UNIT 5

RIEMANN INTEGRATION

Structure

5.1 INTRODUCTION

From earlier classes, you are familiar with sigma notation, e.g.,

$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n$, which is used to sum at the most countable observations in a discrete world. But in the field of statistics and data science you not only need to deal with countable values in a discrete world but you also need to deal with uncountable values in a continuous world. How the role of summation in a discrete world is played by integration in a continuous world and in what way is explained in detail in Sec. 5.2. After understanding the concept that how integration plays the role of summation and in what way, the next thing, being a student of statistics or data scientist that you should know are the two definitions of Riemann integration. These two definitions of Riemann integration and some terms related to the two definitions are explained in detail in Sec. 5.3. Some properties of Riemann integration are stated without proof in Sec. 5.4. From earlier classes you are familiar with three important ideas continuity, differentiation and integration. These three concepts have some connections which are required to study some important ideas of probability theory. So, some such connections are also discussed in Sec. 5.4.

What we have discussed in this unit is summarised in Sec. 5.5. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit some more questions based on the entire unit are given in Sec. 5.6 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 5.7.

In the next unit, you will study to write the equation of hyperplane.

Expected Learning Outcomes

After completing this unit, you should be able to:

- ❖ explain how and in what way the role of summation in the discrete world is played by integration in the continuous world;
- ❖ explain the fact that integration gives an area bounded by four things: (i) the function which we have to integrate (ii) the lower limit of integration (iii) the upper limit of integration and (iv) the axis of integration;
- ❖ apply both definitions of Riemann integration to solve examples; and
- ❖ state some properties and theorems on Riemann integration. Prove some theorems on Riemann integration which will be used in understanding some concepts of probability theory in the course MST-012.

5.2 A REVIEW OF INTEGRATION

In school mathematics, you have seen integration as an antiderivative of a function. For example, derivative of $\sin x + c$ is $\frac{d}{dx}(\sin x + c) = \cos x$ and integration of $\cos x$ is $\int \cos x \, dx = \sin x + c.$ From earlier classes you also know one important application of definite integration is that it gives you an area under the function to which we are integrating and bounded by lower limit, upper limit and axis of integration. For example, the integration

$$\int_1^3 x(5-x)\,dx = \int_1^3 \left(5x - x^2\right)dx = \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_1^3 = \left(\frac{45}{2} - \frac{27}{3} - \frac{5}{2} + \frac{1}{3}\right)$$

$$= \frac{45}{2} - 9 - \frac{5}{2} + \frac{1}{3} = \frac{135 - 54 - 15 + 2}{6} = \frac{68}{6} = \frac{34}{3}\text{unit}^2$$

gives the area of the shaded region shown in blue colour in Fig. 5.1 (a), which is bounded by one curve and three lines mentioned as follows: … (5.1)

Curve : $y = x(5-x)$ (corresponding to the function we are integrating),

First Line : $x = 1$ (corresponding to equation of lower limit of integration),

Second Line : $x = 3$ (corresponding to equation of upper limit of integration),

Third line: $y = 0$ (corresponding to equation of axis of variable of integration)

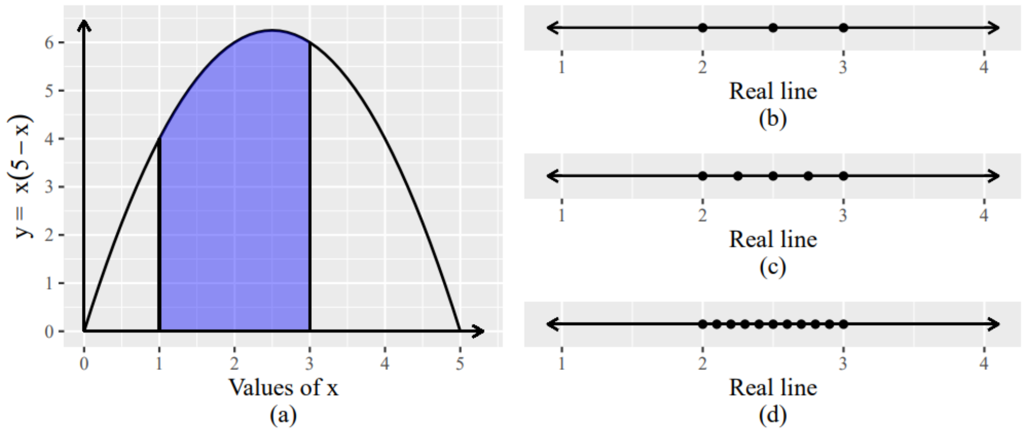

Fig. 5.1: (a) Visualisation of area under the curve of the function x(5 – x) in blue colour and visualisation of points between 2 and 3 including 2 and 3 (b) 3 points (c) 5 points (d) 11 points

In this section we will discuss two things:

(1) How the job of summation in a discrete world is done by integration in a continuous world and in what way? Understanding this idea is very important for a statistician and a data scientist because whenever we want to obtain a statistical measure in a continuous world, we just replace summation by integration in the corresponding formula of the discrete world. For example, the expected value of a discrete random variable X is given by $E(X) = \sum_{x=1}^{\infty} x\,p(x)$ while expected value of a continuous random variable X is given by $E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx,$ where p(x) is the probability mass function of the discrete random variable X and f(x) is the probability density function of continuous random variable X. You will study about this and other statistical measures in detail in Units 6 to 14 of the course MST-012. … (5.2)

(2) How integration gives area bounded by four things (i) under the curve of the function to which we are integrating (ii) equation corresponding to the lower limit (iii) equation corresponding to the upper limit (iv) equation corresponding to the axis of integration. Understanding this idea is also very important for a statistician and a data scientist because the idea of probability in the case of continuous distributions is based on this important mathematical idea. … (5.3)

We will explain both points (1) and (2) mentioned in (5.2) and (5.3) in the following two subsections.

5.2.1 How the Job of Summation in the Discrete World is done by Integration in the Continuous World and in What Way

From the discussion of Sec. 2.3 in Unit 2 of this course you know the meaning of a countable set. If we have at the most countable values then we use summation to deal with their sum. For example, if we have

(a) only 3 numbers 2, 2.5, 3 between 2 and 3 including 2 and 3, then to add all of them we can use summation notation as follows.

$\sum_{x=1}^{3} x_i = x_1 + x_2 + x_3 = 2 + 2.5 + 3 = 7.5,$ where $x_1 = 2,\ x_2 = 2.5,\ x_3 = 3.$ To see it geometrically refer to Fig. 5.1 (b). … (5.4)

(b) only 5 numbers 2, 2.25, 2.5, 2.75, 3 between 2 and 3 including 2 and 3, then to add all of them we can use summation notation as follows.

$\sum_{x=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5 = 2 + 2.25 + 2.5 + 2.75 + 3 = 12.5,$ where $x_1 = 2,\ x_2 = 2.25,\ x_3 = 2.5,\ x_4 = 2.75,\ x_5 = 3.$ To see it geometrically refer to Fig. 5.1 (c). … (5.5)

(c) only 11 numbers 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3 between 2 and 3 including 2 and 3, then to add all of them we can use summation notation as follows.

$$\sum_{x=1}^{11} x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11}$$

$$= 2 + 2.1 + 2.2 + 2.3 + 2.4 + 2.5 + 2.6 + 2.7 + 2.8 + 2.9 + 3 = 27.5,$$

where $x_1 = 2,\ x_2 = 2.1,\ x_3 = 2.2,\ x_4 = 2.3,\ x_5 = 2.4,\ x_6 = 2.5,\ x_7 = 2.6,$ $x_8 = 2.7,\ x_9 = 2.8,\ x_{10} = 2.9,\ x_{11} = 3.$ To see it geometrically refer to Fig. 5.1 (d). … (5.6)

(d) In this way if we go on increasing numbers between 2 and 3 including 2 and 3 then we can add them using summation if they are at the most countable numbers. But if we want to add all the numbers between 2 and 3 including 2 and 3 then you cannot do so using summation because they are not countable. In fact, you even cannot tell what is the next number after 2. If you say 2.1 is the next real number after 2, no it is not the next real number after 2 because 2.01 is between 2 and 2.1. If you say 2.01 is the next real number after 2, no it is not the next real number after 2 because 2.001 is between 2 and 2.01. Similarly, if you say 2.001 is the next real number after 2, no it is not the next real number after 2 because 2.0001 is between 2 and 2.001 and so on. **So, summation notation fails to work in the case of uncountable numbers**. **So, what is the solution? The solution is integration**. But integration does not add the individual numbers like the way we add in summation. **Integration actually measures the size of the shape constituted by all the points of interest**. In the present case, all the points between 2 and 3 including 2 and 3 constitute line segments joining point 2 and point 3 on the real line. So, the shape constituted by all possible numbers between 2 and 3 including 2 and 3 is a line segment. So, integration should give us the size of this line segment. We know that measure of a line segment is its length. So, the length of the line segment joining points 2 and 3 on the real line is 3 – 2 = 1. So, we expect integration should give us value 1. Let us evaluate the integration and verify it. … (5.7)

$$\int_2^3 dx = [x]_2^3 = 3 - 2 = 1. \quad \text{… (5.8)}$$

Great integration has done the job that we expected from it.

Now, our next target is to geometrically visualise LHS of (5.8) so that understanding of the idea of integration becomes clear to you. First of all, let us denote points 2 and 3 on real line by points A and B respectively. Look at Fig. 5.2 (a) and consider small portion of the line segment AB which starts from point A and denote it by dx. We are using variable x because generally horizontal axis is represented by variable X. The length dx which is shown in Fig. 5.2 (a) is very large compared to what actually it represents mathematically. Mathematically it is very small which even cannot be seen by eyes. But to understand the idea we are representing it much longer than what it actually means mathematically. Now if we join such very small portions dx end to end (refer Fig. 5.2 (b) and (c)) $1/dx$ many times we will finally reach at point B. For example, in particular, (i) if dx = 0.001 then $\frac{1}{dx} = \frac{1}{0.001} = 1000$ so we have to join 1000 such dx end to end to reach at the point B. (ii) if dx = 0.0001 then $\frac{1}{dx} = \frac{1}{0.0001} = 10000$ so we have to join 10000 such dx end to end to reach at the point B. (iii) if dx = 0.00001 then $\frac{1}{dx} = \frac{1}{0.00001} = 100000$ so we have to join 100000 such dx end to end to reach at the point B and so on. So, mathematically it is possible by starting from point A to reach at the point B by taking dx > 0 no matter how small it is. So, you can imagine that these small pieces of length each equal to dx will ultimately constitute the complete line segment AB where you need $1/dx$ such small pieces which is mathematically possible. Before the entry of integration in to picture one more clarification is required. We saw that: if we take dx = 0.001 then we need to join 1000 such small pieces, if we take dx = 0.0001 then we need to join 10000 such small pieces, if we take dx = 0.00001 then we need to join 100000 such small pieces. But then steps fall in the category of countable and so we can use summation. Now, here the idea of calculus enters into the picture which forces us to replace summation with integration. This idea of calculus is $dx \to 0^+$ and you know the meaning of something tends to 0 from right from your school mathematics. Conceptually it is so small that we actually cannot specify its value, e.g., you can think of its value like this $dx = 0.\underbrace{00000000 \ldots 00}_{\text{very very large no of times}}1$. So, we have to apply the appropriate tool of calculus to handle this mathematically. The appropriate tool in calculus to obtain a measure of the shape constituted by joining such tends to zero things end to end is integration. That is the explanation of the symbol used in LHS of (5.8).

… (5.9)

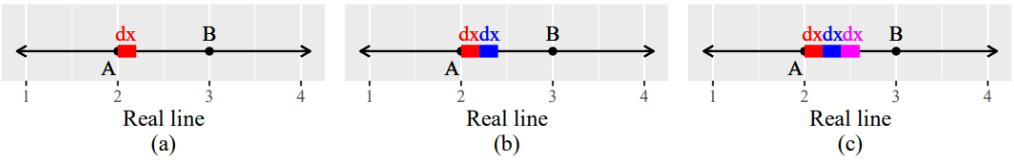

Fig. 5.2: Visualisation of small piece of length dx (a) first piece in red colour (b) one more dx in blue colour is added (c) one more dx in magenta colour is added

This completes the discussion of the point (1) where we explained how the job of summation in discrete word is done by integration in continuous word and in what way.

5.2.2 How Integration gives Area Bounded by Four Things: Curve of the Function, Lower and Upper Limits of Integration and the Axis of Integration

To explain it we need four things. (i) one function which is to be integrated (ii) lower limit of the integration (iii) upper limit of the integration (iv) variable of integration. You know that limits of integration may be $-\infty$ or ∞. But to understand the idea more clearly, we will consider both lower and upper limits of integration some finite real numbers. Let us start the explanation of point (2) mentioned in (5.3) by considering the required four things as follows.

Function: y = f(x) = x + 1 … (5.10)

Lower limit: x = 1 … (5.11)

Upper limit: x = 3 … (5.12)

Variable of integration: x … (5.13)

First of all, let us plot these four things mentioned in (5.10) to (5.13) and shad the region in blue colour bounded by the function given by (5.10) and three lines given by (5.11) to (5.13) in Fig. 5.3 (a).

