UNIT 2

TYPES, CONTINUITY AND DIFFERENTIABILITY OF FUNCTION OF A SINGLE VARIABLE

Structure

2.1 INTRODUCTION

In Unit 1 you have studied different types of intervals, the definition of a function of a single variable and its domain and range. We have also defined many functions by correlating them with daily life examples. A relationship between daughters and mothers explained both the conditions required for a rule to become a function. Here we have further classified functions known as types of a function in Sec. 2.2.

After understanding the meaning of one-one correspondence in Sec. 2.2 we will use this concept to define an important idea of a countable set in Sec. 2.3. Composition of two functions and algebra of functions are discussed in Sec. 2.4. Another two important concepts in mathematics that will be used in some courses of this programme are continuity and differentiability. You have already studied continuity and differentiability in earlier classes. A brief overview of these concepts is given in Secs. 2.5 and 2.6 respectively.

What we have discussed in this unit is summarised in Sec. 2.7. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit, some more questions based on the entire unit are given in Sec. 2.8 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1, solutions of all the SAQs and Terminal Questions are given in Sec. 2.9.

In the next unit, you will study the set function and the distance function.

Expected Learning Outcomes

After completing this unit, you should be able to:

- ❖ explain one-one, onto and one-to-one correspondence with their geometrical interpretation;
- ❖ define the concept of countable and uncountable sets;
- ❖ explain the idea of the composition of two functions and the algebra of functions;
- ❖ define the continuity of a function at a point and will be able to check whether the given function is continuous or not at a given point; and
- ❖ define the differentiability of a function at a point and will be able to check whether the given function is differentiable or not at a given point.

2.2 TYPES OF FUNCTION

In Unit 3 of the course MST-012 you will study the discrete random variable. To define a discrete random variable, we will use the world countable. So, to understand what we mean by the discrete random variable you should know the mathematical meaning of the word countable. But to understand the mathematical meaning of the word countable, you should have a good understanding of one-one correspondence or bijective map or one-one and onto function. This section is devoted to explain all these concepts.

One-One Function

If a machine gives different outputs for different inputs, then such a machine behaves like a one-one function. Mathematically, the one-one function is defined as follows.

A function $f : X \to Y$ is said to be 1-1 or **injective** function if distinct elements of X are associated with distinct elements of Y under f. … (2.1)

i.e., if $x_1, x_2 \in X$ and $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ … (2.2)

or whenever $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ … (2.3)

If we compare this definition with an example of daughters and mothers then one-one function means different daughters must have different mothers, i.e., there cannot be two daughters having the same mother for a function to be one-one.

For example, the function f shown in Fig. 2.1 (a) is not one-one because two different elements x_1, x_2 of X have the same image y_1. But the function f shown in Fig. 2.1 (b) is one-one because all the three elements of X have distinct images in Y.

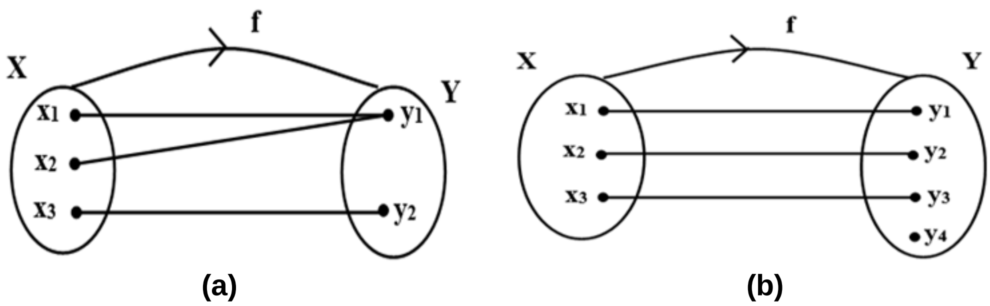

Fig. 2.1 (a) Not one-one function (b) A function which is one-one

Remark 1

If we want to show that a function f is one-one then do the following step. For $x_1, x_2 \in X$ take $f(x_1) = f(x_2)$ and show that $x_1 = x_2$.

For example,

(i) Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7x + 5$ is 1-1 function.

Solution: Let $x_1, x_2 \in \mathbb{R}$ be such that

$f(x_1) = f(x_2) \Rightarrow 7x_1 + 5 = 7x_2 + 5 \Rightarrow 7x_1 = 7x_2 \Rightarrow x_1 = x_2$

$\Rightarrow f$ is 1-1 function

(ii) Check whether the function $f: R \to R$ defined by $f(x) = x^2$ is 1-1 or not.

Solution: $f(x) = x^2$

Let $x_1 = 2,\ x_2 = -2$ then $x_1 \neq x_2$

But $f(x_1) = f(2) = (2)^2 = 4$ and also $f(x_2) = f(-2) = (-2)^2 = 4$

$\therefore f(2) = f(-2)$, i.e., there exists x_1 and x_2 such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

$\Rightarrow f$ is not 1-1 function.

Onto Function

A function $f: X \to Y$ is said to be onto or **surjective** if each element of Y has at least one pre image in X. … (2.4)

i.e., for each y $\in Y$, there exists at least one $x \in X$ such that $f(x) = y$ ... (2.5)

If we compare this definition with example of daughters and mothers then onto function means each mother must have at least one daughter.

For example, the function f shown in Fig. 2.2 (a) is not onto because $y_4 \in Y$ but there is no $x \in X$ such that $y_4 = f(x)$. However, the function f shown in Fig. 2.2 (b) is onto because each element of Y has at least one pre image in X, i.e., y_1 has two pre-images and y_2 has three pre-images.

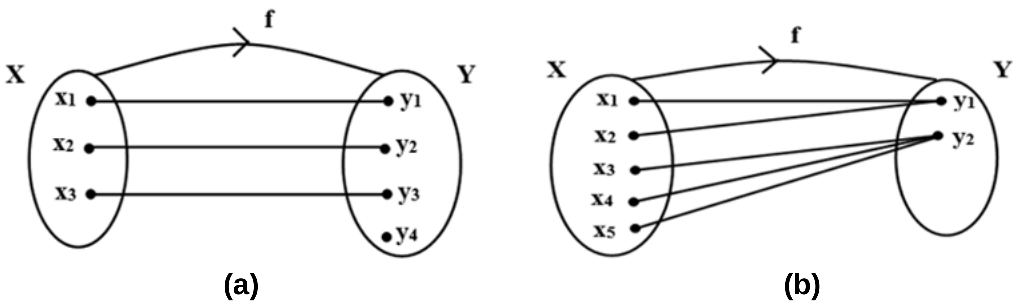

Fig. 2.2 (a) Not onto function (b) A function which is onto

Remark 2

If we want to show that a function $f(x)$ is onto then first we take an element y in Y and we have to show that there exists an element x is X such that $f(x) = y$

For example, show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = 7x+5$ is an onto function.

Solution: Here X = $\mathbb{R}$, Y = $\mathbb{R}$

Let $y \in Y = \mathbb{R}$, then $\frac{y-5}{7} \in X = \mathbb{R}$ be such that $\left[\begin{array}{l} \text{Why } \frac{y-5}{7} \text{ because} \\ y = f(x) = 7x+5 \\ \Rightarrow x = \frac{y-5}{7} \end{array}\right]$

$$f\left(\frac{y-5}{7}\right) = 7\left(\frac{y-5}{7}\right) + 5 = (y-5)+5 = y$$

Therefore, we have proved that for each $y \in Y = \mathbb{R}$ there exists $\frac{y-5}{7} \in X = \mathbb{R}$ such that $f\left(\frac{y-5}{7}\right) = y$

Hence, f is an onto function.

One-One and Onto Function

A function $f: X \to Y$ is said to be one-one and onto or **bijective** or **one-one correspondence** if f is both one-one as well as onto. … (2.6)

If we compare this definition with example of daughters and mothers then one-one and onto function means each mother has exactly one daughter and there is no mother who does not have any daughter. The function f shown in Fig. 2.3 represents a situation of one-one and onto function from X = {1, 2, 3, 4} to Y = {2, 3, 4, 5} where f(x) = x + 1, x $\in$ X.

