UNIT 1

FUNCTION OF A SINGLE VARIABLE

Structure

1.1 INTRODUCTION

In statistics, we use parameters for population characteristics such as mean, variance, etc. Constants and variables are also used in statistics. So, it becomes important to understand the distinction between fixed constant, parameter and variable which is discussed in Sec. 1.2. Another important thing that we are going to use on many occasions in different courses is the interval. In Sec. 1.3 different types of intervals are discussed. To understand many concepts of probability and statistics such as probability mass function, probability density function, cumulative distribution function, likelihood function, etc. you should have a good understanding of function, therefore, a brief overview of the function is given in Sec. 1.4. There are some special functions which have their applications in statistics, probability and machine learning such as linear, polynomial, logarithm, exponential, modulus functions, logistic sigmoid function, etc. which are discussed in Sec. 1.5. In the course MST-024 on machine learning you will also use many other functions such as loss function, distance function (Manhattan distance, Euclidean distance, Chebyshev distance), etc. The distance function is discussed in Unit 3.

What we have discussed in this unit is summarised in Sec. 1.6. Self Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit, some more questions based on the entire unit are given in Sec. 1.7 under the heading Terminal Questions. One of the characteristics of IGNOU study material is that it is self-contained. To make it self-contained regarding solutions of all the SAQs and Terminal Questions we have given solutions of all the SAQs and Terminal Questions of this unit in Sec. 1.8. To best utilise this section, it is recommended that after going through all the concepts discussed in the unit first you should try solutions of all the SAQ's and Terminal Questions yourself. In the case, your answer does not match with the answer given in the solution of the same you may give it another try if still not matching you may refer to the given solution and identify the concept where you need to improve.

In the next unit, you will study types of functions, countable sets, continuity and differentiability of a function.

Expected Learning Outcomes

After completing this unit, you should be able to:

- ❖ define different types of constants, variables, and intervals, and can give examples of each;
- ❖ define a function and obtain its domain and range;
- ❖ evaluate the value of a function at given points; and
- ❖ plot a graph of a function and identify its graphical behaviour.

1.2 CONSTANT AND VARIABLE QUANTITY

Suppose someone wants to know about you then perhaps he/she should know your name, the background of the family you belong to, your educational qualifications, how good as a human being you are, other achievements, etc. Similarly, if you want to know/study the subject statistics then it requires that first, you should have an understanding of some rules of mathematics. Understanding these rules of mathematics will make the journey of studying tools/techniques of the subject of statistics easy as well as interesting. Understanding which tools/techniques of the subject statistics will apply in a given situation is a long journey. Purpose of different courses of this master's degree programme M.Sc. Applied Statistics is to start this journey smoothly and in a sequence. Understanding the mathematics rules required to study statistics is not hard if you start from the basics. In fact, when you start from basic, learning any subject becomes interesting and creates a hunger to learn more and more. Like your name, the subject that we are going to discuss here also has its name called mathematics. Unlike your one family learning mathematics starts from knowing about its many families called families of numbers. You know about these families of numbers from your school mathematics. Let us recall these well-known families of numbers mentioned as follows.

Family of natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ … (1.1)

Family of whole numbers: $W = \{0, 1, 2, 3, 4, \ldots\}$ … (1.2)

Family of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ … (1.3)

Family of rational numbers: $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0, (p, q) = 1\right\}$ … (1.4)

where (p, q) represents the greatest common divisor (gcd) or highest common factor (hcf) of p and q

Family of irrational numbers: A number on the real line which cannot be expressed in p/q form where p and q are integers and q is not equal to zero is called an irrational number. In books on mathematics generally, no notation is specified for the set of all irrational numbers like $\mathbb{Q}$ is used for the set of all rational numbers. … (1.5)

Family of real numbers: $\mathbb{R} = \mathbb{Q} \cup \mathrm{I},$ where I represents the set of all irrational numbers. … (1.6)

Family of complex numbers: $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\},$ where the symbol i is read as iota and denotes the square root of – 1, i.e., $i = \sqrt{-1},$ with $i^2 = -1.$ (1.7)

Note 1: In higher mathematics family of natural, integer, rational, real and complex numbers are denoted by the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R},$ and $\mathbb{C}$ respectively.

After getting the idea about these different families of numbers, the next thing that you should be familiar with is the meaning of constant and variable. Actually, you have used them so many times in school mathematics. Here, we are discussing them because we want to further classify them. In fact, they are a particular type of quantity so the story begins by defining the quantity itself as follows.

Quantity

In mathematics and statistics, quantity means those things on which four basic mathematical operations: addition, subtraction, multiplication and division can be applied. For example: (i) 2, 5 (ii) population mean (μ) (iii) number of children per family (iv) all real numbers between 2 and 3 including 2 and 3, etc. all are examples of different types of quantity which you will come across during your study of different courses of M.Sc. Applied Statistics programme.

A quantity can be classified as a constant or a variable quantity. Constant quantity can be further classified as fixed constant and arbitrary constant/parameter. Variable quantity can be further classified as discrete and continuous. All these classifications are shown in Fig. 1.1.

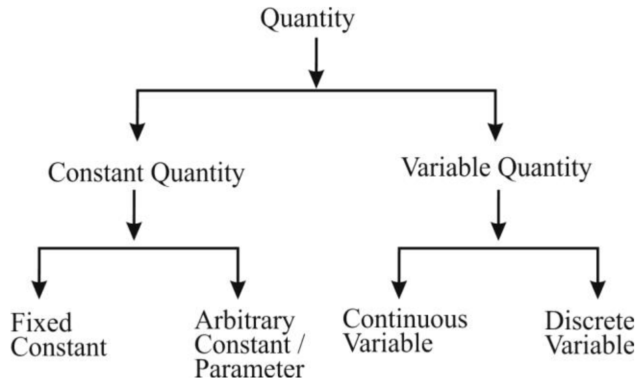


Fig. 1.1: Types of quantity in mathematics and statistics

Constant Quantity

A quantity which remains same (unchanged) throughout a particular problem or distribution is known as a constant quantity. A constant quantity can be further classified as a fixed constant and arbitrary constant (also known as a parameter) which are defined as follows:

Fixed Constant: Those types of constants which always remain the same (unchanged) independent of the place and time are known as fixed constants.

For example: 2, $\frac{3}{2}$, $\sqrt{13}$, $-\sqrt{17}$, π, etc. are fixed constants because the number 2 will remain the same whether you use it in the morning or evening or in India or in any other country. Similarly, the number π which is equal to the quotient of the circumference of a circle ($2\pi r$) and its diameter ($2r$) is a fixed constant. Value of π does not depend on the length of the radius of the circle. Also, remember 22/7 is not the exact value of π it is just an approximate value.

$$\text{Exact value of } \pi = \frac{\text{Length of circumference of the circle}}{\text{Length of diameter of the same circle}}. \qquad \text{... (1.8)}$$

Arbitrary Constant or Parameter: Those types of constants which remain the same in one problem but may vary from problem to problem or distribution to distribution are known as arbitrary constants or parameters and are generally denoted by a, b, c, l, m, n, α, β, γ, μ, σ^2, ρ, etc.

For example, from school mathematics, you know that the equation $y = x + c$ represents a straight line which makes an angle of 45^0 ($\because$ slope is 1 = coefficient of x if we compare it with $y = mx + c$) with a positive direction of the x-axis. For each real value of c, this equation gives a different straight line refer to Fig. 1.2. Here, c behaves like an arbitrary constant because for a particular line, its value is a fixed constant but its value varies from line to line. Similarly, in normal distribution mean (μ) and variance (σ^2) are parameters because they remain fixed constant in a particular normal distribution but vary from normal distribution to normal distribution refer to Fig. 1.3.

Variable

A quantity which may change its value even in a particular problem is known as a variable. For example, in the quadratic equation $x^2 - x - 12 = 0$, x is a variable because it takes two values 4 and – 3.

$$\left[\because x^2 - x - 12 = 0 \;\Rightarrow (x-4)(x+3) = 0 \;\Rightarrow x = 4 \text{ or } -3\right]$$

A variable can be further classified as discrete and continuous explained as follows.

Discrete Variable: If the nature of the quantity is such that its possible values can be written in a sequence then such a quantity is known as **discrete in nature** or **discrete variable**. For example, the number of children in a family in a country may be 0, 1, 2, 3, … . Here, you know that the minimum possible number of children in a family is 0 and after 0 the next possible value for the quantity, the number of children in a family is 1. There is no possibility between 0 and 1 for the quantity, the number of children in a family. Similarly, after 1 the next possible value for the quantity, the number of children in a family is 2 and so on. Here, you can write the possible values in a sequence (0, 1, 2, 3, …) so the quantity, the number of children in a family is a discrete quantity and so is an example of a discrete variable. Other examples of discrete variables are the number of accidents at a particular site on a road, the number of defective items in a packet of 100 items, the outcome of a die when it is thrown once, etc.

Continuous Variable: If the nature of the quantity is such that it can take any possible value between two certain limits, then such a quantity is known as **continuous in nature** or **continuous variable**. In other words, if a quantity is continuous in nature, then you cannot write all its possible values in a sequence. For example, you cannot write all real numbers from 2 to 3 in a

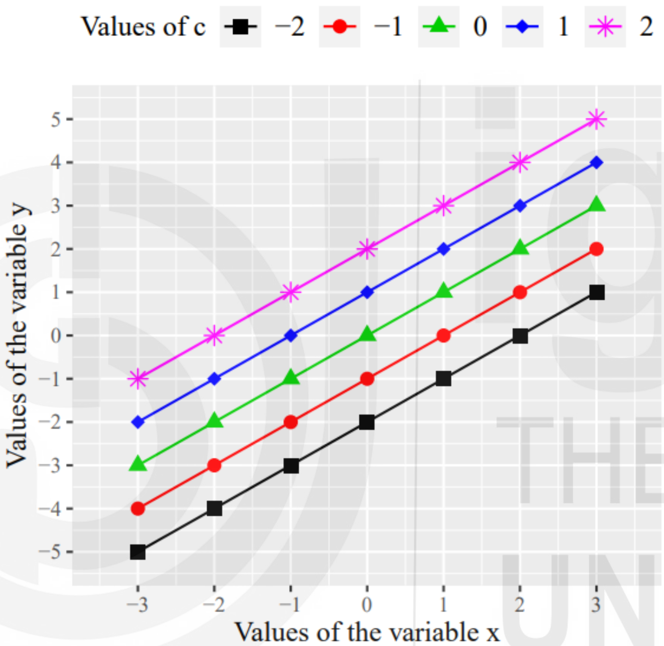


Fig. 1.2: Represents graph of straight lines y = x + c for different values of c

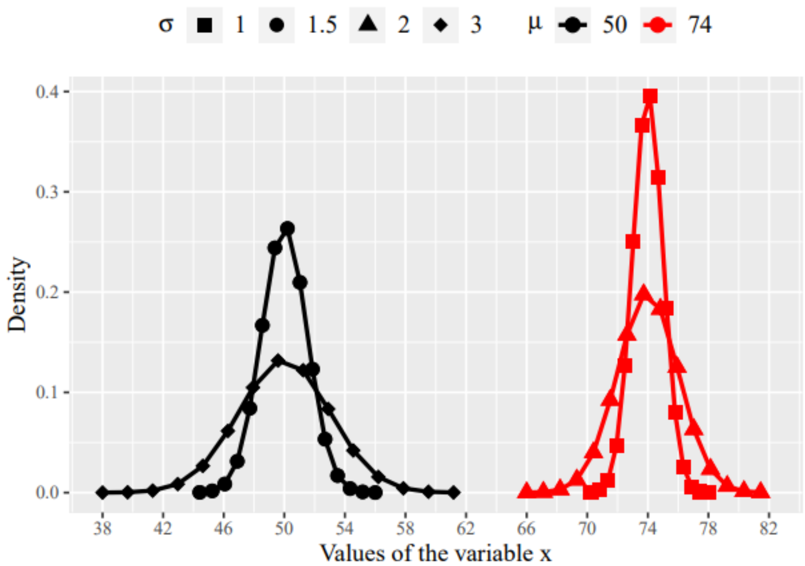


Fig. 1.3: Represents graph of probability density functions of the normal distribution for different values of μ and σ

sequence. In fact, you even cannot tell what is the next possible real number just after 2. If you say 2.1 is the next real number after 2, no it is not the next real number after 2 because 2.01 is between 2 and 2.1. If you say 2.01 is the next real number after 2, no it is not the next real number after 2 because 2.001 is between 2 and 2.01. Similarly, if you say 2.001 is the next real number after 2, no it is not the next real number after 2 because 2.0001 is between 2 and 2.001 and so on. The more advanced mathematical distinction between continuous and discrete quantities can be understood after getting an idea of countable and uncountable sets discussed in Sec. 2.3 of Unit 2 of this course. Other examples of continuous variables are temperature, height, weight, age, etc.

1.3 INTERVAL AND ITS TYPES

Interval

In Sec. 1.2 you have become familiar with well-known families of numbers keep that familiarity in your mind we will recall those families by their names as and when will need.

