UNIT 3

ASSIGNMENT OF PROBABILITIES TO EVENTS IN DISCRETE AND CONTINUOUS WORLDS

Structure

3.1 INTRODUCTION

In the previous unit, you have studied σ-field and some of its properties. We have also explained each member of the triplet $(\Omega, \mathcal{F}, \mathcal{P})$ which is known as probability space. The job of the third member $\mathcal{P}$ of the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is to assign probabilities to the members of $\mathcal{F}$ which are known as events. In Sec. 3.2, we will discuss how probabilities are assigned to events in discrete world of probability theory. In Sec. 3.7 we will discuss how the same job can be done in continuous world of probability theory. In Secs. 3.3 to 3.6 we will discuss some results of measure theory which are required to understand the discussion of Sec. 3.7.

What we have discussed in this unit is summarised in Sec. 3.8. Self-Assessment Questions (SAQs) have been given in some sections which are generally based on the content discussed in that section. But to give you a good practice of what we have discussed in this unit some more questions based on the entire unit are given in Sec. 3.9 under the heading Terminal Questions. Due to the reason mentioned in Sec. 1.1 of Unit 1 of this course, solutions of all the SAQs and Terminal Questions are given Sec. 3.10.

In the next unit, you will study univariate random variables in detail.

Expected Learning Outcomes

After completing this unit, you should be able to:

- answer the question where every subset is an event and where it is not;
- explain the procedure of assigning probabilities in the discrete and continuous worlds of probability theory;
- explain what is Borel σ-field and when it is needed; and
- explain the importance of Caratheodory's extension theorem in probability theory.

3.2 PROBABILITY ASSIGNMENT TO EVENTS IN DISCRETE WORLD

In the previous unit, you have understood all concepts like field, σ-field, some properties of σ-field, probability measure and probability space which are required to discuss the idea of assigning probabilities to events. In this section, we will discuss the procedure of assigning probabilities to the events in a discrete world. From (2.2) of Unit 2 of this course, you know that all the events are always subsets of the sample space but all subsets of the sample space are not always events. However, in the discrete world, all subsets of the sample space are also events. That is in a discrete world, we take the power set of the sample space Ω as σ-field. In a discrete world, sample space may be finite or countably infinite, so we have to consider both the cases which are discussed as follows.

Case I: When Sample Space Ω is Finite

Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$ … (3.1)

As mentioned earlier when sample space is finite, we take the power set of Ω as our σ-field. In probability theory, generally, power set of Ω is denoted by 2^{Ω}. So, we have $\mathcal{F} = 2^{\Omega}$.

Note that (3.1) can be written as a finite union of singleton subsets of Ω:

i.e., $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\} = \{\omega_1\} \cup \{\omega_2\} \cup \{\omega_3\} \cup \ldots \cup \{\omega_n\} = \bigcup_{i=1}^{n} \{\omega_i\}$ … (3.2)

We know that every event is always a subset of Ω and in discrete world every subset of Ω is also an event. So, if E is a non-empty event, then there will exists some i_k, $1 \le i_k \le n$, such that

$E = \{\omega_{i_1}, \omega_{i_2}, \omega_{i_3}, \ldots, \omega_{i_k}\} = \{\omega_{i_1}\} \cup \{\omega_{i_2}\} \cup \{\omega_{i_3}\} \cup \ldots \cup \{\omega_{i_k}\}$ … (3.3)

But here our σ-field is the power set of Ω so all subsets of Ω are members of σ-field and hence all singleton subsets of Ω are also members of the σ-field. This implies if we assign probabilities to all members of σ-field which are singleton subsets of Ω then probabilities will be assigned to each member of the σ-field due to the equation (3.3) and finite additivity of probability measure for disjoint events you may refer (2.32) of the previous unit. For example, if we assign probabilities $p_1, p_2, p_3, \ldots, p_n$ to the singleton subsets $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \ldots, \{\omega_n\}$ respectively such that

$$p_i \geq 0 \;\forall\; i = 1, 2, 3, 4, \ldots, n \text{ and } \sum_{i=1}^{n} p_i = 1 \qquad \ldots (3.4)$$

then using finite additivity of probability measure for disjoint events (refer 2.32), we have

$$\mathcal{P}(E) = \mathcal{P}\left(\left\{\omega_{i_1}, \omega_{i_2}, \omega_{i_3}, \ldots, \omega_{i_n}\right\}\right) = \mathcal{P}\left(\left\{\omega_{i_1}\right\} \cup \left\{\omega_{i_2}\right\} \cup \left\{\omega_{i_3}\right\} \cup \ldots \cup \left\{\omega_{i_k}\right\}\right)$$

$$= \mathcal{P}\left(\left\{\omega_{i_1}\right\}\right) + \mathcal{P}\left(\left\{\omega_{i_2}\right\}\right) + \ldots + \mathcal{P}\left(\left\{\omega_{i_n}\right\}\right) \quad \left[\because \mathcal{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}(A_i)\right]$$

$$= p_{i_1} + p_{i_2} + \ldots + p_{i_n} \qquad \ldots (3.5)$$

In particular, if $p_1 = p_2 = p_3 = \ldots = p_n = \dfrac{1}{n}$, … (3.6)

then we are assuming that all the outcomes of the experiment are equally likely and the probability measure defined by $\mathcal{P}\left(\{\omega_i\}\right) = \dfrac{1}{n}, \; 1 \leq i \leq n$ is known as **uniform probability measure**. … (3.7)

Recall classical approach to probability theory discussed in Unit 1 of this course where you were actually using uniform probability measure. … (3.8)

Case II: **When Sample Space Ω is Countably Infinite**

Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$ … (3.9)

Remember when sample space is countably infinite then like the case of finite sample space, we take power set of Ω as our σ-field. So, we have $\mathcal{F} = 2^{\Omega}$.

Note that (3.9) can be written as countable union of singleton subsets of Ω:

i.e., $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\} = \{\omega_1\} \cup \{\omega_2\} \cup \{\omega_3\} \cup \ldots = \bigcup_{i=1}^{\infty} \{\omega_i\}$ … (3.10)

- **Event vs subset of a Sample Space**: We know that every event is always a subset of Ω whether we are working in a discrete world or continuous world but in a discrete world every subset of Ω is also an event. But in a continuous world a subset of the sample space may or may not be an event.

So, in the case where sample space is countably infinite an event E may be finite or infinite, here infinite means countably infinite. So, event E may be union of finite or countably infinite singleton subsets $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \ldots$. We have seen how we assign probability to a finite event in Case I. So, let us consider E as countably infinite event. So, if E is a non-empty countably infinite event, then there will exists a sequence $n_1, n_2, n_3, \ldots$ such that

$$E = \left\{\omega_{n_1}, \omega_{n_2}, \omega_{n_3}, \ldots\right\} = \left\{\omega_{n_1}\right\} \cup \left\{\omega_{n_2}\right\} \cup \left\{\omega_{n_3}\right\} \cup \ldots = \bigcup_{k=1}^{\infty} \omega_{n_k} \qquad \ldots (3.11)$$

But here our σ-field is the power set of Ω so all subsets of Ω are members of σ-field and hence all singleton subsets of Ω are also members of the σ-field.

This implies that if we assign probabilities to all the members of σ-field which are singleton subsets of Ω then probability will be assigned to each member of the σ-field due to the equation (3.11) and countable additivity of probability measure for disjoint events given by (2.28) in the previous unit. If we assign probabilities $p_1, p_2, p_3, \ldots$ to all the singleton subsets $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \ldots$ respectively of Ω, where

$$p_i \geq 0 \ \forall\ i = 1, 2, 3, \ldots \text{ and } \sum_{i=1}^{\infty} p_i = 1 \qquad \ldots (3.12)$$

then using countable additivity of probability measure for disjoint events (refer 2.28), we have

$$\mathcal{P}(E) = \mathcal{P}\left(\left\{\omega_{n_1}, \omega_{n_2}, \omega_{n_3}, \ldots\right\}\right) = \mathcal{P}\left(\left\{\omega_{n_1}\right\} \cup \left\{\omega_{n_2}\right\} \cup \left\{\omega_{n_3}\right\} \cup \ldots\right)$$

$$= \mathcal{P}\left(\left\{\omega_{n_1}\right\}\right) + \mathcal{P}\left(\left\{\omega_{n_2}\right\}\right) + \mathcal{P}\left(\left\{\omega_{i_n}\right\}\right) + \ldots \quad \left[\because \mathcal{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}(A_i)\right]$$

$$= p_{n_1} + p_{n_2} + p_{n_3} + \ldots \qquad \ldots (3.13)$$

So, the moral of the story is in a discrete world, we only need to assign probabilities to each singleton member of the σ-field (= 2^{Ω} = power set of Ω) and then using finite or countable additivity of probability measure, probability of any event can be obtained as explained in (3.5) and (3.13). … (3.14)

Remark 1: Remember, we assign probabilities to the members of σ-field not the members of sample space. … (3.15)

Remark 2: Remember uniform probability law does not work for countably infinite sample spaces because whatever small probability (p) we assign to each singleton event then we know sum of any positive quantity added countably infinite times will give sum greater than 1. For example, (i) if p = 0.01 then sum of 101 such p will be 1.01 > 1 (ii) if p = 0.001 then sum of 1001 such p will be 1.001 > 1 and so on. So, in countably infinite sample spaces, we use the general case as explained in (3.12). For example, if $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$ then probabilities $p_1, p_2, p_3, \ldots$ are assigned to the singleton events or subsets $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \ldots$ respectively of Ω such that

$$p_i \geq 0 \ \forall\ i = 1, 2, 3, \ldots \text{ and } \sum_{i=1}^{\infty} p_i = 1$$

Let us explain this important point by considering some possible values of p_i.

(i) Consider values of p_i as $\mathcal{P}\left(\left\{\omega_n\right\}\right) = p_n = \dfrac{1}{2^n},\ n = 1, 2, 3, \ldots$ … (3.16)

Verification: Obviously each $p_i = \dfrac{1}{2^i} > 0$ for $n = 1, 2, 3, \ldots$ and

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1/2}{1-1/2} \left[\because \text{Sum of infinite GP} = \frac{a}{1-r}\right]$$

$$= \frac{1/2}{1/2} = 1$$

(ii) Consider values of p_i as $\mathcal{P}\left(\{\omega_n\}\right) = p_n = 5\left(\frac{1}{6}\right)^n$, $n = 1, 2, 3, \ldots \ldots$ (3.17)

Verification: Obviously each $p_n = 5\left(\frac{1}{6}\right)^n > 0$ for $n = 1, 2, 3, \ldots$

and $$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} 5\left(\frac{1}{6}\right)^n = \frac{5}{6} + \frac{5}{6^2} + \frac{5}{6^3} + \ldots = \frac{5/6}{1-1/6} \left[\begin{array}{l}\because \text{sum of infinite} \\ \text{GP} = \dfrac{a}{1-r}\end{array}\right]$$

$$= \frac{5/6}{5/6} = 1$$

(iii) Consider values of p_i as $\mathcal{P}\left(\{\omega_n\}\right) = p_n = \frac{e^{-\lambda}\lambda^n}{\lfloor n}$, $n = 1, 2, 3, \ldots \ldots$ (3.18)

where $\lambda > 0$, etc.

