Supplementary Materials of ....

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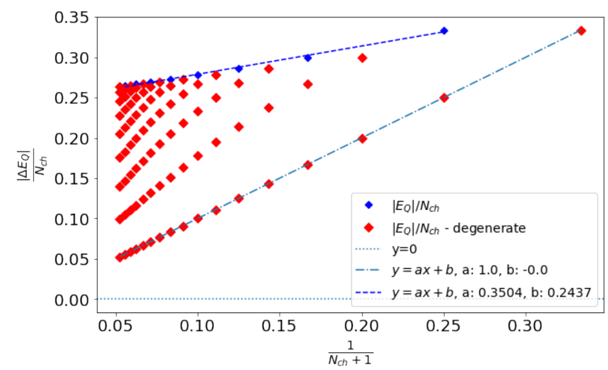

FIG. 1. This shows the variation of quantum energy per channel with $1/N$, where $N = N_{ch} + 1$ is the total number of spins in the systems including the impurity spins.

I. ADDITIONAL PROPERTIES OF THE STAR GRAPH

A. Energy lowering by quantum fluctuations

In the stargraph model Hamiltonian there is a classical ising term ($S_d^z S^z$) and a quantum fluctuation term. Here we are interested in the energy expectation value arising from the quantum fluctuation part of the Hamiltonian (eq.**??**). We define ising enregy E_{ising} and quantum energy E_Q in the ground state $|\psi_g\rangle$ as,

$$E_{ising} = |\langle\psi_g|\mathcal{H}_0^c|\psi_g\rangle|, E_Q = |\langle\psi_g|\mathcal{H}_0^Q|\psi_g\rangle| = |E_g| - |E_{ising}| \ , \\ = \alpha S_d(S+1) - |E_{ising}|. \quad (1)$$

This quantum energy E_Q is generated by the spin-flips between the impurity spin and the outer spin. Our goal here to understand the nature of this E_Q as we go to the large K limit, whether there are some remenant quantum fluctuation even in the thermodynamic limit. To answer that questio we intrduce another important parameter, quantum enrgy per channel, $e_Q = |E_Q|/K$. We know in the gorund state $K = 2S$ thus,

$$e_Q = \frac{\alpha S_d(S+1)}{K} - \frac{|E_{ising}|}{K} \ , \\ = \frac{\alpha S_d}{2} - \frac{1}{K}\bigg(|E_{ising}| - \alpha S_d\bigg) \ , \quad (2)$$

Thus from the above equation of quantum energy per channel one can see that in the large channel limit

$$e_Q = \frac{\alpha S_d}{2} - \frac{|E_{ising}|}{K} < \frac{\alpha S_d}{2} \ , \ K \to \infty, \quad (3)$$

As shown in the above Fig.1 for the case of $S_d = 1/2$, for a particular channel K the maximum quantum energy corresponds to the $|J^z|_{min}$ state and the minimum is related to the $|J^z| = J$ state. In the large channel limit one can see the quatnum energy associated with the $|J^z| = J$ vanishes showing the classical nature of this state where there is no quantum fluctuation present. Whereas, in the state $|J^z|_{min}$ there are finite non-zero quantum fluctuation present in it, showing the true quantum nature of this macroscpic singlet state as we have already discussed.

B. Measure of quantum fluctuations

Here we want to calculate various correlation functions in the ground state. We know that J^z is a good quantum number, can take values $2J+1$ possible values. Here we are interested in calculating the quantum fluctuation present in the ground state by measuring the expectation value of $\mathcal{Q} \equiv \langle\psi_g|(J^x)^2 + (J^y)^2|\psi_g\rangle$. We can see that $\mathcal{Q} = \langle\psi_g|J^2 - (J^z)^2|\psi_g\rangle$. For a general S_d impurity spin problem, $J = |K/2 - S_d|$ in the ground state, thus $J^2 = J(J+1) = (|K/2 - S_d|)(|K/2 - S_d| + 1)$. We define a quantity $\Delta = (K/2 - S_d)$, where $\Delta = 0$ represents the exactly screened problem, $\Delta > 0$ and $\Delta < 0$ represents the overscreened and underscreened problem respectively. For the cases where $|\Delta|$ is integer $J^z_{min} = 0$ and when $|\Delta|$ is half integer $J^z_{min} \pm 1/2$. Here we are intersted in calculating the maxmum quantum fluctuation present in the multi-channel zero mode ground state.