In this section, we will study interval and its types. Interval makes sense only in the family of real numbers and is defined as follows:

Let $\mathbb{R}$ be the set of all real numbers. Then a set $I \subseteq \mathbb{R}$ is said to be an **interval** if whenever $a, b \in I$ and $a < x < b$ then $x \in I$. For example, the set of all real numbers satisfying $2 \le x \le 3$ is an interval where x can take any real value between 2 and 3 including 2 and 3. … (1.9)

Now, you have gone through the definition of interval so we can explain the point why interval makes sense only in the family of real numbers. For example, consider the family of rational numbers: as $1, 2 \in \mathbb{Q}$ then we cannot form an interval in the family of rational numbers with endpoints 1 and 2 including 1 and 2 because you know that value of $\sqrt{2}$ up to three decimal places is 1.414 and so $1 < \sqrt{2} < 2$ but $\sqrt{2}$ is an irrational number, not a rational number and hence $\sqrt{2} \notin [1, 2]$ because we are assuming the interval [1, 2] in the family of rational numbers. That is why interval makes sense only in the family of real numbers.

Types of Intervals

Intervals are classified on the basis of two criteria:

(a) Whether it contains its endpoints or does not contain its endpoints

(b) Whether its length is finite or infinite

Based on these criteria different types of intervals are defined as follows:

(a) Different Types of Intervals based on the Criteria of Whether it contains its endpoints or does not contain its endpoints

On the basis of endpoints criterion, an interval may be classified as an open or a closed or one side open other side closed interval defined as follows:

Open Interval: An interval $I \subseteq \mathbb{R}$ with end points a and b (a < b) is called an open interval if it contains all real numbers between a and b but does not contain end points a and b. An open interval with endpoints a and b is denoted by (a, b) and mathematically is defined as follows:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\} \qquad \ldots (1.10)$$

For example, open interval (2, 5) contains each real number lying between 2 and 5 but does not contain 2 and 5. To represent an open interval on real line we make open dots at both endpoints. Fig. 1.4 (a) indicates it on the real line.

i.e., $x \in (2, 5) \Rightarrow 2 < x < 5$

Closed Interval: An interval $I \subseteq \mathbb{R}$ with endpoints a and b (a < b) is called closed interval if it contains all real numbers between a and b and also contains endpoints a and b. A closed interval with endpoints a and b is denoted by [a, b] and mathematically is defined as follows:

$$[a, b] = \{x \in \mathbb{R} : a \le x \le b\} \qquad \ldots (1.11)$$

For example, closed interval [2, 5] contains each real number lying between 2 and 5 including 2 and 5. To represent a closed interval on a real line we make solid dots at both endpoints. Fig. 1.4 (b) indicates it on the real line.

i.e., $x \in [2, 5] \Rightarrow 2 \le x \le 5$

Left Open and Right Closed Interval: An interval $I \subseteq \mathbb{R}$ with endpoints a and b (a < b) is called a left open and right closed interval if it contains all real numbers between a and b and also includes only one endpoint b. A left open and right closed interval with endpoints a and b is denoted by (a, b] and mathematically is defined as follows:

$$(a, b] = \{x \in \mathbb{R} : a < x \le b\} \qquad \ldots (1.12)$$

For example, left open and right closed interval (2, 5] contain each real number lying between 2 and 5 including end point 5 but does not contain end point 2. To represent it on real line we have to make an open dot at 2 and a solid dot at 5. Fig. 1.4 (c) indicates it on the real line.

i.e., $x \in (2, 5] \Rightarrow 2 < x \le 5$

Left Closed and Right Open Interval: An interval $I \subseteq \mathbb{R}$ with endpoints a and b (a < b) is called a left closed and right open interval if it contains all real numbers between a and b and also includes only one end point a. A left closed and right open interval with endpoints a and b is denoted by [a, b) and mathematically is defined as follows:

$$[a, b) = \{x \in \mathbb{R} : a \le x < b\} \qquad \ldots (1.13)$$

For example, left closed and right open interval [2, 5) contain each real number lying between 2 and 5 including endpoint 2 but does not contain endpoint 5. To represent it on a real line we have to make a solid dot at 2 and an open dot at 5. Fig. 1.4 (d) indicates it on the real line.

i.e., $x \in [2, 5) \Rightarrow 2 \le x < 5$

(b) Different Types of Intervals based on the Criteria Whether its length is finite or infinite

On the basis of the criterion of length of an interval it may be classified as finite or infinite interval defined as follows:

Length of an Interval: Length of each of the intervals (a, b), [a, b], (a, b], [a, b) is defined as $b - a,\ a < b$.

i.e., length of the interval = difference of the end points … (1.14)

For example, if I = (2, 7) then $\ell(I) = 7 - 2 = 5,$, where $\ell(I)$ denotes the length of the interval I.

Finite Interval: An interval is said to be finite if its length is finite. (1.15)

For example, if I = (– 3, 5) then $\ell(I)$ = 5 – (– 3) = 5 + 3 = 8 which is finite.

$\therefore$ interval I = (– 3, 5) is finite.

Infinite Interval: An interval is said to be infinite interval if its length is not finite. … (1.16)

For example,

(i) The set $\{x \in \mathbb{R} : x > a,\ a \in \mathbb{R}\}$ is an infinite interval and is denoted by (a, ∞). Fig. 1.4 (e) Indicates the infinite interval $(2, \infty)$ on real line.

(ii) The set $\{x \in \mathbb{R} : x < a,\ a \in \mathbb{R}\}$ is an infinite interval and is denoted by $(-\infty,\ a)$. Fig. 1.4 (f) indicates the infinite interval $(-\infty,\ 5)$ on real line.

Similarly, infinite intervals $[a, \infty)$, $(-\infty, a]$ are defined as

$[a,\ \infty) = \{x \in R : x \geq a\}$, where 'a' is a fixed real number

$(-\infty,\ a] = \{x \in R : x \leq a\}$, where 'a' is a fixed real number

Also, $(-\infty,\ \infty) = \{x \in R : -\infty < x < \infty\}$, where 'a' is a fixed real number

Fig. 1.4 (g) to (i) indicate the infinite intervals $(-\infty,\ \infty)$, $[2,\ \infty)$, and $(-\infty,\ 5]$ respectively on real line.

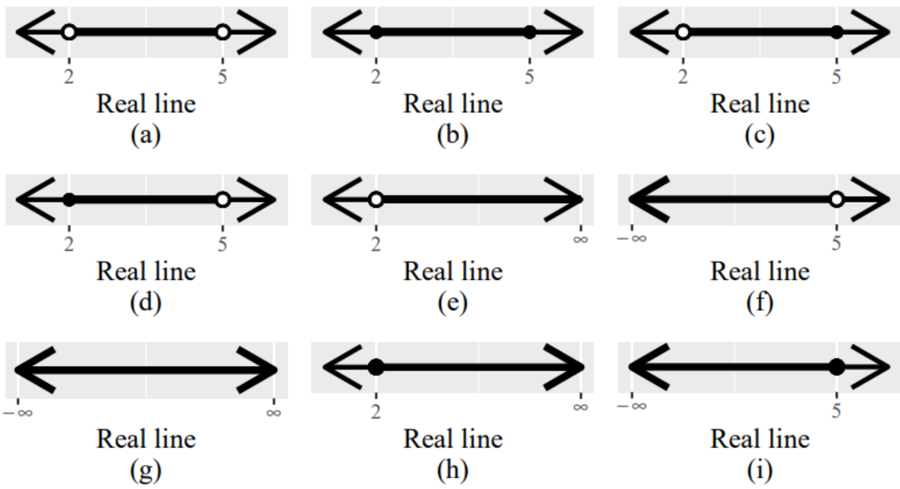


Fig. 1.4: Visualisation of (a) open interval (b) closed interval (c) left open and right closed (d) left closed and right open (e to i) infinite intervals

Remark 1

- Each interval contains infinitely many elements.
- Each interval is an infinite set but an infinite set may or may not be an interval. For example, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are infinite sets but are not intervals.
- A set may or may not be an interval. For example, {2, 7} is a set but not an interval.
- $\mathbb{R}$, set of all real numbers, is an infinite interval given by $\mathbb{R} = (-\infty, \infty)$.
- Remember that $-\infty$ and ∞ are not included in the set of real numbers. Also, if extreme value is ∞ or $-\infty$, then open bracket is used on that side of the interval having ∞ or $-\infty$. In Unit 3 of this course we will extend set of real numbers by including two special numbers $-\infty$ and $+\infty$ and we will denote this new set by $\mathbb{R}^* = (-\infty, \infty) \cup \{-\infty, +\infty\} = \mathbb{R} \cup \{-\infty, +\infty\}$.
- When we say that x is a finite number (real number) it means that $-\infty < x < \infty$.

Now, we are in a position to define a function.

1.4 DEFINITION, NOTATIONS AND PICTORIAL PRESENTATION OF A FUNCTION

Definition of a Function

You know about function from school mathematics. You can treat function as a machine if it works in such a way that (i) corresponding to each input it gives output and (ii) output is unique for each input. Mathematically, a function is defined as follows:

Let X and Y be two non-empty sets. Then a rule which associates each element of X to a unique element of Y is called a function from X to Y. X is called **domain** of the function. Y is called **co-domain** of the function, and the set of only those values of Y for which function is defined is called **range** of the function. That is, the subset $\{y \in Y : y = f(x) \text{ for some } x \in X\}$ of Y is called range of the function. … (1.17)

If we treat function as a machine then the set of all inputs is called domain of the function and the set of all outputs is called range of the function. If we define a function $f : \mathbb{Z} \to \mathbb{N}$ by

$$y = f(x) = x^2 + 1, \quad x \in \mathbb{Z}$$

then its visualisation as a machine is shown in Fig. 1.5.

Usual Notations used for Defining a Function

(i) A function is generally denoted by $f, g, h, \phi, \psi,$ etc., in the case of above definition we write f: $X \to Y$ and read as f is a function from X to Y.

(ii) A function f: $X \to Y$ is generally described by writing $y = f(x), \ x \in X$, where $f(x)$ is an expression in terms of x.

For example, the function used in Fig. 1.5 has expression $f(x) = x^2 + 1$.

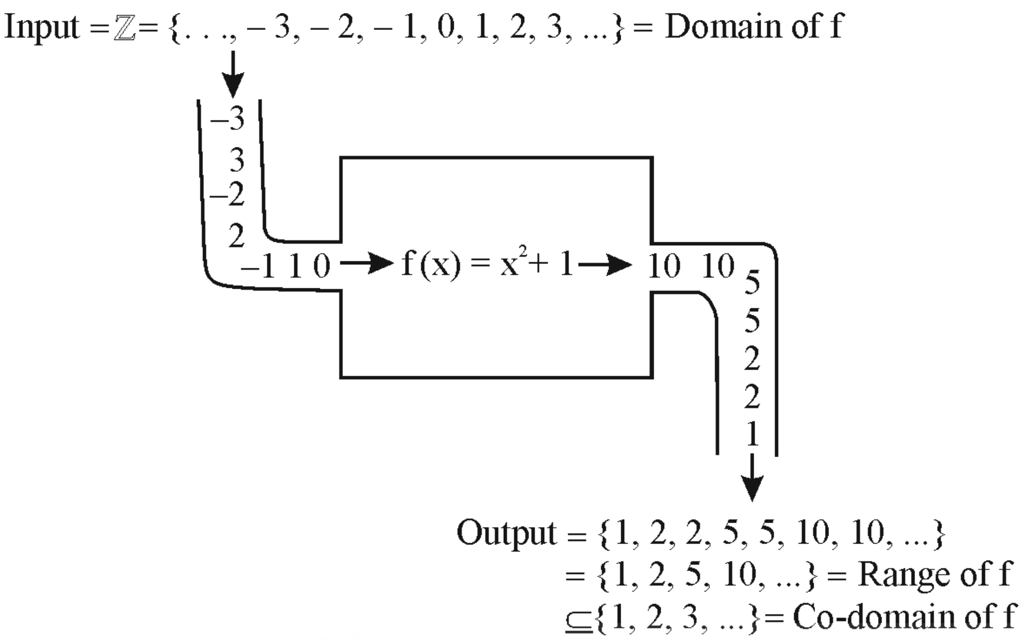


Fig. 1.5: Visualisation of treating function as a machine

Pictorial Presentation of a Function

> **There are two conditions for a rule to be a function.**
>
> (i) Each element of X must be associated to some element of Y.
>
> (ii) There is unique element of Y corresponding to each element of X.

Visualisation of a concept helps in understanding the concept in a better way. Following the same strategy, we are using pictorial presentation of a function which is an attempt to explain the idea of the function through diagram known as pictorial presentation. The box given in the margin of this page specifies the two requirements for a rule to be a function. These two requirements are also satisfied by the association of daughter with her mother. So, to explain the idea of the function through pictorial presentation we will continue with example of daughters and mothers.

Let X = D = set of daughters and Y = M = set of mothers. Then consider the following situations given in Fig. 1.6 (a) to (d) with the help of the diagrams.

