UNIT 9

CHANGE OF ORDER OF SUMMATION AND INTEGRATION

Structure

9.1 INTRODUCTION

From earlier classes you are familiar with the summation notation, e.g., $\sum_{i=1}^{n} x_i$ which is a short notation used for the expression $x_1 + x_2 + x_3 + \ldots + x_n$. Instead of the sum of finite terms if we have a sum of infinite but countable terms, for example, $x_1 + x_2 + x_3 + \ldots$ then in summation notation, it is written as $\sum_{n=1}^{\infty} x_n$. In Sec. 9.2 we will discuss rules to change the origin of the sigma.

After explaining the rules for changing the origin of the sigma in Sec. 9.2, we will explain the meaning of double summation notation in Sec. 9.3. In the same section we will also discuss rules to change the order of two summations. From the discussion of Sec. 5.2 of Unit 5 of this course you know that in the continuous world how the role of summation is played by integration. Similarly, the role of double summation in a continuous world will be played by double integral which is introduced in Sec. 9.4 and the change of order of integration is discussed in Sec. 9.5.

What we have discussed in this unit is summarised in Sec. 9.6. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit some more questions based on the entire unit are given in Sec. 9.7 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 9.8.

The next course will make your entry into the world of uncertainty which is measured by probability. So, the next whole course is devoted to discuss some tools and techniques of the world of probability.

Expected Learning Outcomes

After completing this unit, you should be able to:

- change the origin of the summation;
- change the order of two summations; and
- evaluate double integral and change the order of integration in double integral.

9.2 CHANGE OF ORIGIN OF SUMMATION

In Unit 4 of this course, you have studied sequences and series. Recall that if you have a sequence $a_1, a_2, a_3, a_4, \ldots$ to ∞ then the expression obtained by joining these terms by plus sign, i.e., $a_1 + a_2 + a_3 + a_4 + \ldots$ to ∞ is known as a series and is denoted by $\sum_{n=1}^{\infty} a_n$ using the symbol Σ which is a Greek letter pronounced as **sigma**. Since the notation $\sum_{n=1}^{\infty} a_n$ is used to sum the terms $a_1, a_2, a_3, a_4, \ldots$ to ∞ so it is also known as **summation**. Let us explain some terms related to summation notation as follows. … (9.1)

- The variable n used in the subscript of a is known as the **dummy variable** or **index** of the summation. There is nothing special in n you can use any other dummy variable like k or m or p or r, etc. … (9.2)
- The notation n = 1 written at the bottom of the symbol Σ tells us about the initial or first term of the series. … (9.3)
- The symbol ∞ written at the top of the symbol Σ tells us up to where we have to run the index or dummy variable used in the summation. (9.4)
- The expression a_n written in front of the symbol Σ tells us where you have to replace the dummy variable or index n from 1 to ∞. … (9.5)
- In the notation $\sum_{n=1}^{\infty} a_n$ dummy variable n starts from 1 and goes up to infinity.

  For example,

  - In the notation $\sum_{n=5}^{100} a_n$ dummy variable n starts from 5 and goes up to 100.
  - In the notation $\sum_{n=0}^{99} a_n$ dummy variable n starts from 0 and goes up to 99.

If we call the bottom portion used in the summation notation as **origin of the**

summation then origin of the summation $\sum_{n=0}^{99} a_n$ is at 0 while origin of the summation $\sum_{n=5}^{100} a_n$ is at 5 and origin of the summation $\sum_{n=m}^{100} a_n$ is at m. … (9.6)

To use the notation $\sum_{n=1}^{\infty} a_n$ for the expression $a_1 + a_2 + a_3 + a_4 + \ldots$ to ∞ is convenient and concise. The shorthand or concise notation $\sum_{n=1}^{\infty} a_n$ for the expression $a_1 + a_2 + a_3 + a_4 + \ldots$ to ∞ is known as **sigma notation** or **summation notation**. … (9.7)

Let us now learn how to change origin of the summation. We will discuss it by considering two common situations where the number of terms is infinite and finite.

Change of Origin when we are dealing with Infinite Terms

In the notation $\sum_{n=1}^{\infty} a_n$ we say that the origin of the summation is at n = 1. (9.8)

We can **change this origin** at any other number of our interest without affecting the sum of the terms given by it. Suppose we want to change the origin at m then it can be done by simply doing two things simultaneously:

(1) Identity how much you have to add or subtract from a given origin to get a new origin. Here given origin is at 1. So, to get m (new origin) we have to add m – 1 in the given origin. … (9.9)

(2) Subtract the number m – 1 obtained in the first step from n everywhere in the expression inside the summation. That is just replace n by n – (m – 1) in the expression inside the summation or in the expression written in front of the sigma symbol. … (9.10)

After doing both the steps, we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+(m-1)}^{\infty} a_{n-(m-1)} = \sum_{n=m}^{\infty} a_{n-m+1}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=m}^{\infty} a_{n-m+1} \qquad \ldots (9.11)$$

In left hand side of (9.11) origin of the summation is at n = 1, while in right hand side of (9.11) origin of the summation is at n = m. When you will open these summations, you will get the same expression.

For example, suppose given expression is $\frac{xy^2}{3p^3} + \frac{x^2y^4}{3^2p^4} + \frac{x^3y^6}{3^3p^5} + \frac{x^4y^8}{3^4p^6} + \ldots$ then in summation form it can be written as $\sum_{n=1}^{\infty} \frac{x^n y^{2n}}{3^n p^{n+2}}$. … (9.12)

Origin of the summation in (9.12) is at n = 1. Suppose you want to change it at n = 5, then it can be done as follows. (to make 1 as 5 we need to add 4 in it)

$$\therefore \sum_{n=1}^{\infty}\frac{x^n y^{2n}}{3^n p^{n+2}} = \sum_{n=1+4}^{\infty}\frac{x^{n-4} y^{2(n-4)}}{3^{n-4} p^{(n-4)+2}} = \sum_{n=5}^{\infty}\frac{x^{n-4} y^{2n-8}}{3^{n-4} p^{n-2}}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{x^n y^{2n}}{3^n p^{n+2}} = \sum_{n=5}^{\infty}\frac{x^{n-4} y^{2n-8}}{3^{n-4} p^{n-2}} \qquad \dots (9.13)$$

In the left hand side of (9.13) origin of the summation is at n = 1, while in the right hand side of (9.13) origin of the summation is at n = 5. When you will open these summations, you will get the same expression.

Change of Origin when we are dealing with Finite Terms

In the notation $\sum_{k=1}^{n} a_k$ we say that origin of the summation is at k = 1. … (9.14)

We can **change this origin** at any other number of our interest without affecting the sum of the terms given by it. Suppose we want to change origin at m then it can be done by simply doing three things simultaneously:

(1) Identity how much you have to add or subtract from given origin to get new origin. Here given origin is at k = 1. So, to get m (new origin) we have to add m – 1 in the given origin. … (9.15)

(2) Subtract the number m – 1 obtained in the first step from k everywhere in the expression inside the summation. That is just replace k by k – (m – 1) in the expression inside the summation or in the expression written in front of the sigma symbol. … (9.16)

(3) Add the number m – 1 obtained in the first step in the number n which is written at the top of the summation to change the range of the summation from k = 1 to k = n, to k = m to k = n + (m – 1). … (9.17)

After doing all three steps, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1+(m-1)}^{n+m-1} a_{k-(m-1)} = \sum_{k=m}^{n+m-1} a_{k-m+1}$$

$$\Rightarrow \sum_{k=1}^{n} a_k = \sum_{k=m}^{n+m-1} a_{k-m+1} \qquad \dots (9.18)$$

In the left hand side of (9.18) origin of the summation is at k = 1, and range of the summation is from k = 1 to k = n, while in the right hand side of (9.18) origin of the summation is at k = m and range of the summation is from k = m to k = n + m – 1. When you will open these summations, you will get the same expression.

For example, suppose given expression is

$\frac{xy^2}{3p^3} + \frac{x^2y^4}{3^2p^4} + \frac{x^3y^6}{3^3p^5} + \frac{x^4y^8}{3^4p^6} + \dots + \frac{x^{10}y^{20}}{3^{10}p^{12}}$ then in summation form it can be written as $\sum_{k=1}^{10}\frac{x^k y^{2k}}{3^k p^{k+2}}$. … (9.19)

Origin of the summation in (9.19) is at k = 1 and range of the summation is from k = 1 to k = 10.

(i) Suppose you want to change the origin in (9.19) at k = 5 instead of k = 1, then it can be done as follows. (to make 1 as 5 we need to add 4 in it)

$$\therefore \sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}} = \sum_{k=1+4}^{10+4} \frac{x^{k-4} y^{2(k-4)}}{3^{k-4} p^{(k-4)+2}} = \sum_{n=5}^{14} \frac{x^{k-4} y^{2k-8}}{3^{k-4} p^{k-2}}$$

$$\Rightarrow \sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}} = \sum_{n=5}^{14} \frac{x^{k-4} y^{2k-8}}{3^{k-4} p^{k-2}} \qquad \ldots (9.20)$$

In the left hand side of (9.20) origin of the summation is at k = 1 and range of the summation is from k = 1 to k = 10, while in the right hand side of (9.20) origin of the summation is at k = 5 and range of the summation is from k = 5 to k = 14. When you will open these summations, you will get the same expression.

(ii) Suppose you want to change the origin in (9.19) at k = 0 instead of at k = 1, then it can be done as follows. (to make 1 as 0 we need to subtract 1 from it)

$$\therefore \sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}} = \sum_{k=1-1}^{10-1} \frac{x^{k+1} y^{2(k+1)}}{3^{k+1} p^{(k+1)+2}} = \sum_{n=0}^{9} \frac{x^{k+1} y^{2k+2}}{3^{k+1} p^{k+3}}$$

$$\Rightarrow \sum_{k=1}^{10} \frac{x^k y^{2k}}{3^k p^{k+2}} = \sum_{n=0}^{9} \frac{x^{k+1} y^{2k+2}}{3^{k+1} p^{k+3}} \qquad \ldots (9.21)$$

In the left hand side of (9.21) origin of the summation is at k = 1 and range of the summation is from k = 1 to k = 10, while in the right hand side of (9.21) origin of the summation is at k = 0 and range of the summation is from k = 0 to k = 9. When you will open these summations, you will get the same expression.

Visualisation of the Range of Summation: It is summarised in the following three points and visualised in Fig. 9.1 (a), (b) and (c).

- Range of the dummy variable k used in (9.19) is shown in Fig. 9.1 (a) before changing the origin.
- Range of the dummy variable k after changing the origin at k = 5 refer RHS of (9.20) is shown in Fig. 9.1 (b).
- Range of the dummy variable k after changing the origin at k = 0 is shown in Fig. 9.1 (c).