$$\begin{aligned} \mathcal{Q}_{max} &= \langle\psi_g|J^2 - (J^z_{min})^2|\psi_g\rangle \\ &= (|K/2 - S_d|)(|K/2 - S_d| + 1) - 1/4 \ , \quad 2|\Delta|\text{=odd} \\ &= (|K/2 - S_d|)(|K/2 - S_d| + 1) \ , \quad 2|\Delta|\text{=even} \end{aligned} \quad (4)$$

Then maximum quantum fluctuation per channel is defined as $q_K = \sqrt{\mathcal{Q}_{max}}/K$. We find

$$q_K = \frac{1}{K}\sqrt{|\Delta|(|\Delta|+1)} \quad (5)$$

We can see that at large channel limit is, $\lim_{K\to\infty} q_K = 1/2$. Thus at large channel limit both the overscrened and underscreend ground states has same non-zero quantum fluctuation. Only for the exactly screened cases ($\Delta = 0$) with unique ground state, this measure of quantum fluctuation $\mathcal{Q}_{max}$ vanishes. This shows the duality between overscreened and underscreened cases and it's distinction from the exactly screened cases.

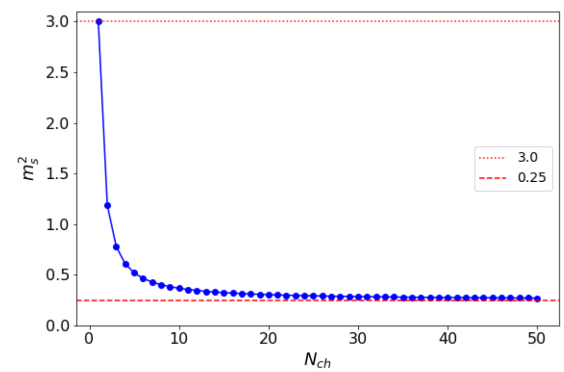

FIG. 2. This shows how the staggered magnetization changes with the number of channels N_{ch}.

C. Staggerred magnetization of the star graph

One can reqrite the Hamiltonian as $\alpha\vec{S}_d.\vec{S} = (\alpha/4)[(\vec{S} + \vec{S}_d)^2 - (\vec{S} - \vec{S}_d)^2]$, this shows that $\vec{J}$ and the staggered magnetization $\vec{M}_s = \vec{S} - \vec{S}_d$ is both good quantum number for in this stargraph model. For single channel case $K = 1$, ground state is an unique 2-spin singlet $|\psi_g\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle = |J = 0, J_z = 0\rangle$, which showns the prefect screening. Our goal here to capture the breakdown of the screening by calculating this staggerd magnetization for a multichannel problem. For a general K cahnnel we find that

$$\begin{aligned} M_s^2 &= \langle\psi_g|(\vec{S}_d - \vec{S})^2|\psi_g\rangle = \langle\psi_g|2(\vec{S}_d^2 + \vec{S}^2) - \vec{J}^2|\psi_g\rangle \\ &= 2S_d(S_d+1) + \frac{K^2}{4} + K + \frac{1}{4} \end{aligned} \quad (6)$$

Thus, staggered magnetization squared per channel,

$$m_s^2 = \frac{M_s^2}{K^2} = \frac{2S_d(S_d+1)}{K^2} + \frac{1}{4} + \frac{1}{K} + \frac{1}{4K^2} \quad (7)$$

Which shows, in the large channel limit this staggered magnetization , $m_s^2 \Rightarrow 1/4$. for any finite S_d. Using the $SU(2)$ property of our problem, we can define m_s for each spatial direction, in 3D there are three independent spatial direction.

$$\langle(m_s^x)^2\rangle = \langle(m_s^y)^2\rangle = \langle(m_s^z)^2\rangle = \frac{1}{3}m_s^2 \ , \quad (8)$$

Just to recall, for the single channel case $\langle(m_s^z)^2\rangle = 1$ showing the perfect screening, on the otherhand in the large channel limit this becomes $\langle(m_s^z)^2\rangle = 1/12$ showing the breakdown of the screening. For the $S_d = 1/2$ case we have shown how this staggered magnetization per-channel decreases in the multichannel case in the Fig.2. This clearly shown that this staggerd magnetization is maximum.

