UNIT 6

HYPERPLANE

Structure

6.1 INTRODUCTION

In earlier classes, you have studied different forms of the equation of a line in 2 and 3-dimension. A brief overview of the equation of a line in 2-dimension is given in Sec. 6.2. To study the equation of a line in 3-dimension you should have an understanding of some basic results of vector algebra which are discussed in Sec. 6.3 and then the equation of a line in 3-dimension is discussed in Sec. 6.4. In the course MST-026 you will study support vector machine (SVM). In SVM sometimes two categories are not linearly separable. So, we go to a higher dimension to find a separating plane. If you work in 3-D then you need a plane which separates two categories and if you work in a higher dimension of more than 3 then we try to identify a hyperplane which separates two categories. These two things plane and hyperplane are discussed in Secs. 6.5 and 6.6 respectively. In some courses of this programme, you will use inequations and so some basic results related to inequation are discussed in Sec. 6.7.

What we have discussed in this unit is summarised in Sec. 6.8. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit some more questions based on the entire unit are given in Sec. 6.9 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 6.10.

In the next unit, you will study about convexity of a set and function.

Expected Learning Outcomes

After completing this unit, you should be able to:

- ❖ obtain the equation of a line in different forms in 2 and 3-dimension;
- ❖ explain the basic results of vector algebra which are required to deal with equations of line, plane and hyperplane;
- ❖ obtain the equation of a plane and hyperplane in different forms; and
- ❖ deal with inequalities in one and two variables.

6.2 EQUATION OF A LINE IN 2-DIMENSION

The slope of a line plays a crucial role in the equation of a straight line in 2-dimension. The slope of a line is directly related to increment. Or we can say that increment plays an important role in the slope of a line. So, before discussing the equation of a line in 2-dimension let us first understand what is increment. And then we will define what is the slope of a line.

Increment: Let an object changes its position from point A(x_1, y_1) to point B(x_2, y_2) then the net changes in the coordinates of point A are known as increments. An increment in the direction of the x-axis is known as **run** and an increment in the direction the of y-axis is known as **rise**. Thus, we have

$$\text{Run} = (\text{x coordinate of point B}) - (\text{x coordinate of point A}) = x_2 - x_1$$

$$\text{Rise} = (\text{y coordinate of point B}) - (\text{y coordinate of point A}) = y_2 - y_1$$

Slope: Let A(x_1, y_1) and B(x_2, y_2) be two points on the line l refer to Fig. 6.1. Draw a line from point A and parallel to the x-axis and another line from point B parallel the to y-axis both intersect at point C. So, we have

$$\text{Run} = AC = x_2 - x_1, \quad \text{Rise} = CB = y_2 - y_1 \qquad \dots (6.1)$$

Consider the ratio $\frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{CB}{AC}$ … (6.2)

Similarly, if we consider two points P and Q on the same line l then in ΔPQR, we have

$$\frac{\text{Rise}}{\text{Run}} = \frac{RQ}{PR} \qquad \dots (6.3)$$

Now, in ΔABC and ΔPQR, we have

$\lfloor 1 = \lfloor 3$ [Corresponding angles]

$\lfloor 2 = \lfloor 4$ [Each 90^0]

$\therefore\ \Delta ABC \sim \Delta PQR$ [AA similarity]

So, $\frac{CB}{AC} = \frac{RQ}{PR}$ [Corresponding parts of similar triangles (cpst)] … (6.4)

In view of equations (6.2), (6.3) and (6.4) we see that if we take any two points on a line then the ratio of Rise and Run remains constant. This constant ratio is known as the **slope** of the line.

Also note that if the line l makes an angle θ with a positive direction of the x-axis then $\angle 1$ and θ being corresponding angles are equal. So, in right-angled $\Delta ABC,$ we have

$$\tan\theta = \frac{CB}{AC} = \frac{Rise}{Run} = \frac{y_2 - y_1}{x_2 - x_1} = \text{Slope of the line l} \qquad \dots (6.5)$$

Also, observe from equation (6.2) that if the coordinates of two points on a line are given then the slope of a line is the ratio of the difference between their y-coordinates to their corresponding x-coordinates.

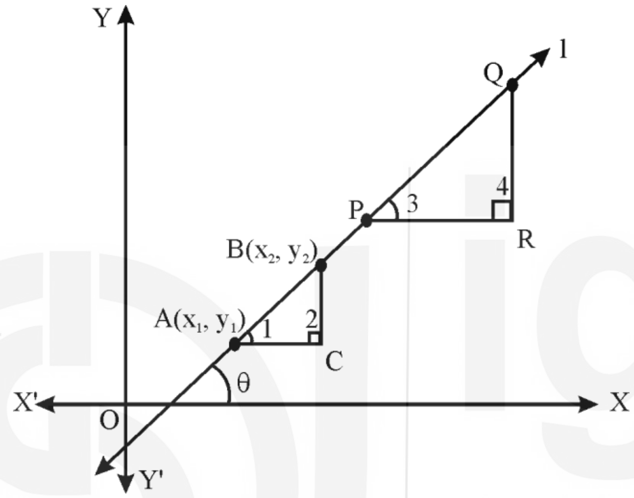

Fig. 6.1: Visualisation of rise, run and concept of slope of a line

Now, we discuss different forms of equation of a line in 2-dimension one at a time.

6.2.1 Point Slope Form

Here we are given two things: a point on the line and slope of the line. Refer Fig. 6.2 (a).

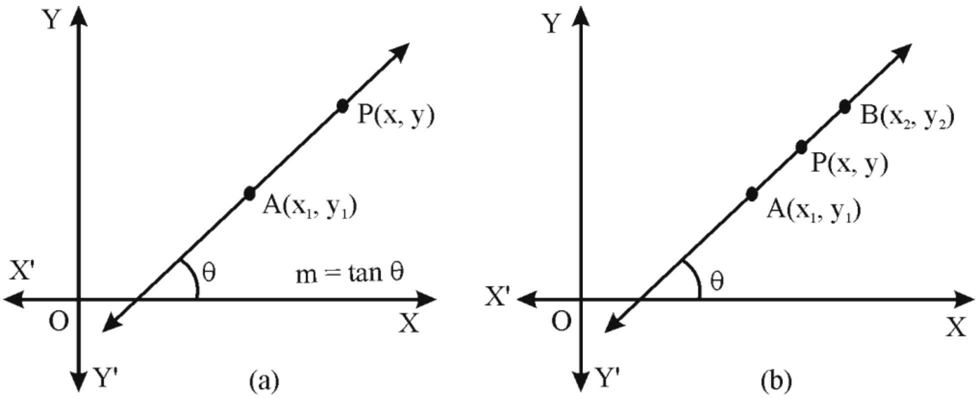

Fig. 6.2: (a) Point slope form (b) Two-point form

Suppose given point is $A(x_1, y_1)$ and slope of the line is m. Let $P(x, y)$ be any point on the line. So, the slope of the line AP is given by $\frac{y - y_1}{x - x_1}$. But the slope of this line is given to us as m. So, equating two slopes we get

$$\frac{y-y_1}{x-x_1}=m$$

$$\Rightarrow y-y_1=m(x-x_1) \qquad \dots (6.6)$$

This is the equation of the line in **point-slope form**. Some authors also call it **one-point form**.

Example 1: Find slope of the line in each of the following cases:

(i) passing through the points (2, 3) and (6, 10).

(ii) making an angle of 60^0 with positive direction of x-axis.

Solution: (i) We know that slope of a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $m=\frac{y_2-y_1}{x_2-x_1}$. So, in our case it is given by

$$m=\frac{10-3}{6-2}=\frac{7}{4}$$

(ii) Slope of the line which makes an angle of 60^0 with positive direction of the x-axis is $m=\tan\theta=\tan 60^0=\sqrt{3}$.

Example 2: Slope of a line is 5/3 and of another line is $-7/9$ interpret both the slopes.

Solution: We know that slope of a line is rise/run. So, slope 5/3 means when a particle moves along this line then it rises 5 units for every run of 3 units. Or we can say that y increases 5 units every time when x increases 3 units. It can also be written as $m=\frac{5}{3}=\frac{5/3}{1}$, so we can say that y increases 5/3 units every time when x increases 1 unit. In the case, slope is $-7/9$ it means y decreases by 7 units every time when x increases by 9 units.

6.2.2 Two Point Form

Here we are given two points on a line and we are interested in the equation of the line passing through these two points refer Fig. 6.2 (b).

Suppose given two points be $A(x_1, y_1)$ and $B(x_2, y_2)$. Let P(x, y) be any point on the line passing through points A and B. Since all the three points A, B and P lie on the same line So, we must have

$$\text{Slope of line AP}=\text{Slope of line AB}$$

$$\Rightarrow \frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1} \qquad [\text{Using (6.5)}]$$

$$\text{or } y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \dots (6.7)$$

$$\text{or } \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad \dots (6.8)$$

$$\text{or } \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\lambda \text{ (say)} \qquad \dots (6.9)$$

$$\Rightarrow x = x_1 + \lambda(x_2 - x_1), \quad y = y_1 + \lambda(y_2 - y_1) \qquad \ldots (6.10)$$

where λ is any real number

Or it can also be written as

$$\Rightarrow x = (1-\lambda)x_1 + \lambda x_2, \quad y = (1-\lambda)y_1 + \lambda y_2, \; -\infty < \lambda < \infty \qquad \ldots (6.11)$$

This line as a set of points can be written as

$$\{(1-\lambda)(x_1, y_1) + \lambda(x_2, y_2) : -\infty < \lambda < \infty\} \qquad \ldots (6.12)$$

Or $\{(1-\lambda)x + \lambda y : -\infty < \lambda < \infty\}$, where $x = (x_1, y_1)$, $y = (x_2, y_2)$ … (6.13)

Equation (6.10) gives us a general point on the line given by (6.8) or (6.9). Note that

(i) when $\lambda = 0$, it gives the point (x_1, y_1).

(ii) when $\lambda = 1$, it gives the point (x_2, y_2).

(iii) when $\lambda = 1/2$, it gives the midpoint of AB, i.e., $\left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}\right)$.

(iv) when $\lambda = 2$, it gives a point 2 units aways from the point A in the same side of position of the point B from the point A.

(v) when $\lambda = -2$, it gives a point 2 units aways from the point A in the opposite side of position of the point B from point A, etc.

(vi) Using (6.9) and on the basis of the discussion done in points (i) to (v), we can say that all the points between points $A(x_1, y_1)$ and $B(x_2, y_2)$ including points A and B (or on the line segment AB) are given by

$$x = x_1 + \lambda(x_2 - x_1), \quad y = y_1 + \lambda(y_2 - y_1), \text{ where } 0 \le \lambda \le 1 \qquad \ldots (6.14)$$

Or you can say that (6.14) represents equation of the line segment AB.

