# UNIT 8

# TRANSFORMATION OF UNIVARIATE AND BIVARIATE RANDOM VARIABLES

## Structure

8.1 Introduction
Expected Learning Outcomes
8.2 Role of Jacobian in Transformations
8.3 Transformation of Univariate and Bivariate Random Variables
8.4 Summary
8.5 Terminal Questions
8.6 Solutions/Answers

## 8.1 INTRODUCTION

In the previous unit, you have studied a function known as moment generating function. With the help of moment generating function, we can obtain many measures of the distribution. Sometimes other than summary statistics of a distribution, we are interested in probability distribution (PMF or PDF or CDF) of some function of random variable(s). To get such probability distributions, we will need the idea of Jacobian. So, Jacobian is discussed in Sec. 8.2 and transformations of univariate and bivariate random variable(s) are discussed in Sec. 8.3.

What we have discussed in this unit is summarised in Sec. 8.4. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit some more questions based on the entire unit are given in Sec. 8.5 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 8.6.

In the next block, you will study some well-known discrete probability distributions.

### Expected Learning Outcomes

After completing this unit, you should be able to:

- find Jacobian of a given transformation; and
- obtain PMF or PDF or CDF of the function of random variable or variables from given PMF or PDF or CDF of random variable or variables.

## 8.2 ROLE OF JACOBIAN IN TRANSFORMATIONS

In Sec. 9.4 of Unit 9 of the course MST-011, you saw that if integrand is 1 then double integral gives area of the region of integration. We will use that idea to explain the role of Jacobian in double integral. To explain it, suppose we want to find area of the square OABC using double integral where coordinates of the vertices of the square OABC are O(0, 0), A(1, 1), B(0, 2) and C( – 1, 1). Square OABC is visualised in Fig. 8.1 (a). Without using double integral, we know that each side of this square is $\sqrt{2}$ so its area will be

$$(\text{side})^2 = \left(\sqrt{2}\right)^2 = 2 \text{ units}^2. \quad \ldots (8.1)$$

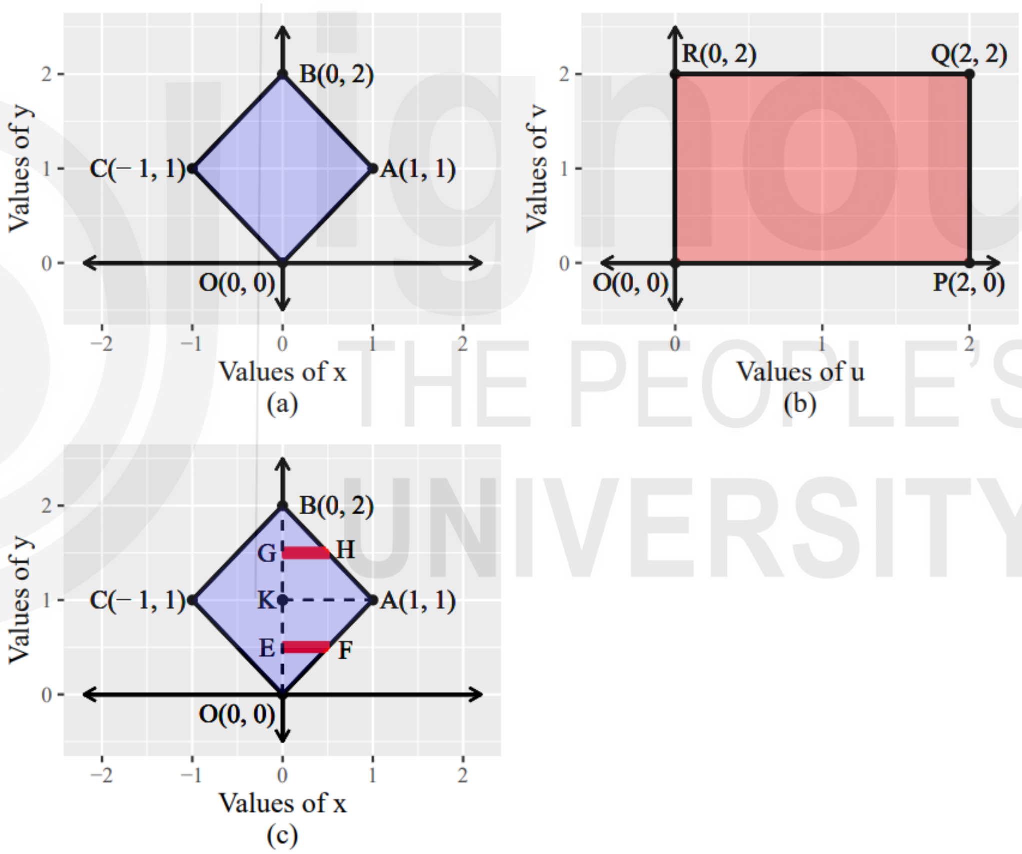


**Fig. 8.1: Visualisation of the given square in (a) xy-plane (b) uv-plane which includes the effect of the transformation (c) in xy-plane with two horizontal strips EF and GH**

But we will also find it using double integral to convey the role of Jacobian. We will find area using double integral one before the transformation and one after making a transformation.

### Finding Area of Square OABC Using Double Integral Directly

To obtain limits of integration, first, we have to find equations of the straight lines OA and AB using two-point form (you may refer to 6.7 of the course MST-011)

Equation of OA: $y-0=\dfrac{1-0}{1-0}(x-0)\Rightarrow y-x=0$ … (8.2)

Equation of AB: $y-1=\dfrac{2-1}{0-1}(x-1)\Rightarrow y=-x+2\Rightarrow x+y=2$ … (8.3)

Now, we can obtain area of the square OABC using double integral as follows.

