex set, you can easily name some sets which are convex sets. Think geometrically what are the sets which contain whole line segment joining any two points on them. First note that all affine sets are convex but converse is not true. But there are many other sets which are convex but not affine the reason for it is **to become a convex set only line segment joining two points should be inside the set but to become affine set whole line joining two points should be inside the set. So, restriction of affine set is more hard compare to convex set**. The name of some sets which are not affine sets but are convex sets includes: circular disk, semicircular disk, square region, rectangle region, region of a parallelogram shape, triangular region, cubic region, half plane, etc. Recall that empty set and the singleton set are affine sets and so convex also…(7.14)

Now, let us do some examples related to convex set.

Example 3: Give an example of a set which is not a convex set.

Solution: We know that equation of a circle having centre at (h, k) and radius r is given by

$$(x-h)^2+(y-k)^2=r^2 \qquad \ldots (7.15)$$

Consider the circle having centre at origin (0, 0) and radius equal to 2. So, its equation is given by

$$(x-0)^2+(y-0)^2=2^2 \text{ or } x^2+y^2=4 \qquad \ldots (7.16)$$

We claim that it is not a convex set. To prove our claim, we have to show that there exist at least two points on (7.16) such that line segment joining those points does not completely lie on (7.16). Consider points A(2, 0) and B(0, 2) and join AB refer Fig. 7.2 (a). Now, midpoint of the line segment AB is $C\left(\frac{2+0}{2}, \frac{0+2}{2}\right)=C(1, 1)$. Obviously, point C does not lie on the circle given by (7.16). $\left[\because 1^2+1^2=4, \text{ i.e., } 2=4 \text{ which is not true}\right]$. Hence, circle given by (7.16) is not a convex set.

Remark 2: If instead of circle we have circular disk say $(x-0)^2+(y-0)^2 \le 2^2$ or $x^2+y^2 \le 4$ then it is a convex set. To see its proof in general case, refer Solution of Terminal Question 2.

Example 4: Prove that the square formed by four lines $x+y=2$, $x-y=-2$, $x+y=-2$, $x-y=2$, is not a convex set.

Solution: Writing equations of 4 sides of the square in intercept form, we get

$$\frac{x}{2}+\frac{y}{2}=1, \ \frac{x}{-2}+\frac{y}{2}=1, \ \frac{x}{-2}+\frac{y}{-2}=1, \ \frac{x}{2}+\frac{y}{-2}=1 \qquad \ldots (7.17)$$

Square ABCD formed by these four lines is shown in Fig. 7.2 (b). Now, points A(2, 0) and $C(-2, 0)$ lie on the square but the midpoint of the line segment AC is the origin O(0, 0) which does not lie on the square ABCD. Hence, the square formed by four lines given by equations (7.17) is not a convex set.

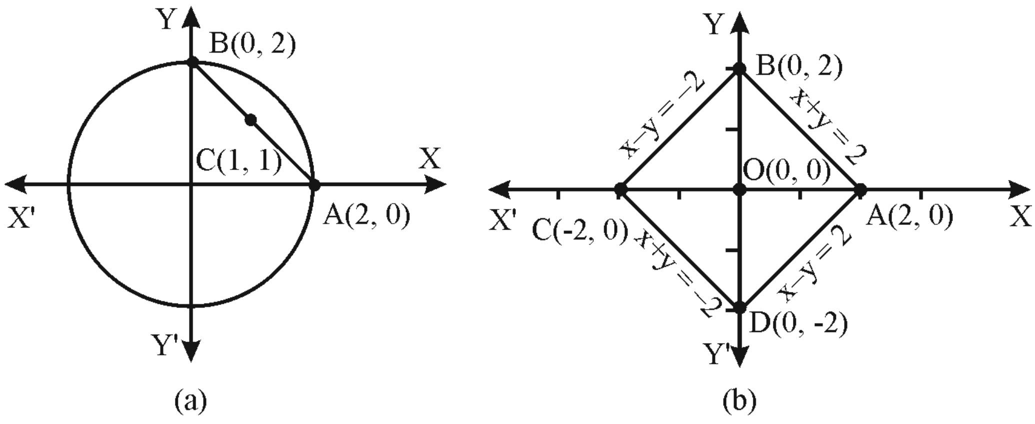

Fig. 7.2: Visualisation of a (a) a circle as an example of a set which is not convex (b) square as an example of a set which is not convex

Now, you can try the following Self-Assessment Question.

SAQ 2

(a) Does the set of natural numbers is a convex set?

(b) Does union of two convex sets is always convex? Give proper justification in support of your answer.

7.4 PROPERTIES OF CONVEX SET

In the previous section you have seen many convex sets. You can generate more convex sets from given convex set(s) using some properties of convex set. In this section we will state and prove three important properties of convex set.

Property 1: Prove that intersection of convex sets is also a convex set.

Solution: Let $C_1, C_2, C_3, \ldots$ be a countable collection of convex sets.

Required to prove $\bigcap_{n=1}^{\infty} C_n$ is a convex set. That is required to prove for each $\alpha \in [0, 1]$

$$(1-\alpha)x + \alpha y \in \bigcap_{n=1}^{\infty} C_n \quad \forall\ x,\ y \in \bigcap_{n=1}^{\infty} C_n \text{ and } \alpha \in [0, 1]$$

Since $x,\ y \in \bigcap_{n=1}^{\infty} C_n$

$\Rightarrow x,\ y \in C_n \quad \forall\ n,\ n = 1, 2, 3, \ldots$

$\Rightarrow (1-\alpha)x + \alpha y \in C_n\ \forall\ n,\ n = 1, 2, 3, \ldots$ and $\forall \alpha \in [0, 1]$ $[\because$ Each C_n is convex$]$

$\Rightarrow (1-\alpha)x + \alpha y \in \bigcap_{n=1}^{\infty} C_n \quad \forall \alpha \in [0, 1]$ and $\forall\ x,\ y \in \bigcap_{n=1}^{\infty} C_n$

Hence, $\bigcap_{n=1}^{\infty} C_n$ is a convex set. … (7.18)

Property 2: If C is a convex set and $\alpha \in \mathbb{R},\ \alpha \geq 0,$ then prove that αC is also a convex set.

Solution: If $\alpha = 0,$ then $\alpha C = \{0\}$ and we know that every singleton set is a convex set and hence αC is also a convex set. Now, let $\alpha \in \mathbb{R}$ and $\alpha > 0,$ then required to prove $(1-\lambda)x + \lambda y \in \alpha C \quad \forall\ x,\ y \in \alpha C$ and $\lambda \in [0, 1]$

Since $x,\ y \in \alpha C$

$\Rightarrow \exists\ u,\ v \in C$ such that $x = \alpha u,\ y = \alpha v$ … (7.19)

Now, $(1-\lambda)x + \lambda y = (1-\lambda)\alpha u + \lambda \alpha v$ $\quad$ [Using (7.19)]

$$= \alpha\left((1-\lambda)u + \lambda v\right) \in \alpha C \quad \left[\begin{array}{l}\because \text{ C is convex so } (1-\lambda)u + \lambda v \in C \\ \Rightarrow \alpha\left((1-\lambda)u + \lambda v\right) \in \alpha C\end{array}\right]$$

Hence, $(1-\lambda)x + \lambda y \in \alpha C \quad \forall\ x,\ y \in \alpha C$ and $\lambda \in [0, 1]$

Hence, αC is a convex set. … (7.20)

Property 3: If C_1 and C_2 are two convex sets then prove that $C_1 + C_2$ is also a convex set.

Solution: By definition of sum of two sets, we know that

$$C_1 + C_2 = \{u + v : u \in C_1, v \in C_2\} \qquad \dots (7.21)$$

In order to prove that $C_1 + C_2$ is also a convex set we have to prove that

$$(1-\alpha)x + \alpha y \in C_1 + C_2 \quad \forall\ x, y \in C_1 + C_2 \text{ and } \alpha \in [0, 1]$$

So, let $x, y \in C_1 + C_2$ then $\exists\ u_1, u_2 \in C_1$ and $v_1, v_2 \in C_2$ such that

$$x = u_1 + v_1 \text{ and } y = u_2 + v_2 \qquad \dots (7.22)$$

Now, $(1-\alpha)x + \alpha y = (1-\alpha)(u_1 + v_1) + \alpha(u_2 + v_2)$ [Using (7.22)]

$$= ((1-\alpha)u_1 + \alpha u_2) + ((1-\alpha)v_1 + \alpha v_2) \in C_1 + C_2$$

$$\left[\begin{array}{l}\because u_1, u_2 \in C_1 \text{ and } v_1, v_2 \in C_2 \text{ and } C_1, C_2 \text{ are convex sets} \\ \text{so } (1-\alpha)u_1 + \alpha u_2 \in C_1 \text{ and } (1-\alpha)v_1 + \alpha v_2 \in C_2\end{array}\right]$$

Hence, $(1-\alpha)x + \alpha y \in C_1 + C_2 \quad \forall\ x, y \in C_1 + C_2$ and $\alpha \in [0, 1]$

Hence, $C_1 + C_2$ is a convex set whenever C_1 and C_2 are convex sets… (7.23)

Now, you can try the following Self-Assessment Question.

SAQ 3

If C_1 and C_2 are two convex sets then which of the following will be definitely convex sets.

(a) $5C_1$ (b) $7C_2 + 9C_1$ (c) $C_1 \cap C_2$ (d) $C_1 \cup C_2$

7.5 DEFINITION OF CONVEX AND CONCAVE FUNCTIONS

Before discussing convex and concave functions let us recall what is local minima, local maxima, global minimum and global maximum which you have studied in earlier classes. Let us explain these taken one at a time.

Local Minima (Maxima): A function y = f(x) is said to have local minima (maxima) at a point x = a in its domain if there exists a $\delta > 0$, such that

$$f(x) \geq f(a) \ \ (f(x) \leq f(a)\ ) \ \forall\ \ x \in (a - \delta, a + \delta) \qquad \dots (7.19)$$

For example, in Fig. 7.3 function y = f(x) has local minima at the points $x_2, x_4,$ and x_6 while this function has local maxima at the points $x_1, x_3,$ and x_5.

Global Minimum (Maximum): A function y = f(x) is said to have global minimum (maximum) at a point x = a in its domain D if

$$f(x) \geq f(a) \ \ (f(x) \leq f(a)\ ) \ \forall\ \ x \in D \qquad \dots (7.20)$$

For example, if $D = [x_1, x_6]$ then graphically (see Fig. 7.3) we see that global minimum of the function is at the point $x = x_2$, while global maximum of the function is at the point $x = x_3$.

So, global minimum means function takes minimum value at that point compare to values of the function at all other points of the domain of the function. Similarly, global maximum means function takes maximum value at that point compare to values of the function at all other points of the domain of the function.

Now, we discuss convex and concave functions. In optimisation problems we are generally interested in global minimum (maximum) instead of local minima (maxima). The main advantage of convex functions is that their local and global minimum (maximum) are the same. This is the reason we prefer to work with convex sets and functions in optimisation problems. We have already discussed about convex sets in the previous section. So, now let us discuss about convex and concave functions.

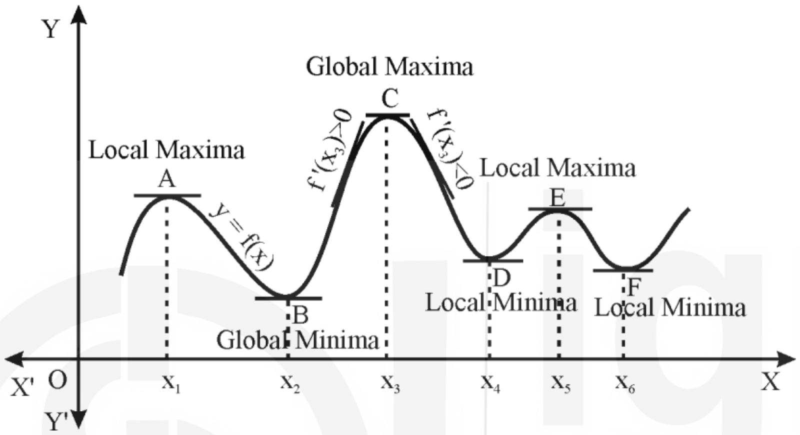

Fig. 7.3: Visualisation of points of local and global minimum and maximum

Convex Function: Let $D \subseteq \mathbb{R}^n,\ n = 1, 2, 3, \ldots, k$ be the convex set. Then a function $f : D \to \mathbb{R}$ is said to be a convex function if

$$f\left((1-\alpha)x + \alpha y\right) \le (1-\alpha)f(x) + \alpha f(y) \quad \forall\ x, y \in D \text{ and } \alpha \in [0, 1] \qquad \ldots (7.21)$$

and it is called concave function if

$$f\left((1-\alpha)x + \alpha y\right) \ge (1-\alpha)f(x) + \alpha f(y) \quad \forall\ x, y \in D \text{ and } \alpha \in [0, 1] \qquad \ldots (7.22)$$

From equations (7.21) and (7.22) note that if

$$f\left((1-\alpha)x + \alpha y\right) = (1-\alpha)f(x) + \alpha f(y) \quad \forall\ x, y \in D \text{ and } \alpha \in [0, 1] \qquad \ldots (7.23)$$

then function f will be both convex and concave.

Before discussing some examples, let us explain geometrically what the definitions of convex and concave functions say.

You know from equation (6.17) in 2-dimension and equation (6.67) in 3-dimension that:

- $(1-\alpha)x + \alpha y,\ \forall\ x, y \in D$ and $\alpha \in [0, 1]$ represents all points on the line segment in D joining points x and y, which are denoted by A and B refer Fig. 7.4 (a) where $D \subset \mathbb{R}$ … (7.24)
- $(1-\alpha)f(x) + \alpha f(y)\ \forall\ x, y \in D$ and $\alpha \in [0, 1]$ represents all points on the line segment joining points f(x) and f(y) in the range set of the function f, refer Fig. 7.4 (a) where points f(x) and f(y) are denoted by points C and D respectively. … (7.25)

Now, equation (7.21) says that the value of the function f obtained on any point of the line segment obtained by joining points x and y is less than or equal to the height of the corresponding point on the chord obtained by joining points (x, f(x)) and (y, f(y)). If you refer to Fig. 7.4 (a) it says that

$$EF \le EG \qquad \dots (7.26)$$

Similarly, equation (7.22) says that the value of the function f obtained on any point of the line segment obtained by joining points x and y is greater than or equal to the height of the corresponding point on the chord obtained by joining points (x, f(x)) and (y, f(y)). If you refer to Fig. 7.4 (b) it says that

$$EF \ge EG \qquad \dots (7.27)$$

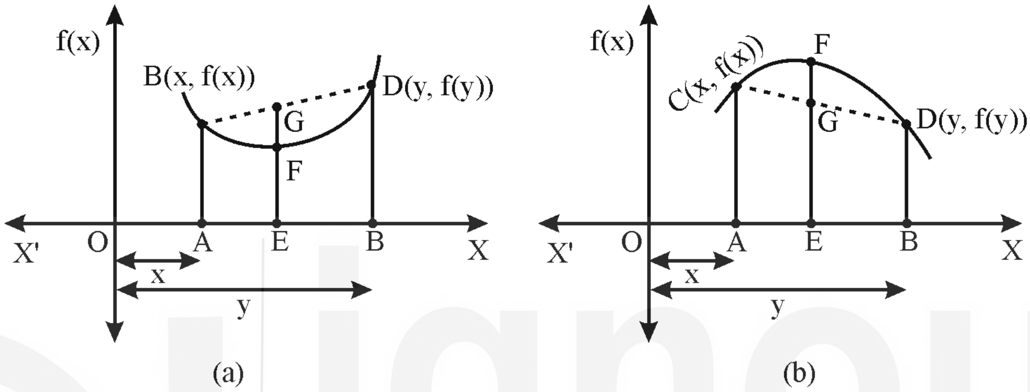

Fig. 7.4: Visualisation of (a) convex function (b) concave function

In other words, you can say that convex functions are open upward, while concave functions are open downward. To connect it with optimisation problems you can say that a convex function has global minimum value while a concave function has global maximum value.

Remark 3: Note that if a function f is convex then – f(x) will be concave function.

Now, let us do some examples.

Example 5: Prove that linear function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = ax + b, \quad a, b, x \in \mathbb{R} \qquad \dots (7.28)$$

is both convex and concave.

Solution: Let $x, \ y \in \mathbb{R} = \text{Domain of } f$ and $\alpha \in [0, 1]$, then required to prove $f\left((1-\alpha)x + \alpha y\right) = (1-\alpha)f(x) + \alpha f(y) \quad \forall \ x, y \in D$ and $\alpha \in [0, 1]$

Since $x, \ y \in \mathbb{R} = \text{Domain of } f$

$\therefore \ f(x) = ax + b$ ... (7.29) and $f(y) = ay + b$ ... (7.30)

Let $\alpha \in [0, 1]$, then

$$\begin{aligned} f\left((1-\alpha)x + \alpha y\right) &= a\left((1-\alpha)x + \alpha y\right) + b && [\text{Using } (7.28)] \\ &= (1-\alpha)ax + \alpha ay + b && \dots (7.31) \end{aligned}$$

$$\begin{aligned} \text{Also, } (1-\alpha)f(x) + \alpha f(y) &= (1-\alpha)\left(ax + b\right) + \alpha\left(ay + b\right) && [\text{Using } (7.28)] \\ &= (1-\alpha)ax + \alpha ay + (1-\alpha+\alpha)b \\ &= (1-\alpha)ax + \alpha ay + b && \dots (7.32) \end{aligned}$$

From equations (7.31) and (7.32), we get

$$f\left((1-\alpha)x+\alpha y\right)=(1-\alpha)f(x)+\alpha f(y) \quad \forall \; x, y \in D \text{ and } \alpha \in [0, 1]$$

Hence, linear function $f(x)=ax+b, \;\; a, b, x \in \mathbb{R}$ is both convex and concave.

Example 6: Explain graphically whether the function $f:[0, \pi] \to [-1, 1]$ defined by $f(x)=\sin x, \;\; x \in [0, \pi]$ … (7.33)

is convex or concave or both or none of them.

Solution: We claim that given function is not a convex function. As per the instruction of statement we have to explain it graphically. So, let us first draw its graph which is shown in Fig. 7.5 (a). Note that graph of this function in the domain $[0, \pi]$ is downward so if you will draw any chord by joining any two points on the graph then graph of the function will lie above the chord between those points. Hence, after observing graph of the function we can say that sine function is concave in the domain $[0, \pi]$.

Example 7: Explain graphically whether the function $f:[\pi, 2\pi] \to [-1, 1]$ defined by $f(x)=\sin x, \;\; x \in [\pi, 2\pi]$ … (7.34)

is convex or concave or both or none of them.

Solution: We claim that given function is a convex function. As per the instruction of statement we have to explain it graphically. So, let us first draw its graph which is shown in Fig. 7.5 (b). Note that graph of this function in the domain $[\pi, 2\pi]$ is upward so if you will draw any chord by joining any two points on the graph then graph of the function will lie below the chord between those points. Hence, after observing graph of the function we can say that sine function is convex in the domain $[\pi, 2\pi]$.

Example 8: Explain graphically whether the function $f:[0, 2\pi] \to [-1, 1]$ defined by $f(x)=\sin x, \;\; x \in [0, 2\pi]$ … (7.35)

is convex or concave or both or none of them.

Solution: We claim that given function is neither convex nor concave. As per the instruction of the statement we have to explain it graphically. Its graph is shown in Fig. 7.5 (c). By Example 6 given function is concave in the domain $[0, \pi]$, while by Example 7 given function is convex in the domain $[\pi, 2\pi]$.

Hence, given function is neither convex nor concave in the entire domain $[0, 2\pi]$.

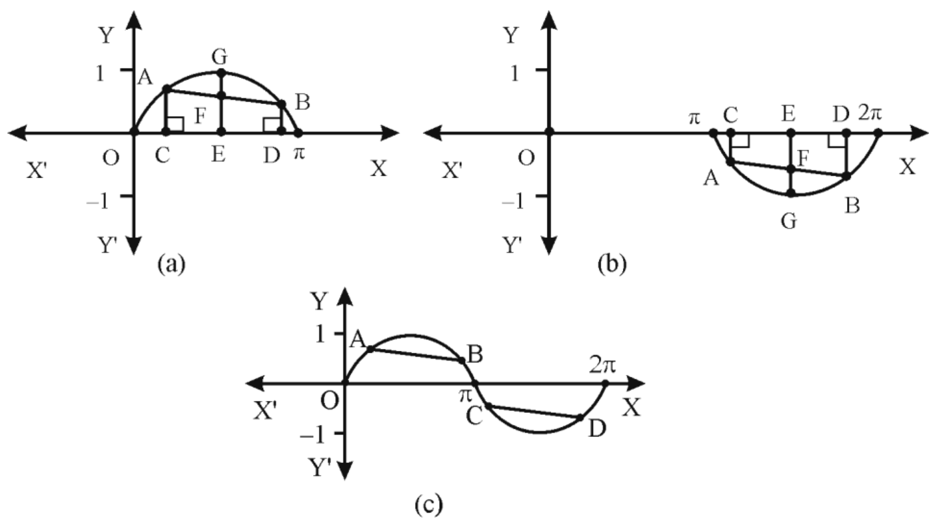

Fig. 7.5: Visualisation of graph of sine function in the domain (a) $[0,\pi]$ (b) $[\pi,2\pi]$ (c) $[0, 2\pi]$

7.6 EPIGRAPH AND PROPERTIES OF CONVEX FUNCTION

One thing that connect convex set with convex function is epigraph of a function via an important property refer Property 5. So, before discussing properties of a convex function let us first define epigraph of a function.

Epigraph: Let us first define epigraph of a function in layman language. The region above and on the graph of a function is known as epigraph of a function. In Fig. 7.6 (a) and (b) shaded regions in light blue colour (you may refer soft copy if this page is printed in black and white in eGyankosh) represent the epigraph of the functions $f(x) = 2 + x^2$ and $f(x) = x(x+2)(x-1)(x-2)$ respectively. Let us explain one more thing before giving definition of epigraph, which will help you to understand definition of epigraph of a function. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 2,\ x \in \mathbb{R}$ the graph of this function is shown in Fig. 7.6 (a).

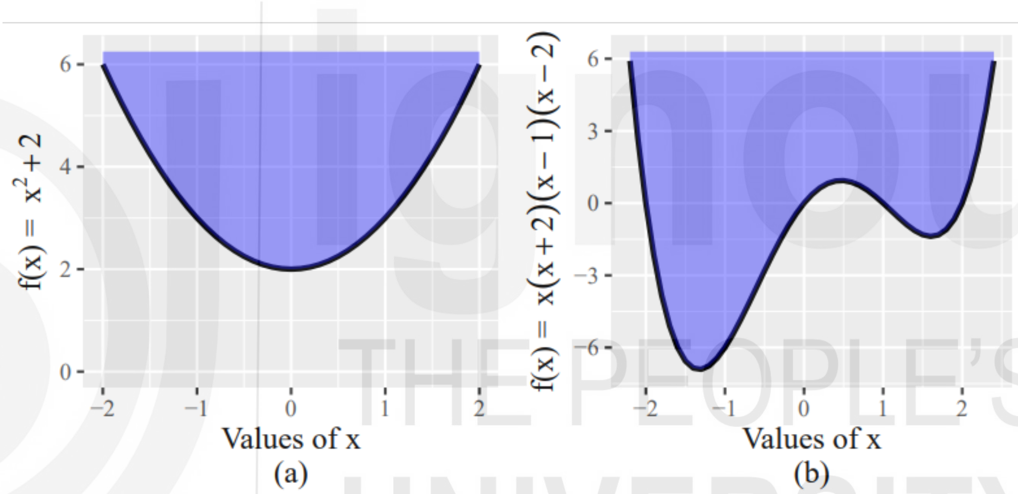

Fig. 7.6: Visualisation of epigraph of a function (a) epigraph of a convex function (b) epigraph of a function which is not convex

Note that: Domain of the function f is: $\mathbb{R}$ which is of 1-dimension but points on its graph are of 2-dimension. For example, $(0, 4) \in \mathbb{R}^2$ is a point in epigraph of this function and is of 2-dimension while domain of f was of 1-dimension. So, in general if $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$ then points in epigraph of the function f will be of n + 1 dimension. That is dimension of the points in the epigraph of a function is one more than the dimension of the domain of the function. Further, if you want to present points on the graph of the function f as a set of points on it then it can be written as follows.