Also, if λ is allowed to take any real value then it will represent equation of line AB and it is known as the parametric equation of the line. So, the parametric equation of the line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$x = x_1 + \lambda(x_2 - x_1), \quad y = y_1 + \lambda(y_2 - y_1), \text{ where } -\infty < \lambda < \infty \qquad \ldots (6.15)$$

Like (6.12) and (6.13), the line segment AB as a set of points can be written as

$$\{(1-\lambda)(x_1, y_1) + \lambda(x_2, y_2) : 0 \le \lambda \le 1\} \qquad \ldots (6.16)$$

Or $\{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}$, where $x = (x_1, y_1)$, $y = (x_2, y_2)$ … (6.17)

Example 3: Find equation of a line passing through the points (4, 5) and $(-3, 7)$. Also find the coordinates of a point on this line which is 100 units away from the point (4, 5) in the direction opposite to the point $(-3, 7)$.

Solution: Using two-point form equation of a line passing through the given points is given by

$$\frac{x-4}{-3-4} = \frac{y-5}{7-5} \Rightarrow \frac{x-4}{-7} = \frac{y-5}{2} \Rightarrow 2x + 7y - 43 = 0$$

Now, coordinates of a point on this line which is 100 units away from the point (4, 5) in the direction opposite to the point $(-3, 7)$ is given by solving following for x and y

$$\frac{x-4}{-7} = \frac{y-5}{2} = -100 \Rightarrow x = 704, \ y = -195$$

So, required point is $(704, -195)$.

6.2.3 Intercept Form

Let us first define what are x and y-intercepts. If a line which is neither horizontal nor vertical intersects x-axis at a point having coordinates (a, 0) and y-axis at a point having coordinates (0, b) then a and b are known as **x and y-intercepts** respectively.

Here we are interested in the equation of a line which has a and b as x and y-intercepts respectively. Refer Fig. 6.3 (a). By definition of intercepts this line passes through the points A(a, 0) and B(0, b). Therefore, using two-point form equation of such a line is given by

$$y-0 = \frac{b-0}{0-a}(x-a) \Rightarrow -ay = bx - ab \Rightarrow bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \quad \ldots (6.18)$$

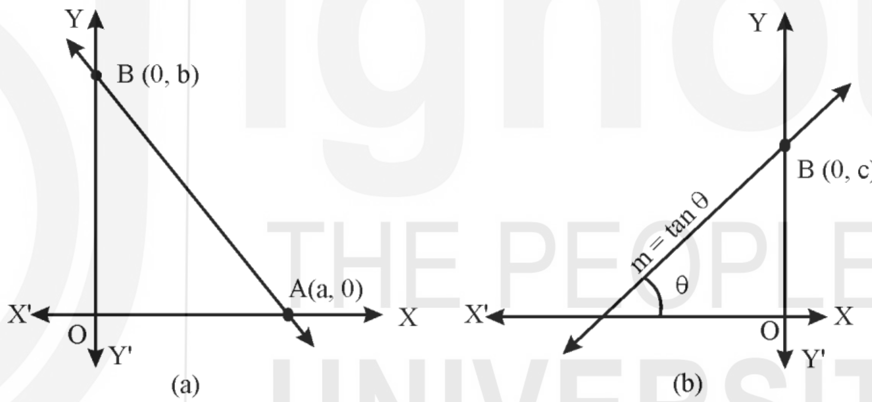

Fig. 6.3: (a) Intercept form (b) Slope intercept form

Example 4: If a line makes -3 and 4 as x and y intercepts respectively. Find equation of the line.

Solution: Using intercept form equation of the required line is given by

$$\frac{x}{-3} + \frac{y}{4} = 1 \Rightarrow 4x - 3y + 12 = 0$$

6.2.4 Slope Intercept Form

Here we are given two things: (i) slope of the line (ii) y-intercept made by the line. We are interested in the equation of such a line. Refer Fig. 6.3 (b).

Let slope of the line be m and c be the y-intercept made by the line. So, coordinates of the point where line intersects y-axis are (0, c). Now, equation of the line using slope point form is given by

$$\Rightarrow y - c = m(x-0) \Rightarrow y = mx + c \quad \ldots (6.19)$$

If c = 0 then line will pass through origin and equation (6.19) reduces to

$$y = mx \quad \ldots (6.20)$$

So, remember equation (6.20) represents line passing through origin and having slope m.

Example 5: If slope of a line is -5 and y-intercept is 2. Find equation of the line.

Solution: Using slope intercept form equation of the required line is given by $y = mx + c \Rightarrow y = -5x + 2$

or $5x + y - 2 = 0$

6.2.5 General Form

All the forms discussed in sub-Secs. 6.2.1 to 6.2.4 after simplification reduces to the form

$$ax + by + c = 0 \qquad \dots (6.21)$$

where not both a and b are zero because if both a = 0 and b = 0 then equation (6.21) will not represent a straight line

The form given in equation (6.21) is known as equation of a line in general form. So, remember graph of any equation of the form given in (6.21) will represent a straight line and equation of any straight line can be represented in the form of equation (6.21).

One more thing to note: you are aware that any two separate points lead to a unique line. But equation (6.21) seems that it has three unknown parameters a, b and c so we need three conditions to obtain equation of a line. Actually, equation (6.21) has only two parameters A and B explained as follows

$$Ax + By = 1 \left[\because ax + by = -c \Rightarrow \frac{a}{-c}x + \frac{b}{-c}y = 1 \Rightarrow Ax + By = 1, A = \frac{a}{-c}, B = \frac{b}{-c}\right]$$

Example 6: If equation of a line is $5x + 3y + 7 = 0,$ then find:

(i) slope of the line and y-intercept.

(ii) intercepts of both x and y.

Solution: Given equation of the line is $5x + 3y + 7 = 0 \qquad \dots (6.22)$

(i) Equation (6.22) can be written as $y = -\frac{5}{3}x - \frac{7}{3} \qquad \dots (6.23)$

Comparing equation (6.23) with equation (6.19), we get $m = -\frac{5}{3}, c = -\frac{7}{3}.$

So, slope of the given line is $m = -\frac{5}{3},$ and y-intercept is $c = -\frac{7}{3}.$

Another way of obtaining slope of a line is $= -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{5}{3}$

But keep in mind that before using this formula make sure that terms of x and y both should be on the same side of equality sign.

(ii) Equation (6.22) can be written as $-\frac{5}{7}x - \frac{3}{7}y = 1$

or $\frac{x}{-7/5} + \frac{y}{-7/3} = 1$ … (6.24)

Comparing equation (6.24) with equation (6.18), we get $a = -\frac{7}{5}, b = -\frac{7}{3}$.

So, x-intercept is $a = -\frac{7}{5}$, and y-intercept is $b = -\frac{7}{3}$.

Also, remember following results related to a line.

Perpendicular Distance of a Point from a Line: Refer Fig. 6.4 (a). If d is the perpendicular distance of a point $A(x_1, y_1)$ from the line ax + by + c = 0 then

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$ … (6.25)

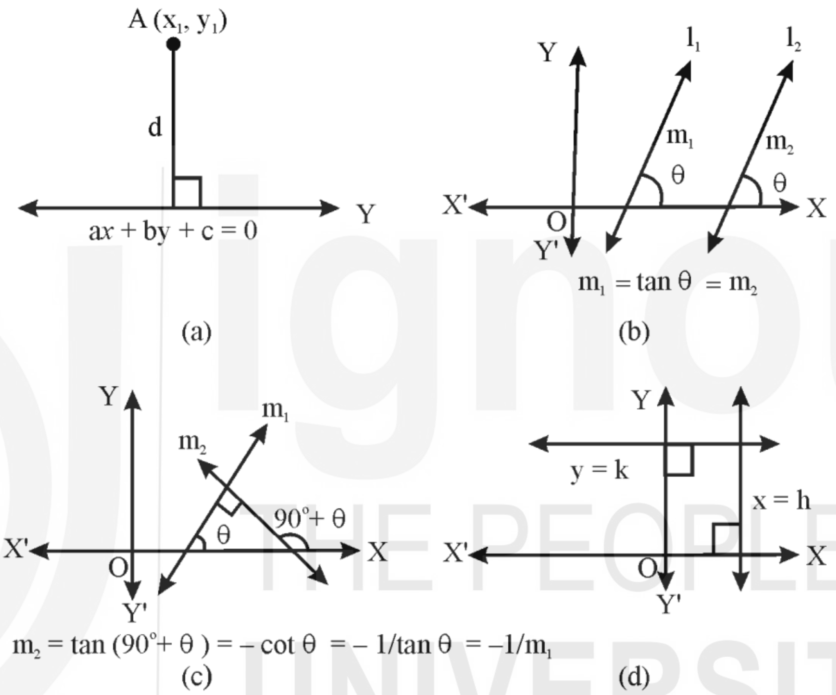

Fig. 6.4: Visualisation of (a) ⊥ distance of a point A from a line (b) Slopes of parallel lines (c) Slopes of ⊥ lines (d) Equation of lines parallel to axes

Slope of a Line Parallel to a Given Line: Refer Fig. 6.4 (b). If m_1 is the slope of a given line l_1, and line l_2 is parallel to line l_1, then the slope m_2 of line l_2 is equal to slope of line l_1. That is $m_1 = m_2$ … (6.26)

Slope of a Line Perpendicular to a Given Line: Refer Fig. 6.4 (c). If m_1 is the slope of a given line l_1, and m_2 be the slope of a line l_2 which is perpendicular to given line l_1, then $m_1 m_2 = -1$. … (6.27)

Except in the special cases slope of horizontal and vertical line. Slope of a horizontal line is zero $\left[\because \tan 0^0 = 0\right]$ and slope of vertical line is not defined. $\left[\because \tan 90^0 = \infty\right]$

Equation of a line Parallel to x-axis or y-axis: Refer Fig. 6.4 (d). For any particular line parallel to x-axis coordinates of each point satisfy the following conditions:

(i) x-coordinates varies from point to point, but

(ii) y-coordinate is a fixed constant.

So, equation of any particular line parallel to x-axis is given by y = k. … (6.28)

Similarly,

equation of any particular line parallel to y-axis is given by x = h. … (6.29)

To obtain value of h or k we need only one condition. Given condition may be: a point is given from which line passes or distance from respective axis is given, and to which side of the respective axis.

Now, you can try the following Self-Assessment Question.

SAQ 1

(a) Find the equation of a line passing through the point (2, 3) and having slope 7/8.

(b) Find the perpendicular distance of the point $(2, -7)$ from the line $3x + 4y + 9 = 0.$

(c) Find the slope of a line which is parallel to the line $5x + 6y + 8 = 0.$

6.3 A REVIEW OF VECTOR ALGEBRA

When we work in higher dimension the use of vector notations makes things simple to write and understand. That is why we are recalling some results of vector algebra from earlier classes before discussing equation of a line and plane in 3-dimension and hyperplane in n-dimension. So, in this section we will review some notations and basic concepts of vector algebra which you have studied in earlier classes. We will use these notations and concepts in the next three sections of this unit and in the courses MST-024 and MST-026. Let us start this journey with definition of a vector.

Vector: A vector is a quantity that has both magnitude and direction. If a vector has initial point A and terminal point B then it is denoted by $\overrightarrow{AB}$. Magnitude of the vector $\overrightarrow{AB}$ is the length of the line segment AB and direction is from point A to point B.

Unit Vector: A vector having magnitude 1 (unity) is called unit vector. A unit vector is denoted by putting a cap on it. Unit vectors along x-axis, y-axis and z-axis are denoted by $\hat{i},\ \hat{j},\ k$ respectively.

