UNIT 2

PROBABILITY SPACE

Structure

2.1 INTRODUCTION

In the previous unit, you have gone through some concepts of probability that you have studied in earlier classes. Here, you will study some basic concepts that you should know to understand the mathematical foundation of probability theory. But keeping the applied nature of this programme in view, we will not focus on theory in detail. However, we will discuss all the basic ideas that a statistics or data science student should know to understand the foundation of probability theory. To define these basic concepts, first of all, you should have familiarity with the idea of an event, class of events and collection of classes which are discussed in Sec. 2.2. The next important idea around which the whole idea of probability theory moves is σ-field which is discussed in Sec. 2.3. To construct more σ-fields, we need some properties or results on σ-field which are discussed in Sec. 2.4. After understanding concepts discussed in Secs. 2.2 to 2.4, you can understand what we mean by probability measure, measurable space and probability space which are discussed in Sec. 2.5.

What we have discussed in this unit is summarised in Sec. 2.6. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit some more problems based on the entire unit are given in Sec. 2.7 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given in Sec. 2.8.

In the next unit, you will study all the technical details of how we assign probabilities to events in discrete and continuous worlds.

Expected Learning Outcomes

After completing this unit, you should be able to:

- explain event, class of events and collection of classes;
- define field, σ-field and some properties of σ-field;
- define probability measure, measurable space and probability space; and
- verify whether a given set function is a probability measure or not.

2.2 EVENT, CLASS OF EVENTS AND COLLECTION OF CLASSES

In Sec. 1.2 of the previous unit, we defined sample space and events related to a random experiment. Let us recall and explain them further.

Sample Space: The set of all possible outcomes of a random experiment is called the sample space of the random experiment. In general, it is denoted by Ω. For example, when a coin is tossed then sample space is $\Omega = \{H, T\}$. Similarly, when a die is thrown then sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. In general, if the possible outcomes of a random experiment are denoted by $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$ then $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$. … (2.1)

Event: A subset of a sample space of a random experiment in which we are **interested** is known as an event. For example, in the random experiment of throwing a die, getting a multiple of 3 is an event and is given by E = {3, 6}. Full sample space, Ω itself and the empty set which is denoted by ϕ are also events. Remember an important point whether you work in a **discrete world** or a **continuous world every event is always a subset of the sample space but every subset of the sample space is an event only in a discrete world**. However, when you work in the **continuous world then like a discrete world every event is always a subset of the sample space but its converse may not hold**. That is in a continuous world there may be subsets of the sample space which will not fall in the class of events you are dealing with. That is why intentionally we have used the word '**interested**' in defining the event. An event is further divided into three categories: an elementary event, an impossible event and a compound event as shown in Fig. 2.1 and defined as follows. … (2.2)

Elementary Event: If the sample space of a random experiment is $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$, then each singleton subset of Ω, i.e., $\{\omega_i\}, 1 \le i \le n$ is called an **elementary event**. For example, in the random experiment of tossing a coin, there are two elementary events {H} and {T}. Similarly, when a die is thrown then there are six elementary events {1}, {2}, {3}, {4}, {5} and {6}. Elementary events of the sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$ are shown in Fig. 2.1. … (2.3)

Remark 1: In Unit 3, you will study the rules of probability in a discrete world. In a discrete world sample space of a random experiment may be finite or at the most countably infinite. However, in the same unit, you will also study the rules of probability in a continuous world where you will see that sample space in a continuous world is always infinite and uncountable. So, the sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$, which is finite is corresponding to some random experiment in a discrete world. Keep this important point in mind which will help you in identifying whether you are working in a discrete world or a continuous world.

Compound Event: An event constituted by a union of two or more than two elementary events is called a compound event. If the sample space of a random experiment is $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$, then events $E = \{\omega_1, \omega_2\} = \{\omega_1\} \cup \{\omega_2\}$ and $F = \{\omega_1, \omega_2, \omega_3\} = \{\omega_1\} \cup \{\omega_2\} \cup \{\omega_3\}$, etc. are compound events because they are formed by respectively union of 2 and 3 elementary events. Events E, F and Ω are shown as the union of elementary events in Fig. 2.1. … (2.4)

Impossible Event: An event is said to be an impossible event if there is no outcome in the sample space of the random experiment which favours the happening of the event. It is denoted by $\{\ \}$ or ϕ. For example, in the random experiment of throwing a standard six faces die, getting a number greater than or equal to 7 is an impossible event. … (2.5)

Sure Event: An event is said to be a sure event if all outcomes of the random experiment favour the happening of the event. Obviously, this can be possible only when we take the sample space itself as an event. So, the sample space of a random experiment is known as a sure event. Sure event is a special compound event which is equal to the union of all elementary events. Fig. 2.1 shows full sample space Ω as the union of all elementary events. … (2.6)

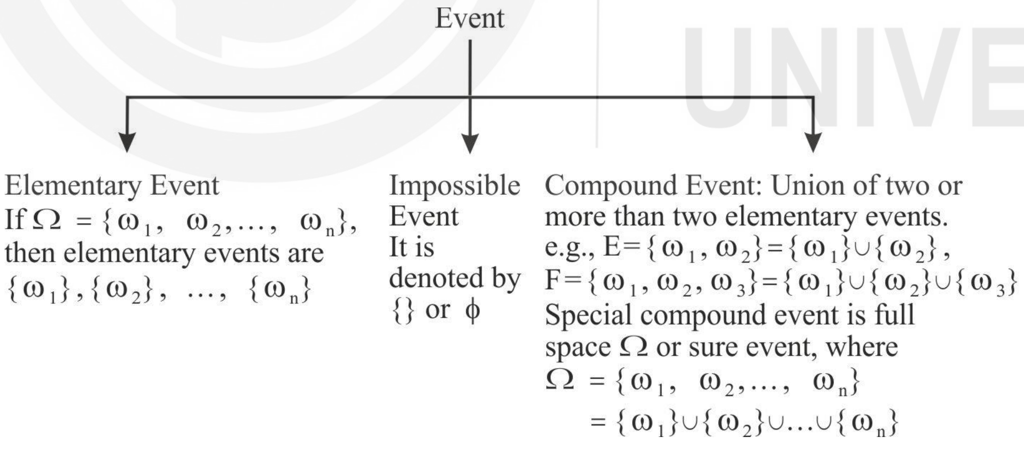

Fig. 2.1: Visualisation of classification of an event

Now, we define what we mean by class of events as follows.

Class of Events: A collection of events is called class of events. Class of events are generally denoted by using letters A to Z in calligraphic font. Letters A to Z in calligraphic font are:

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$

For example, if you consider the experiment of throwing a die then sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Now, consider the following six collections of events.

$$\left.\begin{array}{l} \mathcal{C}_1 = \{\phi, \{1\}, \{2, 3, 4, 5, 6\}, \Omega\}, \quad \mathcal{C}_2 = \{\phi, \{1, 3\}, \{2, 3, 4, 5, 6\}, \Omega\}, \\ \mathcal{C}_3 = \{\phi, \{1, 2\}, \{3, 4, 5, 6\}, \Omega\}, \quad \mathcal{C}_4 = \{\phi, \{1, 6\}, \{2, 3, 4, 5\}, \Omega\}, \\ \mathcal{C}_5 = \{\phi, \{1\}, \{1, 3\}, \{2, 3, 4, 5, 6\}, \{2, 4, 5, 6\}, \{3\}, \{1, 2, 4, 5, 6\}, \Omega\}, \\ \mathcal{C}_6 = \{\phi, \{1, 2\}, \{3, 4\}, \{3, 4, 5, 6\}, \{1, 2, 5, 6\}, \{5, 6\}, \{1, 2, 3, 4\}, \Omega\}. \end{array}\right\} \quad \ldots (2.7)$$

All the six collections $\mathcal{C}_1$ to $\mathcal{C}_6$ are classes of events because their members are events.

Collection of Classes: If a collection is constituted of classes or we can say that members of the collection are classes itself then such a collection is called collection of classes. For example, if you consider the experiment of throwing a die then sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Now, if $\mathcal{C}_1$ to $\mathcal{C}_6$ are the same as defined by (2.7) then each of the following four collections are collections of classes.

$$\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_4\}, \ \{\mathcal{C}_1, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_6\}, \ \{\mathcal{C}_1, \mathcal{C}_4, \mathcal{C}_6\}, \ \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6\}$$

Now, you can try the following Self-Assessment Question.

SAQ 1

Consider the random experiment of tossing a coin three time. Write sample space of this experiment. In each of the following six parts identify whether it is an event, class of events or collection of classes. If it is an event then classify whether it is an elementary or compound or impossible or sure event.