Verification: Obviously each $p_n = \frac{e^{-\lambda}\lambda^n}{\lfloor n} > 0$ for $n = 1, 2, 3, \ldots$ and

$$\sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{\lfloor n} = \frac{e^{-\lambda}\lambda^0}{\lfloor 0} + \frac{e^{-\lambda}\lambda^1}{\lfloor 1} + \frac{e^{-\lambda}\lambda^2}{\lfloor 2} + \frac{e^{-\lambda}\lambda^3}{\lfloor 3} + \ldots$$

$$= e^{-\lambda}\left(\frac{\lambda^0}{\lfloor 0} + \frac{\lambda^1}{\lfloor 1} + \frac{\lambda^2}{\lfloor 2} + \frac{\lambda^3}{\lfloor 3} + \ldots\right) = e^{-\lambda}\left(e^{\lambda}\right) \left[\because e^{-x} = \frac{x^0}{\lfloor 0} + \frac{x^1}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots\right]$$

$$= e^{-\lambda+\lambda} = e^0 = 1$$

This completes the discussion of Remark 2.

Now, we discuss some examples of finite sample spaces.

Example 1: A special coin is tossed where probability of getting head is 3/4 and probability of getting tail is 1/4. Explain the procedure of assigning probability to all events.

Solution: We know that when a coin is tossed then sample space is given by $\Omega = \{H, T\}$. Here sample space is finite. Thus, we consider the largest σ-field on Ω which is given by (remember in the case of finite and countably infinite sample spaces we consider power set as σ-field)

$$2^{\Omega} = \{\phi, \{H\}, \{T\}, \Omega\} \quad \ldots (3.19)$$

Given sample space falls in the category of discrete world and we have already discussed that in discrete world we only need to assign probabilities to singleton members of the σ-field 2^{Ω}. Here we have only two singleton subsets {H} and {T} of Ω. As per the statement their probabilities are given by

$$\mathcal{P}(\{H\}) = \frac{3}{4}, \mathcal{P}(\{T\}) = \frac{1}{4}. \quad \ldots (3.20)$$

Let us denote members of the σ-field 2^{Ω} by $\omega_i, 1 \le i \le 4,$ where

$$\omega_1 = \Omega,\ \omega_2 = \{H\},\ \omega_3 = \{T\},\ \omega_4 = \phi. \quad \ldots (3.21)$$

Using (3.20) and (3.21), we get

$$\mathcal{P}\left(\{\omega_2\}\right) = \mathcal{P}\left(\{H\}\right) = \frac{3}{4},\ \mathcal{P}\left(\{\omega_3\}\right) = \mathcal{P}\left(\{T\}\right) = \frac{1}{4},\ \mathcal{P}\left(\{\omega_4\}\right) = \mathcal{P}(\phi) = 0,$$

$$\begin{aligned}\mathcal{P}\left(\{\omega_1\}\right) = \mathcal{P}(\Omega) = \mathcal{P}\left(\{H, T\}\right) &= \mathcal{P}\left(\{H\} \cup \{T\}\right) \\ &= \mathcal{P}\left(\{H\}\right) + \mathcal{P}\left(\{T\}\right) \quad \begin{bmatrix}\text{Using } (2.32) \text{ from} \\ \text{Unit 2 of this course}\end{bmatrix} \\ &= \frac{3}{4} + \frac{1}{4} = 1\end{aligned}$$

Thus, probabilities have been assigned to all four possible events by following procedure of discrete world of probability theory. You are familiar with how to obtain probabilities of these four events from school level probability studied in mathematics of class 11 and 12. But here we have explained the technical details involved behind such calculations which is must to know at this stage of learning the subject.

Example 2: A special coin is tossed twice where probability of getting head is 3/4 and probability of getting tail is 1/4. Explain the procedure of assigning probability to all events.

Solution: We know that when a coin is tossed then sample space is given by $\Omega = \{H, T\}$. Also, when two coins are tossed simultaneously or a single coin is tossed twice then using idea of cross product of two sets (refer 3.11 of Unit 3 of the course MST-011) sample space is given by

$$\Omega = \{H, T\}^2 = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}. \quad \ldots (3.22)$$

Here sample space is finite, so we consider the largest σ-field on Ω which is given by

$$2^{\Omega} = \left\{\begin{array}{l}\phi, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \\ \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\}, \\ \{HT, TH, TT\}, \Omega\end{array}\right\} (3.23)$$

Given sample space falls in the category of discrete world. We have already discussed that in discrete world, we only need to assign probabilities to singleton members of the σ-field 2^{Ω}.

First of all, we are given that this is a special coin, where

$$\mathcal{P}\left(\{H\}\right) = \frac{3}{4},\ \mathcal{P}\left(\{T\}\right) = \frac{1}{4}. \quad \ldots (3.24)$$

We know that if two events A and B are independent then

$$\mathcal{P}(A \cap B) = \mathcal{P}(A)\mathcal{P}(B) \text{ or } \mathcal{P}(AB) = \mathcal{P}(A)\mathcal{P}(B). \quad \ldots (3.25)$$

First using (3.25) and then (3.24), we get

$$\left.\begin{array}{l}\mathcal{P}(\{HH\}) = \mathcal{P}(H)\mathcal{P}(H) \left[\begin{array}{l}\because \ \{HH\} = \{H\} \cap \{H\}, \text{ and using (3.25).} \\ \text{Also, for simplifying notations, we are writing} \\ \mathcal{P}(H) \text{ in place of } \mathcal{P}(\{H\}), \text{ i.e., } \mathcal{P}(\{H\}) = \mathcal{P}(H)\end{array}\right] \\ \qquad = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16} \ [\text{Using (3.24)}] \\ \text{Similarly, } \mathcal{P}(\{HT\}) = \mathcal{P}(H)\mathcal{P}(T) = \frac{3}{4} \times \frac{1}{4} = \frac{3}{16} \\ \mathcal{P}(\{TH\}) = \mathcal{P}(T)\mathcal{P}(H) = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}, \ \mathcal{P}(\{TT\}) = \mathcal{P}(T)\mathcal{P}(T) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}\end{array}\right\} \quad (3.26)$$

Probabilities of other members of the σ-field 2^{Ω} can be obtained using finite additivity of probability measure. To explain the procedure let us obtain probabilities of two members of σ-field 2^{Ω}.

$$\mathcal{P}(\{HH, HT\}) = \mathcal{P}(\{HH\} \cup \{HT\}) = \mathcal{P}(\{HH\}) + \mathcal{P}(\{HT\}) \quad \left[\begin{array}{l}\text{Using } (2.32) \text{ from} \\ \text{Unit 2 of this course}\end{array}\right]$$

$$= \frac{9}{16} + \frac{3}{16} = \frac{12}{16} = \frac{3}{4} \quad [\text{Using (3.26)}]$$

$$\mathcal{P}(\{HH, HT, TH\}) = \mathcal{P}(\{HH\} \cup \{HT\} \cup \{TH\})$$

$$= \mathcal{P}(\{HH\}) + \mathcal{P}(\{HT\}) + \mathcal{P}(\{HT\}) \quad \left[\begin{array}{l}\text{Using } (2.32) \text{ from} \\ \text{Unit 2 of this course}\end{array}\right]$$

$$= \frac{9}{16} + \frac{3}{16} + \frac{3}{16} = \frac{15}{16} \quad [\text{Using (3.26)}]$$

Similarly, probabilities of remaining events can be obtained and are shown in Table 3.1. But in practice, we have no need to obtain probability of all members of the σ-field 2^{Ω}. We have to obtain probabilities of only some specified events depends on the problem in hand. But important point is being a learner of the master degree programme of statistics, you should know the concept that works behind such calculations. This was the objective to do some such calculations here.

In Unit 3 of the course MST-011, you have studied set function. Probability measure is also a set function from the σ-field 2^{Ω} to the interval [0, 1]. In Unit 1 of the course MST-011, you have seen the pictorial presentation of a function. To give you more clarity of the concept, let us visualise the probability measure discussed here using pictorial presentation in Fig. 3.1. Before that let us denote members of the σ-field 2^{Ω} by $\omega_i, 1 \le i \le 16$ with their corresponding probabilities in Table 3.1 given as follows.

Table 3.1: Members of the σ-field 2^{Ω} with their corresponding probabilities

Outcome	Notation	Probability	Outcome	Notation	Probability
Ω	ω_1	1	{HT, TH, TT}	ω_9	7/16
{HH, HT, TH}	ω_2	15/16	{HT, TH}	ω_{10}	6/16
{HH, HT, TT}	ω_3	13/16	{HT, TT}	ω_{11}	4/16
{HH, TH, TT}	ω_4	13/16	{TH, TT}	ω_{12}	4/16
{HH, HT}	ω_5	12/16	{HT}	ω_{13}	3/16
{HH, TH}	ω_6	12/16	{TH}	ω_{14}	3/16
{HH, TT}	ω_7	10/16	{TT}	ω_{15}	1/16
{HH}	ω_8	9/16	ϕ	ω_{16}	0

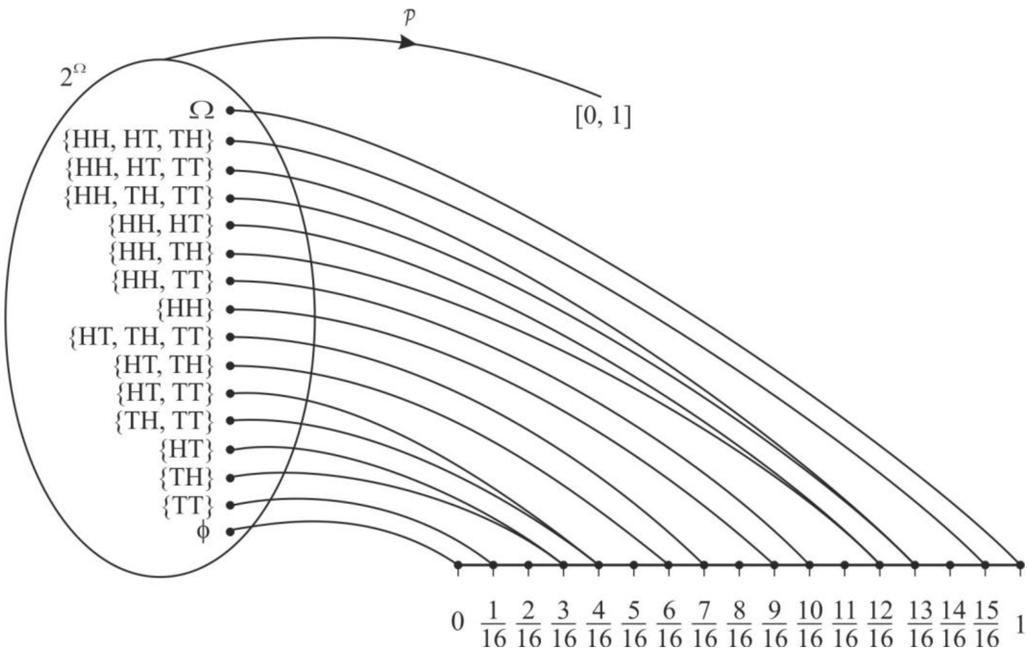

Fig. 3.1: Pictorial presentation of probability measure discussed in Example 2

Example 3: Obtain probabilities given by (2.31) in the previous unit using rules of probability theory under discrete world.