Position Vector of a Point: If O is the origin and A is any point then position vector of the point A is denoted by $\overrightarrow{OA}$ and is a vector having magnitude equal to the length OA and direction is from origin O to the point A. Refer Fig. 6.5 (a). If coordinates of point A are (a_1, b_1, c_1) then you know that $a_1,\ b_1,\ c_1$ are the distances of point A from origin along x-axis, y-axis, z-axis respectively. So, position vector of point A can be written as

$$\overrightarrow{OA} = a_1\,\hat{i} + b_1\,\hat{j} + c_1\,k \quad \ldots (6.30)$$

and magnitude of vector $\overrightarrow{OA}$ is denoted by $\left|\overrightarrow{OA}\right|$ and given by

$$\left|\overrightarrow{OA}\right| = \sqrt{a_1^2 + b_1^2 + c_1^2} \qquad \dots (6.31)$$

where $a_1,\ b_1,\ c_1$ are known as components of the vector $\overrightarrow{OA}$

Keep in mind that throughout this course we will denote the position vector of a general point P(x, y, z) by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. … (6.32)

Remark 1: In the courses MST-018, MST-024 and MST-026 we will denote the vector given by (6.30) in any of the following three ways:

(i) Like a point in 3-dimension as (a_1, b_1, c_1) … (6.33)

(ii) As a column matrix $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ … (6.34)

(iii) Or as a transpose of a row matrix $\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}^T$ … (6.35)

These notations will also be extended to n-dimension in the exactly similar way. So, keep this structure of different notations for the same thing in your mind.

Vector Joining Two Points: If A and B are two points and O is the origin then the vector joining points A and B is denoted by $\overrightarrow{AB}$ and defined as

$$\overrightarrow{AB} = (\text{Position vector of point B}) - (\text{Position vector of point A})$$

$$= \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a} \qquad \dots (6.36)$$

Refer Fig. 6.5 (b).

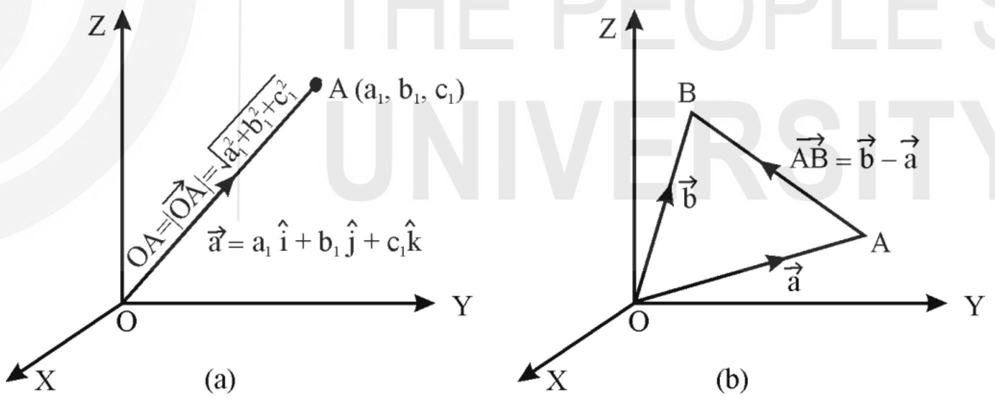

Fig. 6.5: (a) Position vector of a point (b) Vector joining two points

Addition of Vectors: Let $\vec{a}$ and $\vec{b}$ are two vectors refer Fig. 6.6 (a). You know from the knowledge of earlier classes that a vector remains the same if it is shifted parallel to itself anywhere in its plane. Addition of two vectors is based on this property. Let us shift vector $\vec{b}$ parallel to itself such that initial point of $\vec{b}$ coincides with terminal point of $\vec{a}$, refer Fig. 6.6 (b). Join initial point of vector $\vec{a}$ to the terminal point of $\vec{b}$ then the vector thus obtained is known as $\vec{a} + \vec{b}$ refer Fig. 6.6 (c). In particular, using this concept of addition of two vectors in any triangle ABC (say), we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \qquad \dots (6.37)$$

This is known as **triangle law of addition** refer Fig. 6.6 (d).

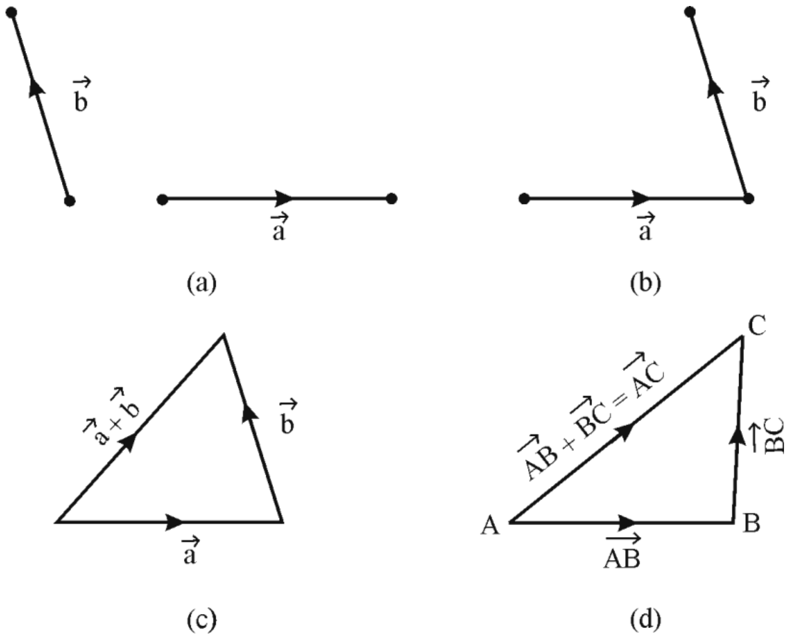

Fig. 6.6: Visualisation of (a) vectors $\vec{a}$ and $\vec{b}$ (b) shifting of vector $\vec{b}$ (c) addition of $\vec{a}$ and $\vec{b}$ (d) triangle law of addition of two vectors

Dot Product of two Vectors: Dot product of two vectors $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as follows.

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \qquad \text{… (6.38)}$$

where θ is the angle between vectors $\vec{a}$ and $\vec{b}$ refer Fig. 6.7 (a).

Some important observations from the definition of dot product are listed as follows.

- If $\vec{a}\perp\vec{b}$ then $\theta = 90^0$, and so $\cos\theta = \cos 90^0 = 0$.

  Hence, if $\vec{a}\perp\vec{b}$ then $\vec{a}.\vec{b} = 0$. … (6.39)

  In particular $\hat{i}.\hat{j} = 0 = \hat{j}.\hat{i},\ \hat{j}.k = 0 = k.\hat{j},\ \ k.\hat{i} = 0 = \hat{i}.k.$ … (6.40)

- If $\vec{a}.\vec{b} = 0$, then either $|\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\vec{a}\perp\vec{b}$. So, if $\vec{a}$ and $\vec{b}$ are non-zero vectors and $\vec{a}.\vec{b} = 0$, then $\vec{a}\perp\vec{b}$. … (6.41)

- If $\vec{a} = \vec{b}$, then $\vec{a}.\vec{a} = |\vec{a}||\vec{a}|\cos 0 = |\vec{a}|^2\ \ [\because \cos 0 = 1]$.

  So, $\vec{a}.\vec{a} = |\vec{a}|^2$. … (6.42)

  In particular $\hat{i}.\hat{i} = 1,\ \ \hat{j}.\hat{j} = 1,\ \ k.k = 1,$ … (6.43)

- So, if $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1 k,\ \vec{b} = a_2\hat{i} + b_2\hat{j} + c_2 k$, then using (6.40) and (6.43), we get

  $\vec{a}.\vec{b} = a_1 a_2 + b_1 b_2 + c_1 c_2$ … (6.44)

- $|\vec{a}.\vec{b}| \le |\vec{a}||\vec{b}|$ or $|\vec{a}.\vec{b}| \le \|\vec{a}\|\|\vec{b}\|$ known as **Cauchy Schwartz Inequality**…(6.45)

  where in higher dimension $|\vec{a}|$ is denoted by $\|\vec{a}\|$ and read as norm of $\vec{a}$

  Its proof is very simple given as follows.

Result holds trivially if either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, because then both sides will become 0 and so equality will hold. So, consider the case that neither $\vec{a}=\vec{0}$ nor $\vec{b}=\vec{0}$. So, when $\vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{0}$, equation (6.38) can be written as

$$\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\cos\theta \Rightarrow \left|\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right|=|\cos\theta| \Rightarrow \frac{|\vec{a}.\vec{b}|}{|\vec{a}||\vec{b}|}\leq 1 \ [\because -1\leq\cos\theta\leq 1\Rightarrow|\cos\theta|\leq 1]$$

$$\Rightarrow |\vec{a}.\vec{b}|\leq|\vec{a}||\vec{b}| \text{ or } |\vec{a}.\vec{b}|\leq\|\vec{a}\|\|\vec{b}\|$$

Some particular cases of Cauchy Schwartz Inequality are given as follows:

In 2-dimension $|a_1b_1+a_2b_2|\leq\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}$, … (6.45a)

where $\vec{a}=a_1\hat{i}+a_2\hat{j}, \vec{b}=b_1\hat{i}+b_2\hat{j}$ or $\vec{a}=(a_1, a_2)$, $\vec{b}=(b_1, b_2)$

In 3-dimension $|a_1b_1+a_2b_2+a_3b_3|\leq\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}$ … (6.45b)

where $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3k, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3k$ or $\vec{a}=(a_1, a_2, a_3), \vec{b}=(b_1, b_2, b_3)$

In n-dimension, we have

$$|a_1b_1+a_2b_2+\ldots+a_nb_n|\leq\sqrt{a_1^2+a_2^2+\ldots+a_n^2}\sqrt{b_1^2+b_2^2+\ldots+b_n^2} \quad \ldots (6.45c)$$

$$\text{or } |a_1b_1+a_2b_2+\ldots+a_nb_n|\leq\|(a_1, a_2, \ldots, a_n)\|\|(b_1, b_2, \ldots, b_n)\| \quad \ldots (6.45d)$$

$$\text{or } |a_1b_1+a_2b_2+\ldots+a_nb_n|\leq\|\vec{a}\|\|\vec{b}\| \quad \ldots (6.45e)$$

where

$\vec{a}=a_1e_1+a_2e_2+\ldots+a_ne_n, \vec{b}=b_1e_1+b_2e_2+\ldots+b_ne_n$ or $\vec{a}=(a_1, a_2, \ldots, a_n)$,

$\vec{b}=(b_1, b_2, \ldots, b_n)$

- $|\vec{a}+\vec{b}|\leq|\vec{a}|+|\vec{b}|$ known as **Triangle Inequality** … (6.46)

  If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then equality will hold. So, consider the case that neither $\vec{a}=\vec{0}$ nor $\vec{b}=\vec{0}$. Using equation (6.42), we have

$$|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=|\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2 \ [\because \vec{a}.\vec{b}=\vec{b}.\vec{a}]$$

$$\leq|\vec{a}|^2+2|\vec{a}.\vec{b}|+|\vec{b}|^2 \quad [\because \vec{a}.\vec{b}\leq|\vec{a}.\vec{b}|]$$

$$\leq|\vec{a}|^2+2|\vec{a}||\vec{b}|+|\vec{b}|^2 \quad [\because \text{Using Cauchy Schwartz Inequality}]$$

$$=(|\vec{a}|+|\vec{b}|)^2$$

$$\Rightarrow|\vec{a}+\vec{b}|^2\leq(|\vec{a}|+|\vec{b}|)^2$$

$$\Rightarrow|\vec{a}+\vec{b}|\leq|\vec{a}|+|\vec{b}| \ [\because \text{if } a>0, b>0 \text{ then } a^2\leq b^2\Rightarrow a\leq b]$$

Projection of a Vector on another Vector: Let $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$ be the position vectors of points A and B respectively. Draw perpendicular from point B to OA which intersects OA at point C then the length $|\overrightarrow{OC}|$ is known as scalar projection of vector $\vec{b}$ on vector $\vec{a}$ and $\overrightarrow{OC}$ is known as vector projection of vector $\vec{b}$ on vector $\vec{a}$ refer Fig. 6.7 (b).