(a) $\{\phi, \{HHH\}, \Omega\}$ (b) $\{HHH, TTH, THH, HTH\}$ (c) $\{TTT\}$ (d) ϕ (e) Ω

(f) $\{\{\phi, \{HHH\}, \Omega\}, \{\{HHH\}, \{TTH\}, \Omega\}\}$

2.3 FIELD AND σ-FIELD

In Sec. 3.3 of Unit 3 of the course MST-011, you have studied σ-field and done some examples on σ-field. So, two questions that may arise in your mind are why we are discussing it again and why it is needed here. Remaining portion of this unit and the entire next unit are devoted to answer these two important questions. In probability theory, first of all, we have an experiment then we specify all its possible outcomes in a set which is known as sample space and is denoted by Ω. After that two situations arise:

- whether your experiment falls in the category of the discrete world or
- it falls in the category of continuous world.

If it falls in the category of the discrete world then mathematics tools permit us to assign probability to each member of the power set of the sample space Ω. But when our sample space falls in the category of the continuous world then it is not feasible to assign probability to each member of the power set of Ω because in that case uncountability of Ω does permit to hold basic three requirements of probability. So, when we work in the continuous world then instead of assigning probability to each member of the power set of $\Omega,$ we identify events of our interest and assign probabilities to only these events of our interest. To do this job of assigning probabilities to the events of our interest, we need the idea of σ-field. … (2.8)

The next unit is devoted to explain the idea of how the job of assigning probabilities to the events of our interest is done using σ-field. But before that, you should have a good understanding of σ-field. So, we are recalling it again. To understand the importance of σ placed in front of field it becomes important to define both field and σ-field which is done as follows. After their definition and examples, we will discuss how σ-field suits the requirements of dealing with the probabilities of events at the end of this section.

Field

Let Ω be a non-empty set and $\mathcal{F}$ be a collection of subsets of Ω then we say that the class $\mathcal{F}$ forms a **field** or **algebra** if it satisfies following three conditions:

(i) $\phi,\ \Omega \in \mathcal{F},$ i.e., empty set and the full space Ω should be members of $\mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c\ (= \Omega \setminus A) \in \mathcal{F},$ i.e., $\mathcal{F}$ is closed with respect to complement

(iii) $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$ for any finite collection $\{A_i\}_{i=1}^{n}$ in $\mathcal{F},$ i.e., $\mathcal{F}$ is closed with respect to finite union. … (2.9)

σ-Field

Let Ω be a non-empty set and $\mathcal{F}$ be a collection of subsets of Ω then we say that the class $\mathcal{F}$ forms a **σ-Field (sigma-field)** or **σ-algebra** (**sigma-algebra**) if it satisfies following three conditions:

(i) $\phi,\ \Omega \in \mathcal{F},$ i.e., empty set and the full space Ω should be members of $\mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c\ (= \Omega \setminus A) \in \mathcal{F},$ i.e., $\mathcal{F}$ is closed with respect to complement

(iii) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ for any countable collection $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{F},$ i.e., $\mathcal{F}$ is closed with respect to countable union. … (2.10)

Remark 2: First requirement that $\phi,\ \Omega \in \mathcal{F},$ can be used to show that $\mathcal{F}$ is non-empty. If $\mathcal{F}$ is non-empty then certainly $\phi,\ \Omega \in \mathcal{F},$ because if any subset E of Ω belongs to $\mathcal{F}$ then $E^c \in \mathcal{F}$ because of (ii) but then $E \cup E^c \in \mathcal{F}$ because of (2.16) of this unit. But $E \cup E^c = \Omega,$ so, $\Omega \in \mathcal{F}$ and $\Omega \in \mathcal{F} \Rightarrow \Omega^c \in \mathcal{F} \Rightarrow \phi \in \mathcal{F}\ \left[\because\ \Omega^c = \phi\right].$ You can also refer (2.20) of this unit because of that $E \cap E^c \in \mathcal{F}.$ But $E \cap E^c = \phi,$ so, $\phi \in \mathcal{F}.$

If the collection $\mathcal{F}$ of subsets of Ω forms a σ-field then we will simply say that $\mathcal{F}$ is a σ-field on Ω instead of saying that $\mathcal{F}$ is a σ-field of subsets of $\Omega.$ Instead of saying σ-field some authors say σ-algebra so both are the same thing. So, you can use any of them. But throughout the course, we will use σ-field instead of σ-algebra. In assigning probabilities to the events in probability theory, we need a σ-field on Ω so, we will focus on σ-field. … (2.11)

Let us consider two examples to understand the concept of σ-field in reference of probability theory.

Example 1: If a coin is tossed twice then Ω = {HH, HT, TH, TT}. What is the

(i) smallest σ-field on Ω

(ii) smallest σ-field containing the set A = {HT} on Ω

(iii) largest σ-field on Ω.

Solution: We know that a collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies the following three conditions.

(a) $\phi,\ \Omega \in \mathcal{F}$ (b) $A \in \mathcal{F} \Rightarrow A^c (= \Omega \setminus A) \in \mathcal{F}$ (c) $\{A_n\}_{n=1}^{\infty}$ is in $\mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

(i) So, the smallest σ-field on Ω is $\mathcal{F} = \{\phi,\ \Omega\}$. It satisfies all three requirements for a collection to be a σ-field. This is known as **trivial** σ-field on Ω.

(ii) Keeping in view the three requirements for a class of events to be a σ-field the smallest σ-field containing a set A on Ω is given by $\mathcal{F} = \{\phi,\ A,\ A^c,\ \Omega\}$

In our case A = {HT} and Ω = {HH, HT, TH, TT}, therefore, required σ-field is given by $\mathcal{F} = \{\phi, \{HT\}, \{HH, TH, TT\},\ \Omega\}$

(iii) By definition of σ-field it is a collection of subsets of Ω which satisfies three conditions. Obviously, from school mathematics you know that the largest collection of subsets of any set is its power set. Further, power set is closed with respect to complement and union. So, the largest σ-field on Ω is the power set of Ω and is given by

$$\mathcal{F} = \left\{\begin{array}{l} \phi, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \\ \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, HT, TT\}, \\ \{HH, TH, TT\}, \{HT, TH, TT\}, \Omega \end{array}\right\}$$

Note 1: We have seen that both $\{\phi,\ \Omega\}$ and $P(\Omega) =$ power set, always form σ-field on Ω so, we can say that both are trivial σ-fields on Ω. Further, $\{\phi,\ \Omega\}$ is the smallest σ-field on Ω, while $P(\Omega) =$ power set of Ω is the largest σ-field on Ω. There may or may not exist σ-fields on Ω other than these two refer SAQ 2 and Terminal Question 1 of this unit. To see one way of constructing σ-fields on Ω, you may refer part (ii) of the previous Example 1 and the next Example 2. Also, refer generalisation given after Example 2. You have also done similar thing in Examples 1 and 2 of Unit 3 of the course MST-011.

Example 2: If a die is thrown then write sample space (Ω) and the smallest σ-field on Ω. Also, write the smallest σ-field containing the events E = {1, 2} and F = {2, 3, 6} on Ω.

Solution: When a die is thrown then sample space is given by

$\Omega = \{1, 2, 3, 4, 5, 6\}$

The smallest σ-fields on Ω is given by $\mathcal{F} = \{\phi,\ \Omega\}$.

We know that a collection of subsets of a set Ω form a σ-field if it contains empty set, full set, closed under complement and closed under countable union. Therefore, a collection containing the events E and F will form a σ-field on Ω if it contains following subsets of Ω

$$\mathcal{F} = \begin{Bmatrix} \phi, E, F, E^c, F^c, E \cap F, E \cap F^c, E^c \cap F, E^c \cap F^c, E \cup F, \\ E \cup F^c, E^c \cup F, E^c \cup F^c, E\Delta F, (E\Delta F)^c, \Omega \end{Bmatrix}$$

where E^c denotes complement of the event E, etc.

In our case E = {1, 2}, F = {2, 3, 6} and $\Omega = \{1, 2, 3, 4, 5, 6\}$ so

$$E^c = \{3, 4, 5, 6\}, F^c = \{1, 4, 5\}, E \cap F = \{2\}, E \cap F^c = \{1\}, E^c \cap F = \{3, 6\},$$
$$E^c \cap F^c = \{4, 5\}, E \cup F = \{1, 2, 3, 6\}, E \cup F^c = \{1, 2, 4, 5\},$$
$$E^c \cup F = \{2, 3, 4, 5, 6\}, E^c \cup F^c = \{1, 3, 4, 5, 6\},$$
$$E\Delta F = (E \cup F) \setminus (E \cap F) = \{1, 3, 6\}, (E\Delta F)^c = \{2, 4, 5\}$$

Hence, required σ-field on Ω after leaving repeated elements (if any) as repetition of elements is not allowed is given by

$$\mathcal{F} = \begin{Bmatrix} \phi, \{1, 2\}, \{2, 3, 6\}, \{3, 4, 5, 6\}, \{1, 4, 5\}, \{2\}, \{1\}, \{3, 6\}, \\ \{4, 5\}, \{1, 2, 3, 6\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \\ \{1, 3, 6\}, \{2, 4, 5\}, \Omega \end{Bmatrix}$$

Generalisation: Every collection of subsets of a non-empty set Ω do not form a σ-field. But we can add more subsets of Ω in the given collection to form a σ-field. How many subsets at the most we need in total to form a σ-field can be understood as follows.