Solution: Sample space in the case of that special die is given by $\Omega = \{1, 2, 3\}$. Here sample space is finite. So, it falls in the category of discrete world. Thus, we consider the largest σ-field on Ω which is given by

$$2^{\Omega} = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \Omega\} \quad \ldots (3.27)$$

We know that in discrete world we only need to assign probabilities to singleton members of the σ-field 2^{Ω}. Here we have only three singleton subsets {1}, {2} and {3} of Ω. As per the statement their probabilities are given by

$$\mathcal{P}(\{1\}) = \frac{3}{6}, \mathcal{P}(\{2\}) = \frac{2}{6}, \mathcal{P}(\{3\}) = \frac{1}{6} \quad \ldots (3.28)$$

Probabilities of other events can be obtained using (3.28) and finite additivity of probability measure (refer 2.32 in the previous unit) as follows.

$$\mathcal{P}(\{1, 2\}) = \mathcal{P}(\{1\} \cup \{2\}) = \mathcal{P}(\{1\}) + \mathcal{P}(\{2\}) = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

$$\mathcal{P}(\{1, 3\}) = \mathcal{P}(\{1\} \cup \{3\}) = \mathcal{P}(\{1\}) + \mathcal{P}(\{3\}) = \frac{3}{6} + \frac{1}{6} = \frac{4}{6}$$

$$\mathcal{P}(\{2, 3\}) = \mathcal{P}(\{2\} \cup \{3\}) = \mathcal{P}(\{2\}) + \mathcal{P}(\{3\}) = \frac{2}{6} + \frac{1}{6} = \frac{3}{6}$$

$$\mathcal{P}(\Omega) = \mathcal{P}(\{1, 2, 3\}) = \mathcal{P}(\{1\} \cup \{2\} \cup \{3\}) = \mathcal{P}(\{1\}) + \mathcal{P}(\{2\}) + \mathcal{P}(\{3\})$$

$$= \frac{3}{6} + \frac{2}{6} + \frac{1}{6} = 1$$

Here, we have obtained probabilities of all members of the σ-field 2^{Ω}. But as mentioned earlier in practice, we have no need to obtain probability of all members of the σ-field 2^{Ω}. We have to obtain probabilities of only some specified events depends on the problem in hand.

Pictorial presentation of this probability measure has been shown in Fig. 2.2 in the previous unit.

Now, you can try the following Self-Assessment Question.

SAQ 1

An unbiased die is thrown once. How many events are possible in this random experiment? Explain the procedure to assign probabilities to all events.

In this section, we have explained the procedure of assigning probabilities in the discrete world. Our next target is to discuss how the same job can be done in the continuous world of probability theory. To do that job in continuous world we need some tools and techniques of measure theory listed as follows.

- Borel σ-field
- Some Borel measurable sets
- Some results on Lebesgue measure
- Understanding of the statement of Caratheodory's extension theorem

The next four sections of this unit are devoted to discuss these tools and techniques of measure theory. Keeping applied nature of this programme in view we will try to avoid proofs as far as possible. Still, we will discuss proofs of some results to give you flavour of the machinery involved behind probability theory. Being a learner of master degree programme at least this much flavour you should have to understand the things in the right direction.

3.3 BOREL σ-FIELD

In the previous unit and previous section of this unit, you have learned many things about σ-field. In this section, you will study about a special σ-field known as Borel σ-field. Recall that in discrete world, we assigned probabilities to all elementary events, i.e., to all members of the σ-field 2^{Ω} which were singleton subsets of the sample space Ω. In continuous world, we cannot do that because of uncountable sample space and probability measure is countably additive not uncountably additive and due to some other similar reasons.

For example, suppose $\Omega = (0, 1]$ then it contains all real numbers which are > 0 but ≤ 1. Here, it is not possible to list all elements of Ω in a sequence which was possible in discrete world. Not only making list of all elements of Ω is impossible task in continuous world but here you even cannot tell what is the starting point which is > 0. If you say 0.1 is the next real number after 0, no it is not the next real number after 0 because 0.01 is between 0 and 0.1. If you say 0.01 is the next real number after 0, no it is not the next real number after 0 because 0.001 is between 0 and 0.01. Similarly, if you say 0.001 is the next real number after 0, no it is not the next real number after 0 because 0.0001 is between 0 and 0.001 and so on. So, we cannot assign probabilities in a way as we assigned in discrete world. What to do? To meet this challenge, we have to make some compromise. Actually, we wish to assign probability to each member of the largest σ-field 2^{Ω} on the sample space Ω. But it is not possible in continuous world. Solution of this problem is, we have to work on a smaller σ-field compared to 2^{Ω} but that should contain at least all events of our interest if not all possible subsets of Ω.

Let us first discuss the construction of Borel a σ-field on the sample space $\Omega = (0, 1]$. Later on we will construct Borel σ-field by considering entire real line as our sample space. … (3.29)

So, first suppose that the sample space is $\Omega = (0, 1]$. Here, we considered Ω as left open and right closed due to some special technical requirement, what is that special technical requirement to know that you have to wait till we reach (3.50). For the sample space $\Omega = (0, 1]$, events of our interest at least includes all kinds of intervals such as (a, b), (a, b], [a, b), and [a, b] where $a, b \in (0, 1], a < b$. To construct a σ-field which contains these intervals, first, we need to discuss following important and powerful results.

Result 1: If X be a non-empty set and $\mathcal{C}$ be a collection of subsets of X then there exists a unique σ-field on X which contains $\mathcal{C}$ and it is the smallest σ-field on X containing $\mathcal{C}$. … (3.30)

Proof: Let $\mathcal{G} = \{\mathcal{F} : \mathcal{F} \subseteq P(X), \mathcal{F} \text{ is a σ-field}, \mathcal{F} \supseteq \mathcal{C}\}$ and $\mathcal{S} = \bigcap_{\mathcal{F} \in \mathcal{G}} \mathcal{F}$.

We have to prove three things

(i) $\mathcal{S}$ is a σ-field (ii) $\mathcal{S}$ contains $\mathcal{C}$ (iii) $\mathcal{S}$ is the smallest σ-field containing $\mathcal{C}$

(i) Since power set P(X) contains all subsets of X and is always a σ-field on X containing $\mathcal{C}$ and hence $P(X) \in \mathcal{G} \Rightarrow \mathcal{G} \neq \phi$. Hence, $\mathcal{S} \neq \phi$. We know that X, ϕ are always members of every σ-field on X and hence $X, \phi \in \mathcal{F} \;\; \forall \;\; \mathcal{F} \in \mathcal{G} \Rightarrow X, \phi \in \bigcap_{\mathcal{F} \in \mathcal{G}} \mathcal{F} = \mathcal{S}$.

Next, we have to prove that $\mathcal{S}$ is closed with respect to complement. To prove it let $A \in \mathcal{S} = \bigcap_{\mathcal{F} \in \mathcal{G}} \mathcal{F} \Rightarrow A \in \mathcal{F} \;\; \forall \;\; \mathcal{F} \in \mathcal{G}$. But each $\mathcal{F}$ is a σ-field on X. So,

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \;\; \forall \;\; \mathcal{F} \in \mathcal{G} \quad \left[\begin{array}{l}\because \text{Each } \mathcal{F} \text{ is a σ-field and every σ-field} \\ \text{is closed with respect to complement}\end{array}\right]$$

$\Rightarrow A^c \in \bigcap_{\mathcal{F} \in \mathcal{G}} \mathcal{F} = \mathcal{S}$. Hence, $\mathcal{S}$ is closed with respect to complement.

Finally, we have to prove that $\mathcal{S}$ is closed with respect to countable union. To prove it let

$$A_n \in \mathcal{S} = \bigcap_{\mathcal{F} \in \mathcal{G}} \mathcal{F}, \; n = 1, 2, 3, \ldots \Rightarrow A_n \in \mathcal{F} \;\; \forall \;\; \mathcal{F} \in \mathcal{G}, \; n = 1, 2, 3, \ldots$$

But each $\mathcal{F}$ is a σ-field on X. So,

$$A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \;\; \forall \;\; \mathcal{F} \in \mathcal{G} \quad \left[\begin{array}{l}\because \text{Each } \mathcal{F} \text{ is a σ-field and every σ-field} \\ \text{is closed with respect to countable union}\end{array}\right]$$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \bigcap_{\mathcal{F} \in \mathcal{G}} \mathcal{F} = \mathcal{S}$. Hence, $\mathcal{S}$ is closed with respect to countable union.

This completes the proof of (i).

(ii) Since $\mathcal{C} \subseteq \mathcal{F} \;\; \forall \;\; \mathcal{F} \in \mathcal{G} \Rightarrow \mathcal{C} \subseteq \bigcap_{\mathcal{F} \in \mathcal{G}} \mathcal{F} = \mathcal{S}$. This completes proof of (ii).

(iii) In order to prove that $\mathcal{S}$ is the smallest σ-field on X containing $\mathcal{C}$, let $\mathcal{F}$ be any σ-field on X which contains $\mathcal{C}$. By definition of $\mathcal{G}$, we have $\mathcal{F} \in \mathcal{G}$.

Implies this $\mathcal{S} = \bigcap_{\mathcal{F} \in \mathcal{G}} \mathcal{F} \subseteq \mathcal{F}$. Hence, $\mathcal{S}$ is the smallest σ-field on X containing $\mathcal{C}$. This completes proof of (iii).

Remark 3: This smallest σ-field on X containing the collection $\mathcal{C}$ is called the σ-field generated by $\mathcal{C}$ and is denoted by $\sigma(\mathcal{C})$. … (3.31)

Now, come to the problem that we want to solve. Suppose we have the sample space $\Omega = (0, 1]$ and events of our interest includes are all kinds of intervals such as (a, b), (a, b], [a, b), and [a, b] where $a, b \in (0, 1]$. To solve this problem we will move step by step. Let $\mathcal{C}$ be the collection of all open subsets of $\Omega = (0, 1]$. The word open set is a technical word from mathematics point of view. So, before moving further let us first explain what we mean by open set. There are two topics known as metric space and topological space where concept of open sets is used heavily and directly suits our requirement. We will discuss definition of open sets in both metric and topological spaces so that you get broad understanding about the idea of open set. In Sec. 3.6 of Unit 3 of the course MST-011, you have studied definition of metric space. So, we directly go to the definition of open set in a metric space. But before that we have to define two more things namely open sphere and interior point.