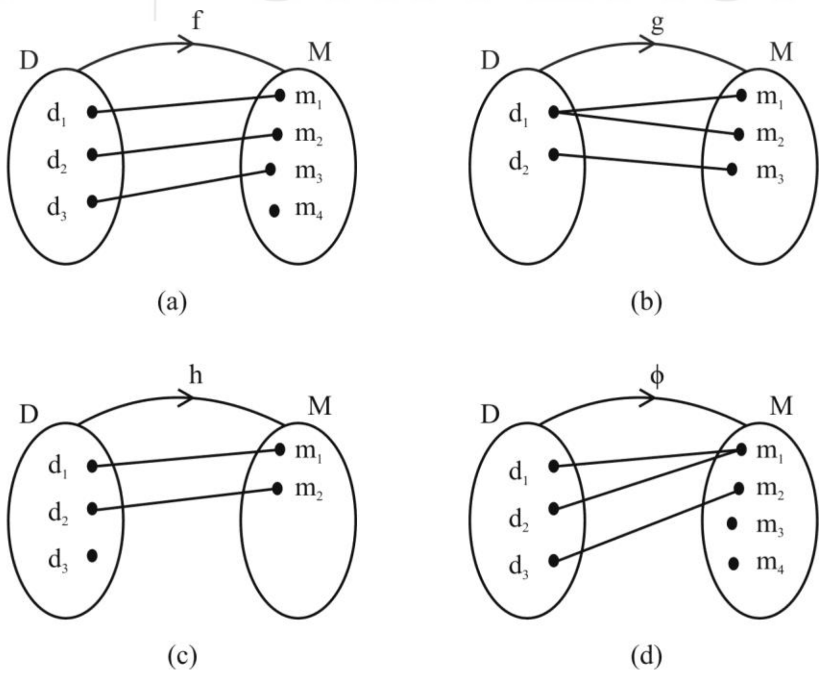


Fig. 1.6: (a) to (d) Pictorial presentation of rules f, g, h, ϕ

The rule f shown in Fig. 1.6 (a) is a function because each daughter has unique mother. [i.e., both conditions mentioned in the box are satisfied] and domain of the function $f =$ set of all daughters $= D = \{d_1, d_2, d_3\}$

co-domain of f = set of all mothers = $M = \{m_1, m_2, m_3, m_4\}$

range of $f =$ set of those mothers who have at least one daughter
$= \{m_1, m_2, m_3\}$

Note 2: One point which may come in your mind is that if m_4 is a mother then there should be at least one daughter of the mother m_4. But as we know that to become a mother it is not necessary that there should be a daughter. A mother may have only one son or only two sons or more than two sons without a daughter.

The rule g shown in Fig. 1.6 (b) is not a function because daughter d_1 has two mothers m_1 and m_2 which is not possible. [i.e., condition (ii) given in the box is not satisfied]

The rule h shown in Fig. 1.6 (c) is not a function because daughter d_3 has no mother. If the daughter d_3 came in this world, then there should be some mother of the daughter d_3. [i.e., condition (i) given in the box is not satisfied]

The rule ϕ shown in Fig. 1.6 (d) is a function because each daughter has unique mother. We see that both daughters d_1, d_2 have the same mother, no problem it is possible. Further mothers m_3, m_4 have no daughters again no problem it is also possible. In this case:

domain of the function $f = \{d_1, d_2, d_3\} = D =$ set of all daughters,

co-domain of the function $\phi = \{m_1, m_2, m_3, m_4\} = M =$ set of all mothers and range of the function $f = \{m_1, m_2\} =$ set of only those mothers who have at least one daughter.

Remark 2

- If $f : X \to Y$ is a function given by $y = f(x)$ then x is known as **pre image** of y and y is known as **image** of x.
- If $y = f(x)$ is a function then values of y depend on values of x. So, y is known as **dependent variable** and x is known as **independent variable**.
- In descriptive statistics we also use dependent and independent variables, especially in correlation and regression. In biostatistics terminology that is generally followed is: independent variable is known as **regressor** and dependent variable is known as **response variable**. In the course MST-024 on machine learning independent variable is known as **feature** and dependent variable is known as **target variable**. So, keep this terminology alert in your mind.

Some Examples of Functions

Example 1: Let $f : \mathbb{N} \to \mathbb{N}$ be defined by

$f(n) = 3n, \quad n \in \mathbb{N}$

Express the function diagrammatically. Also write domain, range and co-domain of the function. Also make the graph of this function. Further, if instead of from $\mathbb{N}$ to $\mathbb{N}$ suppose f is from $\mathbb{R}$ to $\mathbb{R}$ draw its graph.

Solution: $f : \mathbb{N} \to \mathbb{N}$ be defined by

$f(x) = 3n, \quad n \in \mathbb{N}$

$\therefore f(1) = 3, \; f(2) = 6, \; f(3) = 9$ and so on.

Pictorial presentation of the function is shown in Fig. 1.7.

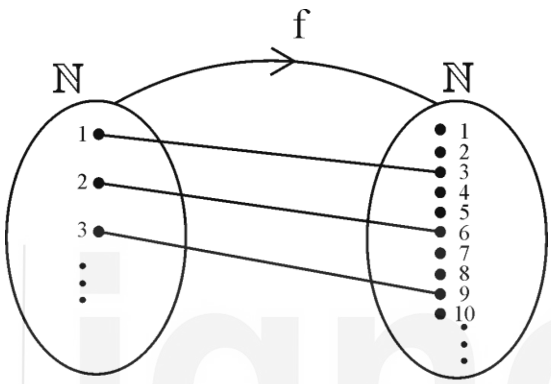


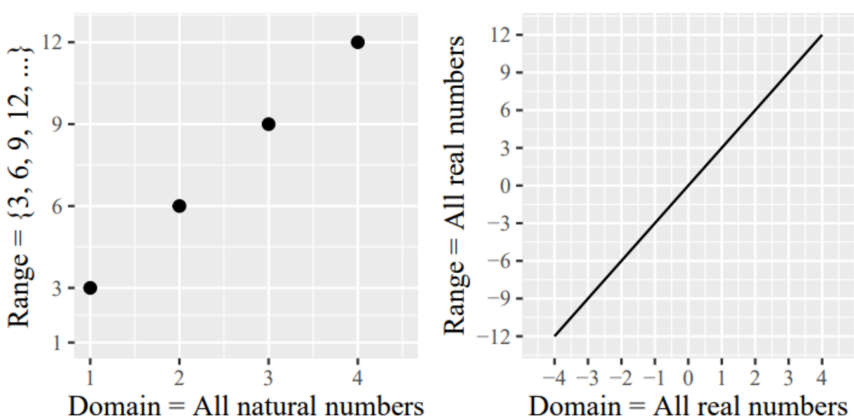
Fig. 1.7: Pictorial presentation of the function

Domain of the function $f = \{1, 2, 3, \ldots\} = \mathbb{N}$

Range of the function f = Set of only those values for which function is defined $= \{3, 6, 9, \ldots\}$

Co-domain $= \{1, 2, 3, \ldots\} = \mathbb{N}$

Graph of the function when both domain and co-domain are set of natural numbers is shown in Fig. 1.8 (a). Graph of the function when both domain and co-domain are set of real numbers is shown in Fig. 1.8 (b). Note the effect of domain and range on the graph of a function. Here mathematical expression of the function is the same but domain and range are different. In Fig. 1.8 (a) only points at (1, 1), (2, 2), (3, 3), (4, 4), … represent its graph while in Fig. 1.8 (b) complete line represent its graph.



(a) (b)

Fig. 1.8: Graph of the function f(x) = 3x (a) when f is from $\mathbb{N}$ to $\mathbb{N}$ (b) when f is from $\mathbb{R}$ to $\mathbb{R}$

Example 2: Find the domain of the function $f:\mathbb{R}\to\mathbb{R}$, defined by

$$f(x)=\sqrt{(x-3)(5-x)}\ ,\quad x\in\mathbb{R}$$

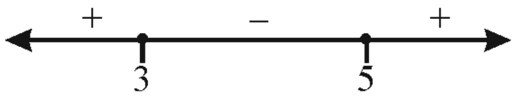
Also evaluate f(3), f(4), f(5), f(7).

Solution: Given function is

$$f(x)=\sqrt{(x-3)(5-x)},\quad x\in\mathbb{R}$$

For $f(x)$ to be real, the quantity under the square root should be non-negative and hence

$$(x-3)(5-x)\geq 0$$



$$\Rightarrow -(x-3)(x-5)\geq 0$$

$$\Rightarrow (x-3)(x-5)\leq 0 \quad \left[\begin{array}{l}\because \text{ when we multiply by } -1 \text{ on both sides of an} \\ \text{inequation then its inequality changes. For more} \\ \text{detail you may refer Sec. 6.7 of Unit 6 of this course}\end{array}\right]$$

Now, if we choose x less than 3, the LHS of this inequality comes out to be (–ve) (–ve) = + ve hence does not satisfy the inequality.

Also, if we choose x greater than 5, the LHS of this inequality comes out to be (+ve) (+ve) = +ve again does not satisfy this inequality.

But, if we take 3 < x < 5, the LHS becomes (+ve) (–ve) = –ve and hence satisfies the inequality. Also, x = 3 and x = 5 satisfy inequality. So, required domain of the given function is [3, 5].

Now, the number 7 does not lie in the domain of f so f(7) cannot be obtained for this function. However, 3, 4 and 5 all lie in the domain of f so values of f at 3, 4 and 5 can be obtained and are given as follows.

$$f(3)=\sqrt{(3-3)(5-3)}\ =\ \sqrt{0\times 2}\ =\ \sqrt{0}=0$$

$$f(4)=\sqrt{(4-3)(5-4)}\ =\ \sqrt{1\times 1}\ =\ 1$$

$$f(5)\ =\ \sqrt{(5-3)(5-5)}\ =\ \sqrt{2\times 0}=\sqrt{0}=0$$

Now, you can try the following Self Assessment Question.

SAQ 1

Find the domain and range of the function $f:\mathbb{R}\to\mathbb{R}$ given by

$f(x) = \dfrac{1}{x-2}$, $x\in\mathbb{R}$. Also, evaluate $f(1)$, $f(3)$, $f(-5)$.

1.5 SOME SPECIAL FUNCTIONS HAVING APPLICATIONS IN STATISTICS, PROBABILITY AND MACHINE LEARNING

In the previous section you have seen that function is a rule (satisfying two conditions mentioned in the box at page number 18) which associates the elements of one set to the elements of another set. In different courses of the programme M.Sc. Applied Statistics (MSCAST) you will used many functions. Some well-known and commonly used functions are discussed in this section. Let us start with the easiest function known as constant function.

Constant Function

Suppose you have a special coin having head on both sides. Then whenever you will throw it, each time its outcome will be head. Ignore the possibility of landing on edge like Sholay film. If X is a variable having value 1 if head appears and 0 otherwise then X will always take value 1. Such type of behaviour is represented by a constant function in mathematics and **degenerate random variable** in probability theory. In mathematics constant function is defined as follows:

Let $X, Y \subseteq \mathbb{R}$, then a function $f : X \to Y$ is said to be a constant function if it associates each element of its domain to a single number in its co-domain and in general is defined as

$f(x) = a, \quad \forall\, x \in \mathbb{R}$, where a is a real constant. … (1.18)

i.e., a function is constant if its range is a singleton set.

For example,

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by

$f(x) = 3, \qquad n \in \mathbb{N}$

is a constant function because all elements of the domain are associated to the single element 3 and its pictorial presentation is shown in the Fig. 1.9 while graphical presentation of this function is shown in Fig. 1.10 (a).

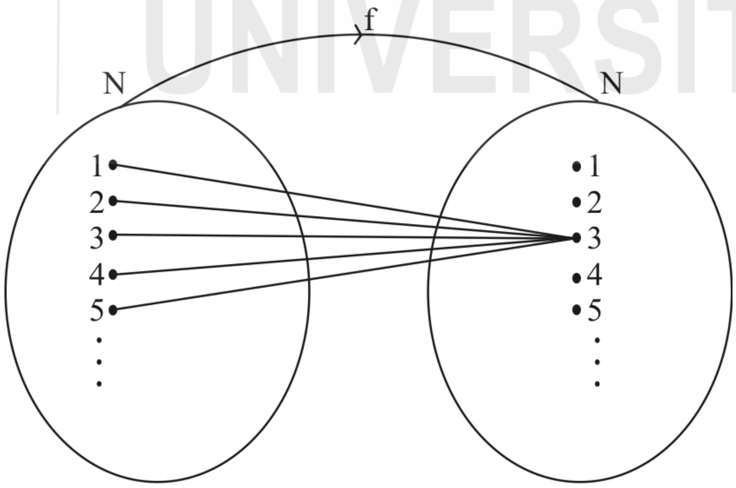
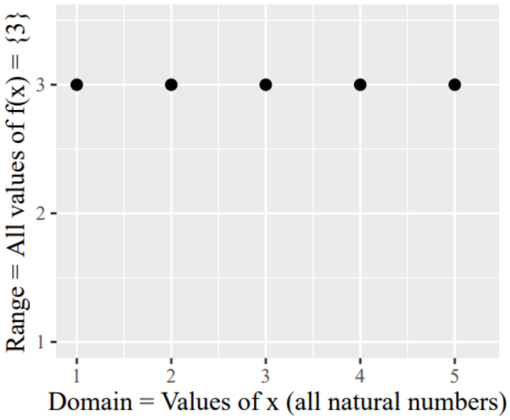


Fig. 1.9: Pictorial presentation of the constant function f(x) = 3

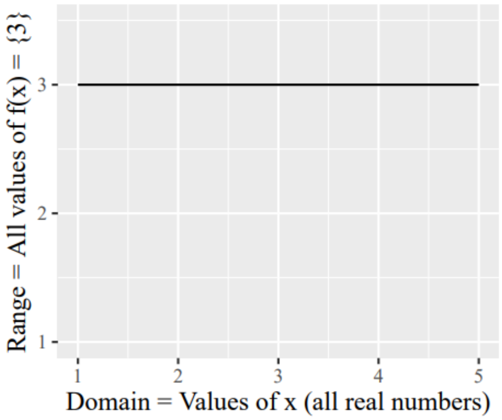
(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by

$f(x) = 3, \; x \in \mathbb{R}$

is also a constant function and its graph is shown in Fig. 1.10 (b).