D. Thermodynamic quantitites

1. Impurity magnetization in terms of parity operators

Just like the complete string operator π^z, the modified string operator $\sigma_d^z\pi^z$ is also a Wilson loop operator that wraps around only the outer nodes of the star graph:

$$\pi_c^z \equiv \sigma_d^z \pi^z = \exp\left[i\frac{\pi}{2}\left(\sum_{l=1}^{K}\sigma_l^z - K\right)\right] \tag{9}$$

The expectation value of the impurity magnetization along a particular direction and in specific ground states can be related to the 't Hooft operator defined under eq. ??. We will work in the state comprised of two adjacent eigenstates of J^z:

$$|g_{J^z}^\theta\rangle \equiv \frac{1}{\sqrt{2}}\left(|J^z\rangle + e^{i\theta}|J^z+1\rangle\right), \quad J^z < \frac{1}{2}(K-1) \tag{10}$$

The expectation value of the impurity magnetization operator σ_d^x can be expressed as

$$\langle\sigma_d^x\rangle \equiv \langle g_{J^z}^\theta|\sigma_d^x|g_{J^z}^\theta\rangle = -\langle J^z+1|\pi_c^x| - J^z\rangle + \text{h.c.} \tag{11}$$

This expression relates the observable impurity magnetization to the topological 't Hooft operator [?]. Evaluating the matrix elements gives

$$\langle\sigma_d^x\rangle = -\frac{\sqrt{K^2-(2J^z+1)^2}}{2(1+K)}\cos\theta \tag{12}$$

Performing a similar calculation reveals that the impurity magnetizations along y and z in the same state are given by

$$\langle\sigma_d^y\rangle = -\frac{\sqrt{K^2-(2J^z+1)^2}}{2(1+K)}\sin\theta, \quad \langle\sigma_d^z\rangle = -\frac{2J^z+1}{(1+K)} \tag{13}$$

Combining eqs. 12 and 13, we find

$$\cos^2\theta\,(\langle\sigma_d^x\rangle)^2 + \sin^2\theta\,(\langle\sigma_d^y\rangle)^2 + \frac{1}{4}(\langle\sigma_d^z\rangle)^2 = \frac{1}{4}\left(\frac{K}{1+K}\right)^2 \tag{14}$$

This relation constrains the values of the magnetization along all the directions: the x and y magnetization values have already been shown to be related to the 't Hooft operators π^x and π^y and the magnetization along z is therefore constrained in terms of the 't Hooft operators and the quantized function on the right-hand side (the function is quantized because K can only take integer values).

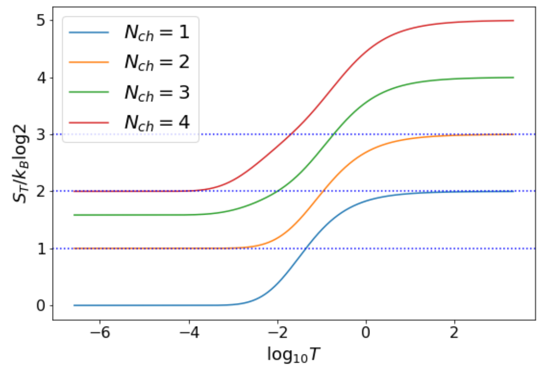

FIG. 3. This shows the variation of thermal entropy with the temperature.

2. Thermal entropy

Thermal entropy is defined as $S = -(\frac{\partial\mathcal{F}}{\partial T})_H$, where H represents a constant magnetic field. In ourcase we will be interested in the zero field case.

$$\mathcal{F} = -k_BT\log Z$$
$$S = -(\frac{\partial\mathcal{F}}{\partial T}) = -k_B\log Z - k_BT\frac{1}{Z}\frac{dZ}{dT} \tag{15}$$

Thus we get

$$S = -k_B\log\sum_\epsilon d(\epsilon)e^{-\beta\epsilon} - \frac{1}{\beta}\frac{k_B\sum_\epsilon \epsilon d(\epsilon)e^{-\beta\epsilon}\beta^2}{\sum_\epsilon d(\epsilon)e^{-\beta\epsilon}}$$

$$\lim_{\beta\to\infty} S = -k_B\log_2 d(\epsilon_G)\,, \quad \lim_{\beta\to 0} S = -k_B\log\sum_\epsilon d(\epsilon)$$