Note that when we changed the origin from k = 1 to k = 5 then range of the dummy variable shifts 4 (= 5 – 1) points towards right. When we changed the origin from k = 1 to k = 0 then range of the dummy variable shifts 1 (0 – 1 = – 1) points towards left.

One more question that may arise in your mind is what happens if the expression inside the sigma notation is independent of the dummy variable (index). In this case we will get the same expression adding as many times as the number of terms in the corresponding series. For example,

(i) $$\sum_{k=1}^{10} 3 = \underbrace{3+3+3+\ldots+3}_{10\text{ times}} = 3\times 10 = 30 \qquad \ldots (9.22)$$

(ii) $\sum_{k=0}^{10} 3 = \underbrace{3+3+3+\ldots+3}_{11 \text{ times}} = 3 \times 11 = 33$ … (9.23)

(iii) $\sum_{k=m}^{n} a = \underbrace{a+a+a+\ldots+a}_{(n-m+1) \text{ times}} = a \times (n-m+1) = (n-m+1)a$ … (9.24)

(iv) $\sum_{k=1}^{10} a = \underbrace{a+a+a+\ldots+a}_{10 \text{ times}} = 10a$ … (9.25)

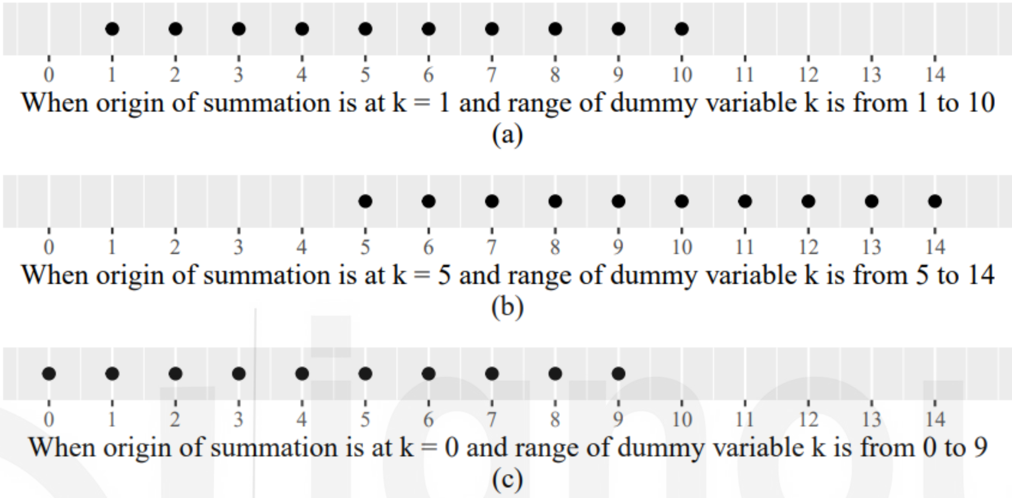

Fig. 9.1: Visualisation of the range of the dummy variable (a) when origin is at k = 1 (b) after changing origin at k = 5 (c) after changing origin at k = 0

Now, you can try the following Self-Assessment Question.

SAQ 1

(a) Choose the correct option for the notation $\sum_{k=1}^{3} k^5$

(A) $1^5 + 3^5$ (B) $(1+3)^5$ (C) $1^5 + 2^5 + 3^5$ (D) $(1+2+3)^5$

(b) Choose the correct option for the notation $\sum_{k=10}^{100} 7$

(A) 770 (B) 630 (C) 700 (D) 637

9.3 CHANGE OF ORDER OF SUMMATION

In the previous section you have seen that if in a given expression subscript or super subscript of only one dummy variable is changing term to term then that expression can be written in short using a single sigma notation. But if instead of one, there are two dummy variables which are changing term to term such expressions can be written in concise form using double sigma notation. For example, let us consider an excel sheet where we have typed collected data on n Variables (V1, V2, V3, …, Vn) and on m Subjects (S1, S2, S3, …, Sm). Excel sheet, in general, for such a data set will look like as given in (9.26) with the help of a screenshot.

	A	B	C	D	E	F
1		**V1**	**V2**	**V3**	**…**	**Vn**
2	**S1**	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	…	$a_{1,n}$
3	**S2**	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	…	$a_{2,n}$
4	**S3**	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	…	$a_{3,n}$
5	⋮	⋮	⋮	⋮	⋱	⋮
6	**Sm**	$a_{m,1}$	$a_{m,2}$	$a_{m,3}$	…	$a_{m,n}$

… (9.26)

This image matches with the look of a matrix of order m by n which you have studied in earlier classes. If you write the same information using a matrix then it will look like as shown in (9.27) given as follows.

$$\begin{array}{cc} & \overbrace{\begin{array}{ccccc} V1 & V2 & V3 & & Vn \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \end{array}}^{\text{Name of the Variable}} \\ \text{Subject number} \begin{bmatrix} S1\rightarrow \\ S2\rightarrow \\ S3\rightarrow \\ \vdots \\ Sm\rightarrow \end{bmatrix} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad \ldots (9.27)$$

Suppose all the Variables have numerical values which can be added. Suppose you want to add all the mn elements of the excel sheet shown in the form of an image given by (9.26) or in the matrix given by 9.27. This job can be done in many ways. But we consider two convenient and common ways of doing this.

(1) First add all the elements of the first row, second row, third row and so on m^{th} row. After that add these m sums of the m rows. If we write this way of getting sum of all mn elements using sigma notation then it can be written as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m}\left(a_{i1}+a_{i2}+a_{i3}+\cdots+a_{in}\right)$$

$$=\left(a_{11}+a_{12}+a_{13}+\cdots+a_{1n}\right)+\left(a_{21}+a_{22}+a_{23}+\cdots+a_{2n}\right)+$$
$$\cdots+\left(a_{m1}+a_{m2}+a_{m3}+\cdots+a_{mn}\right)$$

$$=\left(\text{sum of all elements of } 1^{st} \text{ Row corresponding to Subject 1}\right)$$
$$+\left(\text{sum of all elements of } 2^{nd} \text{ Row corresponding to Subject 2}\right)$$
$$+\left(\text{sum of all elements of } 3^{rd} \text{ Row corresponding to Subject 3}\right)+\cdots$$
$$+\left(\text{sum of all elements of } m^{th} \text{ Row corresponding to Subject m}\right)$$

$$= S_1^{sum}+S_2^{sum}+S_3^{sum}+\cdots+S_m^{sum}, \text{ where } S_i^{sum} \text{ denotes the sum}$$
$$\text{of all elements of } i^{th} \text{ row which is corresponding to Subject i}$$

$$=\sum_{i=1}^{m} S_i^{sum} \qquad \ldots (9.28)$$

(2) First add all the elements of the first column, second column, third column and so on n^{th} column. After that add these n sums of the n columns. If we write this way of getting sum of all mn elements using sigma notation then it can be written as:

$$\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij} = \sum_{j=1}^{n}\left(a_{1j} + a_{2j} + a_{3j} + \cdots + a_{mj}\right)$$

$$= \left(a_{11} + a_{21} + a_{31} + \cdots + a_{m1}\right) + \left(a_{12} + a_{22} + a_{32} + \cdots + a_{m2}\right) + \cdots + \left(a_{1n} + a_{2n} + a_{3n} + \cdots + a_{mn}\right)$$

$$= \left(\begin{array}{l}\text{sum of all elements of } 1^{st} \text{ Column which is} \\ \text{corresponding to Variable 1}\end{array}\right) + \left(\begin{array}{l}\text{sum of all elements of } 2^{nd} \text{ Column which is} \\ \text{corresponding to Variable 2}\end{array}\right) + \left(\begin{array}{l}\text{sum of all elements of } 3^{rd} \text{ Column which is} \\ \text{corresponding to Variable 3}\end{array}\right) + \cdots + \left(\begin{array}{l}\text{sum of all elements of } n^{th} \text{ Column which is} \\ \text{corresponding to Variable n}\end{array}\right)$$

$= V_1^{sum} + V_2^{sum} + V_3^{sum} + \cdots + V_n^{sum}$, where V_i^{sum} denotes the sum of all elements of i^{th} Column which is corresponding to Variable i

$$= \sum_{j=1}^{n} V_j^{sum} \qquad \text{... (9.29)}$$

Equations (9.28) and (9.29) both represents sum of all elements of the excel sheet or matrix given by (9.26) or (9.27) respectively. So,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij} \qquad \text{… (9.30)}$$

This is nothing but known as **change of order of sigmas/summations**. Note that here **range of two dummy variables i and j are constant or independent of each other and the region formed by the points of intersection of the ranges of two dummy variables is a rectangular region**. For particular values 5 and 4 of m and n respectively corresponding region formed by the points of intersection of the ranges of two dummy variables is shown in Fig. 9.2 (a) which is obviously a rectangular region. The 4 rows of this region represent 4 Variables (V1, V2, V3, V4) of the excel sheet given by (9.26) or 4 columns (C_1, C_2, C_3, C_4) of the matrix given by (9.27). While 5 columns of this region represent 5 Subjects (S1, S2, S3, S4, S5) of the excel sheet given by (9.26) or 5 rows $(R_1, R_2, R_3, R_4, R_5)$ of the matrix given by (9.27). Similarly, if values of m and n are infinite then the corresponding region formed by the points of intersection of the ranges of the two dummy variables will be still rectangular in shape as shown in Fig. 9.2 (b). In this case equation (9.30) becomes

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} \qquad \text{… (9.31)}$$

So, moral of the story is if **ranges of two dummy variables are constant or independent of each other, then corresponding region formed by the points of intersection of the ranges of two dummy variables will be rectangular in shape** does not matter whether their ranges are finite or infinite (refer Fig. 9.2 (a) and (b)) and in this case, you can **change the order of two sigmas without any modification in the ranges of the two dummy variables** and in the expressions inside the sigma's as we did in equation (9.30) and (9.31). … (9.32)