We will explain why and how the integration $\int_1^3 (x+1)dx$ represents/gives the area shown in Fig. 5.3 (a) in blue colour. Let us first evaluate the area of the shaded region in Fig. 5.3 (a) without using integration. Note that:

$$\text{Area of the shaded region in Fig. 5.3 (a)} = \text{Area of the trapezium ABCD}$$

$$= \frac{1}{2}(AD + BC)AB \quad \left[\because \text{Area of the trapezium} = \frac{1}{2}(\text{Sum of parallel sides}) \times \text{height}\right]$$

$$= \frac{1}{2}(2+4)(2) = \frac{1}{2}(6)(2) = 6 \text{ unit}^2 \qquad \dots (5.14)$$

Now, we explain how this area can be obtained using integration. Consider a vertical strip of small width dx starting from point A refer to Fig. 5.3 (b). The shaded area in blue colour in Fig. 5.3 (a) is made up by joining such vertical strips end to end. In Fig. 5.3 (c) we have added one more vertical strip of blue colour adjacent to the first red colour strip. In Fig. 5.3 (d) we have added one more vertical strip of magenta colour adjacent to the second strip of blue colour. If we keep on adding strips adjacent to the previous one, one at a time, ultimately, we will reach at point B. So, the required area is equal to the sum of the areas of all these vertical strips. But before obtaining the areas of all such vertical strips we have to consider small pieces of vertical height like we have dx along the horizontal direction. As usual along the vertical axis, we have variable y or values of the function f(x) = x + 1. So, a small portion of the height along the vertical direction is denoted by dy like dx along the horizontal direction. So, every vertical strip is made up of joining such small pieces dy. In Fig. 5.3 (b) to (d) we have shown one small piece of height dy in each vertical strip by dy while in Fig. 5.3 (e) we have shown 3 such pieces within a single

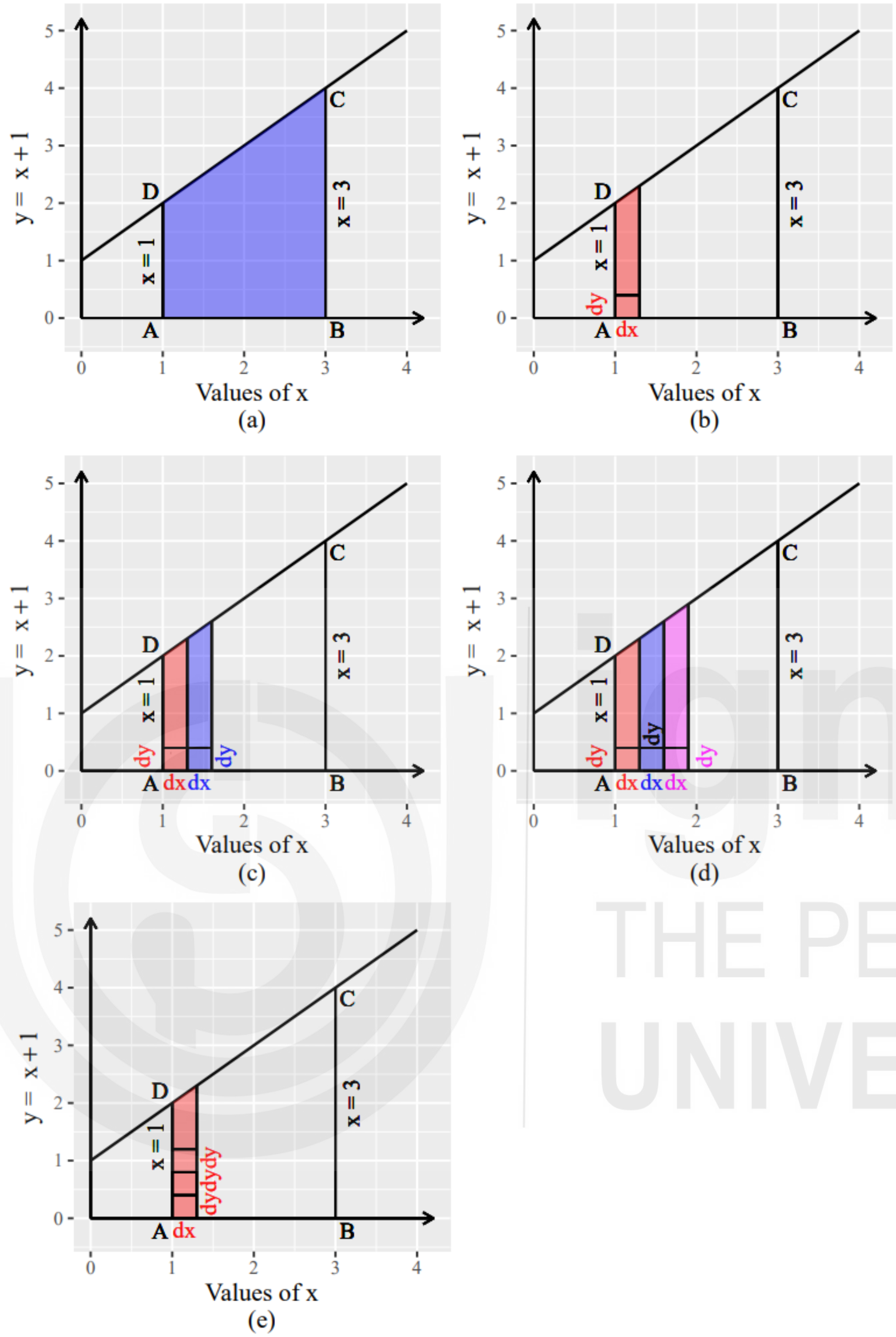

Fig. 5.3: Visualisation of (a) the region bounded by (5.10) to (5.13) (b) the first vertical strips of width dx in red colour (c) one more strip of width dx in blue colour is added (d) one more strip of width dx in magenta colour is added (e) 3 small pieces of height dy within a single strip of width dx

vertical strip each of height dy. So, to obtain area of the shaded region in Fig. 5.3 (a) we have to do two things: (i) first we have to calculate area of one vertical strip and secondly (ii) combine areas of all vertical strips.

To obtain area of one vertical strip is exactly similar to what we have done in explanation of point (1) refer (5.9). Now, as we know dy and dx both are very small, in fact $dy \to 0$ and $dx \to 0$ so the shape constituted by $dy \to 0$ and

$dx \to 0$ is the vertical line segment which starts from the horizontal axis where y = 0 and ends at the curve of the function where y = f(x). So, due to the discussion of point (1) in (5.9) we know that the appropriate tool of calculus to measure the shape constituted by joining such tends to zero things end to end is integration and so:

$$\text{Length of vertical strip} = \int_0^{f(x)} dy \qquad \dots (5.15)$$

Equation (5.15) gives measure of one vertical strip at a general point from x = 1 to x = 3. Thus, the required shaded area is the measure of the shape obtained by joining all such possible vertical line segments (whose measure is given by (5.15)) from x = a = 1 to x = b = 3. But these vertical strips are each of width dx where $dx \to 0$. Again due to the discussion of point (1) in (5.9) we know that the appropriate tool of calculus to measure the shape constituted by joining such tends to zero things end to end is integration and so required shaded area shown in Fig. 5.3 (a) is given by:

$$\text{Area of the shaded region shown in Fig. 5.3 (a)} = \int_{x=a}^{x=b}\left(\int_0^{f(x)} dy\right)dx \qquad \dots (5.16)$$

Note that in equation (5.16) we have two signs of integration known as double integration which is discussed in Unit 9 of this course. But after solving inside integral the expression given in (5.16) can simply be written as follows.

$$\int_{x=a}^{x=b}\left(\int_0^{f(x)} dy\right)dx = \int_{x=a}^{x=b}\left(\left[y\right]_{y=0}^{y=f(x)}\right)dx = \int_{x=a}^{x=b}\left(f(x)-0\right)dx = \int_{x=a}^{x=b} f(x)\,dx \qquad \dots (5.17)$$

The equation (5.17) gives area of the shaded region shown in Fig. 5.3 (a) which is bounded by one curve and three lines mentioned in (5.10) to (5.13) where a = 1 and b = 3. … (5.18)

But in Fig. 5.3 (a), we have particular values of f(x), a and b where f(x) = x + 1, a = 1 and b = 3. So, finally, area of the shaded region shown in Fig. 5.3 (a) is given by just replacing f(x) by x + 1, a by 1 and b by 3 in (5.17). So, we have

$$\text{Area of the shaded region shown in Fig. 5.3 (a)} = \int_{x=a}^{x=b} f(x)\,dx = \int_{x=1}^{x=3} (x+1)\,dx$$

$$= \left[\frac{x^2}{2}+x\right]_1^3 = \frac{9}{2}+3-\frac{1}{2}-1 = 6\,\text{unit}^2 \qquad \dots (5.19)$$

Great! This matches with the value of the area which we obtained without using integration refer (5.14).

This completes the discussion of the point (2) mentioned in equation (5.3) where we explained how integration gives area bounded by four things (i) function or curve which we are integrating (ii) equation corresponding to the lower limit of integration (iii) equation corresponding to the upper limit of integration (iv) equation corresponding to the axis of integration.

5.3 TWO DEFINITIONS OF RIEMANN INTEGRATION

To understand definition of Riemann Integration you should have understanding of following terms.

- Closed and Finite Interval
- Partition of a Finite and Closed Interval
- Norm of a Partition
- Refinement of a Partition
- Least Upper Bound (lub) and Greatest Lower Bound of a Bounded Function over an Interval
- Lower and Upper Sums of a Function over a Partition (Darboux Sums)
- Oscillatory Sum
- Upper and Lower Riemann Integrals

So, let us explain what we mean by these terms one at a time.

Closed and Finite Interval

In equations (1.11) and (1.15) in Unit 1 of this course we have discussed both closed and finite interval respectively. So, before moving ahead just revise what is discussed there.

Partition of a Finite and Closed Interval

If I = [a, b] be a finite and closed interval then by a partition P of I, we mean a finite ordered set $\{x_0, x_1, x_2, x_3, \ldots, x_n\}$ of points of [a, b] such that

$$a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b. \qquad \ldots (5.20)$$

The n + 1 points $x_0, x_1, x_2, x_3, \ldots, x_n$ are called partition points of P and the n sub-intervals

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], I_3 = [x_2, x_3], \ldots, I_i = [x_{i-1}, x_i], \ldots, I_n = [x_{n-1}, x_n] \quad \ldots (5.21)$$

determined by n + 1 points of P are called the segments of the partition P. Fig. 5.4 visualise the positions of n + 1 points $x_0, x_1, x_2, x_3, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n$ and n sub-intervals $I_1, I_2, I_3, \ldots, I_i, I_{i+1}, \ldots, I_n$.

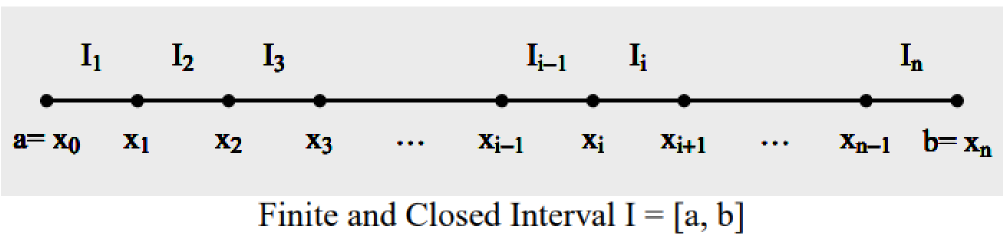

Fig. 5.4: Visualisation of the points of partition of the finite and closed interval I = [a, b] and n sub-intervals $I_1, I_2, I_3, \ldots, I_n$

Notation for Length of Sub-Intervals Formed by Points of a Partition

If I = [a, b] be a finite and closed interval and $P = \{x_0, x_1, x_2, x_3, \ldots, x_n\}$ be a partition of [a, b] such that $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b,$ then n + 1 points $x_0, x_1, x_2, \ldots, x_n$ forms n sub-intervals given by $I_1 = [x_0, x_1], I_2 = [x_1, x_2],$

$I_3 = [x_2, x_3], \ldots, I_i = [x_{i-1}, x_i], \ldots, I_n = [x_{n-1}, x_n].$ The lengths of these n sub-intervals are respectively denoted by $\delta_i,\ i = 1, 2, 3, \ldots, n.$ That is,

$$\delta_i = x_i - x_{i-1},\ i = 1, 2, 3, \ldots, n. \qquad \ldots (5.22)$$

For example, if I = [2, 5] and P = {2, 2.5, 4, 5} then $I_1 = [2, 2.5], I_2 = [2.5, 4],$ $I_3 = [4, 5]$ and so $\delta_1 = 2.5 - 2 = 0.5,\ \delta_2 = 4 - 2.5 = 1.5,\ \delta_3 = 5 - 4 = 1.$

Norm of a Partition

If I = [a, b] be a finite and closed interval and $P = \{x_0, x_1, x_2, x_3, \ldots, x_n\}$ be a partition of [a, b], then norm of the partition P is denoted by $\|P\|$ and defined as follows.

$$\|P\| = \max\{x_i - x_{i-1} : i = 1, 2, 3, 4, \ldots, n\} = \max\{\delta_i : i = 1, 2, 3, 4, \ldots, n\} \qquad \ldots (5.23)$$

For example, if I = [2, 5] and P = {2, 2.5, 4, 5} then $I_1 = [2, 2.5], I_2 = [2.5, 4],$ $I_3 = [4, 5]$ and so $\delta_1 = 2.5 - 2 = 0.5,\ \delta_2 = 4 - 2.5 = 1.5,\ \delta_3 = 5 - 4 = 1.$

Hence, $\|P\| = \max\{\delta_i : i = 1, 2, 3\} = \max\{0.5, 1.5, 1\} = 1.5$

Refinement of a Partition

If I = [a, b] be a finite and closed interval and P_1 be a partition of [a, b], then another partition P_2 of [a, b] is called refinement of the partition P_1 if $P_1 \subset P_2$. That is, every member of P_1 is also a member of P_2. If the partition P_2 is a refinement of partition P_1 then we also say that P_2 is finer than P_1. … (5.24)

For example, if I = [2, 5] and $P_1 = \{2, 2.5, 4, 5\},\ P_2 = \{2, 2.5, 3, 4, 5\}$ are two partitions of I = [2, 5] then P_2 is a refinement of P_1 since all elements of P_1 are also elements of P_2 also partition P_2 has one additional element 3.