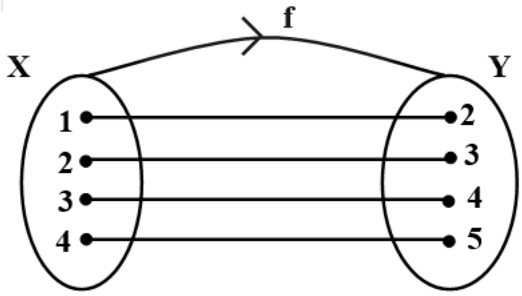

Fig. 2.3: One-one correspondence

Also, the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$f(x) = 7x+5, \quad x \in \mathbb{R}$

is both one-one and onto (already shown) and hence it is a one-one correspondence between set of all real numbers $\mathbb{R}$ to itself.

Remark 3

Definition of a function f from X to Y says that corresponding to each element of X there is a unique element in Y. But one-one correspondence establishes this relationship from Y to X also, i.e., corresponding to each element of Y there is a unique element in X. Application of one-one correspondence lies in the fact that once we have proved one-one correspondence between two sets X and Y then:

- you can say that number of elements in the two sets X and Y are equal in the sense that either both are finite or both are infinite sets.
- you know that multiplicative inverse of a non-zero number a is 1/a. For example, multiplicative inverse of 5 is 1/5. Another advantage of one-one correspondence is that we can talk about inverse of the function. Recall that exponential and logarithm functions are inverse of each other explained in Sec. 1.5 refer (1.29) and (1.33) of the previous unit.

Geometrical Meaning of Injective, Surjective and Bijective Functions

To explain these concepts geometrically we will use graphs of the following four functions:

The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=-x^2$ its graph is shown in Fig. 2.4 (a).

The function $f:\mathbb{R}_0^+\to\mathbb{R}_0$ defined by $f(x)=\frac{1}{x}$ its graph is shown in Fig. 2.4 (b), where $\mathbb{R}_0$ represents set of all real numbers other than zero and $\mathbb{R}_0^+$ represents set of all positive real numbers.

The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x$ its graph is shown in Fig. 2.4 (c).

The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=(x-1)^2(x-3)$ its graph is shown in Fig. 2.4 (d).

Identification of One-One Function using Horizontal Line Test

If a function is 1-1 then geometrically it will satisfy the following condition.

Each horizontal line either does not intersect the graph of the function at all or if it intersects it will intersect exactly at one point. In other words, there should be no horizontal line which intersects the graph of the function at two or more than two points for a function to be one-one. … (2.7)

For example,

(i) Graph shown in Fig. 2.4 (b) is the graph of a one-one function because there is no horizontal line which intersects the graph at more than one point. In fact, each horizontal line above the x-axis intersects the graph at exactly one point. X-axis itself is also a horizontal line and it will not intersect the graph of this function, no problem, because this function will never take zero value. So, it is a one-one function over its domain.

(ii) Graph shown in the Fig. 2.4 (c) is also the graph of a one-one function because each horizontal line intersects the graph exactly at one point.

(iii) But the graph shown in the Fig. 2.4 (a) is not the graph of a one-one function because if we draw any horizontal line below the x-axis, then it will intersect the graph at two points. Similarly, graph shown in Fig. 2.4 (d) is also not the graph of a one-one function because there exist horizontal lines which can intersect its graph at more than one points.

For example, horizontal line $f(x) = -1/2$ will intersect its graph at three points.

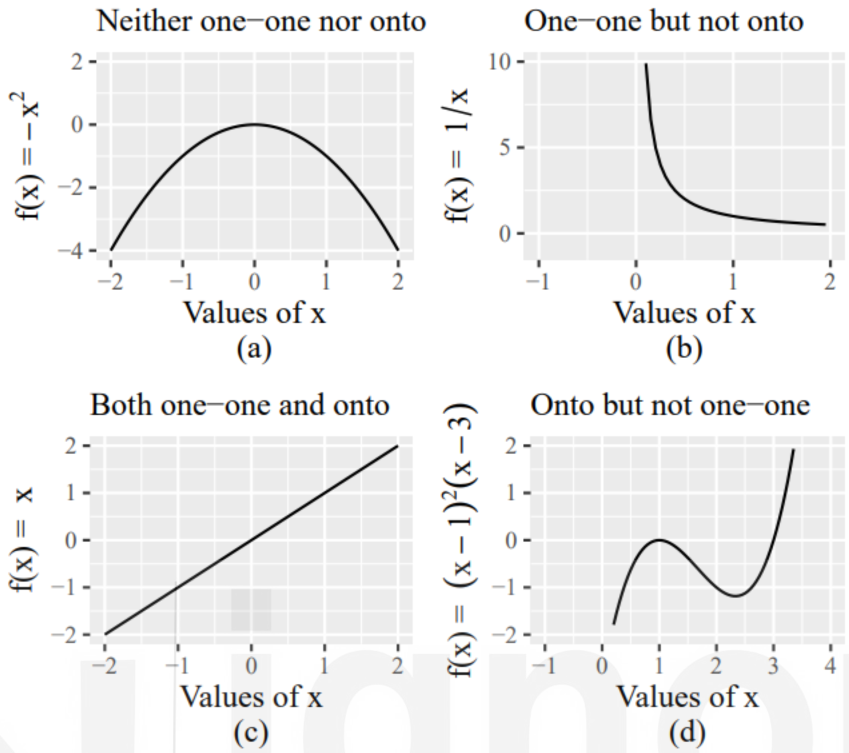

Fig. 2.4: Graphs of some functions to explain idea of one-one and onto functions geometrically

Identification of Onto or Surjective Function using Horizontal Line Test

If a function is onto then geometrically it will satisfy the following condition. **Horizontal line** corresponding to each point of the codomain must intersect the graph of the function at least at one point. … (2.8)

For example,

(i) Graph shown in the Fig. 2.4 (d) is the graph of an onto function because each horizontal line intersects the graph of the function at least at one point. Graph shown in Fig. 2.4 (c) is also onto function because each horizontal line intersects the graph exactly at one point.

(ii) The function having graph shown in the Fig. 2.4 (b) is not an onto function because codomain of this function is whole real line but if we draw any horizontal line below the x-axis then it will not intersect the graph at all.

(iii) Similarly, the function having graph shown in the Fig. 2.4 (a) is not an onto function because codomain of this function is whole real line but if we draw any horizontal line above the x-axis then it will not intersect the graph at all.

Identification of Bijective Function or One-One Correspondence using Horizontal Line Test

If a function is both one-one and onto then it is known as one-one correspondence or bijective function. Therefore, by combining geometrical conditions of both one-one and onto functions we can say that a function will be bijective if **each horizontal line** corresponding to each point of the codomain must intersect the graph of the function exactly at one point. … (2.9)

For example,

(i) The function having graph shown in the Fig. 2.4 (c) is the graph of a one-one and onto function because each horizontal line intersects the graph of the function exactly at one point. So, it is both one-one as well as onto function and therefore it is a bijective function or one-one correspondence.

(ii) The functions having graph shown in the Figs. 2.4 (a) and (b) are not the graph of bijective functions because they are not onto. We have already discussed why they are not onto.

(iii) Similarly, the function having graph shown in the Fig. 2.4 (d) is not the graph of a bijective function because it is not one-one. We have already discussed why it is not one-one.

Total Number of 1-1, Onto and Bijective Functions

Suppose $A, B \subset \mathbb{R}$ are two finite subsets where n(A) = m, n(B) = n then let us try to count total number of functions from A to B and how many of them are 1-1, onto and bijective functions.