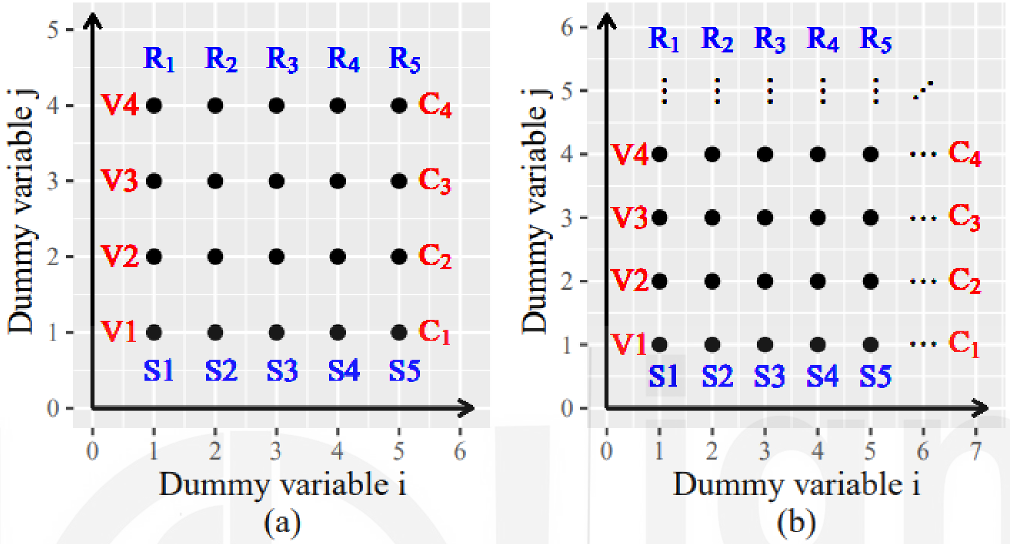

Fig. 9.2: Visualisation of the region formed by the points of intersection of the ranges of two dummy variables i and j (a) range of both i and j is finite, i.e., range of i is 1 to 5 and range of j is 1 to 4 (b) range of both i and j is infinite, i.e., 1 to infinity

So, far in this section we have discussed change of order of summation when ranges of two dummy variables were constant or independent of each other. Now, we will discuss the issue of change of order of two sigma's when limit(s) of one dummy variable either lower or upper or both depends on the other dummy variable. That is either limit(s) of i is(are) in terms of j or limit(s) of j is(are) in terms of i. This is discussed as follows.

Suppose instead of all elements of the excel sheet given by (9.26) or matrix given by (9.27), you want to find out sum of only those elements where row suffix ≥ column suffix.

To handle this problem let us first present the dummy variable which represents row suffix on horizontal axis and the dummy variable which represents column suffix on vertical axis. After doing so the point having coordinates (i, j) will represent the suffixes of the element a_{ij} refer Fig. 9.3 (a) where we have assumed m = 5 and n = 4. In excel sheet we type data when both m and n are finite but sometimes, we also have to deal with the summations where both m and n are infinite such a situation is shown in Fig. 9.3 (b). In fact, after doing so following are some connections between excel sheet or matrix presentation of data and presentation of positions of data in Fig. 9.3.

- first column of excel sheet given by (9.26) or matrix given by (9.27) will lie along the bottom row which is shown by V1 in red colour in Fig. 9.3;

- second column of excel sheet given by (9.26) or matrix given by (9.27) from its second element onward will lie along second row from the bottom which is shown by V2 in red colour in Fig. 9.3;
- third column of excel sheet given by (9.26) or matrix given by (9.27) from its third element onward will lie along third row from the bottom which is shown by V3 in red colour in Fig. 9.3, and so on.

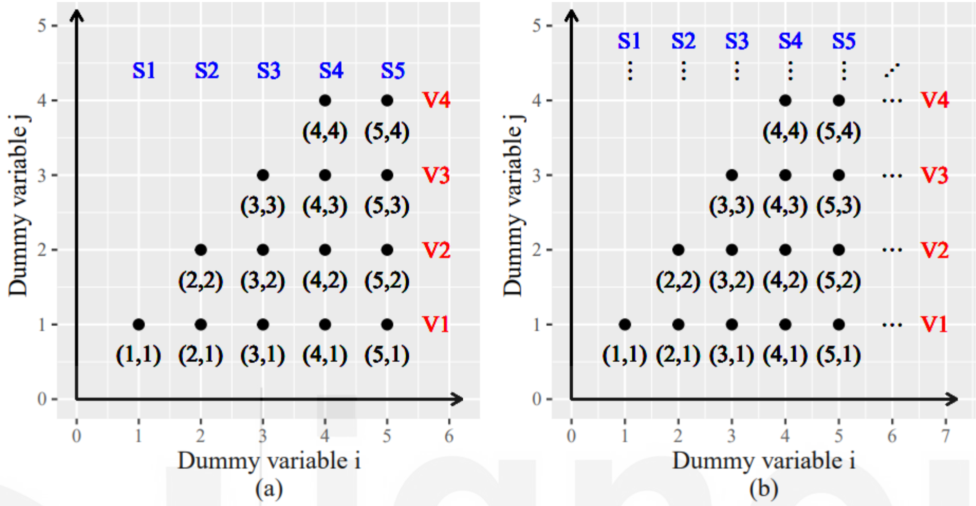

Fig. 9.3: Visualisation of the region formed by the points of intersection of the range of two dummy variables m and n (a) range of both m and n is finite, i.e., range of m is 1 to 5 and range of n is 1 to 4 (b) range of both m and n is infinite, i.e., 1 to infinity

In real data sets generally number of Subjects are more than number of Variables. That is m ≥ n generally. If m = n then the shape of the region formed by the points of intersection of the ranges of two dummy variables will be triangular like triangle ABE in Fig 9.5 (a) if we consider m = 4 and n = 4, however in this figure m = 5 and n = 4. If m > n then the shape of the region formed by the points of intersection of the ranges of two dummy variables will be like trapezium ABCD in Fig. 9.4 (a) where m = 5 and n = 4. So, the challenge is how we can form the limits of two dummy variables when we have region formed by the points of intersection of the ranges of two dummy variables is triangular or trapezium in shape instead of rectangular in shape. In such a situation you cannot take both lower and upper limits of the two dummy variables as constant because if you do so then region will become rectangular in shape which is not the case here.

Here, we will have triangular shape region or trapezium shape region not rectangular shape region. This problem of triangular or trapezium shape region can be solved if we take both lower as well as upper limits of one dummy variable as constant so that entire range along that Variable is covered and decide limits of the other dummy variable by drawing a strip along the direction of the other dummy variable. So, we have to consider two separate cases mentioned as follows.

(1) **Considering both lower and upper limits of the dummy variable j as constant and dummy variable i as variable**: To do so we have to draw a horizontal strip. For example, in the situation of Fig. 9.4 (a) limits of j will be from 1 to 4 while in the situation of Fig. 9.4 (b) limits of j will be from 1

to ∞. To obtain variable limit(s) of the dummy variable i in terms of the dummy variable j we have to draw a horizontal strip PQ refer Fig. 9.4. In this case lower limit of i will be obtained by solving the equation of line PQ for i in term of j. Here, equation of the line PQ is i = j so lower limit of i will be i = j. Upper limit of i will be 5 in the situation of Fig. 9.4 (a) since strip PQ ends at the line where i = 5. Upper limit of i will be ∞ in the situation of Fig. 9.4 (b) since strip PQ ends at the line where i = ∞. We know that generally number of Subjects (S1, S2, S3, …, Sm) are more than number of Variables (V1, V2, V3, …, Vn). So, let us assume that m ≥ n. Therefore, sum of the elements of the excel sheet given by (9.26) or matrix given by (9.27) where row suffix ≥ column suffix in the case m = 5 and n = 4 can be obtained as follows (refer strip PQ shown in trapezium shape region ABCD shown in Fig. 9.4 (a). Remember in the cases where **one dummy variable has constant limit and other dummy variable has variable limit then the sigma corresponding to the dummy variable having variable limits will be inside and the sigma corresponding to the dummy variable having both lower and upper limit as constant will be outside** explained as follows.

$$\sum_{j=1}^{4}\sum_{i=j}^{5} a_{ij} = \sum_{j=1}^{4}\left(a_{jj} + a_{j+1,\,j} + a_{j+2,\,j} + \ldots + a_{5j}\right) \qquad \ldots (9.33)$$

$$= \left(a_{11} + a_{21} + a_{31} + a_{41} + a_{51}\right) + \left(a_{22} + a_{32} + a_{42} + a_{52}\right) +$$

$$+ \left(a_{33} + a_{43} + a_{53}\right) + \left(a_{44} + a_{54}\right)$$

$= (\text{sum of all elements of V1 from first element onward})$

$+ (\text{sum of all elements of V2 from second element onward})$

$+ (\text{sum of all elements of V3 from third element onward})$

$+ (\text{sum of all elements of V4 from fourth element onward})$

$$= V(1, 1) + V(2, 2) + V(3, 3) + V(4, 4) \qquad \ldots (9.34)$$

where V(i, j) denotes the sum of i^{th} Variable from j^{th} elements onward

$$= \sum_{j=1}^{4} V(j, j) \qquad \ldots (9.35)$$

$$\Rightarrow \sum_{j=1}^{4}\sum_{i=j}^{5} a_{ij} = \sum_{j=1}^{4} V(j, j) \qquad \ldots (9.36)$$

(2) **Considering both lower and upper limits of the dummy variable i as constant and dummy variable j as variable**: To do so we have to draw a vertical strip. But here is one more problem which we did not face in the case of horizontal strip. This additional problem is regarding end point of the vertical strip PQ. Look at Fig. 9.5 (a) and note that if you draw a vertical strip within the triangular region ABE then it will always end at the side AB of the triangle ABE, but if you will draw a vertical strip within the rectangular region BCDE then it will always end at the side BC of the rectangle BCDE. So, here we have to deal with the two regions: triangular region ABE and rectangular region BCDE, separately. We know that we have no need to draw a strip for getting limits of a rectangular region. So, we only need to draw a vertical strip for triangular region ABE to get limit

of dummy variable j. For example, in the situation of the Fig. 9.5 (a) limits of i will be from 1 to 4 in the triangular region ABE, while in the situation of Fig. 9.5 (b) limits of i will be from 1 to ∞. To obtain variable limit(s) of the dummy variable j in term of the dummy variable i we have to draw a vertical strip PQ in the triangular region ABE refer Fig. 9.5 (a). In this case lower limit of j will be j = 1 since strip PQ starts from j = 1. Upper limit of j will be obtained by solving the equation of line AB for j in terms of i. Here, equation of the line AB is i = j so upper limit of j will be j = i in the situation of Fig. 9.5 (a). So, finally, limits of the trapezium region ABCD using vertical strip can be written as follows.