From part (ii) of Example 1 note that if a collection has one set other than ϕ and Ω then we need at the most $4 \left(= 2^{2^1}\right)$ subsets of Ω to form a σ-field. From Example 2 note that if a collection has two sets other than ϕ and Ω, then we need at the most $16 \left(= 2^{2^2}\right)$ subsets of Ω to form a σ-field. In general, if a collection has n subsets of Ω other than ϕ and Ω, then we need at the most 2^{2^n} subsets of Ω including ϕ and Ω, to form a σ-field with these given n subsets of Ω. But in practice we have less number of subsets compare to general case. For example, in Example 2 of Unit 3 of the course MST-011, we had only 8 subsets instead of 16. … (2.12)

Why σ-field is suitable to deal with probabilities of Events: The following points explain it.

- In probability theory at many occasions, you need probabilities of empty set and full sample space. Luckily both are included in every σ-field. So, when we assign probabilities to members of σ-field then probabilities will also be associated to these two special events of our interest. So, σ-field qualifies this first requirement to work in the world of probability theory. … (2.13)
- Second requirement of the world of probability theory is whenever you have an event then you also need to deal with its complement. Fortunately, this

requirement is also satisfied by σ-field because σ-field is closed with respect to complement. So, when we assign probabilities to the members of σ-field then probabilities will be assigned to complement of each element of σ-field as well. So, σ-field also qualifies this second requirement to work in the world of probability theory. … (2.14)

- Third and the most important requirement of probability theory is countable additivity, refer Axiom 3 or (1.20) in the previous unit. Which means if you have probabilities of individuals events which are countable as well as disjoint then you can obtain probability of happening of at least one of them using their individual probability as follows

  $$\mathcal{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathcal{P}(E_n)$$

  This requirement of probability world is also satisfied by σ-field because σ-field is closed with respect to countable union. … (2.15)

Thus, we see that σ-field satisfies all basic requirements required/needed to work in the world of probability theory. Due to these silent features of σ-field we discussed it in Unit 3 of the course MST-011 and here so that you can understand its meaning before we build the notion of probability theory on it. In the next section, we will continue the discussion of σ-field and will prove some of its important properties.

Now, you can try the following Self-Assessment Question.

SAQ 2

If a coin is tossed once then write sample space of this random experiment and also write all possible σ-fields on this sample space.

2.4 PROPERTIES OF σ-FIELD

Using definition of σ-field and some results of set theory, we can deduce some properties which will help in doing problems related to σ-field. Some of these properties are stated and proved as follows.

Property 1: If $\mathcal{F}$ is a σ-field on Ω then prove that it is closed with respect to finite union. … (2.16)

Proof: We know that a collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies the following three conditions.

(a) $\phi,\ \Omega \in \mathcal{F}$ (b) $A \in \mathcal{F} \Rightarrow A^c (= \Omega \setminus A) \in \mathcal{F}$ (c) $\{A_n\}_{n=1}^{\infty}$ is in $\mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

Let $A_1, A_2, A_3, \ldots, A_n$ be a finite collection of subsets of Ω which are members of $\mathcal{F}$ then required to prove $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$. Consider the countable collection $A_1, A_2, A_3, \ldots, A_n, \phi, \phi, \phi, \ldots$ where after n all the members are empty sets, i.e. $A_k = \phi$, for all $k \geq n+1$

Now, $\bigcup_{k=1}^{n} A_k = \left(\bigcup_{k=1}^{n} A_k\right) \cup \phi$ $\quad [\because \phi$ contributes nothing in the union$]$

$$= \left(\bigcup_{k=1}^{n} A_k\right) \cup \left(\bigcup_{k=n+1}^{\infty} A_k\right) \quad [\because A_k = \phi, \text{ for all } k \geq n+1]$$

$$= \bigcup_{k=1}^{\infty} A_k \in \mathcal{F} \qquad [\because \text{each } A_k \in \mathcal{F} \text{ and } \mathcal{F} \text{ is a } \sigma\text{-field}]$$

$$\Rightarrow \bigcup_{k=1}^{n} A_k \in \mathcal{F}$$

Property 2: If $\mathcal{F}$ is a σ-field on Ω then prove that it is closed with respect to countable intersection. … (2.17)

Proof: We know that a collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies the following three conditions.

(a) $\phi,\ \Omega \in \mathcal{F}$ (b) $A \in \mathcal{F} \Rightarrow A^c (= \Omega \setminus A) \in \mathcal{F}$ (c) $\{A_n\}_{n=1}^{\infty}$ is in $\mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

Let $A_1, A_2, A_3, \ldots, A_n, \ldots$ be a countable collection of subsets of Ω which are members of $\mathcal{F}$ then required to prove $\bigcap_{k=1}^{\infty} A_k \in \mathcal{F}$.

Since, each $A_k \in \mathcal{F} \Rightarrow$ each $A_k^c \in \mathcal{F}$ $[\because \mathcal{F}$ is a σ-field and using (b)$]$

$$\Rightarrow \bigcup_{k=1}^{\infty} A_k^c \in \mathcal{F} \qquad [\because \mathcal{F} \text{ is a } \sigma\text{-field and using (c)}] \quad \ldots (2.18)$$

Now, $\left(\bigcap_{k=1}^{\infty} A_k\right)^c = \bigcup_{k=1}^{\infty} A_k^c$ [Using De Morgan's law] … (2.19)

From (2.18) and (2.19), we have $\left(\bigcap_{k=1}^{\infty} A_k\right)^c \in \mathcal{F}$

$$\Rightarrow \left(\left(\bigcap_{k=1}^{\infty} A_k\right)^c\right)^c \in \mathcal{F} \qquad [\because \mathcal{F} \text{ is a } \sigma\text{-field and using (b)}]$$

$$\Rightarrow \bigcap_{k=1}^{\infty} A_k \in \mathcal{F} \qquad \left[\because \left(B^c\right)^c = B\right]$$

Hence, proved.

Property 3: If $\mathcal{F}$ is a σ-field on Ω then prove that it is closed with respect to finite intersection. … (2.20)

Proof: We know that a collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies the following three conditions.

(a) $\phi,\ \Omega \in \mathcal{F}$ (b) $A \in \mathcal{F} \Rightarrow A^c (= \Omega \setminus A) \in \mathcal{F}$ (c) $\{A_n\}_{n=1}^{\infty}$ is in $\mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

Let $A_1, A_2, A_3, \ldots, A_n$ be a finite collection of subsets of Ω which are members of $\mathcal{F}$ then required to prove $\bigcap_{k=1}^{n} A_k \in \mathcal{F}$.

Since, each $A_k \in \mathcal{F} \Rightarrow$ each $A_k^c \in \mathcal{F}$ $[\because \mathcal{F}$ is a σ-field and using (b)$]$

$$\Rightarrow \bigcup_{k=1}^{n} A_k^c \in \mathcal{F} \qquad [\because \mathcal{F} \text{ is a } \sigma\text{-field and using property 1}] \quad \ldots (2.21)$$

Now, $$\left(\bigcap_{k=1}^{n} A_k\right)^c = \bigcup_{k=1}^{n} A_k^c \qquad [\text{Using De Morgan's law}] \quad \ldots (2.22)$$

From (2.21) and (2.22), we have $\left(\bigcap_{k=1}^{n} A_k\right)^c \in \mathcal{F}$

$$\Rightarrow \left(\left(\bigcap_{k=1}^{n} A_k\right)^c\right)^c \in \mathcal{F} \qquad [\because \mathcal{F} \text{ is a } \sigma\text{-field and using (b)}]$$

$$\Rightarrow \bigcap_{k=1}^{n} A_k \in \mathcal{F} \qquad \left[\because \left(B^c\right)^c = B\right]$$

Hence, proved.

In the next unit, we will discuss how we assign probabilities to events in discrete and continuous worlds of probability theory. To do that job you need to have good understanding of probability measure and probability space which are discussed in the next section.

2.5 PROBABILITY MEASURE AND PROBABILITY SPACE

As mentioned just before starting of this section, we will discuss how we can assign probabilities to the events in discrete and continuous worlds of probability theory in the next unit. But to do that job, we will need probability measure. So, in this section, first, we will define what is a probability measure. However, in Sec. 3.4 of Unit 3 of the course MST-011 (you may refer (3.23)), we have defined when a set function is called a measure. Since probability measure is a particular case of a measure so let us first recall the definition of a measure.