Open Sphere in a Metric Space: Let (X, d) be a metric space. Let $x_0 \in X$ then open sphere of radius r about x_0 is denoted by $S_r(x_0)$ and is defined by $S_r(x_0) = \{x \in X : d(x_0, x) < r\}$. It is visualised in Fig. 3.2 (a). … (3.32)

Interior Point of a Set in a Metric Space: Let (X, d) be a metric space. Let $A \subseteq X$ be any non-empty subset of X. A point $x_0 \in A$ is said to be an interior point of A if there exists an open sphere $S_r(x_0)$ such that $x_0 \in S_r(x_0) \subseteq A$. It is visualised in Fig. 3.2 (b). … (3.33)

Definition of Open Set in a Metric Space: Let (X, d) be a metric space. Let $A \subseteq X$ then we say that set A is open if every point of A is its interior point. … (3.34)

Remark 4: If $X = \mathbb{R}$ then open sphere $S_r(x_0)$ mean an open interval around x_0, i.e., $S_r(x_0) = (x_0 - r, x_0 + r)$. For example, if X = (0, 1], $x_0 = 0.3, r = 0.1$, then $S_{0.1}(0.3) = (0.3 - 0.1, 0.3 + 0.1) = (0.2, 0.4)$. It is visualised in Fig. 3.2 (c).

Definition of Topological Space: Let X be a non-empty set. A class $\mathcal{T}$ of subsets of X is a topology on X if and only if $\mathcal{T}$ satisfies the following axioms.

(i) $\phi, X \in \mathcal{T}$

(ii) Union of arbitrary members of $\mathcal{T}$ is again a member of $\mathcal{T}$, i.e., if $\{A_i\}_{i \in I}$ is a class of members of $\mathcal{T}$ then $\bigcup_{i \in I} A_i \in \mathcal{T}$

(iii) The intersection of any two members of $\mathcal{T}$ belongs to $\mathcal{T}$, i.e., if $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on a non-empty set X then the pair $(X, \mathcal{T})$ is called a topological space. … (3.35)

For example, if X = {a, b, c, d, e} and

$\mathcal{T}_1 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

$\mathcal{T}_2 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

then $\mathcal{T}_1$ is not a topology on X because

$\{a, c, d\}, \{b, c, d\} \in \mathcal{T}_1$ but $\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\} \notin \mathcal{T}_1$

However, $\mathcal{T}_2$ is a topology on X since it satisfies the necessary three axioms (i), (ii) and (iii).

Definition of Open Set in a Topological Space: Let $(X, \mathcal{T})$ be a topological space then the members of $\mathcal{T}$ are called T-open sets. If we have only one topology on X under consideration then we can simply say that open sets. … (3.36)

For example, we have discussed that if X = {a, b, c, d, e} then

$\mathcal{T}_2 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ is a topology on X. Hence open sets in the topological space $(X, \mathcal{T}_2)$ are ϕ, X, {a}, {c, d}, {a, c, d}, {b, c, d, e}. Also, in a topological space a set A is said to be **closed** if and only if its complement is an open set. … (3.37)

So, closed sets in this topological space are:

$\phi^c = \Omega$, $X^c = \phi$, $\{a\}^c = \{b, c, d, e\}$, $\{c, d\}^c = \{a, b, e\}$, $\{a, c, d\}^c = \{b, e\}$,
$\{b, c, d, e\}^c = \{a\}$.

Thus, closed sets are: Ω, ϕ, {b, c, d, e}, {a, b, e}, {b, e}, {a}.

We see that in this topological space four sets ϕ, Ω, {a}, {b, c, d, e} are both open as well as closed sets. In fact, ϕ, Ω are always open as well as closed sets in every topological space.

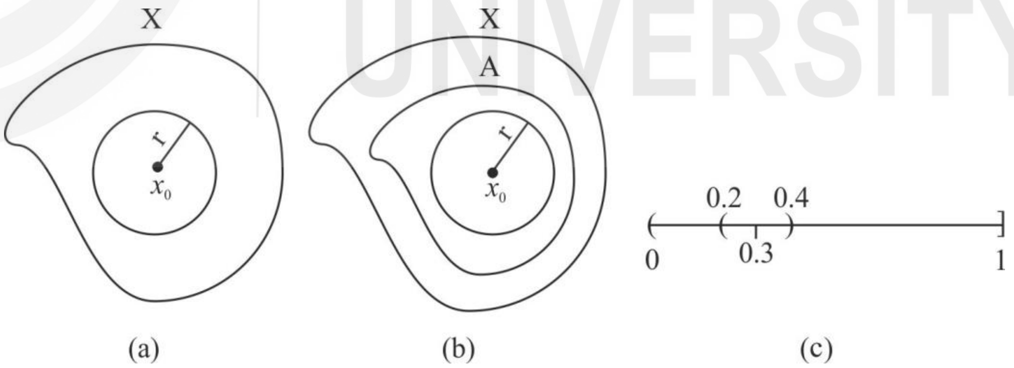

Fig. 3.2: Visualisation of (a) a sphere of radius r about x_0 in the metric space X (b) point x_0 is an interior point of the set A in the metric space X (c) open sphere in $\mathbb{R}$ about 0.3 of radius 0.1 as an open interval

Now, you have understood what is open set in a metric space in general and, in particular, on real line it is simply an open interval. So, let us continue our discussion of explaining what is a Borel σ-field as follows.

So, we have a collection $\mathcal{C}$ of all open subsets of $\Omega = (0, 1]$. Then using Result 1 there will exist the smallest σ-field on $\Omega = (0, 1]$ containing $\mathcal{C}$. This smallest σ-field is called **Borel σ-field** on $\Omega = (0, 1]$ and is denoted by

$\mathcal{B}((0, 1]) = \sigma(\mathcal{C})$. … (3.38)

The members of Borel σ-field are called **Borel sets** or **Borel measurable sets**. … (3.39)

By definition of σ-field the Borel σ-field $\mathcal{B}((0, 1]) = \sigma(\mathcal{C})$ will be closed with respect to complement and countable union. Using these properties, some Borel measurable sets are discussed in the next section.

3.4 SOME BOREL MEASURABLE SETS

In Sec. 3.3, we have seen that the collection $\mathcal{C}$ contains all open intervals in $(0, 1]$ and so all these open intervals are members of Borel σ-field $\mathcal{B}((0, 1])$ because it is generated by the class $\mathcal{C}$. But we know that by definition σ-field is closed with respect to complement. So, the Borel σ-field $\mathcal{B}((0, 1])$ contains complements of all open interval in $(0, 1]$. Also, by definition σ-field is closed with respect to finite as well as countable unions. So, the Borel σ-field $\mathcal{B}((0, 1])$ contains finite and countable union of open interval in $(0, 1]$. Further, by combining the two operations complement and union and using De-morgan's laws we can say that Borel σ-field $\mathcal{B}((0, 1])$ contains finite and countable intersections of its members you may refer (2.32). So, Borel σ-field $\mathcal{B}((0, 1])$ contains all open sets, their complements and any combination of open sets and their complements via finite or countable union or intersection. So, we will see that there are many other interesting sets in $\mathcal{B}((0, 1])$ which are not obvious like complement of open sets. In this section, we will discuss some of them which are discussed one at a time as follows.

Result 2: **Singleton Sets are Borel Measurable Sets**: **Statement**: All the singleton subsets of $\Omega = (0, 1]$ belong to the Borel σ-field of $\Omega = (0, 1]$, i.e., $\{x\} \in \mathcal{B}((0, 1]) \ \forall \ x \in \Omega$. … (3.40)

Proof: Let $x \in \Omega = (0, 1]$, then the singleton set {x} can be written as

$$\{x\} = \bigcap_{n=1}^{\infty}\left(\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \Omega\right) \begin{bmatrix} \text{Here we have taken} \\ \text{intersection with } \Omega = (0, 1], \text{ so} \\ \text{that for each n, } \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \Omega \\ \text{remains inside } \Omega \end{bmatrix} \quad \ldots (3.41)$$

Obviously, for each $n \in \mathbb{N}$, $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \Omega$ is an open interval in $\Omega = (0, 1]$.

So, it is an open set in (0, 1] [$\because$ each open interval is an open set] and hence belongs to the Borel σ-field $\mathcal{B}((0, 1])$. But from (2.17), we know that σ-field is closed with respect to countable intersection. Thus,

$\bigcap_{n=1}^{\infty}\left(\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \Omega\right) \in \mathcal{B}((0, 1])$. But then due to (3.41), we have

$\{x\} \in \mathcal{B}((0, 1])$. Hence, every singleton subset of $\Omega = (0, 1]$ belongs to the Borel σ-field $\mathcal{B}((0, 1])$ of $\Omega = (0, 1]$. Thus, every singleton set in (0, 1] is a Borel measurable set.

Result 3: **All the Finite Subsets of $\Omega = (0, 1]$ are Borel Measurable Sets.**

Statement: All the finite subsets of $\Omega = (0, 1]$ belong to the Borel σ-field of $\Omega = (0, 1]$, i.e., $\{x_1, x_2, x_3, \ldots, x_n\} \in \mathcal{B}((0, 1]) \ \forall \ x_i \in \Omega, i = 1, 2, 3, \ldots, n.$ … (3.42)

Proof: Let $x_i \in \Omega, i = 1, 2, 3, \ldots, n.$ The set $\{x_1, x_2, x_3, \ldots, x_n\}$ can be written as finite union of singleton subsets of Ω as follows.

$$\{x_1, x_2, x_3, \ldots, x_n\} = \{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \ldots \cup \{x_n\} \quad \ldots (3.43)$$

Due to (3.40) each $\{x_i\} \in \mathcal{B}((0, 1]) \ \forall \ i, \ i = 1, 2, 3, \ldots, n.$ But from (2.16) we know that σ-field is closed with respect to finite union. So,

$$\{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \ldots \cup \{x_n\} \in \mathcal{B}((0, 1])$$
$$\Rightarrow \{x_1, x_2, x_3, \ldots, x_n\} \in \mathcal{B}((0, 1])$$

Hence, all the finite subsets of $\Omega = (0, 1]$ belong to the Borel σ-field $\mathcal{B}((0, 1])$ of $\Omega = (0, 1]$. Thus, we can say that a finite subset of (0, 1] is a Borel measurable set.

Result 4: **All the Countably Infinite Subsets of $\Omega = (0, 1]$ are Borel Measurable Sets. Statement**: All the countably infinite subsets of $\Omega = (0, 1]$ belong to the Borel σ-field of $\Omega = (0, 1]$, i.e.,

$$\{x_1, x_2, x_3, \ldots\} \in \mathcal{B}((0, 1]) \ \forall \ x_i \in \Omega, i = 1, 2, 3, \ldots. \quad \ldots (3.44)$$

Proof: Let $x_i \in \Omega, i = 1, 2, 3, \ldots,$ then the set $\{x_1, x_2, x_3, \ldots\}$ is countably infinite. So, it can be written as countable union of singleton subsets of Ω as follows.

$$\{x_1, x_2, x_3, \ldots\} = \{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \ldots \quad \ldots (3.45)$$

Due to (3.40) each $\{x_i\} \in \mathcal{B}((0, 1]) \ \forall \ i, \ i = 1, 2, 3, \ldots$

But we know that σ-field is closed with respect to countable union. So,

$$\{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \ldots \in \mathcal{B}((0, 1])$$
$$\Rightarrow \{x_1, x_2, x_3, \ldots\} \in \mathcal{B}((0, 1])$$

Hence, all the countably infinite subsets of $\Omega = (0, 1]$ belong to the Borel σ-field of $\Omega = (0, 1]$. Thus, we can say that a countably infinite subset of (0, 1] is a Borel measurable set.