Thus at the extreme temperature it is easy to calculate the termal entropy, but it is difficult to visualize for any intermediate temperatures. Thus we plot the thermal entropy (unit $\log 2$) for different tempertures and for different channels in Fig.3. This shows at the extreme temperature the termal entropy saturates and at the intermediate temperature it changes from one to the other like a soliton like solution. At low temperature the thermal enttropy in the unit of $\log 2$ is not always quanlized but at high temperature it is. To understad this let's say at low temperature $S_T/k_B\log T = \Omega = \log_2 d(\epsilon_G)$. Here $d(\epsilon_G)$ is always integer and equal to the channel number K. Thus

$$d(\epsilon_G) = 2^\Omega = K \tag{16}$$

From our study we can see that for some channel number we get $S_T/k_B\log T$ tobe integer. One can easily see that those $K = 2^n$ channel cases, n is an integer will have integer thermal entropy. Thus between m and $m+1$ integer thermal entropy plateau there will be (2^m-1) number of fractional thermal entropy plateau, which is always odd.

Note, The above results for the high temperature holds for spin-1/2 impurity case, which slightly diffes for the case with general S_d impurity spin, though the low temperature behavior only depends on the ground state degeneracy ($d_{\epsilon_G} = (K+1) - 2S_d$) irrespective of the value of S_d. At high temperature the entanglement entropy depends on $\sum_\epsilon d(\epsilon)$, which is the dimension of the Hilbert space, $\sum_\epsilon d(\epsilon) = 2^K \times (2S_d + 1)$. Thus entanglemen entropy at high temperature for a general S_d impurity spin is

$$\lim_{\beta\to 0} S = -k_B \log \sum_\epsilon d(\epsilon) = -k_b \log[2^K(2S_d+1)] \ , \quad (17)$$

One can confirm that for $S_d = 1/2$, $\lim_{\beta\to 0} S = -k_B(K+1)$. Thus only for those S_d values when $2S_d + 1 = 2^Q$, we get $-k_b \log[2^K(2S_d+1)] = -k_b(K+Q)$. Which shows that only some of the half integer impurity problems will have integer entanglement entropy at high temperature limit.

II. ADDITIONAL TOPOLOGICAL FEATURES OF THE LOCAL MOTT LIQUID

Impurity-bath quantum fluctuation resolved all-to-all effective Hamiltonian is written in terms of the spinor spin operators which are individually made out of two electronic degree of freedom. This is defined as $\vec{S}_i = \frac{\hbar}{2} \sum_{\alpha,\beta\in\{\uparrow,\downarrow\}} c^{(i)\dagger}_{0\alpha} \vec{\sigma}_{\alpha\beta} c^{(i)}_{0\beta}$. In the above eq.(**??**) we see the $U(1)$ symmetry of the effective Hamiltonian. Using the spinor repersentation one can see the spin creation operation in terms of the electronic degree of freedom is

$$S_i^+ = \frac{\hbar}{2} \sum_{\alpha,\beta\in\{\uparrow\downarrow\}} c^{(i)\dagger}_{0\alpha} \sigma^+_{\alpha\beta} c^{(i)}_{0\beta} = \frac{\hbar}{2} c^{(i)\dagger}_{0\uparrow} c^{(i)}_{0\downarrow} \quad (18)$$

$$S_i^z = \frac{\hbar}{2} \sum_{\alpha,\beta\in\{\uparrow\downarrow\}} c^{(i)\dagger}_{0\alpha} \sigma^z_{\alpha\beta} c^{(i)}_{0\beta} = \frac{\hbar}{2} (c^{(i)\dagger}_{0\uparrow} c^{(i)}_{0\uparrow} - c^{(i)\dagger}_{0\downarrow} c^{(i)}_{0\downarrow}) \quad (19)$$