Measure: Let Ω be a non-empty set and $\mathcal{F}$ be a collection of subsets of Ω which forms a σ-field then a set function $\mu : \mathcal{F} \to [0, +\infty]$ is called a measure if it satisfies the following two requirements … (2.23)

(i) $\mu(\phi) = 0,$ i.e., measure of an empty set should be zero

(ii) μ is countably additive, i.e., whenever $A_1, A_2, A_3, \ldots \in \mathcal{F}$ with

$$A_m \cap A_n = \phi \text{ for all } m \neq n \text{ then } \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu\left(A_n\right)$$

Since μ is a function from $\mathcal{F}$ to $[0, +\infty]$ so it will assign a unique measure to each member of the σ-field $\mathcal{F}$ and so members of the σ-field $\mathcal{F}$ are known as **measurable sets**. It means each member of $\mathcal{F}$ can be measured using the definition of the measure μ. … (2.24)

As mentioned earlier in this section, we will define a special measure known as probability measure. But before that, you should know when a measure is said to be finite and infinite. So, finite, infinite and probability measures all are defined one at a time as follows.

Finite Measure: A measure μ is called finite if $\mu(\Omega) < \infty$. … (2.25)

Infinite Measure: A measure μ is called infinite if $\mu(\Omega) = \infty$. … (2.26)

Probability Measure: A measure μ is called probability measure if $\mu(\Omega) = 1$. … (2.27)

First of all, probability measure is a measure so its primary job is to assign a number to each member of the σ-field $\mathcal{F}$. But being a probability measure it assigns a number between 0 and 1 including 0 and 1 to events (members of $\mathcal{F}$) which indicate their relative likelihoods. So, proper definition of probability measure is given as follows.

Probability Measure: Let Ω be the sample space of a random experiment and $\mathcal{F}$ is a σ-field of subsets of Ω which are of our interest then the set function $\mathcal{P}: \mathcal{F} \to [0, 1]$ is called a probability measure if it satisfies the following three conditions

(i) $\mathcal{P}(A) \geq 0$ for all $A \in \mathcal{F}$

(ii) $\mathcal{P}(\Omega) = 1$, i.e., measure of the full space is 1

(iii) $\mathcal{P}$ is countably additive, i.e., whenever $A_1, A_2, A_3, \ldots \in \mathcal{F}$ with $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ and $A_m \cap A_n = \phi$ for all $m \neq n$ then $\mathcal{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathcal{P}(A_n)$ … (2.28)

So, you can say that probability measure is a particular case of a finite measure. In probability theory, members of σ-field are known as **events**. Since the σ-field $\mathcal{F}$ contains all events of our interest which we want to measure and can be measured by defining an appropriate probability measure on $\mathcal{F}$. Here by measure of an event, we mean probability of that event. The pair $(\Omega, \mathcal{F})$ indicates that we have a sample space and a σ-field $\mathcal{F}$ on Ω and is known as **measurable space**. Measurable space means all members of the σ-field $\mathcal{F}$ are readily available to get their measure. So, we just need to define some appropriate probability measure on $\mathcal{F}$ to assign a measure to each member of the σ-field $\mathcal{F}$. Further, domain of the probability measure is the σ-field $\mathcal{F}$ so by definition of a function each member of $\mathcal{F}$ will be assigned some measure between 0 and 1 including 0 and 1 so all the members of $\mathcal{F}$ have been assigned a measure or number from 0 to 1. From Unit 1, you know that probability of an event (E) also satisfies the inequality $0 \leq P(E) \leq 1$. So, probability measure ($\mathcal{P}$) assigns probability to each member of the σ-field $\mathcal{F}$ and so the triplet $(\Omega, \mathcal{F}, \mathcal{P})$ is known as **probability space**. So, probability space is a universe where three things Ω, $\mathcal{F}$ and $\mathcal{P}$ are related to each other explained as follows: … (2.29)

- The full space Ω is a set which contains all possible outcomes of the random experiment.
- The second thing of this universe is $\mathcal{F}$ which is a collection of subsets of Ω which are of our interest and having at least two members ϕ and Ω. It is closed under complement and countable union.

- The third thing of this universe is $\mathcal{P}$, a probability measure which assigns probability to each member of $\mathcal{F}$.

Fig. 2.2 further explains each member of the triplet $(\Omega, \mathcal{F}, \mathcal{P})$ with the help of the following example.

Consider a die which is exactly the same in shape like a usual die but instead of 1 to 6 numbers on its six faces:

- on three faces 1 is written,
- on two faces 2 is written and
- on one face 3 is written.

If this die is thrown then probability space will be the triplet $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω, $\mathcal{F}$ and $\mathcal{P}$ are shown in Fig. 2.2. … (2.30)

In this example, applying knowledge of classical probability studied in earlier classes as well as in the previous unit, you can write probabilities of all eight events as follows.

$$\left.\begin{array}{l}\mathcal{P}(\Omega)=1,\ \mathcal{P}(\{1\})=\dfrac{3}{6},\ \mathcal{P}(\{2\})=\dfrac{2}{6},\ \mathcal{P}(\{3\})=\dfrac{1}{6},\\ \mathcal{P}(\{1,\ 2\})=\dfrac{5}{6},\ \mathcal{P}(\{1,\ 3\})=\dfrac{4}{6},\ \mathcal{P}(\{2,\ 3\})=\dfrac{3}{6},\ \mathcal{P}(\phi)=0.\end{array}\right\} \quad \dots (2.31)$$

How, these probabilities are assigned using the idea of σ-field is discussed in Sec. 3.2 of the next unit.

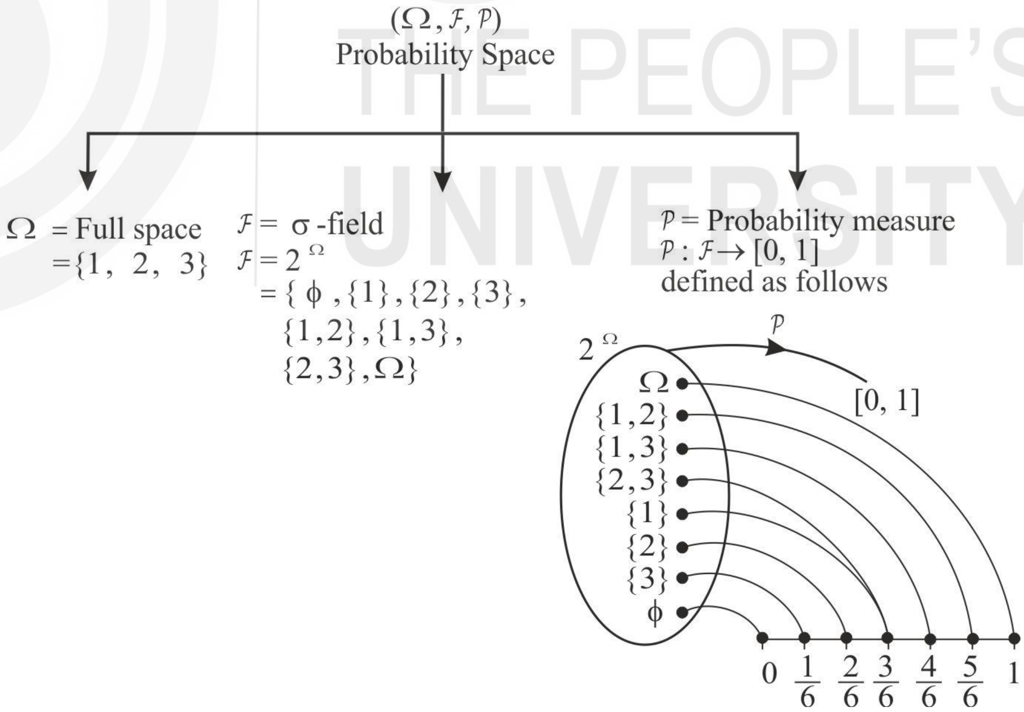

Fig. 2.2: Visualisation of the role of each member of probability space

Before discussing some examples based on probability measure, let us first state and prove some properties of probability measure.

Property 4: If $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space then prove that $\mathcal{P}$ is finitely additive for a disjoint union of events. … (2.32)

Proof: Let $A_1, A_2, A_3, \ldots, A_n \in \mathcal{F}$ with $A_m \cap A_n = \phi$ for all $m \neq n$ then required to prove $\mathcal{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathcal{P}(A_k)$.