Result 5: **All Kinds of Intervals in $\Omega = (0, 1]$ are Borel Measurable Sets.**

Statement: If $\mathcal{B}((0, 1])$ is the Borel σ-field on $\Omega = (0, 1]$, then prove that all intervals of the forms (a, b), (a, b], [a, b) and [a, b] belong to $\mathcal{B}((0, 1])$ $\forall$ a, b $\in \Omega = (0, 1]$. … (3.46)

Proof: From (3.40) we know that all the singleton subsets of $\Omega = (0, 1]$ belong to the Borel σ-field $\mathcal{B}((0, 1])$. So, $a, b \in \Omega = (0, 1] \Rightarrow \{a\}, \{b\} \in \mathcal{B}((0, 1])$ … (3.47)

Also, the Borel σ-field $\mathcal{B}((0, 1])$ is generated by all open subsets of $\Omega = (0, 1]$. Thus, $\forall$ a, b $\in \Omega = (0, 1]$, we have

$$(a, b) \in \mathcal{B}((0, 1]) \quad \ldots (3.48)$$

Due to (3.47), (3.48) and the fact that σ-field is closed with respect to finite union (you may refer 2.16), we have

$(a, b] = (a, b) \cup \{b\} \in \mathcal{B}((0, 1])$

$[a, b) = (a, b) \cup \{a\} \in \mathcal{B}((0, 1])$

$[a, b] = (a, b) \cup \{a\} \cup \{b\} \in \mathcal{B}((0, 1])$

Hence, all intervals of the forms (a, b), (a, b], [a, b) and [a, b] belong to $\mathcal{B}((0, 1]) \quad \forall\ a, b \in \Omega = (0, 1]$. Thus, all kinds of intervals in $\Omega = (0, 1]$ are Borel measurable sets.

Now, we state some more results. Their proofs are beyond the scope of this course. Still, we will prove some of them. We will use these results to explain the idea: how probability is assigned to the members of Borel σ-field.

Result 6: Let $\Omega = (0, 1]$ and $\mathcal{H}$ be the collection of all half open intervals. That is, $\mathcal{H} = \{(a, b] : a, b \in (0, 1], a < b\}$. Let $\mathcal{H}_0$ be the collection of empty set and union of finite number of disjoint members of $\mathcal{H}$, i.e.,

$\mathcal{H}_0 = \{(a_1, b_1] \cup (a_2, b_2] \cup \ldots \cup (a_n, b_n] : 0 \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le 1\} \cup \{\phi\}$.

Prove that $\mathcal{H}_0$ is a (an) field (algebra). … (3.49)

Proof: To prove that it is a field, we have to prove that (i) $\phi, \Omega \in \mathcal{H}_0$ (ii) $\mathcal{H}_0$ is closed with respect to complement, and (iii) $\mathcal{H}_0$ is closed with respect to finite union (you may refer 2.9). But proof of this result is beyond the scope of the course. But one interesting point that you should note is that $\Omega \in \mathcal{H}_0$. This was the **technical reason due to which we considered Ω as left open and right closed interval in our discussion**. … (3.50)

If Ω was not taken as (0, 1] then $\mathcal{H}_0$ will not form a field because one of the requirements of a field is that Ω should be member of that collection which we want to see as a field.

Result 7: Prove that the collection $\mathcal{H}_0$ as defined in Result 6 is not a σ-field. … (3.51a)

Proof: The collection $\mathcal{H}_0$ is a field. To prove that it is not a σ-field we have to show that it is not closed with respect to countable union. To prove it we have to give a counter example where countable union of members of $\mathcal{H}_0$ is not a member of $\mathcal{H}_0$. Let $A_n = \left(0, \frac{n}{n+1}\right]$, $n = 1, 2, 3, \ldots$ then

$$A_1 = \left(0, \frac{1}{2}\right] \in \mathcal{H}_0,\ A_2 = \left(0, \frac{2}{3}\right] \in \mathcal{H}_0,\ A_3 = \left(0, \frac{3}{4}\right] \in \mathcal{H}_0, \ldots$$

So, for each n, $A_n = \left(0, \frac{n}{n+1}\right] \in \mathcal{H}_0$, $n = 1, 2, 3, \ldots$

But $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}\left(0, \frac{n}{n+1}\right] = \left(0, \frac{1}{2}\right] \cup \left(0, \frac{2}{3}\right] \cup \left(0, \frac{3}{4}\right] \cup \ldots = (0, 1) \notin \mathcal{H}_0$.

Hence, the collection $\mathcal{H}_0$ is not a σ-field.

Result 8: Prove that if $\mathcal{H}_0$ is the collection as defined in Result 6, then σ-field generated by the collection $\mathcal{H}_0$ is the Borel σ-field on (0,1]. That is,

$$\sigma\left(\mathcal{H}_0\right) = \mathcal{B}\left((0, 1]\right). \qquad \ldots (3.51b)$$

Proof: Although proof of this result is beyond the scope of this course but keeping its simplicity and conceptually its importance, we are discussing its proof.

You know that if we have two sets A and B and want to prove that $A = B$, then we have to show that $A \subseteq B$ and $B \subseteq A$. To prove $A \subseteq B$ we start by considering one element from A, i.e., $x \in A$ and prove that $x \in B$. Similarly, to prove $B \subseteq A$ we start by considering one element from B, i.e., $x \in B$ and prove that $x \in A$. This is the standard practice followed in set theory. Here we want to prove $\sigma\left(\mathcal{H}_0\right) = \mathcal{B}\left((0, 1]\right)$. So, we have to prove that

$$\sigma\left(\mathcal{H}_0\right) \subseteq \mathcal{B}\left((0, 1]\right) \qquad \ldots (3.52)$$

$$\text{and } \mathcal{B}\left((0, 1]\right) \subseteq \sigma\left(\mathcal{H}_0\right) \qquad \ldots (3.53)$$

To prove (3.52), let $A \in \sigma\left(\mathcal{H}_0\right) \Rightarrow A = (a_1, b_1] \cup (a_2, b_2] \cup \ldots \cup (a_k, b_k]$, where $a_i, b_i \in (0, 1]$. Due to (3.46) each member of this union belongs to $\mathcal{B}\left((0, 1]\right)$. But $\mathcal{B}\left((0, 1]\right)$ is a σ-field so it will be closed with respect to finite as well as countable union hence

$$A = (a_1, b_1] \cup (a_2, b_2] \cup \ldots \cup (a_k, b_k] \in \mathcal{B}\left((0, 1]\right) \Rightarrow A \in \mathcal{B}\left((0, 1]\right)$$

$$\Rightarrow \sigma\left(\mathcal{H}_0\right) \subseteq \mathcal{B}\left((0, 1]\right) \qquad \ldots (3.54)$$

To prove (3.53), it is enough to prove that each open interval (a, b) belongs to $\sigma\left(\mathcal{H}_0\right)$. $\left[\because \mathcal{B}\left((0, 1]\right)\right.$ is generated by collection of all open intervals in $\left.(0, 1]\right]$

Now $(a, b) = \bigcup_{n=1}^{\infty}\left(a, b - \frac{1}{n}\right]$ and $\left(a, b - \frac{1}{n}\right] \in \mathcal{H}_0$ for each n, $n = 1, 2, 3, \ldots$

$$\Rightarrow (a, b) = \bigcup_{n=1}^{\infty}\left(a, b - \frac{1}{n}\right] \in \sigma\left(\mathcal{H}_0\right)$$

$$\Rightarrow \mathcal{B}\left((0, 1]\right) \subseteq \sigma\left(\mathcal{H}_0\right) \qquad \ldots (3.55)$$

Hence from (3.54) and (3.55), we have $\sigma\left(\mathcal{H}_0\right) = \mathcal{B}\left((0, 1]\right)$.

Similarly, we can generate more Borel σ-fields as follows.

Remark 5: Let $\Omega = (a, b]$ and $\mathcal{H}_1$ be the collection of all half open intervals in (a, b]. That is, $\mathcal{H}_1 = \left\{(\alpha, \beta] : \alpha, \beta \in (a, b], \alpha < \beta\right\}$. Let $\mathcal{H}_{01}$ be the collection of empty set and union of finite number of disjoint members of $\mathcal{H}_1$, i.e.,

$\mathcal{H}_{01} = \left\{(a_1, b_1] \cup (a_2, b_2] \cup \ldots \cup (a_n, b_n] : a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b\right\} \cup \{\phi\}$.

Prove that $\mathcal{H}_{01}$ is a (an) field (algebra). … (3.56)

Also, σ-field generated by the collection $\mathcal{H}_{01}$ is the Borel σ-field on (a, b].

That is, $\sigma\left(\mathcal{H}_{01}\right) = \mathcal{B}\left((a, b]\right)$. … (3.57)

All the detail regarding the construction of Borel σ-fields on (0, 1] and (a, b] are now under our belt. Recall (3.29) where we made a promise that after construction of the Borel σ-field on (0, 1] we will discuss the construction of Borel σ-field on the entire real line. Now, it is the time to keep that promise.

Like the case of Borel σ-field on (0, 1] we start by considering a collection $\mathcal{C}_1$ of all open subsets of $\Omega = \mathbb{R} = (-\infty, +\infty)$. Using the Result 1 refer (3.30), there will exist the smallest σ-field on $\Omega = \mathbb{R} = (-\infty, +\infty)$ containing $\mathcal{C}_1$. This smallest σ-field is called **Borel σ-field** on $\Omega = \mathbb{R} = (-\infty, +\infty)$ and is denoted by

$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_1)$. … (3.58)

As usual members of Borel σ-field $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_1)$ are called **Borel sets** or **Borel measurable sets** on the real line. … (3.59)

By definition of σ-field the Borel σ-field $\mathcal{B}(\mathbb{R})$ will be closed with respect to complement and countable union. Using these properties, like the case of the Borel σ-field $\mathcal{B}((0, 1])$ on (0, 1] we can prove results 2 to 5 for the Borel σ-field $\mathcal{B}(\mathbb{R})$. In view of Result 5 all kinds of intervals on $\mathbb{R} = (-\infty, +\infty)$ belong to the Borel σ-field $\mathcal{B}(\mathbb{R})$. However, the intervals of the form $\Omega = (-\infty, x], \ x \in \mathbb{R}$ are of special interest for us. So, let us prove that all intervals of the form $\Omega = (-\infty, x], \ x \in \mathbb{R}$ belong to the Borel σ-field $\mathcal{B}(\mathbb{R})$. Due to result 2 every singleton set belongs to the Borel σ-field $\mathcal{B}(\mathbb{R})$. So,

$\{x\} \in \mathcal{B}(\mathbb{R}) \ \ \forall \ \ x \in \mathbb{R}$. … (3.60)

By definition all open intervals belong to the Borel σ-field $\mathcal{B}(\mathbb{R})$. So,

$(-\infty, x) \in \mathcal{B}(\mathbb{R}) \ \ \forall \ \ x \in \mathbb{R}$. … (3.61)

Therefore, for all $x \in \mathbb{R}$, using (3.60), (3.61) and (2.16), we have

$(-\infty, x] = (-\infty, x) \cup \{x\} \in \mathcal{B}(\mathbb{R})$

$\Rightarrow (-\infty, x] \in \mathcal{B}(\mathbb{R}) \ \ \forall \ x \in \mathbb{R}$ … (3.62)

Hence, for each $x \in \mathbb{R}$ the intervals of the form $(-\infty, x]$ are Borel measurable sets on $\mathcal{B}(\mathbb{R})$. … (3.63)

Result 10: Prove that if $\mathcal{C}_3 = \{(-\infty, x] : x \in \mathbb{R}\}$ then $\sigma(\mathcal{C}_3) = \mathcal{B}(\mathbb{R})$. … (3.64)

Proof: Proof is beyond the scope of this course.