(a)



(b)

Fig. 1.10: Graph of the constant function f(x) = 3 (a) when f is from $\mathbb{N}$ to $\mathbb{N}$ (b) when f is from $\mathbb{R}$ to $\mathbb{R}$

Remark 3

If you connect this function with the example of daughters and mothers then you find that all daughters have 3 as their mothers. But then mother 3 has infinitely many daughters. In real life you will not find a mother who has infinite daughters. So, keep in mind that example of daughters and mothers is given just for better understanding of the two conditions for a rule to become a function. So, limit that example only for understanding the two conditions required for a rule to be a function. The next function that you are going to study is identity function. You will see that identity function associates each element with itself. But in real life a daughter cannot be mother of herself. So, again the same reason example of daughters and mothers is given just for better understanding of the two conditions for a rule to become a function. Keep this point in mind.

Identity Function

You know from school mathematics that when you multiply a real number with 1 outcome will be the same number itself. That is, for any real number a, we have a.1 = 1.a = a. In the universe of functions identity function plays the same role as 1 plays in the family of real numbers. In mathematics identity function is defined as follows:

Let $X \subseteq \mathbb{R}$, a function $f : X \to X$ is said to be an identity function if it associates each element with itself and in general it is defined as

$$f(x) = x, \qquad x \in X \qquad \ldots (1.19)$$

For example,

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = n, \qquad n \in N \qquad \ldots (1.20)$$

is an identity function because it associates each natural number with itself. Pictorial presentation of this function is shown in the Fig. 1.11.

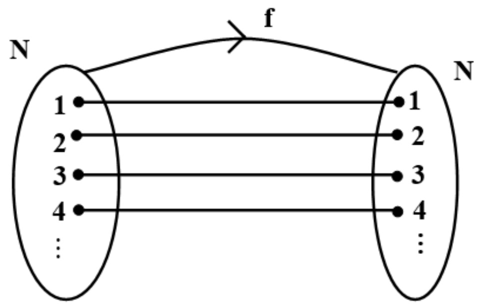


Fig. 1.11: Pictorial presentation of identity function in the world of natural numbers

(ii) Instead of set of natural numbers if you want to define identity function in the set of real numbers then it is defined the same way just replace $\mathbb{N}$ by $\mathbb{R}$ as follows:

$f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x, \quad x \in \mathbb{R} \qquad \dots (1.21)$$

Graphs of identity function in the family of natural numbers and in the family of real numbers are shown in Fig. 1.12 (a) and (b) respectively.

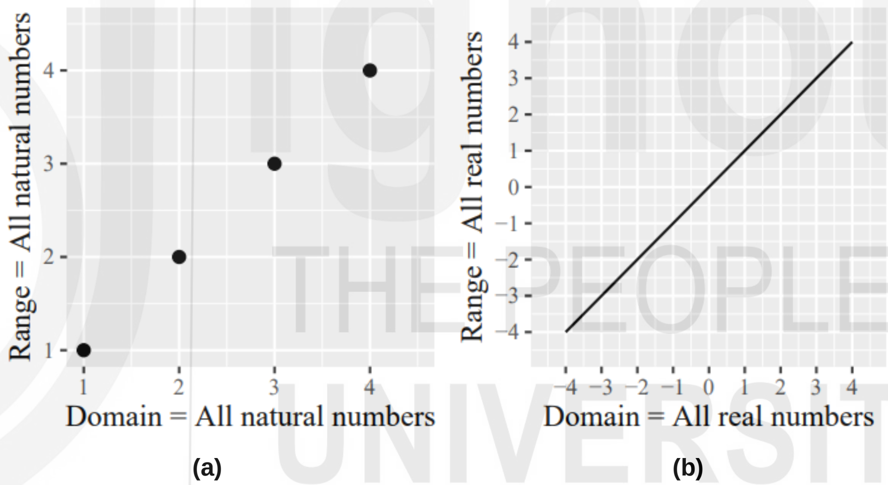


Fig. 1.12: Graph of the identity function f(x) = x (a) when f is from $\mathbb{N}$ to $\mathbb{N}$ (b) when f is from $\mathbb{R}$ to $\mathbb{R}$

Polynomial Function

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a polynomial function of degree n (only possible values of n are 0, 1, 2, 3, 4, …) if it is defined as

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n, x \in \mathbb{R} \qquad \dots (1.22)$$

where $a_0, a_1, a_2, \dots, a_{n-1}, a_n \in \mathbb{R}$, $a_0 \neq 0$ are constants

e.g., f(x) = $2x^3 + x^2 - x + 5$, is a polynomial function of degree 3. But the expression $f(x) = 2x^{3/2} + x^2$ is not a polynomial because the exponent of x is 3/2 in the first term which is not allowed in a polynomial. Similarly, the expression $f(x) = x + \frac{1}{x}$ is not a polynomial because in the second term exponent of x is -1 which is not allowed in a polynomial. Possible values of exponent in a polynomial are from family of whole numbers, i.e., 0, 1, 2, 3, ….

Zero of a Polynomial: If f(x) is a polynomial then those values of x which makes value of polynomial zero are known as zeros of the polynomial. That is if x = a is such that f(a) = 0 then we say that x = a is a zero of the polynomial.

For example, if $f(x) = x^2 - x - 12$ then $f(4) = 4^2 - 4 - 12 = 0$. So, x = 4 is a zero of the polynomial $f(x) = x^2 - x - 12$.

Multiplicity of a Zero of a Polynomial: Multiplicity of a zero tells us about the behaviour of the polynomial near zero. If $f(x) = (x - a)^m(x - b)^n(x - c)$ then we say that multiplicity of zeros a, b, c are m, n, 1 respectively.

For example, if $f(x) = (x - 3)^7(x - 5)^9(x - 13)$ then multiplicity of zeros 3, 5 and 13 are 7, 9 and 1 respectively.

You know that a polynomial of degree 0, 1, 2, 3, 4, 5 are known as **constant**, **linear**, **quadratic**, **cubic**, **quartic** and **quintic polynomial**. In statistics we use polynomials in regression analysis, time series analysis, etc. The shape of a polynomial is dictated by three things (i) degree of the polynomial (ii) sign of the leading coefficient (iii) multiplicity of the zeros of the polynomial. Let us visualise the effect of these three things through Fig. 1.13 (a) to (f) and Fig. 1.14 (a), (b) to Fig. 1.16 (a), (b) which will help you to identify the suitable polynomial which fits the given data. Fig. 1.13 (a) to (f) shows respectively graphs of the following polynomial functions:

$$f(x) = 2,\quad f(x) = -2,\quad f(x) = x + 1,\quad f(x) = -x + 1,$$
$$f(x) = (x + 1)(x - 1) = x^2 - 1,\quad f(x) = -x^2 - 1$$

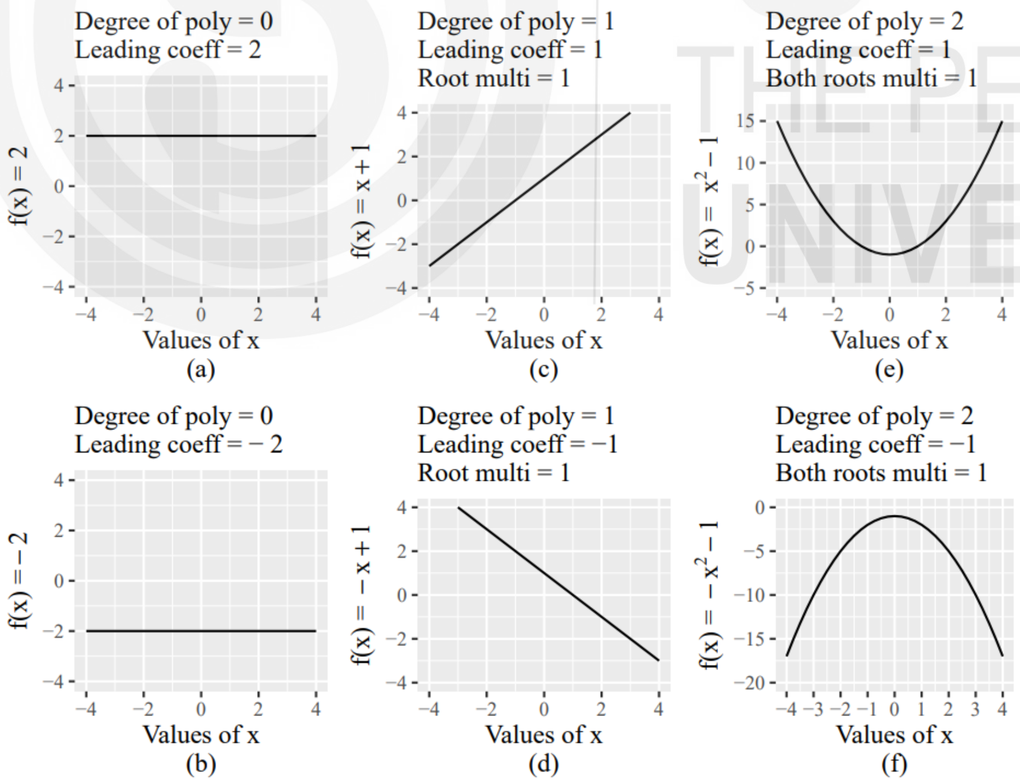


Fig. 1.13: Graphs of the different polynomials of degree 0, 1 and 2 with positive leading coefficient in (a), (c), (e) and negative leading coefficient in (b), (d), (f) and having root(s) multiplicity one

Polynomials shown in Fig. 1.14 (a), (b) to Fig. 1.16 (a), (b) are mentioned in the graphs. From the graphs of these polynomials note that:

- **Even Degree and Positive Leading Coefficient**: End behaviour of the curve on both sides is upward. Refer Fig. 1.16 (a). … (1.23)

- **Even Degree and Negative Leading Coefficient**: End behaviour of the curve on both sides is downward. Refer Fig. 1.16 (b). Note that in the case of even degree either both ends will be upward or downward. … (1.24)

- **Odd Degree and Positive Leading Coefficient**: End behaviour of the curve will be left downward but right upward. Refer Fig. 1.14 (a) and Fig. 1.15 (a). … (1.25)

- **Odd Degree and Negative Leading Coefficient**: End behaviour of the curve will be left upward but right downward. Refer Fig. 1.14 (b) and Fig. 1.15 (b). Note that in the case of odd degree both ends behave opposite to each other, i.e., if one end is upward then another will be downward. … (1.26)

- **Effect of Even Multiplicity of a Zero**: If multiplicity of a zero is even then graph of the polynomial touches the x-axis at that point. Refer Fig. 1.15 (a) and (b) both zeros at $x = -2$ and 2 are of even multiplicity. So, graphs touch x-axis at both points. In Fig. 1.16 (a) and (b) two zeros at $x = 2$ and 3 are of even multiplicity. So, graphs touch x-axis at these points. … (1.27)

- **Effect of Odd Multiplicity of a Zero**: If multiplicity of a zero is odd then graph of the polynomial crosses the x-axis at that point. Further, if multiplicity is 1 then it crosses x-axis at that zero like a straight line (refer Fig. 1.14 (a) and (b) it crosses x-axis at three zeros. Also, in Fig. 1.16 (a) and (b) it crosses x-axis at $x = 3.5$ but if multiplicity is odd other than 1 like 3, 5, 7, etc. then it crosses x-axis at that zero such that it tries to flat at zero. Refer Fig. 1.16 (a) and (b) at the zero $x = 1$. … (1.28)