Consider the countable collection $A_1, A_2, A_3, \ldots, A_n, \phi, \phi, \phi, \ldots$ where after n all members are empty sets, i.e., $A_k = \phi$, for all $k \geq n+1$

Now, $\mathcal{P}\left(\bigcup_{k=1}^{n} A_k\right) = \mathcal{P}\left(\left(\bigcup_{k=1}^{n} A_k\right) \cup \phi\right)$ $\left[\begin{array}{l}\because \phi \text{ contributes nothing} \\ \text{in the union}\end{array}\right]$

$$= \mathcal{P}\left(\left(\bigcup_{k=1}^{n} A_k\right) \cup \left(\bigcup_{k=n+1}^{\infty} A_k\right)\right) \quad [\because A_k = \phi, \text{ for all } k \geq n+1]$$

$$= \mathcal{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mathcal{P}(A_k) \quad [\text{Using (2.28)}]$$

$$= \sum_{k=1}^{n} \mathcal{P}(A_k) + \sum_{k=n+1}^{\infty} \mathcal{P}(A_k)$$

$$= \sum_{k=1}^{n} \mathcal{P}(A_k) \quad [\because \mathcal{P}(\phi) = 0 \text{ and } A_k = \phi \ \forall \ k \geq n+1]$$

$$\Rightarrow \mathcal{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathcal{P}(A_k)$$

Property 5: If $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space then prove that $\mathcal{P}$ is monotonic. That is if $E, F \in \mathcal{F}$ be such that $E \subseteq F$ then $\mathcal{P}(E) \leq \mathcal{P}(F)$. … (2.33)

Proof: Since $E \subseteq F$ so, we can write

$$F = E \cup (F \backslash E) \Rightarrow \mathcal{P}(F) = \mathcal{P}(E \cup (F \backslash E))$$

$$\Rightarrow \mathcal{P}(F) = \mathcal{P}(E) + \mathcal{P}(F \backslash E) \quad [\text{Using (2.32)}] \quad \ldots (2.34)$$

But we know that $\mathcal{P}(A) \geq 0$ for any event A.

So, in particular $\mathcal{P}(F \backslash E) \geq 0$. … (2.35)

Using (2.35) in (2.34), we have $\mathcal{P}(F) \geq \mathcal{P}(E)$ or $\mathcal{P}(E) \leq \mathcal{P}(F)$.

Property 6: If $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $E, F \in \mathcal{F}$ then

$$\left.\begin{array}{l}\mathcal{P}(F \backslash E) = \mathcal{P}(F \cap E^c) = \mathcal{P}(F) - \mathcal{P}(E \cap F) \\ \text{In particular, if } E \subseteq F, \text{ then } \mathcal{P}(F \backslash E) = \mathcal{P}(F \cap E^c) = \mathcal{P}(F) - \mathcal{P}(E)\end{array}\right\} \ldots (2.36)$$

Proof: Any event F can be written as disjoint union of the two events $F \backslash E$ and $E \cap F$, i.e.,

$$F = (F \backslash E) \cup (E \cap F)$$

$$\Rightarrow \mathcal{P}(F) = \mathcal{P}((F \backslash E) \cup (E \cap F))$$

$$= \mathcal{P}(F \backslash E) + \mathcal{P}(E \cap F) \quad [\text{Using (2.32)}]$$

$$\Rightarrow \mathcal{P}(F) = \mathcal{P}(F \backslash E) + \mathcal{P}(E \cap F)$$

$$\Rightarrow \mathcal{P}(F \backslash E) = \mathcal{P}(F) - \mathcal{P}(E \cap F)$$

or $\mathcal{P}(F \cap E^c) = \mathcal{P}(F) - \mathcal{P}(E \cap F)$ $[F \backslash E = F \cap E^c]$

In particular, if

$E \subseteq F$, then $\mathcal{P}(F \setminus E) = \mathcal{P}(F \cap E^c) = \mathcal{P}(F) - \mathcal{P}(E)$ $\quad [\because E \subseteq F \Rightarrow E \cap F = E]$

Property 7: If $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, then for any event $E \in \mathcal{F}$ prove that $\mathcal{P}(E^c) = 1 - \mathcal{P}(E)$. … (2.37)

Proof: For any event E, we have $E \cup E^c = \Omega \Rightarrow \mathcal{P}(E \cup E^c) = \mathcal{P}(\Omega)$

$$\Rightarrow \mathcal{P}(E) + \mathcal{P}(E^c) = 1 \quad [\text{Using (2.32) in LHS and } \mathcal{P}(\Omega) = 1 \text{ in RHS}]$$

$$\Rightarrow \mathcal{P}(E^c) = 1 - \mathcal{P}(E).$$

Property 8: If $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, then for any event $E, F \in \mathcal{F}$ prove that $\mathcal{P}(E \cup F) = \mathcal{P}(E) + \mathcal{P}(F) - \mathcal{P}(E \cap F)$. … (2.38)

Proof: For any two events $E, F \in \mathcal{F}$, we can write

$$E \cup F = (E \setminus F) \cup (E \cap F) \cup (F \setminus E)$$

$$\Rightarrow \mathcal{P}(E \cup F) = \mathcal{P}((E \setminus F) \cup (E \cap F) \cup (F \setminus E))$$

$$= \mathcal{P}((E \setminus F) + \mathcal{P}(E \cap F) + \mathcal{P}(F \setminus E)) \quad [\text{Using (2.32) in RHS}]$$

$$= \mathcal{P}(E) - \mathcal{P}(E \cap F) + \mathcal{P}(E \cap F) + \mathcal{P}(F) - \mathcal{P}(E \cap F) [\text{Using (2.36) in RHS}]$$

$$\Rightarrow \mathcal{P}(E \cup F) = \mathcal{P}(E) + \mathcal{P}(F) - \mathcal{P}(E \cap F).$$

Now, we consider some examples where first, we will define a measure on some σ-field and then check whether the defined measure is a probability measure or not on that σ-field.

Example 3: If Ω be the sample space of the random experiment of tossing a coin and $\mathcal{F}_1, \mathcal{F}_2$ be respectively the smallest and the largest σ-fields on Ω. Does the measure $\mu : \mathcal{F}_1 \to [0, 1]$ is defined by

$$\mu(E) = \begin{cases} 0, & \text{if } E = \phi \\ 1, & \text{if } E \neq \phi \end{cases} \quad E \in \mathcal{F}_1 \qquad \text{… (2.39)}$$

is a probability measure. Also, check whether the same measure defined on the largest σ-field $\mathcal{F}_2$, i.e., $\mu : \mathcal{F}_2 \to [0, 1]$ defined by

$$\mu(E) = \begin{cases} 0, & \text{if } E = \phi \\ 1, & \text{if } E \neq \phi \end{cases} \quad E \in \mathcal{F}_2 \qquad \text{… (2.40)}$$

is a probability measure.

Solution: When a coin is tossed then sample space is given by $\Omega = \{H, T\}$.

The smallest σ-field on Ω is given by $\mathcal{F}_1 = \{\phi, \Omega\}$.

The largest σ-field on Ω is given by $\mathcal{F}_2 = \{\phi, \{H\}, \{T\}, \Omega\}$.

To check whether μ is a probability measure or not we have to check three requirements (i) non-negativity (ii) $\mu_1(\Omega) = 1$ and (iii) countable additivity.

By definition of the set function μ given by (2.39), it gives only two values 0 and 1 as output hence $\mu(E) \geq 0 \;\; \forall \; E \in \mathcal{F}_1 \;\Rightarrow\; \mu$ is non-negative. So, (i) holds.

Since $\Omega \neq \phi$ so by (2.39), we have $\mu(\Omega) = 1.$ Implies this (ii) holds.

Let $E_1, E_2, E_3, \ldots$ be the countable collection of disjoint members of the σ-field $\mathcal{F}_1,$ then countable additivity holds because if Ω presents in this collection then both sides of the expression $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ will be equal to 1 and if Ω does not present then both sides of this expression will be zero. Hence, the set function μ is a probability measure and the triplet $(\Omega, \mathcal{F}_1, \mu)$ is a probability space. … (2.41)

Now, we claim that μ is not a probability measure on $\mathcal{F}_2,$ because $\mu(\Omega) \neq 1$ which is shown as follows.

$$\mu(\Omega) = \mu(\{H, T\}) = \mu(\{H\} \cup \{T\}) = \mu(\{H\}) + \mu(\{T\}) \quad \begin{bmatrix} \text{Using (2.32) in RHS} \\ \text{for a measure} \end{bmatrix}$$

$$= 1 + 1 = 2 \neq 1$$

Hence, the set function μ is not a probability measure on the largest σ-field $\mathcal{F}_2$ on $\Omega,$ while it was a probability measure on the smallest σ-field $\mathcal{F}_1$ on $\Omega.$ So, the triplet $(\Omega, \mathcal{F}_2, \mu)$ is not a probability space. … (2.42)

Note 2: In the case of the smallest σ-field $\mathcal{F}_1$ on $\Omega,$ we cannot write $\Omega = \{H, T\}$ as $\{H, T\} = \{H\} \cup \{T\}$ because $\{H\}, \{T\} \notin \mathcal{F}_1,$ while $\{H\}, \{T\} \in \mathcal{F}_2.$

Example 4: If Ω be the sample space of a random experiment and so it is non-empty. Suppose $\omega \in \Omega$ be a fixed element of Ω and $\mathcal{F}$ be a σ-field on $\Omega.$ Show that the function $\delta_\omega : \mathcal{F} \to [0, 1]$ defined by

$$\delta_\omega(E) = \begin{cases} 0, & \text{if } \omega \notin E \\ 1, & \text{if } \omega \in E \end{cases} \qquad E \in \mathcal{F} \qquad \text{… (2.43)}$$

is a probability measure.