To continue our discussion of how probabilities are assigned in continuous world, we have to define two systems known as π-system and λ-system. Definitions of two systems are given as follows.

π-System: A collection $\mathcal{C}$ of subsets of a set X is said to be a π-system if the collection $\mathcal{C}$ is closed with respect to intersection of any two (finite) members of it. That is, whenever $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$. … (3.65)

λ-System: A collection $\mathcal{C}$ of subsets of a set X is said to be a λ-system if

(i) $X \in \mathcal{C}$

(ii) $\mathcal{C}$ is closed with respect to complement, i.e., $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$

(iii) $\mathcal{C}$ is closed under countable union of it's disjoints members, i.e., whenever $A_1, A_2, A_3, \ldots \in \mathcal{C}$ with $A_i \cap A_j = \phi$ for $i \neq j$ then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$.

… (3.66)

Result 11: Prove that the collection $\mathcal{C}_3 = \{(-\infty, x] : x \in \mathbb{R}\}$ is a π-system. … (3.67)

Proof: Let $A = (-\infty, x], \ B = (-\infty, y]$ where $x, y \in \mathbb{R}$. Required to prove $A \cap B \in \mathcal{C}_3$.

Three cases arise.

Case I: x = y = p (say, refer to Fig. 3.3 a)

In this case $(-\infty, x] = (-\infty, y] \Rightarrow A = B \Rightarrow A \cap B = A = (-\infty, x] \in \mathcal{C}_3$.

Case II: x < y, refer to Fig. 3.3 (b)

In this case $(-\infty, x] \subset (-\infty, y] \Rightarrow A \subset B \Rightarrow A \cap B = A = (-\infty, x] \in \mathcal{C}_3$.

Case III: x > y, refer to Fig. 3.3 (c)

In this case $(-\infty, x] \supset (-\infty, y] \Rightarrow A \supset B \Rightarrow A \cap B = B = (-\infty, y] \in \mathcal{C}_3$.

We see that in all the possible cases $A \cap B \in \mathcal{C}_3$. Hence, $\mathcal{C}_3$ is a π-system.

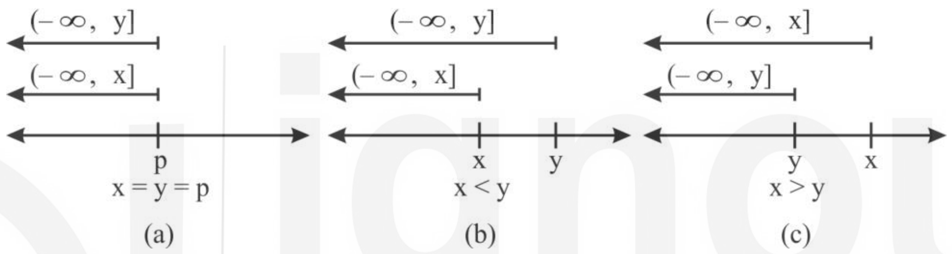

Fig. 3.3: Visualisation of different cases (a) x = y (b) x < y (c) x > y

Result 12: **Dynkin's Theorem:** Let $\mathcal{C}$ be a π-system and $\mathcal{D}$ be a λ-system such that $\mathcal{C} \subset \mathcal{D}$ then $\sigma(\mathcal{C}) \subset \mathcal{D}$. … (3.68)

Proof: Proof is beyond the scope of this course.

We will apply statement of this theorem in our discussion as follows.

We will take $\mathcal{C} = \mathcal{C}_3 = \{(-\infty, x] : x \in \mathbb{R}\}$ which is a π-system already proved.

And $\mathcal{D} = \mathcal{B}(\mathbb{R})$ which is a Borel σ-field and hence a λ-system.

So, applying Dynkin's theorem, we have $\sigma(\mathcal{C}) \subset \mathcal{D}$.

But $\sigma(\mathcal{C}) = \sigma(\mathcal{C}_3) = \mathcal{B}(\mathbb{R}) = \mathcal{D}$. So, in our case $\sigma(\mathcal{C}) = \mathcal{D}$. … (3.69)

Result 13: Let $\mathcal{C}$ be a π-system and $\sigma(\mathcal{C})$ be the σ-field generated by $\mathcal{C}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two probability measures on $\sigma(\mathcal{C})$ then if

$\mathcal{P}_1(A) = \mathcal{P}_2(A) \ \ \forall \ A \in \mathcal{C}$ implies $\mathcal{P}_1(A) = \mathcal{P}_2(A) \ \ \forall \ A \in \sigma(\mathcal{C})$.

i.e., if $\mathcal{P}_1$ and $\mathcal{P}_2$ agrees on $\mathcal{C}$ then they will also agree on $\sigma(\mathcal{C})$. … (3.70)

Proof: Proof is beyond the scope of this course.

We will apply conclusion of this theorem which states that extension of the measure is unique. … (3.71)

3.5 SOME RESULTS OF LEBESGUE MEASURE

In Secs. 3.3 and 3.4, we have discussed some results of measure theory which are required to understand the technical details involved behind the assignment of probability to events in continuous world. In this section, we will discuss some results on Lebesgue measure. We are not defining Lebesgue measure because before defining Lebesgue measure, first we have to define outer measure and before defining outer measure we have to define open cover. But keeping our objective in view we need only statements of some results to continue our discussion of assigning probabilities to the events in continuous world. Statements of these results are straight forward which can be easily understood without understanding the definition of Lebesgue measure and are given as follows.

Result 14: Lebesgue measure of an interval is its length. … (3.72)

Proof: Proof is beyond the scope of the course. For example, Lebesgue measure of the intervals (a, b], [a, b), (a, b) and [a, b] is b – a. In particular Lebesgue measure of the intervals (0.2, 0.9], [0.2, 0.9), (0.2, 0.9) and [0.2, 0.9] is 0.9 – 0.2 = 0.7. So, Lebesgue measure of an interval is the absolute value of the difference of end points of the interval irrespective of whether the interval is both sides open or both sides closed or one side open and other side closed. Lebesgue measure of an interval is denoted by the length function $\lambda,$ to see definition of the length function $\lambda,$ you may refer (3.26) of Unit 3 of the course MST-011.

Result 15: Lebesgue measure is countably additive, i.e., if $E_1, E_2, E_n, \ldots$ be a countably infinite sequence of pairwise disjoint Lebesgue measurable sets then $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n).$ … (3.73)

Proof: Proof is beyond the scope of the course.

Result 16: In particular if $E_i = (a_i, b_i)$ or $E_i = (a_i, b_i]$ or $E_i = [a_i, b_i)$ or $E_i = [a_i, b_i],$ then using Results 14 and 15 or (3.72) and (3.73), we have

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} |b_n - a_n|. \quad \ldots (3.74)$$

Proof: Proof is beyond the scope of the course.

3.6 CARATHEODORY'S EXTENSION THEOREM

We are just one result away to know how probabilities are assigned to events in continuous world. This important result is known as Caratheodory's extension theorem which is stated as follows.

Caratheodory's Extension Theorem: If $\mathcal{H}_0$ be the field on $\Omega = (0, 1]$ and $\mathcal{P}_0 : \mathcal{H}_0 \to [0, 1]$ be such that

(i) $\mathcal{P}_0$ is finitely additive, i.e., whenever we have finite number of disjoint members $A_1, A_2, \ldots, A_n$ of $\mathcal{H}_0$, then $\mathcal{P}_0\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathcal{P}_0(A_k).$

(ii) Whenever we have countably infinite number of disjoint members $A_1, A_2, A_n, \ldots$ of $\mathcal{H}_0$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{H}_0,$ then $\mathcal{P}_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathcal{P}_0(A_n).$

(iii) $\mathcal{P}_0(\Omega) = 1.$

Then the set function $\mathcal{P}_0$ can be extended to a probability measure $\mathcal{P}:\sigma\left(\mathcal{H}_0\right)\to[0, 1]$ on the σ-field $\sigma\left(\mathcal{H}_0\right)$ generated by $\mathcal{H}_0$ such that it agrees with $\mathcal{P}_0$ on $\mathcal{H}_0$, i.e., $\mathcal{P}\left(A\right)=\mathcal{P}_0\left(A\right)\ \ \forall\ A\in\mathcal{H}_0$. Further, if $\mathcal{P}_0$ is finite then its extension is unique. … (3.75)

Proof: Proof is beyond the scope of the course.

Now, we use Caratheodory's extension theorem to do the job of assigning probabilities to the events in continuous world in the next section.

3.7 PROBABILITY ASSIGNMENT IN CONTINUOUS WORLD

In this section, we will combine results discussed in Secs. 3.3 to 3.6 to assign probabilities to the events in continuous world of probability theory. Recall that in discrete world we mainly dealt this issue by classifying probability measure in two categories:

- Uniform probability measure, if you like, you may refer (3.6) and (3.7)
- Non-uniform probability measure, if you like, you may refer (3.4) and (3.5)

Before doing so in continuous world let us first write all the requirements of Caratheodory's extension theorem at one place as follows.

Sample space = $\Omega=(0, 1]$

$\mathcal{H}_0=\left\{(a_1, b_1]\cup(a_2, b_2]\cup\ldots\cup(a_n, b_n]:0\le a_1<b_1\le a_2<b_2\le\ldots\le a_n<b_n\le1\right\}\cup\{\phi\}$

$\mathcal{P}_0:\mathcal{H}_0\to[0, 1]$

Caratheodory's extension theorem will simply extend the probability measure from field $\mathcal{H}_0$ to the σ-field $\sigma\left(\mathcal{H}_0\right)$ so we only need to define it on $\mathcal{H}_0$. How you will define it on $\mathcal{H}_0$ depends on whether we want to assign uniform probability measure or non-uniform probability measure which is discussed as follows.