Total Number of Functions: By definition of a function each element of A must be associated to a unique element of B. So, each element of A has n options to associate an element in B. Therefore, by fundamental principle of multiplication in counting we have n^m functions in total from A to B. … (2.10)

Total Number of One-One Functions: By definition of one-one function different elements of A must be associated to different elements of B. So, to define one-one function from A to B first of all we should have m ≤ n because if m > n then one-one function cannot be defined. Hence, in the case m > n total number of 1-1 functions from A to B is zero. In the case m ≤ n the first element of A can be associated to any of the n elements so has n options. Second element of A has n – 1 option available in B to associate. Similarly, third element has n – 2 options. Finally, m^{th} element of A has n – m + 1 options to get associate in B. Hence, using fundamental principle of multiplication in counting we have n(n – 1)(n – 2)…(n – m + 1) one-one functions in total from A to B. In the case m = n the total number of one-one functions from A to B will be n(n – 1)(n – 2) … 3.2.1, i.e., n!. … (2.11)

Total Number of Onto Functions: By definition of onto function each element of B has at least one pre-image in A. So, to define onto function from A to B first of all we should have m ≥ n because if m < n then onto function cannot be defined. Hence, in the case m < n total number of onto functions from A to B is zero. In the case m ≥ n instead of counting number of onto functions we will count the number of functions which are not onto and then subtract it from the total number of functions to get the number of onto functions. That is here we are going to use the fundamental principle of subtraction for counting. Suppose $B = \{b_1, b_2, b_3, \ldots, b_n\}$ then the function will not be onto in the cases

- either any one of b_i's does not have pre-image or

- any two b_i's do not have pre-images or
- any three b_i's do not have pre-images and so on either
- any n – 1 b_i's do not have pre-images in set A.

But the cases any one b_i does not have pre-images also includes the cases where any two b_i's do not have pre-images in A. Similarly, the cases any two b_i do not have pre-images also includes the cases where any three b_i's do not have pre-images in A and so on. This is exactly the situation of principle of inclusion-exclusion which you have studied for two and three sets in set theory in school mathematics. This principle for n sets states that

$$n(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} n(A_i) - \sum_{i \neq j}^{n} n(A_i \cap A_j) + \sum_{i \neq j \neq k}^{n} n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} n(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Using this principle in our case, number of functions which are not onto are given as follows.

$$\binom{n}{1}(n-1)^m - \binom{n}{2}(n-2)^m + \binom{n}{3}(n-3)^m - \binom{n}{4}(n-4)^m + \ldots + (-1)^{n-1}\binom{n}{n-1}(1)^m$$

Explanation of the first term is: Any one of $b_1, b_2, b_3, \ldots, b_n$ does not have pre-image in A implies there are $\binom{n}{1}$ ways. Now, total number of functions from A to B\{b_i} are $(n-1)^m$ so total number of functions where one b_i's does not have pre-image is $\binom{n}{1}(n-1)^m$. Similarly, other terms can be understood. Now, we use fundamental principle of subtraction of counting:

$$\begin{pmatrix}\text{Total number of onto} \\ \text{functions A to B}\end{pmatrix} = \begin{pmatrix}\text{Total number of} \\ \text{functions A to B}\end{pmatrix} - \begin{pmatrix}\text{Total number of functions} \\ \text{which are not onto from A to B}\end{pmatrix}$$

So, finally, total number of functions from A to B which are onto are given by

$$n^m - \left[\binom{n}{1}(n-1)^m - \binom{n}{2}(n-2)^m + \binom{n}{3}(n-3)^m - \binom{n}{4}(n-4)^m + \ldots + (-1)^{n-1}\binom{n}{n-1}(1)^m\right]$$

$$= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \binom{n}{4}(n-4)^m - \ldots - (-1)^{n-1}\binom{n}{n-1}(1)^m$$

$$= \binom{n}{0}n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \binom{n}{4}(n-4)^m - \ldots - (-1)^{n-1}\binom{n}{n-1}(1)^m$$

$$= \sum_{r=0}^{n-1}\binom{n}{r}(-1)^r (n-r)^m \qquad \ldots (2.12)$$

Total Number of Bijective Functions: By definition of bijective function different elements of A have different images and each element of B has at least one pre-image in A. This is possible only when m = n. Hence, in the case m ≠ n total number of bijective functions from A to B is zero. Like the case of one-one function when m = n total number of bijective functions will be

$$n(n-1)(n-2)\ldots(n-m+1) \qquad \ldots (2.13)$$

But here m = n, so

$$\text{Total number of bijective functions} = n(n-1)(n-2)\ldots(n-n+1)$$
$$= n(n-1)(n-2)\ldots(1) = n! \qquad \ldots (2.14)$$

Now, you can try the following Self-Assessment Question.

SAQ 1

(a) Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=[x]$ is neither 1-1 nor onto.

(b) Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$ is neither 1-1 nor onto.

(c) Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^3$ is both 1-1 and onto.

(d) If n(A) = 5, n(B) = 7 then how many one-one functions are possible from A to B.

2.3 COUNTABLE SET

From school mathematics you know that a set is said to be a **finite set** if either it is an empty set or it has a finite number of elements.

For example, A = {2, 7, 9} is a finite set since it has three elements and 3 is a finite number. In fact, any real number x satisfying $-\infty < x < \infty$ is a finite number.

You also know that a set which is not finite is called an **infinite set**.

For example, B = {5, 10, 15, 20, …} is an infinite set since number of elements in B is not finite.

From earlier classes you know that two finite sets are said to be equivalent if number of elements in the two sets are equal. For example, if A = {a, b, c} and B = {7, 3, m} then $A \sim B$ because n(A) = 3 = n(B), where n(A) denotes cardinality (number of elements in the set) of the set A. But if you have two sets each having infinite number of elements then how you will check whether they are equivalent or not? To answer this question, we will take help of one-one correspondence which has been explained in the previous section of this unit. So, more general definition of equivalents sets in mathematics is given as follows.

Equivalent Sets: Let A and B be two sets either both finite or both infinite then we say that sets A and B are equivalent if either there exists a one-one correspondence from A to B or from B to A and is denoted by $A \sim B$. … (2.15)

For example,

(i) If A = {1, 2, 3, 4} and B = {3, 8, 13, 18}, then $A \sim B$ because there exists a one-one correspondence $f : A \to B$ defined by $f(x) = 5x - 2, \quad x \in A$ between A and B as shown in Fig. 2.5 (a)

(ii) If C = {3, 6, 9, 12, …} and D = {5, 10, 15, 20, …}, then $C \sim D$ because there exists a one-one correspondence $g : C \to D$ defined by $g(x) = \frac{5}{3}x, \quad x \in C$ between C and D as shown in Fig. 2.5 (b)

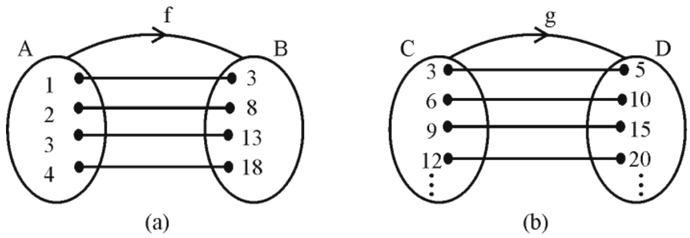

Fig. 2.5: Visualisation of equivalence of two (a) finite sets (b) infinite sets

In the previous unit we have discussed different families of numbers in mathematics. One of those families was the set of natural numbers. This family of natural numbers plays an important role to distinguish between countable and uncountable sets. The next definition specifies this importance.

Enumerable Set: A set E is said to be enumerable, if it is equivalent to the set of natural numbers, i.e., if $\mathbb{N} \sim \mathrm{E}$. … (2.16)
i.e., if there exists a one- one correspondence between $\mathbb{N}$ and E.
An enumerable set is also known as **denumerable** set or **countably infinite set**. … (2.17)

For example,

(i) If $\mathrm{E} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ then E is enumerable because there exists one-one correspondence $f : \mathbb{N} \to \mathrm{E}$ defined by

$$f(n) = \frac{1}{n}, \quad n \in \mathbb{N}$$

The same is shown in Fig. 2.6 (a).