Set of points on the graph of the function f = $\{(x, f(x)) : x \in S\} \subseteq \mathbb{R}^{n+1}$ … (7.36)

Keep this important explanation in mind that **dimension of points on epigraph of a function is one more than the dimension of the domain of the function**. This explanation will help you to understand the definition of the epigraph of the function which is given as follows.

Let $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$ then **epigraph** of the function f is defined as follows.

$$\text{efi}(f) = \{(x, y) \subseteq \mathbb{R}^{n+1} : x \in S,\ y \in \mathbb{R},\ y \geq f(x)\} \quad \text{… (7.37)}$$

For example, for the function f whose epigraph is shown in Fig. 7.6 (a), if $0 \in \mathbb{R}$, then

$E = \left\{(0, y) \subseteq \mathbb{R}^2 : y \in \mathbb{R},\ y \geq f(0)\right\} = \left\{(0, y) \subseteq \mathbb{R}^2 : y \in \mathbb{R},\ y \geq 2\right\}$ which is a vertical line having lowest point at (0, 2).

Similarly, corresponding to each point x_0 of the domain of the function f we get a vertical line which has coordinate of the lowest point as $(x_0, f(x_0))$.

Now, we discuss some properties of convex function.

Properties of Convex Function

Properties of convex function mainly helps us in two ways.

(1) They are used to create more convex functions.

(2) They are used to prove convexity of functions.

Here, first we will list some properties of convex function then we will prove them.

Property 1: If $S \subseteq \mathbb{R}^n$ be a convex set and $f : S \to \mathbb{R},\ g : S \to \mathbb{R}$ are two convex functions then $f + g$ is also a convex function. … (7.38)

Property 2: If $S \subseteq \mathbb{R}^n$ be a convex set and $f : S \to \mathbb{R}$ is a convex function then αf is also a convex function, where $\alpha \in \mathbb{R},\ \alpha \geq 0$. … (7.39)

Property 3: If $S \subseteq \mathbb{R}^n$ be a convex set and $f : S \to \mathbb{R}$ is a convex function then every local minimum of the function f on S is global minimum. … (7.40)

Property 4: **Connection between Convex Set and Convex Function**: If $S \subseteq \mathbb{R}^n$ be a convex set and $f : S \to \mathbb{R}$, then f is a convex function on S if and only if epigraph of the function f is a convex set. … (7.41)

Property 5: If $S \subseteq \mathbb{R}^n$ be a convex set and $f : S \to \mathbb{R},\ g : S \to \mathbb{R}$ are two convex functions then point wise maximum function, i.e., $\max\limits_{x \in S}\left\{f(x), g(x)\right\}$ is also a convex function. … (7.42)

Proof of Property 1: Since functions f and g are two convex functions then by definition of convex function we have

$$f\left((1-\alpha)x + \alpha y\right) \leq (1-\alpha)f(x) + \alpha f(y) \quad \forall \ x, y \in D \text{ and } \alpha \in [0, 1] \qquad \ldots (7.43)$$

$$g\left((1-\alpha)x + \alpha y\right) \leq (1-\alpha)g(x) + \alpha g(y) \quad \forall \ x, y \in D \text{ and } \alpha \in [0, 1] \qquad \ldots (7.44)$$

Required to prove $f + g$ is convex.

$$\left(f+g\right)\left((1-\alpha)x + \alpha y\right) = f\left((1-\alpha)x + \alpha y\right) + g\left((1-\alpha)x + \alpha y\right) \begin{bmatrix}\text{Using definition of} \\ \text{sum of two functions}\end{bmatrix}$$

$$\leq (1-\alpha)f(x) + \alpha f(y) + (1-\alpha)g(x) + \alpha g(y) \ [\text{Using (7.43) and (7.44)}]$$

$$\leq (1-\alpha)\left(f(x) + g(x)\right) + \alpha\left(f(y) + g(x)\right)$$

$$= (1-\alpha)\left(f+g\right)(x) + \alpha\left(f+g\right)(y)$$

$$\left(f+g\right)\left((1-\alpha)x + \alpha y\right) \leq (1-\alpha)\left(f+g\right)(x) + \alpha\left(f+g\right)(y) \quad \forall \ x, y \in D \text{ and } \alpha \in [0, 1]$$

Hence, $f + g$ is also a convex function.

Proof of Property 2: We are given that f is convex function so we have

$$f\left((1-\alpha)x+\alpha y\right) \le (1-\alpha)f(x)+\alpha f(y) \quad \forall \;\; x,\, y \in D \;\text{ and } \alpha \in [0,1] \qquad \dots (7.45)$$

Required to prove

$$(\alpha f)\left((1-\lambda)x+\lambda y\right) \le (1-\lambda)(\alpha f)(x)+\lambda(\alpha f)(y) \;\forall\, x,\, y \in D \text{ and } \lambda \in [0,1],\; \alpha \in \mathbb{R},\; \alpha \ge 0$$

$$\begin{aligned}(\alpha f)\left((1-\lambda)x+\lambda y\right) &= \alpha f\left((1-\lambda)x+\lambda y\right)\\ &\le \alpha\left((1-\lambda)f(x)+\lambda f(y)\right) \qquad [\text{Using } (7.19)]\\ &= (1-\lambda)\alpha f(x)+\lambda\alpha f(y)\\ &= (1-\lambda)(\alpha f)(x)+\lambda(\alpha f)(y)\end{aligned}$$

That is, we have proved that

$$(\alpha f)\left((1-\lambda)x+\lambda y\right) \le (1-\lambda)(\alpha f)(x)+\lambda(\alpha f)(y) \;\forall\, x,\, y \in D \text{ and } \lambda \in [0,1],\; \alpha \in \mathbb{R},\; \alpha \ge 0$$

Hence, αf is a convex function.

Proof of Property 3: We are given that $f : S \to \mathbb{R}$ is a convex function so we have

$$f\left((1-\alpha)x+\alpha y\right) \le (1-\alpha)f(x)+\alpha f(y) \quad \forall \;\; x,\, y \in S \;\text{ and } \alpha \in [0,1] \qquad \dots (7.46)$$

Required to prove every local minimum of the function f on S is global minimum.

Suppose x_0 be the local minimum of the function f having convex domain S. So, by definition of local minimum there will exists a neighbourhood $N_\delta(x_0)$ of x_0 for some $\delta > 0$ such that

$$f(x) \ge f(x_0) \quad \forall \;\; x \in N_\delta(x_0) \cap S \qquad \dots (7.47)$$

Required to prove that $x = x_0$ is global minimum. That is required to prove that

$$f(x) \ge f(x_0) \quad \forall \;\; x \in S \qquad \dots (7.48)$$

Suppose, if possible $x = x_0$ is not global minimum of the function f. So, there will exist at least one point $x = x^*$ such that

$$f(x^*) < f(x_0) \qquad \dots (7.49)$$

Consider the convex combination $(1-\alpha)x^* + \alpha x_0$ of points x^* and x_0. So, all the points of the line segment joining points x^* and x_0 are given by some value of $\alpha \in [0,1]$. Also, $N_\delta(x_0)$ is neighbourhood of the point x_0 so there will exist some $\alpha = \alpha_0 \in (0,1)$ such that

$$(1-\alpha_0)x^* + \alpha_0 x_0 \in N_\delta(x_0) \cap S$$

Let $x^{**} = (1-\alpha_0)x^* + \alpha_0 x_0$.

Now, $f(x^{**}) = f\left((1-\alpha_0)x^* + \alpha_0 x_0\right) \le (1-\alpha_0)f(x^*) + \alpha_0 f(x_0)$, [Using (7.46)]

$$< (1-\alpha_0)f(x_0) + \alpha_0 f(x_0) \quad \text{[Using (7.49)]}$$
$$= f(x_0)$$

i.e., $f(x^{**}) < f(x_0)$, where $x^{**} = (1-\alpha_0)x^* + \alpha_0 x_0 \in N_\delta(x_0) \cap S$

- a contradiction to equation (7.47). So, our supposition that $x = x_0$ is not global minimum of the function f is wrong. Hence, $x = x_0$ is global minimum of the function f. This completes the proof.

Now, you can try the following Self-Assessment Question.

SAQ 4

Prove property 4 of convex function.

7.7 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- A point x is a **linear combination** of points $x_1, x_2, x_3, \ldots, x_k$ if and only if

  $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k, \quad \alpha_i \in \mathbb{R}, \; i = 1, 2, 3, \ldots, k$

- A point x is an **affine combination** of points $x_1, x_2, x_3, \ldots, x_k$ if and only if

  $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k, \; \alpha_i \in \mathbb{R}, \; i = 1, 2, 3, \ldots, k$ and $\sum_{i=1}^{k} \alpha_i = 1\ldots$

- A point x is a **conic combination** of points $x_1, x_2, x_3, \ldots, x_k$ if and only if

  $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k, \quad \alpha_i \in \mathbb{R},$ and $\alpha_i > 0, \; i = 1, 2, 3, \ldots, k$

- A point x is a **convex combination** of points $x_1, x_2, x_3, \ldots, x_k$ if and only if

  $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k, \; \alpha_i \in \mathbb{R}, \; \alpha_i \ge 0, \; i = 1, 2, 3, \ldots, k$ and $\sum_{i=1}^{k} \alpha_i = 1$

- A set S is said to be an **affine set** if affine combination of any two members of it is in the set S.

- A set S is said to be a **convex set** if convex combination of any two members of it is in the set S.

- **Properties of Convex Set**:

  - Intersection of convex sets is also a convex set.
  - If C is a convex set and $\alpha \in \mathbb{R}, \; \alpha \ge 0$, then αC is also a convex set.
  - If C_1 and C_2 are two convex sets then $C_1 + C_2$ is also a convex set.

- Let $D \subseteq \mathbb{R}^n, \; n = 1, 2, 3, \ldots, k$ be the convex set. Then a function $f: D \to \mathbb{R}$ is said to be a **convex function** if

$$f\left((1-\alpha)x+\alpha y\right) \le (1-\alpha)f(x)+\alpha f(y) \quad \forall \ x, y \in D \text{ and } \alpha \in [0, 1]$$

and it is called **concave function** if

$$f\left((1-\alpha)x+\alpha y\right) \ge (1-\alpha)f(x)+\alpha f(y) \quad \forall \ x, y \in D \text{ and } \alpha \in [0, 1]$$

- Let $S \subseteq \mathbb{R}^n$ and $f: S \to \mathbb{R}$ then **epigraph** of the function f is defined as follows $\text{efi}(f) = \left\{(x, y) \subseteq \mathbb{R}^{n+1} : x \in S, y \in \mathbb{R}, \ y \ge f(x)\right\}$

- **Properties of Convex Function**:

  - If $S \subseteq \mathbb{R}^n$ be a convex set and $f: S \to \mathbb{R}, \ g: S \to \mathbb{R}$ are two convex functions then $f+g$ is also a convex function.

  - If $S \subseteq \mathbb{R}^n$ be a convex set and $f: S \to \mathbb{R}$ is a convex function then αf is also a convex function, where $\alpha \in \mathbb{R}, \ \alpha \ge 0.$

  - If $S \subseteq \mathbb{R}^n$ be a convex set and $f: S \to \mathbb{R}$ is a convex function then every local minimum of the function f on S is global minimum.

  - If $S \subseteq \mathbb{R}^n$ be a convex set and $f: S \to \mathbb{R},$ then f is a convex function on S if and only if epigraph of the function f is a convex set.

  - If $S \subseteq \mathbb{R}^n$ be a convex set and $f: S \to \mathbb{R}, \ g: S \to \mathbb{R}$ are two convex functions then point wise maximum function, i.e., $\max\limits_{x \in S}\left\{f(x), g(x)\right\}$ is also a convex function.

7.8 TERMINAL QUESTIONS

1. Does the set {4, 7} a convex set?
2. Prove that the circular disk in $\mathbb{R}^n$ is a convex set.
3. Prove that modulus function is a convex function.
4. Prove property 5 of convex function mentioned in Sec. 7.6.

7.9 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. (a) Here $\alpha_1 = -2, \ \alpha_2 = 3.$ Since $\alpha_1 = -2 < 0,$ so it neither can be a conic nor a convex combination. But $\alpha_1 + \alpha_2 = -2+3 = 1,$ so it is an affine combination.

   (b) Here $\alpha_1 = 3, \ \alpha_2 = 2.$ Since $\alpha_1 + \alpha_2 = 3+2 = 5 \ne 1,$ so it neither can be an affine nor a convex combination. But here $\alpha_1 = 3 > 0, \ \alpha_2 = 2 > 0,$ so it is a conic combination.

   (c) Here $\alpha_1 = 0.3, \ \alpha_2 = 0.7.$ Since $\alpha_1 + \alpha_2 = 0.3+0.7 = 1,$ so it an affine combination. Also $\alpha_1 = 0.3 > 0, \ \alpha_2 = 0.7 > 0,$ so it is a conic combination. Further, $\alpha_1 = 0.3 > 0, \ \alpha_2 = 0.7 > 0$ as well as $\alpha_1 + \alpha_2 = 0.3+0.7 = 1,$ so it is also a convex combination. Hence, it qualifies requirements of all the three types of combinations affine, conic and convex.

2. (a) We know that the set of all natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$. We claim that the set $\mathbb{N}$ is not a convex set. We know that if $\mathbb{N}$ is a convex set, then $(1-\alpha)x + \alpha y \in \mathbb{N}, \forall\ x, y \in \mathbb{N}$ and for each $\alpha \in [0, 1]$.

Let $x = 1,\ y = 2 \in \mathbb{N},\ \alpha = 0.4,$ so $1-\alpha = 0.6$. But
$\alpha 1 + (1-\alpha)2 = (0.4)1 + (0.6)2 = 0.4 + 1.2 = 1.6 \notin \mathbb{N}.$

Hence, the set $\mathbb{N}$ is not a convex set. The above argument is also shown in Fig. 7.7.

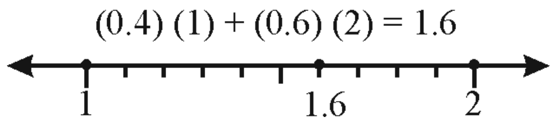

Fig. 7.7: Visualisation of the steps that the set of natural numbers is not a convex set

Remark 4: Similarly, we can show that (i) set of integers (ii) set of rational numbers, and (iii) set of irrational numbers are not convex sets.

2. (b) Union of two convex sets is not always convex. Here we will discuss one example where union of two sets will be convex and one example where union of two sets will not convex. Finally, we will discuss under what restriction union of two sets is always a convex set.

Example where union of two convex sets is a convex set:

Show that the two regions S_1, S_2 each bounded by four sides given as follows are convex sets. Further, show that their union is also a convex set.

$$S_1 = \left\{(x, y) \in \mathbb{R}^2 : x \le 2,\ x \ge 0,\ y \le 2,\ y \ge 0\right\}$$
$$S_2 = \left\{(x, y) \in \mathbb{R}^2 : x \le 3,\ x \ge 2,\ y \le 2,\ y \ge 0\right\}$$

Solution: Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in S_1, refer Fig. 7.8 (a). We shall prove that

$$(1-\alpha)(x_1, y_1) + \alpha(x_2, y_2) \in S_1 \quad \forall\ (x_1, y_1),\ (x_2, y_2) \in S_1 \text{ and } \forall\ \alpha \in [0, 1]$$

Now $(1-\alpha)(x_1, y_1) + \alpha(x_2, y_2) = \left((1-\alpha)x_1 + \alpha x_2,\ (1-\alpha)y_1 + \alpha y_2\right)$

Since $(x_1, y_1),\ (x_2, y_2) \in S_1 \Rightarrow 0 \le x_1, x_2 \le 2$ and $0 \le y_1, y_2 \le 2$

$$\Rightarrow 0 \le (1-\alpha)x_1 + \alpha x_2 \le 2 \text{ and } 0 \le (1-\alpha)y_1 + \alpha y_2 \le 2 \ [\because \alpha \in [0, 1]]$$

$$\therefore (1-\alpha)(x_1, y_1) + \alpha(x_2, y_2) \in S_1 \quad \forall\ (x_1, y_1),\ (x_2, y_2) \in S_1 \text{ and } \forall\ \alpha \in [0, 1]$$

Hence, S_1 is a convex set. Similarly, we can prove that S_2 is also a convex set. Now, we consider their union

$$S_1 \cup S_2 = \left\{(x, y) \in \mathbb{R}^2 : x \le 3,\ x \ge 0,\ y \le 2,\ y \ge 0\right\}$$

We claim that $S_1 \cup S_2$ is also a convex set.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in $S_1 \cup S_2$. We shall prove that
$(1-\alpha)(x_1, y_1) + \alpha(x_2, y_2) \in S_1 \cup S_2$

$$\forall (x_1, y_1), (x_2, y_2) \in S_1 \cup S_2 \text{ and } \forall \alpha \in [0, 1]$$

Now $(1-\alpha)(x_1, y_1)+\alpha(x_2, y_2)=\left((1-\alpha)x_1+\alpha x_2,\ (1-\alpha)y_1+\alpha y_2\right)$

Since $(x_1, y_1),\ (x_2, y_2)\in S_1\cup S_2 \Rightarrow 0\le x_1, x_2\le 3$ and $0\le y_1, y_2\le 2$

$\Rightarrow 0\le(1-\alpha)x_1+\alpha x_2\le 3$ and $0\le(1-\alpha)y_1+\alpha y_2\le 2\ \left[\because \alpha\in[0, 1]\right]$

$\therefore\ (1-\alpha)(x_1, y_1)+\alpha(x_2, y_2)\in S_1\cup S_2$

$$\forall (x_1, y_1), (x_2, y_2)\in S_1\cup S_2 \text{ and } \forall \alpha\in[0, 1]$$

Hence, $S_1\cup S_2$ is a convex set.

Example where union of two convex sets is not a convex set:

Show that the two regions S_1, S_2 bounded by four sides given as follows are convex sets, but their union is not a convex set.

$$S_1=\left\{(x, y)\in\mathbb{R}^2 : x\le 2,\ x\ge 0, y\le 4,\ y\ge 0\right\}$$

$$S_2=\left\{(x, y)\in\mathbb{R}^2 : x\le 6,\ x\ge 4, y\le 4,\ y\ge 0\right\}$$

Solution: Following similar steps as in earlier part we can show that both the sets S_1 and S_2 are convex sets. We claim that their union is not a convex set. Consider the points $P(1, 2), Q(5, 2)\in S_1\cup S_2$, refer Fig. 7.8 (b). Now, the point $R(3, 2)$ lies on the line segment PQ but it does not lie in the region $S_1\cup S_2$. Implies this the region $S_1\cup S_2$ does not contain the whole line segment obtained by joining its two points P and Q. Hence, the region $S_1\cup S_2$ is not a convex set.

Finally, union of two convex sets is also a convex set if either $S_1\subseteq S_2$ or $S_2\subseteq S_1$.

Solution: We are given that both S_1 and S_2 are convex sets.

Case I: $S_1\subseteq S_2$

$\therefore\ S_1\cup S_2=S_2$ which is a convex set. Hence, $S_1\cup S_2$ is a convex set.

Case II: $S_2\subseteq S_1$

$\therefore\ S_1\cup S_2=S_1$ which is a convex set. Hence, $S_1\cup S_2$ is a convex set.

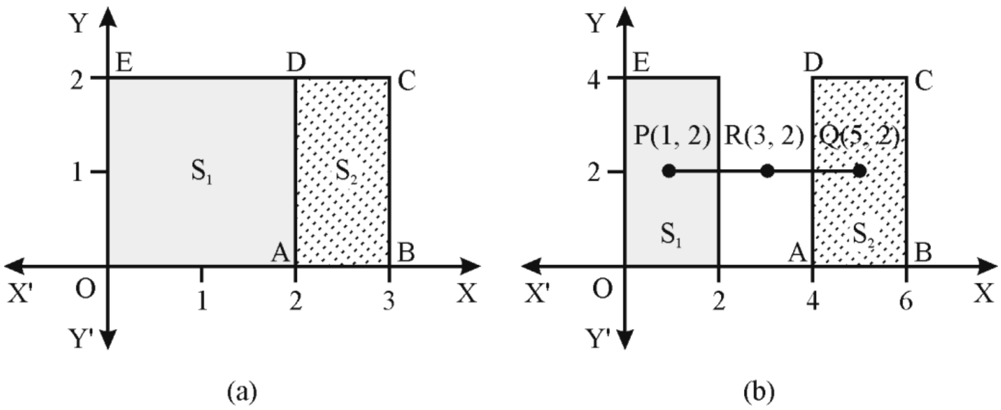

Fig. 7.8: Visualisation of (a) union of two convex sets is convex set (b) union of two convex sets is not a convex set

3. (a) We know that if C is a convex set then $\alpha C, \forall\ \alpha \geq 0$ is also a convex set. In our case $\alpha = 5 > 0$. So, $5C_1$ is a convex set.

(b) We know that if C is a convex set then $\alpha C, \forall\ \alpha \geq 0$ is also a convex set. So, $7C_2$ and $9C_1$ are convex sets. We also know that if C_1 and C_2 are convex sets then their sum $C_1 + C_2$ is also a convex set. Hence, sum of $7C_2$ and $9C_1$, i.e., $7C_2 + 9C_1$ is also a convex set.

(c) We know that if C_1 and C_2 are convex sets then their intersection is also a convex set. Hence, $C_1 \cap C_2$ is a convex set.

(d) We know that in the case $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$ union of two convex sets is convex. But in other cases, nothing can be said about convexity of union of two convex sets with surety. In other cases, union may be convex or may not be convex to see example you may refer solution of SAQ 2 (b).

4. **Proof of Property 4**: Required to prove f is convex function if and only if epigraph of the function f is a convex set. Let efi(f) denote the epigraph of the function f.

First suppose that f is convex function on its domain S. So, we have

$$f\left((1-\alpha)x_1 + \alpha x_2\right) \leq (1-\alpha)f(x_1) + \alpha f(x_2)\ \forall x_1, x_2 \in S \text{ and } \alpha \in [0, 1] \quad \ldots(7.50)$$

Let $(x_1, y_1), (x_2, y_2) \in \text{efi}\left(f\right)$, therefore by definition of epigraph, we have

$$y_1 \geq f(x_1),\ y_2 \geq f(x_2) \quad \ldots (7.51)$$

Consider the convex combination $(1-\alpha)(x_1, y_1) + \alpha(x_2, y_2)$ of points $(x_1, y_1), (x_2, y_2)$ and

$$(1-\alpha)(x_1, y_1) + \alpha(x_2, y_2) = (x, \alpha) \Rightarrow \left((1-\alpha)x_1 + \alpha x_2, (1-\alpha)y_1 + \alpha y_2\right) = (x, \alpha)$$
$$\Rightarrow (1-\alpha)x_1 + \alpha x_2 = x,\ (1-\alpha)y_1 + \alpha y_2 = \alpha \quad \ldots (7.52)$$

Now, $f(x) = f\left((1-\alpha)x_1 + \alpha x_2\right) \leq (1-\alpha)f(x_1) + \alpha f(x_2)$

$$\leq (1-\alpha)y_1 + \alpha y_2 \quad \left[\text{Using } (7.51)\right]$$
$$= \alpha \quad \left[\text{Using } (7.52)\right]$$

$$\Rightarrow f(x) \geq \alpha \quad \Rightarrow x \in \text{efi}\left(f\right)$$

$$\Rightarrow (1-\alpha)x_1 + \alpha x_2 \in \text{efi}\left(f\right)\ \forall\ (x_1, y_1), (x_2, y_2) \in \text{efi}\left(f\right),\ \alpha \in [0, 1]$$

Hence, efi(f) is a convex set.