So, scalar projection is given by

$$OC = OB\cos\theta = \left(\left|\vec{b}\right|\cos\theta\right)\left(\frac{1}{\left|\vec{a}\right|}\right)\left|\vec{a}\right| = \frac{1}{\left|\vec{a}\right|}\left(\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta\right) = \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|} \quad \text{[Using (6.38)]}$$

So, scalar projection of $\vec{b}$ on $\vec{a} = OC = \dfrac{\vec{a}.\vec{b}}{\left|\vec{a}\right|}$ … (6.47)

and vector projection of $\vec{b}$ on $\vec{a} = \overline{OC} = \dfrac{\vec{a}.\vec{b}}{\left|\vec{a}\right|}a$ … (6.48)

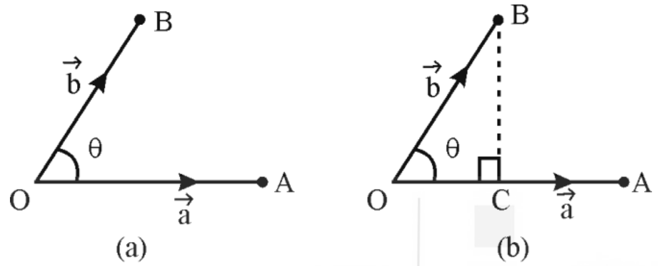

Fig. 6.7: Visualisation of (a) vectors $\vec{a}$ and $\vec{b}$ and angle θ between them (b) projection of $\vec{b}$ on $\vec{a}$

Example 7: Find the scalar projection of the vector $\vec{a} = 2\hat{i} + \hat{j} - 3k$ on the vector $\vec{b} = 2\hat{i} + 3\hat{j} - 6k$.

Solution: Using (6.31) and (6.44), we have

$\left|\vec{a}\right| = \sqrt{(2)^2 + (1)^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14},\ \left|\vec{b}\right| = \sqrt{(2)^2 + (3)^2 + (-6)^2} = \sqrt{49} = 7$

$\vec{a}.\vec{b} = (2)(2) + (1)(3) + (-3)(-6) = 4 + 3 + 18 = 25$

Using (6.47) scalar projection of $\vec{a}$ on $\vec{b}$ is given by $\dfrac{\vec{a}.\vec{b}}{\left|\vec{b}\right|} = \dfrac{25}{7}$

Now, you can try the following Self-Assessment Question.

SAQ 2

Find the vector projection of the vector $\vec{a} = 2\hat{i} + \hat{j} - 3k$ on the vector $\vec{b} = 2\hat{i} + 3\hat{j} - 6k$.

6.4 EQUATION OF A LINE IN 3-DIMENSION

To understand equation of a line in 3-dimension you should be familiar with two things:

(i) basic concepts of vector algebra discussed in Sec. 6.3, and

(ii) direction cosines and direction ratios of a line.

You have already gone through a review of basic concepts of vector algebra in the previous section. So, let us discuss what are direction cosines and direction ratios of a line.

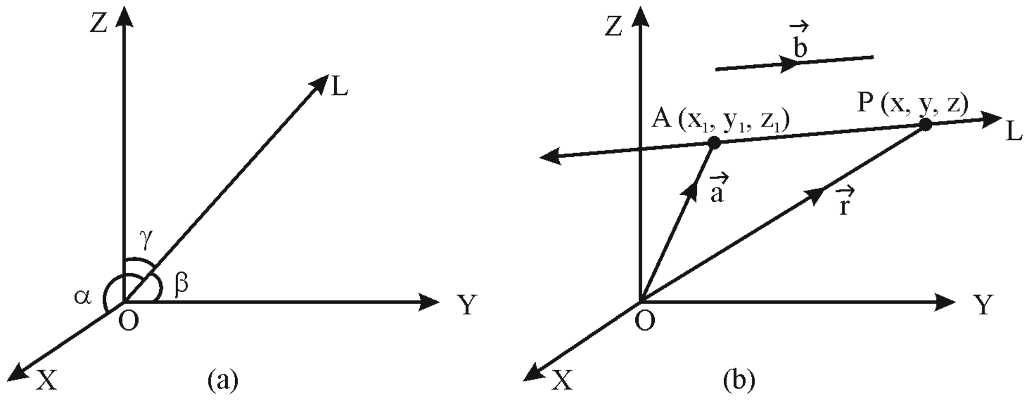

Fig. 6.8: Visualisation of (a) angles of a line with coordinate axes (b) equation of a line passing through a point and parallel to a given vector

Direction Cosines: If a line L makes angles α, β, γ with positive direction of x-axis, y-axis, z-axis respectively refer to Fig. 6.8 (a), then $\cos\alpha, \cos\beta, \cos\gamma$ are known as direction cosines of the line L. For example, x-axis is also a line which makes angles of $0^0, 90^0, 90^0$ with x-axis, y-axis, z-axis respectively, so direction cosines of x-axis are $\cos 0^0, \cos 90^0, \cos 90^0$ or $1, 0, 0$. Direction cosines of a line are generally denoted by l, m, n and written as $< l, m, n >$. So, direction cosines of x-axis can be written as $< 1, 0, 0 >$. Similarly, direction cosines of y-axis and z-axis are $< 0, 1, 0 >$ and $< 0, 0, 1 >$ respectively. If $< l, m, n >$ are the direction cosines of a line then they always satisfy the condition $l^2 + m^2 + n^2 = 1$ … (6.49)

Direction Ratios: Direction ratios are nothing but simply they are proportional to direction cosines. For example, $< \frac{2}{7}, \frac{3}{7}, \frac{6}{7} >$ are direction cosines of a line because $l^2 + m^2 + n^2 = \left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 = \frac{4+9+36}{49} = \frac{49}{49} = 1.$ If you multiply these direction cosines by 7 then they reduce to $< 2, 3, 6 >$ so these are direction ratios of the same line. Instead of 7 if you multiply by 70 then they reduce to $< 20, 30, 60 >$ these are also the direction ratios of the same line. Thus, direction ratios of a line are not unique. In fact, there are infinite number of ways to write direction ratios of the same line. You have seen that direction cosines of a line are generally denoted by l, m, n and written as $< l, m, n >$, but direction ratios of a line are generally denoted by a, b, c and written as $< a, b, c >$.

From earlier classes you also know that direction ratios of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are given by $< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$. … (6.50)

Remark 2: If direction ratios of a line are $< a, b, c >$, then the vector $a\hat{i} + b\hat{j} + ck$ will be parallel to the same line. … (6.51)

Now, we obtain equation of a line in the following two situations.

- Equation of a line passing through a point and parallel to a given line
- Equation of a line passing through two points

Let us discuss these taken one at a time.

Equation of a Line Passing Through a Point and Parallel to a Given Line

Let line L passes through a point A(x_1, y_1, z_1) and parallel to a vector $\vec{b} = a\hat{i} + b\hat{j} + ck$, refer Fig. 6.8 (b). Let P(x, y, z) be any point on line L then $\overrightarrow{AP}$ lies on the line and so is parallel to the vector $\vec{b}$. Therefore,

$$\overrightarrow{AP} = \lambda\,\vec{b} \qquad \left[\because \text{If } \vec{a} \parallel \vec{b} \text{ then } \vec{a} = \lambda\vec{b} \text{ for some real number } \lambda\right]$$

$$\Rightarrow \vec{r} - \vec{a} = \lambda\vec{b} \qquad \left[\text{Using } (6.36) \text{ in LHS}\right] \qquad \ldots (6.52)$$

$$\text{or} \quad \vec{r} = \vec{a} + \lambda\,\vec{b} \qquad \ldots (6.53)$$

This is known as **vector equation** of a line passing through a point having position vector $\vec{a}$ and parallel to the vector $\vec{b}$.

Putting values of $\vec{r}$, $\vec{a}$, $\vec{b}$ in component form in (6.52), we get

$$\Rightarrow x\hat{i} + y\hat{j} + zk - \left(x_1\hat{i} + y_1\hat{j} + z_1k\right) = \lambda\left(a\hat{i} + b\hat{j} + ck\right)$$

$$\Rightarrow (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)k = \lambda a\hat{i} + \lambda b\hat{j} + \lambda ck$$

Equating coefficients of $\hat{i}$, $\hat{j}$, k on both sides, we get

$$(x - x_1) = \lambda a, \quad (y - y_1) = \lambda b, \quad (z - z_1) = \lambda c \;\Rightarrow \frac{x - x_1}{a} = \lambda, \quad \frac{y - y_1}{b} = \lambda, \quad \frac{z - z_1}{c} = \lambda$$

$$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda \qquad \ldots (6.54)$$

$$\text{or} \quad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \ldots (6.55)$$

This is known as **cartesian equation** of a line which passes through a point A(x_1, y_1, z_1) and is parallel to a line having direction ratios $< a, b, c >$.

Any point on line (6.54) is given by

$$\Rightarrow x = x_1 + \lambda a, \quad y = y_1 + \lambda b, \quad z = z_1 + \lambda c \qquad \ldots (6.56)$$

where λ is any real number

Equation (6.56) gives us a general point on the line given by (6.54) or (6.55).

Equation of a Line Passing Through Two Points

Let line L passes through two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) refer Fig. 6.9. Let P(x, y, z) be any point on line L then $\overrightarrow{AP}$ lies on the line. Also, the vector $\overrightarrow{AB}$ lies on the same line and hence we can say that $\overrightarrow{AP} \parallel \overrightarrow{AB}$. Therefore,

$$\overrightarrow{AP} = \lambda\overrightarrow{AB} \qquad \left[\because \text{If } \vec{a} \parallel \vec{b} \text{ then } \vec{a} = \lambda\vec{b} \text{ for some real number } \lambda\right]$$

$$\Rightarrow \vec{r} - \vec{a} = \lambda\left(\vec{b} - \vec{a}\right) \qquad \left[\text{Using equation } (6.36) \text{ in LHS and RHS}\right] \qquad \ldots (6.57)$$

$$\Rightarrow \vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right) \qquad \ldots (6.58)$$

This is known as **vector equation** of a line passing through two points having position vectors $\vec{a}$ and $\vec{b}$.