Least Upper Bound (lub) and Greatest Lower Bound of a Bounded Function over an Interval

In Unit 4 of this course, we have defined both lub and glb refer (4.23) to (4.26) just replace sequence by function, a_n by f(x) and $\mathbb{N}$ by domain of the function. So, before moving ahead just revise what is discussed there.

Lower and Upper Sums of a Function over a Partition (Darboux Sums)

Lower and Upper sums of a bounded function over a finite and closed interval [a, b] are defined as follows.

Let I = [a, b] be a finite and closed interval and $P = \{x_0, x_1, x_2, x_3, \ldots, x_n\}$ be a partition of [a, b]. So, this partition divides the interval [a, b] into n sub-intervals $I_i = [x_{i-1}, x_i],\ i = 1, 2, 3, \ldots, n$ and lengths of these sub-intervals are $\delta_i = x_i - x_{i-1},\ i = 1, 2, 3, \ldots, n.$

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Since f is a bounded function on [a, b] so it will be bounded on each of the sub-intervals $I_i = [x_{i-1}, x_i],\ i = 1, 2, 3, \ldots, n.$

So, lub and glb of f on each $I_i = [x_{i-1}, x_i],\ i = 1, 2, 3, \ldots, n$ will exist. Suppose
$m_i = \text{glb}\{f(x) : x \in [x_{i-1}, x_i]\},\ M_i = \text{lub}\{f(x) : x \in [x_{i-1}, x_i]\},\ i = 1, 2, 3, 4, \ldots, n$ (5.25)

then the sum

$$m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4 + \ldots + m_n\delta_n = \sum_{i=1}^{n} m_i\delta_i \quad \ldots (5.26)$$

is called the **lower Darboux sum** of f over the partition P and is denoted by

$L(f, P)$. So, $L(f, P) = \sum_{i=1}^{n} m_i\delta_i$ … (5.27)

and the sum

$$M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4 + \ldots + M_n\delta_n = \sum_{i=1}^{n} M_i\delta_i$$

is called the **upper Darboux sum** of f over the partition P and is denoted by

$U(f, P)$. So, $U(f, P) = \sum_{i=1}^{n} M_i\delta_i$ … (5.28)

Oscillatory Sum

If we consider the function f and all other terms same as defined in lower and upper Darboux sums then oscillatory sum of function f over a partition P is denoted by $\omega(f, P)$ and defined as follows

$$\omega(f, P) = U(f, P) - L(f, P) = \sum_{i=1}^{n} M_i\delta_i - \sum_{i=1}^{n} m_i\delta_i = \sum_{i=1}^{n} (M_i - m_i)\delta_i = \sum_{i=1}^{n} o_i\delta_i, \quad \ldots (5.29)$$

where $o_i = M_i - m_i$

Experiment: Let us do an experiment where we will obtain lower and upper Darboux sums of the function y = f(x) = x + 1 over three partitions $P_1 = \{1, 2, 4\}$, $P_2 = \{1, 2, 3, 4\}$, and $P_3 = \{1, 2, 3, 3.5, 4\}$ all are partitions of the finite and closed interval [1, 4]. Note that partition P_2 is a refinement of the partition P_1 and partition P_3 is refinement of P_2 and hence of P_1 also. After completing this experiment, we will discuss some interesting observations of the experiment.

Let us divide this experiment into three parts as follows.

Part I: Calculation of L(f, P_1) and U(f, P_1)

Here, y = f(x) = x + 1, $P_1 = \{1, 2, 4\}$, so

$I_1 = [1, 2],\ I_2 = [2, 4],\ \delta_1 = 2 - 1 = 1,\ \delta_2 = 4 - 2 = 2,\ m_1 = f(1) = 2,\ m_2 = f(2) = 3,$

$M_1 = f(2) = 3,\ M_2 = f(4) = 5.$

So, $L(f, P_1) = \sum_{i=1}^{2} m_i\delta_i = (2)(1) + (3)(2) = 8$ … (5.30)

$U(f, P_1) = \sum_{i=1}^{2} M_i\delta_i = (3)(1) + (5)(2) = 13$ … (5.31)

Values of (5.30) and (5.31) are visualised in Fig. 5.5 (a) and (b) respectively. Areas of two solid rectangles (L1 and L2) in blue colour in Fig. 5.5 (a) represent two terms (2 and 6) of RHS of (5.30) respectively. While areas of two solid rectangles (U1 and U2) in red colour in Fig. 5.5 (b) represent two terms (3 and 10) of RHS of (5.31) respectively.

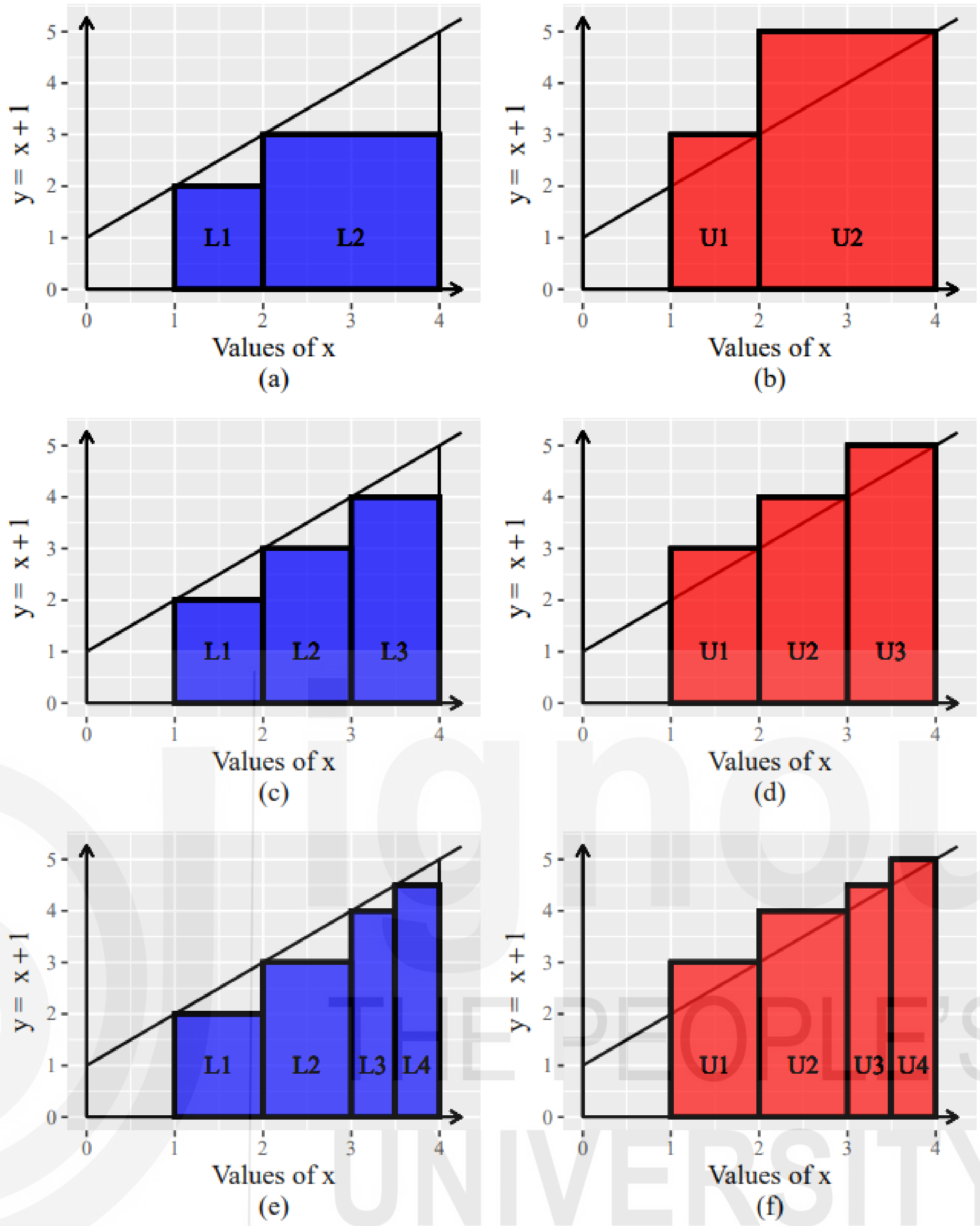

Fig. 5.5: Visualisation of Darboux sums over partition (a) P_1 lower sum (b) P_1 upper sum (c) P_2 lower sum (d) P_2 upper sum (e) P_3 lower sum (f) P_3 upper sum

Part II: Calculation of $L(f, P_2)$ and $U(f, P_2)$

Here, $y = f(x) = x + 1$, $P_2 = \{1, 2, 3, 4\}$, so

$I_1 = [1, 2], I_2 = [2, 3], I_3 = [3, 4], \delta_1 = 2 - 1 = 1, \delta_2 = 3 - 2 = 1, \delta_3 = 4 - 3 = 1,$

$m_1 = f(1) = 2, m_2 = f(2) = 3, m_3 = f(3) = 4, M_1 = f(2) = 3, M_2 = f(3) = 4, M_3 = f(4) = 5.$

$$\text{So, } L(f, P_2) = \sum_{i=1}^{3} m_i \delta_i = (2)(1) + (3)(1) + (4)(1) = 9, \qquad \dots (5.32)$$

$$U(f, P_2) = \sum_{i=1}^{3} M_i \delta_i = (3)(1) + (4)(1) + (5)(1) = 12 \qquad \dots (5.33)$$

Values of (5.32) and (5.33) are visualised in Fig. 5.5 (c) and (d) respectively. Areas of three solid rectangles (L1, L2 and L3) in blue colour in Fig. 5.5 (c) represent three terms (2, 3 and 4) of RHS of (5.32) respectively. While areas of three solid rectangles (U1, U2 and U3) in red colour in Fig. 5.5 (d) represent three terms (3, 4 and 5) of RHS of (5.33) respectively.

Part III: Calculation of $L(f, P_3)$ and $U(f, P_3)$

Here, y = f(x) = x + 1, $P_3 = \{1, 2, 3, 3.5, 4\}$, so

$I_1 = [1, 2], I_2 = [2, 3], I_3 = [3, 3.5], \ I_4 = [3.5, 4], \delta_1 = 2-1=1, \delta_2 = 3-2=1,$

$\delta_3 = 3.5-3 = 0.5, \delta_4 = 4-3.5 = 0.5, m_1 = f(1) = 2, m_2 = f(2) = 3, m_3 = f(3) = 4,$

$m_4 = f(3.5) = 4.5, M_1 = f(2) = 3, M_2 = f(3) = 4, M_3 = f(3.5) = 4.5, M_4 = f(4) = 5.$

$$\text{So, } L(f, P_3) = \sum_{i=1}^{4} m_i\delta_i = (2)(1)+(3)(1)+(4)(0.5)+(4.5)(0.5) = 9.25, \quad \ldots (5.34)$$

$$U(f, P_3) = \sum_{i=1}^{4} M_i\delta_i = (3)(1)+(4)(1)+(4.5)(0.5)+(5)(0.5) = 11.75 \quad \ldots (5.35)$$

Values of (5.34) and (5.35) are visualised in Fig. 5.5 (e) and (f) respectively. Areas of four solid rectangles (L1, L2, L3 and L4) in blue colour in Fig. 5.5 (e) represent four terms (2, 3, 2 and 2.25) of RHS of (5.34) respectively. While areas of four solid rectangles (U1, U2, U3 and U4) in red colour in Fig. 5.5 (f) represent four terms (3, 4, 2.25 and 2.5) of RHS of (5.35) respectively.

Now, as promised let us discuss some interesting observations of the experiment as follows.

- First of all let us calculate the exact area under the curve y = (x) = x + 1 and between the lines x = 1, x = 4 and y = 0 using formula for area of trapezium.

  $$\text{Required area} = \frac{1}{2}(2+5)(3) = \frac{1}{2}(21) = 10.5 \text{ unit}^2 \quad \ldots (5.36)$$

- **Observation 1**: Note that lower sum under estimate the exact area. It can be seen from Fig. 5.5 (a) and equations (5.30), (5.32) and (5.34) where values of lower sum are $L(f, P_1) = 8, L(f, P_2) = 9, L(f, P_3) = 9.25$ which are < 10.5 (exact value). But important and interesting observation is lower sum is increasing and coming closer and closer to exact value as we add more points in the partition. On comparing Fig. 5.5 (a), (c) and (e) you can observe that as number of points in the partition of finite and closed interval [1, 4] increases then lower sum will also increase and each new point in the partition contributes some new area in the old lower sum. No doubt as we add a new point in the partition then lower sum increase but the amount of new area contributed by new point decrease as we go on increasing points in the partition. This can be seen on comparing values of lower sum given by equations (5.30). (5.32) and (5.34). Now, let us first compare values of lower sum given by equations (5.30) and (5.32). The value of lower sum in (5.32) is 1 unit^2 (8 vs 9) more that the value in (5.30) where partition used in (5.32) has one additional point compare to the partition used to evaluate (5.30). On the other hand, the value of lower sum in (5.34) is 0.25 unit^2 (9 vs 9.25) more that the value in (5.32) where partition used in (5.34) has one additional point compare to the partition used to evaluate (5.32). So, moral of the story is addition of a new point in the partition contributes some additional area in the previous lower sum but in most of the cases this contribution decreases except some cases. The reason for such a behaviour is: initially difference between value of lower sum and value of exact area is more compare to later stages. For example, consider four more partitions of finite and closed interval [1, 4] given as follows.