Thus this spinor spins are nothing but the Anderson pseudospin formulation in the spin channel. The spin creation opertor (S_i^+) shows simultaneous creation of an electron-hole pair at the realspace origin of the i^{th} channel. The condensation of such electron-hole pais has already been shown in [CITE: mott]. Thus for this effective Hamiltonian one can define twist-translation oeprations to construct the gauge theory and unveiling any hidden degeneracy. Let's recall the effective Hamiltonian

$$H_{eff} = \frac{\beta_\uparrow(\alpha,\omega_\uparrow)}{4} \left[\sum_{ij} S_i^+ S_j^- \ + \text{h.c.} \right] , \quad (20)$$

where i, j are the channel indices. Due to the all-to-all nature of the connectivity one can draw total $K!$ possible unique closed paths ($\mathcal{C}_\mu$) where each nodes (channels) is being touched only once. One can thus define total $K!$ number of translation operators ($\hat{T}_\mu$) which keeps the Hamiltonian invariant. Let's define one of such translation operator $\hat{T}_\mu$ thich gives a periodic shifts along the closed path $\mathcal{C}_\mu$. Let's define twist operator along the path $\mathcal{C}_\mu$

$$\hat{\mathcal{O}}_\mu = \exp(i \frac{2\pi}{K} \sum_{\substack{j=1 \\ \mathcal{C}_\mu}}^{K} j S_j^z) \ , \quad \hat{T}_\mu = e^{i \hat{P}_\mu^{cm}} \quad (21)$$

The operation of the translation operator $\hat{T}_\mu$ is defined as $\hat{T}_\mu S_j^z = S_{j+1}^z$ where $j+1$ and j are the nearest neighbor on the closed path $\mathcal{C}_\mu$. Then the braiding rule between the twist and translation operators are give as

$$\begin{aligned} \hat{T}_\mu \hat{\mathcal{O}}_\mu \hat{T}_\mu^\dagger \hat{\mathcal{O}}_\mu^\dagger &= \exp\{i[2\pi S_1^z - \frac{2\pi}{K} S^z]\} \\ &= \exp\{i[\pm\pi - \frac{2\pi}{K} S^z]\} = \exp(i\frac{2\pi p}{q}) \end{aligned} \quad (22)$$

The availability of the non-trivial braiding staatistics between these twist and translation operators is possible if $p \neq 0$ and $q \neq \infty$. Further simplification leads to the condition

$$\begin{aligned} \pm\pi - \frac{2\pi}{K} S^z = \frac{2\pi p}{q} \quad &\Rightarrow \pm\frac{1}{2} - \frac{S^z}{K} = \frac{p}{q} \\ &\frac{(\pm K - 2S^z)}{2K} = \frac{p}{q} \ , \end{aligned} \quad (23)$$

p, q are mutual prime, We knwo that the S^z can take values $(\mp K/2 \pm m)$ where m is a integer $0 \leq m \leq K$. Putting this value in the above equation leads to two possible solutions.

$$\frac{(K-m)}{K} = \frac{p}{q} \ , \quad \frac{-m}{K} = \frac{p}{q} \quad (24)$$

For the first case $K - m = p$ we can see that $m = K$ makes p trivial hus the allowed values are $m = 0, \cdots, K-1$, which represents the corresponding S^z eigen values

$$-K/2, -K/2+1, \cdots, K/2-2, K/2-1 \quad (25)$$

Similarly the second case implies, $-m = p$, but $p = 0$ is not allowed as this makes the braiding statistics trivial, thus the possible S^z values are

$$-K/2+1, -K/2+2, \cdots, K/2-1, K/2 \quad (26)$$

Thus we get the general braiding statistics between the twist and the translation

$$\hat{T}_\mu \hat{\mathcal{O}}_\mu \hat{T}_\mu^\dagger \hat{\mathcal{O}}_\mu{}^\dagger = e^{i\frac{2\pi p}{K}} \quad (27)$$

where p corresponds to different S^z states, related as $p = \pm K/2 - S^z$. Thus we can see that there are K possible S^z plaleau states in the all-to-all model where each plateaux is K fold degenerate. This p/K is similar to the filling factor of the fractional quantum Hall effect. There are possibly $K!$ pairs of twist and translation operator corresponding to different closed paths $\mathcal{C}_\mu$ which can probe this degeneracy.