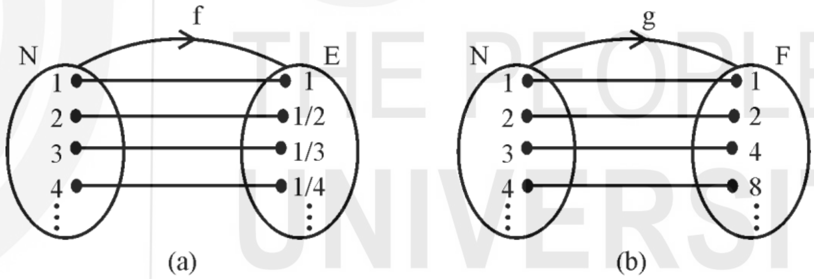

Fig. 2.6: One-one correspondence between (a) $\mathbb{N}$ and E (b) $\mathbb{N}$ and F

(ii) If $\mathrm{F} = \{1, 2, 4, 8, \cdots\}$ then F is enumerable because there exists one-one correspondence $f : \mathbb{N} \to \mathrm{F}$ defined by

$$f(n) = 2^{n-1}, \quad n \in \mathbb{N}$$

The same is shown in Fig. 2.6 (b).

You have studied about one-one correspondence and enumerable set so now you can understand the definition of countable set which is given as follows.

Countable Set: A set is said to be countable if either it is finite or enumerable. … (2.18)

For example,

(i) If A = {} = ϕ, then number of elements in the set A is zero and therefore it is a finite set. So, set A is countable.

(ii) If B = {a, b, c}, then the number of elements in the set B is 3 and therefore it is a finite set. So, set B is countable.

(iii) If $C = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \cdots\right\}$ then the number of elements in the set C is infinite but this set has one-one correspondence with the set of all natural numbers $\mathbb{N}$, $f : \mathbb{N} \to C$ defined by

$$f(n) = \frac{1}{n^2}, \quad n \in \mathbb{N}$$

Therefore, it is enumerable. Hence, it is a countable set.

Remark 4

- In Unit 3 of the Course MST-012, you will meet the word countable in the definition of the discrete random variable. If you want to enjoy the flavour of discrete world it is recommended that you should keep this definition of countable set in your mind when you go will through the definition of discrete random variable.

- Note that when a set is countable then its elements can be written in a sequence. You will study about sequence in Unit 4 of this course in detail. So, all those variables whose values can be written in a sequence are known as discrete random variable. Example 1 explain it further.

- Subset of a countable set is countable.

Example 1: Does the set containing all possible values of 'number of accidents' at a crossing during 4am to 10pm on a particular day is a countable set?

Solution: Answer of this question is yes and the explanation is given as follows.

Number of accidents during 4am to 10pm on a particular day may be 0 or 1 or 2 or 3 and so on. So, if we write these possible values in the set A then we have A = {0, 1, 2, 3, 4, …}. In real life obviously this set A is finite and so is countable. But if theoretically we assume that set A is infinite then also we can define a function $f : \mathbb{N} \to A$ by

$$f(x) = n - 1, \quad n \in \mathbb{N}$$

This function is both 1-1 and onto and hence $\mathbb{N} \sim A.$ So, set A is enumerable and so countable.

Now, you can try the following Self Assessment Question.

SAQ 2

(a) Does the set of all integers is countable?

(b) Does the set of all prime numbers is countable?

Now, we define algebra and composition of functions in the next section.

2.4 COMBINING FUNCTIONS

You know how to add, subtract, multiply and divide in real numbers. You can also perform these operations in the universe of functions known as algebra of functions which will be discussed here. Further, if function f takes you from set A to set B and function g takes you from set B to set C then you can compose functions g and f to go from set A to set C via set B this is known as composition of functions g and f which is also discussed here. Visualisation of composition of two functions f and g is shown in Figs. 2.7 and 2.8.

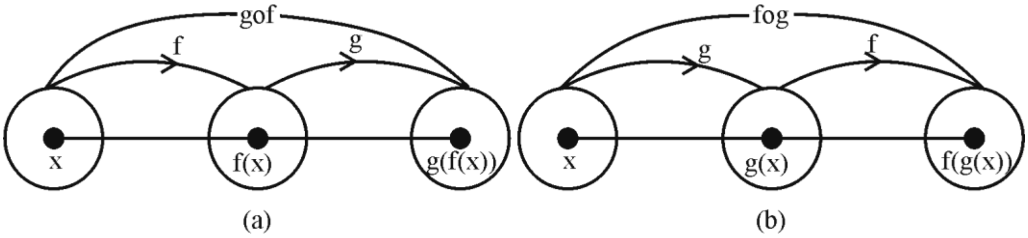

Fig. 2.7: Visualisation of composition of two functions

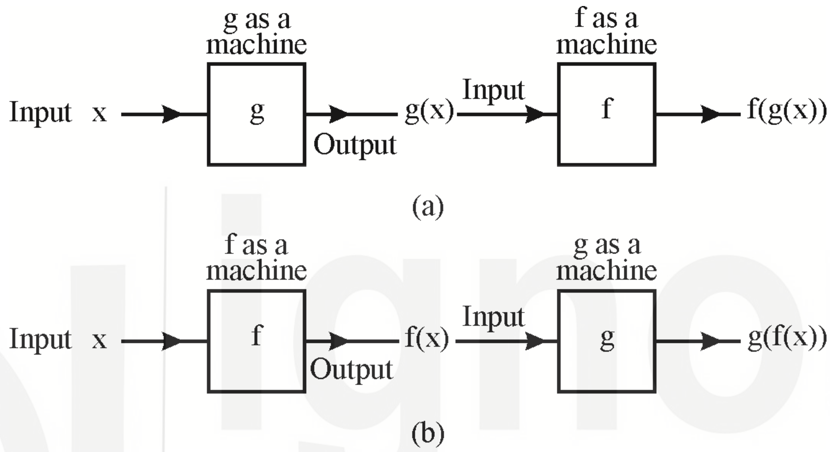

Fig. 2.8: Visualisation of composition of two functions

Let f and g be two functions having domains D_1 and D_2 respectively. Let $D = D_1 \cap D_2$ be their common domain. So, both functions f and g are defined on D hence we can apply both functions on all $x \in D$. This permits us to do algebra on functions. Sum, difference, product and quotient of functions f and g are defined as follows:

Sums of f and g

$$(f+g)(x) = f(x) + g(x), \quad x \in D \qquad \dots (2.19)$$

Differences of f and g

$$(f-g)(x) = f(x) - g(x), \quad x \in D \qquad \dots (2.20)$$

Products of f and g

$$(fg)(x) = f(x)g(x), \quad x \in D \qquad \dots (2.21)$$

Quotients of f and g

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad x \in D, \text{ where } g(x) \neq 0 \text{ and} \qquad \dots (2.22)$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}, \quad x \in D, \text{ where } f(x) \neq 0 \qquad \dots (2.23)$$

Sometimes we also need to multiply a function by a constant. Let c be a real number then product of constant c with the function f is defined as follows:

$$(cf)(x) = c\,f(x), \quad x \in D_1 \qquad \dots (2.24)$$

One more way of combining functions f and g is known as composition of f and g denoted by $g \circ f$. This makes sense if range of f is subset of domain of g and is defined as follows:

$$(g \circ f)(x) = g(f(x)), \quad x \in D_1, \text{ where } \text{Range}(f) \subseteq D_2 \qquad \dots (2.25)$$

Similarly, $f \circ g$ makes sense if range of g is subset of domain of f and is defined as follows:

$$(f \circ g)(x) = f(g(x)), \quad x \in D_2, \text{ where } \text{Range}(g) \subseteq D_1 \qquad \dots (2.26)$$

Let us explain these ideas of combining two functions in Examples 2 and 3 given as follows:

Example 2: Functions f and g are defined by $f(x) = \sqrt{2-x}, \ \ g(x) = \sqrt{x-1}$. Find domains of f and g. Obtain domains where $f+g$, $f-g$, fg and f/g are defined.