$P_4 = \{1, 1.25, 1.5, 1.75, 2, 2.25, 2.5, 2.75, 3, 3.25, 3.5, 3.75, 4\}$,

$P_5 = \{1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2, 2.1, \ldots, 4\}$,

$P_6 = \{1, 1.01, 1.02, 1.03, 1.04, 1.05, 1.06, 1.07, 1.08, 1.09, 1.10, \ldots, 4\}$,

$P_7 = \{1, 1.001, 1.002, 1.003, 1.004, 1.005, 1.006, \ldots, 4\}$

Now, values of lower sums obtained over the partitions P_1 to P_7 are compared with exact value of the area of 10.5 unit^2 look at the Table 5.1 given as follows. Also, lower and upper sums corresponding to partitions P_4 and P_5 are shown in Fig. 5.6 (a), (c) and (b), (d) respectively.

Table 5.1 Comparison of lower sums in different stages with exact area

Stage	Name of the Partition	Exact Area	Lower Sum	Underestimate in Area	Underestimate in Area in %
1	P_1	10.5	8	2.5	$\frac{2.5}{10.5} \simeq 23.81\%$
2	P_2	10.5	9	1.5	$\frac{1.5}{10.5} \simeq 14.29\%$
3	P_3	10.5	9.25	1.25	$\frac{1.25}{10.5} \simeq 11.90\%$
4	P_4	10.5	10.125	0.375	$\frac{0.375}{10.5} \simeq 3.57\%$
5	P_5	10.5	10.35	0.15	$\frac{1.25}{10.5} \simeq 1.43\%$
6	P_6	10.5	10.485	0.015	$\frac{0.015}{10.5} \simeq 0.14\%$
7	P_7	10.5	10.4985	0.0015	$\frac{0.015}{10.5} \simeq 0.01\%$

Note that in the first stage percentage of the difference between values of exact area and lower sum with exact area was 23.81% (approx.) while in second stage this difference reduces to 14.29% (approx.) and in third stage this difference reduces to 11.90% (approx.) and so on in seventh stage when we have 3001 points in the partition P_7 then value of lower sum is just 0.0015 unit^2 less than the exact area and in percentage it is 0.01% (approx.). Thus, we can say that if given function is Riemann integrable then as number of points in the partition will tend to infinity then lower sum will also tend to exact value of the integral. This limiting value if exists is called **lower integral** and is denoted by

$$\underline{\int_1^4} f(x)\,dx = \lim_{n\to\infty} L(f, P) = \lim_{\|P\|\to 0} L(f, P) \qquad \ldots (5.37)$$

whereas n, the number of points in the partition P tends to ∞ then norm of the partition $\|P\| \to 0$.

Observation 2: Note that upper sum overestimates the exact area. It can be seen from Fig. 5.5 (b) and equations (5.31), (5.33) and (5.35) where values of upper sum are $U(f, P_1) = 13, U(f, P_2) = 12, U(f, P_3) = 11.75$ which are > 10.5 (exact value). But important and interesting observation is upper sum is decreasing and coming closer and closer to exact value as we add more points in the partition. On comparing Fig. 5.5 (b), (d) and (f) you can observe that as number of points in the partition of finite and closed interval [1, 4] increases then upper sum decreases and each new point in the partition remove some area from the old upper sum. No doubt as we add a new point in the partition then upper sum decreases but the amount of area removed by new point decreases as we go on increasing points in the partition. This can be seen on comparing values of upper sum given by equations (5.31). (5.33) and (5.35). Now, let us first compare values of upper sum given by equations (5.31) and (5.33). The value of upper sum in (5.33) is 1 unit^2 (12 vs 13) less that the value in (5.31) where partition used in (5.33) has one additional point compare to the partition used to evaluate (5.31). On the other hand, the value of upper sum in (5.35) is 0.25 unit^2 (11.75 vs 12) less than that of the value in (5.33) where partition used in (5.35) has one additional point compare to the partition used to evaluate (5.33). So, moral of the story is addition of a new point in the partition remove some area from the previous upper sum and reduces it but in most of the cases this deletion of area decreases except some cases. The reason for such a behaviour is: initially difference between value of upper sum and the value of the exact area is more compare to later stages. For example, values of upper sums obtained over the partitions P_1 to P_7 are compared with exact value of the area of 10.5 unit^2 look at the Table 5.2 given as follows.

Table 5.2 Comparison of upper sums in different stages with exact area

Stage	Name of the Partition	Exact Area	Upper Sum	Overestimate in Area	Overestimate in Area in %
1	P_1	10.5	13	2.5	$\frac{2.5}{10.5} \simeq 23.81\%$
2	P_2	10.5	12	1.5	$\frac{1.5}{10.5} \simeq 14.29\%$
3	P_3	10.5	11.75	1.25	$\frac{1.25}{10.5} \simeq 11.90\%$
4	P_4	10.5	10.875	0.375	$\frac{0.375}{10.5} \simeq 3.57\%$
5	P_5	10.5	10.65	0.15	$\frac{1.25}{10.5} \simeq 1.43\%$
6	P_6	10.5	10.515	0.015	$\frac{0.015}{10.5} \simeq 0.14\%$
7	P_7	10.5	10.5015	0.0015	$\frac{0.015}{10.5} \simeq 0.01\%$

Note that in the first stage percentage of the difference between values of exact area and upper sum was 23.81% (approx.) while in second stage this difference reduces to 14.29% (approx.) and in third stage this difference reduces to 11.90% (approx.) and so on in seventh stage when we have 3001 points in the partition P_7 then value of upper sum is just 0.0015 unit^2 more than the exact area and in percentage it is 0.01% (approx.). Thus, we can say that if given function is Riemann integrable then as number of points in the partition will tend to infinity then upper sum will also tend to exact value of the integral. This limiting value if exists is called **upper**

integral and is denoted by

$$\overline{\int_1^4} f(x)\,dx = \lim_{n\to\infty} U(f,\, P) = \lim_{\|P\|\to 0} U(f,\, P) \qquad \ldots (5.38)$$

whereas n, the number of points in the partition P tends to ∞ then norm of the partition $\|P\| \to 0$.

- **Observation 3**: $n \to \infty$ and $\|P\| \to 0$ are equivalent … (5.39)

  because as the number of points in the partition increases then norm of the partition either remain the same if additional point(s) does(do) not fall in the sub-interval which decides norm of the partition and it will decrease if additional point(s) falls(fall) in the sub-interval which decides norm of the partition. But as $n \to \infty$ then length of every sub-interval will approach to zero and so $\|P\| \to 0$. This gives us the way to do examples. You will experience it in Example 1.

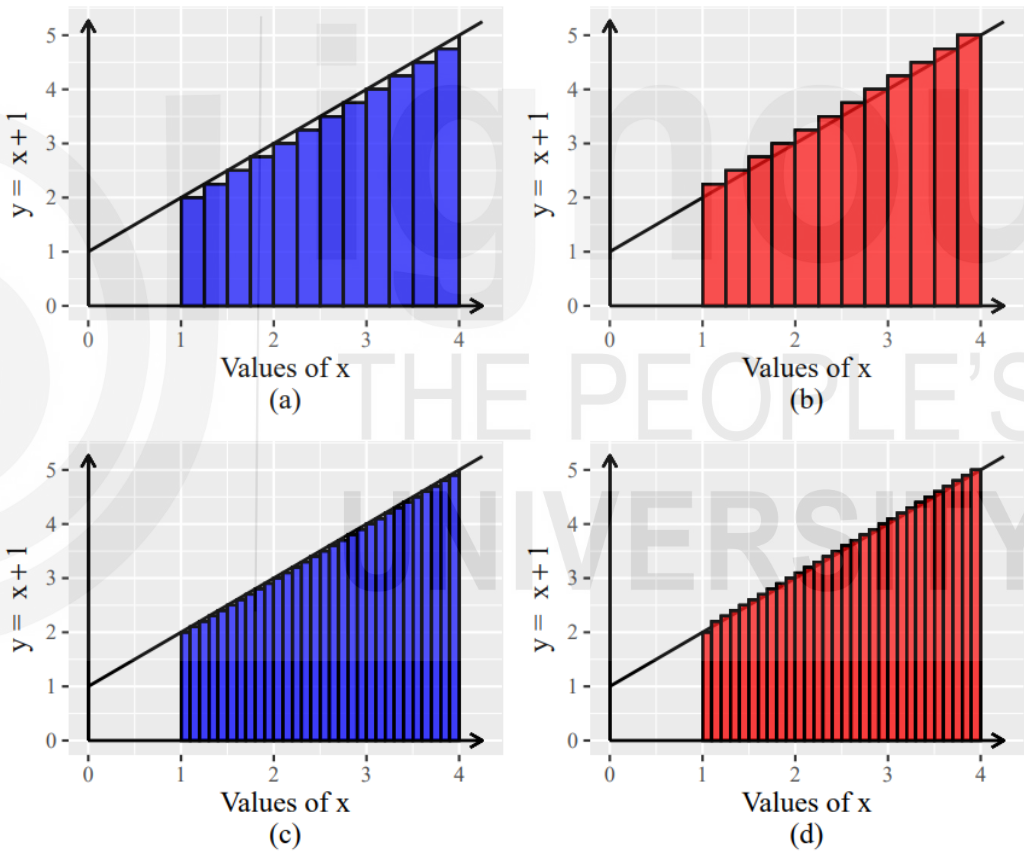

Fig. 5.6: Visualisation of Darboux sums over partition (a) P_4 lower sum (b) P_4 upper sum (c) P_5 lower sum (d) P_5 upper sum

Upper and Lower Riemann Integrals

Before defining lower and upper integral we have to prove a result given as follows.

Result 1: Let [a, b] be a finite and closed interval and P[a, b] denote the set of all partitions over [a, b]. If $f : [a, b] \to \mathbb{R}$ be a bounded function then

$$m(b-a) \le L(f,\, P) \le U(f,\, P) \le M(b-a) \quad \forall\ P \in P[a,\, b] \qquad \ldots (5.40)$$

where m = glb of f and M = lub of f on [a, b].

Proof: Consider the partition $P = \{a = x_0, x_1, x_2, x_3, \ldots, x_n = b\}$ of [a, b], then in usual notations, we have

$I_i = [x_{i-1}, x_i],\ \delta_i = x_i - x_{i-1},\ i = 1, 2, 3, \ldots, n$

Since f is bounded on [a, b] $\Rightarrow$ f is bounded on each sub-interval I_i.

Let $m_i =$ glb of f on I_i and $M_i =$ lub of f on I_i, $i = 1, 2, 3, \ldots, n$.

Therefore, m_i, M_i, m and M will always satisfy the following relation

$$m \le m_i \le M_i \le M\ \ \forall\ i,\ i = 1, 2, 3, \ldots, n$$

$$\Rightarrow m\delta_i \le m_i\delta_i \le M_i\delta_i \le M\delta_i\ \ \forall\ i,\ i = 1, 2, 3, \ldots, n\ [\because \delta_i = \text{length of an interval} > 0]$$

$$\Rightarrow \sum_{i=1}^{n} m\delta_i \le \sum_{i=1}^{n} m_i\delta_i \le \sum_{i=1}^{n} M_i\delta_i \le \sum_{i=1}^{n} M\delta_i$$

$$\Rightarrow m\sum_{i=1}^{n} \delta_i \le \sum_{i=1}^{n} m_i\delta_i \le \sum_{i=1}^{n} M_i\delta_i \le M\sum_{i=1}^{n} \delta_i$$

$$\Rightarrow m(b-a) \le L(f, P) \le U(f, P) \le M(b-a)\ \ \left[\because \sum_{i=1}^{n} \delta_i = b - a\right]$$

Hence, we have proved that

$$m(b-a) \le L(f, P) \le U(f, P) \le M(b-a)\ \ \forall\ P \in P[a, b]$$

Now, we can define lower and upper Riemann integrals as follows.

Let [a, b] be a finite and closed interval and $f : [a, b] \to \mathbb{R}$ be a bounded function then we know that

$$m(b-a) \le L(f, P) \le U(f, P) \le M(b-a)\ \ \forall\ P \in P[a, b] \qquad \ldots (5.41)$$

From (5.41), we have

$$L(f, P) \le M(b-a)\ \ \forall\ P \in P[a, b] \text{ and } U(f, P) \ge m(b-a)\ \ \forall\ P \in P[a, b]$$

$\Rightarrow$ the set $\{L(f, P)\}_{P \in P[a, b]}$ of all lower sums is bounded above by M(b – a) and therefore has the lub. Similarly, the set $\{U(f, P)\}_{P \in P[a, b]}$ of all upper sums is bounded below by m(b – a) and therefore has the glb.

where m = glb of f and M = lub of f.

Now, **lower Riemann integral** of f on [a, b] is denoted by $\int_{\underline{a}}^{b} f(x)\,dx$ and is

defined by $$\int_{\underline{a}}^{b} f(x)\,dx = \text{lub}\{L(f, P)\}_{P \in P[a, b]} \qquad \ldots (5.42)$$

and **upper Riemann integral** of f on [a, b] is denoted by $\int_{a}^{\bar{b}} f(x)\,dx$ and is

defined by $$\int_{a}^{\bar{b}} f(x)\,dx = \text{glb}\{U(f, P)\}_{P \in P[a, b]} \qquad \ldots (5.43)$$

So, far we have discussed all the terms required to understand the two definitions of Riemann integration given as follows.

Definition 1: Let [a, b] be a finite and closed interval and $f:[a, b] \to \mathbb{R}$ be a bounded function, then we say that f is **Riemann integrable** if lower Riemann integral and upper Riemann integral are equal. i.e., if

$$\int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{\bar{b}} f(x)\,dx \qquad \dots (5.44)$$

The common value of these integrals is called the Riemann integral of f on [a, b] and is denoted by $\int_a^b f(x)\,dx$.

$$\text{i.e., } \int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{\bar{b}} f(x)\,dx = \int_a^b f(x)\,dx. \qquad \dots (5.44\ D1)$$

Definition 2: Let [a, b] be a finite and closed interval and $f:[a, b] \to \mathbb{R}$ be a bounded function, then we say that f is **Riemann integrable** if

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\xi_i)\delta_i \qquad \dots (5.45)$$

exists and is independent of the choice of sub-interval $[x_{i-1}, x_i]$ and of the point $\xi_i \in [x_{i-1}, x_i]$ … (5.46)

If limit given by (5.45) exists then value of this limit is called the Riemann integral of f on [a, b] and is denoted by $\int_a^b f(x)\,dx$.

$$\text{i.e., } \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\xi_i)\delta_i = \int_a^b f(x)\,dx. \qquad \dots (5.46\ D2)$$

Now, let us do an example using both the definitions so that you become familiar with the procedure of evaluating integral using definition of Riemann integration.