Conversely: Suppose that efi(f) is a convex set. Let $(x_1, f(x_1)), (x_2, f(x_2)) \in \text{efi}\left(f\right)$. Since efi(f) is a convex set, so for all $\alpha \in [0, 1]$,

$$(1-\alpha)(x_1, f(x_1)) + \alpha(x_2, f(x_2)) \in \text{efi}\left(f\right)\ \forall\ (x_1, f(x_1)), (x_2, f(x_2)) \in \text{efi}\left(f\right)$$

$$\Rightarrow \left((1-\alpha)x_1 + \alpha x_2,\ (1-\alpha)f(x_1) + \alpha f(x_2)\right) \in \text{efi}(f)$$

$$\Rightarrow f\left((1-\alpha)x_1 + \alpha x_2\right) \le (1-\alpha)f(x_1) + \alpha f(x_2) \ \left[\text{By definition of epigraph}\right]$$

i.e., $f\left((1-\alpha)x_1 + \alpha x_2\right) \le (1-\alpha)f(x_1) + \alpha f(x_2) \ \forall x_1, x_2 \in S$ and $\alpha \in [0, 1]$

Hence, f is a convex function.

Terminal Questions

1. Let C = {4, 7}. We claim that the set C is not a convex set. We know that if C is a convex set, then $(1-\alpha)x + \alpha y \in C,\ \forall\ x, y \in C$ and for each $\alpha \in [0, 1]$.

   Let $\alpha = 0.4$, then $1-\alpha = 0.6$. But $\alpha 4 + (1-\alpha)7 = (0.4)4 + (0.6)7 = 1.6 + 4.2 = 5.8 \notin C.$

   Hence, the set C is not a convex set.

2. Let us consider a circular disk with centre at origin and radius r, refer Fig. 7.9 to see shape of a circular disk in $\mathbb{R}^2$. Therefore, its equation is given by

   $$x_1^2 + x_2^2 + \ldots + x_n^2 \le r^2, \text{ where } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,\ r \in \mathbb{R},\ r > 0 \quad \ldots (7.53)$$

   Let $C = \left\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \ldots + x_n^2 \le r^2\right\}$

   Required to prove: The set C is convex. That is required to prove $(1-\alpha)x + \alpha y \in C,\ \forall\ x, y \in C$ and for each $\alpha \in [0, 1]$

   Let $x, y \in C$ then $x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n)$ and will satisfy (7.53)

   $$\therefore\ x_1^2 + x_2^2 + \ldots + x_n^2 \le r^2 \quad \ldots (7.54)$$

   $$\text{and } y_1^2 + y_2^2 + \ldots + y_n^2 \le r^2 \quad \ldots (7.55)$$

   Let $z = (1-\alpha)x + \alpha y = (1-\alpha)(x_1, x_2, \ldots, x_n) + \alpha(y_1, y_2, \ldots, y_n)$

   $$= \left((1-\alpha)x_1 + \alpha y_1,\ (1-\alpha)x_2 + \alpha y_2,\ \ldots,\ (1-\alpha)x_n + \alpha y_n\right),\ \alpha \in [0, 1]$$

   Now, z is any point on the line segment joining points x and y including x and y. So, we will prove that point z lies on the circular disk given by (7.53) for all values of $\alpha \in [0, 1]$. Let us consider

   $$\left((1-\alpha)x_1 + \alpha y_1\right)^2 + \left((1-\alpha)x_2 + \alpha y_2\right)^2 + \ldots + \left((1-\alpha)x_n + \alpha y_n\right)^2$$

   $$= (1-\alpha)^2\left[x_1^2 + x_2^2 + \ldots + x_n^2\right] + \alpha^2\left[x_1^2 + x_2^2 + \ldots + x_n^2\right]$$
   $$+ 2\alpha(1-\alpha)\left[x_1y_1 + x_2y_2 + \ldots + x_ny_n\right]$$
   $$\le (1-\alpha)^2\left[r^2\right] + \alpha^2\left[r^2\right] \qquad \left[\text{Using } (7.54) \text{ and } (7.55)\right]$$
   $$+ 2\alpha(1-\alpha)\left[x_1y_1 + x_2y_2 + \ldots + x_ny_n\right]$$
   $$\le \left[(1-\alpha)r\right]^2 + \left[\alpha r\right]^2 + 2\alpha(1-\alpha)\left|x_1y_1 + x_2y_2 + \ldots + x_ny_n\right| \ \left[\because x.y \le |x.y|\right]$$

$$\le [(1-\alpha)r]^2 + [\alpha r]^2 + 2\alpha(1-\alpha)\|x\|\|y\| \quad [\text{Using } (6.45e)]$$

$$\le [(1-\alpha)r]^2 + [\alpha r]^2 + 2\alpha(1-\alpha)(r)(r) \quad \begin{bmatrix} \because \|x\| = \|x-0\| \le r \text{ and} \\ \|y\| = \|y-0\| \le r \end{bmatrix}$$

$$= \left[(1-\alpha)^2 + \alpha^2 + 2\alpha(1-\alpha)\right] r^2 = (1-\alpha+\alpha)^2 r^2 = r^2$$

i.e., $((1-\alpha)x_1 + \alpha y_1)^2 + ((1-\alpha)x_2 + \alpha y_2)^2 + \ldots + ((1-\alpha)x_n + \alpha y_n)^2 \le r^2$

$\Rightarrow z = (1-\alpha)x + \alpha y \in C, \ \forall \ x, y \in C$ and for each $\alpha \in [0, 1]$

Hence, circular disk in $\mathbb{R}^n$ is a convex set.

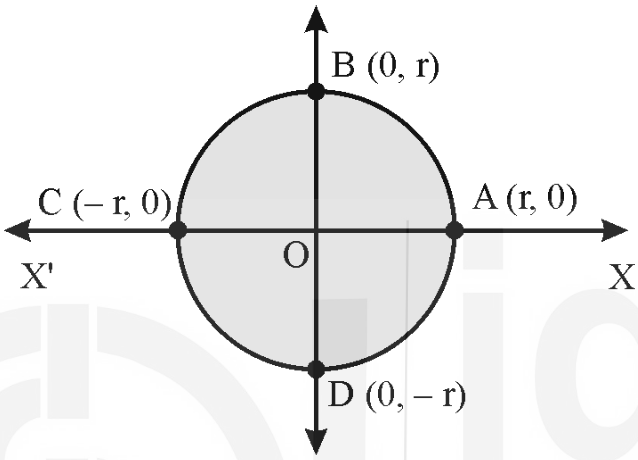

Fig. 7.9: Visualisation of a circular disk in two dimensions

3. We know that modulus function is defined as follows. $f : \mathbb{R} \to \mathbb{R}$ defined by
$f(x) = |x|, \quad x \in \mathbb{R}$ … (7.56)

Let $x, \ y \in \mathbb{R}$ and $\alpha \in [0, 1]$, then required to prove

$f((1-\alpha)x + \alpha y) \le (1-\alpha)f(x) + \alpha f(y) \quad \forall \ x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$

Now,

$$f((1-\alpha)x + \alpha y) = |(1-\alpha)x + \alpha y| \quad [\text{Using } (7.56)]$$

$$\le |(1-\alpha)x| + |\alpha y| \quad [\because |a+b| \le |a| + |b|]$$

$$= (1-\alpha)|x| + \alpha|y| \quad [\because 1-\alpha \ge 0, \alpha \ge 0 \text{ so } |1-\alpha| = 1-\alpha \text{ and } |\alpha| = \alpha]$$

$$= (1-\alpha)f(x) + \alpha f(y) \quad [\text{Again using } (7.56)]$$

i.e., $f((1-\alpha)x + \alpha y) \le (1-\alpha)f(x) + \alpha f(y) \quad \forall \ x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$

Hence, modulus function is a convex function. … (7.57)

4. **Proof of Property 5**: To prove this property we will use Property 4 already proved in SAQ 4. So, we know that

Function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex on S iff efi(f) is a convex set. … (7.58)

Since functions f and g are convex functions, so efi(f) and efi(g) are convex sets. [Using (7.58) for functions f and g]

$\Rightarrow \text{efi}(f) \cap \text{efi}(g)$ is a convex set [Using (7.18)] … (7.59)

But using (7.37), we have

$$\Rightarrow \text{efi}(f) \cap \text{efi}(g) = \{(x, y) \subseteq \mathbb{R}^{n+1} : x \in S, y \in \mathbb{R}, \ y \geq f(x), \ y \geq g(x)\}$$

$$= \left\{(x, y) \subseteq \mathbb{R}^{n+1} : x \in S, y \in \mathbb{R}, \ y \geq \max_{x \in S}\{f(x), g(x)\}\right\} \quad \text{... (7.60)}$$

In view of (7.59) and (7.60), we have

$\left\{(x, y) \subseteq \mathbb{R}^{n+1} : x \in S, y \in \mathbb{R}, \ y \geq \max_{x \in S}\{f(x), g(x)\}\right\}$ is a convex set.

$\Rightarrow \max_{x \in S}\{f(x), g(x)\}$ is a convex function. [Using (7.58)]

UNIT 8

BETA AND GAMMA FUNCTIONS

Structure

8.1 Introduction

Expected Learning Outcomes

8.2 Gamma Function and its Graphical Behaviour

8.3 Beta Function and its Graphical Behaviour

8.4 Properties of Gamma and Beta Functions

8.5 Summary

8.6 Terminal Questions

8.7 Solutions/Answers

8.1 INTRODUCTION

You know about factorial function from earlier classes. Factorial function works for non-negative integers. For example, $\underline{|0} = 1$, $\underline{|1} = 1$, $\underline{|2} = 2 \times 1 = 2$, $\underline{|3} = 3 \times 2 \times 1 = 6$, $\ldots$. In this unit you will study a function known as gamma function which is related to factorial function by the relation gamma(n+1) = factorial(n), where n is non-negative integer, but other than this it also interpolates factorial function for non-integers values. Gamma function and its graphical behaviour are discussed in Sec. 8.2. Another function which is related to the gamma function is beta function which is discussed in Sec. 8.3. Graphical behaviour of the beta function is also discussed in the same section. Some properties of gamma and beta functions are discussed in Sec 8.4.

What we have discussed in this unit is summarised in Sec. 8.5. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit, some more questions based on the entire unit are given in Sec. 8.6 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 8.7.

In the next unit, you will study about change of order of Sigma and integration.

Expected Learning Outcomes

After completing this unit, you should be able to:

- define gamma and beta functions;
- explain the effect of parameters of gamma and beta functions on their shapes; and
- establish some properties of gamma and beta functions.

8.2 GAMMA FUNCTION AND ITS GRAPHICAL BEHAVIOUR

You know that the factorial function is defined as follows.

$$\lfloor n = n(n-1)(n-2)(n-3)\ldots 3.2.1 \qquad \ldots (8.1)$$

Consider the function $\overline{\lfloor n} = \int_0^\infty x^{n-1} e^{-x}\, dx,\ n > 0 \qquad \ldots (8.2)$

which is known as the gamma function.

After replacing n by n + 1 in equation (8.2), we get

$$\overline{\lfloor n+1} = \int_0^\infty x^{n} e^{-x}\, dx,\ n+1 > 0 \qquad \ldots (8.3)$$

If you evaluate values of the functions given by (8.1) and (8.3) at $n = 1, 2, 3, 4, 5, \ldots$. You will see that $\overline{\lfloor n+1} = \lfloor n$. If you evaluate the value of the gamma function given in (8.2) or (8.3) manually then it will be time-consuming. You can do it easily in R. In the course MSTL-011 you have studied factorial() and gamma() functions to evaluate values of (8.1) and (8.2) respectively in R. The screenshot of R codes and their outputs in the R console is shown as follows.

```
> factorial(0:5) # to get factorial of the numbers 0, 1, 2, 3, 4 and 5
[1]   1   1   2   6  24 120
> gamma(1:6) # to get values of gamma function at 1, 2, 3, 4, 5 and 6
[1]   1   1   2   6  24 120
```

Equation (8.1) makes sense only when $n = 1, 2, 3, 4, 5, \ldots$. By convention we also define $\lfloor 0 = 1$. But equation (8.2) gives finite values for each real number n > 0. Again using R you can obtain values of the gamma function at any value of n > 0. Values of the Gamma Function at $n = 0.05, 0.15, 0.25, 0.50$ using R are given by running the following code on R console.

```
> gamma(c(0.05, 0.15, 0.25, 0.5))
[1] 19.470085  6.220273  3.625610  1.772454
```
… (8.4)

To see the similarity in the values of the functions (8.1) and (8.3) for $n = 1, 2, 3, 4, 5, \ldots$ graphically, you may refer to Fig. 8.1 (a) and (b). Note that the domain of the factorial function given by (8.1) is $\{0, 1, 2, 3, 4, 5, \ldots\}$ while the domain of the gamma function given by (8.3) is $(0, \infty)$. To see the effect of the domain on a function you may refer to Fig. 1.8 in Unit 1 of this course.

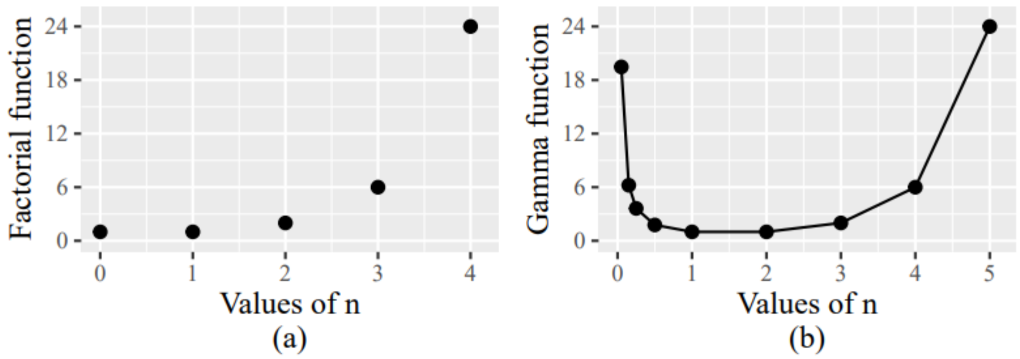

Fig. 8.1: Visualisation of (a) factorial function (b) gamma function

Graphical Behaviour of the Function $y = f(x) = x^n e^{-x}$ and its Integral known as Gamma Function for different values of n

From the discussion of Sec. 5.2 of Unit 5 of this course you know that single integral $\int_a^b f(x)\,dx,$ represents area bounded by two vertical lines $x = a, x = b,$ one horizontal line x-axis itself and the curve $y = f(x),$ refer (5.1). So, equation (8.3) gives area bounded by two vertical lines $x = a, x \to \infty,$ one horizontal line x-axis itself and the curve $y = f(x) = x^n e^{-x}$. To understand the effect of n on the shape of the function $y = f(x) = x^n e^{-x}$ and value of the area given by equation (8.3), let us visualise both in Fig. 8.2 (a) to (f) for n = 1, 2, 3, 4, 5, 6 respectively. From the graph of the function $y = f(x) = x^n e^{-x}$ we observe that:

- **To the Left Side of Origin**
  - the graph goes downward when n is odd refer to Fig. 8.2 (a), (c) and (e). It happens so due to the reason that e^{-x} being exponential function is always positive and x^n will be negative when x is negative and n is odd. So, the product of a positive number and a negative number is negative. So, $x^n e^{-x}$ is negative to the left side of the origin. … (8.5)
  - the graph goes upward when n is even refer to Fig. 8.2 (b), (d) and (f). It happens so due to the reason that e^{-x} being exponential function is always positive and x^n will be positive when x is negative and n is even. So, the product of two positive numbers is positive. So, $x^n e^{-x}$ is positive to the left side of the origin.
- **To the Right Side of Origin**
  - the graph initially increases and attains its maximum value after attaining maximum value it starts decreasing and goes on decreasing and ultimately it tends to zero as x tends to infinity. You may refer to any of the Fig. 8.2 (a) to (f). It happens so due to the reason that initially the function x^n maintains growth over the function e^{-x} but after a certain value of x, the function e^{-x} starts dominating the function x^n and ultimately the value of the product $x^n e^{-x}$ is dictated by the exponential function e^{-x} which tends to zero as x tends to infinity.
  - the area under the curve $y = f(x) = x^n e^{-x}$ increases as n increases but the important point is, it remains finite for all finite values of n. The actual value of the area under the curve is given by $\overline{|n+1}$. For example, Table

8.1 shows areas of shaded regions in Fig. 8.2 (a) to (f) which is given as follows.

Table 8.1: Actual values of areas of shaded regions in Fig. 8.2 (a) to (f)

Value of n	Figure number	Shaded area in terms of gamma function	Shaded area in terms of factorial function	Actual shaded area
1	8.2 (a)	$\Gamma 2$	$\lfloor 1$	1
2	8.2 (b)	$\Gamma 3$	$\lfloor 2$	2
3	8.2 (c)	$\Gamma 4$	$\lfloor 3$	6
4	8.2 (d)	$\Gamma 5$	$\lfloor 4$	24
5	8.2 (e)	$\Gamma 6$	$\lfloor 5$	120
6	8.2 (f)	$\Gamma 7$	$\lfloor 6$	720

- The shape of the function $y = f(x) = x^n e^{-x}$ to the right of zero looks like normal density which is discussed in Unit 12 of the course MST-12.

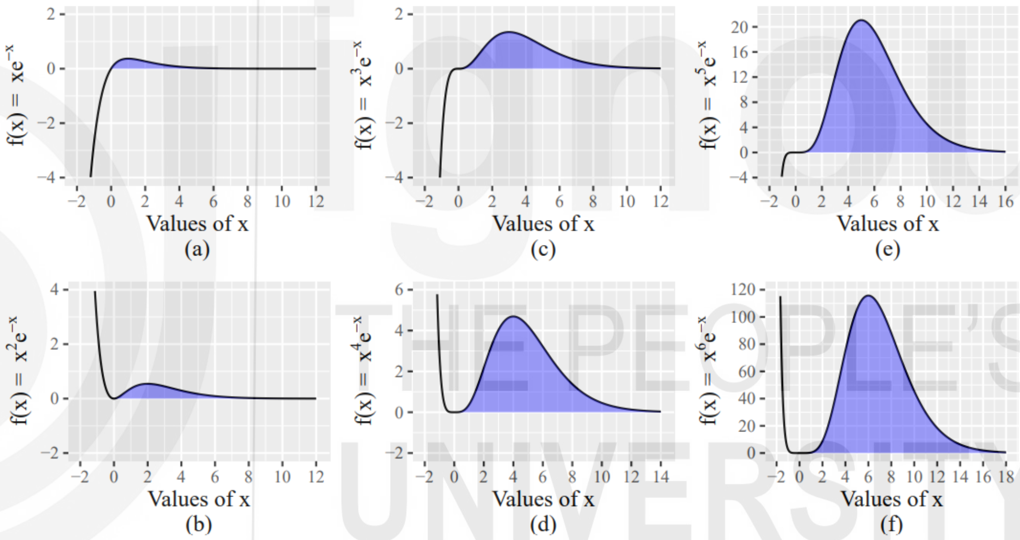

Fig. 8.2: Visualisation of the function $y = f(x) = x^{n-1} e^{-x}$ by curve and $\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx$ by shaded region for different values of n (a) n = 1 (b) n = 2 (c) n = 3 (d) n = 4 (e) n = 5 (f) n = 6

So far in this section we have compared gamma function with factorial function and we have also seen effect of n on the shape of the function $y = f(x) = x^n e^{-x}$ which is integrand in the gamma function. Let us now formally define gamma function as follows.

Definition of Gamma Function: A function $\Gamma : (0, \infty) \to (0, \infty)$ defined by

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx, \ n \in (0, \infty)$$ is called gamma function.

Graph of gamma function for n > 0 is shown in Fig. 8.3 given as follows. Note that:

- It is a convex function since if you join any two points on it then the graph of the function will lie below the chord joining two points. … (8.6)

- Global minimum is at the point 1.461632145. It cannot be obtained graphically. To obtain it using calculus is beyond the scope of this course. You can verify it in R. Global minimum value at the point 1.461632145 is 0.8856032. … (8.7)

In the next section you will study similar analysis about beta function.

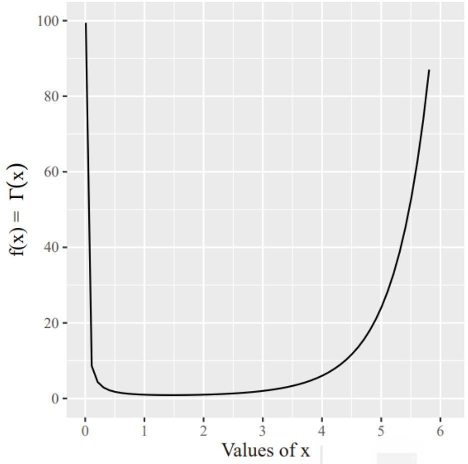

Fig. 8.3: Visualisation of gamma function

Now, you can try the following Self-Assessment Question.

SAQ 1

Does $\overline{|8} = \underline{|8}$?

8.3 BETA FUNCTION AND ITS GRAPHICAL BEHAVIOUR

In Sec. 8.2 you have studied about graphical behaviour of:

- gamma function $\overline{|n} = \int_0^\infty x^{n-1} e^{-x} dx,\ n \in (0, \infty)$;
- integrand of gamma function $y = f(x) = x^n e^{-x}$

for different values of n. We have also seen effect of n on the shape of the function $y = f(x) = x^n e^{-x}$ which is integrand in the gamma function.

In this section we will do similar study for beta function and its integrand. Let us start with the definition of beta function.

Definition of Beta Function: The integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0, n > 0$ is known as beta function and is denoted by B(m, n).

i.e., $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0, n > 0$ … (8.8)

Let us first evaluate value of this function manually for some particular values of m and n as follows.

$$B(1, 1) = \int_0^1 x^{1-1}(1-x)^{1-1} dx = \int_0^1 1 dx \quad \left[\begin{array}{l}\because x^0 = 1,\ (1-x)^0 = 1.\ \text{In fact, if a is any} \\ \text{finite number other than zero then } a^0 = 1\end{array}\right]$$

$$= [x]_0^1 = 1 - 0 = 1 \qquad \text{... (8.9)}$$

$$B(2, 1) = \int_0^1 x^{2-1}(1-x)^{1-1}\,dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}(1-0) = \frac{1}{2} \qquad \text{... (8.10)}$$

$$B(1, 2) = \int_0^1 x^{1-1}(1-x)^{2-1}\,dx = \int_0^1 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = \left(1 - \frac{1}{2} - 0 + 0\right) = \frac{1}{2} \qquad \text{... (8.11)}$$

$$B(2, 2) = \int_0^1 x^{2-1}(1-x)^{2-1}\,dx = \int_0^1 x(1-x)\,dx = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3} - 0 + 0\right) = \frac{1}{6} \qquad \text{... (8.12)}$$

Like this you can obtain values of beta function B(m, n) manually for different values of m and n. You can also calculate these values in R using beta() function. Screenshot of R codes and their outputs in R console is shown as follows.

```
> beta(1,1)    # m = 1, n = 1
[1] 1
> beta(2,1)    # m = 2, n = 1
[1] 0.5
> beta(1,2)    # m = 1, n = 2
[1] 0.5
> beta(2,2)    # m = 2, n = 2
[1] 0.1666667
```

… (8.13)

Remark 1: Note that values of B(2, 1) and B(1, 2) are equal. This did not happen by chance. In Sec. 8.4 you will prove that B(m, n) = B(n, m) for all values of m and n where m > 0, n > 0. This is known as **symmetric property of beta function**.