$$\text{Area of square OABC} = 2\ \text{Area of the region OABKO}$$

$$\begin{bmatrix}\text{Due to symmetry of square OABC about}\\ \text{the diagonal OB. Refer Fig. 8.1(c)}\end{bmatrix}$$

$$= 2\ \text{Area of the regions}\left(\text{OAKO}+\text{KABK}\right)$$

$$=2\left(\int_{y=0}^{y=1}\int_{x=0}^{x=y} dx\,dy+\int_{y=1}^{y=2}\int_{x=0}^{x=2-y} dx\,dy\right)$$

$$\begin{bmatrix}\because\ \text{Horizontal strip EF starts from } x=0\\ \text{and ends at } x=y.\ \text{Similar argument}\\ \text{works for strip GH. Refer Fig.8.1(c)}\end{bmatrix}$$

$$=2\left(\int_{y=0}^{y=1}[x]_{x=0}^{x=y}\,dy+\int_{y=1}^{y=2}[x]_{x=0}^{x=2-y}\,dy\right)$$

$$=2\left(\int_{y=0}^{y=1} y\,dy+\int_{y=1}^{y=2}(2-y)\,dy\right)=2\left(\left[\frac{y^2}{2}\right]_{y=0}^{y=1}+\left[2y-\frac{y^2}{2}\right]_{y=1}^{y=2}\right)$$

$$=2\left(\left[\frac{1}{2}-0\right]+\left[4-2-2+\frac{1}{2}\right]\right)=2(1)=2 \qquad \text{... (8.4)}$$

As expected, both (8.1) and (8.4) match.

Now, we find the area of the square OABC by making use of a transformation.

**Finding Area of Square OABC Making Transformation in Double Integral**

Like equations of OA and AB, we can find equations of the straight lines OC and BC which respectively are given by

$x+y=0$ … (8.5)

$y-x=2$ … (8.6)

Combining (8.2), (8.3), (8.5) and (6.6) square OABC can be expressed as follows.

Square OABC in xy-plane $=\{(x,\ y):0\le x+y\le 2,\ 0\le y-x\le 2\}$ … (8.7)

Let us make the transformation

$x+y=u$ … (8.8)

$y-x=v$ … (8.9)

Now, square OABC in uv-plane $=\{(u,\ v):0\le u\le 2,\ 0\le v\le 2\}$ … (8.10)

which is visualised in Fig. 8.1 (b).

Note that under the transformation given by (8.8) and (8.9), we have

Point O(0, 0) remains unchanged. $\left[\because u = 0 + 0 = 0, \text{ and } v = 0 - 0 = 0\right]$

Point A(1, 1) transfers to P(2, 0). $\left[\because u = 1 + 1 = 2, \text{ and } v = 1 - 1 = 0\right]$

Point B(0, 2) transfers to Q(2, 2). $\left[\because u = 0 + 2 = 2, \text{ and } v = 2 - 0 = 2\right]$

Point $C(-1, 1)$ transfers to R(0, 2). $\left[\because u = -1 + 1 = 0, \text{ and } v = 1 - (-1) = 2\right]$

So, square OABC in xy-plane transferred to square OPQR in uv-plane.

Now, we obtain the area of the square OPQR using the double integral. Note that sides of square OPQR are parallel to u and v axes so as explained in Unit 9 of the course MST-011, we have no need to draw any kind of horizontal or vertical strip to obtain limits of integration in double integral. So, using the double integral area of square OPQR is given by

$$\text{Area of square OPQR} = \int_{v=0}^{v=2}\int_{u=0}^{u=2} du\,dv = \int_{v=0}^{v=2} \left[u\right]_{u=0}^{u=2} dv = \int_{v=0}^{v=2} 2\,dv = 2\left[v\right]_{v=0}^{v=2}$$

$$= 2\left(2 - 0\right) = 4\,\text{unit}^2 \qquad \text{... (8.11)}$$

If we compare (8.4) and (8.11), then we see that the values of two areas do not match. Why two areas are not matching? The answer to this question is, we did not take into account a factor known as **Jacobian**. Actually, when we make a transformation to move our coordinate system into another coordinate system to make our problem either comparatively easy or sometimes solvable which is not solvable in the original coordinate system, etc. we have to take into account a factor (stretching or shrinking) to establish equality of the two results. As said earlier this factor which ensures the equality of two results is known as **Jacobian** of the transformation. Note that the area in the uv-plane is coming twice the area in xy-plane. So, we expect that the value of Jacobian for this transformation should be 1/2. … (8.12)

Formula for Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad \text{… (8.13)}$$

It is generally denoted by $\frac{\partial(x, y)}{\partial(u, v)}$. So, $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ … (8.14)

Before obtaining value of (8.14), first we have to obtain x and y as functions of u and v.

(8.8) – (8.9) gives $2x = u - v \Rightarrow x = \frac{1}{2}u - \frac{1}{2}v$ … (8.15)

(8.8) + (8.9) gives $2y = u + v \Rightarrow y = \frac{1}{2}u + \frac{1}{2}v$ … (8.16)

Now, partial derivatives are given by (refer 6.107, 6.107a and 6.107b in Unit 6)

$$\frac{\partial x}{\partial u} = \frac{1}{2},\ \frac{\partial x}{\partial v} = -\frac{1}{2},\ \frac{\partial y}{\partial u} = \frac{1}{2},\ \frac{\partial y}{\partial v} = \frac{1}{2} \qquad \ldots (8.17)$$

$$\text{Now } J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \qquad \ldots (8.18)$$

Refer (8.12), we were expecting value of Jacobian should be 1/2 which meets our expectation. So, whenever we make transformation then we should use Jacobian as an adjusting factor. Thus, we have

$$\iint\limits_{\substack{\text{Original region}\\ \text{of integration in}\\ \text{xy-plane}}} dx\,dy = \iint\limits_{\substack{\text{Transformed region}\\ \text{of integration in}\\ \text{uv-plane}}} |J|\,du\,dv \qquad \ldots (8.19)$$

Now, we discuss one example to evaluate Jacobian of a transformation.

**Example 1**: Find the value of Jacobian if transformation is done from cartesian coordinates system to polar coordinates system.

**Solution**: We know that transformation from cartesian coordinates system to polar coordinates system is given by

$$x = r\cos\theta,\ y = r\sin\theta \qquad \ldots (8.20)$$

Now, required Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r(1) = r \qquad \ldots (8.21)$$

Now, you can try the following Self-Assessment Question.