Example 1: If $f:[2, 10] \to \mathbb{R}$ be a function defined by $f(x) = 4x + 3,\ x \in [2, 10]$. Show that f is Riemann integrable using definition 1 and definition 2 and $\int_2^{10} f(x)\,dx = 216$.

Solution: In usual notations we are given

a = 2, b = 10, f(x) = 4x + 3, length of the interval [2, 10] is 10 – 2 = 8 = b – a.

For the finite and closed interval [a, b] we form the partition

$$P = \{a = a + 0h,\ a + h,\ a + 2h,\ a + 3h,\ \dots,\ a + nh = b\},\ h = \frac{b-a}{n} \qquad \dots (5.47)$$

So, in our case partition will be

$$P = \{2 = 2 + 0h,\ 2 + h,\ 2 + 2h,\ 2 + 3h,\ \dots,\ 2 + nh = 10\},\ \text{where } h = \frac{10-2}{n} = \frac{8}{n}$$

Therefore, i^{th} sub interval $= [2 + (i-1)h,\ 2 + ih]$, so

$$x_{i-1} = 2 + (i-1)h = 2 + \frac{8(i-1)}{n},\ x_i = 2 + ih = 2 + \frac{8i}{n},\ \text{as } h = \frac{8}{n} \text{ and } \delta_i = h = \frac{8}{n}$$

Now, we proceed to apply definition 1 and definition 2 of Riemann integration as follows.

Using Definition 1

Let $m_i = \text{glb of } f \text{ on } I_i = \left[2+\frac{8(i-1)}{n},\ 2+\frac{8i}{n}\right],\ i = 1, 2, 3, \ldots, n$

$M_i = \text{lub of } f \text{ on } I_i = \left[2+\frac{8(i-1)}{n},\ 2+\frac{8i}{n}\right],\ i = 1, 2, 3, \ldots, n$

Since f is an increasing function over the interval [2, 10] and hence over each sub-interval $I_i = \left[2+\frac{8(i-1)}{n},\ 2+\frac{8i}{n}\right],\ i = 1, 2, 3, \ldots, n,$ therefore

$m_i = f\left(2+\frac{8(i-1)}{n}\right),\ M_i = f\left(2+\frac{8i}{n}\right),\ i = 1, 2, 3, \ldots, n,$

So,

$$L(f, P) = \sum_{i=1}^{n} m_i\,\delta_i = \sum_{i=1}^{n} f\left(2+\frac{8(i-1)}{n}\right)\frac{8}{n} = 8\sum_{i=1}^{n}\left[4\left(2+\frac{8(i-1)}{n}\right)+3\right]\frac{1}{n}$$

$$= 8\sum_{i=1}^{n}\left[\frac{32}{n^2}(i-1)+\frac{11}{n}\right] = \frac{256}{n^2}\sum_{i=1}^{n}(i-1)+\frac{8}{n}\sum_{i=1}^{n}11$$

$$= \frac{256}{n^2}(0+1+2+3+\ldots+n-1)+\frac{8}{n}(11n) = \frac{256}{n^2}\frac{n(n-1)}{2}+88 = 128\left(1-\frac{1}{n}\right)+88$$

$$\left[\because 1+2+3+\ldots+n = \frac{n(n+1)}{2},\ \text{so } 1+2+3+\ldots+n-1 = \frac{n(n-1)}{2}\right]$$

$$U(f, P) = \sum_{i=1}^{n} M_i\,\delta_i = \sum_{i=1}^{n} f\left(2+\frac{8i}{n}\right)\frac{8}{n} = 8\sum_{i=1}^{n}\left[4\left(2+\frac{8i}{n}\right)+3\right]\frac{1}{n}$$

$$= 8\sum_{i=1}^{n}\left[\frac{32}{n^2}i+\frac{11}{n}\right] = \frac{256}{n^2}\sum_{i=1}^{n}i+\frac{8}{n}\sum_{i=1}^{n}11$$

$$= \frac{256}{n^2}(1+2+3+\ldots+n)+\frac{8}{n}(11n) = \frac{256}{n^2}\frac{n(n+1)}{2}+88 = 128\left(1+\frac{1}{n}\right)+88$$

By definition of lower Riemann integration refer (5.42), we have

$$\underline{\int_2^{10}} f(x)\,dx = \text{lub}\{L(f, P)\}_{P\in P[2,10]} = \lim_{\|P\|\to 0}\left(128\left(1-\frac{1}{n}\right)+88\right)$$

$$= \lim_{n\to\infty}\left(128\left(1-\frac{1}{n}\right)+88\right) \qquad [\text{Using (5.39)}]$$

$$= 128(1-0)+88 = 216$$

By definition of upper Riemann integration refer (5.43), we have

$$\overline{\int_2^{10}} f(x)\,dx = \text{glb}\{U(f, P)\}_{P\in P[2,10]} = \lim_{\|P\|\to 0}\left(128\left(1+\frac{1}{n}\right)+88\right) = \lim_{n\to\infty}\left(128\left(1+\frac{1}{n}\right)+88\right)$$

$$= 128(1+0)+88 = 216$$

Since $\underline{\int_2^{10}} f(x)\,dx = 216 = \overline{\int_2^{10}} f(x)\,dx,$ so by definition 1 of Riemann integration f is Riemann integrable and $\int_2^{10} f(x)\,dx = 216.$

Using Definition 2:

Let $\xi_i = 2 + \frac{8i}{n}, \ \ i = 1, 2, 3, \ldots, n$ [Refer (5.46)]

Now,

$$\lim_{\|P\|\to 0}\sum_{i=1}^{n} f(\xi_i)\delta_i = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f\left(2+\frac{8i}{n}\right)\frac{8}{n} \qquad \left[\because\ \xi_i = 2+\frac{8i}{n} \text{ and } \delta_i = \frac{8}{n}\right]$$

$$= 8\lim_{\|P\|\to 0}\sum_{i=1}^{n}\left[4\left(2+\frac{8i}{n}\right)+3\right]\frac{1}{n} = 8\lim_{\|P\|\to 0}\left(\sum_{i=1}^{n}\left[\frac{32}{n^2}i+\frac{11}{n}\right]\right) = \lim_{\|P\|\to 0}\left(\frac{256}{n^2}\sum_{i=1}^{n} i + \frac{8}{n}\sum_{i=1}^{n} 11\right)$$

$$= \lim_{\|P\|\to 0}\left(\frac{256}{n^2}(1+2+3+\ldots+n)+\frac{8}{n}(11n)\right) = \lim_{n\to\infty}\left(\frac{256}{n^2}\frac{n(n+1)}{2}+88\right) \quad \left[\begin{array}{l}\text{Using}\\ (5.39)\end{array}\right]$$

$$= \lim_{n\to\infty}\left(128\left(1+\frac{1}{n}\right)+88\right) = 128(1+0)+88 = 216.$$

Since $\lim\limits_{\|P\|\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\delta_i = 216$ so by definition 2 of Riemann integration f is Riemann integrable and $\int_2^{10} f(x)\,dx = 216$.

Remark 1: From school mathematics you know that

$\int_2^{10} f(x)\,dx = \int_2^{10}(4x+3)\,dx = \left[2x^2+3x\right]_2^{10} = 200+30-8-6 = 230-14 = 216.$ But do not evaluate integral like this if it is given that evaluate using definition 1 or definition 2 of Riemann integration.

Example 2: Give an example of a function which is not Riemann integrable.

Solution: Let us consider the Dirichlet function on the interval [0, 1] given as follows.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}\cap[0, 1] \\ 0, & x \notin \mathbb{Q}\cap[0, 1] \end{cases}$$

In usual notations we are given

a = 0, b = 1, f(x) as defined, length of the interval [0, 1] is 1 – 0 = 1 = b – a.

For the finite and closed interval [a, b] we form the partition

$$P = \{a = a+0h,\ a+h,\ a+2h,\ a+3h,\ \ldots,\ a+nh = b\},\ h = \frac{b-a}{n}$$

So, in our case partition will be

$$P = \{0 = 0+0h,\ 0+h,\ 0+2h,\ 0+3h,\ \ldots,\ 0+nh = 1\},\ \text{ where } h = \frac{1-0}{n} = \frac{1}{n}$$

$$\Rightarrow P = \{0h,\ h,\ 2h,\ 3h,\ \ldots,\ nh = 1\},\ \text{ where } h = \frac{1}{n}$$

Therefore, i^{th} sub interval $= [(i-1)h, ih]$, so

$$x_{i-1} = (i-1)h = \frac{(i-1)}{n},\ x_i = ih = \frac{i}{n},\ \text{as } h = \frac{1}{n} \text{ and } \delta_i = h = \frac{1}{n}$$

We claim that given function is not Riemann integrable. To prove our claim, we shall show that value of two limits used in definition 1 of Riemann integration do not match.

Let $m_i = \text{glb of } f \text{ on } I_i = \left[\frac{(i-1)}{n}, \frac{i}{n}\right], \ i = 1, 2, 3, \ldots, n$

$M_i = \text{lub of } f \text{ on } I_i = \left[\frac{(i-1)}{n}, \frac{i}{n}\right], \ i = 1, 2, 3, \ldots, n$

Since every sub-interval $\left[\frac{(i-1)}{n}, \frac{i}{n}\right], \ i = 1, 2, 3, \ldots, n$ contains rational as well as irrational numbers so, we have

$m_i = 0, \ M_i = 1, \ i = 1, 2, 3, \ldots, n,$

So, $L(f, P) = \sum_{i=1}^{n} m_i \delta_i = \sum_{i=1}^{n} (0)\frac{1}{n} = \sum_{i=1}^{n} 0 = 0$

and $U(f, P) = \sum_{i=1}^{n} M_i \delta_i = \sum_{i=1}^{n} (1)\frac{1}{n} = \sum_{i=1}^{n} \frac{1}{n} = \frac{1}{n}(n) = 1$

By definition of lower Riemann integration refer (5.42), we have

$$\underline{\int_{0}^{1}} f(x)\,dx = \text{lub}\{L(f, P)\}_{P \in P[0, 1] \cap \mathbb{Q}} = \lim_{\|P\| \to 0} (0) = 0 \qquad \ldots (5.48)$$

By definition of upper Riemann integration refer (5.43), we have

$$\overline{\int_{0}^{1}} f(x)\,dx = \text{glb}\{U(f, P)\}_{P \in P[0, 1] \cap \mathbb{Q}} = \lim_{\|P\| \to 0} (1) = \lim_{n \to \infty} (1) = 1 \qquad \ldots (5.49)$$

Since $\underline{\int_{0}^{1}} f(x)\,dx \neq \overline{\int_{0}^{1}} f(x)\,dx$. Hence, given function is not Riemann integrable.

Before doing the next example let us define a notation.

Notation: We shall denote the set of all Riemann integrable functions over the closed and finite interval [a, b] by R[a, b].

Example 3: If c is a fixed real number and $f : [a, b] \to \mathbb{R}$ is defined by

$f(x) = c, \ x \in [a, b],$ then prove that $f \in R[a, b],$ and $\int_{a}^{b} f(x)\,dx = c(b-a).$ (5.50)

Hence, evaluate $\int_{10}^{1000} 100\,dx.$

Solution: In usual notations we are given

a = a, b = b, f(x) = c, length of the interval [a, b] is b – a.

For the finite and closed interval [a, b] we form the partition

$P = \{a = a + 0h, a + h, a + 2h, a + 3h, \ldots, a + nh = b\}, h = \frac{b-a}{n}$

Therefore, i^{th} sub interval $= [a + (i-1)h, a + ih],$ so

$x_{i-1} = a + (i-1)h, \ x_i = a + ih,$ as $h = \frac{b-a}{n}$ and $\delta_i = h = \frac{b-a}{n}$

Since f is a constant function so it will take value c at all points of its domain. Hence, if

$m_i = \text{glb of } f \text{ on } I_i = [a+(i-1)h, \, a+ih], \quad i = 1, 2, 3, \ldots, n$ and

$M_i = \text{lub of } f \text{ on } I_i = [a+(i-1)h, \, a+ih], \quad i = 1, 2, 3, \ldots, n$ then

$m_i = c, \quad M_i = c, \quad i = 1, 2, 3, \ldots, n,$

So, $L(f, P) = \sum_{i=1}^{n} m_i \, \delta_i = \sum_{i=1}^{n} ch = \sum_{i=1}^{n} c\left(\frac{b-a}{n}\right) = c\left(\frac{b-a}{n}\right)n = c(b-a)$

and $U(f, P) = \sum_{i=1}^{n} M_i \, \delta_i = \sum_{i=1}^{n} ch = \sum_{i=1}^{n} c\left(\frac{b-a}{n}\right) = c\left(\frac{b-a}{n}\right)n = c(b-a)$

By definition of lower Riemann integration refer (5.42), we have

$$\underline{\int_a^b} f(x)\, dx = \text{lub}\{L(f, P)\}_{P \in P[a, b]} = \lim_{\|P\| \to 0} (c(b-a)) = c(b-a) \qquad \ldots (5.51)$$

By definition of upper Riemann integration refer (5.43), we have

$$\overline{\int_a^b} f(x)\, dx = \text{glb}\{U(f, P)\}_{P \in P[a, b]} = \lim_{\|P\| \to 0} (c(b-a)) = c(b-a) \qquad \ldots (5.52)$$

Since due to (5.51) and (5.52) $\underline{\int_a^b} f(x)\, dx = c(b-a) = \overline{\int_a^b} f(x)\, dx$. Hence, given function is Riemann integrable and $\int_a^b f(x)\, dx = c(b-a)$.