Solution: To check whether δ_ω is a probability measure or not we have to check three requirements (i) non-negativity (ii) $\delta_\omega(\Omega) = 1$ and (iii) countable additivity.

By definition of the set function δ_ω given by (2.43), it gives only two numbers 0 and 1 as output hence $\delta_\omega(E) \geq 0 \; \forall \; E \in \mathcal{F} \;\Rightarrow \delta_\omega$ is non-negative. So, (i) holds.

Since $\omega \in \Omega$ so by (2.43), we have $\delta_\omega(\Omega) = 1.$ Implies this (ii) holds.

Let $E_1, E_2, E_3, \ldots$ be the countable collection of disjoint members of the σ-field $\mathcal{F},$ then we claim that countable additivity holds. To prove our claim, we have to consider two cases.

Case I: $\omega \in \bigcup_{n=1}^{\infty} E_n$

Since $\omega \in \bigcup_{n=1}^{\infty} E_n$ so by (2.43), we have $\delta_\omega\left(\bigcup_{n=1}^{\infty} E_n\right) = 1$ … (2.44)

Also, $\omega \in \bigcup_{n=1}^{\infty} E_n \Rightarrow \omega \in E_i$ for exactly one i as E_i's are pairwise disjoint. Let

$\omega \in E_{i_0}$ for some fixed i_0, i_0 may be any one among 1, 2, 3, 4, …

$\Rightarrow \delta_{\omega}\left(E_{i_0}\right) = 1$ and $\delta_{\omega}\left(E_j\right) = 0, \ \forall$ j other than i_0

Hence, $\sum_{n=1}^{\infty} \delta_{\omega}(E_n) = 1$ … (2.45)

From (2.44) and (2.45), we have $\delta_{\omega}\left(\bigcup_{n=1}^{\infty} E_n\right) = 1 = \sum_{n=1}^{\infty} \delta_{\omega}(E_n)$.

Case II: $\omega \notin \bigcup_{n=1}^{\infty} E_n$

Since $\omega \notin \bigcup_{n=1}^{\infty} E_n$ so by (2.43), we have $\delta_{\omega}\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$ … (2.46)

Also, $\omega \notin \bigcup_{n=1}^{\infty} E_n \Rightarrow \omega \notin E_i$ for any i $\left[\because x \notin A \cup B \Rightarrow x \notin A \text{ and } x \notin B\right]$

Since $\omega \notin E_n$ for any $n = 1, 2, 3, 4, \ldots$

$\Rightarrow \delta_{\omega}(E_n) = 0 \quad \forall\ n = 1, 2, 3, 4, \ldots$

$$\Rightarrow \sum_{n=1}^{\infty} \delta_{\omega}(E_n) = 0 \quad \left[\begin{array}{l}\underbrace{0+0+0+0+\ldots}_{\text{Infinite number of times}} = 0.\infty \text{ and we know that} \\ \text{in extended real number system } 0.\infty = 0\end{array}\right] \quad \ldots (2.47)$$

From (2.46) and (2.47), we have $\delta_{\omega}\left(\bigcup_{n=1}^{\infty} E_n\right) = 0 = \sum_{n=1}^{\infty} \delta_{\omega}(E_n)$.

We see that in both cases δ_{ω} satisfies additivity condition. Hence, δ_{ω} defined by (3.43) is a probability measure and the triplet $\left(\Omega, \mathcal{F}, \delta_{\omega}\right)$ is a probability space. Observe that in this probability space whole mass (probability) is sitting at one point ω.

This probability measure has a special name known as **Dirac delta measure**. … (2.48)

Remark 3: Note that if Ω be the sample space of the random experiment of throwing a die then $\Omega = \{1, 2, 3, 4, 5, 6\}$. Suppose we are working in the largest σ-field 2^{Ω}. Suppose E, F and G be the events defined as follows:

E: getting multiple of 3

F: getting even number

G: getting a number greater than 2

then E = {3, 6}, F = {2, 4, 6}, G = {3, 4, 5, 6}.

We now apply measures discussed in Examples 3 and 4 on these events. Applying the measure discussed in Example 3, we have

$\mu(E) = 1, \ \mu(F) = 1, \ \mu(G) = 1.$

Further, if we choose $\omega = 6$, then using the Dirac delta measure discussed in Example 4, we have

$\delta_6(E) = 1, \ \delta_6(F) = 1, \ \delta_6(G) = 1.$

If we look at cardinality of the events E, F and G, then we find that n(E) = 2, n(F) = 3, and $n(G) = 4.$ The measure μ discussed in Example 3 and Dirac delta measure discussed in Example 4 both do not take into account the cardinality of the events in assigning measure to the events. According to the measure μ discussed in Example 3 every non-empty event has measure 1. Similarly, according to Dirac delta measure whole mass is sitting at one point ω. But obviously, in discrete world if we see from probability point of view event F is more likely to happen compare to event E and event G is more likely to happen compare to events E and F. Such a measure which takes into account the cardinality of an event in view, before assigning probability to it, is discussed in Example 5.

Example 5: Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$ be the sample space of a random experiment having finite number of outcomes. If $\mathcal{F} = 2^{\Omega}$ = power set of Ω, then $\mathcal{F}$ will be the largest σ-field on Ω. Show that the set function $\mathcal{P} : \mathcal{F} \to [0, 1]$ defined by

$$\left.\begin{aligned} &\mathcal{P}(E) = \frac{n(E)}{n(\Omega)}, \text{ where } n(.) \text{ denotes cardinality of an event} \\ &\text{or } \mathcal{P}(E) = \frac{n(E)}{n}, \quad E \in \mathcal{F} \end{aligned}\right\} \quad \ldots (2.49)$$

is a probability measure.

Solution: To show that $\mathcal{P}$ is a probability measure, we have to prove three requirements (i) non-negativity (ii) $\mathcal{P}(\Omega) = 1$ and (iii) countable additivity.

By definition of the set function $\mathcal{P}$, we have

$$\mathcal{P}(\phi) = \frac{n(\phi)}{n(\Omega)} = \frac{0}{n} = 0, \text{ while for any non-empty event E, n(E) > 0 and so}$$

$$\mathcal{P}(E) = \frac{n(E)}{n(\Omega)} = \frac{\text{a number} > 0}{\text{a finite positive number}} > 0 \quad \forall\ E(\neq \phi) \in \mathcal{F}$$

Hence, $\mathcal{P}(E) \geq 0 \ \forall\ E \in \mathcal{F}$. So, non-negativity requirement holds.

$$\mathcal{P}(\Omega) = \frac{n(\Omega)}{n(\Omega)} = \frac{n}{n} = 1. \text{ So, (ii) requirement } \mathcal{P}(\Omega) = 1 \text{ also holds.}$$

To prove (iii) requirement of countable additivity. Let $E_k \in \mathcal{F},\ k = 1, 2, 3, \ldots$ be a countable sequence of disjoint events in $\mathcal{F}$. Since $\mathcal{F}$ is a σ-field so $\bigcup_{k=1}^{\infty} E_k \in \mathcal{F}$.

$$\text{Now, } \mathcal{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \frac{n\left(\bigcup_{k=1}^{\infty} E_k\right)}{n} = \frac{\sum_{k=1}^{\infty} n(E_k)}{n} \quad [\because E_i\text{'s are disjoints}]$$

$$= \sum_{k=1}^{\infty} \frac{n(E_k)}{n} = \sum_{k=1}^{\infty} \mathcal{P}(E_k)$$

which proves that $\mathcal{P}$ satisfies countable additivity condition. Hence, $\mathcal{P}$ is a probability measure.