You can also obtain these values in a single command using c() function. Screenshot of R code and its output in R console is shown as follows.

```
> beta(c(1,2,1,2),c(1,1,2,2))
[1] 1.0000000 0.5000000 0.5000000 0.1666667
```

… (8.14)

So far we have obtained values of B(m, n) at positive integer values of m and n. But this function gives finite value for any values of m and n which are greater than zero. You can see it using R. Screenshot of R codes and their outputs in R console is shown as follows.

```
> beta(1/2, 1/2)
[1] 3.141593
> beta(1/2, 3/2)
[1] 1.570796
> beta(11/3, 7/6)
[1] 0.1987252
> beta(1/10, 1/3)
[1] 12.4657
```

Now, like gamma function let us analyse graphical behaviour of beta function and its integrand $y = f(x) = x^{m-1}(1-x)^{n-1}$ for different values of m and n. You may refer to Fig. 8.4 (a) to (j). In beta function, range of integral is 0 to 1 so it will represent area of the region bounded by two vertical lines x = 0, x = 1 one horizontal line x-axis itself and the function $y = f(x) = x^{m-1}(1-x)^{n-1}$. This area is shaded by light blue colour in the Fig. 8.4 (a) to (j).

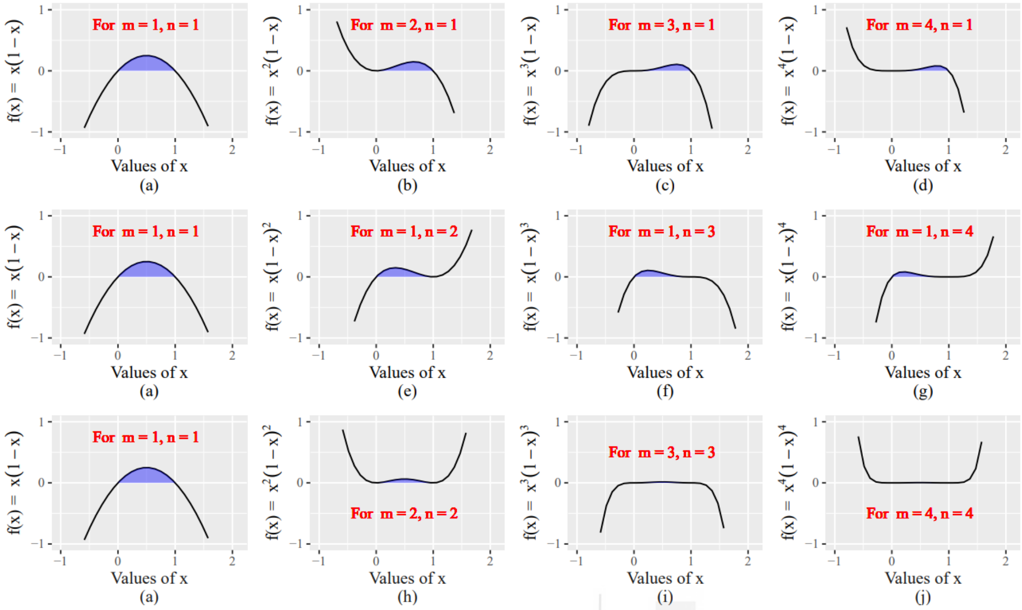

Fig. 8.4: Visualisation of the function $y = f(x) = x^{m-1}(1-x)^{n-1}$ by curve and $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ by shaded region for different values of m and n (a) m = 1, n = 1 (b) m = 2, n = 1 (c) m = 3, n = 1 (d) m = 4, n = 1 (e) m = 1, n = 2 (f) m = 1, n = 3 (g) m = 1, n = 4 (h) m = 2, n = 2 (i) m = 3, n = 3 (j) m = 4, n = 4

Graphical Behaviour of the Function $y = f(x) = x^n e^{-x}$ and its Integral known as Beta Function for different values of m and n

Before studying graphical behaviour of the function $y = f(x) = x^{m-1}(1-x)^{n-1}$ and its integral first of all we make a remark.

Remark 2: In Table 8.2 we shall use three properties of beta and gamma functions which will be discussed in the next Sec. 8.4. So, for the time being assume that they hold. These properties are: (1) $\overline{|1} = 1$ (2) $\overline{|n} = \underline{|n-1}$

(3) $B(m, n) = \dfrac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$.

Now, from the Fig. 8.4 (a) to (j) we note the following points about the graphical behaviour of the function $y = f(x) = x^{m-1}(1-x)^{n-1}$ and its integral which is known as beta function and is denoted by B(m, n).

- When m is odd then curve is left downward, refer to Fig. 8.4 (a), (c), (e), (f). (g) and (i). … (8.15)
- When m is even then curve is left upward, refer to Fig. 8.4 (b), (d), (h) and (j). … (8.16)
- When n is odd then curve is right downward, refer to Fig. 8.4 (a) to (d), (f) and (i). … (8.17)
- When n is even then curve is right upward, refer to Fig. 8.4 (e), (g), (h) and (j). … (8.18)
- Area under the curve decreases as we keep one of m and n fix and increase other. This point is explained as follows. … (8.19)

For example: (1) In Fig. 8.4 (a) to (d) n = 1 is fix and m increases so area under the curve decreases. To see numerical values of area, refer to column 6 and row 1 to row 4 of the Table 8.2.

(2) In Fig. 8.4 (a), (e) to (g) m = 1 is fix and n increases so area under the curve decreases. To see numerical values of area, refer to column 6 and row 1, row 5 to row 8 of the Table 8.2.

- Area under the curve decreases as both m and n increase. … (8.20)

For example, in Fig. 8.4 (a), (h) to (j) both m and n increase so area under the curve decreases. To see numerical values of area, refer to column 6 and row 1, row 8 to row 10 of the Table 8.2.

Table 8.2: Actual values of areas of shaded regions in Fig. 8.4 (a) to (j)

Value of m	Value of n	Figure number	Shaded area in terms of beta function	Shaded area in terms of gamma function using property 3	Actual shaded area	
1	1	8.4 (a)	B(1, 1)	$\frac{\Gamma 1\,\Gamma 1}{\Gamma(1+1)} = \frac{0!\,0!}{1!} = 1$	1	Decreasing ↓
2	1	8.4 (b)	B(2, 1)	$\frac{\Gamma 2\,\Gamma 1}{\Gamma(2+1)} = \frac{1!\,0!}{2!} = \frac{1}{2}$	0.5	
3	1	8.4 (c)	B(3, 1)	$\frac{\Gamma 3\,\Gamma 1}{\Gamma(3+1)} = \frac{2!\,0!}{3!} = \frac{1}{3}$	$0.\overline{3}$	
4	1	8.4 (d)	B(4, 1)	$\frac{\Gamma 4\,\Gamma 1}{\Gamma(4+1)} = \frac{3!\,0!}{4!} = \frac{1}{4}$	0.25	
1	2	8.4 (e)	B(1, 2)	$\frac{\Gamma 1\,\Gamma 2}{\Gamma(1+2)} = \frac{0!\,1!}{2!} = \frac{1}{2}$	0.5	Decreasing ↓
1	3	8.4 (f)	B(1, 3)	$\frac{\Gamma 1\,\Gamma 3}{\Gamma(1+3)} = \frac{0!\,2!}{3!} = \frac{1}{3}$	$0.\overline{3}$	
1	4	8.4 (g)	B(1, 4)	$\frac{\Gamma 1\,\Gamma 4}{\Gamma(1+4)} = \frac{0!\,3!}{4!} = \frac{1}{4}$	0.25	
2	2	8.4 (h)	B(2, 2)	$\frac{\Gamma 2\,\Gamma 2}{\Gamma(2+2)} = \frac{1!\,1!}{3!} = \frac{1}{6}$	$0.1\overline{6}$	Decreasing ↓
3	3	8.4 (i)	B(3, 3)	$\frac{\Gamma 3\,\Gamma 3}{\Gamma(3+3)} = \frac{2!\,2!}{5!} = \frac{1}{30}$	$0.0\overline{3}$	
4	4	8.4 (j)	B(4, 4)	$\frac{\Gamma 4\,\Gamma 4}{\Gamma(4+4)} = \frac{3!\,3!}{7!} = \frac{1}{140}$	$\simeq 0.007$	

In the next section you will study some properties of gamma and beta functions.

Now, you can try the following Self-Assessment Question.

SAQ 2

Without calculating values of B(4, 2) and B(5, 2) give reason whether B(4, 2) > B(5, 2) is true or false?

8.4 PROPERTIES OF GAMMA AND BETA FUNCTIONS

Properties of gamma function help us to find value of the gamma function $\overline{|n}$ at various values of n with some known values of it. First, we will list some properties of gamma function which are required in different courses of this programme and then we will discuss their proofs.

Property 1: If $n > 0$ then recurrence relation for gamma function is $\overline{|n+1} = n\overline{|n}$. … (8.21p1)

Property 2: Prove that $\overline{|1} = 1$. … (8.21p2)

Property 3: If n is a positive integer, then prove that $\overline{|n} = \underline{|n-1}$. … (8.21p3)

Property 4: For n > 0, a > 0, prove that $\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\overline{|n}}{a^n}$ … (8.21p4)

Property 5: **Relation between beta and gamma functions**: For $m > 0,\ n > 0$, prove that $B(m, n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$. … (8.21p5)

Remark 3: In the proof of this property, we need idea of double integral which is discussed in the next unit, i.e., Unit 9. So, we will prove it in the next unit. So, right now assume that this result holds.

Property 6: For $p > -1,\ q > -1$, prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\overline{\left|\frac{p+1}{2}\right.}\ \overline{\left|\frac{q+1}{2}\right.}}{2\,\overline{\left|\frac{p+q+2}{2}\right.}}$.
… (8.21p6)

Property 7: Prove that $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$. … (8.21p7)

Property 8: Prove that beta function is symmetric in m and n, i.e., prove that B(m, n) = B(n, m). … (8.21p8)

Property 9: For m > 0, n > 0, prove that $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ … (8.21p9.1)

Hence, prove that $B(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$. … (8.21p9.2)

and $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$. … (8.21p9.3)

Property 10: Prove that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. … (8.21p10)

Property 11: Prove that $\int_{-\infty}^{0} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. … (8.21p11)

Property 12: Prove that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. … (8.21p12)

Property 13: Prove that

(i) $B(p, q) = B(p+1, q) + B(p, q+1)$ ... (8.21p13.1)

(ii) $\dfrac{B(p, q+1)}{q} = \dfrac{B(p+1, q)}{p} = \dfrac{B(p, q)}{p+q}$ ... (8.21p13.2)

Property 14: **Duplication Formula**: For m > 0 prove that

$$\overline{|m}\,\overline{\left|m+\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{2^{2m-1}}\overline{|2m}. \qquad \dots (8.21p14)$$

Property 15: Prove that $\overline{\left|\frac{1}{4}\right.}\,\overline{\left|\frac{3}{4}\right.} = \sqrt{2}\,\pi.$ … (8.21p15)

Property 16: Prove that $\overline{|n}\,\overline{|1-n} = \dfrac{\pi}{\sin(n\pi)},\ n \notin \mathbb{Z}$ … (8.21p16)

Let us now prove these properties one at a time except property 16 whose proof is beyond the scope of this course.

Proof of Property 1: By definition of gamma function, we have

$$\overline{|n+1} = \int_0^\infty x^{n+1-1}e^{-x}dx,\ n \in (0, \infty)$$

$$= \int_0^\infty x^n e^{-x}dx$$

Integrating by parts keeping x^n as the first function and e^{-x} as the second function, we get

$$\overline{|n+1} = \left[x^n \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty n\,x^{n-1}\frac{e^{-x}}{-1}dx$$

$$= -(0-0) + n\int_0^\infty x^{n-1}e^{-x}dx \qquad \left[\begin{array}{l}\because \text{as } x \to \infty \text{ then } x^n \to \infty \text{ and } e^{-x} \to 0 \text{ but} \\ x^n e^{-x} \to 0 \text{ since } e^{-x} \text{ goes faster to zero} \\ \text{than } x^n \text{ goes to } \infty\end{array}\right]$$

$$= n\overline{|n} \qquad \left[\because \overline{|n} = \int_0^\infty x^{n-1}e^{-x}dx\right]$$

Hence, $\overline{|n+1} = n\overline{|n}$ … (8.22)

Proof of Property 2: We know that

$$\overline{|n} = \int_0^\infty x^{n-1}e^{-x}dx,\ n \in (0, \infty)$$

Replacing n by 1, we get

$$\overline{|1} = \int_0^\infty x^{1-1}e^{-x}dx = \int_0^\infty e^{-x}dx \quad [\because x^0 = 1]$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = -(0-1) = 1 \quad [\because e^{-x} \to 0 \text{ as } x \to \infty]$$

Hence, $\overline{|1} = 1.$ … (8.23)

Proof of Property 3: We know that (refer 8.22 and 8.23)

$$\overline{|n+1} = n\overline{|n} \quad \ldots (8.24)$$

$$\overline{|1} = 1 \quad \ldots (8.25)$$

Replacing n by n – 1 in (8.24), we get

$$\overline{|n} = (n-1)\overline{|n-1} \quad \ldots (8.26)$$

Replacing n by n – 1 in (8.26), we get

$$\overline{|n-1} = (n-2)\overline{|n-2} \quad \ldots (8.27)$$

Using (8.27) in RHS of (8.26), we get

$$\overline{|n} = (n-1)(n-2)\overline{|n-2} \quad \ldots (8.28)$$

Replacing n by n – 1 in (8.27), we get

$$\overline{|n-2} = (n-3)\overline{|n-3} \quad \ldots (8.29)$$

Using (8.29) in RHS of (8.28), we get

$$\overline{|n} = (n-1)(n-2)(n-3)\ \overline{|n-3} \quad \ldots (8.30)$$

Continuing in this way after n – 4 more steps, we get

$$\overline{|n} = (n-1)(n-2)(n-3)\ldots 3.2.1\,\overline{|1}$$

$$= (n-1)(n-2)(n-3)\ldots 3.2.1 \quad \left[\because \overline{|1} = 1 \text{ using } (8.25)\right]$$

$$= \underline{|n-1}$$

Proof of Property 4: Let $I = \int_0^\infty x^{n-1} e^{-ax} dx,\ n \in (0, \infty)$ … (8.31)

Let us put ax = y so that (8.31) reduces to the form of gamma function

Differentiating, we get a dx = dy

Also, when x = 0, then y = 0 and

when $x \to \infty \Rightarrow y \to \infty \quad [\because a > 0]$

$\therefore$ (8.31) becomes

$$I = \int_0^\infty \left(\frac{y}{a}\right)^{n-1} e^{-y} \frac{1}{a} dy = \frac{1}{a^n} \int_0^\infty y^{n-1} e^{-y} dy = \frac{1}{a^n}\overline{|n} \quad \left[\because \int_0^\infty x^{n-1} e^{-x} dx = \overline{|n}\right]$$

Hence, $\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\overline{|n}}{a^n}$. … (8.32)

Proof of Property 6: We know that $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0, n > 0$

… (8.33)

Having a look at the required relation, we have to put $x = \sin^2\theta$

Differentiating, we get $dx = 2\sin\theta\cos\theta\, d\theta$

Also, when x = 0, then $\theta = 0$ and

when $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore$ (8.33) becomes

$$B(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\; d\theta$$

So, we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\; d\theta = \frac{1}{2}B(m, n)$$

$$= \frac{1}{2}\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad [\text{Using } (8.21p5)] \qquad \ldots (8.34)$$

As per the need of the required form, we have to replace 2m – 1 by p and 2n – 1 by q in (8.34). After doing so we have

$$\int_0^{\pi/2} \sin^{p}\theta\cos^{q}\theta\; d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} \qquad \left[\begin{array}{l} \because 2m-1=p,\ 2n-1=q \\ \Rightarrow m=\dfrac{p+1}{2} \text{ and } n=\dfrac{q+1}{2} \\ \text{Also, } m>0, n>0 \Rightarrow p>-1, q>-1 \end{array}\right]$$

Hence, $$\int_0^{\pi/2} \sin^{p}\theta\cos^{q}\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)},\ p>-1,\ q>-1. \qquad \ldots (8.35)$$

Proof of Property 7: We know that

$$\int_0^{\pi/2} \sin^{p}\theta\cos^{q}\theta\; d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)},\ p>-1,\ q>-1 \qquad \ldots (8.36)$$

Putting p = 0, q = 0 in (8.36), we get

$$\int_0^{\pi/2} \sin^{0}\theta\cos^{0}\theta\; d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{0+0+2}{2}\right)} \Rightarrow \int_0^{\pi/2} 1\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{2}\right)} \Rightarrow [\theta]_0^{\pi/2} = \frac{1}{2}\frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(1)}$$

$$\Rightarrow \frac{\pi}{2} - 0 = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 \quad [\because \Gamma(1) = 1 \text{ using } (8.23)]$$

$$\Rightarrow \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \ldots (8.37)$$

Proof of Property 8: We know that

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m>0, n>0 \qquad \ldots (8.38)$$

Putting $x = 1 - y$ in (8.38),

Differentiating, we get $dx = -dy$

Also, when x = 0, then $y = 1$ and

when $x = 1 \Rightarrow y = 0$

$\therefore$ (8.38) becomes

$$B(m, n) = \int_1^0 (1-y)^{m-1} (1-(1-y))^{n-1} (-dy), \; m > 0, n > 0$$

$$= -\int_1^0 (1-y)^{m-1} y^{n-1} dy$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} dy \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \right]$$

$$= B(n, m)$$

Hence, B(m, n) = B(n, m) … (8.39)

Proof of Property 9: We know that

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \; m > 0, n > 0 \qquad \dots (8.40)$$

Putting $x = \frac{t}{1+t}$ in (8.40), … (8.41)

Differentiating, we get $dx = \frac{(1+t).dt - t(0+dt)}{(1+t)^2} \Rightarrow dx = \frac{1}{(1+t)^2} dt$

Also, (8.41) implies $t = \frac{x}{1-x}$ and so when x = 0, then $t = 0$ and when $x \to 1 \Rightarrow t \to \infty$

Further, $1 - x = 1 - \frac{t}{1+t} = \frac{1+t-t}{1+t} = \frac{1}{1+t}$, i.e., $1 - x = \frac{1}{1+t}$

$\therefore$ (8.40) becomes

$$B(m, n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1} \left(\frac{1}{1+t}\right)^{n-1} \frac{1}{(1+t)^2} dt, \; m > 0, n > 0$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt, \; m > 0, n > 0$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \qquad \left[\because \int_a^b f(u)\,du = \int_a^b f(v)\,dv \right]$$

Hence, $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx, \; m > 0, n > 0$ … (8.42)

So, using (8.42), we get $B(n, m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx, \; m > 0, n > 0$ … (8.43)

But B(m, n) = B(n, m) [using (8.39)]

Hence, $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx, \; m > 0, n > 0$ … (8.44)

Proof of Property 10: Let $I = \int_0^\infty e^{-x^2} dx$ … (8.45)

Putting $x = \sqrt{t}$ in (8.45),

Differentiating, we get $dx = \frac{1}{2\sqrt{t}} dt$

Also, when x = 0, then $t = 0$ and

when $x \to \infty \Rightarrow t \to \infty$

$\therefore$ (8.45) becomes

$$I = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}} dt = \frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/2}\, dt = \frac{1}{2}\int_0^\infty e^{-t}\, t^{1/2-1}\, dt = \frac{1}{2}\left|\!\overline{\frac{1}{2}}\right. \quad \left[\because \left|\!\overline{n}\right. = \int_0^\infty x^{n-1} e^{-x} dx,\ n > 0\right]$$

$$= \frac{1}{2}\sqrt{\pi} \qquad \left[\because \left|\!\overline{\frac{1}{2}}\right. = \sqrt{\pi} \text{ refer (8.37)}\right]$$

Hence, $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$ … (8.46)

Proof of Property 11: Let $I = \int_{-\infty}^{0} e^{-x^2} dx$ … (8.47)

Putting $x = -t$ in (8.47),

Differentiating, we get $dx = -dt$

Also, when x = 0, then $t = 0$ and

when $x \to -\infty \Rightarrow t \to \infty$

$\therefore$ (8.47) becomes

$$I = \int_\infty^0 e^{-t^2}(-dt) = -\int_\infty^0 e^{-t^2}\, dt = \int_0^\infty e^{-t^2}\, dt \quad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \frac{1}{2}\sqrt{\pi} \qquad \left[\because \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi} \text{ refer (8.46)}\right]$$

Hence, $\int_{-\infty}^{0} e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$ … (8.48)

Proof of Property 12:

$$\text{Let } I = \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{0} e^{-x^2} dx + \int_0^\infty e^{-x^2} dx \quad \left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$$

$$= \frac{1}{2}\sqrt{\pi} + \frac{1}{2}\sqrt{\pi} \qquad [\text{Using (8.46) and (8.48)}]$$

$$= \sqrt{\pi}$$

Hence, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ … (8.49)

Proof of Property 13: (i)

$$RHS = B(p+1, q) + B(p, q+1) = \frac{\left|\!\overline{p+1}\right.\left|\!\overline{q}\right.}{\left|\!\overline{p+q+1}\right.} + \frac{\left|\!\overline{p}\right.\left|\!\overline{q+1}\right.}{\left|\!\overline{p+q+1}\right.} \quad [\text{Using (8.21p5)}]$$

$$= \frac{p\left|\!\overline{p}\right.\left|\!\overline{q}\right.}{(p+q)\left|\!\overline{p+q}\right.} + \frac{\left|\!\overline{p}\right. q\left|\!\overline{q}\right.}{(p+q)\left|\!\overline{p+q}\right.} \qquad [\text{Using (8.22)}]$$

$$= \frac{(p+q)\Gamma p \Gamma q}{(p+q)\Gamma(p+q)} = \frac{\Gamma p \Gamma q}{\Gamma(p+q)} = B(p,\, q) = \text{LHS} \qquad [\text{Using } (8.21p5)]$$

(ii) $$\frac{B(p,\, q+1)}{q} = \frac{\Gamma p\, \Gamma(q+1)}{q\,\Gamma(p+q+1)} \qquad [\text{Using } (8.21p5)]$$

$$= \frac{\Gamma p\, q\, \Gamma q}{q\,(p+q)\Gamma(p+q)} \qquad [\text{Using } (8.22)]$$

$$= \frac{\Gamma p\, \Gamma q}{(p+q)\Gamma(p+q)} = \frac{B(p,\, q)}{p+q} \qquad \ldots (8.50)$$

Also,

$$\frac{B(p+1,\, q)}{p} = \frac{\Gamma(p+1)\Gamma q}{p\,\Gamma(p+q+1)} \qquad [\text{Using } (8.21p5)]$$

$$= \frac{p\,\Gamma p\, \Gamma q}{p\,(p+q)\Gamma(p+q)} \qquad [\text{Using } (8.22)]$$

$$= \frac{\Gamma p\, \Gamma q}{(p+q)\Gamma(p+q)} = \frac{B(p,\, q)}{p+q} \qquad \ldots (8.51)$$

From (8.50) and (8.51), we get $\dfrac{B(p,\, q+1)}{q} = \dfrac{B(p+1,\, q)}{p} = \dfrac{B(p,\, q)}{p+q}$ … (8.52)

Proof of Property 14: **Proof of Duplication Formula**: We know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)},\ p > -1,\ q > -1 \qquad \ldots (8.53)$$

Putting p = q in (8.53), we get

$$\int_0^{\pi/2} \sin^p\theta \cos^p\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{2\,\Gamma\left(\frac{p+p+2}{2}\right)} \Rightarrow \frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{2\,\Gamma(p+1)} = \int_0^{\pi/2} \left(\frac{2\sin\theta\cos\theta}{2}\right)^p d\theta$$

$$\Rightarrow \frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{2\,\Gamma(p+1)} = \frac{1}{2^p}\int_0^{\pi/2} (\sin 2\theta)^p\, d\theta \qquad \ldots (8.54)$$

Putting $2\theta = t$ in (8.54),

Differentiating, we get $2\,d\theta = dt$

Also, when $\theta = 0$, then $t = 0$ and

when $\theta = \dfrac{\pi}{2} \Rightarrow t \to \pi$

$\therefore$ (8.54) becomes

$$\frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{2\,\Gamma(p+1)} = \frac{1}{2^p}\int_0^{\pi} (\sin t)^p \frac{1}{2}\, dt = \frac{1}{2}\frac{1}{2^p}\int_0^{\pi} \sin^p t\, dt$$

$$= \frac{1}{2}\frac{1}{2^p}\, 2\int_0^{\pi/2} \sin^p t\,dt \quad \left[\begin{array}{l} \because \text{ If } f(2a-x)=f(x), \text{ then } \displaystyle\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \\ \text{Here, } a=\pi/2,\ f(t)=\sin^p t, \text{ and} \\ f(\pi-t)=\sin^p(\pi-t)=\sin^p t = f(t) \end{array}\right]$$

$$= \frac{1}{2^p}\int_0^{\pi/2} \sin^p t \cos^0 t\,dt$$

$$= \frac{1}{2^p}\, \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{p+0+2}{2}\right)} \qquad [\text{Using } (8.53)]$$

$$\Rightarrow \frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{\Gamma(p+1)} = \frac{1}{2^p}\, \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+0+2}{2}\right)}$$

$$\Rightarrow \frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)}{\Gamma(p+1)} = \frac{1}{2^p}\, \frac{\sqrt{\pi}}{\Gamma\left(\frac{p+2}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\ [\text{Using } (8.37)]\right] \qquad \ldots (8.55)$$

Putting $\frac{p+1}{2} = m$ in (8.55), we get

$$\Rightarrow \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\, \frac{\sqrt{\pi}}{\Gamma\left(\frac{2m+1}{2}\right)} \qquad \left[\because \frac{p+1}{2} = m \Rightarrow p+1 = 2m \Rightarrow \frac{p+2}{2} = \frac{2m+1}{2}\right]$$

$$\Rightarrow \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m) \qquad \ldots (8.56)$$

Hence, proved.