$$\underbrace{\sum_{\substack{i=1\\ i\geq j}}^{4}\sum_{j=1}^{i} a_{ij}}_{\substack{\text{triangular}\\ \text{region ABE}}} + \underbrace{\sum_{i=5}^{5}\sum_{j=1}^{4} a_{ij}}_{\substack{\text{Rectangular}\\ \text{region BCDE}}} = \sum_{\substack{i=1\\ i\geq j}}^{4}\left(a_{i1}+a_{i2}+a_{i3}+\ldots+a_{ii}\right)$$

$$+\sum_{i=5}^{5}\left(a_{i1}+a_{i2}+a_{i3}+a_{i4}\right) \quad \ldots (9.37)$$

$$= \underbrace{\left(a_{11}\right)+\left(a_{21}+a_{22}\right)+\left(a_{31}+a_{32}+a_{33}\right)+\left(a_{41}+a_{42}+a_{43}+a_{44}\right)}_{\text{These terms are obtained from the first summation}}$$

$$+ \underbrace{\left(a_{51}+a_{52}+a_{53}+a_{54}\right)}_{\text{Obtained from the second summation}}$$

$$= (\text{sum of all elements of S1 upto it's first element})$$
$$+(\text{sum of all elements of S2 upto it's second element})$$
$$+(\text{sum of all elements of S3 upto it's third element})$$
$$+(\text{sum of all elements of S4 upto it's fourth element})$$
$$+(\text{sum of all elements of S5 upto it's fourth element})$$

$$= S(1, 1) + S(2, 2) + S(3, 3) + S(4, 4) + S(5, 4) \quad \ldots (9.38)$$

where S(i, j) denotes the sum of i^{th} Subject upto it's j^{th} element

$$= \sum_{i=1}^{4} S(i, i) + \sum_{i=5}^{5} (i, 4) \quad \ldots (9.39)$$

$$\Rightarrow \underbrace{\sum_{\substack{i=1\\ i\geq j}}^{4}\sum_{j=1}^{i} a_{ij}}_{\substack{\text{triangular}\\ \text{region ABE}}} + \underbrace{\sum_{i=5}^{5}\sum_{j=1}^{4} a_{ij}}_{\substack{\text{Rectangular}\\ \text{region BCDE}}} = \underbrace{\sum_{i=1}^{4} S(i, i)}_{\substack{\text{triangular}\\ \text{region ABE}}} + \underbrace{\sum_{i=5}^{5} (i, 4)}_{\substack{\text{Rectangular}\\ \text{region BCDE}}} \quad \ldots (9.40)$$

Both (9.33) and (9.37) or (9.34) and (9.38) or (9.36) and (9.40) give sum of the same elements of the excel sheet given by (9.26) or matrix given by (9.27) where row suffix ≥ column suffix. So, we have

$$\sum_{j=1}^{4}\sum_{i=j}^{5} a_{ij} = \sum_{j=1}^{4} V(j, j) = \underbrace{\sum_{\substack{i=1\\ i\geq j}}^{4}\sum_{j=1}^{i} a_{ij}}_{\substack{\text{triangular}\\ \text{region ABE}}} + \underbrace{\sum_{i=5}^{5}\sum_{j=1}^{4} a_{ij}}_{\substack{\text{Rectangular}\\ \text{region BCDE}}} = \sum_{i=1}^{4} S(i, i) + + \sum_{i=5}^{5} (i, 4) \quad \ldots (9.41)$$

$$\sum_{j=1}^{4}\sum_{i=j}^{5} a_{ij} = \underbrace{\sum_{i=1}^{4}\sum_{j=1}^{i} a_{ij}}_{\substack{\text{triangular}\\ \text{region ABE}}} + \underbrace{\sum_{i=5}^{5}\sum_{j=1}^{4} a_{ij}}_{\substack{\text{Rectangular}\\ \text{region BCDE}}} \quad \ldots (9.42)$$

So, moral of the story is when **region formed by the points of intersection of the ranges of two dummy variables is either triangular (when m = n) or trapezium shape (when m ≠ n) then both limits of one dummy variable**

will be constant and written as outside summation while limit(s) of the other dummy variable either lower or upper or both will depend on the first dummy variable whose both limits we have taken as constant. To identify limits of the second dummy variable we draw a strip (PQ say) parallel to the axis of the second dummy variable. Lower limit of the other dummy variable will be decided by the equation of the line from where strip PQ starts and upper limit will be decided by the equation of the line where strip PQ ends. … (9.43)

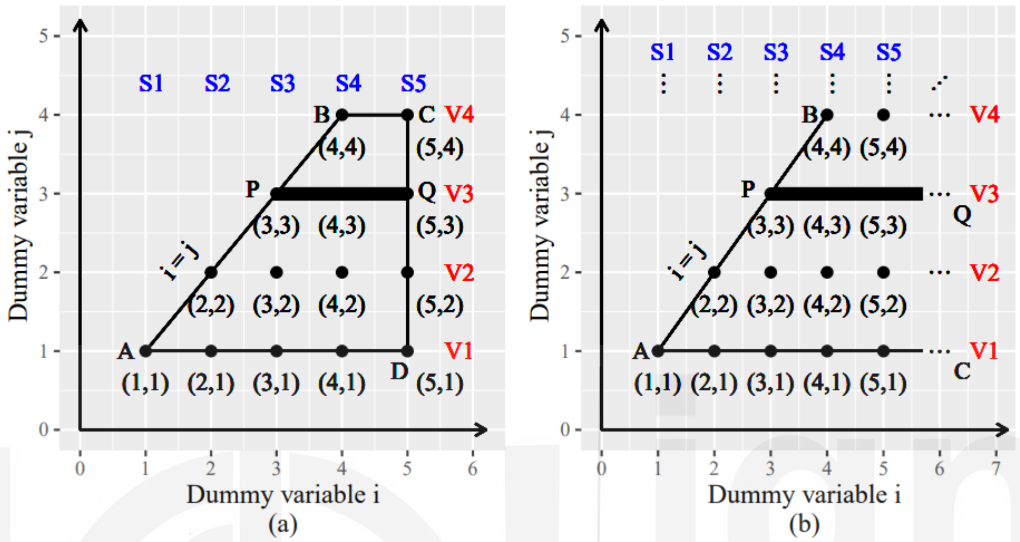

Fig. 9.4: Visualisation of the horizontal strip when (a) ranges of both i and j are finite, i.e., range of i is 1 to 5 and range of j is 1 to 4 (b) indirectly ranges of both i and j are infinite, but they are written as range of j from 1 to infinity and lower range of i is obtained from the equation of line AB in terms of j, here it is i = j and upper range of i is infinity

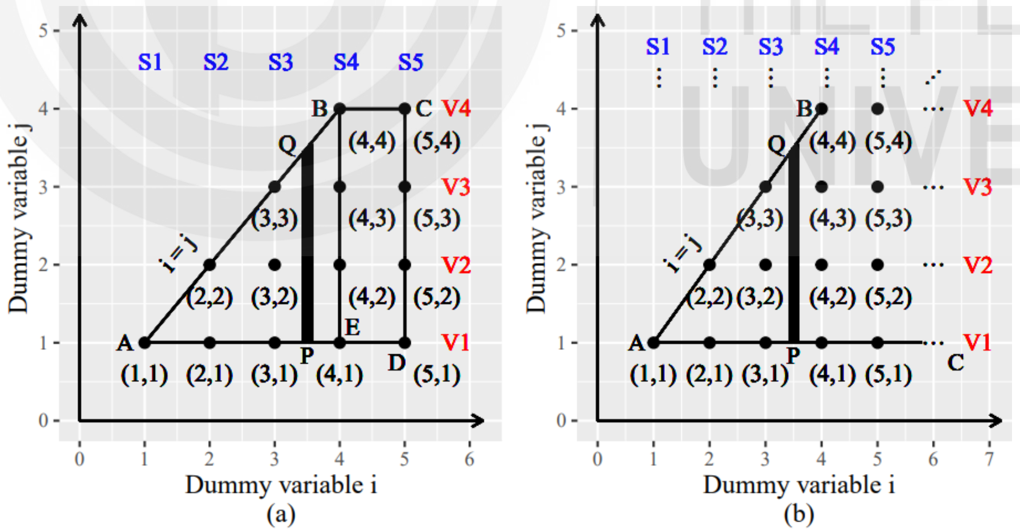

Fig. 9.5: Visualisation of the vertical strip when region is triangular in shape (a) ranges of both i and j are finite, i.e., range of i is 1 to 5 and range of j is 1 to i (b) indirectly ranges of both i and j are infinite, but they are written as range of i from 1 to infinity and lower limit of j is 1 and upper limit of j is obtained from the equation of line AB in terms of i. Here it is j = i

In the cases where limit(s) of one dummy variable is(are) in term of other dummy variable then the sigma corresponding to variable limit(s) is written inside the other sigma. The sigma corresponding to the dummy variable having both lower and upper limits as constant is written outside refer left hand sides of (9.33) and (9.37). … (9.44)

Let us do an example.

Example 1: In real data sets number of Subjects and number of Variables are finite in an excel sheet. But for this example, assume that number of Subjects (m) and number of Variables (n) are infinite. Suppose you are interested in the sum of all those elements of the excel sheet where row suffix ≥ column suffix. Write sum of the elements of our interest using double summation notation and keeping both the limits of the dummy variable which represents number of Variables as constant.

Solution: To obtain limits of the required type let us first present the dummy variable which represents suffix of the subjects on horizontal axis and the dummy variable which represents suffix of the variables on vertical axis. After doing so the point having coordinates (i, j) will represent the suffixes of the element $a_{i\,j}$ refer Fig. 9.4 (b) where we have assumed m = ∞ and n = ∞. The region formed by the positions of the points of our interest is triangular in shape and is given by the triangle ABC in Fig. 9.4 (b). In this region limits of dummy variable j are from 1 to ∞. To obtain limits of the dummy variable i we have to draw a horizontal strip PQ as shown in Fig. 9.4 (b). This strip starts from the line AB where equation of the line AB is i = j. So, lower limit of i is i = j. Strip PQ end where j is infinity. Thus, upper limit of j will be j = ∞. Hence, the sum of the elements of our interest using double sigma's and keeping both the limits of the dummy variable which represents number of Variables as constant can be written as.

$$\sum_{j=1}^{\infty}\sum_{i=j}^{\infty} a_{i\,j} \qquad \text{… (9.45)}$$

Now, you can try the following Self-Assessment Question.

SAQ 2

In real data sets number of Subjects and number of Variables are finite in an excel sheet. But for this SAQ, assume that number of Subjects (m) and number of Variables (n) are infinite. Suppose you are interested in the sum of all those elements of an excel sheet where row suffix ≥ column suffix. Write sum of the elements of our interest using double summation notation and keeping both the limits of the dummy variable which represents number of subjects as constant.