Probability Assignment in Continuous world Using Uniform Measure

Members of $\mathcal{H}_0$ are disjoint intervals. Now think what is an appropriate uniform measure of an interval. It is not a big problem, you can easily observe that length of the interval is the answer of this question. Also note that whole space $\Omega=(0, 1]$ is an interval and its length is 1 – 0 = 1 which is also one of the requirements of probability measure you may refer (2.27). Another requirement of a probability measure is $\mu(\phi)=0,$ you may refer (2.23) and (2.24). There is one more important feature that we expect from a probability measure to satisfy is that it should satisfy translation invariance property. This property states that probability measure of an event A remains same if we translate it by some fixed number. That is probability measure of an event E and E + x should be the same. This requirement is also satisfied by length function of an interval. For example, if $E=(0.25, 0.37]$ and x = 0.42, then $E+x=(0.25+0.42, 0.37+0.42]=(0.67, 0.79].$ Now, length of E is

$0.37 - 0.25 = 0.12$ and also length of E + x is $0.79 - 0.67 = 0.12$. This is visualised in Fig. 3.4. So, this requirement is also satisfied by length function. Actually, this property states that whole process should be independent of where you are measuring. … (3.76)

Now, we have all detail how should we define $\mathcal{P}_0$. Thus, we define $\mathcal{P}_0 : \mathcal{H}_0 \to [0, 1]$ by

$$\mathcal{P}_0(E) = \begin{cases} 0, & \text{if } E = \phi \\ \sum_{k=1}^{n} (b_k - a_k), & \text{if } E = \bigcup_{k=1}^{n} (a_k, b_k] \end{cases} \quad E \in \mathcal{H}_0 \qquad \dots (3.77)$$

where $0 \le a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n \le 1$ … (3.78)

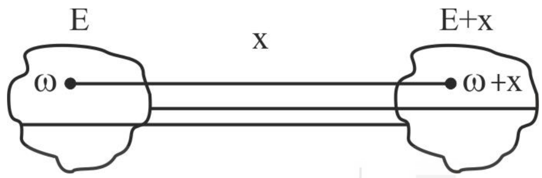

Fig. 3.4: Visualisation of the translation invariant property

Since the measure $\mathcal{P}_0$ is defined on a field $\mathcal{H}_0$ and all other requirements of Caratheodory's extension theorem also hold. So, by Caratheodory's extension theorem there exists a unique probability measure $\mathcal{P}$ on $\mathcal{B}((0, 1])$ such that

$\mathcal{P}(E) = \mathcal{P}_0(E) \ \forall E \in \mathcal{H}_0$, i.e., $\mathcal{P}$ agrees with $\mathcal{P}_0$ on $\mathcal{H}_0$. … (3.79)

This extended probability measure $\mathcal{P}$ on $\mathcal{B}((0, 1])$ is known as **Lebesgue measure** or **length function**. … (3.80)

If instead of $\Omega = (0, 1]$, we have $\Omega = (a, b]$ then

$$\mathcal{H}_0 = \{(a_1, b_1] \cup (a_2, b_2] \cup \dots \cup (a_n, b_n] : a \le a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n \le b\} \cup \{\phi\}$$

where $a, b \in \mathbb{R}$.

In this case we define uniform probability measure $\mathcal{P}_0$ on $\mathcal{H}_0$, i.e., $\mathcal{P}_0 : \mathcal{H}_0 \to [0, 1]$ by

$$\mathcal{P}_0(E) = \begin{cases} 0, & \text{if } E = \phi \\ \frac{1}{b-a} \sum_{k=1}^{n} (b_k - a_k), & \text{if } E = \bigcup_{k=1}^{n} (a_k, b_k] \end{cases} \quad E \in \mathcal{H}_0 \qquad \dots (3.81)$$

Like the case of $\Omega = (0, 1]$. If the measure $\mathcal{P}_0$ is defined on a field $\mathcal{H}_{01}$ and all other requirements of Caratheodory's extension theorem also hold, then by Caratheodory's extension theorem there exists a unique probability measure $\mathcal{P}$ on $\mathcal{B}((a, b])$ such that … (3.82)

$\mathcal{P}(E) = \mathcal{P}_0(E) \ \forall E \in \mathcal{H}_{01}$, i.e., $\mathcal{P}$ agrees with $\mathcal{P}_0$ on $\mathcal{H}_{01}$. … (3.83)

This extended probability measure $\mathcal{P}$ on $\mathcal{B}((a, b])$ is known as **Lebesgue measure** or **length function**. … (3.84)

In the case $\Omega = \mathbb{R}$ then obviously $\mathbb{R}$ is an interval $(-\infty, +\infty)$ of infinite length and so $\mathcal{P}_0(\mathbb{R}) = \infty.$ So, in this case $\mathcal{P}_0$ cannot be converted into a uniform probability measure. Since for a probability measure, we should have $\mathcal{P}_0(\Omega) = 1.$ However, in the case $\Omega = \mathbb{R}$ we can have non-uniform probability measure which is discussed next. … (3.85)

Before discussing non-uniform probability measure let us prove an important result. You know that all singleton subsets of Ω are members of the Borel σ-field. So, we will need measure of singletons in our discussion of probability theory. Following result discuss it.

Result 17: Prove that Lebesgue measure of a singleton set is zero. … (3.86)

Proof: Let $x \in \Omega = (a, b],$ then we have

$$\{x\} = \bigcap_{n=1}^{\infty}\left(\left(x - \frac{1}{n}, x\right] \cap (a, b]\right)$$

Also, for each n, $\left(x - \frac{1}{n}, x\right] \cap (a, b] \in \mathcal{B}((a, b])$ $\quad [\because \text{Using } (3.46)]$

But $\mathcal{B}((a, b])$ is a Borel σ-field. Hence, it is closed with respect to countable intersection. If you want you may refer to (2.17). So,
$\{x\} = \bigcap_{n=1}^{\infty}\left(\left(x - \frac{1}{n}, x\right] \cap (a, b]\right) \in \mathcal{B}((a, b]).$

But due to (3.81) to (3.83) $\mathcal{P}$ is a probability measure on $\mathcal{B}((a, b]).$ So, we can apply $\mathcal{P}$ on $\bigcap_{n=1}^{\infty}\left(\left(x - \frac{1}{n}, x\right] \cap (a, b]\right).$

Now,

$$\begin{aligned}
\mathcal{P}(\{x\}) &= \mathcal{P}\left(\bigcap_{n=1}^{\infty}\left(\left(x - \frac{1}{n}, x\right] \cap (a, b]\right)\right) \\
&= \lim_{n\to\infty} \mathcal{P}\left(\left(x - \frac{1}{n}, x\right] \cap (a, b]\right) \qquad [\text{Using } (4.49)] \\
&\le \lim_{n\to\infty} \mathcal{P}\left(x - \frac{1}{n}, x\right] \qquad [\because A \cap B \subset A \text{ and using } (2.33)] \\
&= \lim_{n\to\infty}\left(x - x + \frac{1}{n}\right) \qquad [\text{Using } (3.77)] \\
&= \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0
\end{aligned}$$

Hence, in continuous world probability of singleton event is zero. … (387)

Now, we discuss the case of non-uniform probability measure in the continuous world of probability as follows.

Probability Assignment in Continuous world Using Non-Uniform Measure

In the case probability is not uniformly distributed over the sample space $\Omega = (0, 1],$ then obviously some regions of the sample space of the same length will have higher probabilities compare to the other region of the same length. Refer Fig. 3.5 where probability over the interval a to b is more compare to the interval c to d though length of the two intervals is the same.

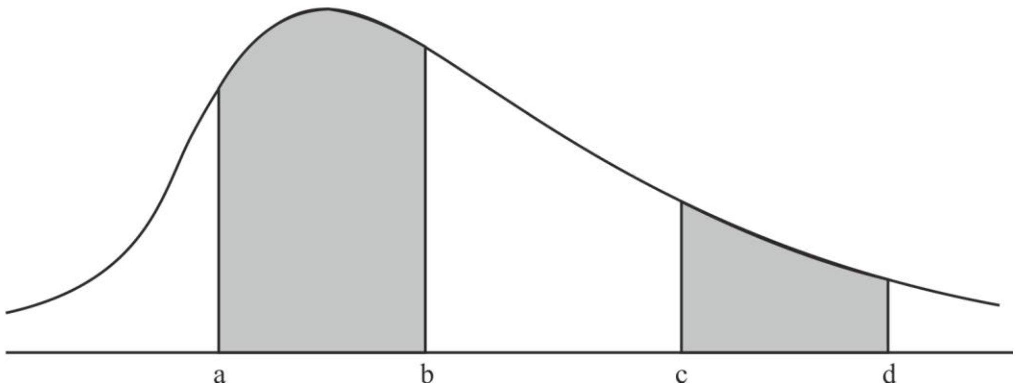

Fig. 3.5: Visualisation of the situation where probabilities are not distributed uniformly over the entire region of our interest

So, we need a function which can capture this relation of varying probability. You can imagine importance of this function because our whole discussion will now move via this function. You were thinking that, keeping depth of the concept in view definitely this function will have a special name. You are absolutely right this function not only have a special name but also has some special properties. The name of this function is **cumulative distribution function (CDF)** which is denoted by F and defined from $\mathbb{R}$ to [0, 1], i.e., $F:\mathbb{R}\to[0, 1].$ This function has the following special properties. … (3.88)

(a) $F(-\infty)=\lim\limits_{x\to-\infty}F(x)=0.$ … (3.89)

(b) $F(\infty)=\lim\limits_{x\to\infty}F(x)=1.$ … (3.90)

(c) F is increasing or non-decreasing. … (3.91)

(d) F is right continuous. … (3.92)

You will get proof of these properties in Sec. 4.5 of the next unit. In that section you will study this function in detail. Here we will just use this function in explaining the procedure of assigning probabilities to events in continuous world.

Now, we can define non-uniform probability measure in terms of CDF in the cases where we have $\Omega=(0, 1]$ or $\Omega=(a, b]$ or $\Omega=\mathbb{R}$ as follows.

In the case $\Omega=(0, 1]$, we assign probabilities to the events as follows.

$\mathcal{P}_0:\mathcal{H}_0\to[0, 1]$ by

$$\mathcal{P}_0(E)=\begin{cases}0, & \text{if } E=\phi\\ \sum\limits_{k=1}^{n}\left(F(b_k)-F(a_k)\right), & \text{if } E=\bigcup\limits_{k=1}^{n}(a_k, b_k]\in\mathcal{H}_0\end{cases} \quad \dots (3.93)$$

where $0\le a_1<b_1\le a_2<b_2\le\ldots\le a_n<b_n\le 1$ … (3.94)

In the case $\Omega=(a, b]$, we assign probabilities to the events as follows.