Proof of Property 15: We know that (refer (8.56))

$$\Rightarrow \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m) \qquad \ldots (8.57)$$

Putting $m = \frac{1}{4}$ in (8.57), we get

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{1/2-1}}\,\Gamma\left(2\left(\frac{1}{4}\right)\right) \Rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}}{2^{-1/2}}\,\Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\sqrt{\pi}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\ [\text{Using } (8.37)]\right]$$

$$= \sqrt{2}\,\pi$$

Hence, $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$ … (8.58)

Proof of Property 16: Beyond the scope of this course.

Example 1: Evaluate $\int_0^\infty \frac{x^3(1+x^4)}{(1+x)^{12}}dt$ using properties of gamma and beta functions.

Solution:

$$\int_0^\infty \frac{x^3(1+x^4)}{(1+x)^{12}}dx = \int_0^\infty \frac{x^3}{(1+x)^{12}}dx + \int_0^\infty \frac{x^7}{(1+x)^{12}}dx = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+8}}dx + \int_0^\infty \frac{x^{8-1}}{(1+x)^{8+4}}dx$$

$$= B(4, 8) + B(8, 4) \quad \left[\because B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx\right]$$

$$= 2B(4, 8) \qquad [\because B(m, n) = B(n, m)]$$

$$= 2\frac{\overline{|4}\,\overline{|8}}{\overline{|4+8}} \qquad \left[\because B(m, n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}\right]$$

$$= 2\frac{\underline{|3}\,\underline{|7}}{\underline{|11}} = 2\frac{6\underline{|7}}{11.10.9.8.\underline{|7}} = \frac{1}{660}$$

Now, you can try the following Self-Assessment Question.

SAQ 3

Evaluate $\int_0^\infty x^4 e^{-2x}dx$ using properties of beta and gamma function.

8.5 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- **Factorial function** is defined as $\underline{|n} = n(n-1)(n-2)(n-3)\ldots3.2.1.$ In R it can be obtained by using the built-in function factorial(n).
- **Gamma function** is defined as $\overline{|n} = \int_0^\infty x^{n-1}e^{-x}dx,\ n > 0.$ In R it can be obtained by using the built-in function gamma(n).
- **Beta function** is defined as $\int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m > 0, n > 0$ In R it can be obtained by using the built-in function beta(m, n).
- **Some properties of gamma and beta** functions are:
  - $\overline{|1} = 1$
  - $\overline{|n+1} = n\overline{|n}$
  - $\overline{|n} = \underline{|n-1}$
  - $\int_0^\infty x^{n-1}e^{-ax}dx = \frac{\overline{|n}}{a^n}$
  - $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$
  - $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$
  - $\int_{-\infty}^0 e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$
  - $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$
  - $\overline{\left|\frac{1}{4}\right.}\,\overline{\left|\frac{3}{4}\right.} = \sqrt{2}\,\pi$
  - $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

- $B(n, m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$
- $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$
- $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\left|\frac{p+1}{2}\right.\left|\frac{q+1}{2}\right.}{2\left|\frac{p+q+2}{2}\right.}$
- $B(p, q) = B(p+1, q) + B(p, q+1)$
- $\frac{B(p, q+1)}{q} = \frac{B(p+1, q)}{p} = \frac{B(p, q)}{p+q}$
- B(m, n) = B(n, m) (This is known as symmetric property of beta function)
- $B(m, n) = \frac{\left|m\right.\left|n\right.}{\left|m+n\right.}$ (This is known as the relation between gamma and beta functions)
- $\left|m\right.\left|m+\frac{1}{2}\right. = \frac{\sqrt{\pi}}{2^{2m-1}}\left|2m\right.$ (This is known as duplication formula)
- $\left|n\right.\left|1-n\right. = \frac{\pi}{\sin(n\pi)}, \; n \notin \mathbb{Z}$ (This is known as **Euler Reflection formula**)

8.6 TERMINAL QUESTIONS

1. Express the given integral in beta function:
   $\int_0^b x^{p-1}(b-x)^{q-1} dx, \; p > 0, q > 0.$
2. Express the given integral in beta function:
   $\int_0^1 x^{p-1}(1-x^a)^{q-1} dx, \; p > 0, q > 0.$
3. Evaluate using properties of beta and gamma functions: $\int_0^\infty \frac{1}{1+x^4} dx.$

8.7 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. Answer is no, since $\left|n\right. = \underline{|n-1}$, so $\left|8\right. = \underline{|7}$.
2. We know that area under the curve of beta function decreases as we keep one of m and n fix and increase other. Here value of n is fix and equal to 2 but value m varies. Hence, B(4, 2) < B(5, 2) is false because 5 > 4 so value of B(4, 2) will be greater than value of B(5, 2).
3. We know that $\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\left|n\right.}{a^n}$

In our case n = 5, a = 2, so

$$\int_0^\infty x^4 e^{-2x}\,dx = \int_0^\infty x^{5-1} e^{-2x}\,dx = \frac{\Gamma 5}{2^5} = \frac{\lfloor 4}{32} = \frac{24}{32} = \frac{3}{4}$$

Terminal Questions

1. Let $I = \int_0^b x^{p-1}(b-x)^{q-1}\,dx,\ p > 0, q > 0$ … (8.59)

Putting x = bt

Differentiating, we get $dx = b\,dt$

Also, when $x = 0$, then $t = 0$ and

when $x = b \Rightarrow t = 1$

$\therefore$ (8.59) becomes

$$I = \int_0^1 (bt)^{p-1}(b-bt)^{q-1}\,b\,dt = \int_0^1 b^{p-1}t^{p-1}b^{q-1}(1-t)^{q-1}\,b\,dt$$

$$= b^{p+q-1}\int_0^1 t^{p-1}(1-t)^{q-1}\,dt,\ p > 0,\ q > 0$$

$$= b^{p+q-1}B(p,\ q)\quad \left[\because B(m,\ n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m > 0,\ n > 0\right]$$

2. Let $I = \int_0^1 x^{p-1}(1-x^a)^{q-1}\,dx,\ p > 0, q > 0$ … (8.60)

Putting $x^a = t \Rightarrow x = t^{1/a}$

Differentiating, we get $dx = \frac{1}{a}t^{a-1}\,dt$

Also, when $x = 0$, then $t = 0$ and

when $x = 1 \Rightarrow t = 1$

$\therefore$ (8.60) becomes

$$I = \int_0^1 \left(t^{1/a}\right)^{p-1}(1-t)^{q-1}\frac{1}{a}t^{1/a-1}\,dt = \frac{1}{a}\int_0^1 t^{\frac{1}{a}(p-1+1)-1}\ (1-t)^{q-1}\,dt$$

$$= \frac{1}{a}\int_0^1 t^{\frac{p}{a}-1}\ (1-t)^{q-1}\,dt,\ p > 0,\ q > 0$$

$$= \frac{1}{a}B\left(\frac{p}{a},\ q\right)\quad \left[\because B(m,\ n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m > 0,\ n > 0\right]$$

3. Let $I = \int_0^\infty \frac{1}{1+x^4}\,dx$ … (8.61)

Putting $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$

Differentiating, we get $dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta$

Also, when $x = 0$, then $\theta = 0$ and when $x \to \infty \Rightarrow \theta \to \frac{\pi}{2}$

$\therefore$ (8.61) becomes

$$I = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta}\,\frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sec^2\theta}\,\frac{1}{\sqrt{\tan\theta}}\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}}\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\,d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\cos^{1/2}\theta\,d\theta$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{-1/2+1}{2}\right)\Gamma\left(\frac{1/2+1}{2}\right)}{2\Gamma\left(\frac{-1/2+1/2+2}{2}\right)} \quad \left[\because \int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)},\ p > -1, q > -1\right]$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) \quad \left[\because \Gamma(1) = 1\right]$$

$$= \frac{1}{4}\,\frac{\pi}{\sin(\pi/4)} \qquad \left[\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)},\ n \notin \mathbb{Z}\right]$$

$$= \frac{\sqrt{2}\,\pi}{4} \qquad \left[\because \sin(\pi/4) = \frac{1}{\sqrt{2}}\right]$$

UNIT 9

CHANGE OF ORDER OF SUMMATION AND INTEGRATION

Structure

9.1 Introduction

Expected Learning Outcomes

9.2 Change of Origin of Summation

9.3 Change of Order of Summation

9.4 Double Integration

9.5 Change of Order of Integration

9.6 Summary

9.7 Terminal Questions

9.8 Solutions/Answers

9.1 INTRODUCTION

From earlier classes you are familiar with the summation notation, e.g., $\sum_{i=1}^{n} x_i$ which is a short notation used for the expression $x_1 + x_2 + x_3 + \ldots + x_n$. Instead of the sum of finite terms if we have a sum of infinite but countable terms, for example, $x_1 + x_2 + x_3 + \ldots$ then in summation notation, it is written as $\sum_{n=1}^{\infty} x_n$. In Sec. 9.2 we will discuss rules to change the origin of the sigma.

After explaining the rules for changing the origin of the sigma in Sec. 9.2, we will explain the meaning of double summation notation in Sec. 9.3. In the same section we will also discuss rules to change the order of two summations. From the discussion of Sec. 5.2 of Unit 5 of this course you know that in the continuous world how the role of summation is played by integration. Similarly, the role of double summation in a continuous world will be played by double integral which is introduced in Sec. 9.4 and the change of order of integration is discussed in Sec. 9.5.

What we have discussed in this unit is summarised in Sec. 9.6. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit some more questions based on the entire unit are given in Sec. 9.7 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 9.8.

The next course will make your entry into the world of uncertainty which is measured by probability. So, the next whole course is devoted to discuss some tools and techniques of the world of probability.

Expected Learning Outcomes

After completing this unit, you should be able to:

- change the origin of the summation;
- change the order of two summations; and
- evaluate double integral and change the order of integration in double integral.

9.2 CHANGE OF ORIGIN OF SUMMATION

In Unit 4 of this course, you have studied sequences and series. Recall that if you have a sequence $a_1, a_2, a_3, a_4, \ldots$ to ∞ then the expression obtained by joining these terms by plus sign, i.e., $a_1 + a_2 + a_3 + a_4 + \ldots$ to ∞ is known as a series and is denoted by $\sum_{n=1}^{\infty} a_n$ using the symbol Σ which is a Greek letter pronounced as **sigma**. Since the notation $\sum_{n=1}^{\infty} a_n$ is used to sum the terms $a_1, a_2, a_3, a_4, \ldots$ to ∞ so it is also known as **summation**. Let us explain some terms related to summation notation as follows. … (9.1)

- The variable n used in the subscript of a is known as the **dummy variable** or **index** of the summation. There is nothing special in n you can use any other dummy variable like k or m or p or r, etc. … (9.2)
- The notation n = 1 written at the bottom of the symbol Σ tells us about the initial or first term of the series. … (9.3)
- The symbol ∞ written at the top of the symbol Σ tells us up to where we have to run the index or dummy variable used in the summation. (9.4)
- The expression a_n written in front of the symbol Σ tells us where you have to replace the dummy variable or index n from 1 to ∞. … (9.5)
- In the notation $\sum_{n=1}^{\infty} a_n$ dummy variable n starts from 1 and goes up to infinity.

  For example,

  - In the notation $\sum_{n=5}^{100} a_n$ dummy variable n starts from 5 and goes up to 100.
  - In the notation $\sum_{n=0}^{99} a_n$ dummy variable n starts from 0 and goes up to 99.

If we call the bottom portion used in the summation notation as **origin of the**

summation then origin of the summation $\sum_{n=0}^{99} a_n$ is at 0 while origin of the summation $\sum_{n=5}^{100} a_n$ is at 5 and origin of the summation $\sum_{n=m}^{100} a_n$ is at m. … (9.6)

To use the notation $\sum_{n=1}^{\infty} a_n$ for the expression $a_1 + a_2 + a_3 + a_4 + \ldots$ to ∞ is convenient and concise. The shorthand or concise notation $\sum_{n=1}^{\infty} a_n$ for the expression $a_1 + a_2 + a_3 + a_4 + \ldots$ to ∞ is known as **sigma notation** or **summation notation**. … (9.7)

Let us now learn how to change origin of the summation. We will discuss it by considering two common situations where the number of terms is infinite and finite.

Change of Origin when we are dealing with Infinite Terms

In the notation $\sum_{n=1}^{\infty} a_n$ we say that the origin of the summation is at n = 1. (9.8)

We can **change this origin** at any other number of our interest without affecting the sum of the terms given by it. Suppose we want to change the origin at m then it can be done by simply doing two things simultaneously:

(1) Identity how much you have to add or subtract from a given origin to get a new origin. Here given origin is at 1. So, to get m (new origin) we have to add m – 1 in the given origin. … (9.9)

(2) Subtract the number m – 1 obtained in the first step from n everywhere in the expression inside the summation. That is just replace n by n – (m – 1) in the expression inside the summation or in the expression written in front of the sigma symbol. … (9.10)

After doing both the steps, we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+(m-1)}^{\infty} a_{n-(m-1)} = \sum_{n=m}^{\infty} a_{n-m+1}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=m}^{\infty} a_{n-m+1} \quad \ldots (9.11)$$

In left hand side of (9.11) origin of the summation is at n = 1, while in right hand side of (9.11) origin of the summation is at n = m. When you will open these summations, you will get the same expression.

For example, suppose given expression is $\frac{xy^2}{3p^3} + \frac{x^2y^4}{3^2p^4} + \frac{x^3y^6}{3^3p^5} + \frac{x^4y^8}{3^4p^6} + \ldots$ then in summation form it can be written as $\sum_{n=1}^{\infty} \frac{x^n y^{2n}}{3^n p^{n+2}}$. … (9.12)

Origin of the summation in (9.12) is at n = 1. Suppose you want to change it at n = 5, then it can be done as follows. (to make 1 as 5 we need to add 4 in it)

$$\therefore \sum_{n=1}^{\infty} \frac{x^n y^{2n}}{3^n p^{n+2}} = \sum_{n=1+4}^{\infty} \frac{x^{n-4} y^{2(n-4)}}{3^{n-4} p^{(n-4)+2}} = \sum_{n=5}^{\infty} \frac{x^{n-4} y^{2n-8}}{3^{n-4} p^{n-2}}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n y^{2n}}{3^n p^{n+2}} = \sum_{n=5}^{\infty} \frac{x^{n-4} y^{2n-8}}{3^{n-4} p^{n-2}} \qquad \ldots (9.13)$$

In the left hand side of (9.13) origin of the summation is at n = 1, while in the right hand side of (9.13) origin of the summation is at n = 5. When you will open these summations, you will get the same expression.

Change of Origin when we are dealing with Finite Terms

In the notation $\sum_{k=1}^{n} a_k$ we say that origin of the summation is at k = 1. … (9.14)

We can **change this origin** at any other number of our interest without affecting the sum of the terms given by it. Suppose we want to change origin at m then it can be done by simply doing three things simultaneously:

(1) Identity how much you have to add or subtract from given origin to get new origin. Here given origin is at k = 1. So, to get m (new origin) we have to add m – 1 in the given origin. … (9.15)

(2) Subtract the number m – 1 obtained in the first step from k everywhere in the expression inside the summation. That is just replace k by k – (m – 1) in the expression inside the summation or in the expression written in front of the sigma symbol. … (9.16)

(3) Add the number m – 1 obtained in the first step in the number n which is written at the top of the summation to change the range of the summation from k = 1 to k = n, to k = m to k = n + (m – 1). … (9.17)

After doing all three steps, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1+(m-1)}^{n+m-1} a_{k-(m-1)} = \sum_{k=m}^{n+m-1} a_{k-m+1}$$

$$\Rightarrow \sum_{k=1}^{n} a_k = \sum_{k=m}^{n+m-1} a_{k-m+1} \qquad \ldots (9.18)$$

In the left hand side of (9.18) origin of the summation is at k = 1, and range of the summation is from k = 1 to k = n, while in the right hand side of (9.18) origin of the summation is at k = m and range of the summation is from k = m to k = n + m – 1. When you will open these summations, you will get the same expression.

For example, suppose given expression is

$\frac{xy^2}{3p^3} + \frac{x^2y^4}{3^2p^4} + \frac{x^3y^6}{3^3p^5} + \frac{x^4y^8}{3^4p^6} + \ldots + \frac{x^{10}y^{20}}{3^{10}p^{12}}$ then in summation form it can be written as $\sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}}$. … (9.19)

Origin of the summation in (9.19) is at k = 1 and range of the summation is from k = 1 to k = 10.

(i) Suppose you want to change the origin in (9.19) at k = 5 instead of k = 1, then it can be done as follows. (to make 1 as 5 we need to add 4 in it)

$$\therefore \sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}} = \sum_{k=1+4}^{10+4} \frac{x^{k-4} y^{2(k-4)}}{3^{k-4} p^{(k-4)+2}} = \sum_{n=5}^{14} \frac{x^{k-4} y^{2k-8}}{3^{k-4} p^{k-2}}$$

$$\Rightarrow \sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}} = \sum_{n=5}^{14} \frac{x^{k-4} y^{2k-8}}{3^{k-4} p^{k-2}} \qquad \ldots (9.20)$$

In the left hand side of (9.20) origin of the summation is at k = 1 and range of the summation is from k = 1 to k = 10, while in the right hand side of (9.20) origin of the summation is at k = 5 and range of the summation is from k = 5 to k = 14. When you will open these summations, you will get the same expression.

(ii) Suppose you want to change the origin in (9.19) at k = 0 instead of at k = 1, then it can be done as follows. (to make 1 as 0 we need to subtract 1 from it)

$$\therefore \sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}} = \sum_{k=1-1}^{10-1} \frac{x^{k+1} y^{2(k+1)}}{3^{k+1} p^{(k+1)+2}} = \sum_{n=0}^{9} \frac{x^{k+1} y^{2k+2}}{3^{k+1} p^{k+3}}$$

$$\Rightarrow \sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}} = \sum_{n=0}^{9} \frac{x^{k+1} y^{2k+2}}{3^{k+1} p^{k+3}} \qquad \ldots (9.21)$$

In the left hand side of (9.21) origin of the summation is at k = 1 and range of the summation is from k = 1 to k = 10, while in the right hand side of (9.21) origin of the summation is at k = 0 and range of the summation is from k = 0 to k = 9. When you will open these summations, you will get the same expression.

Visualisation of the Range of Summation: It is summarised in the following three points and visualised in Fig. 9.1 (a), (b) and (c).

- Range of the dummy variable k used in (9.19) is shown in Fig. 9.1 (a) before changing the origin.
- Range of the dummy variable k after changing the origin at k = 5 refer RHS of (9.20) is shown in Fig. 9.1 (b).
- Range of the dummy variable k after changing the origin at k = 0 is shown in Fig. 9.1 (c).

Note that when we changed the origin from k = 1 to k = 5 then range of the dummy variable shifts 4 (= 5 – 1) points towards right. When we changed the origin from k = 1 to k = 0 then range of the dummy variable shifts 1 (0 – 1 = – 1) points towards left.

One more question that may arise in your mind is what happens if the expression inside the sigma notation is independent of the dummy variable (index). In this case we will get the same expression adding as many times as the number of terms in the corresponding series. For example,

(i) $$\sum_{k=1}^{10} 3 = \underbrace{3+3+3+\ldots+3}_{10 \text{ times}} = 3 \times 10 = 30 \qquad \ldots (9.22)$$

(ii) $\sum_{k=0}^{10} 3 = \underbrace{3+3+3+\ldots+3}_{11 \text{ times}} = 3 \times 11 = 33$ ... (9.23)

(iii) $\sum_{k=m}^{n} a = \underbrace{a+a+a+\ldots+a}_{(n-m+1) \text{ times}} = a \times (n-m+1) = (n-m+1)a$ ... (9.24)

(iv) $\sum_{k=1}^{10} a = \underbrace{a+a+a+\ldots+a}_{10 \text{ times}} = 10a$ ... (9.25)

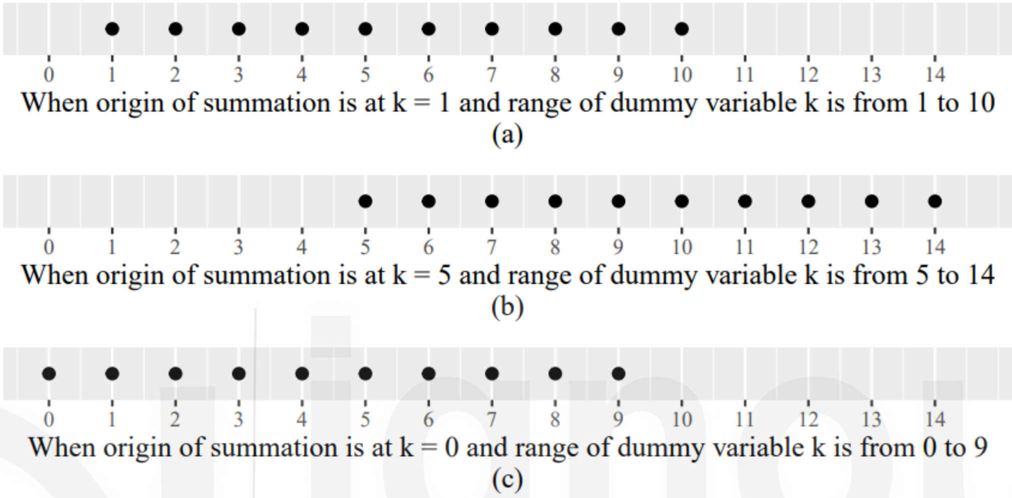

Fig. 9.1: Visualisation of the range of the dummy variable (a) when origin is at k = 1 (b) after changing origin at k = 5 (c) after changing origin at k = 0

Now, you can try the following Self-Assessment Question.