1. Action on the Hamiltonian

Now we findout the action of these twist operators on the Hamiltonian. The Hamiltonian is written as

$$H_{eff} = \frac{\beta_\uparrow(\alpha,\omega_\uparrow)}{2}(S^{x2}+S^{y2}) \tag{28}$$

Due the all-to-all nature of the effetive Hamiltonian one can find $K!$ possible relative arrangement of those K channels which keeps the Hamiltonian invariant. Here we briefly discuss the choice of the closed loop $\mathcal{C}_\mu$ and the insertion of the flux. As shown in the Fig.??(b) we have chosen a particular closed path which crosses all the outer spin only once. We embed that closed loop on a plane and put the flux perpendiculr to the plane through the closed loop. One can find a different closed loop where the ordering of the outer spins will be different. The action of the translation operator shifts the outer spins along this closed curve by one step.

The action of the twist operator on the S^x is determined in the follwoing calculation

$$\begin{aligned}\hat{\mathcal{O}}_\mu S^x \hat{\mathcal{O}}_\mu^\dagger &= \exp(i\frac{2\pi}{K}\sum_{\substack{j=1\\ \mathcal{P}_\mu}}^{K} jS_j^z)S^x\exp(-i\frac{2\pi}{K}\sum_{\substack{j=1\\ \mathcal{P}_\mu}}^{K} jS_j^z)\\ &= e^X S^x e^{-X}\end{aligned} \tag{29}$$

For simplicity of thte calculation we define $i\frac{2\pi}{K}=\Omega$, thus $X=\Omega\sum_j jS_j^z$. Thus we get

$$\begin{aligned}e^X S^x e^{-X} &= S^x + [X,S^x] + \frac{1}{2!}[X,[X,S^x]]+\cdots\\ &= \sum_l (S_l^x\cos\theta_l - S_l^y\sin\theta_l)\ , \quad \theta_l=\frac{2\pi l}{K} \quad (30)\\ e^X S^y e^{-X} &= \sum_l (S_l^y\cos\theta_l + S_l^x\sin\theta_l) \quad (31)\end{aligned}$$

As already defined, $\theta_l=\frac{2\pi l}{K}=\frac{2\pi(K-n)}{K}$, where n is an integer. We can see that in the large channel limit θ_l becomes inter multiple of 2π. Thus in the large channel limit

$$\begin{aligned}&\lim_{K\to\infty}\hat{\mathcal{O}}_\mu H_{eff}\hat{\mathcal{O}}_\mu^\dagger\\ &=\frac{\beta_\uparrow(\alpha,\omega_\uparrow)}{2}\left[\hat{\mathcal{O}}_\mu S^x\hat{\mathcal{O}}_\mu^\dagger\hat{\mathcal{O}}_\mu S^x\hat{\mathcal{O}}_\mu^\dagger+\hat{\mathcal{O}}_\mu S^y\hat{\mathcal{O}}_\mu^\dagger\hat{\mathcal{O}}_\mu S^y\hat{\mathcal{O}}_\mu^\dagger\right]\\ &=\frac{\beta_\uparrow(\alpha,\omega_\uparrow)}{2}\left[S^{x2}+S^{y2}\right]=H_{eff}\end{aligned}$$

$$\lim_{K\to\infty}[\hat{\mathcal{O}},H_{eff}]=0 \tag{32}$$

The translation operator $\hat{T}_\mu=e^{i\hat{\mathcal{P}}_\mu}$ commutes with the Hamiltonian thus the generator $\hat{\mathcal{P}}_\mu$ also commutes with the Hamiltonian. We can label the j^{th} state with the eigenvalues of this operator $\hat{\mathcal{P}}_\mu$ as $|p_\mu^j\rangle$. We here discuss the action of these twist and translation operators on these states.

$$\begin{aligned}\hat{T}_\mu|p_\mu^j\rangle &= e^{i\hat{\mathcal{P}}_\mu}|p_\mu^j\rangle = e^{ip_\mu^j}|p_\mu^j\rangle\\ \hat{T}_\mu\hat{\mathcal{O}}_\mu|p_\mu^j\rangle &= \hat{\mathcal{O}}_\mu\hat{T}_\mu e^{i\frac{2\pi m}{K}}|p_\mu^j\rangle\ ,\end{aligned}$$

here m signifies different S^z plateaux.