**SAQ 1**

If $J = \frac{\partial(x,\ y)}{\partial(u,\ v)}$ and $J' = \frac{\partial(u,\ v)}{\partial(x,\ y)}$ then $JJ' = 1.$ Verify this property for cartesian and polar coordinates systems discussed in Example 1.

## 8.3 TRANSFORMATION OF UNIVARIATE AND BIVARIATE RANDOM VARIABLES

In Sec. 7.2 of Unit 7, we discussed and obtained expected value of the function g(X) of a random variable X when PMF or PDF of X is given to us refer (7.13), (7.15), (7.17) and (7.19). But sometimes, we are interested in getting PMF or PDF or CDF of g(X) instead of only its expected value when we are given PMF or PDF or CDF of the random variable X. In this section, we will discuss how we can obtain them. We will classify our discussion under the following two categories.

- Transformation in Univariate Random Variables
- Transformation in Bivariate Random Variables

In Sec. 2.2 of Unit 2 of the course MST-011, you have studied types of functions such as one-one (injective), onto (surjective) and both one-one as well as onto (bijective) also known as one-one correspondence or one to one. When we work in the world of transformations, we may or may not need inverse of the given transformation. If you need inverse of the transformation then make sure that given transformation should be both one-one and onto. Here we are going to deal with transformations, so one-one and onto will play its role. Let us discuss transformations in univariate and bivariate random variables taken one at a time as follows.

## 8.3.1 Transformation in Univariate Random Variables

In univariate case, we will discuss transformation of both discrete and continuous random variables one at a time as follows.

- **Transformation of Discrete Univariate Random Variables**

Discrete case is simple. Suppose we are given PMF of random variable X and want to find PMF of g(X). Here we follow the following steps.

**Step 1**: Find value of g(X) for each value of X. … (8.22)

**Step 2**: If all values of g(X) are distinct then function g is one-one. In this case probabilities of values of g(X) will remain the same as they were corresponding to the values of X. Write values of g(X) with their corresponding probabilities in tabular form. The table so formed will give you PMF of g(X). … (8.23)

**Step 3**: But if all the values of g(X) are not distinct then function g will not one-one. In this case, we have to add all probabilities which are corresponding to the same value of g(X), and for distinct values of g(X), we will follow the procedure as explained in Step 2. … (8.24)

Let us explain it through an example.

**Example 2**: If PMF of the random variable X is given by Table 8.1 as follows.

**Table 8.1: Probability mass function of the random variable X**

| **Values of X** | 0 | 1 | 2 | 3 | Total |
|---|---|---|---|---|---|
| **Probabilities ($p_X(x)$)** | 0.1 | 0.4 | 0.3 | 0.2 | 1 |

Find the PMF of the random variables (a) 2X (b) 2X + 5 (c) visualise PMF of X, 2X and 2X + 5 and explain the relationship between them intuitively.

**Solution**: (a) Here the function 2X takes different values corresponding to different values of X so it is one-one. So, to obtain PMF of 2X, we have to just multiply values of X by 2 but corresponding probabilities for values of X will remain unchanged. Hence, PMF of 2X is given in Table 8.2 as follows.

**Table 8.2: Probability mass function of the random variable 2X**

| Values of 2X | 0 | 2 | 4 | 6 | Total |
|---|---|---|---|---|---|
| Probabilities ($p_X(x)$) | 0.1 | 0.4 | 0.3 | 0.2 | 1 |

(b) Similarly, the function 2X + 5 is one-one. So, to obtain PMF of 2X + 5, we have to just multiply values of X by 2 and then add 5 but corresponding probabilities for values of X will remain unchanged. Hence, PMF of 2X + 5 is given in Table 8.3 as follows.

**Table 8.3: Probability mass function of the random variable 2X + 5**

| Values of 2X + 5 | 5 | 7 | 9 | 11 | Total |
|---|---|---|---|---|---|
| Probabilities ($p_X(x)$) | 0.1 | 0.4 | 0.3 | 0.2 | 1 |