SAQ 1

(a) Choose the correct option for the notation $\sum_{k=1}^{3} k^5$

(A) $1^5 + 3^5$ (B) $(1+3)^5$ (C) $1^5 + 2^5 + 3^5$ (D) $(1+2+3)^5$

(b) Choose the correct option for the notation $\sum_{k=10}^{100} 7$

(A) 770 (B) 630 (C) 700 (D) 637

9.3 CHANGE OF ORDER OF SUMMATION

In the previous section you have seen that if in a given expression subscript or super subscript of only one dummy variable is changing term to term then that expression can be written in short using a single sigma notation. But if instead of one, there are two dummy variables which are changing term to term such expressions can be written in concise form using double sigma notation. For example, let us consider an excel sheet where we have typed collected data on n Variables (V1, V2, V3, …, Vn) and on m Subjects (S1, S2, S3, …, Sm). Excel sheet, in general, for such a data set will look like as given in (9.26) with the help of a screenshot.

	A	B	C	D	E	F
1		**V1**	**V2**	**V3**	**…**	**Vn**
2	**S1**	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	…	$a_{1,n}$
3	**S2**	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	…	$a_{2,n}$
4	**S3**	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	…	$a_{3,n}$
5	⋮	⋮	⋮	⋮	⋱	⋮
6	**Sm**	$a_{m,1}$	$a_{m,2}$	$a_{m,3}$	…	$a_{m,n}$

… (9.26)

This image matches with the look of a matrix of order m by n which you have studied in earlier classes. If you write the same information using a matrix then it will look like as shown in (9.27) given as follows.

$$\begin{array}{ll} & \overbrace{\begin{array}{ccccc} V1 & V2 & V3 & & Vn \end{array}}^{\text{Name of the Variable}} \\ & \begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \cdots & \downarrow \end{array} \\ \text{Subject number} \left[\begin{array}{c} S1 \rightarrow \\ S2 \rightarrow \\ S3 \rightarrow \\ \vdots \\ Sm \rightarrow \end{array}\right. & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array} \quad \ldots (9.27)$$

Suppose all the Variables have numerical values which can be added. Suppose you want to add all the mn elements of the excel sheet shown in the form of an image given by (9.26) or in the matrix given by 9.27. This job can be done in many ways. But we consider two convenient and common ways of doing this.

(1) First add all the elements of the first row, second row, third row and so on m^{th} row. After that add these m sums of the m rows. If we write this way of getting sum of all mn elements using sigma notation then it can be written as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m}\left(a_{i1} + a_{i2} + a_{i3} + \cdots + a_{in}\right)$$

$$= \left(a_{11} + a_{12} + a_{13} + \cdots + a_{1n}\right) + \left(a_{21} + a_{22} + a_{23} + \cdots + a_{2n}\right) + \cdots + \left(a_{m1} + a_{m2} + a_{m3} + \cdots + a_{mn}\right)$$

$$= \left(\text{sum of all elements of } 1^{st} \text{ Row corresponding to Subject 1}\right)$$
$$+ \left(\text{sum of all elements of } 2^{nd} \text{ Row corresponding to Subject 2}\right)$$
$$+ \left(\text{sum of all elements of } 3^{rd} \text{ Row corresponding to Subject 3}\right) + \cdots$$
$$+ \left(\text{sum of all elements of } m^{th} \text{ Row corresponding to Subject m}\right)$$

$$= S_1^{sum} + S_2^{sum} + S_3^{sum} + \cdots + S_m^{sum}, \text{ where } S_i^{sum} \text{ denotes the sum of all elements of } i^{th} \text{ row which is corresponding to Subject i}$$

$$= \sum_{i=1}^{m} S_i^{sum} \quad \ldots (9.28)$$

(2) First add all the elements of the first column, second column, third column and so on n^th column. After that add these n sums of the n columns. If we write this way of getting sum of all mn elements using sigma notation then it can be written as:

$$\sum_{j=1}^{n}\sum_{i=1}^{m} a_{i\,j} = \sum_{j=1}^{n}\left(a_{1\,j}+a_{2\,j}+a_{3\,j}+\cdots+a_{m\,j}\right)$$

$$=\left(a_{11}+a_{21}+a_{31}+\cdots+a_{m1}\right)+\left(a_{12}+a_{22}+a_{32}+\cdots+a_{m2}\right)+\cdots+\left(a_{1n}+a_{2n}+a_{3n}+\cdots+a_{mn}\right)$$

$$=\begin{pmatrix}\text{sum of all elements of } 1^{st} \text{ Column which is}\\ \text{corresponding to Variable 1}\end{pmatrix}$$
$$+\begin{pmatrix}\text{sum of all elements of } 2^{nd} \text{ Column which is}\\ \text{corresponding to Variable 2}\end{pmatrix}$$
$$+\begin{pmatrix}\text{sum of all elements of } 3^{rd} \text{ Column which is}\\ \text{corresponding to Variable 3}\end{pmatrix}+\cdots$$
$$+\begin{pmatrix}\text{sum of all elements of } n^{th} \text{ Column which is}\\ \text{corresponding to Variable n}\end{pmatrix}$$

$= V_1^{sum}+V_2^{sum}+V_3^{sum}+\cdots+V_n^{sum}$, where V_i^{sum} denotes the sum of all elements of i^{th} Column which is corresponding to Variable i

$$=\sum_{j=1}^{n} V_j^{sum} \qquad \text{... (9.29)}$$

Equations (9.28) and (9.29) both represents sum of all elements of the excel sheet or matrix given by (9.26) or (9.27) respectively. So,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i\,j} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{i\,j} \qquad \text{… (9.30)}$$

This is nothing but known as **change of order of sigmas/summations**. Note that here **range of two dummy variables i and j are constant or independent of each other and the region formed by the points of intersection of the ranges of two dummy variables is a rectangular region**. For particular values 5 and 4 of m and n respectively corresponding region formed by the points of intersection of the ranges of two dummy variables is shown in Fig. 9.2 (a) which is obviously a rectangular region. The 4 rows of this region represent 4 Variables (V1, V2, V3, V4) of the excel sheet given by (9.26) or 4 columns (C_1, C_2, C_3, C_4) of the matrix given by (9.27). While 5 columns of this region represent 5 Subjects (S1, S2, S3, S4, S5) of the excel sheet given by (9.26) or 5 rows $(R_1, R_2, R_3, R_4, R_5)$ of the matrix given by (9.27). Similarly, if values of m and n are infinite then the corresponding region formed by the points of intersection of the ranges of the two dummy variables will be still rectangular in shape as shown in Fig. 9.2 (b). In this case equation (9.30) becomes

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i\,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i\,j} \qquad \text{… (9.31)}$$

So, moral of the story is if **ranges of two dummy variables are constant or independent of each other, then corresponding region formed by the points of intersection of the ranges of two dummy variables will be rectangular in shape** does not matter whether their ranges are finite or infinite (refer Fig. 9.2 (a) and (b)) and in this case, you can **change the order of two sigmas without any modification in the ranges of the two dummy variables** and in the expressions inside the sigma's as we did in equation (9.30) and (9.31). … (9.32)

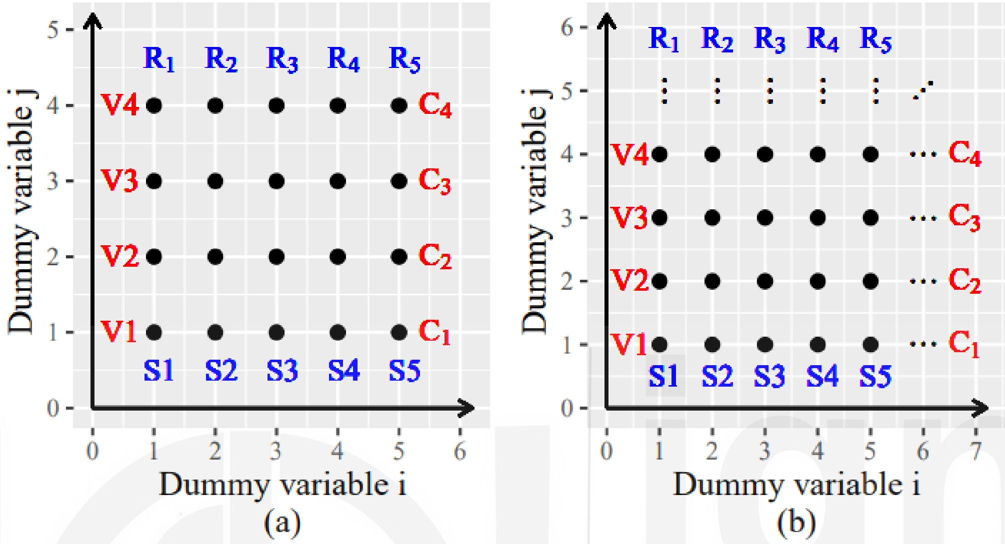

Fig. 9.2: Visualisation of the region formed by the points of intersection of the ranges of two dummy variables i and j (a) range of both i and j is finite, i.e., range of i is 1 to 5 and range of j is 1 to 4 (b) range of both i and j is infinite, i.e., 1 to infinity

So, far in this section we have discussed change of order of summation when ranges of two dummy variables were constant or independent of each other. Now, we will discuss the issue of change of order of two sigma's when limit(s) of one dummy variable either lower or upper or both depends on the other dummy variable. That is either limit(s) of i is(are) in terms of j or limit(s) of j is(are) in terms of i. This is discussed as follows.

Suppose instead of all elements of the excel sheet given by (9.26) or matrix given by (9.27), you want to find out sum of only those elements where row suffix ≥ column suffix.

To handle this problem let us first present the dummy variable which represents row suffix on horizontal axis and the dummy variable which represents column suffix on vertical axis. After doing so the point having coordinates (i, j) will represent the suffixes of the element a_{ij} refer Fig. 9.3 (a) where we have assumed m = 5 and n = 4. In excel sheet we type data when both m and n are finite but sometimes, we also have to deal with the summations where both m and n are infinite such a situation is shown in Fig. 9.3 (b). In fact, after doing so following are some connections between excel sheet or matrix presentation of data and presentation of positions of data in Fig. 9.3.

- first column of excel sheet given by (9.26) or matrix given by (9.27) will lie along the bottom row which is shown by V1 in red colour in Fig. 9.3;

- second column of excel sheet given by (9.26) or matrix given by (9.27) from its second element onward will lie along second row from the bottom which is shown by V2 in red colour in Fig. 9.3;
- third column of excel sheet given by (9.26) or matrix given by (9.27) from its third element onward will lie along third row from the bottom which is shown by V3 in red colour in Fig. 9.3, and so on.

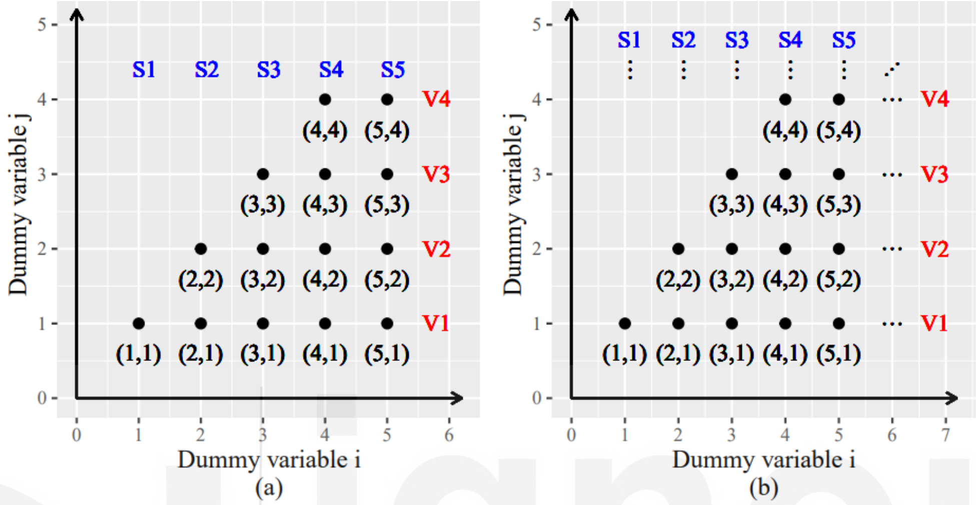

Fig. 9.3: Visualisation of the region formed by the points of intersection of the range of two dummy variables m and n (a) range of both m and n is finite, i.e., range of m is 1 to 5 and range of n is 1 to 4 (b) range of both m and n is infinite, i.e., 1 to infinity

In real data sets generally number of Subjects are more than number of Variables. That is m ≥ n generally. If m = n then the shape of the region formed by the points of intersection of the ranges of two dummy variables will be triangular like triangle ABE in Fig 9.5 (a) if we consider m = 4 and n = 4, however in this figure m = 5 and n = 4. If m > n then the shape of the region formed by the points of intersection of the ranges of two dummy variables will be like trapezium ABCD in Fig. 9.4 (a) where m = 5 and n = 4. So, the challenge is how we can form the limits of two dummy variables when we have region formed by the points of intersection of the ranges of two dummy variables is triangular or trapezium in shape instead of rectangular in shape. In such a situation you cannot take both lower and upper limits of the two dummy variables as constant because if you do so then region will become rectangular in shape which is not the case here.

Here, we will have triangular shape region or trapezium shape region not rectangular shape region. This problem of triangular or trapezium shape region can be solved if we take both lower as well as upper limits of one dummy variable as constant so that entire range along that Variable is covered and decide limits of the other dummy variable by drawing a strip along the direction of the other dummy variable. So, we have to consider two separate cases mentioned as follows.

(1) **Considering both lower and upper limits of the dummy variable j as constant and dummy variable i as variable**: To do so we have to draw a horizontal strip. For example, in the situation of Fig. 9.4 (a) limits of j will be from 1 to 4 while in the situation of Fig. 9.4 (b) limits of j will be from 1

to ∞. To obtain variable limit(s) of the dummy variable i in terms of the dummy variable j we have to draw a horizontal strip PQ refer Fig. 9.4. In this case lower limit of i will be obtained by solving the equation of line PQ for i in term of j. Here, equation of the line PQ is i = j so lower limit of i will be i = j. Upper limit of i will be 5 in the situation of Fig. 9.4 (a) since strip PQ ends at the line where i = 5. Upper limit of i will be ∞ in the situation of Fig. 9.4 (b) since strip PQ ends at the line where i = ∞. We know that generally number of Subjects (S1, S2, S3, …, Sm) are more than number of Variables (V1, V2, V3, …, Vn). So, let us assume that m ≥ n. Therefore, sum of the elements of the excel sheet given by (9.26) or matrix given by (9.27) where row suffix ≥ column suffix in the case m = 5 and n = 4 can be obtained as follows (refer strip PQ shown in trapezium shape region ABCD shown in Fig. 9.4 (a). Remember in the cases where **one dummy variable has constant limit and other dummy variable has variable limit then the sigma corresponding to the dummy variable having variable limits will be inside and the sigma corresponding to the dummy variable having both lower and upper limit as constant will be outside** explained as follows.

$$\sum_{j=1}^{4}\sum_{i=j}^{5} a_{ij} = \sum_{j=1}^{4}\left(a_{jj} + a_{j+1,\,j} + a_{j+2,\,j} + \ldots + a_{5j}\right) \qquad \text{... (9.33)}$$

$$= \left(a_{11} + a_{21} + a_{31} + a_{41} + a_{51}\right) + \left(a_{22} + a_{32} + a_{42} + a_{52}\right) + $$

$$+ \left(a_{33} + a_{43} + a_{53}\right) + \left(a_{44} + a_{54}\right)$$

$$= \left(\text{sum of all elements of V1 from first element onward}\right)$$

$$+ \left(\text{sum of all elements of V2 from second element onward}\right)$$

$$+ \left(\text{sum of all elements of V3 from third element onward}\right)$$

$$+ \left(\text{sum of all elements of V4 from fourth element onward}\right)$$

$$= V(1, 1) + V(2, 2) + V(3, 3) + V(4, 4) \qquad \text{... (9.34)}$$

where V(i, j) denotes the sum of i^{th} Variable from j^{th} elements onward

$$= \sum_{j=1}^{4} V(j, j) \qquad \text{... (9.35)}$$

$$\Rightarrow \sum_{j=1}^{4}\sum_{i=j}^{5} a_{ij} = \sum_{j=1}^{4} V(j, j) \qquad \text{… (9.36)}$$

(2) **Considering both lower and upper limits of the dummy variable i as constant and dummy variable j as variable**: To do so we have to draw a vertical strip. But here is one more problem which we did not face in the case of horizontal strip. This additional problem is regarding end point of the vertical strip PQ. Look at Fig. 9.5 (a) and note that if you draw a vertical strip within the triangular region ABE then it will always end at the side AB of the triangle ABE, but if you will draw a vertical strip within the rectangular region BCDE then it will always end at the side BC of the rectangle BCDE. So, here we have to deal with the two regions: triangular region ABE and rectangular region BCDE, separately. We know that we have no need to draw a strip for getting limits of a rectangular region. So, we only need to draw a vertical strip for triangular region ABE to get limit

of dummy variable j. For example, in the situation of the Fig. 9.5 (a) limits of i will be from 1 to 4 in the triangular region ABE, while in the situation of Fig. 9.5 (b) limits of i will be from 1 to ∞. To obtain variable limit(s) of the dummy variable j in term of the dummy variable i we have to draw a vertical strip PQ in the triangular region ABE refer Fig. 9.5 (a). In this case lower limit of j will be j = 1 since strip PQ starts from j = 1. Upper limit of j will be obtained by solving the equation of line AB for j in terms of i. Here, equation of the line AB is i = j so upper limit of j will be j = i in the situation of Fig. 9.5 (a). So, finally, limits of the trapezium region ABCD using vertical strip can be written as follows.

$$\underbrace{\sum_{\substack{i=1\\ i\geq j}}^{4}\sum_{j=1}^{i} a_{ij}}_{\text{triangular region ABE}} + \underbrace{\sum_{i=5}^{5}\sum_{j=1}^{4} a_{ij}}_{\text{Rectangular region BCDE}} = \sum_{\substack{i=1\\ i\geq j}}^{4}\left(a_{i1}+a_{i2}+a_{i3}+\ldots+a_{ii}\right)$$

$$+\sum_{i=5}^{5}\left(a_{i1}+a_{i2}+a_{i3}+a_{i4}\right) \qquad \ldots (9.37)$$

$$=\underbrace{\left(a_{11}\right)+\left(a_{21}+a_{22}\right)+\left(a_{31}+a_{32}+a_{33}\right)+\left(a_{41}+a_{42}+a_{43}+a_{44}\right)}_{\text{These terms are obtained from the first summation}}$$

$$+\underbrace{\left(a_{51}+a_{52}+a_{53}+a_{54}\right)}_{\text{Obtained from the second summation}}$$

$$=(\text{sum of all elements of S1 upto it's first element})$$
$$+(\text{sum of all elements of S2 upto it's second element})$$
$$+(\text{sum of all elements of S3 upto it's third element})$$
$$+(\text{sum of all elements of S4 upto it's fourth element})$$
$$+(\text{sum of all elements of S5 upto it's fourth element})$$

$$=S(1,1)+S(2,2)+S(3,3)+S(4,4)+S(5,4) \qquad \ldots (9.38)$$

where S(i, j) denotes the sum of i^{th} Subject upto it's j^{th} element

$$=\sum_{i=1}^{4}S(i,i)+\sum_{i=5}^{5}(i,4) \qquad \ldots (9.39)$$

$$\Rightarrow \underbrace{\sum_{\substack{i=1\\ i\geq j}}^{4}\sum_{j=1}^{i} a_{ij}}_{\text{triangular region ABE}} + \underbrace{\sum_{i=5}^{5}\sum_{j=1}^{4} a_{ij}}_{\text{Rectangular region BCDE}} = \underbrace{\sum_{i=1}^{4}S(i,i)}_{\text{triangular region ABE}} + \underbrace{\sum_{i=5}^{5}(i,4)}_{\text{Rectangular region BCDE}} \qquad \ldots (9.40)$$

Both (9.33) and (9.37) or (9.34) and (9.38) or (9.36) and (9.40) give sum of the same elements of the excel sheet given by (9.26) or matrix given by (9.27) where row suffix ≥ column suffix. So, we have

$$\sum_{j=1}^{4}\sum_{i=j}^{5} a_{ij} = \sum_{j=1}^{4}V(j,j) = \underbrace{\sum_{\substack{i=1\\ i\geq j}}^{4}\sum_{j=1}^{i} a_{ij}}_{\text{triangular region ABE}} + \underbrace{\sum_{i=5}^{5}\sum_{j=1}^{4} a_{ij}}_{\text{Rectangular region BCDE}} = \sum_{i=1}^{4}S(i,i)++\sum_{i=5}^{5}(i,4) \qquad \ldots (9.41)$$

$$\sum_{j=1}^{4}\sum_{i=j}^{5} a_{ij} = \underbrace{\sum_{i=1}^{4}\sum_{j=1}^{i} a_{ij}}_{\text{triangular region ABE}} + \underbrace{\sum_{i=5}^{5}\sum_{j=1}^{4} a_{ij}}_{\text{Rectangular region BCDE}} \qquad \ldots (9.42)$$

So, moral of the story is when **region formed by the points of intersection of the ranges of two dummy variables is either triangular (when m = n) or trapezium shape (when m ≠ n) then both limits of one dummy variable**

will be constant and written as outside summation while limit(s) of the other dummy variable either lower or upper or both will depend on the first dummy variable whose both limits we have taken as constant. To identify limits of the second dummy variable we draw a strip (PQ say) parallel to the axis of the second dummy variable. Lower limit of the other dummy variable will be decided by the equation of the line from where strip PQ starts and upper limit will be decided by the equation of the line where strip PQ ends. … (9.43)

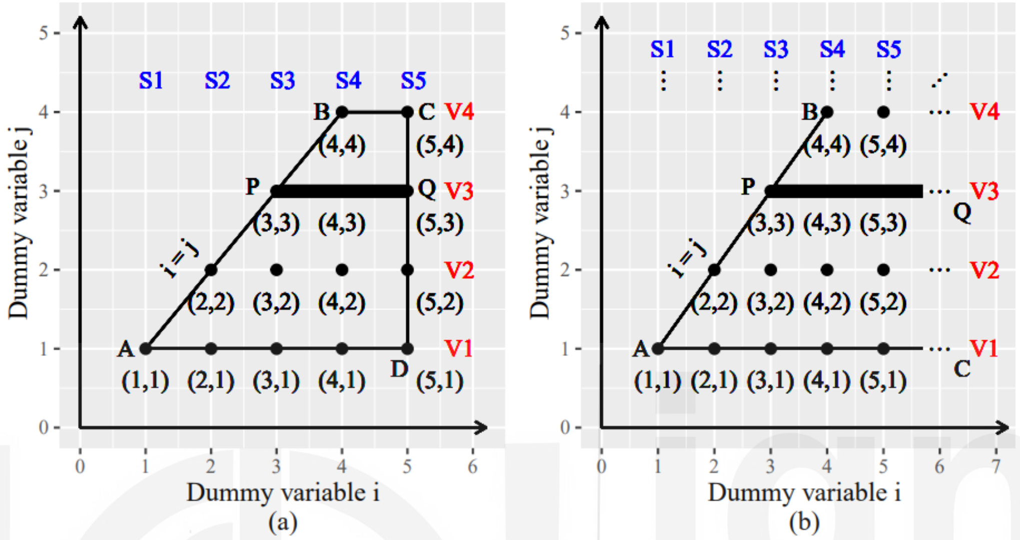

Fig. 9.4: Visualisation of the horizontal strip when (a) ranges of both i and j are finite, i.e., range of i is 1 to 5 and range of j is 1 to 4 (b) indirectly ranges of both i and j are infinite, but they are written as range of j from 1 to infinity and lower range of i is obtained from the equation of line AB in terms of j, here it is i = j and upper range of i is infinity

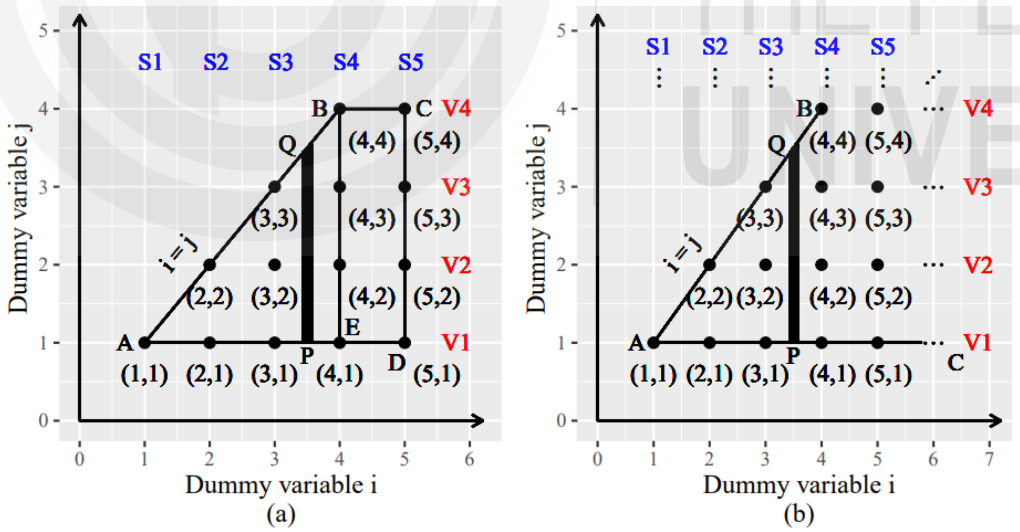

Fig. 9.5: Visualisation of the vertical strip when region is triangular in shape (a) ranges of both i and j are finite, i.e., range of i is 1 to 5 and range of j is 1 to i (b) indirectly ranges of both i and j are infinite, but they are written as range of i from 1 to infinity and lower limit of j is 1 and upper limit of j is obtained from the equation of line AB in terms of i. Here it is j = i

In the cases where limit(s) of one dummy variable is(are) in term of other dummy variable then the sigma corresponding to variable limit(s) is written inside the other sigma. The sigma corresponding to the dummy variable having both lower and upper limits as constant is written outside refer left hand sides of (9.33) and (9.37). … (9.44)

Let us do an example.