Also obtain (i) $(f+g)(2)$ (ii) $(f-g)(1)$ (iii) $(fg)(3/2)$ (iv) $\left(\frac{f}{g}\right)(3/2)$

(v) $(f+g)(3)$ (vi) $\left(\frac{g}{f}\right)(2)$ (vii) $\left(\frac{g}{f}\right)(1.8)$ (viii) $\left(\frac{f}{g}\right)(1.4)$ (ix) $\left(\frac{f}{g}\right)(1)$.

Solution: Domains of f and g are $(-\infty, 2]$ and $[1, \infty)$ respectively. Now, sums, differences and products are defined on $(-\infty, 2] \cap [1, \infty) = [1, 2]$, while f/g is defined on (1, 2] and g/f is defined on [1, 2).

(i) $(f+g)(2) = f(2) + g(2) = \sqrt{2-2} + \sqrt{2-1} = 0 + 1 = 1$

(ii) $(f-g)(1) = f(1) - g(1) = \sqrt{2-1} - \sqrt{1-1} = 1 - 0 = 1$

(iii) $(fg)(3/2) = f(3/2)g(3/2) = \sqrt{2-3/2}\sqrt{3/2-1} = \sqrt{1/2}\sqrt{1/2} = 1/2$

(iv) $\left(\frac{f}{g}\right)(3/2) = \frac{f(3/2)}{g(3/2)} = \frac{\sqrt{2-3/2}}{\sqrt{3/2-1}} = \frac{\sqrt{1/2}}{\sqrt{1/2}} = 1$

(v) Since domain of f + g is [1, 2] and $3 \notin [1, 2]$, so (f + g)(3) is not defined.

(vi) Since domain of g/f is [1, 2) and $2 \notin [1, 2)$ so (g/f)(2) is not defined.

(vii) $\left(\frac{g}{f}\right)(1.8) = \frac{g(1.8)}{f(1.8)} = \frac{\sqrt{1.8-1}}{\sqrt{2-1.8}} = \frac{\sqrt{0.8}}{\sqrt{0.2}} = \sqrt{\frac{0.8}{0.2}} = \sqrt{4} = 2$

(viii) $\left(\frac{f}{g}\right)(1.4) = \frac{f(1.4)}{g(1.4)} = \frac{\sqrt{2-1.4}}{\sqrt{1.4-1}} = \frac{\sqrt{0.6}}{\sqrt{0.4}} = \sqrt{\frac{0.6}{0.4}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$

(ix) Since domain of f/g is (1, 2] and $1 \notin (1, 2]$ so (f/g)(1) is not defined.

Example 3: Functions f and g are defined by $f(x) = \sqrt{x+3}, \ \ g(x) = x-2$. Obtain (i) $(f \circ g)(x)$ (ii) $(g \circ f)(x)$ (iii) $(f \circ f)(x)$ (iv) $(g \circ g)(x)$ (v) $(f \circ g)(6)$ (vi) $(g \circ f)(6)$ (vii) $(f \circ f)(6)$ (viii) $(g \circ g)(6)$.

Solution: By definition of composition of two functions, we have

(i) $(f \circ g)(x) = f(g(x)) = f(x-2) = \sqrt{x-2+3} = \sqrt{x+1}$

(ii) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x+3}) = \sqrt{x+3} - 2$

(iii) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x+3}) = \sqrt{\sqrt{x+3}+3}$

(iv) $(g o g)(x) = g(g(x)) = g(x-2) = x-2-2 = x-4$

(v) Since $(f o g)(x) = \sqrt{x+1}$ so, $(f o g)(6) = \sqrt{6+1} = \sqrt{7}$

(vi) Since $(g o f)(x) = \sqrt{x+3} - 2$ so, $(g o f)(6) = \sqrt{6+3} - 2 = \sqrt{9} - 2 = 3 - 2 = 1$

(vii) Since $(f o f)(x) = \sqrt{\sqrt{x+3}+3}$ so,

$(f o f)(6) = \sqrt{\sqrt{6+3}+3} = \sqrt{\sqrt{9}+3} = \sqrt{3+3} = \sqrt{6}$

(viii) Since $(g o g)(x) = x - 4$ so, $(g o g)(6) = 6 - 4 = 2$

Now, you can try the following Self Assessment Question.

SAQ 3

Consider functions f and g as defined in Example 3. Obtain domains of the functions (i) f (ii) g (iii) $(f o g)(x)$ (iv) $(g o f)(x)$ (v) $(f o f)(x)$ (vi) $(g o g)(x)$.

2.5 CONTINUITY OF A FUNCTION

Limit and Continuity at a Point: In school mathematics you have studied what we mean by limit and continuity of a function at a point. You know that a function f is continuous at x = a if left hand limit (LHL), right hand limit (RHL) and value of the function f at x = a exist and, all the three are equal. i.e.,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a). \quad \ldots (2.27)$$

You know the meaning of $x \to a^-$. For example, if $a = 2$, then $x \to 2^-$, means x is coming close and close to 2 but remains less than 2. For understanding purpose, you can suppose that x is taking values 1.9, 1.98, 1.998, 1.9998, 1.99998, and so on. In real mathematical sense this string of 9 is huge beyond your imagination. It can be realised by assuming that suppose someone go on writing 9 in 1.9999… whole life then his/her child do the same job and suppose it continue for 10 (nothing special in 10 just to specify some number for understanding the concept, you can consider another number) generations and tenth generation person put 8 at the end then this number is still less than 2 but very close to 2. Similarly, meaning of $x \to a^+$, can be understood by assuming a = 2 then $x \to 2^+$, means x is coming close and close to 2 but remains greater than 2. For understanding purpose, you can suppose that x is taking values 2.1, 2.01, 2.001, 2.0001, 2.00001, and so on. So, remember $x \to a$ means either x is slightly less than a or slightly greater than a but x ≠ a. You also know that at x = a if LHL = RHL ≠ f(a), then we say that $\lim_{x \to a} f(x)$ exists but function f is not continuous at x = a. Further, if both LHL and RHL exist but LHL ≠ RHL then we say that $\lim_{x \to a} f(x)$ does not exist.

Continuous Function: If a function is continuous at each point in its domain, then we say that it is a **continuous function**. … (2.28)

Geometrical Interpretation of the Continuity of a Function at a Point: Geometrically continuity of a function at a point x = a means when we draw the graph of the function with pen or pencil, we do not need to lift the pen or pencil as we cross the point (a, f(a)) on the graph of that function. … (2.29)

For example, (i) all polynomial functions are continuous refer Figs. 1.13 to 1.16 because there is no whole in their graphs, similarly, (ii) exponential function is continuous refer Fig. 1.17 (iii) logarithm function is continuous refer Fig. 1.19 (iv) sigmoid function is continuous refer Fig. 1.20. On the other hand (i) unit step function is not continuous because there is a whole in its graph at the origin refer Fig. 1.22, similarly, (ii) signum function is not continuous refer Fig. 1.23 (a) (iii) greatest integer function is not continuous refer Fig. 1.23 (b). In fact, greatest integer function is not continuous at all integers. So, it has infinite points where it is not continuous. But if a function is continuous at all points of its domain except at one point where it is not continuous then we say that this function is not continuous. So, to say that a function is not continuous it is enough to find out one point in its domain where it is not continuous.

Continuity on an Interval: We have defined that a function f is continuous at a point x = c if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$. Now there may be the situation that $\lim_{x\to c^-} f(x) = f(c) \neq \lim_{x\to c^+} f(x)$ in this case we say that the function f is **left continuous** at x = c. Another situation may be that $\lim_{x\to c^+} f(x) = f(c) \neq \lim_{x\to c^-} f(x)$ in this case we say that function f is **right continuous** at x = c. … (2.30)

If [a, b] lies in the domain of the function f then we say that function f is **continuous on the closed interval** [a, b] if (i) f is continuous at each point in the open interval (a, b) (ii) f is right continuous at x = a, and (iii) f is left continuous at x = b. … (2.31)

Let us now consider one example given as follows.