9.4 DOUBLE INTEGRATION

In Sub Sec. 5.2.2 we have explained the fact that double integral

$$\int_{x=a}^{x=b}\left(\int_{y=0}^{y=f(x)} dy\right)dx \text{ or } \int_{x=a}^{x=b}\left(\int_{0}^{f(x)} dy\right)dx = \int_{x=a}^{x=b}\left(\left[y\right]_{y=0}^{y=f(x)}\right)dx = \int_{x=a}^{x=b}\left(f(x)-0\right)dx = \int_{x=a}^{x=b} f(x)\,dx$$

… (9.46)

represents/gives area bounded by four things, refer RHS of (9.46):

(1) Curve of the Function: $y = f(x)$ … (9.47)

(2) Line corresponding to the Lower Limit of the Integration: $x = a$ … (9.48)

(3) Line corresponding to the Upper Limit of the Integration: $x = b$ … (9.49)

(4) Line corresponding to the Axis of the Variable of Integration: x … (9.50)

From equation (9.46) note that $\int_{x=a}^{x=b}\left(\int_{y=0}^{y=f(x)} dy\right)dx = \int_{x=a}^{x=b} f(x)\,dx$ … (9.51)

Recall equation (5.8) from Unit 5 which is $\int_2^3 dx = [x]_2^3 = 3-2=1.$ … (9.52)

Equation (9.52) gives measure of the line segment AB refer Fig. 9.6 (a). Why it is so? To get the answer of it refer sub Sec. 5.2.1 of Unit 5 of this course.

Both sides of equation (9.51) give area of the shaded region shown in Fig. 9.6 (b). First difference between LHS and RHS of equation (9.51) is that, in RHS function f(x) is mentioned inside the integral while in LHS of (9.51) no function is mentioned inside the integral sign. Second difference between LHS and RHS of equation (9.51) is that in RHS single integral sign is present while in LHS of (9.51) two integral (double integral) signs are present. From here we conclude that presence of a function inside the integral sign is equivalent to give measure of an integral along the direction of the axis of the function.

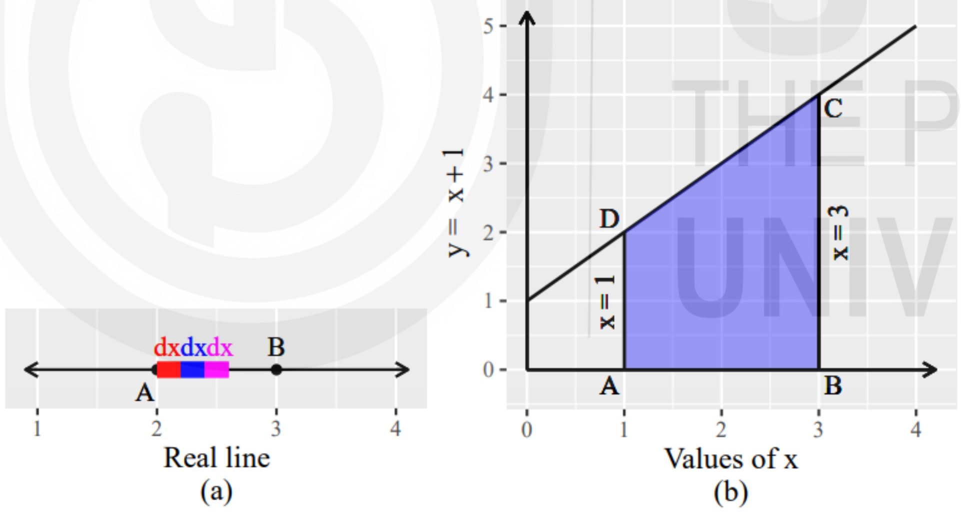

Fig. 9.6: Visualisation of the (a) horizontal strip to obtain length of the line segment AB using integration (b) area bounded by four things mentioned in (9.47) to (9.50) to obtain using integration

9.5 CHANGE OF ORDER OF INTEGRATION

Like change of order of summation change of order of integration also depends on the shape of the region of integration. So, let us consider two cases:

(a) When Region of Integration is Rectangular in Shape

(b) When Region of Integration is not Rectangular in Shape

Let us discuss these two one at a time

(a) When Region of Integration is Rectangular in Shape

If region of integration is rectangular in shape, then like change of order of

summation, change of order of integration is very simple just change the order of two integration without any modification in the limits of the two variables of integration. For example, region of integration in equation (9.53) and (9.54) is rectangular in shape as shown in Fig. 9.7 (a). So, you can change order of integration without any modification in the limits of two integrals which is done as follows.

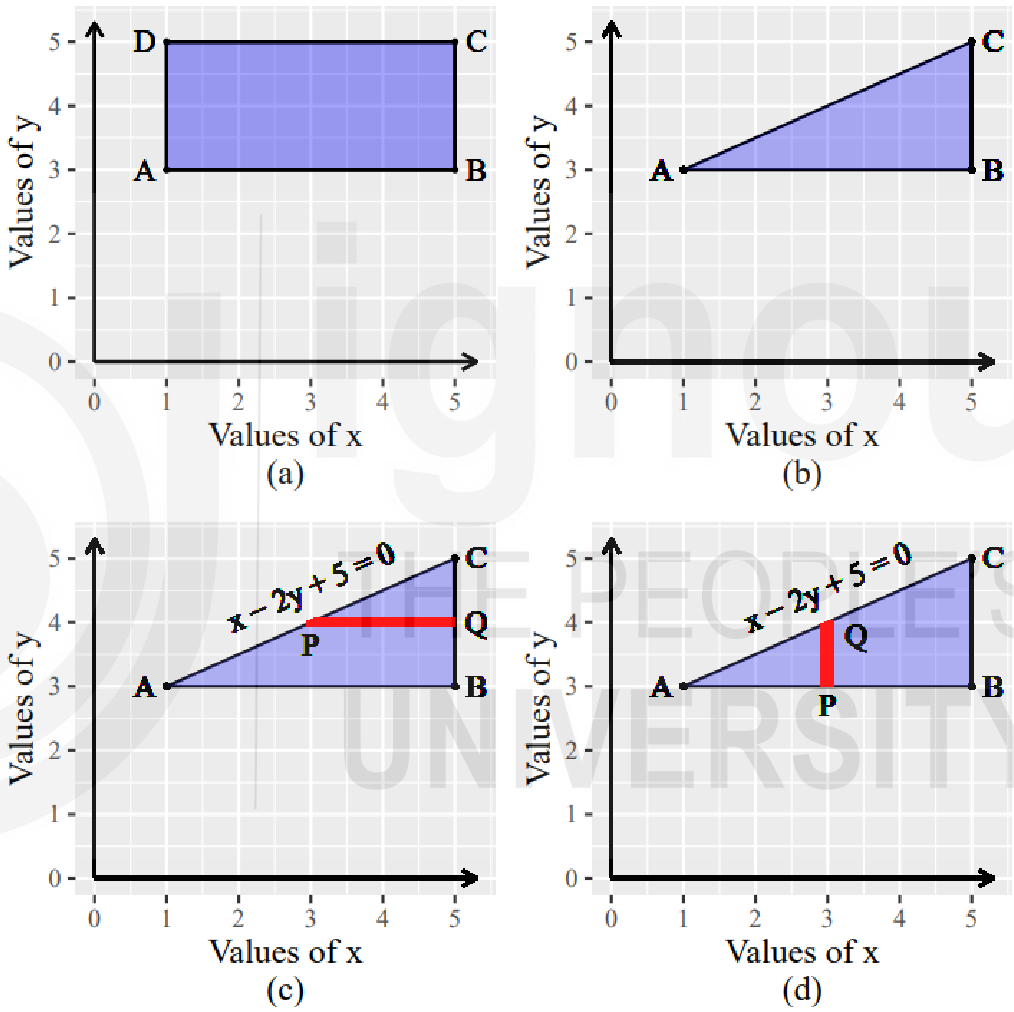

Fig. 9.7: Visualisation of the region of integration used in (a) (9.53) and (9.54) (b) (9.56) and (9.57) (c) horizontal strip PQ (d) vertical strip PQ

First integrate with respect to x and then with respect to y

$$\int_{y=3}^{y=5}\left(\int_{x=1}^{x=5}(x+y)\,dx\right)dy = \int_{y=3}^{y=5}\left[\frac{x^2}{2}+xy\right]_{x=1}^{x=5} dy = \int_{y=3}^{y=5}\left[\frac{25}{2}+5y-\frac{1}{2}-y\right]dy$$

$$= \int_{y=3}^{y=5}\left[4y+12\right]dy = \left[2y^2+12y\right]_3^5$$

$$= 50+60-18-36 = 56 \qquad \text{... (9.53)}$$

First integrate with respect to y and then with respect to x

$$\int_{x=1}^{x=5}\left(\int_{y=3}^{y=5}(x+y)\,dy\right)dx=\int_{x=1}^{x=5}\left[xy+\frac{y^2}{2}\right]_{y=3}^{y=5}dx=\int_{x=1}^{x=5}\left[5x+\frac{25}{2}-3x-\frac{9}{2}\right]dx$$

$$=\int_{x=1}^{x=5}\left[2x+8\right]dx=\left[x^2+8x\right]_1^5$$

$$=25+40-1-8=56 \qquad \text{... (9.54)}$$

From (9.53) and (9.54), we have

$$\int_{y=3}^{y=5}\left(\int_{x=1}^{x=5}(x+y)\,dx\right)dy=\int_{x=1}^{x=5}\left(\int_{y=3}^{y=5}(x+y)\,dy\right)dx \qquad \text{... (9.55)}$$

So, (9.55) is an example which shows that in the case we have region of integration as rectangular then order of integration can be changed without any modification in the limits of two variables.