Example 1: In real data sets number of Subjects and number of Variables are finite in an excel sheet. But for this example, assume that number of Subjects (m) and number of Variables (n) are infinite. Suppose you are interested in the sum of all those elements of the excel sheet where row suffix ≥ column suffix. Write sum of the elements of our interest using double summation notation and keeping both the limits of the dummy variable which represents number of Variables as constant.

Solution: To obtain limits of the required type let us first present the dummy variable which represents suffix of the subjects on horizontal axis and the dummy variable which represents suffix of the variables on vertical axis. After doing so the point having coordinates (i, j) will represent the suffixes of the element $a_{i\,j}$ refer Fig. 9.4 (b) where we have assumed m = ∞ and n = ∞. The region formed by the positions of the points of our interest is triangular in shape and is given by the triangle ABC in Fig. 9.4 (b). In this region limits of dummy variable j are from 1 to ∞. To obtain limits of the dummy variable i we have to draw a horizontal strip PQ as shown in Fig. 9.4 (b). This strip starts from the line AB where equation of the line AB is i = j. So, lower limit of i is i = j. Strip PQ end where j is infinity. Thus, upper limit of j will be j = ∞. Hence, the sum of the elements of our interest using double sigma's and keeping both the limits of the dummy variable which represents number of Variables as constant can be written as.

$$\sum_{j=1}^{\infty}\sum_{i=j}^{\infty} a_{i\,j} \qquad \ldots (9.45)$$

Now, you can try the following Self-Assessment Question.

SAQ 2

In real data sets number of Subjects and number of Variables are finite in an excel sheet. But for this SAQ, assume that number of Subjects (m) and number of Variables (n) are infinite. Suppose you are interested in the sum of all those elements of an excel sheet where row suffix ≥ column suffix. Write sum of the elements of our interest using double summation notation and keeping both the limits of the dummy variable which represents number of subjects as constant.

9.4 DOUBLE INTEGRATION

In Sub Sec. 5.2.2 we have explained the fact that double integral

$$\int_{x=a}^{x=b}\left(\int_{y=0}^{y=f(x)} dy\right)dx \text{ or } \int_{x=a}^{x=b}\left(\int_{0}^{f(x)} dy\right)dx = \int_{x=a}^{x=b}\left(\left[y\right]_{y=0}^{y=f(x)}\right)dx = \int_{x=a}^{x=b}\left(f(x)-0\right)dx = \int_{x=a}^{x=b} f(x)\,dx$$

… (9.46)

represents/gives area bounded by four things, refer RHS of (9.46):

(1) Curve of the Function: $y = f(x)$ … (9.47)

(2) Line corresponding to the Lower Limit of the Integration: $x = a$ … (9.48)

(3) Line corresponding to the Upper Limit of the Integration: $x = b$ … (9.49)

(4) Line corresponding to the Axis of the Variable of Integration: x … (9.50)

From equation (9.46) note that $\int_{x=a}^{x=b}\left(\int_{y=0}^{y=f(x)} dy\right)dx = \int_{x=a}^{x=b} f(x)\,dx$ … (9.51)

Recall equation (5.8) from Unit 5 which is $\int_2^3 dx = [x]_2^3 = 3 - 2 = 1.$ … (9.52)

Equation (9.52) gives measure of the line segment AB refer Fig. 9.6 (a). Why it is so? To get the answer of it refer sub Sec. 5.2.1 of Unit 5 of this course.

Both sides of equation (9.51) give area of the shaded region shown in Fig. 9.6 (b). First difference between LHS and RHS of equation (9.51) is that, in RHS function f(x) is mentioned inside the integral while in LHS of (9.51) no function is mentioned inside the integral sign. Second difference between LHS and RHS of equation (9.51) is that in RHS single integral sign is present while in LHS of (9.51) two integral (double integral) signs are present. From here we conclude that presence of a function inside the integral sign is equivalent to give measure of an integral along the direction of the axis of the function.

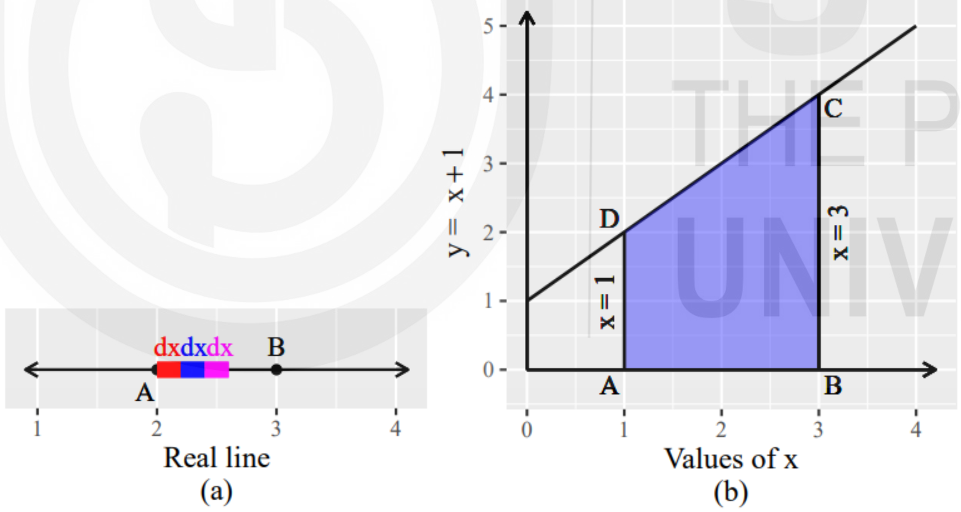

Fig. 9.6: Visualisation of the (a) horizontal strip to obtain length of the line segment AB using integration (b) area bounded by four things mentioned in (9.47) to (9.50) to obtain using integration

9.5 CHANGE OF ORDER OF INTEGRATION

Like change of order of summation change of order of integration also depends on the shape of the region of integration. So, let us consider two cases:

(a) When Region of Integration is Rectangular in Shape

(b) When Region of Integration is not Rectangular in Shape

Let us discuss these two one at a time

(a) When Region of Integration is Rectangular in Shape

If region of integration is rectangular in shape, then like change of order of

summation, change of order of integration is very simple just change the order of two integration without any modification in the limits of the two variables of integration. For example, region of integration in equation (9.53) and (9.54) is rectangular in shape as shown in Fig. 9.7 (a). So, you can change order of integration without any modification in the limits of two integrals which is done as follows.

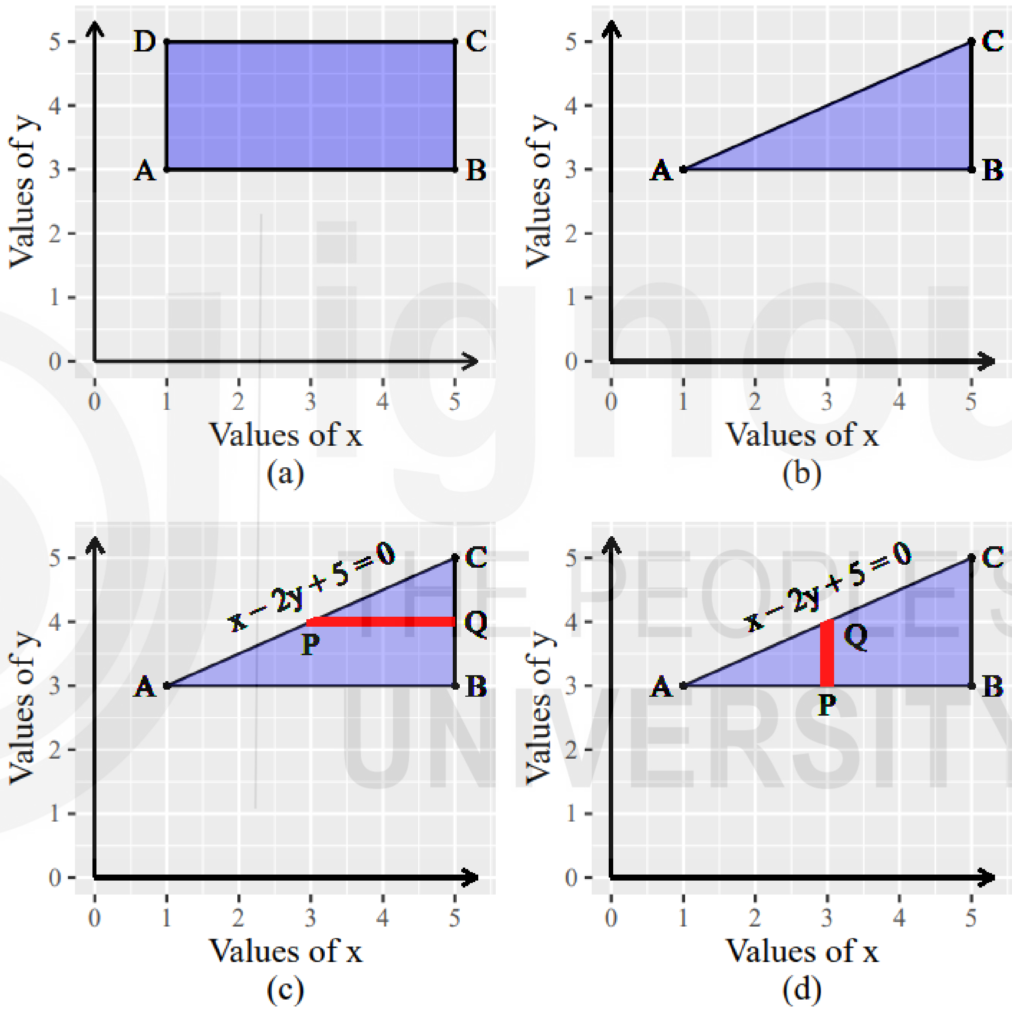

Fig. 9.7: Visualisation of the region of integration used in (a) (9.53) and (9.54) (b) (9.56) and (9.57) (c) horizontal strip PQ (d) vertical strip PQ

First integrate with respect to x and then with respect to y

$$\int_{y=3}^{y=5}\left(\int_{x=1}^{x=5}(x+y)\,dx\right)dy = \int_{y=3}^{y=5}\left[\frac{x^2}{2}+xy\right]_{x=1}^{x=5}dy = \int_{y=3}^{y=5}\left[\frac{25}{2}+5y-\frac{1}{2}-y\right]dy$$

$$= \int_{y=3}^{y=5}[4y+12]\,dy = \left[2y^2+12y\right]_3^5$$

$$= 50+60-18-36 = 56 \qquad \dots (9.53)$$

First integrate with respect to y and then with respect to x

$$\int_{x=1}^{x=5}\left(\int_{y=3}^{y=5}(x+y)\,dy\right)dx = \int_{x=1}^{x=5}\left[xy+\frac{y^2}{2}\right]_{y=3}^{y=5}dx = \int_{x=1}^{x=5}\left[5x+\frac{25}{2}-3x-\frac{9}{2}\right]dx$$

$$= \int_{x=1}^{x=5}[2x+8]\,dx = \left[x^2+8x\right]_1^5$$

$$= 25+40-1-8 = 56 \qquad \text{... (9.54)}$$

From (9.53) and (9.54), we have

$$\int_{y=3}^{y=5}\left(\int_{x=1}^{x=5}(x+y)\,dx\right)dy = \int_{x=1}^{x=5}\left(\int_{y=3}^{y=5}(x+y)\,dy\right)dx \qquad \text{... (9.55)}$$

So, (9.55) is an example which shows that in the case we have region of integration as rectangular then order of integration can be changed without any modification in the limits of two variables.

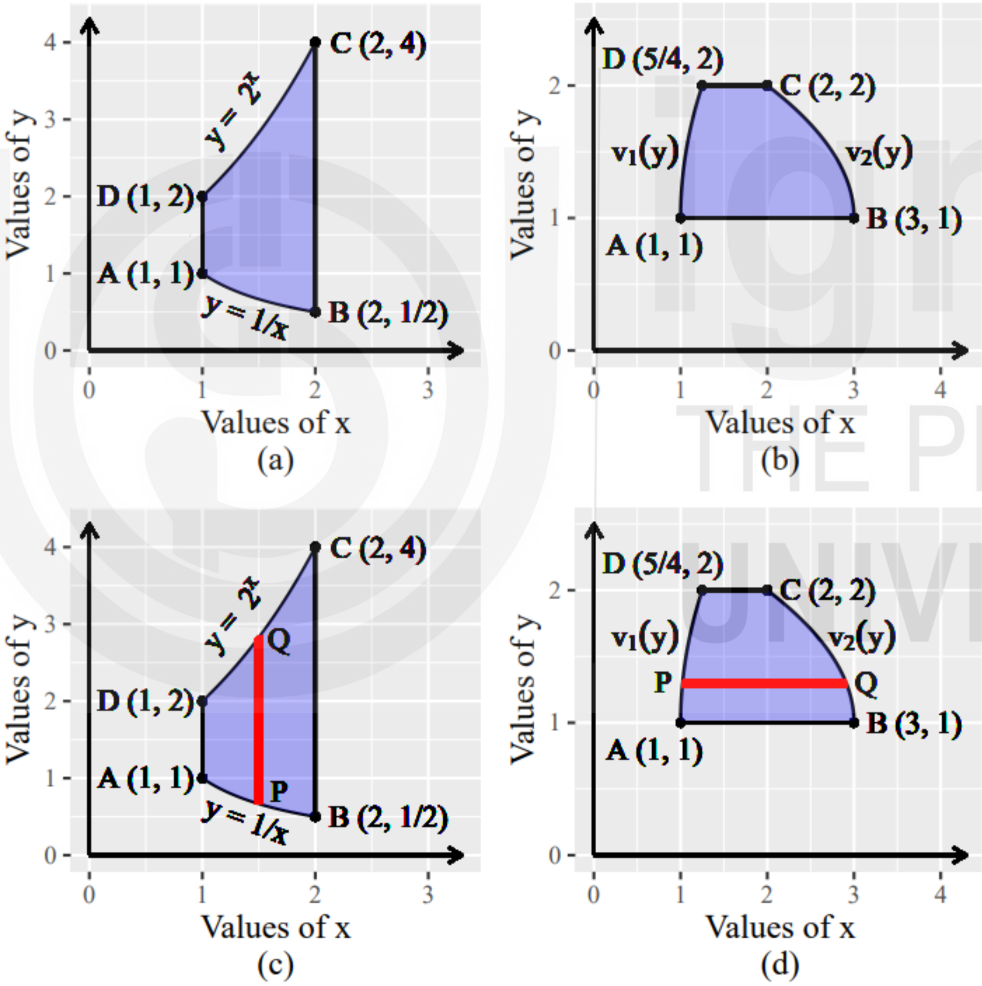

Fig. 9.8: Visualisation of the region of (a) Type I (b) Type II (c) vertical strip PQ in region of Type I (d) horizontal strip PQ in region of Type II

Now, we consider the case when the region of integration is not rectangular in shape.

(b) When Region of Integration is not Rectangular in Shape

Before discussing the problem of change of order of integration when region of integration is not in rectangular shape, first we have to define two terms known as Regions of Type I and Type II. These two terms are defined as follows.

Region of Type I

A region D in x-y plane is said to be of Type I if it is bounded by two vertical lines x = a and x = b and two functions $h_1 : [a, b] \to \mathbb{R}$ and $h_2 : [a, b] \to \mathbb{R}$ both continuous on [a, b] such that $h_1(x) \le h_2(x) \quad \forall\ x \in [a, b]$

In other words, if $[a, b] \subset \mathbb{R}$ and $h_1 : [a, b] \to \mathbb{R}$ and $h_2 : [a, b] \to \mathbb{R}$ be two continuous functions then a region D defined as follows:

$D = \{(x, y) \in \mathbb{R}^2 : a \le x \le b, h_1(x) \le y \le h_2(x) \quad \forall\ x \in [a, b]\}$ is called region of Type I. Region shown in Fig. 9.8 (a) is an example of region of Type I. In this example, a = 1, b = 2, $h_1(x) = \frac{1}{x}, h_2(x) = 2^x$. … (9.56)

Region of Type II

A region D in x-y plane is said to be of Type II if it is bounded by two horizontal lines y = c and y = d and two functions $v_1 : [c, d] \to \mathbb{R}$ and $v_2 : [c, d] \to \mathbb{R}$ both continuous on [c, d] such that $v_1(y) \le v_2(y) \quad \forall\ y \in [c, d]$

In other words, if $[c, d] \subset \mathbb{R}$ and $v_1 : [c, d] \to \mathbb{R}$ and $v_2 : [c, d] \to \mathbb{R}$ be two continuous functions then a region D defined as follows:

$D = \{(x, y) \in \mathbb{R}^2 : v_1(y) \le x \le v_2(y),\ c \le y \le d \quad \forall\ y \in [c, d]\}$ is called region of Type II. Region shown in Fig. 9.8 (b) is an example of region of Type II. In this example, c = 1, d = 2, $v_1(y) = 1 + \frac{1}{4}(y-1)^2,\ v_2(y) = 3 - (y-1)^2$. … (9.57)

After defining region of Type I and Type II, now we discuss how we can change the order of integration when region of integration is not rectangular in shape.

From the discussion of change of order of summation, we know that in such cases we take both lower and upper limits of one dummy variable as constant so that entire region along the axis of that variable is covered. Limits of the other dummy variable are obtained by drawing a strip PQ parallel to the axis of that dummy variable. Lower limit will be decided by solving the equation of the line or curve from where strip PQ starts for that dummy variable in terms of the other dummy variable. Similarly, upper limit will be decided by solving the equation of the line or curve where strip PQ ends. Keep following three points in mind before doing so.

(1) In the case when region of integration is of Type I then both lower and upper limits of the variable x will be constant and limits of variable y will be obtained by drawing a vertical strip in the region of integration. … (9.58)

(2) In the case when region of integration is of Type II then both lower and upper limits of the variable y will be constant and limits of variable x will be obtained by drawing a horizontal strip in the region of integration. … (9.59)

(3) In the case when region of integration is of both types: Type I as well as Type II then you have both options as mentioned in the first two points. … (9.60)

Let us do some examples to explain the working procedure.

Example 2: Evaluate the area of the triangular region shown in Fig. 9.7 (b) without using integration.

Solution: The triangular region ABC shown in Fig. 9.7 (b) is a right-angled triangle. We know that area of a right-angled triangle is $\frac{1}{2}\text{base} \times \text{height.}$ Here base = AB = 4 units and height = BC = 2 units.

Hence, required area = $\frac{1}{2}(4)(2) = 4\,\text{unit}^2.$ … (9.61)

Example 3: Evaluate the area of the triangular region shown in Fig. 9.7 (b) using double integration.

Solution: Region shown in Fig. 9.7 (b) is of both types: Type I and Type II. Let us first evaluate it by treating it as Type I region refer Fig. 9.7 (d). So, here both lower and upper limits of the variable x will be constant and given by x = 1 and x = 5 refer Fig. 9.7 (d). Limits of variable y will be obtained by drawing a vertical strip PQ refer Fig. 9.7 (d). This vertical strip PQ starts from y = 3 and ends at y = (x+5)/2. Hence, using double integration

$$\text{required area} = \int_{x=1}^{x=5}\left(\int_{y=3}^{y=(x+5)/2} dy\right)dx = \int_{x=1}^{x=5} [y]_3^{(x+5)/2}\, dx$$

$$= \int_{x=1}^{x=5}\left(\frac{x+5}{2} - 3\right)dx = \int_{x=1}^{x=5}\frac{x-1}{2}dx = \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^5$$

$$= \frac{1}{2}\left[\frac{25}{2} - 5 - \frac{1}{2} + 1\right] = 4\,\text{unit}^2 \quad \text{... (9.62)}$$

Let us now evaluate it by treating it as Type II region refer Fig. 9.7 (c). So, here both lower and upper limits of the variable y will be constant and given by y = 3 and y = 5 refer Fig. 9.7 (c). Limits of variable x will be obtained by drawing a horizontal strip PQ refer Fig. 9.7 (c). This horizontal strip PQ starts from x = 2y – 5 and ends at x = 5. Hence, using double integration

$$\text{required area} = \int_{y=3}^{y=5}\left(\int_{x=2y-5}^{x=5} dx\right)dy = \int_{y=3}^{y=5} [x]_{2y-5}^{5}\, dy$$

$$= \int_{y=3}^{y=5}(5-2y+5)\,dy = \int_{y=3}^{y=5}(10-2y)\,dy$$

$$= \left[10y - y^2\right]_3^5 = \left[50 - 25 - 30 + 9\right] = 4\,\text{unit}^2 \quad \text{... (9.63)}$$

Remark 1: Compare (9.61), (9.62) and (9.63) you see that all the three are equal. Remember that this did not happen by chance. But it is a general result and holds always. So, double integral gives area of the region of integration when we have 1 as the function inside the integration. Also, if we equate the two integrals used in (9.62) and (9.63) then, we have

$$\int_{x=1}^{x=5}\left(\int_{y=3}^{y=(x+5)/2} dy\right)dx = \int_{y=3}^{y=5}\left(\int_{x=2y-5}^{x=5} dx\right)dy \quad \text{… (9.64)}$$

the relation given by (9.64) is known as change of order of integration in triangular region ABC given in Fig. 9.7 (b).