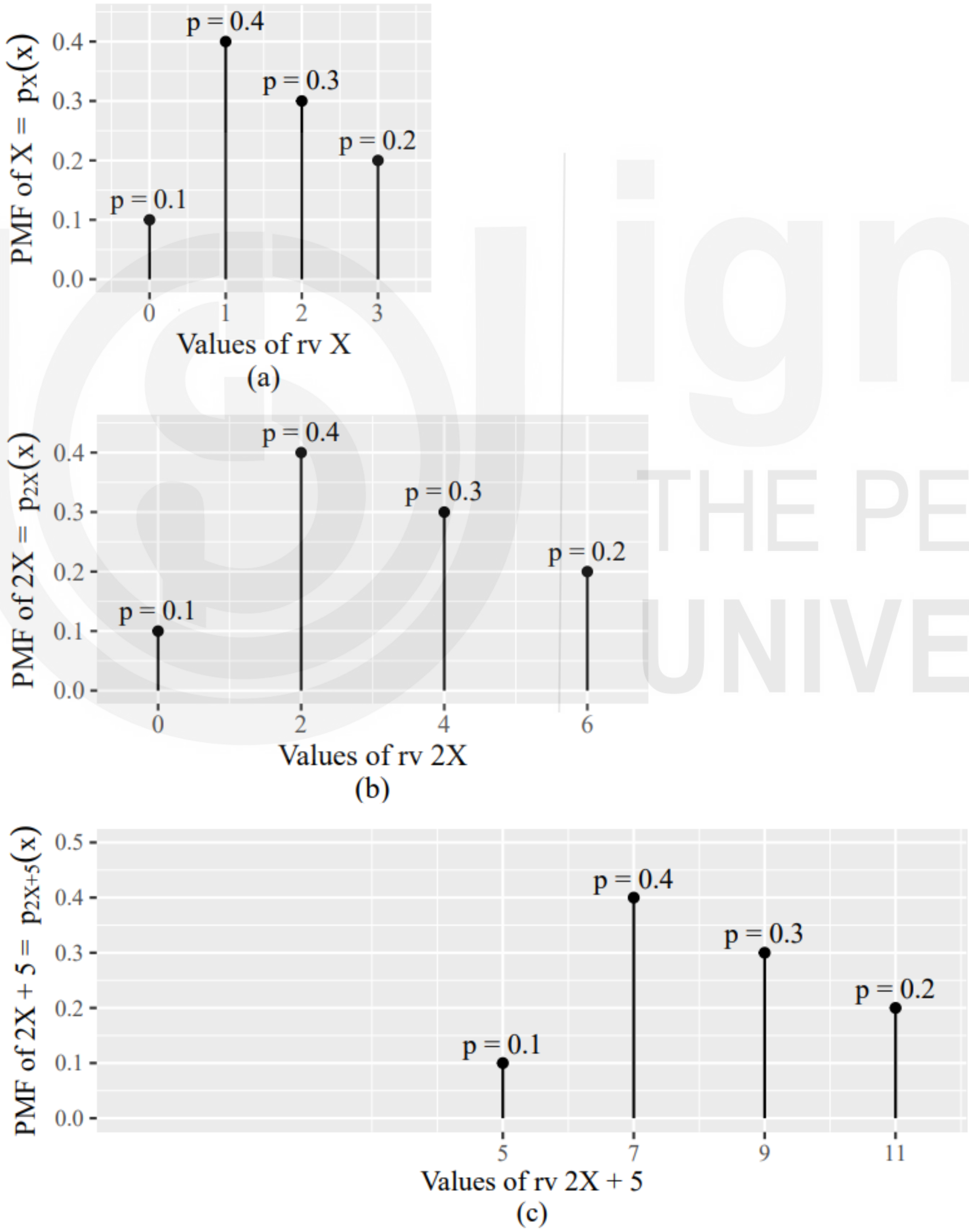


**Fig. 8.2: Visualisation of the PMF of (a) X (b) 2X (c) 2X + 5**

(c) Probability mass functions of X, 2X and 2X + 5 are visualised in Fig. 8.2 (a), (b) and (c), respectively. Note that to visualise this information, we have used one principle of data visualisation which states that if we want to compare horizontal shift of two or more distributions then we have to plot graphs vertically one above the other. Also, refer to Fig. 9.1 in the next unit to see another similar principle of data visualisation.

After a look at the PMF's of X, 2X and 2X + 5 intuitively, we observe the following.

- Effect of 2 on PMF of 2X compared to PMF of X is that it has shifted the PMF of X twice of horizontal units. There is no effect of 2 on probabilities. … (8.32)

- Effect of 5 on PMF of 2X + 5 compared to PMF of 2X is that it has shifted the PMF of 2X to right on horizontal axis by 5 units. There is no effect of 5 on probabilities. … (8.33)

- **Transformation of Continuous Univariate Random Variables**

When we deal a new concept in continuous world of probability theory, we should be very thankful to CDF because it always comes in front to rescue us from any kind of problem we are facing. Solution of the present problem will also be obtained using the concept of CDF. Recall that to obtain CDF of the function $Y = g(X)$ from given CDF or PDF of random variable X, we need to find $\mathcal{P}\left(Y \le y\right)$ or $\mathcal{P}\left(g(X) \le y\right)$. But to convert this CDF of $Y = g(X)$ in terms of CDF of X, we need some information about the function g.

- The first requirement is we need the inverse of g. But you know from the learning of Unit 2 of the course MST-011 that for the existence of inverse, the function g should be one-one and onto at least in the domain of our interest. … (8.34)
- To make the decision about the inequality sign in the CDF at the time of replacing function g by its inverse the nature of g whether it is increasing or decreasing plays a crucial role which is visualised in Fig. 8.3 (a) and (b) and used in obtaining expression in (8.38) and (8.40). … (8.35)

Now, let us explain the working procedure in both the situations when g is increasing and decreasing as follows.

**First suppose g is 1-1 and onto and increasing**: If g is 1-1 and onto then its inverse will exist. Let h be the inverse of g. Therefore, for any given y in the range of the function g, we have

$$g(x) = y \Rightarrow x = g^{-1}(y) \Rightarrow x = h(y) \text{ as h is the inverse of g} \quad \dots (8.36)$$

Therefore, CDF of Y = g(X) is given by

$$F_Y(y) = \mathcal{P}(Y \le y) = \mathcal{P}\left(g(X) \le y\right) = \mathcal{P}\left(X \le g^{-1}(y)\right) \quad \left[\begin{array}{l}\because \text{ g is increasing refer} \\ \text{to Fig. 8.3 (a). So, inequality} \\ \text{will remain the same}\end{array}\right]$$

$$= \mathcal{P}\left(X \le h(y)\right) \quad \left[\because \text{Using } (8.27)\right]$$

$$= F_X(h(y)) \quad \left[\text{By definition of CDF}\right]$$

$$\Rightarrow F_Y(y) = F_X(h(y)) \quad \dots (8.37)$$

If we are interested in PDF of Y = g(X), then it can be obtained by differentiating (8.37) with respect to y as follows

$$\frac{d}{dy}\left(F_Y(y)\right)=\frac{d}{dy}\left(F_X(h(y))\right)$$

Or $f_Y(y)=F_X'(h(y))\frac{d}{dy}\left(h(y)\right)$ $\quad$ [Using chain rule of differentiation in RHS]

Hence, PDF of

$$Y=g(X)\text{ is given by }f_Y(y)=\begin{cases}F_X'(h(y))\dfrac{d}{dy}\left(h(y)\right)\\ \text{or } f_X(x)\dfrac{d}{dy}\left(h(y)\right),\text{ as }F_X'(h(y))=f_X(x)\end{cases} \quad \ldots (8.38)$$

Now, we consider the case when the function g is decreasing.