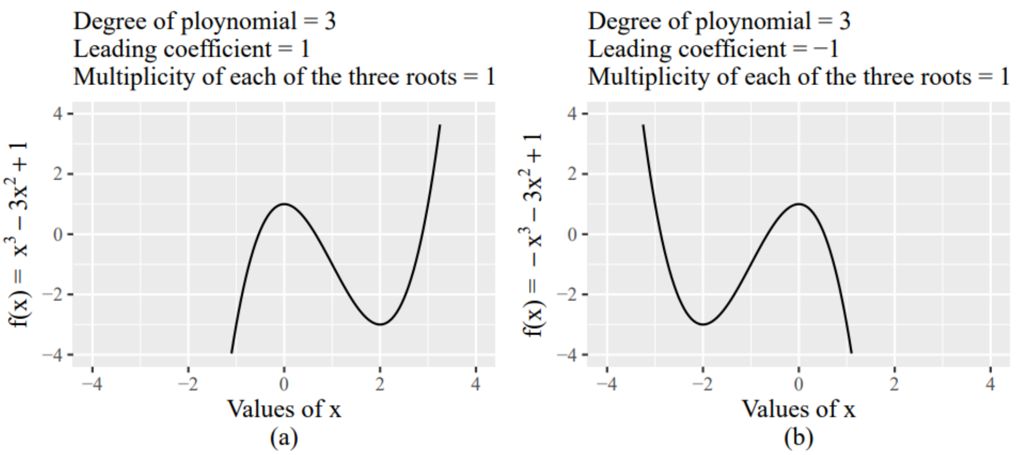


Fig. 1.14: Graphs of the cubic polynomials (a) with positive leading coefficient, and (b) negative leading coefficient

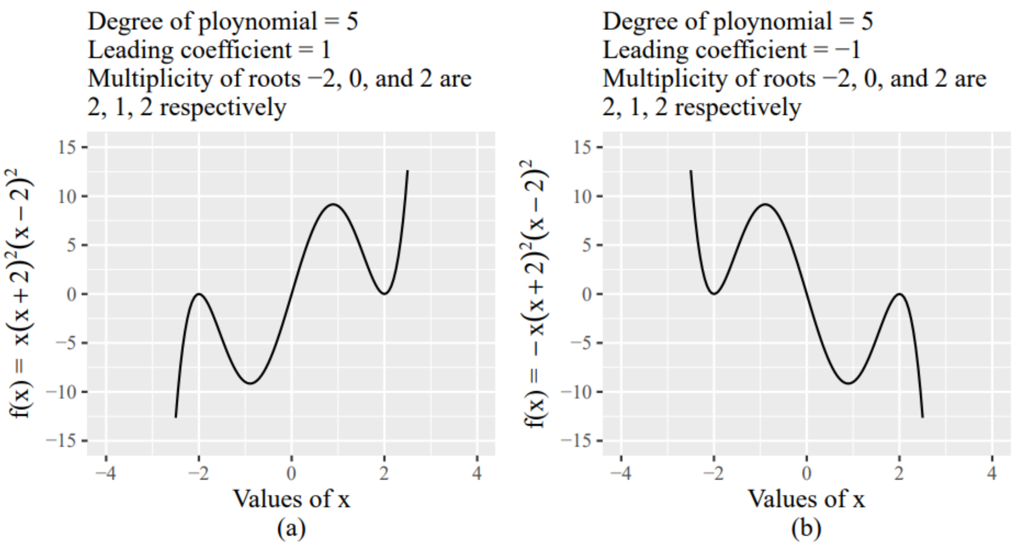


Fig. 1.15: Graphs of the polynomials of degree 5 (a) with positive leading coefficient, and (b) negative leading coefficient

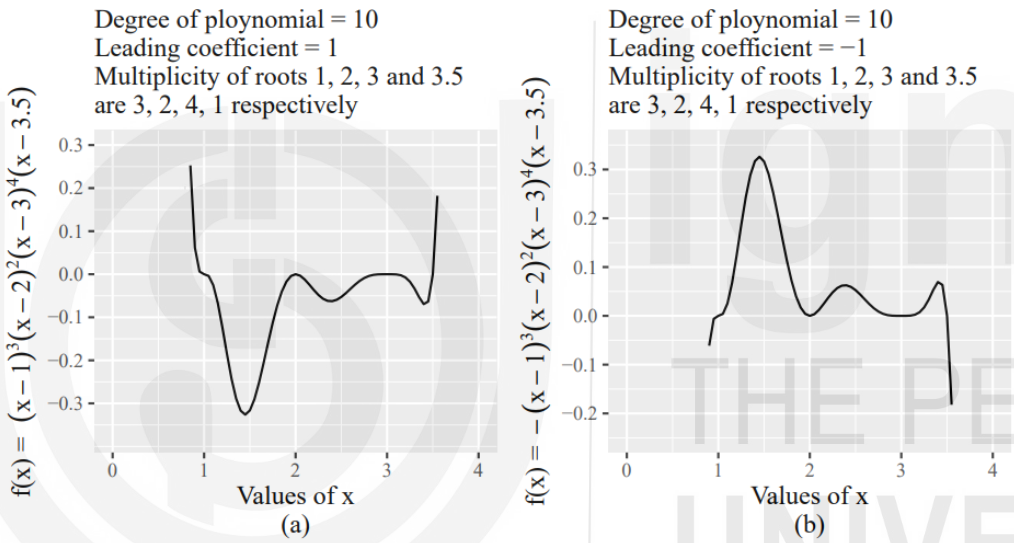


Fig. 1.16: Graphs of the 10-degree polynomials having four roots 1, 2, 3 and 3.5 with multiplicities 3, 2, 4, 1 respectively (a) with positive leading coefficient, and (b) negative leading coefficient

Exponential Function

From school mathematics you know that if you invest Rs P at r% per annum for a period of n years compounded annually then amount A after n years is given by

$$A = P\left(1 + \frac{r}{100}\right)^n$$

Suppose P = Rs 1, r = 100% per annum then formula of amount reduces to

$A = 2^n$, where n is time and 2 represents growth rate.

So, if you invest Rs 1 for a period of 4 years and growth rate is 2 then after 4 years your amount will be $A = 2^4 = \text{Rs } 16.$ Such type of functions where we have a constant in the base and variable (here time) in the power are known as exponential functions. Mathematically, an exponential function is defined as follows:

A function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=a^x,\quad x\in\mathbb{R},\qquad a>0,\ a\neq 1 \qquad \ldots (1.29)$$

is called an exponential function.

i.e., the nature of an exponential function is of the form = $(\text{constant})^{\text{variable}}$

For example, $f(x)=2^x,\ g(x)=10^x,\ h(x)=\left(\frac{1}{2}\right)^x$, etc. all are exponential functions.

Domain and Range of the Exponential Function: In exponential function variable x (say) is sitting in exponent of a constant so function is defined for all real values of the variable x and hence **domain** of an exponential function is the set of all real numbers. Suppose $f(x)=2^x$ then note that values of this function will never be negative and zero for every real value of x

$$\left[\begin{array}{l}\because 2^0=1,\ 2^1=2,\ 2^2=4,\ 2^3=8,\ \text{etc.}\\ 2^{-1}=\frac{1}{2}=0.5,\ 2^{-2}=\frac{1}{2^2}=\frac{1}{4}=0.25,\ 2^{-3}=\frac{1}{8}=0.125,\ \text{etc.}\end{array}\right]$$

and hence its range is the set of positive real numbers $\mathbb{R}^+=(0,\infty)$. This is not only true for this particular exponential function but it is true in general for all the exponential functions so **range** of an exponential function is $\mathbb{R}^+=(0,\infty)$. … (1.30)

Graphical Behaviour of Exponential Function: Steepness of an exponential function is dictated by its rate. Further, if rate is < 1 then with increase in time its value will decay, and when its rate is > 1 then its value will increase with time. Fig. 1.17 (a) shows decay as a = 1/2 < 1 while Fig. 1.17 (b) shows growth as a = 2 > 1. Note that value of each exponential function at x = 0 is 1. So, graph of every exponential function will pass through the point (0, 1) so this special point is also indicated in the graph.

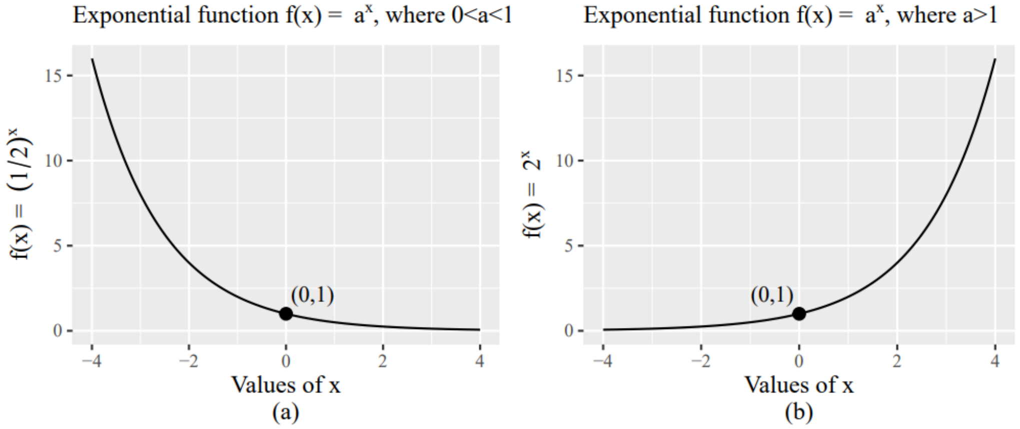


Fig. 1.17: Graph of exponential function (a) $f(x) = (1/2)^x$ (b) $f(x) = 2^x$

Example 3: In the following exponential functions name the rate, time and amount.

(i) $A = 3^n$ (ii) $B = 10^m$ (iii) $a^k = x$ (iv) $8^{(4/3)} = 16$

Solution: We know that an exponential function can be interpreted as $\text{Amount} = (\text{rate})^{\text{time}}$

Therefore, we have

(i) Amount = A, rate = 3, time = n

(ii) Amount = B, rate = 10, time = m

(iii) Amount = x, rate = a, time = k

(iv) Amount = 16, rate = 8, time = 4/3

Logarithm Function

In Sec. 1.4 you have seen that function can be treated as a machine (refer Fig. 1.5) which gives unique output corresponding to each input. You know that in an exponential function if rate is 10 and time is 3 then output (amount) will be 1000. If you are interested in another machine (function) which works in such a way that if output of an exponential function is given as input in this new machine, then its output should be corresponding input of the exponential function such a machine (function) is known as logarithm function. In this new machine rate of exponential function is known as base of the logarithm function. Table 1.1 explains this inverse relationship between exponential and logarithm functions and Fig. 1.18 shows both functions as machines simultaneously.

Table 1.1: **Exponential and logarithm functions are inverse of each other**

Exponential Function			**Logarithm Function**		
	Function as a machine			Function as a machine	
Rate	Input	Output	Base	Input	Output
2	5	$2^5 = 32$	2	32	$\log_2 32 = 5$
10	3	$10^3 = 1000$	10	1000	$\log_{10} 1000 = 3$
1/2	4	$(1/2)^4 = 1/16$	1/2	1/16	$\log_{1/2}(1/16) = 4$
2	-5	$2^{-5} = 1/32$	2	1/32	$\log_2(1/32) = -5$
10	-3	$10^{-3} = 1/1000$	10	1/1000	$\log_{10}(1/1000) = -3$
1/2	-4	$(1/2)^{-4} = 16$	1/2	16	$\log_{1/2} 16 = -4$
2	0	$2^0 = 1$	2	1	$\log_2 1 = 0$
e	x	e^x	e	e^x	$\log e^x = x$

Note that when output of an exponential function is given as input to the logarithm function then output of the logarithm function matches with corresponding input of the exponential function and vice versa. In the last row of the Table 1.1 the number e is a special number in mathematic. You have studied expansion of e^x in school mathematics and the same is given as follows.

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \dots \qquad \dots (1.31)$$

So, value of e can be obtained just replacing x by 1 in this expansion of e^x and the same is given by the infinite series (you will study about infinite series in Sec. 4.4 of Unit 4 of this course)

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 5} + \dots \qquad \dots (1.32)$$

You know that value of e up-to 3-decimal place is 2.718. In fact, it is an irrational number. In the course MSTL-001 you will see that this value of e can be obtained in R by typing exp(1) on R console and hitting enter.

Mathematically, logarithm function is defined as follows:

A function $f : \mathbb{R}^+ \to \mathbb{R}$ is said to be logarithm function if it is defined as

$$y = f(x) = \log_a x, \quad x \in \mathbb{R}^+ = \text{set of all positive real numbers} \qquad \dots (1.33)$$

where a > 0 and $a \neq 1$

You have seen that if $a^m = n$ then in terms of logarithm we write it as $\log_a n = m$.