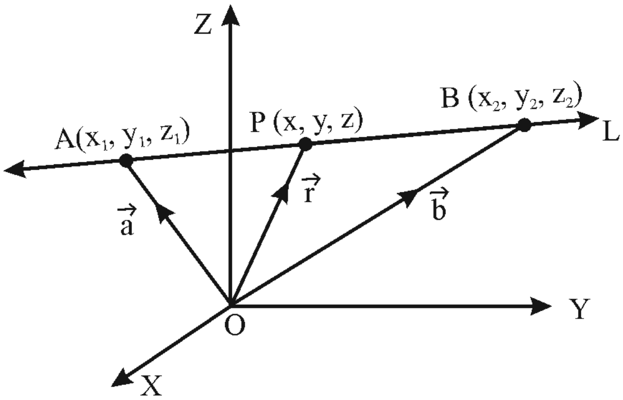

Fig. 6.9: Visualisation of equation of a line passing through two points

Putting values of $\vec{r},\ \vec{a},\ \vec{b}$ in component form in equation (6.57), we get

$$\Rightarrow x\hat{i} + y\hat{j} + zk - \left(x_1\hat{i} + y_1\hat{j} + z_1k\right) = \lambda\left(\left(x_2\hat{i} + y_2\hat{j} + z_2k\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1k\right)\right)$$

$$\Rightarrow (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)k = \lambda(x_2 - x_1)\hat{i} + \lambda(y_2 - y_1)\hat{j} + \lambda(z_2 - z_1)k$$

Equating coefficients of $\hat{i},\ \hat{j},\ k$ on both sides, we get

$$(x - x_1) = \lambda(x_2 - x_1),\quad (y - y_1) = \lambda(y_2 - y_1),\quad (z - z_1) = \lambda(z_2 - z_1)$$

$$\Rightarrow \frac{x - x_1}{x_2 - x_1} = \lambda,\quad \frac{y - y_1}{y_2 - y_1} = \lambda,\quad \frac{z - z_1}{z_2 - z_1} = \lambda$$

$$\text{or}\quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda \qquad \dots (6.59)$$

$$\text{or}\quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad \dots (6.60)$$

This is known as **cartesian equation** of a line which passes through two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2).

Any point on line (6.59) is given by

$$\Rightarrow x = x_1 + \lambda(x_2 - x_1),\quad y = y_1 + \lambda(y_2 - y_1),\quad z = z_1 + \lambda(z_2 - z_1) \qquad \dots (6.61)$$

where λ is any real number

Like equation (6.14) and in view of equation (6.59) we can say that all the points between points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) including points A and B (or on the line segment AB) are given by

$$x = x_1 + \lambda(x_2 - x_1),\ y = y_1 + \lambda(y_2 - y_1),\ z = z_1 + \lambda(z_2 - z_1) \qquad \dots (6.62)$$

where $0 \le \lambda \le 1$

$$\text{Or}\ \ x = (1-\lambda)x_1 + \lambda x_2,\ \ y = (1-\lambda)y_1 + \lambda y_2,\ \ z = (1-\lambda)z_1 + \lambda z_2, \qquad \dots (6.63)$$

where $0 \le \lambda \le 1$

This line as a set of points can be written as

$$\left\{\left((1-\lambda)x_1 + \lambda x_2,\ (1-\lambda)y_1 + \lambda y_2,\ (1-\lambda)z_1 + \lambda z_2\right) : -\infty < \lambda < \infty\right\}$$

$$\left\{(1-\lambda)(x_1, y_1, z_1) + \lambda(x_2, y_2, z_2) : -\infty < \lambda < \infty\right\} \qquad \dots (6.64)$$

Or $\left\{(1-\lambda)x + \lambda y : -\infty < \lambda < \infty\right\}$, where $x = (x_1, y_1, z_1)$, $y = (x_2, y_2, z_2)$ … (6.65)

Like (6.16) and (6.17), the line segment AB as a set of points can be written as

$$\{(1-\lambda)(x_1, y_1, z_1)+\lambda(x_2, y_2, z_2): 0 \le \lambda \le 1\} \quad \ldots (6.66)$$

Or $\{(1-\lambda)x+\lambda y: 0 \le \lambda \le 1\}$, where $x=(x_1, y_1, z_1)$, $y=(x_2, y_2, z_2)$ … (6.67)

Example 8: If a line passes through two points $A(2, 5, -4)$ and $B(1, 6, 4)$, then find:

(i) Direction ratios of the line passing through the points A and B, and also direction cosines of this line.

(ii) Find the cartesian equation of the line passing through points A and B.

Solution: Given points are $A(2, 5, -4)$ and $B(1, 6, 4)$.

(i) Direction ratios of the line passing through the points $A(2, 5, -4)$ and $B(1, 6, 4)$ are given by

$$< x_2-x_1, y_2-y_1, z_2-z_1 > = < 1-2, 6-5, 4-(-4) > = < -1, 1, 8 >$$

Direction cosine of the same line are given by

$$< \frac{-1}{\sqrt{(-1)^2+(1)^2+(8)^2}}, \frac{1}{\sqrt{(-1)^2+(1)^2+(8)^2}}, \frac{8}{\sqrt{(-1)^2+(1)^2+(8)^2}} >$$

$$= < \frac{-1}{\sqrt{66}}, \frac{1}{\sqrt{66}}, \frac{8}{\sqrt{66}} >$$

(ii) Cartesian equation of the line passing through the points $A(2, 5, -4)$ and $B(1, 6, 4)$ using (6.60) is given by

$$\frac{x-2}{1-2}=\frac{y-5}{6-5}=\frac{z-(-4)}{4-(-4)}$$

or $\dfrac{x-2}{-1}=\dfrac{y-5}{1}=\dfrac{z+4}{8}$

Now, you can try the following Self-Assessment Question.

SAQ 3

If a line passes through two points $A(2, 5, -4)$ and $B(1, 6, 4)$, then find the vector equation of the line.

6.5 EQUATION OF A PLANE

If you are working in two dimension and want to divide the cartesian plane in two parts then it can be done by drawing a line. If you are working in three dimension and want to divide the 3-dimensional space in two parts then it can be done by drawing a plane. If you are working in more than three-dimension n (say) and want to divide the n dimensional space in to two parts then it can be done by drawing a hyperplane. In this section we will discuss equation of a plane and in the next section we will discuss equation of a hyperplane. Now, a natural question that may arise in your mind is why we want to divide plane or space in two parts? You will understand its importance when you will study support vector machine (SVM) algorithm and binary classification problem in the course MST-026: Introduction to Machine Learning.

Here, we will obtain equation of a plane in the following two situations.

- When normal to the plane and perpendicular distance of the plane from the origin are given (known as normal form)
- When normal to the plane and a point on the plane are given

Let us discuss these taken one at a time.

- **When Normal to the Plane and Perpendicular Distance of the Plane from the Origin are Given (Normal Form)**

Let us first define what is normal to the plane. A vector which is perpendicular to the plane is known as normal to the plane. A unit vector in the direction of the normal is known as unit normal vector.

Let π be the given plane and d be the perpendicular distance of the plane from origin. Refer Fig. 6.10 (a). Draw perpendicular from origin to the plane and let it intersects the plane π at the point N then $\overrightarrow{ON}$ will be normal to the plane. Let n be the unit normal vector along the normal $\overrightarrow{ON}$ so $\overrightarrow{ON} = dn$. Let P be any point on the plane and $\vec{r}$ be the position vector of P then $\overrightarrow{NP}$ will lie on the plane therefore

$\overrightarrow{NP} \perp \overrightarrow{ON}$ $\quad \left[\because \overrightarrow{ON} \text{ is normal to the plane}\right]$

$\overrightarrow{NP}.\overrightarrow{ON} = 0$ $\quad \left[\because \text{Dot product of two perpendicular vectors is zero}\right]$

$\Rightarrow (\vec{r} - dn).(dn) = 0$ $\quad \left[\text{Using equation (6.36) for } \overrightarrow{NP}\right]$

$\Rightarrow (\vec{r} - dn).n = 0$ $\quad \left[\because d \neq 0 \text{ so dividing by d on both sides}\right]$

$\Rightarrow \vec{r}.n - d = 0$ $\quad \left[\because n.n = 1\right]$

$$\text{or } \vec{r}.n = d \qquad \ldots (6.68)$$

This is known as **vector equation** of the plane which is at a perpendicular distance of d unit from the origin and have n as unit normal vector to the plane.

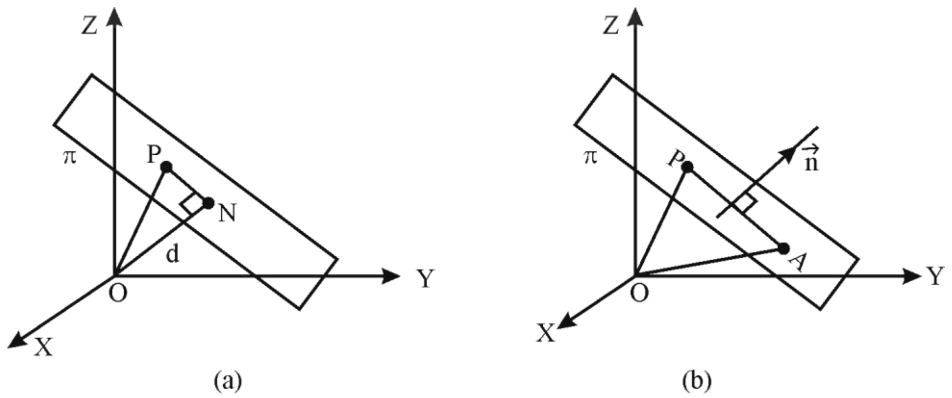

Fig. 6.10: Visualisation of equation of a plane in (a) normal form (b) passing through a point and normal to the plane is given

If $n = l\hat{i} + m\hat{j} + nk$ then putting values of $\vec{r}$ and n in equation (6.68), we get

$$\Rightarrow (x\hat{i} + y\hat{j} + zk).(l\hat{i} + m\hat{j} + nk) = d$$

$$\Rightarrow lx + my + nz = d \qquad \ldots (6.69)$$

This is known as **cartesian equation** of the plane which is at a perpendicular distance of d unit from the origin and direction cosines of the line normal to the plane are $< l, m, n >$.

Remark 3: It is important to note that direction cosines of the normal to the plane are coefficients of x, y, z in equation (6.69). It did not happen by chance. Remember this important fact it is true for every equation of a plane.

Normal to the Plane and a Point on the Plane are Given

Let π be the given plane and $A(x_1, y_1, z_1)$ be the point on the plane. Let $\vec{n}$ be normal to the plane. Let $\vec{r}$ be the position vector of any point P on the plane and $\vec{a}$ be the position vector of the point A. Both the points A and P lie on the plane so normal vector $\vec{n}$ will be perpendicular to the vector $\overrightarrow{AP}$.

$\therefore \overrightarrow{AP} \perp \vec{n}$ $\quad \left[\because \vec{n} \text{ is normal to the plane}\right]$

$\overrightarrow{AP}.\vec{n} = 0$ $\quad \left[\because \text{ if } \vec{a} \perp \vec{b} \text{ then } \vec{a}.\vec{b} = 0\right]$

$\Rightarrow \left(\vec{r} - \vec{a}\right).\vec{n} = 0$ $\quad \left[\because \overrightarrow{AP} = \vec{r} - \vec{a}\right]$ .. (6.70)

This is known as **vector equation** of the plane passing through a point having position vector $\vec{a}$ and $\vec{n}$ is the normal to the plane.