Example 4: Evaluate the area of the region of Type I shown in Fig. 9.8 (a) using double integration.

Solution: Region shown in Fig. 9.8 (a) is of Type I because it lies between two vertical lines. So, here both lower and upper limits of the variable x will be constant and given by x = 1 and x = 2 refer Fig. 9.8 (a) or (c). Limits of the variable y will be obtained by drawing a vertical strip PQ refer Fig. 9.8 (c). This vertical strip PQ starts from y = 1/x and ends at y = 2^x. Hence, using double integration

$$\text{required area} = \int_{x=1}^{x=2}\left(\int_{y=1/x}^{y=2^x} dy\right)dx = \int_{x=1}^{x=2}[y]_{y=1/x}^{y=2^x}\,dx$$

$$= \int_{x=1}^{x=2}\left(2^x - \frac{1}{x}\right)dx = \left[\frac{2^x}{\log 2} - \log x\right]_1^2 = \frac{4}{\log 2} - \log 2 - \frac{2}{\log 2} + \log 1$$

$$= \left(\frac{2}{\log 2} - \log 2\right)\text{unit}^2 \quad [\because \log 1 = 0] \qquad \text{... (9.65)}$$

Now, you can try the following Self-Assessment Question.

SAQ 3

Evaluate the area of the region of Type II shown in Fig. 9.8 (b) using double integration.

9.6 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- The variable n used in subscript of a in the sigma notation $\sum_{n=1}^{\infty} a_n$ is known as **dummy variable** or **index** of the summation. There is nothing special in n you can use any other dummy variable like k or m or p or r, etc. .

- In the notation $\sum_{n=5}^{100} a_n$ dummy variable n starts from 5 and goes up to 100.

  So, origin of the summation is at 5, while in the summation $\sum_{n=7}^{50} a_n$ origin of the summation is at 7.

- We can **change the origin** of a given summation at any other number of our interest without affecting the sum of the terms given by it. For example,

  $$\sum_{n=1}^{\infty} a_n = \sum_{n=m}^{\infty} a_{n-m+1} \quad \text{and} \quad \sum_{k=1}^{n} a_k = \sum_{k=m}^{n+m-1} a_{k-m+1}$$

  Also, $\sum_{k=m}^{n} a = \underbrace{a + a + a + \ldots + a}_{(n-m+1)\text{ times}} = a \times (n - m + 1) = (n - m + 1)a$

- If range of two dummy variables are **constant or independent of each other**, then corresponding region formed by the points of intersection of the ranges of two dummy variables will be **rectangular in shape** does not matter whether their ranges are finite or infinite and in this case, you can change the order of two sigma's without any modification in the ranges of the two dummy variables or in the expressions inside the sigma's. For example,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij} \text{ and } \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

- Double integration $\iint_D dx\,dy$ gives area of the region D. If the region D is a rectangle, then order of integration can be changed without any modification in the limits of the two variables of integration. But if region D is not rectangular then order of integration can also be changed but we have to do modification in the limits depending upon whether it is region of Type I or Type II as discussed in Sec. 9.5.

9.7 TERMINAL QUESTIONS

1. Choose the correct option for the notation $\sum_{k=1}^{3}\frac{t^k}{n}$

   (A) $\frac{1^1}{1}+\frac{2^2}{2}+\frac{3^3}{3}$ (B) $\frac{t^1}{n}+\frac{t^2}{n}+\frac{t^3}{n}$ (C) $\frac{1^k}{n}+\frac{2^k}{n}+\frac{3^k}{n}$ (D) $\frac{t^k}{1}+\frac{t^k}{2}+\frac{t^k}{3}$

2. Choose the correct option for the notation $\sum_{k=4}^{300}\left(\frac{2}{3}\right)$

   (A) $300\times(2/3)$ (B) $296\times(2/3)$ (C) $304\times(2/3)$ (D) $297\times(2/3)$

3. Specify one application of change of order of integration with suitable example.

4. Find the area/volume of the region formed by $0 \le x \le 5$, $0 \le y \le 5$, $0 \le z \le x + y$ (a) intuitively and (b) using double integral.

9.8 SOLUTIONS/ANSWERS

<u>Self-Assessment Questions (SAQs)</u>

1. (a) We know that $\sum_{k=1}^{n} k^p = 1^p + 2^p + 3^p + \ldots + n^p$. Here, p = 5, n = 3, so (C) is the correct option.

   (b) We know that $\sum_{k=m}^{n} a = \underbrace{a+a+a+\ldots+a}_{(n-m+1)\text{ times}} = (n-m+1)a$ Here, a = 7, n = 100, m = 10, so $\sum_{k=10}^{100} 7 = (100-10+1)(7) = 91\times 7 = 637$. Hence, (D) is the correct option.

2. To obtain limits of the required type let us first present the dummy variable which represents suffix of the Subjects on horizontal axis and the dummy variable which represents suffix of the Variables on vertical axis. After doing so the point having coordinates (i, j) will represent the

suffixes of the element a_{ij} refer Fig. 9.5 (b) where we have assumed m $= \infty$ and n $= \infty$. The region formed by the positions of the points of our interest is triangular in shape and is given by the triangle ABC in Fig. 9.5 (b). In this region limits of dummy variable i are from 1 to ∞. To obtain limits of the dummy variable j we have to draw a vertical strip PQ as shown in Fig. 9.5 (b). This strip starts from the line AC where j = 1. So, lower limit of j is j = 1. Strip PQ ends on the line AB where i = j. So, upper limit of j will be j = i. Hence, the sum of the elements of our interest using double sigma's can be written as.

$$\sum_{\substack{i=1\\ i\geq j}}^{\infty}\sum_{j=1}^{i} a_{ij}$$

3. Region shown in Fig. 9.8 (b) is of Type II because it lies between two horizontal lines. So, here both lower and upper limits of the variable y will be constant and given by y = 1 and y = 2 refer Fig. 9.8 (b) or (d). Limits of the variable x will be obtained by drawing a horizontal strip PQ refer Fig. 9.8 (d). This horizontal strip PQ starts from the curve where $x = 1 + \frac{(y-1)^2}{4} = v_1(y)$ and ends at $x = 3-(y-1)^2 = v_2(y)$. Hence, using double integration required area is given by

$$\int_{y=1}^{y=2}\left(\int_{x=1+\frac{(y-1)^2}{4}}^{x=3-(y-1)^2} dx\right)dy = \int_{y=1}^{y=2}\left[x\right]_{x=1+\frac{(y-1)^2}{4}}^{x=3-(y-1)^2} dy = \int_{y=1}^{y=2}\left[3-(y-1)^2-1-\frac{(y-1)^2}{4}\right]dy$$

$$= \int_{y=1}^{y=2}\left[2-\frac{5}{4}(y-1)^2\right]dy = \left[2y - \frac{5}{4}\frac{(y-1)^3}{3}\right]_1^2 = 4 - \frac{5}{12}(1)^2 - 2 + 0$$

$$= \frac{19}{12}\text{unit}^2$$

Terminal Questions

1. Here dummy variable is k so only value of k will change term to term and all variables/unknowns other than k will remain unchanged in each term. So, (B) is the correct option.

2. We know that $\sum_{k=m}^{n} a = \underbrace{a+a+a+\ldots+a}_{(n-m+1)\text{ times}} = (n-m+1)a$. Here, $a = \frac{2}{3}$, n = 300, m = 4, so $\sum_{k=4}^{300}\frac{2}{3} = (300-4+1)\left(\frac{2}{3}\right) = 297\times\frac{2}{3}$. Hence, (D) is the correct option.

3. One important application of change of order of integration is that sometimes an integral is difficult or not possible to evaluate in given form but it can be easily evaluated by the change of the order of integration. To realise it let us consider the following example.

Example: Evaluate the integral $\int_0^2\int_{y/2}^1 e^{x^2}\,dx\,dy$.

Solution: Note that we do not know the integral of e^{x^2}. So, in the given form it cannot be integrated. But let us change the order of integration. To change the order of integration first we have to plot the region of integration. Given region of integration is bounded by the four lines:

x = y/2, x = 1, y = 0 and y = 2.

Region bounded by these four lines is shown in Fig. 9.9 (a). In the given integral limits of the variable y are constant and limits of the variable x are in term of variable y. To change the order of integration we have to do its reverse. That is, we will form limits of integration in such a way that limits of the variable x as constant and limits of the variable y in term of the variable x. To do so we have to draw a vertical strip PQ refer Fig. 9.9 (b). This strip PQ starts from the x-axis where y = 0 and ends at the line OB. Equation of line OB is y = 2x, so upper limit of y will be y = 2x. Hence, after changing order of integration given integral can be written as:

$$\int_0^1\int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1\left[y\,e^{x^2}\right]_0^{2x} dx = \int_0^1\left(2x\,e^{x^2}\right)dx$$

Putting $x^2 = t$

Differentiating, we get $2x\,dx = dt$

Also, when $x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t = 1$

$$\therefore \int_0^1\left(2x\,e^{x^2}\right)dx = \int_0^1 e^t\,dt = \left[e^t\right]_0^1 = e^1 - e^0 = e - 1$$

Thus, we saw that in given form double integral was not integrable, but after changing the order of integration it not only becomes integrable but very simple also. So, this example is an evidence of at least one application of the change of order of integration.

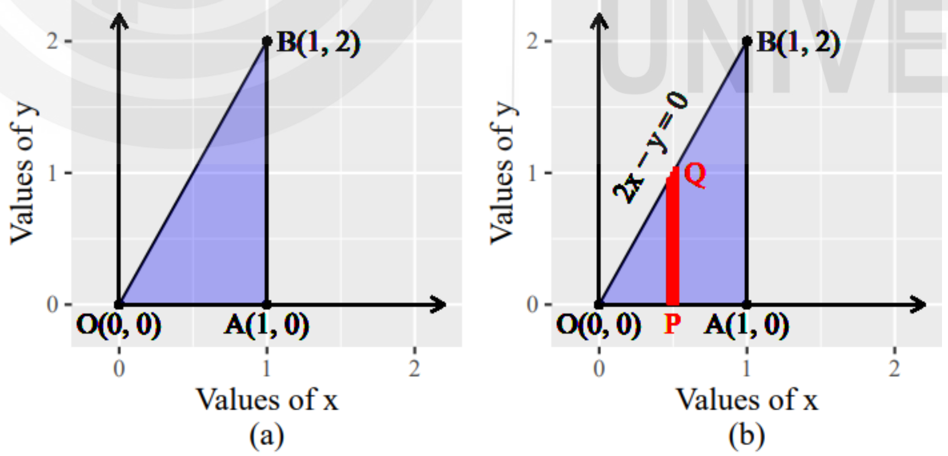

Fig. 9.9: Visualisation of the (a) region of given integration (b) vertical strip PQ to obtain limit of the variable y in term of the variable x

4. Given region is

$0 \le x \le 5,\ 0 \le y \le 5,\ 0 \le z \le x + y$ … (9.66)

This region is bounded by following six planes having equations

x = 0, x = 5, y = 0, y = 5, z = 0, and z = f(x, y) = x + y … (9.67)

The plane having equation z = x + y is shown in Fig. 9.10 in black colour.

Two different views of the complete region bounded by six planes given by (9.67) are shown in Figs. 9.11 and 9.12.

(a) Obtaining volume of the given region using Intuitive Way

The region shown in Fig. 9.11 or 9.12 is a part of a cuboid of dimension 5 units along x-axis, 5 units along y-axis and 10 units along z-axis. So, volume of this cuboid is

$$\text{lbh} = 5 \times 5 \times 10 = 250\,\text{unit}^3 \qquad \dots (9.68)$$

But given region is exactly half of it. Hence, volume of the region is

$$\frac{250}{2} = 125\,\text{unit}^3 \qquad \dots (9.69)$$

This completes the solution using intuitive way.

(b) Now, we obtain this volume using double integral as follows.

$$\text{Required volume} = \int_0^5\int_0^5 (x+y)\,dy\,dx = \int_0^5 \left[xy + \frac{y^2}{2}\right]_0^5 dx = \int_0^5 \left(5x + \frac{25}{2} - 0\right) dx$$

$$= \int_0^5 \left(5x + \frac{25}{2}\right) dx = \left[\frac{5x^2}{2} + \frac{25}{2}x\right]_0^5 = \frac{125}{2} + \frac{125}{2} - 0 = 125\,\text{unit}^3 \qquad \dots (9.70)$$

From (9.69) and (9.70) note that both the ways give the same answer,

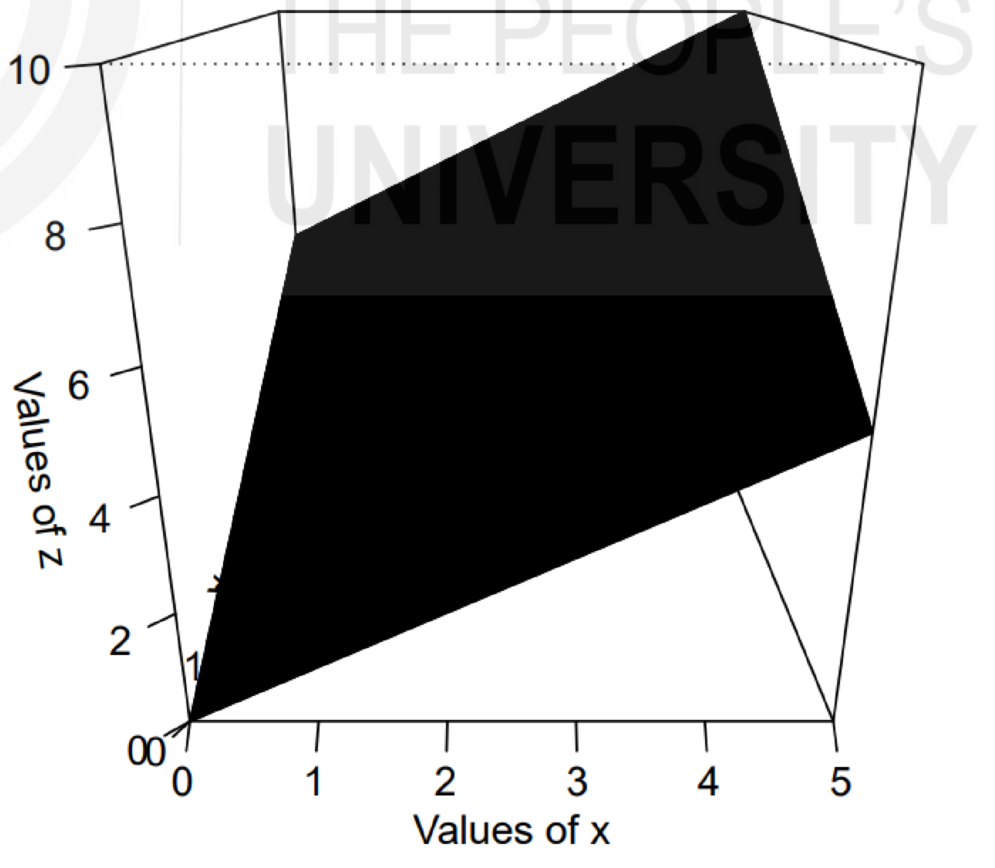

Fig. 9.10: Visualisation of the plane z = x + y

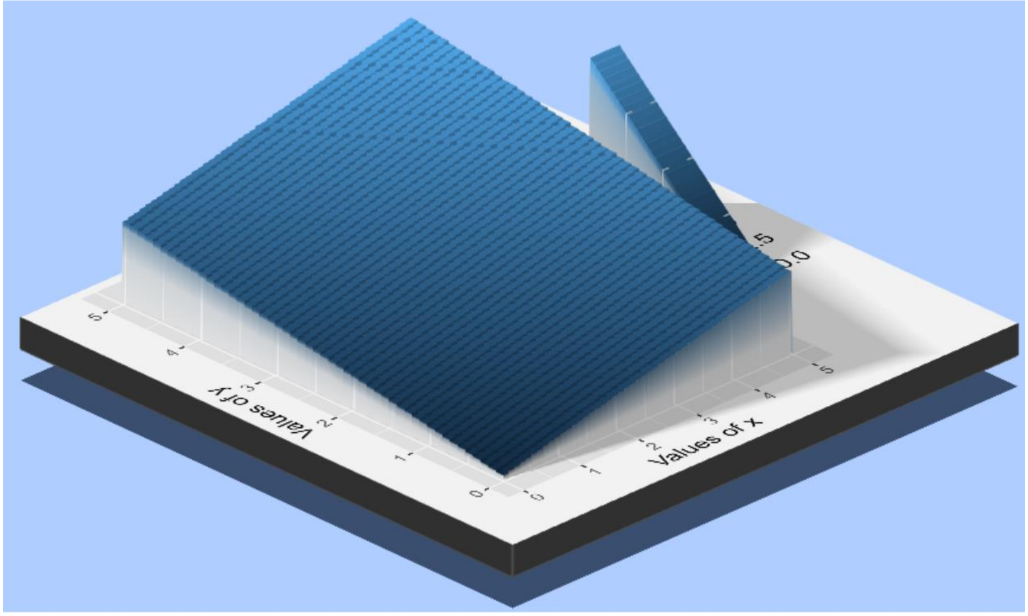

Fig. 9.11: Visualisation of the region bounded by six planes given by (9.67)

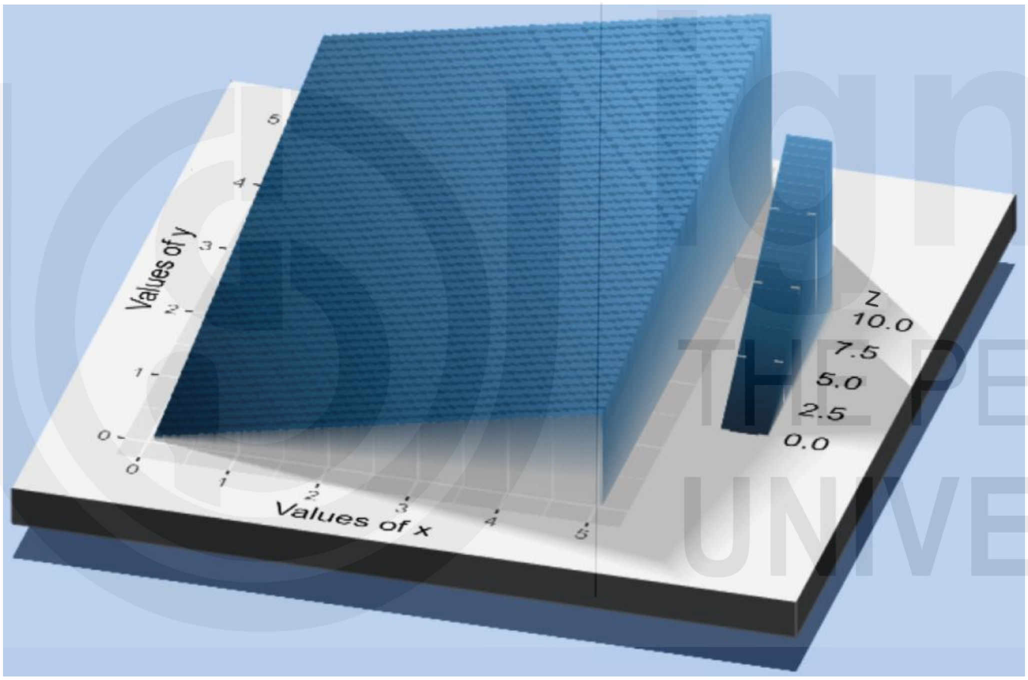

Fig. 9.12: Visualisation of the different view of the region bounded by six planes given by (9.67)

In Remark 3 of Unit 8 we promised that property 5 of gamma and beta functions will be proved in Unit 9. This is the time to keep that promise.

Proof of Property 5 of Unit 8: We know by definition of gamma function that

$$\overline{|n} = \int_0^\infty x^{n-1} e^{-x} dx \qquad \ldots (9.71)$$

We also know that, refer (8.32)

$$\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\overline{|n}}{a^n} \qquad \ldots (9.72)$$

$$\Rightarrow \boxed{n} = a^n \int_0^\infty x^{n-1} e^{-ax} dx \qquad \ldots (9.73)$$

Replacing a by z in equation (9.73), we get

$$\overline{|n} = z^n \int_0^\infty x^{n-1} e^{-zx} dx \qquad \ldots (9.74)$$

Multiplying on both sides of (9.74) by $z^{m-1} e^{-z}$, we get

$$z^{m-1} e^{-z} \overline{|n} = z^{m-1} e^{-z} z^n \int_0^\infty x^{n-1} e^{-zx} dx$$

$$\Rightarrow z^{m-1} e^{-z} \overline{|n} = z^{m+n-1} e^{-z} \int_0^\infty x^{n-1} e^{-zx} dx$$

Integrating on both sides with respect to z from 0 to ∞, we get

$$\overline{|n} \int_0^\infty x^{m-1} e^{-z} dz = \int_0^\infty \left[z^{m+n-1} e^{-z} \int_0^\infty x^{n-1} e^{-zx} dx \right] dz$$

$$= \int_0^\infty \left[\int_0^\infty z^{m+n-1} e^{-z} x^{n-1} e^{-zx} dx \right] dz$$

$$= \int_0^\infty \left[\int_0^\infty z^{m+n-1} e^{-z} x^{n-1} e^{-zx} dz \right] dx \quad \left[\begin{array}{l} \text{Changing order of integration as} \\ \text{limits of both variables are constant} \end{array} \right]$$

$$= \int_0^\infty \left[x^{n-1} \int_0^\infty z^{m+n-1} e^{-z(1+x)} dz \right] dx$$

$$= \int_0^\infty \left[x^{n-1} \frac{\overline{|m+n}}{(1+x)^{m+n}} \right] dx \quad [\text{Using } (9.71), \text{ where } a = (1 + x), \text{ n is } m+n]$$

$$\Rightarrow \overline{|n} \int_0^\infty x^{m-1} e^{-z} dz = \overline{|m+n} \int_0^\infty \left[\frac{x^{n-1}}{(1+x)^{m+n}} \right] dx$$

$$\Rightarrow \overline{|n}\, \overline{|m} = \overline{|m+n} \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \quad [\text{Using } (9.71), \text{ in LHS}]$$

$$\Rightarrow \overline{|n}\, \overline{|m} = \overline{|m+n}\, B(n, m) \qquad [\text{Using } (8.43), \text{ in RHS}]$$

$$\Rightarrow \overline{|n}\, \overline{|m} = \overline{|m+n}\, B(m, n) \qquad [\because B(m, n) = B(n, m) \;\; \text{Using } (8.39)]$$

$$\Rightarrow B(m, n) = \frac{\overline{|m}\, \overline{|n}}{\overline{|m+n}}$$

Hence, proved.

Your feedback pertaining to this course will help us undertake maintenance and timely revision of the course. You may give your feedback regarding SLM of this course.

FEEDBACK FORM LINK IS: https://forms.gle/Hf5kvZth9M8CxbXU9