Domain and Range of the Logarithm Function: Since logarithm function is inverse of exponential function so range of exponential function will be domain of logarithm function and domain of exponential function will be range of the logarithm function. Hence, domain of the logarithm function is $\mathbb{R}^+ = (0, \infty)$ and range of the logarithm function is $\mathbb{R} = (-\infty, \infty)$. … (1.34)

Graphical Behaviour of Logarithm Function: If base is < 1 then logarithm function is decreasing and when its base is > 1 then it is an increasing function. Note that $\log_{10} 10 = 1, \log_{10} 100 = 2, \log_{10} 1000 = 3, \log_{10} 10000 = 4.$ See the difference between the input numbers (10, 100 = 10^2, 1000 = 10^3, 10000 = 10^4) and output numbers (1, 2, 3, 4). We observe that variation between the numbers 10, 100, 1000 and 10000 is huge compare to the variation of the output numbers 1, 2, 3 and 4. This is one of the silent features of logarithm function and due to this it facilitates graphical represents of the numbers with large variations. For example, if you want to compare the sizes of the populations of all Asian countries in the same plot then you should plot values of population after taking logarithm of population sizes of each country. Without logarithm plot will not be more informative due presence of two large populations figures of India and China. Fig. 1.19 (a) and (b) shows graphical behaviour of logarithm function when a = 1/2 < 1 and when a = 2 > 1. In the first case logarithm function is decreasing and in the second case it is increasing. Like an exponential function there is a special point in the graph of logarithm function. Refer second last row of the Table 1.1 which says that $\log_2 1 = 0.$ There is nothing special in base 2 you can take any base according to the definition of logarithm function its value at 1 will always be 0. So, graph of every logarithm function will pass through the point (1, 0) which is also shown in the graph.

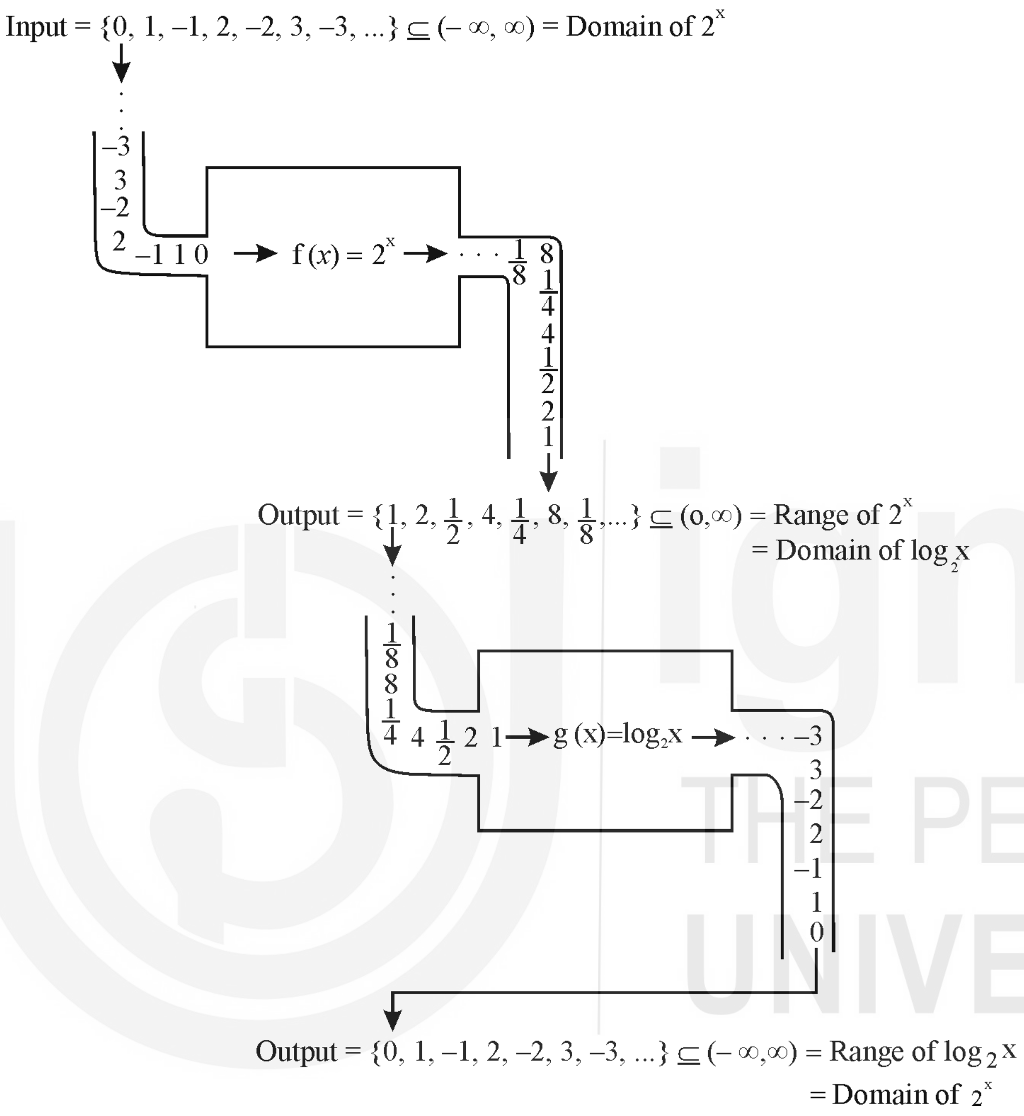


Fig. 1.18: Exponential and logarithm functions are machines which undo effect of each other

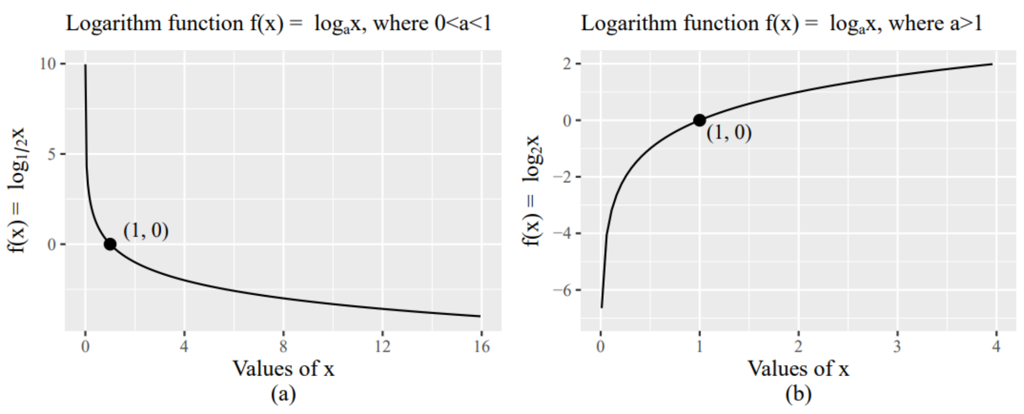


Fig. 1.19: Graph of logarithm function (a) f(x) = $\log_{1/2} x$ (b) f(x) = $\log_2 x$

Finally, following are some important laws of logarithm function

1. $\log_a mn = \log_a m + \log_a n$ … (1.35)

2. $\log_a \frac{m}{n} = \log_a m - \log_a n$ … (1.36)

3. $\log_a m^n = n \log_a m$ … (1.37)

4. $a^{\log_a m} = m$ … (1.38)

5. $\log_a a = 1$ … (1.39)

6. $\log_a b = \frac{1}{\log_b a}$ … (1.40)

7. $\log_a b = \frac{\log_n b}{\log_n a}$ this is known as base change formula, in fact we can take any base in place of n. … (1.41)

Remark 4

- If base of the logarithm is 10 then it is known as **common logarithm**.
- If base of the logarithm is e then it is known as **natural logarithm** and some time is written as ln x instead of log x.
- When we write log x it means base is e. That is, in most of the cases base is mentioned only when it is other than e.

Example 4: Solve for x in each part:

(i) $\log_2 64 = x$ (ii) $\log_4 64 = x$ (iii) $\log_8 64 = x$ (iv) $\log_{1/4} 64 = x$

(v) $\log_8 16 = x$ (vi) $\log_{\sqrt{3}} 81 = x$ (vii) $\log_{16} x = \frac{3}{2}$ (viii) $\log_2 x = -3$

Solution:

(i) $\log_2 64 = x \Rightarrow 2^x = 64 \quad [\because \log_a b = m \Rightarrow a^m = b]$
$\Rightarrow 2^x = 2^6 \Rightarrow x = 6 \quad [\because a^m = a^n \Rightarrow m = n]$

Similarly,

(ii) $\log_4 64 = x \Rightarrow 4^x = 64 \Rightarrow 4^x = 4^3 \Rightarrow x = 3$

(iii) $\log_8 64 = x \Rightarrow 8^x = 64 \Rightarrow 8^x = 8^2 \Rightarrow x = 2$

(iv) $\log_{1/4} 64 = x \Rightarrow (1/4)^x = 64 \Rightarrow (1/4)^x = 4^3$
$\Rightarrow (1/4)^x = (1/4)^{-3} \Rightarrow x = -3$

(v) $\log_8 16 = x \Rightarrow 8^x = 16 \Rightarrow (2^3)^x = 2^4 \Rightarrow 2^{3x} = 2^4 \Rightarrow 3x = 4 \Rightarrow x = \frac{4}{3}$

(vi) $\log_{\sqrt{3}} 81 = x \Rightarrow (\sqrt{3})^x = 81 \Rightarrow 3^{x/2} = 3^4 \Rightarrow \frac{x}{2} = 4 \Rightarrow x = 8$

(vii) $\log_{16} x = \frac{3}{2} \Rightarrow (16)^{3/2} = x \Rightarrow (4^2)^{3/2} = x \Rightarrow 4^3 = x \Rightarrow x = 64$

(viii) $\log_2 x = -3 \Rightarrow 2^{-3} = x \Rightarrow \frac{1}{2^3} = x \Rightarrow \frac{1}{8} = x \Rightarrow x = \frac{1}{8}$

Sigmoid Logistic Function

In the course MST-012 you will study about probability and random variables. One of the silent features of probability is that its value lies between 0 and 1. The value zero means impossible event and 1 means sure event. Output of sigmoid logistic function also lies between 0 and 1. So whenever there is a requirement to predict value as probability, sigmoid function is one of candidates you can think about. Because of this nature, in the course MST-026 we will use it in neural network as an activation function. Mathematically, sigmoid logistic function $\sigma : \mathbb{R} \to \mathbb{R}$ is defined by

$$\sigma(x) = \frac{1}{1+e^{-x}}, \quad x \in \mathbb{R} \qquad \ldots (1.42)$$

Domain and Range of the Sigmoid Logistic Function: Here x is sitting in exponent of e (constant) so function is defined for all real values of the variable x and hence **domain** of sigmoid logistic function is the set of all real numbers. That is domain is $(-\infty, \infty)$. Also note that e^{-x} being exponential function will always take positive values for all real values of x and hence the value of the expression $\frac{1}{1+e^{-x}}$ or $\frac{e^x}{e^x+1}$ will always lie between 0 and 1, i.e., $0 < \frac{1}{1+e^{-x}} < 1$ or $0 < \frac{e^x}{e^x+1} < 1$ and hence its range is $(0, 1)$. Range suggest that whenever you have to predict value as probability p (say) you can think about this function. … (1.43)

Graphical Behaviour of Sigmoid Logistic Function: Note that value of this function is very close to zero when x is – 5 or less. It approaches to 0 as x approaches to $-\infty$. At x = 0 its value is 0.5. It approaches to 1 as x approaches to ∞. Due to similarity of its shape like S it is known as Sigmoid function.

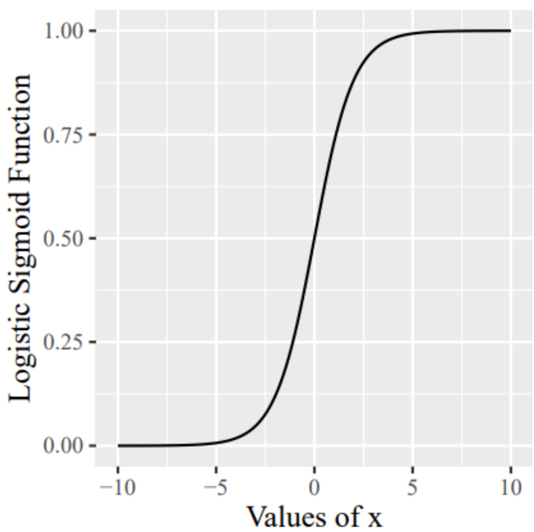


Fig. 1.20: Graph of sigmoid logistic function

Piecewise Function

Before defining it first we have to understand what we mean by partition of a set.

Partition of a Set: Let X be a non-empty set then a collection $C = \{A_1, A_2, A_3, \ldots, A_n\}$ of non-empty subsets of X is called a partition of X if it satisfies following two conditions:

(i) $X = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

(ii) $A_i \cap A_j = \phi$ for $i \neq j$ … (1.44)

For example, if X = {1, 3, 7, 9, 12, 13} then the collection of the non-empty subsets $A_1 = \{1, 3, 7\}$, $A_2 = \{9\}$, $A_3 = \{12, 13\}$ of X form a partition of X. Another partition of X may be formed by the non-empty subsets $B_1 = \{1, 3,\}$, $B_2 = \{7, 9, 12, 13\}$ of X. Similarly, more partitions can be formed.