**Suppose g is 1-1 and onto but decreasing instead of increasing**: If g is 1-1 and onto then its inverse will exist. Let h be the inverse of g. Therefore, for any given y in the range of the function g, we have

$$g(x)=y\Rightarrow x=g^{-1}(y)\Rightarrow x=h(y)\text{ as h is the inverse of g} \quad \ldots (8.39)$$

Therefore, CDF of Y = g(X) is given by

$$\begin{aligned}F_Y(y)=\mathcal{P}(Y\le y)=\mathcal{P}\left(g(X)\le y\right)&=\mathcal{P}\left(X\ge g^{-1}(y)\right)\quad\begin{bmatrix}\because\text{ g is decreasing, so}\\ \text{inequality will reverse}\\ \text{refer Fig. 8.3(b)}\end{bmatrix}\\ &=\mathcal{P}\left(X\ge h(y)\right)\quad\left[\because\text{ Using }(8.30)\right]\\ &=1-\mathcal{P}\left(X<h(y)\right)\quad\begin{bmatrix}\because\ \mathcal{P}(X\ge\alpha)+\mathcal{P}(X<\alpha)=1\\ \Rightarrow\mathcal{P}(X\ge\alpha)=1-\mathcal{P}(X<\alpha)\end{bmatrix}\\ &=1-\mathcal{P}\left(X\le h(y)\right)\quad\begin{bmatrix}\because\text{ Random variable X}\\ \text{is continuous, so}\\ \mathcal{P}\left(X=h(y)\right)=0\end{bmatrix}\\ &=1-F_X(h(y))\quad\left[\text{By definition of CDF}\right]\end{aligned}$$

$$\Rightarrow F_Y(y)=1-F_X(h(y)) \quad \ldots (8.40)$$

If we are interested in PDF of Y = g(X), then it can be obtained by differentiating (8.40) with respect to y as follows.

$$\frac{d}{dy}\left(F_Y(y)\right)=\frac{d}{dy}\left(1-F_X(h(y))\right)$$

Or $f_Y(y)=-F_X'(h(y))\frac{d}{dy}\left(h(y)\right)$ $\quad$ [Using chain rule of differentiation in RHS]

Hence, PDF of

$$Y = g(X) \text{ is given by } f_Y(y) = \begin{cases} -F_X'(h(y))\dfrac{d}{dy}(h(y)) \\ \text{or } -f_X(x)\dfrac{d}{dy}(h(y)), \text{ as } F_X'(h(y)) = f_X(x) \end{cases} \quad \ldots (8.41)$$

Since h(y) is a decreasing function so its derivative with respect to y will be negative. Keeping this fact in view (8.41) can be written as

$$\text{PDF of } Y = g(X) \text{ is given by } f_Y(y) = \begin{cases} F_X'(h(y))\left|\dfrac{d}{dy}(h(y))\right| \\ \text{or } f_X(x)\left|\dfrac{d}{dy}(h(y))\right|, \text{ as } F_X'(h(y)) = f_X(x) \end{cases}$$

… (8.42)

In (8.38) function h(y) is increasing so its derivative with respect to y will be positive. So, (8.38) and (8.42) can be combined in a single result as follows.

$$\text{PDF of } Y = g(X) \text{ is given by } f_Y(y) = F_X'(h(y))\left|\frac{d}{dy}(h(y))\right| \quad \ldots (8.43)$$

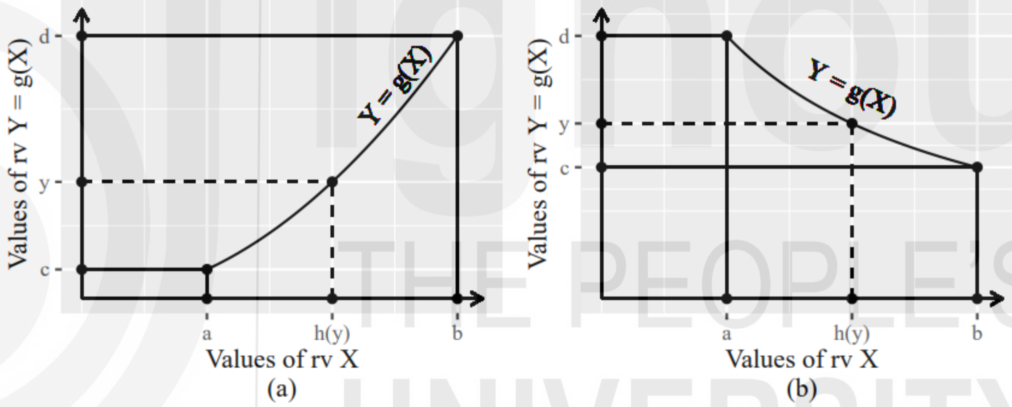


**Fig. 8.3: Visualisation of the corresponding domains of the functions Y = g(X) and X = h(Y) in the cases when (a) g is an increasing (b) g is a decreasing function**

So, in continuous case, we follow the following two steps.

**Step 1**: Find CDF of Y = g(X) keeping in view whether the function g is increasing or decreasing as well as limits of g(X) by following the procedure explained here. … (8.44)

**Step 2**: If only CDF is required then Step 1 is sufficient. But if PDF is required then use (8.43) to obtain PDF of Y = g(X). … (8.45)

Let us explain it through an example.

**Example 3**: Suppose you are deriving at a speed of X km/h where X follows uniform probability distribution between 80 km/h to 90 km/h. Find the probability density function of the random variable Y which represents time taken by you to travel a distance of 100 km. Hence, find expected time of your journey.

**Solution**: According to the statement of this example, PDF of the random variable X is given by (you may refer to Unit 13 of this course)

$$f_X(x) = \begin{cases} \dfrac{1}{90-80}, & 80 < X < 90 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Or } f_X(x) = \begin{cases} \dfrac{1}{10}, & 80 < X < 90 \\ 0, & \text{otherwise} \end{cases} \qquad \ldots (8.46)$$

CDF of the random variable X when $80 \le x < 90$ is given by

$$F_X(x) = \int_{80}^{x} f_X(x)\,dx = \int_{80}^{x} \frac{1}{10}\,dx = \left[\frac{1}{10}x\right]_{80}^{x} = \frac{x-80}{10}$$

Hence, CDF of the random variable X is given by

$$F_X(x) = \begin{cases} 0, & \text{if } x < 80 \\ \dfrac{x-80}{10}, & \text{if } 80 \le x < 90 \\ 1, & \text{if } x \ge 90 \end{cases} \qquad \ldots (8.47)$$

We know that $\text{Time} = \dfrac{\text{Distance travelled}}{\text{Speed}}$

$$\therefore \; Y = \frac{100}{X} = g(X) \qquad \ldots (8.48)$$

$$\Rightarrow X = \frac{100}{Y} = g^{-1}(Y) = h(Y) \qquad \ldots (8.49)$$

Now, we can solve this problem using two approaches discussed as follows.