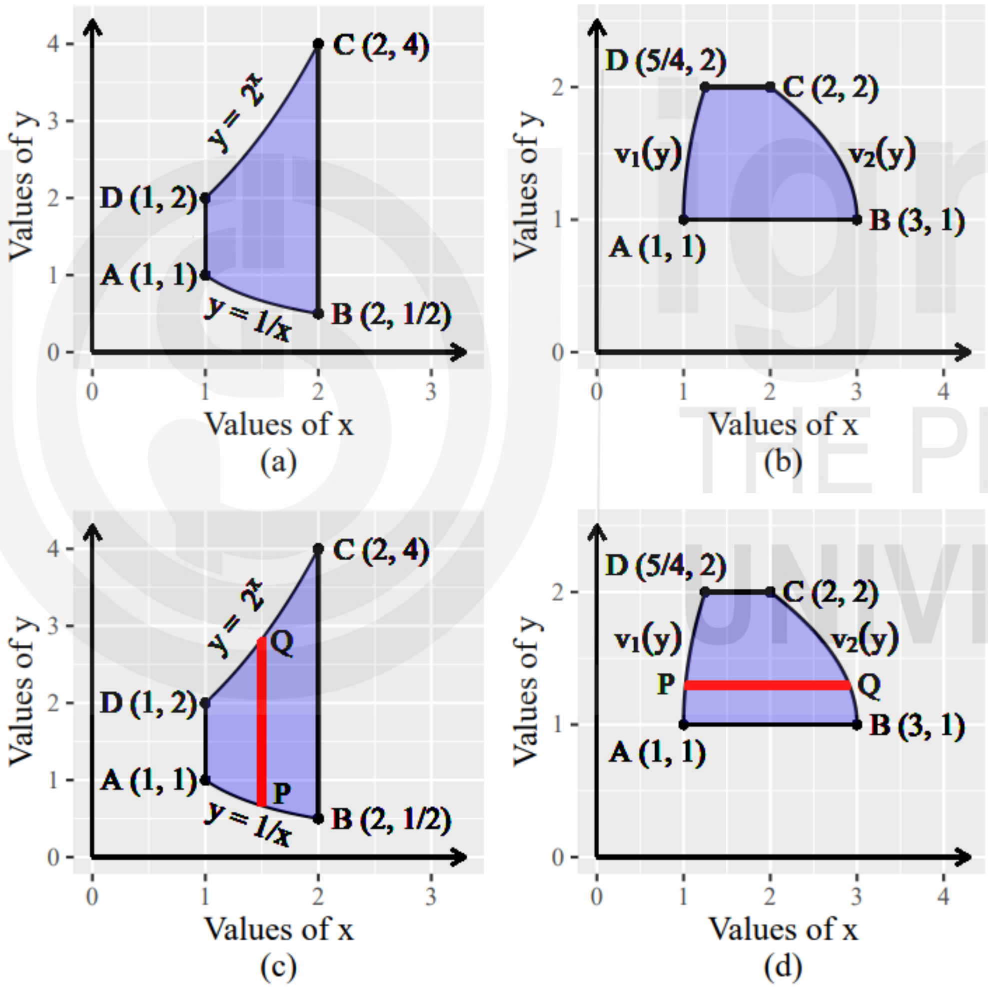

Fig. 9.8: Visualisation of the region of (a) Type I (b) Type II (c) vertical strip PQ in region of Type I (d) horizontal strip PQ in region of Type II

Now, we consider the case when the region of integration is not rectangular in shape.

(b) When Region of Integration is not Rectangular in Shape

Before discussing the problem of change of order of integration when region of integration is not in rectangular shape, first we have to define two terms known as Regions of Type I and Type II. These two terms are defined as follows.

Region of Type I

A region D in x-y plane is said to be of Type I if it is bounded by two vertical lines x = a and x = b and two functions $h_1 : [a, b] \to \mathbb{R}$ and $h_2 : [a, b] \to \mathbb{R}$ both continuous on [a, b] such that $h_1(x) \le h_2(x) \quad \forall\ x \in [a, b]$

In other words, if $[a, b] \subset \mathbb{R}$ and $h_1 : [a, b] \to \mathbb{R}$ and $h_2 : [a, b] \to \mathbb{R}$ be two continuous functions then a region D defined as follows:

$D = \left\{(x, y) \in \mathbb{R}^2 : a \le x \le b, h_1(x) \le y \le h_2(x) \quad \forall\ x \in [a, b]\right\}$ is called region of Type I. Region shown in Fig. 9.8 (a) is an example of region of Type I. In this example, a = 1, b = 2, $h_1(x) = \frac{1}{x}, h_2(x) = 2^x$. … (9.56)

Region of Type II

A region D in x-y plane is said to be of Type II if it is bounded by two horizontal lines y = c and y = d and two functions $v_1 : [c, d] \to \mathbb{R}$ and $v_2 : [c, d] \to \mathbb{R}$ both continuous on [c, d] such that $v_1(y) \le v_2(y) \quad \forall\ y \in [c, d]$

In other words, if $[c, d] \subset \mathbb{R}$ and $v_1 : [c, d] \to \mathbb{R}$ and $v_2 : [c, d] \to \mathbb{R}$ be two continuous functions then a region D defined as follows:

$D = \left\{(x, y) \in \mathbb{R}^2 : v_1(y) \le x \le v_2(y),\ c \le y \le d \quad \forall\ y \in [c, d]\right\}$ is called region of Type II. Region shown in Fig. 9.8 (b) is an example of region of Type II. In this example, c = 1, d = 2, $v_1(y) = 1 + \frac{1}{4}(y-1)^2, v_2(y) = 3 - (y-1)^2$. … (9.57)

After defining region of Type I and Type II, now we discuss how we can change the order of integration when region of integration is not rectangular in shape.

From the discussion of change of order of summation, we know that in such cases we take both lower and upper limits of one dummy variable as constant so that entire region along the axis of that variable is covered. Limits of the other dummy variable are obtained by drawing a strip PQ parallel to the axis of that dummy variable. Lower limit will be decided by solving the equation of the line or curve from where strip PQ starts for that dummy variable in terms of the other dummy variable. Similarly, upper limit will be decided by solving the equation of the line or curve where strip PQ ends. Keep following three points in mind before doing so.

(1) In the case when region of integration is of Type I then both lower and upper limits of the variable x will be constant and limits of variable y will be obtained by drawing a vertical strip in the region of integration. … (9.58)

(2) In the case when region of integration is of Type II then both lower and upper limits of the variable y will be constant and limits of variable x will be obtained by drawing a horizontal strip in the region of integration. … (9.59)

(3) In the case when region of integration is of both types: Type I as well as Type II then you have both options as mentioned in the first two points. … (9.60)

Let us do some examples to explain the working procedure.

Example 2: Evaluate the area of the triangular region shown in Fig. 9.7 (b) without using integration.

Solution: The triangular region ABC shown in Fig. 9.7 (b) is a right-angled triangle. We know that area of a right-angled triangle is $\frac{1}{2}\text{base}\times\text{height}.$ Here base = AB = 4 units and height = BC = 2 units.

Hence, required area = $\frac{1}{2}(4)(2)=4\,\text{unit}^2.$ … (9.61)

Example 3: Evaluate the area of the triangular region shown in Fig. 9.7 (b) using double integration.

Solution: Region shown in Fig. 9.7 (b) is of both types: Type I and Type II. Let us first evaluate it by treating it as Type I region refer Fig. 9.7 (d). So, here both lower and upper limits of the variable x will be constant and given by x = 1 and x = 5 refer Fig. 9.7 (d). Limits of variable y will be obtained by drawing a vertical strip PQ refer Fig. 9.7 (d). This vertical strip PQ starts from y = 3 and ends at y = (x+5)/2. Hence, using double integration

$$\text{required area} = \int_{x=1}^{x=5}\left(\int_{y=3}^{y=(x+5)/2} dy\right)dx = \int_{x=1}^{x=5}[y]_3^{(x+5)/2}\,dx$$

$$= \int_{x=1}^{x=5}\left(\frac{x+5}{2}-3\right)dx = \int_{x=1}^{x=5}\frac{x-1}{2}dx = \frac{1}{2}\left[\frac{x^2}{2}-x\right]_1^5$$

$$= \frac{1}{2}\left[\frac{25}{2}-5-\frac{1}{2}+1\right] = 4\,\text{unit}^2 \qquad \text{... (9.62)}$$

Let us now evaluate it by treating it as Type II region refer Fig. 9.7 (c). So, here both lower and upper limits of the variable y will be constant and given by y = 3 and y = 5 refer Fig. 9.7 (c). Limits of variable x will be obtained by drawing a horizontal strip PQ refer Fig. 9.7 (c). This horizontal strip PQ starts from x = 2y – 5 and ends at x = 5. Hence, using double integration

$$\text{required area} = \int_{y=3}^{y=5}\left(\int_{x=2y-5}^{x=5} dx\right)dy = \int_{y=3}^{y=5}[x]_{2y-5}^{5}\,dy$$

$$= \int_{y=3}^{y=5}(5-2y+5)\,dy = \int_{y=3}^{y=5}(10-2y)\,dy$$

$$= \left[10y-y^2\right]_3^5 = \left[50-25-30+9\right] = 4\,\text{unit}^2 \qquad \text{... (9.63)}$$

Remark 1: Compare (9.61), (9.62) and (9.63) you see that all the three are equal. Remember that this did not happen by chance. But it is a general result and holds always. So, double integral gives area of the region of integration when we have 1 as the function inside the integration. Also, if we equate the two integrals used in (9.62) and (9.63) then, we have

$$\int_{x=1}^{x=5}\left(\int_{y=3}^{y=(x+5)/2} dy\right)dx = \int_{y=3}^{y=5}\left(\int_{x=2y-5}^{x=5} dx\right)dy \qquad \text{… (9.64)}$$

the relation given by (9.64) is known as change of order of integration in triangular region ABC given in Fig. 9.7 (b).

Example 4: Evaluate the area of the region of Type I shown in Fig. 9.8 (a) using double integration.

Solution: Region shown in Fig. 9.8 (a) is of Type I because it lies between two vertical lines. So, here both lower and upper limits of the variable x will be constant and given by x = 1 and x = 2 refer Fig. 9.8 (a) or (c). Limits of the variable y will be obtained by drawing a vertical strip PQ refer Fig. 9.8 (c). This vertical strip PQ starts from y = 1/x and ends at y = 2^x. Hence, using double integration

$$\text{required area} = \int_{x=1}^{x=2}\left(\int_{y=1/x}^{y=2^x} dy\right)dx = \int_{x=1}^{x=2}[y]_{y=1/x}^{y=2^x}dx$$

$$= \int_{x=1}^{x=2}\left(2^x - \frac{1}{x}\right)dx = \left[\frac{2^x}{\log 2} - \log x\right]_1^2 = \frac{4}{\log 2} - \log 2 - \frac{2}{\log 2} + \log 1$$

$$= \left(\frac{2}{\log 2} - \log 2\right)\text{unit}^2 \quad [\because \log 1 = 0] \qquad \text{... (9.65)}$$

Now, you can try the following Self-Assessment Question.

SAQ 3

Evaluate the area of the region of Type II shown in Fig. 9.8 (b) using double integration.

9.6 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- The variable n used in subscript of a in the sigma notation $\sum_{n=1}^{\infty} a_n$ is known as **dummy variable** or **index** of the summation. There is nothing special in n you can use any other dummy variable like k or m or p or r, etc. .
- In the notation $\sum_{n=5}^{100} a_n$ dummy variable n starts from 5 and goes up to 100.