Partition of Domain of a Function: Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. If the collection of sub-domains $D_1, D_2, D_3, \ldots, D_n$ satisfies $D = \bigcup_{i=1}^{n} D_i$ and $D_i \cap D_j = \phi$ for $i \neq j$ then we say that the collection containing the n sub-domains $D_1, D_2, D_3, \ldots, D_n$ partition the domain D of the function f. (1.45)

For example, suppose $f : [0, 5] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x + 2, & 0 \leq x < 2 \\ 2x, & 2 \leq x < 3 \\ x - 1, & x = 3 \\ x^2, & 3 < x \leq 5 \end{cases}$$

then $D = [0, 5], D_1 = [0, 2), D_2 = [2, 3), D_3 = \{3\}, D_4 = (3, 5]$

The function defined like this in pieces (here we have four pieces) is known as **piecewise function**. Often sub-domains are mutually disjoint but, in some cases if $D_i \cap D_j \neq \phi$ for some i and j then values of i^{th} and j^{th} pieces of the function at the common point is the same. For example, if in the present example D_1 also includes 2, i.e., $D_1 = [0, 2]$ then $D_1 \cap D_2 = \{2\}$. Now value of f obtained using the first piece and the second piece at x = 2 should match. This can be seen as follows: … (1.46)

Using the first piece: $f(2) = 2 + 2 = 4$ $\left[\because 2 \in [0, 2] \text{ so } f(x) = x + 2\right]$

Using the second piece: $f(2) = 2(2) = 4$ $\left[\because \text{ also } 2 \in [2, 3) \text{ so } f(x) = 2x\right]$

For Indian citizens having age less than 60 years and taxable income Rs x (rounded in rupees) then calculation of income tax for FY 2020-21 under old tax regime is a piecewise function given as follows:

$$f(x)=\begin{cases}0, & x\le 250000\\ 5\% \text{ of } (x-250000), & 250001\le x\le 500000\\ 12500+20\% \text{ of } (x-500000), & 500001\le x\le 1000000\\ 112500+30\% \text{ of } (x-1000000), & x\ge 1000001\end{cases}$$

There are some piecewise functions which are used in mathematics and statistics. Here, we will discuss only four of them namely: modulus function, unit step function (both having two pieces), Signum function (having three pieces), and greatest integer function (having infinite pieces).

Absolute Value Function or Modulus Function

If a machine behaves in such a way that its output is exactly the same number as was the input in the case its input is zero or positive number, and its output is negative times the input in the case input is a negative number, then behaviour of such a machine matches with absolute value function or modulus function. Mathematically, modulus function is defined as follows:

A function $f:\mathbb{R}\to\mathbb{R}^+\cup\{0\}$ defined by

$$f(x)=|x|=\begin{cases}x, & x\ge 0\\ -x, & x<0\end{cases}, \quad x\in R \qquad \ldots (1.47)$$

is called absolute value function.

Note that this function has two pieces one piece corresponding to non-negative values of x and other corresponding to negatives values of x.

Domain and Range of the Modulus Function: In modulus function neither x appears in denominator nor under the square root sign so it is defined for all real numbers and hence its domain is $\mathbb{R}=(-\infty,\infty)$. But this function does not give negative number as output so range of the modulus function is

$\mathbb{R}^+\cup\{0\}=[0,\infty)$. … (1.48)

Graphical Behaviour of Modulus Function: Graph of modulus function is always of v shape. Graph of the function of the form |x – a| is always symmetrical about the vertical line x = a and vertex of the v is at x = a on the x-axis. Since values of modulus function is always ≥ 0 so its graph will never go below x-axis. Graphs of three modulus functions |x + 8|, |x| and |x – 8| are shown in Fig. 1.21.

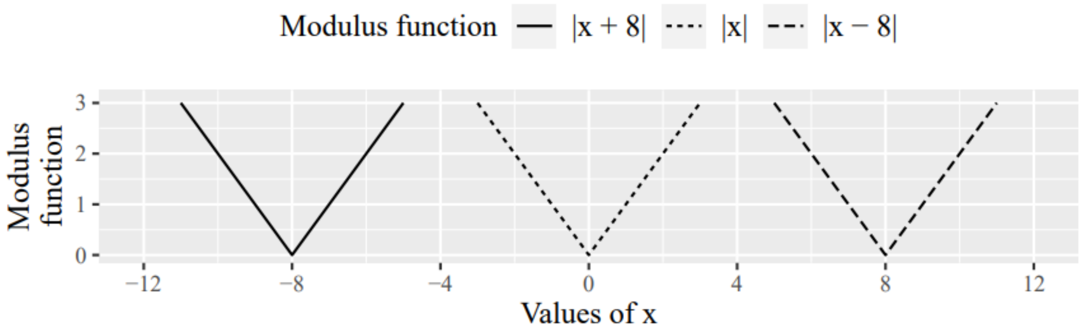


Fig. 1.21: Graph of three modulus functions |x + 8|, |x| and |x – 8|

Unit Step Function

If a machine behaves in such a way that its output is 1 in the case its input is zero or positive number, and its output is 0 in the case input is a negative number, then behaviour of such a machine matches with unit step function. Mathematically, unit step function is defined as follows:

A function $f : \mathbb{R} \to \{0, 1\}$ defined by

$$f(x) = U(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}, \; x \in R \qquad \ldots (1.49)$$

is called unit step function.

Note that like modulus function this function also has two pieces one piece corresponding to non-negative values of x and other corresponding to negative values of x.

Domain and Range of the Unit Step Function: Unit step function is defined for all real values of x hence its domain is $\mathbb{R} = (-\infty, \infty)$. But this function takes only two values 0 and 1 so range of the unit step function is {0, 1} … (1.50)

Graphical Behaviour of Unit Step Function: Unit step function takes only two values 0 and 1 so its graph is represented by two horizontal straight lines y = 0 and y = 1 and is shown in Fig. 1.22. Note the point why we have made open dot at the position (0, 0) and solid dot at the position (0, 1). Refer Fig. 1.4 where we have said about it.

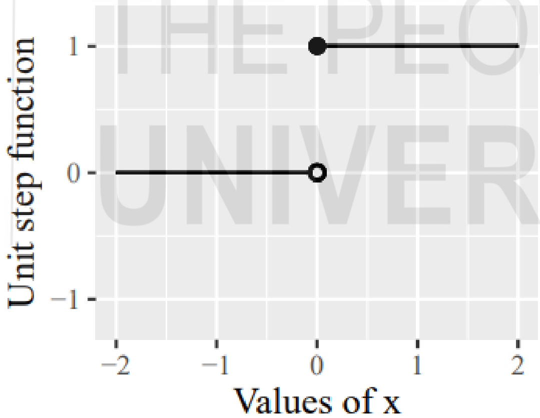


Fig. 1.22: Graph of unit step function U(x)

Example 5: If $f(x) = |x| - |1-x| - 3|2x-5|$ then evaluate $f(2), f(-2), f(3), f(-3), f(1/2)$.

Solution:

$$f(2) = |2| - |1-2| - 3|2(2)-5| = 2 - |-1| - 3|-1| = 2 - 1 - 3 = -2$$

$$\left[\begin{array}{l} \because 2 > 0, \text{ so by definition of modulus function } |2| = 2 \\ \text{Also } -1 < 0, \text{ so } |-1| = -(-1) = 1 \end{array}\right]$$

Similarly,

$$f(-2) = |-2| - |1+2| - 3|2(-2)-5| = 2 - |3| - 3|-9| = 2 - 3 - 27 = -28$$

$$f(3) = |3| - |1-3| - 3|2(3)-5| = 3 - |-2| - 3|1| = 3 - 2 - 3 = -2$$

$$f(-3) = |-3| - |1+3| - 3|2(-3)-5| = 3 - |4| - 3|-11| = 3 - 4 - 33 = -34$$

$$f\left(\frac{1}{2}\right) = \left|\frac{1}{2}\right| - \left|1-\frac{1}{2}\right| - 3\left|2\left(\frac{1}{2}\right)-5\right| = \frac{1}{2} - \left|\frac{1}{2}\right| - 3|-4| = \frac{1}{2} - \frac{1}{2} - 12 = -12$$

Example 6: Solve the equation $|1-2x| = 5$.

Solution:

$|1-2x| = 5$

$\Rightarrow 1-2x = \pm\, 5 \;\left[\because \text{ if } |x| = a \Rightarrow x = \pm\, a \text{ refer (6.92) in Unit 6}\right]$

$\Rightarrow 1-2x = 5 \quad \text{or} \quad 1-2x = -5$

$\Rightarrow -2x = 5-1 \quad \text{or} \quad -2x = -5-1$

$\Rightarrow -2x = 4 \quad \text{or} \quad -2x = -6$

$\Rightarrow x = -2 \quad \text{or} \quad x = 3$

Signum Function

The piecewise function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases} \qquad x \in \mathbb{R} \qquad \ldots (1.51)$$

is called signum function.

Sometimes this function is also defined as follows:

$$f(x) = \operatorname{sgn}(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad \ldots (1.52)$$

Note that this function has three pieces one piece corresponding to negative values of x, one piece corresponding to x = 0 and third piece corresponding to positive values of x.

Domain and Range of the Signum Function: This function is defined for all real values of x and hence its domain is $\mathbb{R} = (-\infty, \infty)$. But this function takes only three values -1, 0 and 1 so range of the signum function is $\{-1, 0, 1\}$. … (1.53)

Graphical Behaviour of Signum Function: Being piecewise function it has three pieces two of them are lines parallel to x-axis and third piece is a single point at origin. Its graph is shown in Fig, 1.23 (a). Note that here we have made two open dots one at the points (0, 1) and other at the point (0, -1) and one solid dot at the point (0, 0), i.e., at the origin because at x = 0 function takes value 0 not 1 or $-$ 1.

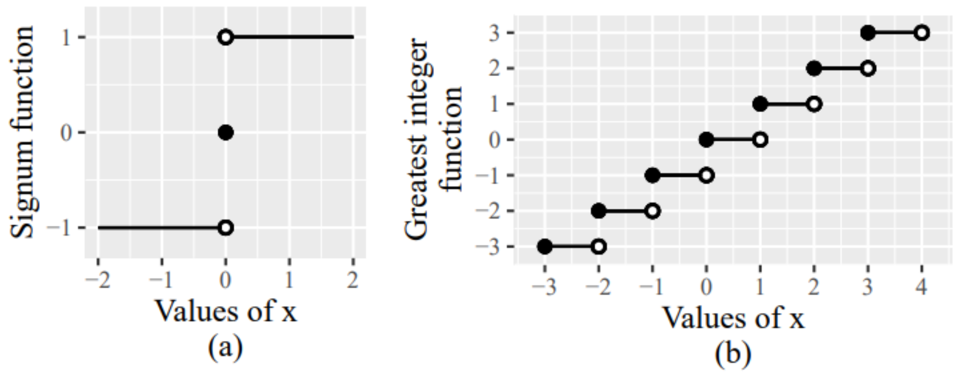


Fig. 1.23: Graph of (a) Signum function (b) Greatest integer function

Greatest Integer Function

If for a given real number x, we have a machine which gives output as the greatest integer less than or equal to x then such a machine is known as greatest integer function in mathematics. Greatest integer function is denoted by [x]. Thus, the piecewise function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=[x]=\begin{cases}\vdots & \\ -3, & -3\le x<-2\\ -2, & -2\le x<-1\\ -1, & -1\le x<0\\ 0, & 0\le x<1\\ 1, & 1\le x<2\\ 2, & 2\le x<3\\ 3, & 3\le x<4\\ \vdots & \end{cases} \quad \ldots (1.54)$$

is called greatest integer function.

Note that this function has infinite pieces where value in each piece is an integer. In general, if n is an integer and n ≤ x < n + 1 then [x] = n.

Domain and Range of the Greatest Integer Function: This function is defined for all real values of x and hence its domain is $\mathbb{R}=(-\infty,\infty)$. Since this function takes only integral values so range of the greatest integer function is $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. … (1.55)

Graphical Behaviour of Greatest Integer Function: We have seen that it is a piecewise function having infinite number of pieces. Its graphical behaviour is shown in Fig. 1.23 (b). Consider the interval [1, 2], note that here we have made solid dot at the point (1, 1) because at x = 1 function takes value 1. But at the point (2, 1) we have made an open dot because at x = 2 function takes value 2 not 1 hence solid dot at the point (2, 2) and open dot at the point (2, 1). Similarly, graph is plotted in other intervals.

Note 3: In Unit 4 of the course MST-012 you will study cumulative distribution function (cdf) of a discrete random variable. There you will see that graphical behaviour of cdf of a discrete random variable is similar to this function. Height at each step will be dictated by the probability at the corresponding value of the random variable. This is one of the reasons we discussed this function here so that understanding of cdf becomes easy for you.