Comparing $\int_{10}^{1000} 100\, dx$ with $\int_a^b c\, dx$, we get a = 10, b = 1000, c = 100, hence

$$\int_{10}^{1000} 100\, dx = 100(1000-10) = 100(990) = 99000.$$

Example 4: If [a, b] is finite and closed interval and $f : [a, b] \to \mathbb{R}$ is defined by $f(x) = x, \; x \in [a, b]$, i.e., f is an identity function then prove that $f \in R[a, b]$.

Solution: In usual notations we are given

a = a, b = b, f(x) = x, length of the interval [a, b] is b – a.

For the finite and closed interval [a, b] we form the partition

$P = \{a = a + 0h, \, a+h, \, a+2h, \, a+3h, \, \ldots, \, a+nh = b\}, \; h = \frac{b-a}{n}$

Therefore, i^{th} sub interval $= [a+(i-1)h, \, a+ih]$, so

$x_{i-1} = a+(i-1)h, \; x_i = a+ih, \;$ as $\; h = \frac{b-a}{n}$ and $\delta_i = h = \frac{b-a}{n}$

Since f is an identity function so it will take value equal to its input from the domain. Hence, if

$m_i = \text{glb of } f \text{ on } I_i = [a+(i-1)h, \, a+ih], \quad i = 1, 2, 3, \ldots, n$ and

$M_i = \text{lub of } f \text{ on } I_i = [a+(i-1)h, \, a+ih], \quad i = 1, 2, 3, \ldots, n$ then

$m_i = a + (i-1)h, \quad M_i = a + ih, \quad i = 1, 2, 3, \ldots, n,$

So,

$$L(f, P) = \sum_{i=1}^{n} m_i \delta_i = \sum_{i=1}^{n} \left(a + (i-1)h\right)h = \sum_{i=1}^{n} \left(ah + (i-1)h^2\right) = \sum_{i=1}^{n} ah + \sum_{i=1}^{n} (i-1)h^2$$

$$= nah + h^2(0 + 1 + 2 + 3 + \ldots + n - 1) = anh + h^2(1 + 2 + 3 + \ldots + n - 1)$$

$$= anh + h^2 \frac{n(n-1)}{2} \qquad \left[\begin{array}{l} \because 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}, \text{ so} \\ 1 + 2 + 3 + \ldots + n - 1 = \dfrac{n(n-1)}{2} \end{array}\right]$$

$$= a\left(b-a\right) + \left(\frac{b-a}{n}\right)^2 \frac{n(n-1)}{2} \quad \left[\because h = \frac{b-a}{n}\right]$$

$$\Rightarrow L(f, P) = a\left(b-a\right) + \frac{1}{2}\left(b-a\right)^2\left(1 - \frac{1}{n}\right)$$

$$U(f, P) = \sum_{i=1}^{n} M_i \delta_i = \sum_{i=1}^{n} \left(a + ih\right)h = \sum_{i=1}^{n} \left(ah + ih^2\right) = \sum_{i=1}^{n} ah + \sum_{i=1}^{n} ih^2$$

$$= nah + h^2(1 + 2 + 3 + \ldots + n)$$

$$= anh + h^2 \frac{n(n+1)}{2} \qquad \left[\because 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}\right]$$

$$= a\left(b-a\right) + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2} \quad [\text{Using } (5.52)]$$

$$\Rightarrow U(f, P) = a\left(b-a\right) + \frac{1}{2}\left(b-a\right)^2\left(1 + \frac{1}{n}\right)$$

By definition of lower Riemann integration refer (5.42), we have

$$\underline{\int_a^b} f(x)\,dx = \text{lub}\left\{L(f, P)\right\}_{P \in P[a, b]} = \lim_{\|P\| \to 0}\left(a\left(b-a\right) + \frac{1}{2}\left(b-a\right)^2\left(1 - \frac{1}{n}\right)\right)$$

$$= \lim_{n \to \infty}\left(a\left(b-a\right) + \frac{1}{2}\left(b-a\right)^2\left(1 - \frac{1}{n}\right)\right) \quad [\text{Using } (5.39)]$$

$$= a\left(b-a\right) + \frac{1}{2}\left(b-a\right)^2(1-0) = ab - a^2 + \frac{1}{2}b^2 + \frac{1}{2}a^2 - ab = \frac{1}{2}\left(b^2 - a^2\right)$$

By definition of upper Riemann integration refer (5.43), we have

$$\overline{\int_a^b} f(x)\,dx = \text{glb}\left\{U(f, P)\right\}_{P \in P[a, b]} = \lim_{\|P\| \to 0}\left(a\left(b-a\right) + \frac{1}{2}\left(b-a\right)^2\left(1 + \frac{1}{n}\right)\right)$$

$$= \lim_{n \to \infty}\left(a\left(b-a\right) + \frac{1}{2}\left(b-a\right)^2\left(1 + \frac{1}{n}\right)\right) \quad \left[\because h = \frac{b-a}{n}\right]$$

$$= a\left(b-a\right) + \frac{1}{2}\left(b-a\right)^2(1+0) = ab - a^2 + \frac{1}{2}b^2 + \frac{1}{2}a^2 - ab = \frac{1}{2}\left(b^2 - a^2\right)$$

Since $\underline{\int_a^b} f(x)\,dx = \frac{1}{2}\left(b^2 - a^2\right) = \overline{\int_a^b} f(x)\,dx$. Hence, given function is Riemann integrable and $\int_a^b f(x)\,dx = \frac{1}{2}\left(b^2 - a^2\right)$.

Now, you can try the following Self-Assessment Question.

SAQ 1

If $f:[1, 3] \to \mathbb{R}$ be a function defined by $f(x) = x^2 + 1,\ x \in [1, 3],$ then using definition 1 and definition 2 show that f is Riemann integrable and $\int_1^3 f(x)\,dx = \frac{32}{3}.$

5.4 PROPERTIES OF RIEMANN INTEGRATION

In the previous section you have understood two definitions of Riemann integration and learnt to evaluate integral using them. In the same section we have proved that constant and identity functions are Riemann integrable. There is a long list of functions which are Riemann integrable. But due to scope of the course we did not discuss their proof. For example, every continuous function is Riemann integrable, we have stated this as a theorem refer (5.63). You know that polynomial, exponential, logarithm, sine, cosine, etc. all are continuous in their domain and hence Riemann integrable. Properties of Riemann integration help in creating more Riemann integrable functions from given Riemann integrable functions. In this section we will list some properties of Riemann integration. Proofs of these properties are beyond the scope of this course. Before listing these properties let define a notation. We will denote the set of all Riemann integrable function on the finite and closed interval [a, b] by R[a, b].

Property 1: Let $f, g \in R[a, b]$ such that $f(x) \le g(x)\ \ \forall\ x \in [a, b]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \qquad \ldots (5.53)$$

Also, if $f(x) \ge 0\ \ \forall\ x \in [a, b]$ then $\int_a^b f(x)\,dx \ge 0.$ … (5.54)

Property 2: Let $f \in R[a, b]$ and c is any real number then

$$\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx \qquad \ldots (5.55)$$

Property 3: Let $f, g \in R[a, b]$ and $\alpha, \beta \in \mathbb{R},$ then

(a) $f + g \in R[a, b]$ and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ … (5.56)

(b) $f - g \in R[a, b]$ and $\int_a^b (f-g)(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ … (5.57)

(c) $\alpha f + \beta g \in R[a, b]$ and $\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$ … (5.58)

(d) $fg \in R[a, b]$ … (5.59)

(e) $f^2 \in R[a, b]$ … (5.60)

(f) $|f| \in R[a, b]$ and $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f|(x)\,dx$ … (5.61)

Property 4: For some $c \in [a, b]$, $f \in R[a, b]$, if and only if

$f \in R[a, c]$ and $f \in R[c, b]$. Also, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ … (5.62)

We will end this section by proving four important theorems (Theorems 4 to 7) which will not only give you inside of calculus, connection between differentiation and integration but will also help you in understanding many concepts of probability theory of continuous world. But to prove these four theorems we need three more theorems. Due to the scope of this course, we are just stating these three theorems. After stating these three theorems we will state and prove four theorems as mentioned earlier. So, let us start discussion of last four results of this unit.

Theorem 1: Let [a, b] be a finite and closed interval and $f : [a, b] \to \mathbb{R}$ be a continuous function then f is integrable or $f \in R[a, b]$. … (5.63)

Theorem 2: If a function is continuous on a finite and closed interval [a, b] then f is bounded. … (5.64)

Theorem 3: Let [a, b] be a finite and closed interval and $f : [a, b] \to \mathbb{R}$ be a bounded and integrable function on [a, b] then $|f|$ is also bounded and integrable function on [a, b]. Moreover, $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f|(x)\,dx$. … (5.65)

Theorem 4: Let [a, b] be a finite and closed interval and $f : [a, b] \to \mathbb{R}$ be a continuous function and $F(x) = \int_a^x f(t)\,dt$, where $x \in [a, b]$, then F is continuous on [a, b]. … (5.66)

Proof: We are given that $F(x) = \int_a^x f(t)\,dt$ where $x \in [a, b]$ … (5.67)

It is a function of x. As x increases F(x) also increases refer Fig. 5.7 (a) to (d).

Also, it is given that f is a continuous function on finite and closed interval [a, b].

$\Rightarrow$ f is a bounded function on [a, b]. … (5.68)

$\Rightarrow \exists$ a positive real number k such that

$|f(t)| \le k \qquad \forall\ t \in [a, b]$ … (5.69)

Let $x, y \in [a, b]$ such that $a \le x < y \le b$ then due to (5.67), we have

$$F(x) = \int_a^x f(t)\,dt \quad \text{and} \quad F(y) = \int_a^y f(t)\,dt$$

$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt \quad [\because x < y]$$

$$= \int_x^y f(t)\,dt$$

$$\therefore \ |F(y)-F(x)| = \left|\int_x^y f(t)\,dt\right|$$

$$\le \int_x^y |f(t)|\,dt \qquad \left[\begin{array}{l}\because \text{ If } f \text{ is bounded and integrable on } [a,\ b] \text{ then} \\ |f| \text{ is also bounded and integrable and} \\ \left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt\end{array}\right]$$

$$\therefore \ |F(y)-F(x)| \le \int_x^y k\,|dt| \qquad [\text{Using } (5.69)]$$

$$= k\int_x^y |dt| = k|y-x| < \varepsilon \quad \text{whenever } |y-x| < \frac{\varepsilon}{k}$$

$$\Rightarrow |F(y)-F(x)| < \varepsilon \quad \text{whenever } |y-x| < \frac{\varepsilon}{k}$$

$\Rightarrow$ F is uniformly continuous on [a, b]. [Refer (2.35)]

Hence, F is continuous on [a, b]. $[\because$ Uniform continuity $\Rightarrow$ continuity$]$

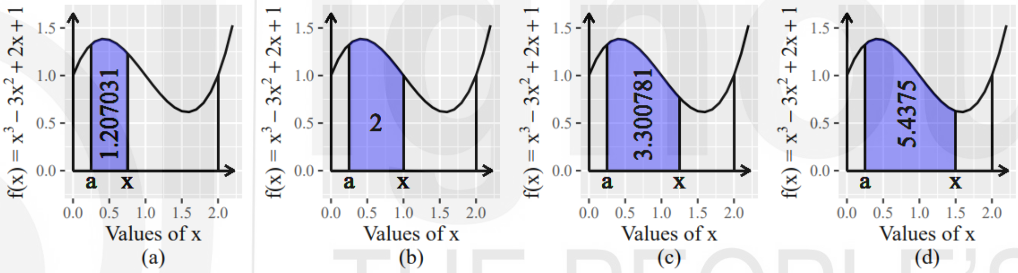

Fig. 5.7: Visualisation of F(x) increases as x increases (a) to (d)

Theorem 5: Let [a, b] be a finite and closed interval and $f:[a,\ b]\to\mathbb{R}$ be a continuous function then there exists $c\in[a,\ b]$, such that $\int_a^b f(x)\,dx = (b-a)f(c)$.

… (5.70)

Proof: It is given that f is a continuous function on finite and closed interval [a, b]. … (5.71)

$\Rightarrow$ f is a bounded function on [a, b]. … (5.72)

Also, continuity of f implies f is Riemann integrable. … (5.73)

Due to (5.72) and (5.40), we have

$$m(b-a) \le L(f,\ P) \le U(f,\ P) \le M(b-a) \quad \forall\ P \in P[a,\ b] \qquad \dots (5.74)$$

We know that lower and upper integrals are the lub and glb respectively of the lower and upper sums, therefore

$$L(f,\ P) \le \int_{\underline{a}}^{b} f(x)\,dx = \int_a^b f(x)\,dx \qquad [\text{Using } (5.63) \text{ and } (5.71)] \qquad \dots (5.75)$$

$$U(f,\ P) \ge \int_a^{\bar{b}} f(x)\,dx = \int_a^b f(x)\,dx \qquad [\text{Using } (5.63) \text{ and } (5.71)] \qquad \dots (5.76)$$

Combining (5.74), (5.75) and (5.76), we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\Rightarrow m \le \frac{1}{(b-a)}\int_a^b f(x)\,dx \le M$$

$$\Rightarrow m \le \lambda \le M, \quad \text{where} \quad \lambda = \frac{1}{(b-a)}\int_a^b f(x)\,dx$$

Or $\frac{1}{(b-a)}\int_a^b f(x)\,dx = \lambda, \quad m \le \lambda \le M$

$$\Rightarrow \int_a^b f(x)\,dx = \lambda(b-a), \quad \lambda \in [m, M] \qquad \dots (5.77)$$

Since f is continuous on [a, b], so it will attain every value between its bounds m and M.