Remark 4: The probability measure discussed in Example 5 assigns uniform probability $\frac{1}{n}$ to each member of the sample space Ω. So, it is known as **uniform probability law**. … (2.50)

In Remark 3, we specify three events E, F and G on the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let us obtain their measures using the measure discussed in Example 5. So, measures of events E, F and G are given by

$$\mathcal{P}(E) = \frac{n(E)}{n} = \frac{2}{6} = \frac{1}{3},\ \mathcal{P}(F) = \frac{n(F)}{n} = \frac{3}{6} = \frac{1}{2},\ \mathcal{P}(G) = \frac{n(G)}{n} = \frac{4}{6} = \frac{2}{3}.$$

These measures are simply probabilities of events E, F and G as per uniform probability law. In Unit 1 of this course, you have studied classical approach to probability theory where we were actually using uniform probability law because we were assuming that all outcomes are equally likely. As we have discussed in Unit 1 that one of the drawbacks of the classical approach to probability was that it fails when number of elements in the sample space is infinite. **So, remember uniform probability law does not work on countably infinite sample space**.

Now, let us connect two probability measures discussed in Examples 4 and 5 as follows.

Connection of Measures discussed in Examples 4 and 5: Consider the same experiment of throwing a die and the same events E, F and G as discussed in Remark 3. Now, we express the measure discussed in Example 5 in terms of the Dirac delta measure discussed in Example 4. Before doing so let us first specify $\omega_1 = 1, \omega_2 = 2, \omega_3 = 3, \omega_4 = 4, \omega_5 = 5, \omega_6 = 6.$ Now note that we can write

$$\mathcal{P}(E) = \frac{n(E)}{n} = \frac{2}{6} = \frac{0+0+1+0+0+1}{6}$$

$$= \frac{\delta_1(E) + \delta_2(E) + \delta_3(E) + \delta_4(E) + \delta_5(E) + \delta_6(E)}{6} \quad \left[\begin{array}{l} \because 3, 6 \in E,\ \text{so}\ \delta_3(E) = \delta_6(E) = 1 \\ \text{but}\ 1, 2, 4, 5 \notin E,\ \text{so} \\ \delta_1(E) = \delta_2(E) = \delta_4(E) = \delta_6(E) = 0 \end{array}\right]$$

$$= \frac{1}{6}\sum_{i=1}^{6} \delta_i(E)$$

Similarly, for event F, it can be written as

$$\mathcal{P}(F) = \frac{n(F)}{n} = \frac{3}{6} = \frac{0+1+0+1+0+1}{6}$$

$$= \frac{\delta_1(F) + \delta_2(F) + \delta_3(F) + \delta_4(F) + \delta_5(F) + \delta_6(F)}{6} \quad \left[\begin{array}{l} \because 2, 4, 6 \in F,\ \text{so}\ \delta_2(F) = \delta_4(F) = \delta_6(F) = 1 \\ \text{but}\ 1, 3, 5 \notin F,\ \text{so} \\ \delta_1(F) = \delta_3(F) = \delta_5(F) = 0 \end{array}\right]$$

$$= \frac{1}{6}\sum_{i=1}^{6} \delta_i(F)$$

In general, if $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$ then

$$\mu(E) = \frac{1}{n}\sum_{i=1}^{n}\delta_{\omega_i}(E) \quad \forall \ E \in 2^{\Omega} \qquad \ldots (2.51)$$

We can generate more probability measures from given probability measures. In Unit 7 of the course MST-011, you have studied about convex combination. The idea of convex combination can be used to generate more probability measures from given probability measures.

Convex Combination of Probability Measures is Again a Probability Measure

Property 1: If we have n probability measures $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots, \mathcal{P}_n$ on the measurable space $(\Omega, \mathcal{F})$ and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be non-negative real numbers such that $\sum_{i=1}^{n}\alpha_i = 1,$ then prove that the convex combination $\sum_{i=1}^{n}\alpha_i \mathcal{P}_i$ is also a probability measure. … (2.52)

Proof: To prove that $\sum_{i=1}^{n}\alpha_i \mathcal{P}_i$ is a probability measure we have to prove that following three requirements hold:

(i) non-negativity (ii) $\left(\sum_{i=1}^{n}\alpha_i \mathcal{P}_i\right)(\Omega) = 1$ and (iii) countable additivity.

Since each $\mathcal{P}_i, \ i = 1, 2, 3, \ldots, n$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$ so

(i) $\mathcal{P}_i(\phi) = 0$

(ii) $\mathcal{P}_i(\Omega) = 1$

(iii) $\mathcal{P}_i\left(\bigcup_{k=1}^{\infty}E_k\right) = \sum_{k=1}^{\infty}\mathcal{P}_i(E_k)$

Now, $\left(\sum_{i=1}^{n}\alpha_i \mathcal{P}_i\right)(\phi) = \sum_{i=1}^{n}\alpha_i \mathcal{P}_i(\phi) = \sum_{i=1}^{n}\alpha_i(0) = \sum_{i=1}^{n}0 = 0$ [Using (i)]

Also, each
$\alpha_i \geq 0$ and each $\mathcal{P}_i$ is a probability measure so $\mathcal{P}_i(E) \geq 0 \quad \forall \ E \in \mathcal{F}$

$$\Rightarrow \alpha_i \mathcal{P}_i(E) \geq 0 \quad \forall \ i, \ i = 1, 2, 3, 4, \ldots, n$$

$$\Rightarrow \sum_{i=1}^{n}\alpha_i \mathcal{P}_i(E) \geq 0 \Rightarrow \left(\sum_{i=1}^{n}\alpha_i \mathcal{P}_i\right)(E) \geq 0 \quad \forall \ E \in \mathcal{F}$$

Now,

$$\left(\sum_{i=1}^{n}\alpha_i \mathcal{P}_i\right)(\Omega) = \sum_{i=1}^{n}\alpha_i \mathcal{P}_i(\Omega) = \sum_{i=1}^{n}\alpha_i(1) \qquad \text{[Using (ii)]}$$

$$= \sum_{i=1}^{n}\alpha_i = 1 \qquad \text{[Given in the statement]}$$

Finally, let $E_k \in \mathcal{F}, \ k = 1, 2, 3, \ldots$ be a countable sequence of disjoint events in $\mathcal{F}$. Since $\mathcal{F}$ is a σ-field so $\bigcup_{k=1}^{\infty}E_k \in \mathcal{F}.$

$$\left(\sum_{i=1}^{n}\alpha_i \mathcal{P}_i\right)\left(\bigcup_{k=1}^{\infty}E_k\right) = \sum_{i=1}^{n}\alpha_i\left(\mathcal{P}_i\left(\bigcup_{k=1}^{\infty}E_k\right)\right)$$

$$= \sum_{i=1}^{n}\alpha_i\left(\sum_{k=1}^{\infty}\mathcal{P}_i\left(E_k\right)\right) \quad \text{[Using (iii)]}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{i=1}^{n}\alpha_i \mathcal{P}_i\left(E_k\right)\right) = \sum_{k=1}^{\infty}\left(\left(\sum_{i=1}^{n}\alpha_i \mathcal{P}_i\right)\left(E_k\right)\right)$$

So, the convex combination $\sum_{i=1}^{n}\alpha_i \mathcal{P}_i$ of the probability measures $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots, \mathcal{P}_n$ is also a probability measure on the same measurable space $(\Omega, \mathcal{F})$.

Now, you can try the following Self-Assessment Question.

SAQ 3

If we have a countable sequence of probability measures $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$ on the measurable space $(\Omega, \mathcal{F})$ and $\alpha_1, \alpha_2, \alpha_3, \ldots$ be non-negative real numbers such that $\sum_{i=1}^{\infty}\alpha_i = 1$, then prove that the convex combination $\sum_{i=1}^{\infty}\alpha_i \mathcal{P}_i$ is also a probability measure.

2.6 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- **Sample Space**: Set of all possible outcomes of a random experiment is called sample space of the random experiment.

- **Event**: A subset of a sample space of a random experiment in which we are **interested** is known as an event.

- **Elementary Event**: If the sample space of a random experiment is $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$, then each singleton subset of Ω, i.e., $\{\omega_i\}, 1 \le i \le n$ is called an **elementary event**.

- **Compound Event**: An event constituted by union of two or more than two elementary events is called a compound event.

- **Impossible Event**: An event is said to be impossible event if there is no outcome in the sample space of the random experiment which favours the happening of the event. It is denoted by $\{\ \}$ or ϕ.

- **Sure Event**: An event is said to be a sure event if all outcomes of the experiment favour the happening of the event.

- **Class of Events**: A collection of events is called class of events.

- **Collection of Classes**: If a collection is constituted of classes or we can say that members of the collection are classes itself then such collections are called collection of classes.

- Let Ω be a non-empty set and $\mathcal{F}$ be a collection of subsets of Ω then we say that the class $\mathcal{F}$ forms a **field** or **algebra** if it satisfies following three conditions: (i) $\phi,\ \Omega \in \mathcal{F}$, (ii) $A \in \mathcal{F} \Rightarrow A^c\ (= \Omega \setminus A) \in \mathcal{F}$, (iii) $\mathcal{F}$ is closed with respect to finite union.