Example 4: Discuss the continuity of signum function at x = 0.

Solution: By definition of signum function (already discuss in the previous unit also refer Fig. 1.23(a)), we know that

$$f(x) = \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases} \qquad x \in \mathbb{R}$$

Now,

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-1) \quad \left[\because x \to 0^- \Rightarrow x \text{ is slightly less than 0, so } f(x) = -1\right]$$

$$= -1$$

Similarly, $\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1) = 1$

Since, LHL ≠ RHL, so $\lim_{x\to 0} f(x)$ does not exist and hence f is not continuous at x = 0. $\left[\because \text{For continuity limit should exists as well as it must be equal to } f(0)\right]$

Epsilon (ε) and Delta (δ) Definition of Limit of a Function at a Point

Let $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be such that (c – r, c) and (c, c + r) are contained in D for some r > 0, and let $f : D \to \mathbb{R}$. Then $\lim_{x\to c} f(x)$ exists if and only if there is L such that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever

$$|x - c| < \delta, \text{ where } x \in D \;\Rightarrow\; |f(x) - L| < \varepsilon \qquad \dots (2.32)$$

Epsilon (ε) and Delta (δ) Definition of Continuity of a Function at a Point

Let $D \subseteq \mathbb{R}$ then the function $f : D \to \mathbb{R}$ is said to be continuous at a point $c \in D$ if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever

$$|x - c| < \delta, \text{ where } x \in D \Rightarrow |f(x) - f(c)| < \varepsilon \quad \dots (2.33)$$

In this definition of continuity $\varepsilon > 0$ depends on both point c as well as $\delta > 0$.

… (2.34)

If in this definition of continuity $\varepsilon > 0$ depends only on $\delta > 0$ not on the point c then we say that function is **uniformly continuous**. … (2.35)

Example 5: Using ε and δ definition show that $\lim_{x \to 5}(2x - 4) = 6.$

Solution: In usual notations we are given f(x) = 2x – 4, c = 5, L = 6.

Now, let $\varepsilon > 0$ be given then

$$|f(x) - L| < \varepsilon \Rightarrow |2x - 4 - 6| < \varepsilon \Rightarrow |2x - 10| < \varepsilon \Rightarrow 2|x - 5| < \varepsilon \Rightarrow |x - 5| < \frac{\varepsilon}{2}$$

Let $\delta = \frac{\varepsilon}{2}$, then we have for given $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{2}$, such that

whenever $|x - 5| < \delta = \frac{\varepsilon}{2}$, $\Rightarrow |(2x - 4) - 6| < \varepsilon$, so $\lim_{x \to 5}(2x - 4) = 6.$

Now, you can try the following Self-Assessment Question.

SAQ 4

For given $\varepsilon = 0.1$ find $\delta > 0$ such that whenever $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$, where $f(x) = \sqrt{5x - 1}$, $L = 3$, $c = 2$.

2.6 DIFFERENTIABILITY OF A FUNCTION AT A POINT

In earlier classes you have studied a lot about differentiability and its applications. Let y = f(x) be a function then average rate of change of y with respect to x over the interval [a, a + h] is defined by

$$\frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}$$

If $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ exists, then it is known as instantaneous rate of change of y with respect to x at the point a, and is known as derivative of y with respect to x at the point a and is denoted by $f'(a)$. … (2.36)

Example 6: If f(x) = $2x^2$ – 4x + 5 then find $f'(3)$ using definition.

Solution: By definition of derivative, we know that

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{2(3+h)^2 - 4(3+h) + 5 - \left[2(3)^2 - 4(3) + 5\right]}{h}$$

$$= \lim_{h \to 0} \frac{2(9 + h^2 + 6h) - (12 + 4h) + 5 - [18 - 12 + 5]}{h} = \lim_{h \to 0} \frac{2h^2 + 8h}{h}$$

$$= \lim_{h \to 0} (2h + 8) = 8$$

Now, you can try the following Self-Assessment Question.

SAQ 5

If $f(x) = |x - 5|$ then discuss its differentiability at x = 5.

2.7 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- A function $f : X \to Y$ is said to be **1-1** or **injective** function if distinct elements of X are associated to distinct elements of Y under f. If a function is 1-1 then **geometrically** it will satisfy the following condition.

  Each horizontal line either does not intersect the graph of the function at all or if it intersects it will intersect exactly at one point. In other words, there should be no horizontal line which intersects the graph of the function at two or more than two points.

- A function $f : X \to Y$ is said to be onto or **surjective** if each element of Y has at least one pre image in X. That is for each y $\in Y$, there exists at least one $x \in X$ such that $f(x) = y$. If a function is onto then **geometrically** it will satisfy the following condition.

  Horizontal line corresponding to each point of the codomain must intersect the graph of the function at least at one point.

- A function $f : X \to Y$ is said to be one-one and onto or **bijective** or **one-one correspondence** if f is both one-one as well as onto. That is if a function is both one-one and onto then it is known as one-one correspondence or bijective function. Therefore, by combining geometrical conditions of both one-one and onto functions we can say that a function will be bijective if each horizontal line corresponding to each point of the codomain must intersect the graph of the function exactly at one point.

- A set is said to be a **finite set** if either it is an empty set or it has a finite number of elements. A set which is not finite is called an **infinite set**.

- Let A and B be two sets either both finite or both infinite then we say that sets A and B are **equivalent** if either there exists a one-one correspondence from A to B or from B to A and is denoted by $A \sim B$.

- A set E is said to be **enumerable**, if it is equivalent to the set of natural numbers, i.e., if $\mathbb{N} \sim E$.

- A set is said to be countable if either it is finite or enumerable.

- Sum, difference, product and quotient of functions f and g having D as their common domain are defined as follows:

  $(f+g)(x) = f(x) + g(x), \quad x \in D$

  $(f-g)(x) = f(x) - g(x), \quad x \in D$

  $(fg)(x) = f(x)g(x), \quad x \in D$

  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad x \in D,$ where $g(x) \neq 0$ and

  $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}, \quad x \in D,$ where $f(x) \neq 0$

- Sometimes we also need to multiply a function by a constant. Let c be a real number then product of constant c with the function f is defined as follows:

  $(cf)(x) = c\,f(x), \quad x \in D_1$

- Composition of functions f and g is denoted by $g o f$. This makes sense if range of f is subset of domain of g and is defined as follows:

  $(g o f)(x) = g(f(x)), \quad x \in D_1,$ where $\text{Range}(f) \subseteq D_2$

  Similarly, $f o g$ makes sense if range of g is subset of domain of f and is defined as follows:

  $(f o g)(x) = f(g(x)), \quad x \in D_2,$ where $\text{Range}(g) \subseteq D_1$

- If at x = a, LHL = RHL ≠ f(a) then we say that $\lim\limits_{x \to a} f(x)$ exists but function f is not continuous at x = a.

- A function f is **continuous** at x = a if left hand limit (LHL), right hand limit (RHL) and value of the function f at x = a exist and all the three are equal, i.e., $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a).$

- Differentiability of a function y = f(x) with respect to x at the point x = a is denoted by $f'(a)$ and is defined by

  $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

2.8 TERMINAL QUESTIONS

1 (a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is neither 1-1 nor onto.

(b) Show that the function $f : \mathbb{R}\backslash\{5\} \to \mathbb{R}\backslash\{2\}$ defined by $f(x) = \frac{2x-3}{x-5}$ is both 1-1 and onto.

(c) If n(A) = 5, n(B) = 7 then how many functions are possible from A to B.

(d) If n(A) = 5, n(B) = 3 then how many onto functions are possible from A to B.