If $\vec{n} = a\hat{i} + b\hat{j} + ck$ then putting values of $\vec{r}$ and $\vec{a}$ in equation (6.70), we get

$$\Rightarrow \left(\left(x\hat{i} + y\hat{j} + zk\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1k\right)\right).\left(a\hat{i} + b\hat{j} + ck\right) = 0$$

$$\Rightarrow \left((x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)k\right).\left(a\hat{i} + b\hat{j} + ck\right) = 0$$

$$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad \ldots (6.71)$$

This is known as **cartesian equation** of the plane passing through a point $A(x_1, y_1, z_1)$ and having direction ratios of the normal to the plane as $< a, b, c >$.

Note that after simplification equation (6.71) can be written as

$$ax + by + cx + d = 0, \text{ where } d = -(ax_1 + by_1 + cz_1) \qquad \ldots (6.72)$$

Example 9: Find the vector equation of a plane which passes through the point A(2, 5, 4) and perpendicular to the line BC where B(1, 4, 6) and C(7, 2, 10).

Solution: Direction ratios of line BC are

$< x_2 - x_1, y_2 - y_1, z_2 - z_1 > = < 7 - 1, 2 - 4, 10 - 6 > = < 6, -2, 4 > = < 3, -1, 2 >$

Since plane is perpendicular to the line BC so normal to the plane will be parallel to the line BC and so normal vector to the plane is

$$\vec{n} = 3\hat{i} - \hat{j} + 2k$$

Also, position vector of point A is $\vec{a} = 2\hat{i} + 5\hat{j} + 4k$

Now, vector equation of the plane passing through a point having position vector $\vec{a}$ and $\vec{n}$ as the normal to the plane is given by

$\left(\vec{r} - \vec{a}\right).\vec{n} = 0$ $\quad \left[\text{Using (6.70)}\right]$

$$\Rightarrow \left(\left(x\hat{i}+y\hat{j}+zk\right)-\left(2\hat{i}+5\hat{j}+4k\right)\right).\left(3\hat{i}-\hat{j}+2k\right)=0 \qquad \ldots (6.73)$$

Which is required vector equation of the plane.

Now, you can try the following Self-Assessment Question.

SAQ 4

Find the cartesian form of the plane given in Example 9.

6.6 EQUATION OF HYPERPLANE

You can geometrically visualise:

- a line, For example, real line.
- two perpendicular lines. For example, x-axis and y-axis, and
- three mutually perpendicular lines. For example, x-axis, y-axis and z-axis.

But you cannot geometrically visualise four or more mutually perpendicular lines. But theory of three mutually perpendicular axes can be extended to any dimension n (say). For example, direction cosines of x-axis, y-axis and z-axis in 3-dimension are $<1, 0, 0>, <0, 1, 0>, <0, 0, 1>$ respectively. This idea of direction cosines says that we can write direction cosines of 4, 5, 6, 7, … mutually perpendicular lines. For example, direction cosines of four lines $<1, 0, 0, 0>, <0, 1, 0, 0>, <0, 0, 1, 0>, <0, 0, 0, 1>$ are such that they are mutually perpendicular which can be seen by taking their dot product as follows:

$$(1, 0, 0, 0).(0, 1, 0, 0)=(1)(0)+(0)(1)+(0)(0)+(0)(0)=0$$
$$(1, 0, 0, 0).(0, 0, 1, 0)=(1)(0)+(0)(0)+(0)(1)+(0)(0)=0$$
$$(1, 0, 0, 0).(0, 0, 0, 1)=(1)(0)+(0)(0)+(0)(0)+(0)(1)=0$$
$$(0, 1, 0, 0).(0, 0, 1, 0)=(0)(0)+(1)(0)+(0)(1)+(0)(0)=0$$
$$(0, 1, 0, 0).(0, 0, 0, 1)=(0)(0)+(1)(0)+(0)(0)+(0)(1)=0$$
$$(0, 0, 1, 0).(0, 0, 0, 1)=(0)(0)+(0)(0)+(1)(0)+(0)(1)=0$$

So, if we denote unit vectors along these four directions by e_1, e_2, e_3, e_4 respectively then because their dot products are zero so using equation (6.41) we can say that there exists four directions which are mutually perpendicular. So, in general, if we denote the unit vectors $e_1, e_2, e_3, e_4, \ldots, e_n$ along the lines having direction cosines given as follows:

$$<\underbrace{1, 0, 0, 0, \ldots, 0}_{\text{n positions}}>, <\underbrace{0, 1, 0, 0, \ldots, 0}_{\text{n positions}}>, <\underbrace{0, 0, 1, 0, \ldots, 0}_{\text{n positions}}>, \ldots, <\underbrace{0, 0, 0, 0, \ldots, 1}_{\text{n positions}}>,$$

then we can say that there exist n directions which are mutually perpendicular.

Now, you have gotten the idea why theoretically there exists four or more mutually perpendicular directions. In 2-dimensional geometry the geometric entity of one less dimension, i.e., 2 – 1 = 1 dimension is known as straight line which we have discussed in Sec. 6.2. In 3-dimensional geometry the geometric entity of one less dimension, i.e., 3 – 1 = 2 dimension is known as plane which we have discussed in Sec. 6.5. In n-dimensional geometry the geometric entity of one less dimension, i.e., n – 1 dimension is known as **hyperplane** which we will discuss in this section.

Let us recall general equation of a line given by (6.21) which is

$$ax + by + c = 0$$

General equation of the plane given by (6.72) is

$$ax + by + cx + d = 0$$

With the notations a, b, c, d and x, y, z it will be difficult to write equation of a plane in n dimension. So, to write equation of a plane in n dimension which is called hyperplane we have to change these notations as follows.

Equation (6.21) of a line in 2-dimension can be written as

$$\omega_1 x_1 + \omega_2 x_2 + \omega_0 = 0, \omega_i \in \mathbb{R}, i = 0, 1, 2 \quad \ldots (6.74)$$

where not all $\omega_i = 0, i = 1, 2$

Equation (6.72) of a plane in 3-dimension can be written as

$$\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_0 = 0, \omega_i \in \mathbb{R}, \ i = 0, 1, 2, 3 \quad \ldots (6.75)$$

where not all $\omega_i = 0, \ i = 1, 2, 3$

Similarly, equation of a plane in n-dimension called hyperplane can be written as

$$\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \ldots + \omega_n x_n + \omega_0 = 0, \omega_i \in \mathbb{R}, \ i = 0, 1, 2, 3, \ldots, n \quad \ldots (6.76)$$

where not all $\omega_i = 0, \ i = 1, 2, 3, \ldots, n$

After extending theory of 3-dimension and in view of Remark 3 we can say that direction ratios of the normal to the hyperplane given by (6.76) are $< \omega_1, \omega_2, \omega_3, \ldots, \omega_n >$.

In view of the notations mentioned in equations (6.34) and (6.35) under Remark 1 and using rule of matrix multiplication, equations (6.74), (6.75) and (6.76) all can be written as

$$\omega^T x + b = 0, \quad \ldots (6.77)$$

where

for equation (6.74), we have

$$\omega^T = [\omega_1 \quad \omega_2], \ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \quad x_2]^T, \ b \in \mathbb{R} \quad \ldots (6.78)$$

for equation (6.75), we have

$$\omega^T = [\omega_1 \quad \omega_2 \quad \omega_3], \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1 \quad x_2 \quad x_3]^T, \ b \in \mathbb{R} \quad \ldots (6.79)$$

for equation (6.76), we have

$$\omega^T = [\omega_1 \quad \omega_2 \quad \omega_3 \quad \ldots \quad \omega_n], \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_n]^T, \; b \in \mathbb{R} \quad \ldots (6.80)$$

Remark 4: All the results of the plane discussed in Sec. 6.5 can be extended to n-dimensional hyperplane. For example, if $\omega_0 = 0$ then the point (0, 0, 0, …, 0) will satisfy hyperplane given by (6.76) and so it will pass through the origin. Also, equation of the hyperplane in vector form which passes through a point having position vector $\vec{a}$ and $\vec{n}$ is the normal to the hyperplane is given by

$$(\vec{r} - \vec{a}).\vec{n} = 0 \quad \ldots (6.81)$$

where

$$\vec{r} = x_1 e_1 + x_2 e_2 + x_3 e_3 + \ldots + x_n e_n, \; \vec{a} = a_1 e_1 + a_2 e_2 + a_3 e_3 + \ldots + a_n e_n, \quad \ldots (6.82)$$

$$\vec{n} = n_1 e_1 + n_2 e_2 + n_3 e_3 + \ldots + n_n e_n \quad \ldots (6.83)$$

$$\text{where } \vec{e}_i = (0, 0, \ldots, 0, \underset{i^{th}\text{ position}}{1}, 0, 0, \ldots, 0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ position} \quad \ldots (6.84)$$

Example 10: Find the equation of the hyperplane in $\mathbb{R}^5$ in cartesian form (or scalar form) which passes through the point $A(2, 1, -1, 3, -2)$ and perpendicular to the direction having direction ratios $< 3, 1, 2, 5, 6 >$.

Solution: We know that equation of a hyperplane having normal vector ω is given by

$$\omega^T x + b = 0$$

Here $\omega^T = [3 \quad 1 \quad 2 \quad 5 \quad 6]$, $x = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T$, so we have

$$[3 \quad 1 \quad 2 \quad 5 \quad 6] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + b = 0 \Rightarrow 3x_1 + x_2 + 2x_3 + 5x_4 + 6x_5 + b = 0$$

But it passes through the point $A(2, 1, -1, 3, -2)$, we have

$$3(2) + 1 + 2(-1) + 5(3) + 6(-2) + b = 0 \Rightarrow 6 + 1 - 2 + 15 - 12 + b = 0 \Rightarrow b = -8$$

Putting value of b, we get

$$3x_1 + x_2 + 2x_3 + 5x_4 + 6x_5 - 8 = 0$$

6.7 INEQUATION IN ONE AND TWO VARIABLES

You know that when two expressions are equated to each other than resulting relation is called equation in mathematics. For example, $4x + 5 = 7 - x$ is an equation in one variable x and $3x + y = 4 + 3y - 2x$ is also an equation but it is in two variables x and y. Now, in place of equality sign $(=)$ if we have any of the four $\leq, <, \geq,$ or $>$ inequality signs then resulting relation is called an **inequation**.

For example, (i) $2x < 5$ (ii) $2x - (3 - x) \leq 8$ (iii) $3x + 2 > 8 - x$ (iv) $3x \geq 2 + x$ all are examples of inequations.

Like equations you can also solve inequations for x. But there are some rules which you should keep in mind while solving inequations. If a, b, c, d are real numbers then following are some rules which you should be familiar before start working with inequalities.