**First Approach**

CDF of Y = g(X) is given by

$$\begin{aligned} F_Y(y) = \mathcal{P}(Y \le y) = \mathcal{P}\left(g(X) \le y\right) &= \mathcal{P}\left(X \ge g^{-1}(y)\right) && [\because \text{g is decreasing}] \\ &= \mathcal{P}\left(X \ge h(y)\right) && [\text{Using (8.49)}] \\ &= 1 - \mathcal{P}\left(X \le h(y)\right) && \left[\begin{array}{l} \because \text{ Random variable X} \\ \text{is continuous, so} \\ \mathcal{P}\left(X = h(y)\right) = 0 \end{array}\right] \\ &= 1 - F_X(h(y)) && [\text{By definition of CDF}] \\ &= 1 - F_X\left(\frac{100}{y}\right) && [\text{Using (8.49)}] \quad \ldots (8.50) \end{aligned}$$

Using (8.47) in (8.50), we have

$$F_Y(y) = 1 - \begin{cases} 0, & \text{if } \dfrac{100}{y} < 80 \\ \dfrac{\dfrac{100}{y} - 80}{10}, & \text{if } 80 \le \dfrac{100}{y} < 90 \\ 1, & \text{if } \dfrac{100}{y} \ge 90 \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} 1, & \text{if } \frac{10}{8} < y \\ 9 - \frac{10}{y}, & \text{if } \frac{10}{9} < y \le \frac{10}{8} \\ 0, & \text{if } \frac{10}{9} \ge y \end{cases}$$

Y is a continuous random variable so including or excluding a single point has no effect on its CDF. Hence, CDF of Y can be written as

$$F_Y(y) = \begin{cases} 0, & \text{if } y < \frac{10}{9} \\ 9 - \frac{10}{y}, & \text{if } \frac{10}{9} \le y < \frac{10}{8} \\ 1, & \text{if } y \ge \frac{10}{8} \end{cases} \qquad \dots (8.51)$$

Now, PDF of Y can be obtained by differentiating (8.51) with respect to y.

$$f_Y(y) = \begin{cases} \frac{d}{dy}(0), & \text{if } y < \frac{10}{9} \\ \frac{d}{dy}\left(9 - \frac{10}{y}\right), & \text{if } \frac{10}{9} \le y < \frac{10}{8} \\ \frac{d}{dy}(1), & \text{if } y \ge \frac{10}{8} \end{cases}$$

$$\text{Or } f_Y(y) = \begin{cases} \frac{10}{y^2}, & \text{if } \frac{10}{9} < y < \frac{10}{8} \\ 0, & \text{otherwise} \end{cases} \qquad \dots (8.52)$$

**Second Approach**:

We can find PDF of Y using (8.43) as follows.

$$f_Y(y) = F_X'(h(y))\left|\frac{d}{dy}(h(y))\right|$$

$$f_Y(y) = \frac{1}{10}\left|-\frac{100}{y^2}\right| \qquad \left[\begin{array}{l} \because \text{From (8.47), } F_X'(x) = \frac{1}{10} \text{ also} \\ h(y) = \frac{100}{y} \Rightarrow \frac{d}{dy}(h(y)) = -\frac{100}{y^2} \end{array}\right]$$

$$= \frac{10}{y^2}$$

Since Y is continuous random variable so not including both end points or including one end point or including both end points has no effect.

Hence, PDF of Y can be written either of the following two ways if end points are finite real numbers.

$$f_Y(y) = \begin{cases} \frac{10}{y^2}, & \text{if } \frac{10}{9} < y < \frac{10}{8} \\ 0, & \text{otherwise} \end{cases} \quad \text{or} \quad f_Y(y) = \begin{cases} \frac{10}{y^2}, & \text{if } \frac{10}{9} \le y \le \frac{10}{8} \\ 0, & \text{otherwise} \end{cases} \qquad \dots (8.53)$$

Now, we can obtain CDF of Y as follows.

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)\,dy = \int_{10/9}^{y} \frac{10}{y^2}\,dy = \left[\frac{-10}{y}\right]_{10/9}^{y} = -10\left[\frac{1}{y} - \frac{9}{10}\right] = 9 - \frac{10}{y}$$

Hence, CDF of Y is given by

$$F_Y(y) = \begin{cases} 0, & \text{if } y < \frac{10}{9} \\ 9 - \frac{10}{y}, & \text{if } \frac{10}{9} \le y < \frac{10}{8} \\ 1, & \text{if } y \ge \frac{10}{8} \end{cases} \qquad \dots (8.54)$$

We have obtained PDF of Y so we can obtain expected value of Y as follows.

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_{10/9}^{10/8} y \frac{10}{y^2}\,dy = \int_{10/9}^{10/8} \frac{10}{y}\,dy = [10 \log y]_{10/9}^{10/8}$$

$$= 10\left[\log\left(\frac{10}{8}\right) - \log\left(\frac{10}{9}\right)\right] = 10 \log\left[\frac{10}{8} \times \frac{9}{10}\right] = 10 \log(9/8) \simeq 1.18 \text{ hours}$$

Also, note that expected value of random variable X is

$$\frac{80+90}{2} = 85 \text{ km/h} \qquad [\text{Refer (13.24) of this course}]$$

So, expected time will be $\frac{100}{85} \simeq 1.18$ hour.

Now, we discuss bivariate case.