(e) If n(A) = 5, n(B) = 5 then how many bijective functions are possible from A to B.

2 If a coin is tossed till head appears then write the set of all possible outcomes. Is this set countable?

3. Discuss the continuity of the greatest integer function at x = n, where n is an integer. Also, comment on left and right continuity of the function at n.

4. Discuss the continuity of the greatest integer function at x = m, where m is not an integer.

2.9 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1 (a) We have to show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=[x],\quad x\in\mathbb{R}$$

is neither 1-1 nor onto. By definition of greatest integer function, we have

$f(2.3)=[2.3]=2$ and also $f(2.4)=[2.4]=2$

So, $f(2.3)=f(2.4)$ but $2.3\neq 2.4$ $\Rightarrow$ it is not a one-one function.

Also, $2.7\in\mathbb{R}$ (codomain) but there is no $x\in\mathbb{R}$ (domain) such that

$$f(x)=2.7\quad\left[\begin{array}{l}\because \text{ Output of greatest integer function is always}\\ \text{an integer and hence cannot be } 2.7\end{array}\right]$$

$\Rightarrow$ it is not an onto function.

Hence, the given function f is neither 1-1 nor onto.

(b) We have to show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=x^2,\quad x\in\mathbb{R}$$

is neither 1-1 nor onto.

Consider $-2, 2\in\mathbb{R}$ (domain)

Now, $f(-2)=(-2)^2=4$ and $f(2)=(2)^2=4$. So, $f(-2)=f(2)$ but $-2\neq 2$

$\Rightarrow$ it is not a one-one function.

Also, $-2\in\mathbb{R}$ (codomain) but there is no $x\in\mathbb{R}$ (domain) such that

$$f(x)=x^2=-2\quad[\because \text{ Square of a real number can never be equal to } -2]$$

$\Rightarrow$ it is not an onto function.

Hence, the given function f is neither 1-1 nor onto.

(c) We have to show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) = x^3, \quad x \in \mathbb{R}$$

is both 1-1 and onto.

Consider $x_1, x_2 \in \mathbb{R}$ (domain) be such that

$$f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$$

$\Rightarrow$ it is a one-one function.

Also, for each $y \in \mathbb{R}$ (codomain) there exists $\sqrt[3]{y} \in \mathbb{R}$ (domain) such that

$$f(\sqrt[3]{y}) = \left(\sqrt[3]{y}\right)^3 = y$$

$\Rightarrow$ it is an onto function.

Hence, the given function f is both 1-1 and onto.

(d) Here n(A) = 5 = m and n(B) = 7 = n, so total number of 1-1 functions from A to B are given by

n(n – 1)(n – 2)…(n – m + 1) = 7(7 – 1)(7 – 2)(7 – 3)(7 – 4) = 7.6.5.4.3

= 2520

2 (a) We have to show that the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is countable. Define a function $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(x) = \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd} \\ -\dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases} \qquad x \in \mathbb{N}$$

We claim that f is 1-1

Let $n_1, n_2 \in \mathbb{N}$ be such that

$$f(n_1) = f(n_2) \qquad \ldots (2.1)$$

Case I: Both n_1, n_2 are odd, then (2.1) gives

$$\frac{n_1 - 1}{2} = \frac{n_2 - 1}{2} \Rightarrow n_1 - 1 = n_2 - 1 \Rightarrow n_1 = n_2$$

$\Rightarrow$ In this case f is 1-1.

Case II: Both n_1, n_2 are even, then (2.1) gives

$$-\frac{n_1}{2} = -\frac{n_2}{2} \Rightarrow -n_1 = -n_2 \Rightarrow n_1 = n_2$$

$\Rightarrow$ In this case f is 1-1.

Case III: One of n_1, n_2 is even and other is odd. Without loss of generality suppose n_1 is odd and n_2 is even, then (2.1) gives

$\dfrac{n_1 - 1}{2} = -\dfrac{n_2}{2} \Rightarrow n_1 - 1 = -n_2 \Rightarrow n_1 + n_2 = 1$ a contradiction because both n_1 and n_2 are ≥ 1 and hence $n_1 + n_2 \geq 2$.

This implies if $n_1 \neq n_2$ then $f(n_1) \neq f(n_2)$

Hence, in all cases, we have f is 1-1.

Next, we claim that f is onto.

Let n be an integer. Two cases arise:

Case I: n is a negative integer $\Rightarrow -2n$ is an even positive integer and so $-2n \in \mathbb{N}$ is such that

$$f(-2n) = -\frac{-2n}{2} = n$$

So, in this case f is onto.

Case II: n (≥ 0) is a non-negative integer $\Rightarrow$ 2n + 1 is an odd positive integer.

$\Rightarrow 2n+1 \in \mathbb{N}$ is an odd natural number such that

$$f(2n+1) = \frac{2n+1-1}{2} = \frac{2n}{2} = n$$

So, in this case f is onto.

Hence, f is onto.

We have proved that f is 1-1 and onto hence $\mathbb{N} \sim \mathbb{Z} \Rightarrow \mathbb{Z}$, the set of all integers is a countable set.

(b) We know that an integer > 1 is called a prime number if it has only two positive divisors 1 and the number itself. Therefore, if we denote the set

of all prime numbers by P then

P = {2, 3, 5, 7, 11, 13, 17, 19, 23, …}

Note that this set P is a subset of $\mathbb{N}$, the set of all natural numbers. We know that subset of a countable set is countable. Hence, P is countable because set of all natural numbers is countable and $P \subseteq \mathbb{N}$.

3. We are given

$$f(x) = \sqrt{x+3}, \quad g(x) = x-2, \quad (f\,o\,g)(x) = \sqrt{x+1}, \quad (g\,o\,f)(x) = \sqrt{x+3} - 2$$

$$(f\,o\,f)(x)) = \sqrt{\sqrt{x+3}+3}, \quad (g\,o\,g)(x) = x-4$$

(i) Function f will take real values if $x+3 \geq 0 \Rightarrow x \geq -3$. So, domain of f is $[-3, \infty)$.

(ii) Function g takes real values for all real values of x. Hence, domain of g is $\mathbb{R}$.

(iii) Function $(f\,o\,g)(x) = \sqrt{x+1}$ will take real values if $x+1 \geq 0 \Rightarrow x \geq -1$. Hence, domain of $f\,o\,g$ is $[-1, \infty)$.

(iv) Function $(g\,o\,f)(x) = \sqrt{x+3} - 2$ will take real values if $x+3 \geq 0 \Rightarrow x \geq -3$. Hence, domain of $g\,o\,f$ is $[-3, \infty)$.

(v) Function $(f\,o\,f)(x) = \sqrt{\sqrt{x+3}+3}$ will take real values if $x+3 \geq 0 \Rightarrow x \geq -3$. Hence, domain of $f\,o\,f$ is $[-3, \infty)$.

(vi) Function $(gog)(x) = x - 4$ takes real values for all real values of x. Hence, domain of gog is $\mathbb{R}$.

4. We are given $f(x) = \sqrt{5x-1}, L = 3, \varepsilon = 0.1, c = 2$

Now, $|f(x) - L| < \varepsilon \Rightarrow \left|\sqrt{5x-1} - 3\right| < 0.1$

$\Rightarrow -0.1 < \sqrt{5x-1} - 3 < 0.1 \quad [\because |x| < a \Rightarrow -a < x < a, \text{ refer } (6.95)]$

$\Rightarrow 2.9 < \sqrt{5x-1} < 3.1 \Rightarrow (2.9)^2 < 5x - 1 < (3.1)^2$

$\Rightarrow 8.41 + 1 < 5x < 9.61 + 1 \Rightarrow 9.41 < 5x < 10.61 \Rightarrow 1.882 < x < 2.122$

Now, $\delta = \min\{2 - 1.882, 2.122 - 2\} = \min\{0.118, 0.122\} = 0.118$

Hence, $\delta = 0.118$. In fact, all values between 0 and 0.118 excluding 0 will work as values of δ for given $\varepsilon = 0.1$.