  So, origin of the summation is at 5, while in the summation $\sum_{n=7}^{50} a_n$ origin of the summation is at 7.
- We can **change the origin** of a given summation at any other number of our interest without affecting the sum of the terms given by it. For example,

  $$\sum_{n=1}^{\infty} a_n = \sum_{n=m}^{\infty} a_{n-m+1} \text{ and } \sum_{k=1}^{n} a_k = \sum_{k=m}^{n+m-1} a_{k-m+1}$$

  Also, $\sum_{k=m}^{n} a = \underbrace{a + a + a + \ldots + a}_{(n-m+1)\text{ times}} = a \times (n-m+1) = (n-m+1)a$
- If range of two dummy variables are **constant or independent of each other**, then corresponding region formed by the points of intersection of the ranges of two dummy variables will be **rectangular in shape** does not matter whether their ranges are finite or infinite and in this case, you can

change the order of two sigma's without any modification in the ranges of the two dummy variables or in the expressions inside the sigma's. For example,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij} \text{ and } \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

- Double integration $\iint_D dx\,dy$ gives area of the region D. If the region D is a rectangle, then order of integration can be changed without any modification in the limits of the two variables of integration. But if region D is not rectangular then order of integration can also be changed but we have to do modification in the limits depending upon whether it is region of Type I or Type II as discussed in Sec. 9.5.

9.7 TERMINAL QUESTIONS

1. Choose the correct option for the notation $\sum_{k=1}^{3} \frac{t^k}{n}$

   (A) $\frac{1^1}{1}+\frac{2^2}{2}+\frac{3^3}{3}$ (B) $\frac{t^1}{n}+\frac{t^2}{n}+\frac{t^3}{n}$ (C) $\frac{1^k}{n}+\frac{2^k}{n}+\frac{3^k}{n}$ (D) $\frac{t^k}{1}+\frac{t^k}{2}+\frac{t^k}{3}$

2. Choose the correct option for the notation $\sum_{k=4}^{300}\left(\frac{2}{3}\right)$

   (A) $300\times(2/3)$ (B) $296\times(2/3)$ (C) $304\times(2/3)$ (D) $297\times(2/3)$

3. Specify one application of change of order of integration with suitable example.

4. Find the area/volume of the region formed by $0 \le x \le 5$, $0 \le y \le 5$, $0 \le z \le x + y$ (a) intuitively and (b) using double integral.

9.8 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. (a) We know that $\sum_{k=1}^{n} k^p = 1^p + 2^p + 3^p + \ldots + n^p$. Here, p = 5, n = 3, so (C) is the correct option.

   (b) We know that $\sum_{k=m}^{n} a = \underbrace{a+a+a+\ldots+a}_{(n-m+1)\text{ times}} = (n-m+1)a$ Here, a = 7, n = 100, m = 10, so $\sum_{k=10}^{100} 7 = (100-10+1)(7) = 91\times 7 = 637$. Hence, (D) is the correct option.

2. To obtain limits of the required type let us first present the dummy variable which represents suffix of the Subjects on horizontal axis and the dummy variable which represents suffix of the Variables on vertical axis. After doing so the point having coordinates (i, j) will represent the

suffixes of the element a_{ij} refer Fig. 9.5 (b) where we have assumed m $= \infty$ and n = ∞. The region formed by the positions of the points of our interest is triangular in shape and is given by the triangle ABC in Fig. 9.5 (b). In this region limits of dummy variable i are from 1 to ∞. To obtain limits of the dummy variable j we have to draw a vertical strip PQ as shown in Fig. 9.5 (b). This strip starts from the line AC where j = 1. So, lower limit of j is j = 1. Strip PQ ends on the line AB where i = j. So, upper limit of j will be j = i. Hence, the sum of the elements of our interest using double sigma's can be written as.

$$\sum_{\substack{i=1 \\ i\geq j}}^{\infty} \sum_{j=1}^{i} a_{ij}$$

3. Region shown in Fig. 9.8 (b) is of Type II because it lies between two horizontal lines. So, here both lower and upper limits of the variable y will be constant and given by y = 1 and y = 2 refer Fig. 9.8 (b) or (d). Limits of the variable x will be obtained by drawing a horizontal strip PQ refer Fig. 9.8 (d). This horizontal strip PQ starts from the curve where $x = 1 + \frac{(y-1)^2}{4} = v_1(y)$ and ends at $x = 3-(y-1)^2 = v_2(y)$. Hence, using double integration required area is given by

$$\int_{y=1}^{y=2} \left(\int_{x=1+\frac{(y-1)^2}{4}}^{x=3-(y-1)^2} dx \right) dy = \int_{y=1}^{y=2} [x]_{x=1+\frac{(y-1)^2}{4}}^{x=3-(y-1)^2} dy = \int_{y=1}^{y=2} \left[3-(y-1)^2 - 1 - \frac{(y-1)^2}{4} \right] dy$$

$$= \int_{y=1}^{y=2} \left[2 - \frac{5}{4}(y-1)^2 \right] dy = \left[2y - \frac{5}{4}\frac{(y-1)^3}{3} \right]_1^2 = 4 - \frac{5}{12}(1)^2 - 2 + 0$$

$$= \frac{19}{12}\text{unit}^2$$

Terminal Questions

1. Here dummy variable is k so only value of k will change term to term and all variables/unknowns other than k will remain unchanged in each term. So, (B) is the correct option.

2. We know that $\sum_{k=m}^{n} a = \underbrace{a + a + a + \ldots + a}_{(n-m+1)\text{ times}} = (n-m+1)a.$ Here, $a = \frac{2}{3}$, n = 300, m = 4, so $\sum_{k=4}^{300} \frac{2}{3} = (300-4+1)\left(\frac{2}{3}\right) = 297 \times \frac{2}{3}.$ Hence, (D) is the correct option.

3. One important application of change of order of integration is that sometimes an integral is difficult or not possible to evaluate in given form but it can be easily evaluated by the change of the order of integration. To realise it let us consider the following example.

Example: Evaluate the integral $\int_0^2 \int_{y/2}^1 e^{x^2} dx\, dy.$

Solution: Note that we do not know the integral of e^{x^2}. So, in the given form it cannot be integrated. But let us change the order of integration. To change the order of integration first we have to plot the region of integration. Given region of integration is bounded by the four lines:

x = y/2, x = 1, y = 0 and y = 2.

Region bounded by these four lines is shown in Fig. 9.9 (a). In the given integral limits of the variable y are constant and limits of the variable x are in term of variable y. To change the order of integration we have to do its reverse. That is, we will form limits of integration in such a way that limits of the variable x as constant and limits of the variable y in term of the variable x. To do so we have to draw a vertical strip PQ refer Fig. 9.9 (b). This strip PQ starts from the x-axis where y = 0 and ends at the line OB. Equation of line OB is y = 2x, so upper limit of y will be y = 2x. Hence, after changing order of integration given integral can be written as:

$$\int_0^1\int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1\left[y\,e^{x^2}\right]_0^{2x} dx = \int_0^1\left(2x\,e^{x^2}\right)dx$$

Putting $x^2 = t$

Differentiating, we get $2x\,dx = dt$

Also, when $x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t = 1$

$$\therefore \int_0^1\left(2x\,e^{x^2}\right)dx = \int_0^1 e^t\,dt = \left[e^t\right]_0^1 = e^1 - e^0 = e - 1$$

Thus, we saw that in given form double integral was not integrable, but after changing the order of integration it not only becomes integrable but very simple also. So, this example is an evidence of at least one application of the change of order of integration.

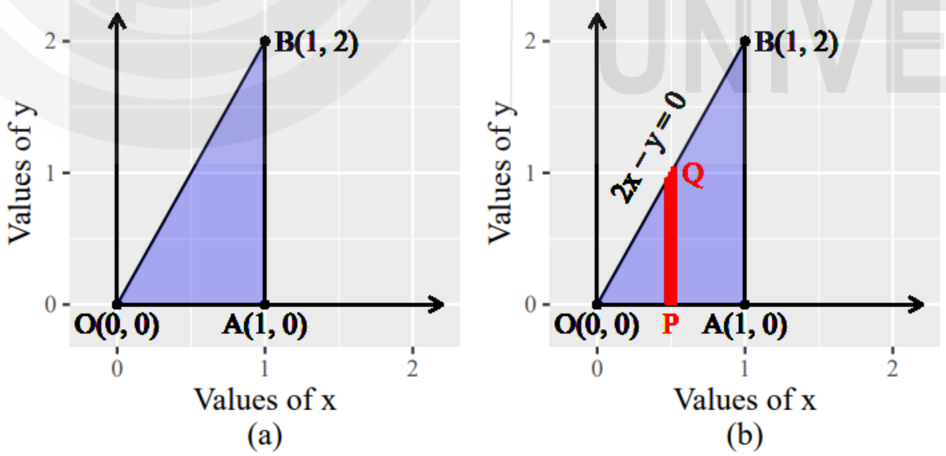

Fig. 9.9: Visualisation of the (a) region of given integration (b) vertical strip PQ to obtain limit of the variable y in term of the variable x

4. Given region is

$0 \le x \le 5,\ 0 \le y \le 5,\ 0 \le z \le x + y$ … (9.66)

This region is bounded by following six planes having equations

$x = 0,\ x = 5,\ y = 0,\ y = 5,\ z = 0$, and $z = f(x, y) = x + y$ … (9.67)

The plane having equation z = x + y is shown in Fig. 9.10 in black colour.

Two different views of the complete region bounded by six planes given by (9.67) are shown in Figs. 9.11 and 9.12.

(a) Obtaining volume of the given region using Intuitive Way

The region shown in Fig. 9.11 or 9.12 is a part of a cuboid of dimension 5 units along x-axis, 5 units along y-axis and 10 units along z-axis. So, volume of this cuboid is

$$lbh = 5 \times 5 \times 10 = 250\,\text{unit}^3 \qquad \ldots (9.68)$$

But given region is exactly half of it. Hence, volume of the region is

$$\frac{250}{2} = 125\,\text{unit}^3 \qquad \ldots (9.69)$$

This completes the solution using intuitive way.