$\therefore \lambda \in [m, M] \Rightarrow \exists$ a number $c \in [a, b]$ such that

$$f(c) = \lambda \qquad \dots (5.78)$$

Using (5.78) in (5.77), we get

$$\int_a^b f(x)\,dx = (b-a)f(c). \text{ where } c \in [a, b]$$

Remark 2: If we replace b by a + h in above integral then we have

$$\int_a^{a+h} f(x)\,dx = h\,f(a+\theta h), \text{ where } a+\theta h \in [a, a+h], \ 0 \le \theta \le 1 \qquad \dots (5.79)$$

Theorem 6: Let [a, b] be a finite and closed interval and $f:[a, b] \to \mathbb{R}$ be a continuous function and $F(x) = \int_a^x f(t)\,dt,$ where $x \in [a, b],$ then F is differentiable on [a, b] and $F' = f.$ … (5.80)

Proof: It is given that f is a continuous function on finite and closed interval [a, b]. … (5.81)

Also, $F(x) = \int_a^x f(t)\,dt, \ x \in [a, b]$ … (5.82)

Let x_0 be an arbitrary point of [a, b] and h be a small positive real number such that $x_0 + h \in [a, b].$ So, due to (5.82), we have

$$F(x_0) = \int_a^{x_0} f(t)\,dt \quad \text{and} \quad F(x_0+h) = \int_a^{x_0+h} f(t)\,dt$$

$$\therefore F(x_0+h) - F(x_0) = \int_a^{x_0+h} f(t)\,dt - \int_a^{x_0} f(t)\,dt$$

$$= \int_a^{x_0} f(t)\,dt + \int_{x_0}^{x_0+h} f(t)\,dt - \int_a^{x_0} f(t)\,dt \quad [\text{Using (5.62) in the first integral}]$$

$$= \int_{x_0}^{x_0+h} f(t)\,dt$$

$$= hf(x_0 + \theta h), \text{ where } 0 \le \theta \le 1 \quad [\text{Using (5.79)}]$$

$$\Rightarrow \frac{F(x_0+h) - F(x_0)}{h} = f(x_0 + \theta h), \text{ where } 0 \le \theta \le 1$$

$$\Rightarrow \lim_{h\to 0} \frac{F(x_0+h) - F(x_0)}{h} = \lim_{h\to 0} f(x_0 + \theta h), \text{ where } 0 \le \theta \le 1$$

$$\Rightarrow F'(x_0) = f(x_0), \; x_0 \in [a, b]$$

Since x_0 is an arbitrary point of [a, b] so F is differentiable on [a, b] and $F' = f$.

Theorem 7: (**Fundamental Theorem of Calculus**): Let [a, b] be a finite and closed interval and $f : [a, b] \to \mathbb{R}$ be Riemann integrable (or we can simply say integrable) on [a, b] and F is antiderivative of the function f then

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad \ldots (5.83)$$

Proof: Since F is antiderivative of f on [a, b], therefore

$$F'(x) = f(x) \;\; \forall \; x \in [a, b] \qquad \ldots (5.84)$$

Consider the partition $P = \{a = x_0, x_1, x_2, x_3, \ldots, x_n = b\}$ of [a, b], then in usual notations, we have

$$I_i = [x_{i-1}, x_i], \; \delta_i = x_i - x_{i-1}, \; i = 1, 2, 3, \ldots, n \qquad \ldots (5.85)$$

Since F is differentiable on [a, b] $\Rightarrow$ F is continuous on [a, b].

$\Rightarrow$ F is continuous on each sub-interval $I_i = [x_{i-1}, x_i], i = 1, 2, 3, \ldots, n$

Applying Lagrange's mean value theorem to F on each sub-interval $I_i = [x_{i-1}, x_i], i = 1, 2, 3, \ldots, n$ we have

$$F(x_i) - F(x_{i-1}) = (x_i - x_{i-1})F'(\xi_i) \;\; \xi_i \in [x_{i-1}, x_i]$$

$$= f(\xi_i)\delta_i \qquad [\text{Using (5.84) and (5.85)}]$$

or $f(\xi_i)\delta_i = F(x_i) - F(x_{i-1})$

$$\Rightarrow \sum_{i=1}^{n} f(\xi_i)\delta_i = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big)$$

$$= \big(F(x_1) - F(x_0)\big) + \big(F(x_2) - F(x_1)\big) + \big(F(x_3) - F(x_2)\big) + \ldots + \big(F(x_n) - F(x_{n-1})\big)$$

$$\Rightarrow \sum_{i=1}^{n} f(\xi_i)\delta_i = F(x_n) - F(x_0) \quad \text{All other terms in RHS cancel out in pairs}$$

$$\Rightarrow \sum_{i=1}^{n} f(\xi_i)\delta_i = F(b) - F(a) \qquad [\because x_0 = a, \; x_n = b]$$

$$\Rightarrow \lim_{\|P\|\to 0} \left(\sum_{i=1}^{n} f(\xi_i)\delta_i \right) = \lim_{\|P\|\to 0} \big(F(b) - F(a)\big)$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a) \qquad [\text{Using (5.46 D2) in LHS}]$$

Now, you can try the following Self Assessment Question.

SAQ 2

Find $\frac{d}{dx}(F(x))$, where $F(x) = \int_5^x e^{t^2} dt$.

5.5 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- Summation notation fails to work in the case of uncountable numbers. So, what is the solution? Solution is **integration**.

- Definite integration gives area bounded by four things (i) under the curve of the function to which we are integrating (ii) equation corresponding to the lower limit of integration (iii) equation corresponding to the upper limit of integration (iv) equation corresponding to the axis of integration.

- If I = [a, b] be a finite and closed interval then by a **partition P of I**, we mean a finite ordered set $\{x_0, x_1, x_2, x_3, \ldots, x_n\}$ of points of [a, b] such that

  $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b.$

  The n + 1 points $x_0, x_1, x_2, x_3, \ldots, x_n$ are called partition points of P and the n sub-intervals
  $I_1 = [x_0, x_1], I_2 = [x_1, x_2], I_3 = [x_2, x_3], \ldots, I_i = [x_{i-1}, x_i], \ldots, I_n = [x_{n-1}, x_n]$
  determined by n + 1 points of P are called the segments of the partition P.

- The norm of the partition P is denoted by $\|P\|$ and defined as follows.

  $\|P\| = \max\{x_i - x_{i-1} : i = 1, 2, 3, 4, \ldots, n\}$

- If I = [a, b] be a finite and closed interval and P_1 be a partition of [a, b], then another partition P_2 of [a, b] is called refinement of the partition P_1 if $P_1 \subset P_2$.

- **Lower Darboux sum** of f over the partition P is denoted by $L(f, P)$ and given by $L(f, P) = \sum_{i=1}^{n} m_i \delta_i$, where $m_i = \text{glb of } f \text{ on } [a, b]$

- **Upper Darboux sum** of f over the partition P is denoted by $U(f, P)$ and given by $U(f, P) = \sum_{i=1}^{n} M_i \delta_i$. where $M_i = \text{lub of } f \text{ on } [a, b]$

- **Oscillatory sum** of f over the partition P is denoted by $\omega(f, P)$ and given by

  $\omega(f, P) = \sum_{i=1}^{n} (M_i - m_i)\delta_i.$

- On addition of a new point in a given partition it contributes some additional area in the previous lower sum. So, addition of a new point in a given partition increases its lower sum.

- On addition of a new point in a given partition it deletes some area from the previous upper sum. So, addition of a new point in a given partition decreases its upper sum.

- **Lower Riemann integral** of f on [a, b] is denoted by $\underline{\int_a^b} f(x)\,dx$ and is defined by $\underline{\int_a^b} f(x)\,dx = \text{lub}\{L(f, P)\}_{P \in P[a, b]}$

- **Upper Riemann integral** of f on [a, b] is denoted by $\overline{\int_a^b} f(x)\,dx$ and is defined by $\overline{\int_a^b} f(x)\,dx = \text{glb}\{U(f, P)\}_{P \in P[a, b]}$

- **Definition 1**: We say that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if lower Riemann integral and upper Riemann integral are equal. i.e., if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

- **Definition 2**: We say that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if

$\lim\limits_{\|P\| \to 0} \sum\limits_{i=1}^{n} f(\xi_i)\delta_i$ exists and is independent of the choice of sub-interval $[x_{i-1}, x_i]$ and of the point $\xi_i \in [x_{i-1}, x_i]$. If this limit exists then value of this limit is called the Riemann integral of f on [a, b] and is denoted by $\int_a^b f(x)\,dx$.

- We have also listed some properties of Riemann integration. Some theorems on Riemann integration are also discussed in this unit.

5.6 TERMINAL QUESTIONS

1. If $f : [0, 4] \to \mathbb{R}$ be a function defined by $f(x) = x^3 + 7,\ x \in [0, 4].$ Show that f is Riemann integrable using definition 1 and definition 2 and $\int_0^4 f(x)\,dx = 92.$

2. Without integrating, evaluate $\int_5^{25} 30\,dx$.

3. If $f : [0, 0.5] \to \mathbb{R}$ be a function defined by $f(x) = 2x + 3,\ x \in [0, 0.5]$ and let P = {0, 0.1, 0.4, 0.5} be the partition of [0, 0.5] then find $L(f, P)$ and $U(f, P)$.

5.7 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. In usual notations we are given

   a = 1, b = 3, f(x) = x^2 + 1, length of the interval [1, 3] is 3 – 1 = 2 = b – a.

   For the finite and closed interval [a, b] we form the partition

$$P = \{a = a + 0h,\ a + h,\ a + 2h,\ a + 3h,\ \ldots,\ a + nh = b\},\ h = \frac{b-a}{n}$$

So, in our case partition will be

$$P = \{1 + 0h,\ 1 + h,\ 1 + 2h,\ 1 + 3h,\ \ldots,\ 1 + nh = 3\},\ \text{ where } h = \frac{3-1}{n} = \frac{2}{n}$$

Therefore, i^{th} sub interval $= [1 + (i-1)h,\ 1 + ih]$, so

$$x_{i-1} = 1 + (i-1)h = 1 + \frac{2(i-1)}{n},\ x_i = 1 + ih = 1 + \frac{2i}{n},\ \text{as } h = \frac{2}{n} \text{ and } \delta_i = h = \frac{2}{n}$$

Now, we proceed to apply definition 1 and definition 2 of Riemann integration as follows.

Using Definition 1

Let $m_i = \text{glb of } f \text{ on } I_i = \left[1 + \frac{2(i-1)}{n}, 1 + \frac{2i}{n}\right],\ i = 1, 2, 3, \ldots, n$

$M_i = \text{lub of } f \text{ on } I_i = \left[1 + \frac{2(i-1)}{n}, 1 + \frac{2i}{n}\right],\ i = 1, 2, 3, \ldots, n$

Since f is an increasing function over the interval [1, 3] and hence over each sub-interval $I_i = \left[1 + \frac{2(i-1)}{n}, 1 + \frac{2i}{n}\right],\ i = 1, 2, 3, \ldots, n,$ therefore

$$m_i = f\left(1 + \frac{2(i-1)}{n}\right),\ M_i = f\left(1 + \frac{2i}{n}\right),\ i = 1, 2, 3, \ldots, n,$$

So,

$$L(f, P) = \sum_{i=1}^{n} m_i\,\delta_i = \sum_{i=1}^{n} f\left(1 + \frac{2(i-1)}{n}\right)\frac{2}{n} = 2\sum_{i=1}^{n}\left[\left(1 + \frac{2(i-1)}{n}\right)^2 + 1\right]\frac{1}{n}$$

$$= 2\sum_{i=1}^{n}\left[\left(1 + \frac{2^2(i-1)^2}{n^2} + 4\frac{i-1}{n}\right) + 1\right]\frac{1}{n} = \frac{8}{n^3}\sum_{i=1}^{n}(i-1)^2 + \frac{8}{n^2}\sum_{i=1}^{n}(i-1) + \frac{2}{n}\sum_{i=1}^{n}2$$

$$= \frac{8}{n^3}\left(0^2 + 1^2 + 2^2 + \ldots + (n-1)^2\right) + \frac{8}{n^2}(0 + 1 + 2 + \ldots + n - 1) + \frac{2}{n}(2n)$$

$$= \frac{8}{n^3}\left(1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\right) + \frac{8}{n^2}(1 + 2 + 3 + \ldots + n - 1) + 4$$

$$= \frac{8}{n^3}\frac{n(n-1)(2n-1)}{6} + \frac{8}{n^2}\frac{n(n-1)}{2} + 4$$

$$\left[\begin{array}{l} \because 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2},\ \text{so } 1 + 2 + 3 + \ldots + n - 1 = \dfrac{n(n-1)}{2} \\ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6},\ \text{so} \\ 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6} \end{array}\right]$$

$$\Rightarrow L(f, P) = \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + 4\left(1 - \frac{1}{n}\right) + 4$$

$$U(f, P) = \sum_{i=1}^{n} M_i\,\delta_i = \sum_{i=1}^{n} f\left(1 + \frac{2i)}{n}\right)\frac{2}{n} = 2\sum_{i=1}^{n}\left[\left(1 + \frac{2i}{n}\right)^2 + 1\right]\frac{1}{n}$$

$$= 2\sum_{i=1}^{n}\left[\left(1+\frac{2^2 i^2}{n^2}+4\frac{i}{n}\right)+1\right]\frac{1}{n} = \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 2$$

$$= \frac{8}{n^3}\left(1^2+2^2+3^2+\ldots+n^2\right)+\frac{8}{n^2}(1+2+3+\ldots+n)+\frac{2}{n}(2n)$$

$$= \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{8}{n^2}\frac{n(n+1)}{2}+4$$

$$\Rightarrow U(f, P) = \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)+4$$

By definition of lower Riemann integration refer (5.42), we have

$$\underline{\int_1^3} f(x)\,dx = \text{lub}\{L(f, P)\}_{P\in P[1, 3]} = \lim_{\|P\|\to 0}\left(\frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+4\left(1-\frac{1}{n}\right)+4\right)$$

$$= \lim_{n\to\infty}\left(\frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+4\left(1-\frac{1}{n}\right)+4\right) \qquad [\text{Using } (5.39)]$$

$$= \frac{4}{3}(1-0)(2-0)+4(1-0)+4 = \frac{8}{3}+8 = \frac{32}{3}$$

By definition of upper Riemann integration refer (5.43), we have

$$\overline{\int_1^3} f(x)\,dx = \text{lub}\{L(f, P)\}_{P\in P[1, 3]} = \lim_{\|P\|\to 0}\left(\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)+4\right)$$

$$= \lim_{n\to\infty}\left(\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)+4\right) \qquad [\text{Using } (5.39)]$$

$$= \frac{4}{3}(1+0)(2+0)+4(1+0)+4 = \frac{8}{3}+8 = \frac{32}{3}$$

Since $\underline{\int_1^3} f(x)\,dx = \frac{32}{3} = \overline{\int_1^3} f(x)\,dx$, so by definition 1 of Riemann integration f is Riemann integrable and $\int_1^3 f(x)\,dx = \frac{32}{3}$.