## 8.3.2 Transformation in Bivariate Random Variables

In Sub Sec. 8.3.1, we saw that discrete case is straight forward. So, here we will discuss only transformation in continuous bivariate random variables. Following the similar steps like univariate case if we are given the joint PDF $f_{X,Y}(x, y)$ of the jointly continuous random variable (X, Y) and we are interested in joint PDF of jointly continuous random variable (U, V) where (X, Y) and (U, V) are connected by the 1-1 and onto transformations

$$g_1(X, Y) = U \qquad \dots (8.55)$$

$$g_2(X, Y) = V \qquad \dots (8.56)$$

such that (8.55) and (8.56) can be solved to obtain X and Y in terms of U and V. Let after solving, we get

$$h_1(U, V) = X \qquad \dots (8.57)$$

$$h_2(U, V) = Y \qquad \dots (8.58)$$

Then like (8.43) joint PDF of (U, V) is given by

$$\text{Joint PDF of } (U, V) = f_{U,V}(u, v) = f_{X,Y}(h_1(x, y), h_2(x, y))|J|, \qquad \dots (8.59)$$

$$\text{where } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad \dots (8.60)$$

So, to obtain joint PDF of (U, V) we have to follow the following five steps.

**Step 1**: Check whether you are given one or two relation(s) of the form (8.55) or (8.56). If only one is given then you have to suppose other from your side such that (8.57) and (8.58) are possible as well as J should not be zero.

**Step 2**: After Step 1 obtain values of X and Y in terms of U and V as shown in (8.57) and (8.58).

**Step 3**: Find J, the Jacobian of the transformation.

**Step 4**: Put values of $h_1(x, y)$, $h_2(x, y)$ and $J$ in RHS of (8.59).

**Step 5**: Find limits of U and V keeping in view that the limits of X and Y and relations among U, V and X, Y. Combine all steps and if necessary, use discussion of Unit 6 regarding marginal PDF's or any other concept.

Let us explain it through an example.

**Example 4**: Joint PDF of the jointly continuous random variable (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} \lambda_1 \lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}, & \text{if } x, y > 0 \\ 0, & \text{otherwise} \end{cases} \quad \dots (8.61)$$

Find the PDF of the random variable $\frac{X+Y}{2}$.

**Solution**: Let $U = \frac{X+Y}{2}$ … (8.62)

If we follow the five steps procedure mentioned before this example then we note that we have to suppose a relation for V in terms of X only or Y only or in terms of both X and Y keeping in view the requirement mentioned in Step 1.

So, let $V = \frac{X-Y}{2}$ … (8.63)

Solving (8.62) and (8.63) for X and Y, we get

$$X = U + V \quad \dots (8.64)$$

$$Y = U - V \quad \dots (8.65)$$

Now, $$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \quad \dots (8.66)$$

Using (8.61), (8.64) to (8.66) in (8.59), we get

$$f_{U,V}(u, v) = \begin{cases} |-2| \lambda_1 \lambda_2 e^{-(\lambda_1 (u+v) + \lambda_2 (u-v))}, & \text{if } u+v > 0, u-v > 0 \\ 0, & \text{otherwise} \end{cases}$$

Note that to keep both $u + v > 0, u - v > 0$ true u may take any real value > 0. But v cannot be greater than u because if v > u then $u - v > 0$ will not hold. Similarly, if $v < -u$ then $u + v > 0$ will not hold. Hence, joint PDF of (U, V) is given by

$$f_{U,V}(u,v)=\begin{cases}2\lambda_1\lambda_2 e^{-((\lambda_1+\lambda_2)u+(\lambda_1-\lambda_2)v)}, & \text{if } u>0, -u<v<u\\ 0, & \text{otherwise}\end{cases} \quad \ldots (8.67)$$

But here we were not interested in joint PDF of jointly continuous random variable (U, V) but we were interested in PDF of $\frac{X+Y}{2}=U.$ So, using the concept of marginal PDF discussed in Unit 6, we can obtain required PDF as follows.

$$f_U(u)=\int_{-u}^{u} f_{U,V}(u,v)\,dv=\int_{-u}^{u} 2\lambda_1\lambda_2 e^{-((\lambda_1+\lambda_2)u+(\lambda_1-\lambda_2)v)}\,dv=\left[\frac{2\lambda_1\lambda_2 e^{-((\lambda_1+\lambda_2)u+(\lambda_1-\lambda_2)v)}}{-(\lambda_1-\lambda_2)}\right]_{-u}^{u}$$

$$=\frac{2\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[e^{-((\lambda_1+\lambda_2)u+(\lambda_1-\lambda_2)u)}-e^{-((\lambda_1+\lambda_2)u-(\lambda_1-\lambda_2)u)}\right]=\frac{2\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[e^{-2\lambda_1 u}-e^{-2\lambda_2 u}\right]$$

$$\therefore\ f_U(u)=\frac{2\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[e^{-2\lambda_1 u}-e^{-2\lambda_2 u}\right],\ u>0 \quad \ldots (8.68)$$

Hence, PDF of $U=\frac{X+Y}{2}$ is given by

$$\therefore\ f_U(u)=\begin{cases}\frac{2\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[e^{-2\lambda_1 u}-e^{-2\lambda_2 u}\right], & u>0\\ 0, & \text{otherwise}\end{cases}$$

Now, you can try the following Self-Assessment Question.

**SAQ 2**

If the random variable X follows uniform probability distribution over the interval $(-2, 6)$ then find PDF of the random variable $Y=e^X$.

## 8.4 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- Jacobian of x, y with respect to u, v is given by $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$
- Jacobian of u, v with respect to x, y is given by $J'=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$
- $JJ'=1$
- Probability density function of $Y=g(X)$ is $f_Y(y)=F'_X(h(y))\left|\frac{d}{dy}(h(y))\right|$
- Joint pdf of $(U,V)=f_{U,V}(u,v)=f_{X,Y}(h_1(x,y),h_2(x,y))|J|,$

## 8.5 TERMINAL QUESTIONS

1. PMF of random variable X is given by $p_X(x) = \begin{cases} \frac{2x+1}{25}, & x = 0, 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$

   Find PMF of the random variable $Y = \frac{2X+3}{5}$.

2. The joint PDF of the jointly continuous random variable (X, Y) is given by

   $f_{X,Y}(x, y) = \begin{cases} 10\,e^{-2x-5y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$

   Find the joint PDF of (U, V) where U = 2X and V = Y – X.