- Let Ω be a non-empty set and $\mathcal{F}$ be a collection of subsets of Ω then we say that the class $\mathcal{F}$ forms a **sigma-field** (**σ-Field**) or **sigma-algebra (σ-algebra)** if it satisfies following three conditions: (i) $\phi,\ \Omega \in \mathcal{F}$,
(ii) $A \in \mathcal{F} \Rightarrow A^c\ (= \Omega \setminus A) \in \mathcal{F}$, (iii) $\mathcal{F}$ is closed with respect to countable union.

- **Properties of σ-field**

  - If $\mathcal{F}$ is a σ-field on Ω then prove that it is closed with respect to finite union.

  - If $\mathcal{F}$ is a σ-field on Ω then prove that it is closed with respect to countable intersection.

  - If $\mathcal{F}$ is a σ-field on Ω then prove that it is closed with respect to finite intersection.

- **Measure**: Let Ω be a non-empty set and $\mathcal{F}$ be a collection of subsets of Ω which forms a σ-field then a set function $\mu : \mathcal{F} \to [0, +\infty]$ is called a measure if it satisfies the following two requirements

  (i) $\mu(\phi) = 0$, i.e., measure of an empty set should be zero

  (ii) μ is countably additive, i.e., whenever $A_1, A_2, A_3, \ldots \in \mathcal{F}$ with

  $A_m \cap A_n = \phi$ for all $m \neq n$ then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

- **Finite Measure**: A measure μ is called finite if $\mu(\Omega) < \infty$.

- **Infinite Measure**: A measure μ is called infinite if $\mu(\Omega) = \infty$.

- **Probability Measure**: A measure is said to be a probability measure if $\mathcal{P}(\Omega) = 1$, i.e., if measure of the full space is 1 under $\mathcal{P}$.

- The σ-field $\mathcal{F}$ contains all the events of our interest which we want to measure and can be measured by defining an appropriate probability measure on $\mathcal{F}$ so the pair $(\Omega, \mathcal{F})$ is known as **measurable space**.

- Probability measure assigns probability to each member of the σ-field $\mathcal{F}$ and so the triplet $(\Omega, \mathcal{F}, \mathcal{P})$ is known as **probability space**. So, probability space is a universe where three things Ω, $\mathcal{F}$ and $\mathcal{P}$ are related to each other.

2.7 TERMINAL QUESTIONS

1. If Ω = {a, b, c, d} then what is the smallest σ-field containing the set A = {a, b} on Ω.

2. If Ω = {a, b, c, d} then what is the smallest σ-field containing the sets A = {a, b} and B = {b, c} on Ω.

2.8 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. Sample space of the random experiment of tossing a coin three times is given by

$$\Omega = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}$$

(a) It is a class of events because it has three members $\phi, \{\text{HHH}\}, \Omega$ which are events.

(b) Here our sample space is finite so falls in the category of discrete world. We know that in discrete world of probability every event is always a subset of the sample space and vice-versa. So, being a subset of the sample space, it is an event. In fact, it is a compound event because it is union of four elementary events $\{\text{HHH}\}, \{\text{TTH}\}, \{\text{THH}\}, \{\text{HTH}\}$.

(c) Due to the same reason mentioned in part (b) being subset of the sample space it is also an event. However, it is an elementary event because it contains single individual outcome TTT.

(d) Due to the same reason mentioned in part (b) being subset of the sample space it is also an event. Another explanation for the same is: it is an empty subset of the sample space and we are always interested in empty set. So, ϕ being the subset of the sample space in which we are interested is an event. It is a special event known as impossible event.

(e) Due to the same reason mentioned in part (b) being subset of the sample space it is also an event. Like part (d) another reason may be: sample space itself is always of our interest so being subset of the sample space in which we are interested is an event. But this special event includes all outcomes of the experiment so it is known as the sure event.

(f) This collection has two members $\{\phi, \{\text{HHH}\}, \Omega\}$ and $\{\{\text{HHH}\}, \{\text{TTH}\}, \Omega\}$ which are itself classes of events. So, it a collection of classes.

2. When a coin is tossed then sample space is given by $\Omega = \{\text{H, T}\}$. The smallest and the largest σ-fields on Ω are given by

$$\mathcal{F}_1 = \{\phi,\ \Omega\},\ \mathcal{F}_2 = \{\phi,\ \{\text{H}\},\ \{\text{T}\},\ \Omega\}$$

There is no other σ-field on Ω. Hence, on this sample space only these two σ-fields are possible.

3. To check whether $\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i$ is a probability measure or not we have to check three requirements (i) non-negativity (ii) $\left(\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i\right)(\Omega) = 1$ and (iii) countable additivity.

Since each $\mathcal{P}_i$, $i = 1, 2, 3, \ldots$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$ so

(i) $\mathcal{P}_i(\phi) = 0$

(ii) $\mathcal{P}_i(\Omega) = 1$

(iii) $\mathcal{P}_i\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathcal{P}_i(E_k)$

Now, $\left(\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i\right)(\phi) = \sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i(\phi) = \sum_{i=1}^{\infty} \alpha_i (0) = 0$ [Using (i)]

Also, each $\alpha_i \geq 0$ and each $\mathcal{P}_i$ is a measure so $\mathcal{P}_i(E) \geq 0 \quad \forall \; E \in \mathcal{F}$

$\Rightarrow \alpha_i \mathcal{P}_i(E) \geq 0 \quad \forall \; i, i = 1, 2, 3, 4, \ldots$

$\Rightarrow \sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i(E) \geq 0 \Rightarrow \left(\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i\right)(E) \geq 0 \quad \forall \; E \in \mathcal{F}$

$$\text{Also, } \left(\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i\right)(\Omega) = \sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i(\Omega) = \sum_{i=1}^{\infty} \alpha_i (1) \quad [\text{Using (ii)}]$$

$$= \sum_{i=1}^{\infty} \alpha_i = 1 \qquad [\text{Given in the statement}]$$

Finally, let $E_k \in \mathcal{F}$, $k = 1, 2, 3, \ldots$ be a countable sequence of disjoint events in $\mathcal{F}$. Since $\mathcal{F}$ is a σ-field so $\bigcup_{k=1}^{\infty} E_k \in \mathcal{F}$.

$$\therefore \left(\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i\right)\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{i=1}^{\infty} \alpha_i \left(\mathcal{P}_i\left(\bigcup_{k=1}^{\infty} E_k\right)\right)$$

$$= \sum_{i=1}^{\infty} \alpha_i \left(\sum_{k=1}^{\infty} \mathcal{P}_i(E_k)\right) \quad [\text{Using (iii)}]$$

$$= \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i(E_k)\right) = \sum_{k=1}^{\infty} \left(\left(\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i\right)(E_k)\right)$$

So, the convex combination $\sum_{i=1}^{\infty} \alpha_i \mathcal{P}_i$ of the probability measures $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$ is also a probability measure on the same measurable space $(\Omega, \mathcal{F})$.

Terminal Questions

1. We know that the smallest σ-field containing the set A other than ϕ and

Ω on Ω is given by

$$\mathcal{F} = \{\phi,\ A,\ A^c,\ \Omega\}$$

In our case A = {a, b} and $\Omega = \{a, b, c, d\}$ so it becomes

$$\mathcal{F} = \{\phi,\ \{a, b\},\ \{c, d\},\ \Omega\}$$

i.e., $\mathcal{F} = P(\Omega)$ = the smallest σ-field on Ω containing A.

2. We know that the smallest σ-field containing the sets A and B other than ϕ and Ω on Ω is given by

$$\mathcal{F} = \left\{\begin{array}{l}\phi,\ A,\ B,\ A^c,\ B^c,\ A \cap B,\ A \cap B^c,\ A^c \cap B,\ A^c \cap B^c,\ A \cup B, \\ A \cup B^c,\ A^c \cup B,\ A^c \cup B^c,\ A \Delta B,\ (A \Delta B)^c,\ \Omega\end{array}\right\}$$

In our case A = {a, b}, B = {b, c} and $\Omega = \{a, b, c, d\}$ so it becomes

$$\mathcal{F} = \left\{\begin{array}{l}\phi,\ \{a, b\},\ \{b, c\},\ \{c, d\},\ \{a, d\},\ \{b\},\ \{a\},\ \{c\},\ \{d\},\ \{a, b, c\}, \\ \{a, b, d\},\ \{b, c, d\},\ \{a, c, d\},\ \{a, c\},\ \{b, d\},\ \{a, b, c, d\}\end{array}\right\}$$

i.e., $\mathcal{F} = P(\Omega)$ = the largest σ-algebra on Ω