5. We are given $f(x) = |x - 5|$

We have already studied about absolute value function or modulus function in Unit 1 of this course refer (1.47). Here, first we have to find out left hand derivative and right hand derivative.

$$\text{LHD}_{\text{at } x=5} = \lim_{h\to 0^-} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0^-} \frac{|5+h-5| - |5-5|}{h} = \lim_{h\to 0^-} \frac{|h| - |0|}{h}$$

$$= \lim_{h\to 0^-} \frac{-h-0}{h} \quad \begin{bmatrix} \because h \to 0^- \text{ means h is slightly less than 0,} \\ \text{so } |h| = -h. \text{ Also, } |0| = 0, \text{ refer } (1.47) \end{bmatrix}$$

$$= \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1$$

$$\text{RHD}_{\text{at } x=5} = \lim_{h\to 0^+} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0^+} \frac{|5+h-5| - |5-5|}{h} = \lim_{h\to 0^+} \frac{|h| - |0|}{h}$$

$$= \lim_{h\to 0^+} \frac{h-0}{h} \quad \begin{bmatrix} \because h \to 0^+ \text{ means h is slightly greater than 0,} \\ \text{so } |h| = h. \text{ Also, } |0| = 0, \text{ refer } (1.47) \end{bmatrix}$$

$$= \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} (1) = 1$$

Since, $\text{LHD}_{\text{at } x=5} = -1 \neq \text{RHD}_{\text{at } x=5} = 1$. Hence, f is not differentiable at x = 5.

Terminal Questions

1 (a) We have to show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$f(x) = |x|, \quad x \in \mathbb{R}$

is neither 1-1 nor onto. By definition of modulus function, we know that

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}, \quad x \in R$$

$\therefore f(-2) = |-2| = -(-2) = 2$ and $f(2) = |2| = 2$

So, $f(-2) = f(2)$ but $-2 \neq 2$ $\Rightarrow$ it is not a one-one function.

Also, $-2 \in \mathbb{R}$ (codomain) but there is no $x \in \mathbb{R}$ (domain) such that

$$f(x) = -2 \quad \begin{bmatrix} \because \text{ Output of modulus function can} \\ \text{never be a negative number} \end{bmatrix}$$

$\Rightarrow$ it is not an onto function.

Hence, the given function f is neither 1-1 nor onto.

(b) We have to show that the function $f : \mathbb{R}\backslash\{5\} \to \mathbb{R}\backslash\{2\}$ defined by

$$f(x) = \frac{2x-3}{x-5}, \quad x \in \mathbb{R}$$

is both 1-1 and onto.

Consider $x_1, x_2 \in \mathbb{R}$ (domain) be such that

$$f(x_1) = f(x_2) \Rightarrow \frac{2x_1 - 3}{x_1 - 5} = \frac{2x_2 - 3}{x_2 - 5}$$

$$\Rightarrow 2x_1x_2 - 10x_1 - 3x_2 + 15 = 2x_1x_2 - 3x_1 - 10x_2 + 15$$
$$\Rightarrow -10x_1 - 3x_2 = -3x_1 - 10x_2 \quad \Rightarrow -7x_1 = -7x_2 \quad \Rightarrow x_1 = x_2$$

$\Rightarrow$ it is a one-one function.

Also, for each $y \in \mathbb{R}$ (codomain) there exists $\dfrac{5y-3}{y-2} \in \mathbb{R}$ (domain) s.t.

$$f\left(\frac{5y-3}{y-2}\right) = \frac{2\left(\frac{5y-3}{y-2}\right) - 3}{\frac{5y-3}{y-2} - 5} = \frac{10y - 6 - 3y + 6}{5y - 3 - 5y + 10} = \frac{7y}{7} = y$$

$\Rightarrow$ it is an onto function.

Hence, the given function f is both 1-1 and onto.

(c) Here n(A) = 5 = m and n(B) = 7 = n, so total number of functions from A to B are given by $n^m = 7^5 = 16807$

(d) Here n(A) = 5 = m and n(B) = 3 = n, so total number of onto functions from A to B are given by

$$= \binom{3}{0}(-1)^0(3-0)^5 + \binom{3}{1}(-1)^1(3-1)^5 + \binom{3}{2}(-1)^2(3-2)^5$$

$$= 243 - 96 + 3 = 150$$

(e) Here n(A) = 5 = m and n(B) = 5 = n, so total number of bijective functions from A to B are given by n! = 5! = 120.

2. If the outcomes head and tail are represented by H, T respectively and S denotes the set of all possible outcomes of this experiment then

S = {H, TH, TTH, TTTH, TTTTH, …}

Let X counts number of failures preceding the first success then X takes values 0, 1, 2, 3, 4, … corresponding to the elements of the set S and if we write values of X in a set then

X = {0, 1, 2, 3, 4, …}

Note that corresponding to each element of S there is exactly one element in X and vice versa so n(S) = n(X). So, in order to prove that S is countable it is enough to prove that X is countable. Define a function $f:\mathbb{N}\to X$ by

$$f(n)=n-1,\ \ n\in\mathbb{N}$$

We claim that f is 1-1 and onto. Let us first prove that f is 1-1. Let $n_1, n_2\in\mathbb{N}$ be such that

$$f(n_1)=f(n_2)\ \Rightarrow n_1-1=n_2-1\Rightarrow n_1=n_2$$

$\Rightarrow$ f is a one-one function.

Also, for each $n\in X$ (codomain) there exists $n+1\in\mathbb{N}$ (domain) such that

$$f(n+1)=n+1-1=n$$

$\Rightarrow$ f is an onto function.

Hence, the given function f is both 1-1 and onto. Thus, $\mathbb{N}\sim X\ \Rightarrow$ X is enumerable and so X is countable. Countability of X implies S is countable because n(X) = n(S) $\Rightarrow X\sim S$.

3. We claim that the greatest integer function at x = n, where n is an integer is not continuous. Let us calculate LHL and RHL at x = n.

$$\text{LHL}=\lim_{x\to n^-}f(x)=\lim_{x\to n^-}[x]=n-1\quad\left[\because x\to n^-\text{ means }n-1<x<n,\text{ so }[x]=n-1\right]$$

$$\text{Similarly, RHL}=\lim_{x\to n^+}f(x)=\lim_{x\to n^+}[x]=n\quad\left[\because x\to n^+\text{ means }n<x<n+1,\text{ so }[x]=n\right]$$

$\therefore$ LHL ≠ RHL $\Rightarrow$ f is not continuous at x = n, where n is an integer.

Left and Right continuity of f at n: Obviously, f(n) = [n] = n.

Since $\text{LHL}_{\text{at }x=n}=n-1\neq n=f(n)\Rightarrow$ f is not left continuous at x = n.

However, $\text{RHL}_{\text{at }x=n}=f(n)=n\Rightarrow$ f is right continuous at x = n.

4. We claim that the greatest integer function is continuous at x = m, where m is not an integer. Let us calculate LHL and R HL at x = m. Suppose k is an integer such that k < m < k + 1.

$$\text{LHL}=\lim_{x\to m^-}f(x)=\lim_{x\to m^-}[x]=k\quad\left[\because x\to m^-\text{ and }k<m<k+1\text{ means }k<x<k+1,\text{ so }[x]=k\right]$$

$$\text{Similarly, RHL}=\lim_{x\to m^+}f(x)=\lim_{x\to m^+}[x]=k\quad\left[\because x\to m^+\text{ and }k<m<k+1\Rightarrow k<x<k+1,\text{ so }[x]=k\right]$$

Also, f(m) = [m] = k $\left[\because k<m<k+1,\text{ where k is an integer}\Rightarrow[m]=k\right]$

$\therefore$ LHL = RHL = f(m) $\Rightarrow$ f is continuous at x = m, where m is not an integer.