- If $a < b,$ then $a + c < b + c.$ For example, $2 < 5 \Rightarrow 2 + 3 < 5 + 3.$
- If $a < b,$ then $a - c < b - c.$ For example, $2 < 5 \Rightarrow 2 - 3 < 5 - 3.$
- If $c > 0$ and $a < b,$ then $ac < bc.$ … (6.85)
  For example, $2 < 5 \Rightarrow 2(3) < 5(3),$ i.e., $6 < 15$
- If $c < 0$ and $a < b,$ then $ac > bc$ … (6.86)
  For example, $2 < 5 \Rightarrow 2(-3) > 5(-3),$ i.e., $-6 > -15.$
- If $a > 0,$ then
  $ax < b \Rightarrow x < \frac{b}{a}, ax \leq b \Rightarrow x \leq \frac{b}{a}, ax > b \Rightarrow x > \frac{b}{a}, ax \geq b \Rightarrow x \geq \frac{b}{a}$ … (6.87)
  For example, $6 \times 4 < 30 \Rightarrow 4 < \frac{30}{6},$ i.e., $4 < 5.$
- If $a < 0,$ then
  $ax < b \Rightarrow x > \frac{b}{a}, ax \leq b \Rightarrow x \geq \frac{b}{a}, ax > b \Rightarrow x < \frac{b}{a}, ax \geq b \Rightarrow x \leq \frac{b}{a}$ … (6.88)
  For example, $(-6)4 < 30 \Rightarrow 4 > \frac{30}{-6},$ i.e., $4 > -5.$
- If $a > 0 \Rightarrow \frac{1}{a} > 0.$ For example, $5 > 0 \Rightarrow \frac{1}{5} = 0.2 > 0.$
- If $a < 0 \Rightarrow \frac{1}{a} < 0.$ For example, $-5 < 0 \Rightarrow \frac{1}{-5} = -0.2 < 0.$
- If $a > 0, b > 0$ and $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}.$ … (6.89)

  For example, $5 > 0, 2 > 0.$ So, $5 > 2 \Rightarrow \frac{1}{5} < \frac{1}{2},$ i.e., $0.2 < 0.5.$
- If $a < 0, b < 0$ and $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}.$ … (6.90)

  For example, $-5 < 0, -2 < 0.$ So, $-2 > -5 \Rightarrow \frac{1}{-2} < \frac{1}{-5},$ i.e., $-0.5 < -0.2.$
- If $a < b, c < d$ then $a + c < b + d.$ … (6.91)
  For example, $2 < 5,$ and $6 < 7 \Rightarrow 2 + 6 < 5 + 7,$ i.e., $8 < 12.$

Further, you have already studied modulus function in Unit 1 of this course. In some courses you will also need to deal with both modulus function and its appearance in inequalities. If a is a real number, then following are some rules which you should keep in mind while working with inequalities and modulus function simultaneously.

- $|x| = a \Leftrightarrow x = \pm a.$ For example, $|x| = 2 \Leftrightarrow x = \pm 2.$ … (6.92)
- $|x| \le a \Leftrightarrow -a \le x \le a.$ For example, $|x| \le 2 \Leftrightarrow -2 \le x \le 2.$ … (6.93)
- $|x| \ge a \Leftrightarrow x \ge a$ or $x \le -a$ For example, $|x| \ge 2 \Leftrightarrow x \ge 2$ or $x \le -2$ … (6.94)
- $|x| < a \Leftrightarrow -a < x < a.$ For example, $|x| < 2 \Leftrightarrow -2 < x < 2.$ … (6.95)
- $|x| > a \Leftrightarrow x > a$ or $x < -a.$ For example, $|x| > 2 \Leftrightarrow x > 2$ or $x < -2.$ … (6.96)

Some inequalities when square of variable is involved are given as follows:

- If $a > 0$ then $x^2 \le a \Leftrightarrow -\sqrt{a} \le x \le \sqrt{a}.$ … (6.97)
  For example, $|x| \le 2 \Leftrightarrow -\sqrt{2} \le x \le \sqrt{2}.$
- If $a > 0$ then $x^2 \ge a \Leftrightarrow x \ge \sqrt{a}$ or $x \le -\sqrt{a}$ … (6.98)
  For example, $x^2 \ge 2 \Leftrightarrow x \ge \sqrt{2}$ or $x \le -\sqrt{2}$

Let us now do some examples to apply these inequalities.

Example 11: Solve for x: $3x \le 15$

Solution: $3x \le 15 \Rightarrow x \le \frac{15}{3} \Rightarrow x \le 5.$ $\quad$ $\left[\text{Using } (6.87)\right]$

So, required solution set is $(-\infty, 5].$

Example 12: Solve for x: $-3x \le 15$

Solution: $-3x \le 15 \Rightarrow x \ge \frac{15}{-3}$ $\left[\begin{array}{l}\text{Using } (6.88) \\ \text{as } -3 < 0, \text{ So, inequality sign has reversed}\end{array}\right]$

$\Rightarrow x \ge -5.$ So, required solution set is $[-5, \infty).$

Example 13: Solve for x: $8 < -2(3x - 2) \le 20$

Solution: $8 < -2(3x - 2) \le 20$

$$\Rightarrow \frac{8}{-2} > 3x - 2 \ge \frac{20}{-2} \quad [\because -2 < 0, \text{ So, both inequalities signs have reversed}]$$

$$\Rightarrow -4 > 3x - 2 \ge -10 \Rightarrow -4 + 2 > 3x \ge -10 + 2 \Rightarrow -2 > 3x \ge -8$$

$$\Rightarrow \frac{-2}{3} > x \ge \frac{-8}{3} \qquad \left[\text{Using } (6.87)\right]$$

So, required solution set is $\left[-\frac{8}{3}, -\frac{2}{3}\right).$

Example 14: Write solution set of: $3x + 2y \le 6.$ Also, show its solution graphically with shaded region.

Solution: $3x + 2y \le 6$ … (6.99)

Solution set of the inequation (6.99) is given by $\{(x, y) \in \mathbb{R}^2 : 3x + 2y \le 6\}$.

Using (6.87), inequation (6.99) can be written as

$$\frac{x}{2} + \frac{y}{3} \le 1 \qquad \dots (6.100)$$

Equation corresponding to inequation (6.100) is

$$\frac{x}{2} + \frac{y}{3} = 1 \qquad \dots (6.101)$$

which is equation of a line in intercept form having 2 and 3 as its x and y intercepts respectively, refer equation (6.18). So, graph of the equation (6.101) not inequation (6.100) is given in Fig. 6.11 (a).

Now, to show solution of inequation (6.100) graphically with shaded region, we have to check on which side (above or below) of line given by (6.100) its solution lies. This can be decided by putting coordinates of any point in (6.100) which is not on the line given by (6.101). If a line does not pass through the origin in that case to save calculation work, we put coordinates of origin in the inequation. So, putting x = 0, y = 0 in inequation (6.99), we get $0 + 0 \le 6$ which is true. Hence, solution set of inequation (6.99) or (6.100) contains origin. So, graphical solution of inequation (6.99) is all the region below the line given by (6.101) including the line itself and have been shaded refer to Fig. 6.11 (c).

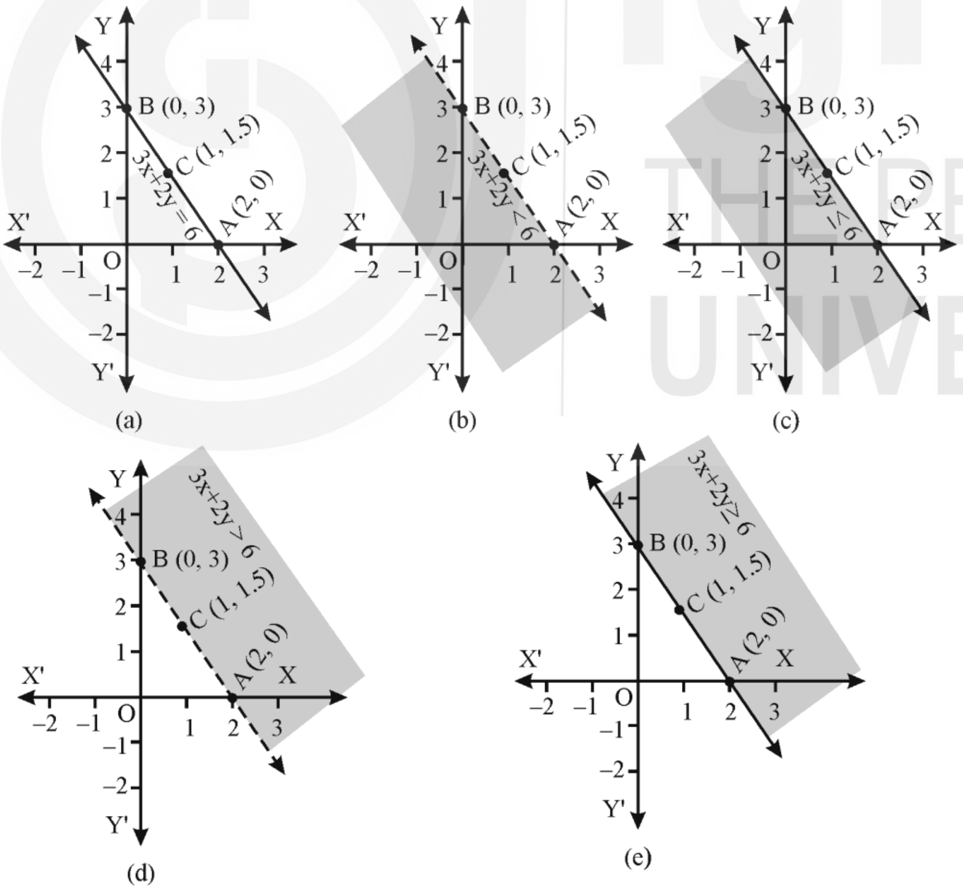

Fig. 6.11: Visualisation of solution of (a) line given by (6.101) (b) the inequation 3x + 2y < 6 by shaded region (c) (6.99) or (6.100) by shaded region (d) 3x + 2y > 6 by shaded region (e) 3x + 2y ≥ 6 by shaded region

Remark 5: Line given by (6.101) divides the plane in two parts known as **half planes**. One half plane is the shaded region and the other half plane is unshaded region. So, all inequations of the types $ax + b \geq c$, $ax + b > c$, $ax + b \leq c$, $ax + b < c$, where $a, b, c \in \mathbb{R}$ represents half planes. If we have $\leq$ or $\geq$ signs then straight line which divides xy-plane into two halves is included in the shaded region refer Fig. 6.11 (c) and (e). While in the case we have $<$ or $>$ signs then straight line which divides xy-plane into two halves is included in the unshaded region not in shaded region refer Fig. 6.11 (b) and (d). Keep this in mind regarding notion of half planes.

Now, you can try the following Self-Assessment Question.

SAQ 5

(a) Solve for x: $5 - 3x \geq 17$

(b) Solve the given inequations graphically and represent the solution set by shaded region: $3x + 2y \geq 6$, $x + 3y \leq 9$, $x - y \leq 1$.