## 8.6 SOLUTIONS/ANSWERS

### Self-Assessment Questions (SAQs)

1. In Example 1, we have already obtained $J = \frac{\partial(x, y)}{\partial(r, \theta)} = r.$ … (8.69)

   Now, before obtaining $J'$ first we have to obtain r and $\theta$ in terms of x and y. So, squaring and adding two equations given by (8.20), we have

   $x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) \Rightarrow x^2 + y^2 = r^2 \quad \left[\because\ \cos^2\theta + \sin^2\theta = 1\right]$

   $\Rightarrow r = \sqrt{x^2 + y^2}$ … (8.70)

   Dividing two equations given by (8.20), we have

   $\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \Rightarrow \tan\theta = \frac{y}{x}$

   $\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$ … (8.71)

   $$\text{Now, } J' = \frac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+\frac{y^2}{x^2}} & \frac{1/x}{1+\frac{y^2}{x^2}} \end{vmatrix}$$

   $$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{\left(x^2+y^2\right)^{3/2}} + \frac{y^2}{\left(x^2+y^2\right)^{3/2}}$$

   $$= \frac{x^2+y^2}{\left(x^2+y^2\right)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r} \qquad \text{... (8.72)}$$

   From (8.69) and (8.72), we have $JJ' = r\left(\frac{1}{r}\right) = 1$

   Which verify the property that $JJ' = 1$.

2. We are given that random variable X follows uniform probability distribution over the interval $(-2, 6)$, so PDF of X is given by (you may refer to 13.8 of this course)

$$f_X(x) = \begin{cases} \dfrac{1}{6-(-2)}, & -2 < X < 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Or } f_X(x) = \begin{cases} \dfrac{1}{8}, & -2 < X < 6 \\ 0, & \text{otherwise} \end{cases} \qquad \ldots (8.73)$$

CDF of the random variable X when $-2 \le x < 6$ is given by

$$F_X(x) = \int_{-2}^{x} f_X(x)\,dx = \int_{-2}^{x} \frac{1}{8}\,dx = \left[\frac{1}{8}x\right]_{-2}^{x} = \frac{x+2}{8} \qquad \ldots (8.74)$$

Hence, CDF of the random variable X is given by

$$F_X(x) = \begin{cases} 0, & \text{if } x < -2 \\ \dfrac{x+2}{8}, & \text{if } -2 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases} \qquad \ldots (8.75)$$

Now, $Y = e^X = g(X) \Rightarrow X = \log Y = h(Y)$ … (8.76)

Using (8.43) PDF of $Y = e^X$ is given by

$$f_Y(y) = F_X'(h(y))\left|\frac{d}{dy}(h(y))\right|$$

$$f_Y(y) = \frac{1}{8}\left|\frac{1}{y}\right| \qquad \left[\begin{array}{l} \because \text{From (8.75), } F_X'(x) = \dfrac{1}{8} \text{ also} \\ h(y) = \log y \Rightarrow \dfrac{d}{dy}(h(y)) = \dfrac{1}{y} \end{array}\right]$$

$$= \frac{1}{8y}$$

Hence, PDF of Y is given by

$$f_Y(y) = \begin{cases} \dfrac{1}{8y}, & \text{if } -2 < \log y < 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Or } f_Y(y) = \begin{cases} \dfrac{1}{8y}, & \text{if } e^{-2} < y < e^{6} \\ 0, & \text{otherwise} \end{cases} \qquad \ldots (8.77)$$

## Terminal Questions

1. We are given PMF of X

$$p_X(x) = \begin{cases} \dfrac{2x+1}{25}, & x = 0, 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases} \qquad \ldots (8.78)$$

We have to find PMF of $Y = \dfrac{2X+3}{5}$ … (8.79)

Let us first find

$$\mathcal{P}(Y = y) = \mathcal{P}\left(\frac{2X+3}{5} = y\right) \quad [\text{Using } (8.79)]$$

$$= \mathcal{P}(2X + 3 = 5y) = \mathcal{P}(2X = 5y - 3) = \mathcal{P}\left(X = \frac{5y-3}{2}\right)$$

$$= \frac{2\left(\frac{5y-3}{2}\right)+1}{25} \quad [\text{Using } (8.78)]$$

$$= \frac{5y-3+1}{25} = \frac{5y-2}{25}$$

Also, when x = 0, 1, 2, 3, 4 then

$$Y = \frac{2(0)+3}{5}, \frac{2(1)+3}{5}, \frac{2(2)+3}{5}, \frac{2(3)+3}{5}, \frac{2(4)+3}{5}$$

$$= \frac{3}{5}, \frac{5}{5}, \frac{7}{5}, \frac{9}{5}, \frac{11}{5} = 0.6, 1, 1.4, 1.8, 2.2$$

Thus, PMF of Y is given by $p_Y(y) = \begin{cases} \dfrac{5y-2}{25}, & y = 0.6, 1, 1.4, 1.8, 2.2 \\ 0, & \text{otherwise} \end{cases}$

2. We are given joint PDF of the jointly continuous random variable (X, Y)

$$f_{X,Y}(x, y) = \begin{cases} 10\,e^{-2x-5y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases} \quad \text{… (8.80)}$$

We have to find out the joint PDF of (U, V) where

U = 2X … (8,81)

V = Y – X … (8.82)

Let us follow the five steps procedure to obtain joint PDF of (U, V).

Solving (8.81) and (8.82) for X and Y, we get

$X = U/2$ … (8.83)

$Y = V + U/2$ … (8.84)

$$\text{Now, } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} - 1 = \frac{1}{2}. \quad \text{… (8.85)}$$

Using (8.80), (8.83) to (8.65) in (8.59), we get

$$f_{U,V}(u, v) = \begin{cases} \left|\frac{1}{2}\right| 10e^{-2(u/2)-5(v+u/2)}, & \text{if } 0 < u/2 < u/2 + v < \infty \\ 0, & \text{otherwise} \end{cases}$$

Hence, joint PDF of (U, V) is given by

$$f_{U,V}(u, v) = \begin{cases} 5\,e^{-7/2u-5v}, & \text{if } u > 0, 0 < v < \infty \\ 0, & \text{otherwise} \end{cases} \quad \text{… (8.86)}$$