(b) Now, we obtain this volume using double integral as follows.

$$\text{Required volume} = \int_0^5\int_0^5 (x+y)\,dy\,dx = \int_0^5 \left[xy + \frac{y^2}{2}\right]_0^5 dx = \int_0^5 \left(5x + \frac{25}{2} - 0\right) dx$$

$$= \int_0^5 \left(5x + \frac{25}{2}\right) dx = \left[\frac{5x^2}{2} + \frac{25}{2}x\right]_0^5 = \frac{125}{2} + \frac{125}{2} - 0 = 125\,\text{unit}^3 \qquad \ldots (9.70)$$

From (9.69) and (9.70) note that both the ways give the same answer,

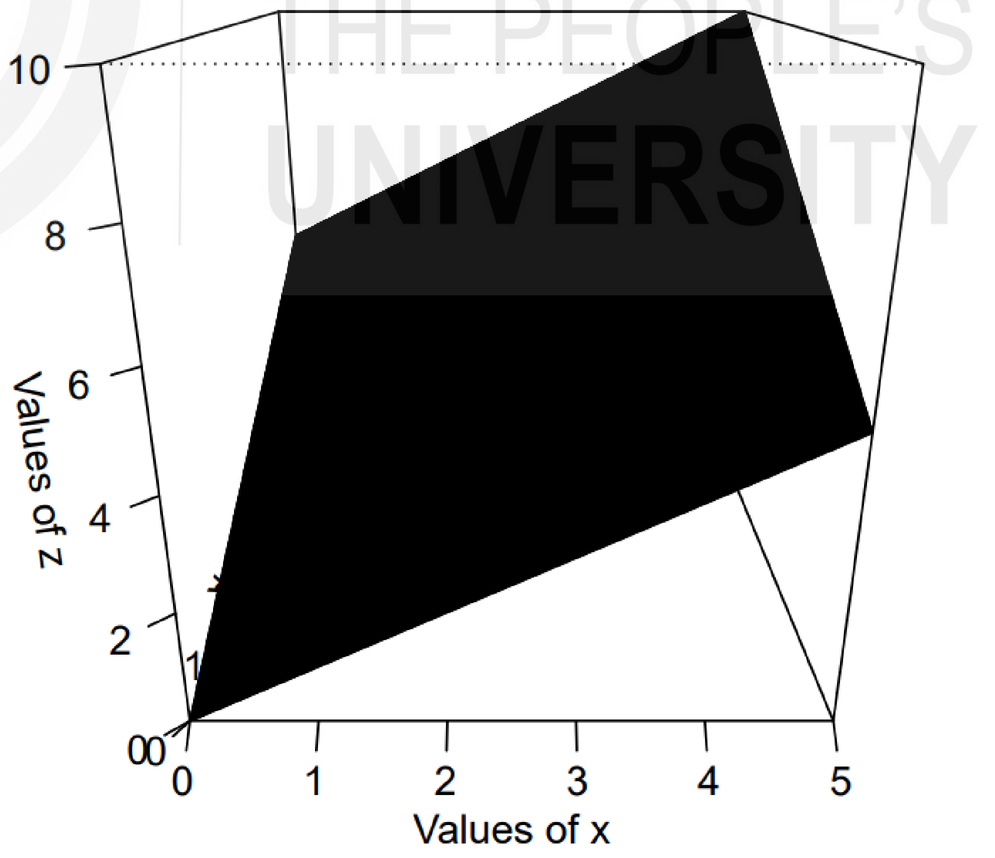

Fig. 9.10: Visualisation of the plane z = x + y

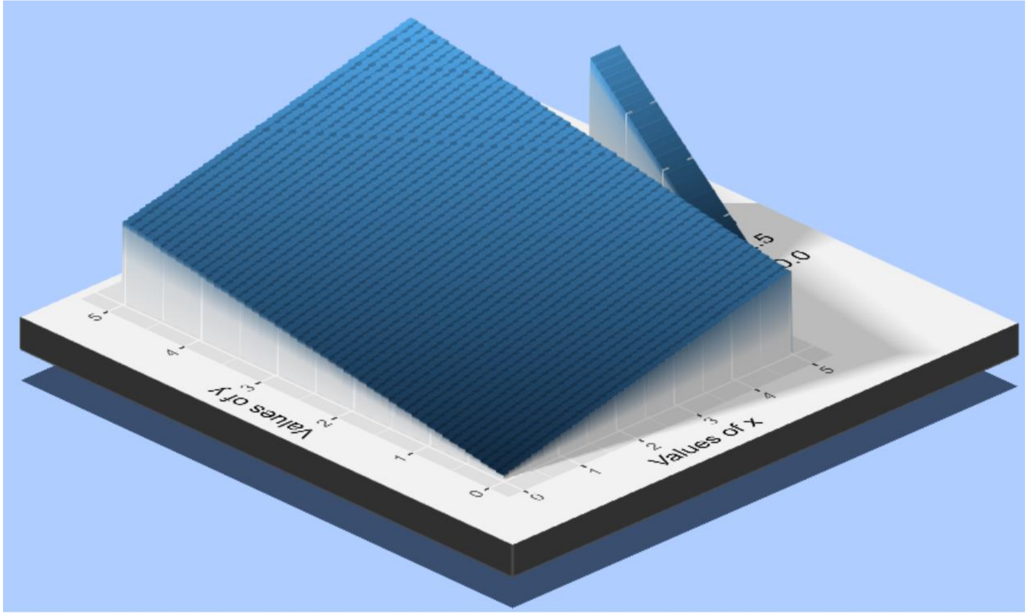

Fig. 9.11: Visualisation of the region bounded by six planes given by (9.67)

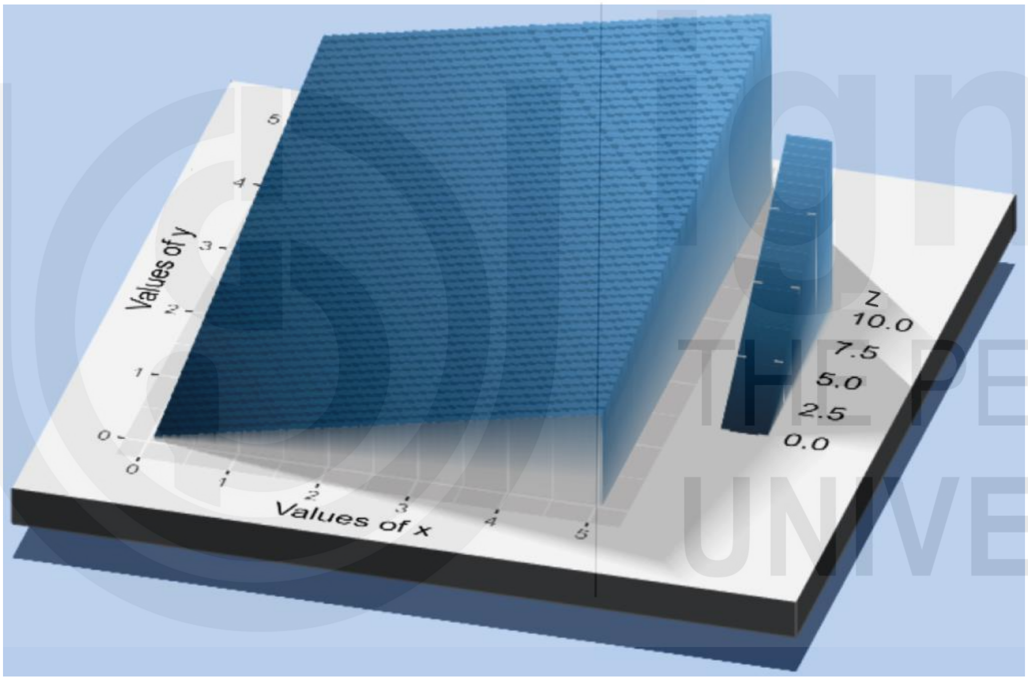

Fig. 9.12: Visualisation of the different view of the region bounded by six planes given by (9.67)

In Remark 3 of Unit 8 we promised that property 5 of gamma and beta functions will be proved in Unit 9. This is the time to keep that promise.

Proof of Property 5 of Unit 8: We know by definition of gamma function that

$$\overline{|n} = \int_0^\infty x^{n-1} e^{-x} dx \qquad \dots (9.71)$$

We also know that, refer (8.32)

$$\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\overline{|n}}{a^n} \qquad \dots (9.72)$$

$$\Rightarrow \lceil n = a^n \int_0^\infty x^{n-1} e^{-ax} dx \qquad \ldots (9.73)$$

Replacing a by z in equation (9.73), we get

$$\lceil n = z^n \int_0^\infty x^{n-1} e^{-zx} dx \qquad \ldots (9.74)$$

Multiplying on both sides of (9.74) by $z^{m-1} e^{-z}$, we get

$$z^{m-1} e^{-z} \lceil n = z^{m-1} e^{-z} z^n \int_0^\infty x^{n-1} e^{-zx} dx$$

$$\Rightarrow z^{m-1} e^{-z} \lceil n = z^{m+n-1} e^{-z} \int_0^\infty x^{n-1} e^{-zx} dx$$

Integrating on both sides with respect to z from 0 to ∞, we get

$$\lceil n \int_0^\infty x^{m-1} e^{-z}\, dz = \int_0^\infty \left[z^{m+n-1} e^{-z} \int_0^\infty x^{n-1} e^{-zx}\, dx \right] dz$$

$$= \int_0^\infty \left[\int_0^\infty z^{m+n-1} e^{-z}\, x^{n-1} e^{-zx}\, dx \right] dz$$

$$= \int_0^\infty \left[\int_0^\infty z^{m+n-1} e^{-z}\, x^{n-1} e^{-zx}\, dz \right] dx \quad \begin{bmatrix} \text{Changing order of integration as} \\ \text{limits of both variables are constant} \end{bmatrix}$$

$$= \int_0^\infty \left[x^{n-1} \int_0^\infty z^{m+n-1} e^{-z(1+x)}\, dz \right] dx$$

$$= \int_0^\infty \left[x^{n-1} \frac{\lceil m+n}{(1+x)^{m+n}} \right] dx \quad [\text{Using } (9.71), \text{ where } a = (1 + x), \text{ n is } m+n]$$

$$\Rightarrow \lceil n \int_0^\infty x^{m-1} e^{-z}\, dz = \lceil m+n \int_0^\infty \left[\frac{x^{n-1}}{(1+x)^{m+n}} \right] dx$$

$$\Rightarrow \lceil n \lceil m = \lceil m+n \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \quad [\text{Using } (9.71), \text{ in LHS}]$$

$$\Rightarrow \lceil n \lceil m = \lceil m+n\ B(n, m) \qquad [\text{Using } (8.43), \text{ in RHS}]$$

$$\Rightarrow \lceil n \lceil m = \lceil m+n\ B(m, n) \qquad [\because B(m, n) = B(n, m) \text{ Using } (8.39)]$$

$$\Rightarrow B(m, n) = \frac{\lceil m \lceil n}{\lceil m+n}$$

Hence, proved.

Your feedback pertaining to this course will help us undertake maintenance and timely revision of the course. You may give your feedback regarding SLM of this course**.**

FEEDBACK FORM LINK IS: https://forms.gle/Hf5kvZth9M8CxbXU9