6.8 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- **Increment**: Let an object changes it position from the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$ then the net changes in the coordinates of point A are known as increments. Increment in the direction of x-axis is known as **run** and increment in the direction of y-axis is known as **rise**.
- **Slope**: If we take any two points on a line then the ratio of Rise and Run remains constant. This constant ratio is known as **slope** of the line.
- Equation of the line in **point slope form** is: $y - y_1 = m(x - x_1)$
- Equation of the line in **two point form** is: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
- Line as a set of points $\{(1-\lambda)(x_1, y_1) + \lambda(x_2, y_2) : -\infty < \lambda < \infty\}$
- Line segment as a set of points $\{(1-\lambda)(x_1, y_1) + \lambda(x_2, y_2) : 0 \leq \lambda \leq 1\}$
- Equation of the line in **intercept form** is: $\frac{x}{a} + \frac{y}{b} = 1$
- Equation of the line in **slope intercept form** is: $y = mx + c$
- Equation of the line **through origin** is: $y = mx$
- Equation of the line in **general form** is: $ax + by + c = 0$ or $\omega_1 x_1 + \omega_2 x_2 + \omega_0 = 0$
- **Perpendicular Distance of a Point from a Line**: $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$
- Slopes of two **parallel lines are equal**.
- **Product** of slopes of two **perpendicular lines is** -1.
- Equation of any particular line **parallel to x-axis** is given by y = k.
- Equation of any particular line **parallel to y-axis** is given by x = h.

- **Position vector of a point** $A(a_1, b_1, c_1)$ is written as $\overrightarrow{OA} = a_1\hat{i} + b_1\hat{j} + c_1 k$
- **Magnitude** of vector $\overrightarrow{OA}$ is $\left|\overrightarrow{OA}\right| = \sqrt{a_1^2 + b_1^2 + c_1^2}$
- **Vector Joining Two Points**: If A and B are two points and O is the origin then

$\overrightarrow{AB} =$ Position vector of point B $-$ Position vector of point A $= \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

- **Triangle law of addition** is $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- **Dot Product of two Vectors** $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as: $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- If $\vec{a} \perp \vec{b}$ then $\vec{a}.\vec{b} = 0$.
- Scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$
- Vector projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}a$
- If $< l, m, n >$ are the **direction cosines** of a line then $l^2 + m^2 + n^2 = 1$.
- **Direction Ratios** are proportional to direction cosines.
- Equation of a line **passing through a point** having position vector $\vec{a}$ and **parallel to a vector** $\vec{b}$ is

  $\vec{r} - \vec{a} = \lambda\vec{b}$ or $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

- **Equation of a line passing through two points** having position vectors $\vec{a}$ and $\vec{b}$ **is** $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$ or $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$
- **Equation of a plane in normal form is** $\vec{r}.n = d$ or $lx + my + nz = d$
- **Equation of a plane when normal to the plane and a point on the plane are given is** $\left(\vec{r} - \vec{a}\right).\vec{n} = 0$ or $a\left(x - x_1\right) + b\left(y - y_1\right) + c\left(z - z_1\right) = 0$
- Equation of **hyperplane** is $\omega^T x + b = 0,$ where

$$\omega^T = \left[\omega_1 \quad \omega_2 \quad \omega_3 \quad \dots \quad \omega_n\right], \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \left[x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n\right]^T, \; b \in \mathbb{R}$$

6.9 TERMINAL QUESTIONS

1. Find the slope of a line which is perpendicular to the line $2x - 3y + 7 = 0$.
2. Write the direction ratios of the normal to the hyperplane:
   $2x_1 - 3x_2 + 7x_3 + 5x_4 + 4x_5 - 9x_6 + 10 = 0.$
   Also, write direction cosines of the normal to this hyperplane.

6.10 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. (a) We know that equation of a line passing through a point A(x_1, y_1) and having slope m is given by

$$y - y_1 = m(x - x_1)$$

$\therefore$ required equation is given by

$$y - 3 = \frac{7}{8}(x-2) \Rightarrow 8y - 24 = 7x - 14 \Rightarrow 7x - 8y + 10 = 0$$

(b) If d is the perpendicular distance of the point $A(2, -7)$ from the line $3x + 4y + 9 = 0,$ then

$$d = \frac{|3(2) + 4(-7) + 9|}{\sqrt{(3)^2 + (4)^2}} = \frac{|6 - 28 + 9|}{\sqrt{25}} = \frac{|-13|}{5} = \frac{13}{5}$$

(c) Slope of the line $5x + 6y + 8 = 0$ is $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\dfrac{5}{6}.$ We know that slopes of two parallel lines are equal. So, slope of any line parallel to the given line is $-5/6$.

2. In Example 7 we have already obtained $|\vec{b}| = 7,\ \vec{a}.\vec{b} = 25.$

Also, unit vector along $\vec{b}$ is given by $b = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{2\hat{i} + 3\hat{j} - 6k}{7}.$

So, using (6.48) vector projection of $\vec{a}$ on $\vec{b}$ is given by

$$\frac{\vec{a}.\vec{b}}{|\vec{b}|} b = \frac{25}{7}\left(\frac{2\hat{i} + 3\hat{j} - 6k}{7}\right) = \frac{25}{49}\left(2\hat{i} + 3\hat{j} - 6k\right)$$

3. Using equation (6.58) required vector equation of the line is given by

$$\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right) = 2\hat{i} + 5\hat{j} - 4k + \lambda\left(\left(\hat{i} + 6\hat{j} + 4k\right) - \left(2\hat{i} + 5\hat{j} - 4k\right)\right)$$

$$= 2\hat{i} + 5\hat{j} - 4k + \lambda\left(-\hat{i} + \hat{j} + 8k\right), \text{ where } \lambda \in \mathbb{R}$$

4. Using (6.71) required equation of the plane is given by

$$3(x-2) - (y-5) + 2(z-4) = 0$$

$$3x - y + 2z - 9 = 0$$

5. (a) $5 - 3x \geq 17 \Rightarrow -3x \geq 17 - 5 \Rightarrow -3x \geq 12$

$$\Rightarrow x \leq \frac{12}{-3} \quad [\because -3 < 0, \text{ So, inequality sign has changed}]$$

$$\Rightarrow x \leq -4$$

So, required solution set is $(-\infty, -4]$.

(b) Given inequations are

$3x + 2y \geq 6$ ... (6.102) $x + 3y \leq 9$ ... (6.103) $x - y \leq 1$ ...(6.104)

Equations corresponding to inequations (6.102) to (6.104) are

$3x + 2y = 6$ ... (6.105) $x + 3y = 9$ ... (6.106) $x - y = 1$ ...(6.107)

Intercept forms of equations (6.105) to (6.107) are given by

$\frac{x}{2} + \frac{y}{3} = 1$ ... (6.108) $\frac{x}{9} + \frac{y}{3} = 1$ ... (6.109) $\frac{x}{1} + \frac{y}{-1} = 1$ ...(6.110)

So, graphs of the equations (6.108) to (6.110) not inequations (6.102) to (6.104) are given in Fig. 6.12 (a).

Like Example 14 to show solution of inequations (6.102) to (6.104) graphically with shaded region, we have to check for each inequation on which side (above or below) of line its solution lies. None of the three lines passes through origin so like Example 14 we put coordinates of origin in each inequation given by (6.102) to (6.104).

From inequation (6.102), we get $0 + 0 \geq 6$ which is false, so solution set of inequation (6.102) does not contain origin. Similarly, from inequations (6.103) and (6.104), we get $0 + 0 \leq 9$ and $0 - 0 \leq 1$ both are true, so solution sets of inequations (6.103) and (6.104) contain origin. Hence, the solution set which is common to all the three inequations (6.102) to (6.104) is shown by shaded region in Fig. 6.12 (b).

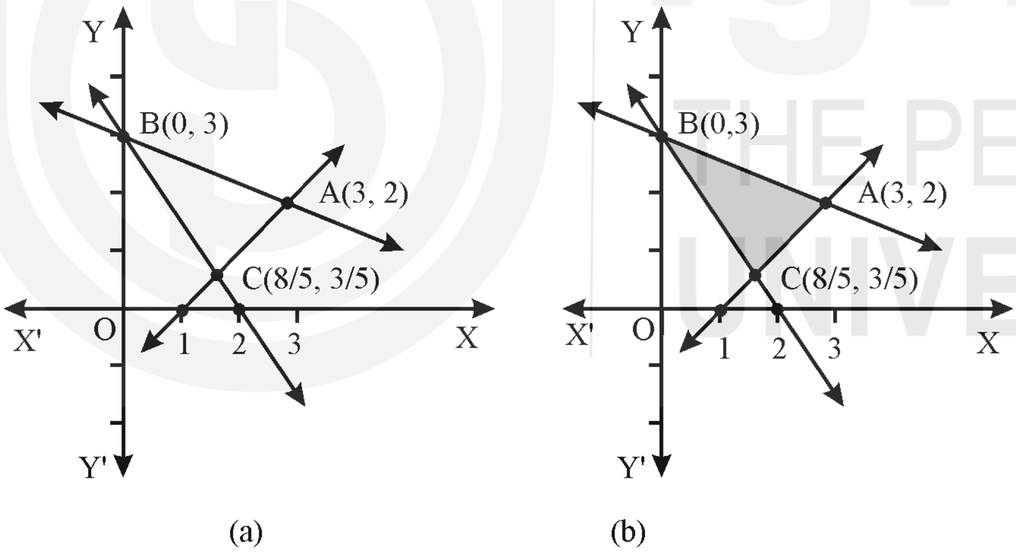

Fig. 6.12: Visualisation of (a) three lines given by (6.108) to (6.110) (b) solution of the inequations (6.102) to (6.104) by shaded region

Terminal Questions

1. Slope of the line $2x - 3y + 7 = 0.$ is $m_1 = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{2}{-3} = \frac{2}{3}$. If m_2 be the slope of a line perpendicular to the given line then we know that $m_1 m_2 = -1 \Rightarrow (2/3)m_2 = -1 \Rightarrow m_2 = -3/2$.

2. We know that coefficients of $x_1, x_2, x_3, x_4, x_5, x_6$ in the equation of the hyperplane are the direction ratios of the normal to the hyperplane.

Hence, direction ratios of the normal to the given hyperplane are: $< 2, -3, 7, 5, 4, -9 >$.

We know that if $< a_1, a_2, a_3, \cdots, a_n >$ are the direction ratios of the normal to a hyperplane then direction cosines of the normal are given by

$$< \frac{a_1}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}, \frac{a_2}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}, \cdots, \frac{a_n}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}} >$$

Hence, required direction cosines are given by

$$< \frac{2}{2\sqrt{46}}, \frac{-3}{2\sqrt{46}}, \frac{7}{2\sqrt{46}}, \frac{5}{2\sqrt{46}}, \frac{4}{2\sqrt{46}}, \frac{-9}{2\sqrt{46}} >$$

$$\text{or } < \frac{1}{\sqrt{46}}, \frac{-3}{2\sqrt{46}}, \frac{7}{2\sqrt{46}}, \frac{5}{2\sqrt{46}}, \frac{2}{\sqrt{46}}, \frac{-9}{2\sqrt{46}} >$$

Since

$$\begin{aligned}\sqrt{2^2 + (-3)^2 + 7^2 + 5^2 + 4^2 + (-9)^2} &= \sqrt{4 + 9 + 49 + 25 + 16 + 81} \\ &= \sqrt{184} = \sqrt{2 \times 2 \times 46} = 2\sqrt{46}\end{aligned}$$