Remark 5

- In the course MSTL-011 you will study **floor()** function in R. The floor() function in R is nothing but is simply greatest integer function that is discussed here.
- In the course MSTL-011 you will study another function namely **ceiling()** function in R. The ceiling() function in R gives us the smallest integer greater than or equal to x. It is generally denoted by $\lceil x \rceil$. For example, $\lceil 4.75 \rceil = 5$, $\lceil 4.999 \rceil = 5$, $\lceil 4.0001 \rceil = 5$, $\lceil -4.75 \rceil = -4$, $\lceil -0.15 \rceil = 0$, $\lceil -100.75 \rceil = -100$.

Example 7: Evaluate: (i) $\text{sgn}(-1.76)$ (ii) $\text{sgn}(3.89)$ (iii) $\text{sgn}(0)$ (iv) $\text{sgn}(5)$ (v) $[-1.76]$ (vi) $[3.89]$ (vii) $[0]$ (viii) $[5]$ (ix) $\lceil -1.76 \rceil$ (x) $\lceil 3.89 \rceil$ (xi) $\lceil 0 \rceil$ (xii) $\lceil 5 \rceil$

Solution: We know that

$$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}, \quad [x] = \begin{cases} \vdots \\ -3, & -3 \le x < -2 \\ -2, & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \\ \vdots \end{cases}, \quad \lceil x \rceil = \begin{cases} \vdots \\ -2, & -3 < x \le -2 \\ -1, & -2 < x \le -1 \\ 0, & -1 < x \le 0 \\ 1, & 0 < x \le 1 \\ 2, & 1 < x \le 2 \\ 3, & 2 < x \le 3 \\ 4, & 3 < x \le 4 \\ \vdots \end{cases}$$

Therefore,

(i) $\text{sgn}(-1.76) = -1$ (ii) $\text{sgn}(3.89) = 1$ (iii) $\text{sgn}(0) = 0$ (iv) $\text{sgn}(5) = 1$

(v) $[-1.76] = -2$ (vi) $[3.89] = 3$ (vii) $[0] = 0$ (viii) $[5] = 5$

(ix) $\lceil -1.76 \rceil = -1$ (x) $\lceil 3.89 \rceil = 4$ (xi) $\lceil 0 \rceil = 0$ (xii) $\lceil 5 \rceil = 5$

Even and Odd Functions

Even Function: If a function is such that it takes the same value at x and – x then it is known as even function. So, even function is defined as follows.

A function $f(x)$ is said to be even function if it satisfies

$f(-x) = f(x)$, for all points x of the domain of the function f. … (1.56)

For example,

(i) $f(x) = x^6 + x^4 + x^2$ is an even function since

$$f(-x) = (-x)^6 + (-x)^4 + (-x)^2 = x^6 + x^4 + x^2 = f(x)$$

(ii) $f(x) = |x|$

$f(-x) = |-x| = |(-1)(x)| = |(-1)||x| = (1)|x|$ as $|-1| = -(-1) = 1$

$= |x| = f(x)$

$\therefore$ modulus function is an even function

(iii) $f(x) = x^2 + x^3$

$f(-x) = (-x)^2 + (-x)^3 = x^2 - x^3 \neq f(x)$

$\therefore$ it is not an even function.

Odd Function: A function $f(x)$ is said to be odd function if it satisfies $f(-x) = -f(x)$, for all points x of the domain of the function f, i.e., the value of the function at $-x$ becomes negative times the value of the function at x.

… (1.57)

For example,

(i) $f(x) = x^5 + x^3 + x$ is an odd function since

$$f(-x) = (-x)^5 + (-x)^3 + (-x) = -x^5 - x^3 - x = -\left(x^5 + x^3 + x\right) = -f(x)$$

(ii) $f(x) = \dfrac{1}{x^3}$ is also an odd function since

$$f(-x) = \frac{1}{(-x)^3} = \frac{1}{-x^3} = -\frac{1}{x^3} = -f(x)$$

(iii) $f(x) = x^2 + x^3$ is not an odd function since

$f(-x) = (-x)^2 + (-x)^3 = x^2 - x^3$

$-f(x) = -\left(x^2 + x^3\right) \neq f(-x)$

We see that if $f(x) = x^2 + x^3$ then neither $f(-x) = f(x)$ nor $f(-x) = -f(x)$

$\therefore f(x) = x^2 + x^3$ is neither even nor odd function.

Graphical Behaviour of Even and Odd Functions: Graph of an even function is symmetrical about y-axis or a line parallel to y-axis. Graph of odd function is symmetrical about origin and if you rotate its graph by 180^0 then it remains unchanged. Fig. 1.24 (a), (b) show graphs of even and odd functions respectively.

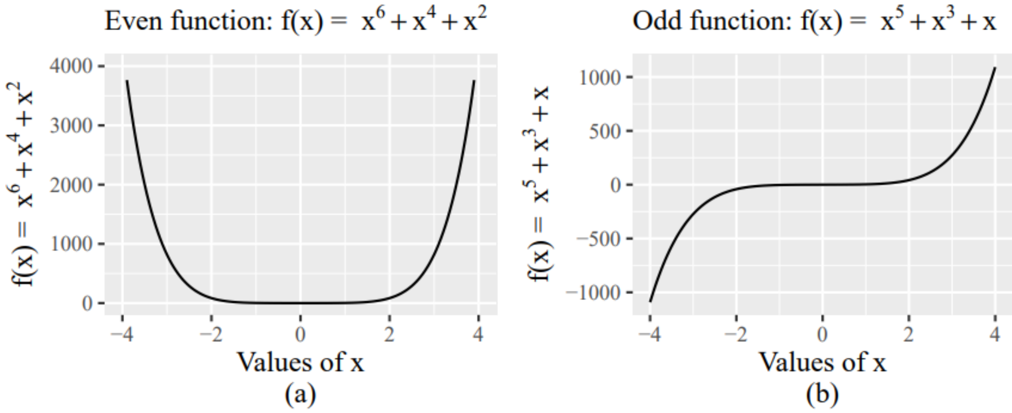


Fig. 1.24: (a) Graph of an even function (b) Graph of an odd function

Now, you can try the following Self-Assessment Question.

SAQ 2

If $f(x) = 5 - |x-3| + |x+1|$ then evaluate $f(2), f(-2), f(6), f(-5), f(12)$.

1.6 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- In mathematics and statistics **quantity** means those things on which four basic mathematical operations addition, subtraction, multiplication and division can be applied.
- A quantity which remains same (unchanged) throughout a particular problem or distribution is known as a **constant quantity**. Those types of constants which always remain the same (unchanged) independent of the place and time are known as **fixed constant**. Those types of constants which remain same in one problem but may vary from problem to problem or distribution to distribution are known as **arbitrary constants** or **parameters.**
- A quantity which may change its value even in a particular problem is known as **variable**. If the nature of the quantity is such that its possible values can be written in a sequence then such a quantity is known as discrete in nature or **discrete variable**. If the nature of the quantity is such that it can take any possible value between two certain limits, then such a quantity is known as continuous in nature or **continuous variable**.
- Let $\mathbb{R}$ be the set of all real numbers. Then a set $I \subseteq \mathbb{R}$ is said to be an **interval** if whenever $a, b \in I$ and $a < x < b$ then $x \in I$.
- An interval $I \subseteq R$ with end points a and b (a < b) is called **open interval** if it contains all real numbers between a and b but does not contain end points a and b. If the interval also includes end points a and b then it is called **closed interval**. If interval I includes only one end point b not the other end point a then it is known as **left open and right closed interval**. If interval I includes only one end point a not the other end point b then it is known as **left closed and right open interval**.
- **Length** of each of the intervals (a, b), [a, b], (a, b], [a, b) is defined as $b - a,\ a < b$.
- An interval is said to be **finite** if its length is finite. An interval is said to be **infinite** if its length is not finite.
- Let X and Y be two non-empty sets. Then a rule which associates each element of X to a unique element of Y is called a **function** from X to Y. X is called **domain** of the function. Y is called **co-domain** of the function and the set of only those values of Y for which function is defined is called **range** of the function. That is, subset

$\{y \in Y : y = f(x) \text{ for some } x \in X\}$ of Y is called range of the function.

- If $f : X \to Y$ is a function given by $y = f(x)$ then x is known a **pre image** of y and y is known as **image** of x.
- If $y = f(x)$ is a function then values of y depend on values of x. So, y is known as **dependent variable** and x is known as **independent variable**.
- In different courses of the programme M.Sc. Applied Statistics (MSCAST) you will use many functions. Some well-known and commonly used functions are discussed in this unit such as constant, identity, polynomial, exponential, logarithm, sigmoid, modulus, signum, greatest integer, even and odd functions.

1.7 TERMINAL QUESTIONS

1. Find the domain and range of the function $f : \mathbb{R} \to \mathbb{R}$ given by

   $f(x) = \dfrac{1}{\sqrt{x+5}},\ x \in \mathbb{R}$. Also, evaluate $f(0), f(4), f(-1), f(-5)$ if possible.

2. If $5^4 = 625$ then express it in logarithm form.

3. If $f(x) = \dfrac{1}{1+e^{-x}}$ then evaluate $f(0), f(-\log x), f(\log x)$.

4. If $f(x) = (x+4)^3(x-3)^4(x+6)^5(x-5)^6$ then which xero has multiplicity equal to 5.

5. If $f(x) = (x+4)^3(x-3)^4(x+6)^5(x-5)^6$ then which xero has multiplicity equal to 4.

1.8 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. If we put x = 2 in the given function then denominator becomes zero and hence function is not defined at x = 2. Other than 2 it is defined for all real values of x. That is function takes real values for all real values of x except x = 2. Hence domain of the function f is $\mathbb{R} \setminus \{2\}$.

   We note that f(x) cannot be zero for any value of x in its domain. However, other than 0, f(x) can assume any real value so it's range is $\mathbb{R} \setminus \{0\}$

   Now, $f(1) = \dfrac{1}{1-2} = \dfrac{1}{-1} = -1,\ f(3) = \dfrac{1}{3-2} = \dfrac{1}{1} = 1$ and

   $f(-5) = \dfrac{1}{-5-2} = \dfrac{1}{-7} = -\dfrac{1}{7}$

2. $f(x) = 5 - |x-3| + |x+1|$

$$f(2) = 5 - |2-3| + |2+1| = 5 - |-1| + |3|$$
$$= 5 - (-(-1)) + 3 \quad [\because -1 < 0, \text{ so } |-1| = -(-1), \text{ but } 3 > 0, \text{ so } |3| = 3]$$
$$= 5 - 1 + 3 = 7$$

Similarly,

$$f(-2) = 5 - |-2-3| + |-2+1| = 5 - |-5| + |-1|$$
$$= 5 - (-(-5)) + (-(-1)) = 5 - 5 + 1 = 1$$

$$f(6) = 5 - |6-3| + |6+1| = 5 - |3| + |7| = 5 - 3 + 7 = 9$$

$$f(-5) = 5 - |-5-3| + |-5+1| = 5 - |-8| + |-4|$$
$$= 5 - (-(-8)) + (-(-4)) = 5 - 8 + 4 = 1$$

$$f(12) = 5 - |12-3| + |12+1| = 5 - |9| + |13| = 5 - 9 + 13 = 9$$

Terminal Questions

1. First of all x + 5 is under the square root so value of $\sqrt{x+5}$ will be real if $x + 5 \geq 0$. But $\sqrt{x+5}$ lies in the denominator so it cannot be zero. Hence, given function will assume real values only when $x + 5 > 0$. So, domain of the given function is $(-5, \infty)$.

   We know that value of a square root function is always ≥ 0. So, value of $\sqrt{x+5}$ will be ≥ 0. Also note that the function $\frac{1}{\sqrt{x+5}}$ cannot assume zero value for any value of x in its domain. Hence, range of the function is $(0, \infty)$.

   Now,

$$f(0) = \frac{1}{\sqrt{0+5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5},\ f(4) = \frac{1}{\sqrt{4+5}} = \frac{1}{\sqrt{9}} = \frac{1}{3},\ f(-1) = \frac{1}{\sqrt{-1+5}} = \frac{1}{2}$$

   But value of the function at x = − 5 cannot be calculated because it is not a point in the domain of the function.

2. We know that if $a^m = b$ then $\log_a b = m$, therefore

   $5^4 = 625 \Rightarrow \log_5 625 = 4$

3. $$f(0) = \frac{1}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$$

$$f(-\log x) = \frac{1}{1+e^{-(-\log x)}} = \frac{1}{1+e^{\log x}} = \frac{1}{1+x} \quad [\because e^{\log f(x)} = f(x)]$$

$$f(\log x) = \frac{1}{1+e^{-\log x}} = \frac{1}{1+e^{\log x^{-1}}} = \frac{1}{1+e^{\log\left(\frac{1}{x}\right)}} = \frac{1}{1+\frac{1}{x}} = \frac{x}{x+1}$$

4. Exponent of the factor (x + 6) is 5, hence required zero is obtained by solving x + 6 = 0 which gives x = − 6. Hence, multiplicity of the zero − 6 is 5.

5. Exponent of the factor $(x - 3)$ is 4, hence required zero is obtained by solving $x - 3 = 0$ which gives $x = 3$. Hence, multiplicity of the zero 3 is 4.