Using Definition 2

Let $\xi_i = 1+\frac{2i}{n}, \; i = 1, 2, 3, \ldots, n$ [Refer (5.46)]

Now,

$$\lim_{\|P\|\to 0}\sum_{i=1}^{n} f(\xi_i)\delta_i = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n} \qquad \left[\because\ \xi_i = 1+\frac{2i}{n} \text{ and } \delta_i = \frac{2}{n}\right]$$

$$= 2\lim_{\|P\|\to 0}\sum_{i=1}^{n}\left[\left(1+\frac{2i}{n}\right)^2+1\right]\frac{1}{n} = 2\lim_{\|P\|\to 0}\sum_{i=1}^{n}\left[\left(1+\frac{2^2 i^2}{n^2}+4\frac{i}{n}\right)+1\right]\frac{1}{n}$$

$$= \frac{8}{n^3}\lim_{\|P\|\to 0}\sum_{i=1}^{n} i^2 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 2$$

$$= \lim_{\|P\|\to 0}\left(\frac{8}{n^3}\left(1^2+2^2+3^2+\ldots+n^2\right)+\frac{8}{n^2}(1+2+3+\ldots+n)+\frac{2}{n}(2n)\right)$$

$$= \lim_{\|P\|\to 0}\left(\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{8}{n^2}\frac{n(n+1)}{2}+4\right)$$

$$\Rightarrow \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(\xi_i)\delta_i = \lim_{\|P\|\to 0}\left(\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)+4\right)$$

$$= \lim_{n\to\infty}\left(\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)+4\right) \quad [\text{Using } (5.39)]$$

$$= \frac{4}{3}(1+0)(2+0)+4(1+0)+4 = \frac{8}{3}+8 = \frac{32}{3}$$

Since $\lim_{\|P\|\to 0}\sum_{i=1}^{n} f(\xi_i)\delta_i = \frac{32}{3}$ so by definition 2 of Riemann integration f is Riemann integrable and $\int_1^3 f(x)\,dx = \frac{32}{3}$.

2. We are given $F(x) = \int_5^x e^{t^2}\,dt$.

Let $f(t) = e^{t^2}$ which is an exponential function. We know that exponential function is continuous. Therefore, f(t) is a continuous function. So, using (5.80), we get

$$\frac{d}{dx}(F(x)) = f(x) = e^{x^2}.$$

Terminal Questions

1. In usual notations we are given

$a = 0$, $b = 4$, $f(x) = x^3 + 7$, length of the interval [0, 4] is $4 - 0 = 4 = b - a$.

For the finite and closed interval [a, b] we form the partition

$$P = \{a = a + 0h,\ a + h,\ a + 2h,\ a + 3h,\ \ldots,\ a + nh = b\},\ h = \frac{b-a}{n}$$

So, in our case partition will be

$$P = \{0 + 0h,\ 0 + h,\ 0 + 2h,\ 0 + 3h,\ \ldots,\ 0 + nh = 4\},\ \text{ where } h = \frac{4-0}{n} = \frac{4}{n}$$

$$\Rightarrow P = \{0h,\ h,\ 2h,\ 3h,\ \ldots,\ nh = 4\},\ \text{ where } h = \frac{4}{n}$$

Therefore, i^{th} sub interval $= [(i-1)h, ih]$, so

$$x_{i-1} = (i-1)h = \frac{4(i-1)}{n},\ x_i = ih = \frac{4i}{n},\ \text{as } h = \frac{4}{n} \text{ and } \delta_i = h = \frac{4}{n}$$

Now, we proceed to apply definition 1 and definition 2 of Riemann integration as follows.

Using Definition 1

Let $m_i = \text{glb of } f \text{ on } I_i = \left[\frac{4(i-1)}{n}, \frac{4i}{n}\right],\ i = 1, 2, 3, \ldots, n$

$$M_i = \text{lub of } f \text{ on } I_i = \left[\frac{4(i-1)}{n}, \frac{4i}{n}\right],\ i = 1, 2, 3, \ldots, n$$

Since f is an increasing function over the interval [0, 4] and hence over each sub-interval $I_i = \left[\frac{4(i-1)}{n}, \frac{4i}{n}\right],\ i = 1, 2, 3, \ldots, n,$ therefore

$$m_i = f\left(\frac{4(i-1)}{n}\right), \quad M_i = f\left(\frac{4i}{n}\right), \quad i = 1, 2, 3, \ldots, n,$$

So,

$$L(f, P) = \sum_{i=1}^{n} m_i\,\delta_i = \sum_{i=1}^{n} f\left(\frac{4(i-1)}{n}\right)\frac{4}{n} = 4\sum_{i=1}^{n}\left[\left(\frac{4(i-1)}{n}\right)^3 + 7\right]\frac{1}{n}$$

$$= 4\sum_{i=1}^{n}\left(\frac{4^3(i-1)^3}{n^3} + 7\right)\frac{1}{n} = \frac{256}{n^4}\sum_{i=1}^{n}(i-1)^3 + \frac{4}{n}\sum_{i=1}^{n}7$$

$$= \frac{256}{n^4}\left(0^3 + 1^3 + 2^3 + \ldots + (n-1)^3\right) + \frac{4}{n}(7n)$$

$$= \frac{256}{n^4}\left(1^3 + 2^3 + 3^3 + \ldots + (n-1)^3\right) + 28$$

$$= \frac{256}{n^4}\left(\frac{n(n-1)}{2}\right)^2 + 28$$

$$\left[\begin{array}{l} \because 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2, \text{ so} \\ 1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 = \left(\dfrac{n(n-1)}{2}\right)^2 \end{array}\right]$$

$$\Rightarrow L(f, P) = 64\left(1 - \frac{1}{n}\right)^2 + 28$$

$$U(f, P) = \sum_{i=1}^{n} M_i\,\delta_i = \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\frac{4}{n} = 4\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^3 + 7\right]\frac{1}{n}$$

$$= 4\sum_{i=1}^{n}\left(\frac{4^3 i^3}{n^3} + 7\right)\frac{1}{n} = \frac{256}{n^4}\sum_{i=1}^{n} i^3 + \frac{4}{n}\sum_{i=1}^{n}7$$

$$= \frac{256}{n^4}\left(1^3 + 2^3 + \ldots + n^3\right) + \frac{4}{n}(7n) = \frac{256}{n^4}\left(\frac{n(n+1)}{2}\right)^2 + 28$$

$$\Rightarrow U(f, P) = 64\left(1 + \frac{1}{n}\right)^2 + 28$$

By definition of lower Riemann integration refer (5.42), we have

$$\underline{\int_0^4} f(x)\,dx = \text{lub}\{L(f, P)\}_{P \in P[0, 4]} = \lim_{\|P\| \to 0}\left(64\left(1 - \frac{1}{n}\right)^2 + 28\right)$$

$$= \lim_{n \to \infty}\left(64\left(1 - \frac{1}{n}\right)^2 + 28\right) \qquad [\text{Using } (5.39)]$$

$$= 64(1 - 0) + 28 = 92$$

By definition of upper Riemann integration refer (5.43), we have

$$\overline{\int_0^4} f(x)\,dx = \text{glb}\{U(f, P)\}_{P \in P[0, 4]} = \lim_{\|P\| \to 0}\left(64\left(1 + \frac{1}{n}\right)^2 + 28\right)$$

$$= \lim_{n \to \infty}\left(64\left(1 + \frac{1}{n}\right)^2 + 28\right) \qquad [\text{Using } (5.39)]$$

$$= 64(1 + 0) + 28 = 92$$

Since $\underline{\int_0^4} f(x)\,dx = 92 = \overline{\int_0^4} f(x)\,dx,$ so by definition 1 of Riemann integration f is Riemann integrable and $\int_0^4 f(x)\,dx = 92.$

Using Definition 2

Let $\xi_i = \dfrac{4i}{n},\ \ i = 1, 2, 3, \ldots, n$ [Refer (5.46)]

Now,

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\xi_i)\delta_i = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\frac{4}{n} \qquad \left[\because\ \xi_i = \frac{4i}{n} \text{ and } \delta_i = \frac{4}{n}\right]$$

$$= 4 \lim_{\|P\|\to 0} \sum_{i=1}^{n} \left[\left(\frac{4i}{n}\right)^3 + 7\right]\frac{1}{n} = 4 \lim_{\|P\|\to 0} \sum_{i=1}^{n} \left(\frac{4^3 i^3}{n^3} + 7\right)\frac{1}{n}$$

$$= \lim_{\|P\|\to 0} \left(\frac{256}{n^4}\sum_{i=1}^{n} i^3 + \frac{4}{n}\sum_{i=1}^{n} 7\right)$$

$$= \lim_{\|P\|\to 0} \left(\frac{256}{n^4}\left(1^3 + 2^3 + 3^3 + \ldots + n^3\right) + \frac{4}{n}(7n)\right)$$

$$= \lim_{\|P\|\to 0} \left(\frac{256}{n^4}\left(\frac{n(n+1)}{2}\right)^2 + 28\right)$$

$$\Rightarrow \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\xi_i)\delta_i = \lim_{\|P\|\to 0} \left(64\left(1 + \frac{1}{n}\right)^2 + 28\right)$$

$$= \lim_{n\to\infty} \left(64\left(1 + \frac{1}{n}\right)^2 + 28\right) \quad [\text{Using } (5.39)]$$

$$= 64(1+0)^2 + 28 = 64 + 28 = 92$$

Since $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\xi_i)\delta_i = 92,$ so by definition 2 of Riemann integration f is Riemann integrable and $\int_0^4 f(x)\,dx = 92.$

2. Comparing $\int_5^{25} 30\,dx$ with $\int_a^b c\,dx,$ we get

a = 5, b = 25, c = 30, length of the interval [5, 25] is 25 – 5 = 20.

Now, using (5.50), we get

$$\int_5^{25} 30\,dx = 30(25 - 5) = 30(20) = 600.$$

3. In usual notations we are given

a = 0, b = 0.5, f(x) = 2x + 3, length of the interval [0, 0.5] is 0.5 – 0 = 0.5.

Given partition of the finite and closed interval [0, 0.5] is

$P = \{0, 0.1, 0.4, 0.5\}$

Sub-intervals are $I_1 = [0, 0.1],\ \ I_2 = [0.1, 0.4],\ \ I_3 = [0.4, 0.5].$

Length of these sub-intervals are
$\delta_1 = 0.1 - 0 = 0.1, \ \ \delta_2 = 0.4 - 0.1 = 0.3, \ \ \delta_2 = 0.5 - 0.4 = 0.1.$

Let $m_i = \text{glb of } f \text{ on } I_i, \ M_i = \text{lub of } f \text{ on } I_i, \ i = 1, 2, 3.$

Since f is an increasing function over the interval [0, 0.5] and hence over each sub-interval $I_i, \ \ i = 1, 2, 3.$ therefore

$m_1 = f(0) = 2(0) + 3 = 3$, Similary, $m_2 = f(0.1) = 3.2, \ m_3 = f(0.4) = 3.8.$
$M_1 = f(0.1) = 2(0.1) + 3 = 3.2$, Similary, $M_2 = f(0.4) = 3.8, \ M_3 = f(0.5) = 4$

Now, we find L(f, P) and U(f, P) as follows.

$$L(f, P) = \sum_{i=1}^{3} m_i \delta_i = m_1\delta_1 + m_2\delta_2 + m_3\delta_3$$
$$= (3)(0.1) + (3.2)(0.3) + (3.8)(0.1) = 1.64$$
$$U(f, P) = \sum_{i=1}^{3} M_i \delta_i = M_1\delta_1 + M_2\delta_2 + M_3\delta_3$$
$$= (3.2)(0.1) + (3.8)(0.3) + (4)(0.1) = 1.86$$

Remark 3: By Fundamental Theorem of Calculus, we know that

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x), \ a = 0, b = 0.5$

Now, $F'(x) = f(x) \Rightarrow F(x) = \int (2x+3)\,dx = x^2 + 3x$

$\therefore F(b) = F(0.5) = (0.5)^2 + 3(0.5) = 0.25 + 1.5 = 1.75$

and $F(a) = F(0) = (0)^2 + 3(0) = 0$

So, $\int_0^{0.5} (2x+3)\,dx = F(b) - F(a) = 1.75 - 0 = 1.75.$

Note that lower sum is 1.64 which is ≤ 1.75 and

upper sum is 1.86 which is ≥ 1.75 as expected.