$\mathcal{P}_0 : \mathcal{H}_{01} \to [0, 1]$ by

$$\mathcal{P}_0(E) = \begin{cases} 0, & \text{if } E = \phi \\ \sum_{k=1}^{n} \left(F(b_k) - F(a_k)\right), & \text{if } E = \bigcup_{k=1}^{n} (a_k, b_k] \in \mathcal{H}_{01} \end{cases} \quad \ldots (3.95)$$

where $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$ … (3.96)

If we have $a \le a_1 < b_1 = a_2 < b_2 = a_3 < b_3 = a_4 < \ldots < b_{n-1} = a_n < b_n \le b$ then (3.97) reduces to

$$\mathcal{P}_0(E) = F(b_n) - F(a_1), \qquad \text{where } E = [a_1, b_n] \quad \ldots (3.98)$$

$$\left[\begin{aligned} \because \sum_{k=1}^{n} \left(F(b_k) - F(a_k)\right) &= \left(F(b_1) - F(a_1)\right) + \left(F(b_2) - F(a_2)\right) + \left(F(b_3) - F(a_3)\right) \\ &\quad + \ldots + \left(F(b_{n-1}) - F(a_{n-1})\right) + \left(F(b_n) - F(a_n)\right) \\ &= \left(F(a_2) - F(a_1)\right) + \left(F(a_3) - F(a_2)\right) + \left(F(a_4) - F(a_3)\right) + \left(F(a_5) - F(a_4)\right) \\ &\quad + \ldots + \left(F(a_n) - F(a_{n-1})\right) + \left(F(b_n) - F(a_n)\right) \left[\because b_i = a_{i+1},\ i = 1, 2, 3, \ldots, n-1\right] \\ &= F(b_n) - F(a_1) \quad \left[\begin{array}{l}\text{All other terms cancel out because} \\ \text{it is a telescoping sum}\end{array}\right] \end{aligned}\right]$$

In general, if $E = (\alpha, \beta]$, $\alpha, \beta \in (a, b]$ then

$$\mathcal{P}_0((\alpha, \beta]) = F(\beta) - F(\alpha) \quad \ldots (3.99)$$

Note that we defined probability measure $\mathcal{P}_0$ on $\mathcal{H}_0$ and $\mathcal{H}_0$ is a field so by Caratheodory's extension theorem it can be uniquely extended to the σ-field $\sigma(\mathcal{H}_0)$ but we know that $\sigma(\mathcal{H}_0) = \mathcal{B}((0, 1])$ or $\sigma(\mathcal{H}_0) = \mathcal{B}((a, b])$ depending upon whether we start with the sample space $\Omega = (0, 1]$ or $\Omega = (a, b]$. So, finally, we have successfully assigned probabilities to the events of our interest or we can say to all the members of the Borel σ-field using Caratheodory's extension theorem.

Last observation when we have $\Omega = (a, b]$ where $a \to -\infty$ and $b \to +\infty$ then $\Omega = \mathbb{R}$, and Borel σ-field in this case will be $\mathcal{B}(\mathbb{R})$. But we know that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, where $\mathcal{C} = \mathcal{C}_3 = \{(-\infty, x] : x \in \mathbb{R}\}$, you may refer (3.69). So, using (3.98), we have

$$\mathcal{P}_0(E) = F(x) - F(-\infty), \text{ where } E = (-\infty, x] \in \mathcal{B}(\mathbb{R}) \quad \ldots (3.100)$$

$$\Rightarrow \mathcal{P}_0((-\infty, x]) = F(x) - 0, \ \left[\because F(-\infty) = 0 \text{ due to (3.89)}\right]$$

$$\Rightarrow \mathcal{P}_0((-\infty, x]) = F(x) \quad \ldots (3.101)$$

In view of (3.70), F(x) defined here is unique. … (3.102)

This completes the discussion of a long journey in the search of how probabilities are assigned to the events of our interest in the continuous world of probability theory. The main objective of this long discussion was to give you a flavour of the kind of mathematics machinery works behind the development of the probability theory as a subject. Understanding of this long discussion will open doors of the world of probability theory to you to become a good statistician and data scientist.

In rest of this course, we will use the findings of this long discussion directly to explain some concept as and when we required. Especially, equations (3.77), (3.79) to (3.87), (3.93) to (3.95) and (3.98) to (3.101).

3.8 SUMMARY

A brief summary of what we have covered in this unit is given as follows:

- **Event vs subset of a Sample Space**: Every event is always a subset of Ω whether we are working in a discrete world or continuous world but in a discrete world every subset of Ω is also an event. But in a continuous world a subset of the sample space may or may not be an event.

- **Assignment of Probabilities to Events in Discrete World**: In a discrete world we only need to assign probabilities to each singleton member of the σ-field (= 2^{Ω} = power set of Ω) and then using finite or countable additivity of probability measure, probability of any event can be obtained.

- **Assignment of Probabilities to Events not Outcomes of the Random Experiment**: We assign probabilities to the members of σ-field not the members of the sample space.

- **Uniform Probability Law for Finite Sample Spaces**: If $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$ then uniform probability law assigns probabilities to singleton subsets of the sample space as follows. $\mathcal{P}(\{\omega_i\}) = \frac{1}{n},\ 1 \le i \le n.$

- **Limitation of Uniform Probability Law for Countably Infinite or Uncountable Sample Spaces**: Uniform probability law does not work for countably infinite sample spaces. In such cases we use non uniform probability laws.

- If $\mathcal{C}$ is the collection of all open subsets of $\Omega = (0, 1],$ then the smallest σ-field on $\Omega = (0, 1]$ containing $\mathcal{C}$ is known as **Borel σ-field** on $\Omega = (0, 1]$ and is denoted by $\mathcal{B}((0, 1]) = \sigma(\mathcal{C}).$

- The members of Borel σ-field are called **Borel sets** or **Borel measurable sets**.

- **Some Borel Measurable Sets**: All singleton, finite, countably infinites sets are Borel Measurable Sets. Also, all kinds of intervals (a, b), (a, b], [a, b) and [a, b] are Borel measurable sets.

- **Uniform Probability Law in Continuous World**: It is defined by

$$\mathcal{P}_0 : \mathcal{H}_0 \to [0, 1]$$

$$\mathcal{P}_0(E) = \begin{cases} 0, & \text{if } E = \phi \\ \sum_{k=1}^{n}(b_k - a_k), & \text{if } E = \bigcup_{k=1}^{n}(a_k, b_k] \end{cases} \quad E \in \mathcal{H}_0$$

where $0 \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le 1$

- $\mathcal{P}_0(E)$ is defined in a similar way when we work on (a, b] instead of (0, 1] and the same is given by

$$\mathcal{P}_0(E) = \begin{cases} 0, & \text{if } E = \phi \\ \frac{1}{b-a}\sum_{k=1}^{n}(b_k - a_k), & \text{if } E = \bigcup_{k=1}^{n}(a_k, b_k] \end{cases} \quad E \in \mathcal{H}_{01}$$

- **Non-Uniform Probability Law in Continuous World**: It is defined by

$\mathcal{P}_0 : \mathcal{H}_0 \to [0, 1]$ by

$$\mathcal{P}_0(E) = \begin{cases} 0, & \text{if } E = \phi \\ \sum_{k=1}^{n}\left(F(b_k) - F(a_k)\right), & \text{if } E = \bigcup_{k=1}^{n}(a_k, b_k] \in \mathcal{H}_0 \end{cases}$$

where $0 \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le 1$ or $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$.

- If $a \le a_1 < b_1 = a_2 < b_2 = \ldots = a_n < b_n \le b$ and $E = (\alpha, \beta]$, $\alpha, \beta \in (a, b]$ then

$\mathcal{P}_0((\alpha, \beta]) = F(\beta) - F(\alpha)$.

In particular, if $\Omega = \mathbb{R}$ and so $(-\infty, x] \in \mathcal{B}(\mathbb{R})$ thus,

$\mathcal{P}_0((-\infty, x]) = F(x)$

3.9 TERMINAL QUESTION

1. A special coin is tossed thrice where probability of getting head is 2/3 and probability of getting tail is 1/3. Explain the procedure of assigning probability to all the events.

3.10 SOLUTIONS/ANSWERS

Self-Assessment Questions (SAQs)

1. We know that when a die is thrown then sample space is given by

$\Omega = \{1, 2, 3, 4, 5, 6\}$.

Here sample space is finite having 6 outcomes. So, we consider the largest σ-field 2^{Ω} on Ω which will have $2^6 = 64$ members and given by

$$2^{\Omega} = \begin{Bmatrix} \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \ldots, \{1, 2, 3, 4, 5\}, \\ \{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \\ \{2, 3, 4, 5, 6\}, \Omega \end{Bmatrix} \quad \ldots (3.103)$$

Given sample space falls in the category of discrete world. We know that in discrete world we only need to assign probabilities to singleton members of the σ-field 2^{Ω}.

It is given that die is unbiased so using uniform probability law, we have

$$\mathcal{P}(\{1\}) = \mathcal{P}(\{2\}) = \mathcal{P}(\{3\}) = \mathcal{P}(\{4\}) = \mathcal{P}(\{5\}) = \mathcal{P}(\{6\}) = \frac{1}{6}. \quad \ldots (3.104)$$

$$\mathcal{P}(\{1, 2\}) = \mathcal{P}(\{1\} \cup \{2\}) = \mathcal{P}(\{1\}) + \mathcal{P}(\{2\}) \quad [\text{Using } (2.16)]$$
$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \quad [\text{Using } (3.104)]$$

Similarly,

probability of all the 15 events having two outcomes is 2/6,

probability of all the 20 events having three outcomes is 3/6,

probability of all the 15 events having four outcomes is 4/6,

probability of all the 6 events having five outcomes is 5/6, and finally

probability of the single event Ω having all the six outcomes is 6/6 = 1.

Terminal Question

1. We know that when a coin is tossed then sample space is given by $\Omega = \{H, T\}$. Also, when three coins are tossed simultaneously or a single coin is tossed thrice then using idea of cross product of two sets (refer to 3.11 of Unit 3 of the course MST-011) sample space is given by

$$\Omega = \{H, T\}^3 = \{H, T\} \times \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\} \times \{H, T\}$$
$$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \quad \ldots (3.105)$$

Here sample space is finite having 8 outcomes. So, we consider the largest σ-field 2^{Ω} on Ω which will have $2^8 = 256$ members and given by

$$2^{\Omega} = \{\phi, \{HHH\}, \{HHT\}, \{HTH\}, \{HTT\}, \{THH\}, \{THT\}, \{TTH\}, \{TTT\}, \ldots, \Omega\} \quad \ldots (3.106)$$

Given sample space falls in the category of discrete world. We know that in discrete world we only need to assign probabilities to singleton members of the σ-field 2^{Ω}.

We are given that this is a special coin, where

$$\mathcal{P}(\{H\}) = \frac{2}{3}, \ \mathcal{P}(\{T\}) = \frac{1}{3}. \quad \ldots (3.107)$$

We know that if two events A and B are independent then

$$\mathcal{P}(A \cap B) = \mathcal{P}(A)\mathcal{P}(B) \text{ or } \mathcal{P}(AB) = \mathcal{P}(A)\mathcal{P}(B). \qquad \ldots (3.108)$$

First using (3.108) and then (3.107), we get

$$\mathcal{P}(\{HHH\}) = \mathcal{P}(\{H\})\mathcal{P}(\{H\})\mathcal{P}(\{H\}) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27},$$

$$\mathcal{P}(\{HHT\}) = \mathcal{P}(\{H\})\mathcal{P}(\{H\})\mathcal{P}(\{T\}) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$$

Similarly, probabilities of other members of the σ-field 2^{Ω} can be obtained using finite additivity of probability measure.