$$\begin{aligned}&= \hat{\mathcal{O}}_\mu e^{i(\frac{2\pi m}{K}+p_\mu^j)}|p_\mu^j\rangle\ , \quad \frac{2\pi m}{K}\equiv p_\mu^m\\ \hat{T}_\mu\left(\hat{\mathcal{O}}_\mu|p_\mu^j\rangle\right) &= e^{i(p_\mu^m+p_\mu^j)}\left(\hat{\mathcal{O}}_\mu|p_\mu^j\rangle\right)\end{aligned} \tag{33}$$

Thus we can see that $\hat{\mathcal{O}}_\mu|p_\mu^j\rangle$ is a state of the traslation operator with different eiven value than $|p_\mu^j\rangle$, thus orthogonal to each other. Simiarly one can generally show that $\langle p_\mu^j|\hat{\mathcal{O}}_\mu^q|p_\mu^j\rangle=0$, where q is any integer. Also in the large K limit we can see from the eq.(32) that these differnt twisted states has same energy. Which shows that at each plateau state labeled by the S^z eigenvalue has K fold degenerate eigenstates labeled by the eigenvalue of the translation operators. Thus the CSCO is formed by $H, S^z, \hat{T}$. Thus states can be lebeled as $|E, S_j^z, P_\mu^j\rangle$.

III. EFFECT OF CONDUCTION BATH EXCITATIONS ON THE FIXED POINT THEORY

A. Non-Fermi liquid signatures in momentum space for 2-channel Kondo

Obtaining the effective Hamiltonian involves obtaining the low energy excitations on top of the ground state of the star graph. The large-energy excitations are ones that involve spin flips. This guides the separation of the Hamiltonian into a diagonal and an off-diagonal piece:

$$H = H_d + V = \underbrace{H_0 + JS_d^z s_{\text{tot}}^z}_{H_d} + \underbrace{\frac{J}{2}S_d^+ s_{\text{tot}}^- + \text{h.c.}}_{V+V^\dagger} \tag{34}$$

We define V as the interaction term that decreases s_{tot}^z by 1: $V\,|s_{\text{tot}}^z\rangle\to|s_{\text{tot}}^z-1\rangle$. Similarly, we define $V^\dagger\,|s_{\text{tot}}^z\rangle\to|s_{\text{tot}}^z+1\rangle$. The effective Hamiltonian that has the states $|S_d^z, s_{\text{tot}}, s_{\text{tot}}^z\rangle$ as eigenstates are

$$H_{\text{eff}} = H_d + V\frac{1}{E_{\text{gs}}-H_d}V = H_d + \frac{J}{2}S_d^+ s_{\text{tot}}^-\frac{1}{E_{\text{gs}}-JS_d^z s_{\text{tot}}^z - H_0}\frac{J}{2}S_d^- s_{\text{tot}}^+ + \frac{J}{2}S_d^- s_{\text{tot}}^+\frac{1}{E_{\text{gs}}-JS_d^z s_{\text{tot}}^z - H_0}\frac{J}{2}S_d^+ s_{\text{tot}}^- \tag{35}$$

This is obtained from the Schrodinger equation for the ground state. If we expand the ground state in terms of $|S_d^z, s_{\rm tot}, s_{\rm tot}^z\rangle$, we have $|\Psi_{\rm gs}\rangle = \sum_{S_d^z, s_{\rm tot}, s_{\rm tot}^z} C_{S_d^z, s_{\rm tot}, s_{\rm tot}^z} |S_d^z, s_{\rm tot}, s_{\rm tot}^z\rangle$. The Schrodinger equation for the ground state can be written as

$$E_{\rm gs} |\Psi_{\rm gs}\rangle = H |\Psi_{\rm gs}\rangle = (H_d + V) |\Psi_{\rm gs}\rangle \implies (E_{\rm gs} - H_d) \sum C_{S_d^z, s_{\rm tot}, s_{\rm tot}^z} |S_d^z, s_{\rm tot}, s_{\rm tot}^z\rangle = V \sum C_{S_d^z, s_{\rm tot}, s_{\rm tot}^z} |S_d^z, s_{\rm tot}, s_{\rm tot}^z\rangle \tag{36}$$

$E_{\rm gs}$ is the ground state energy, and can be replaced by the star graph ground state energy if we remove the kinetic energy cost via normal ordering: $E_{\rm gs} = -\frac{J}{2}\left(\frac{K}{2}+1\right)$. Since the interaction part V only changes $S_d^z \to -S_d^z$ and $s_{\rm tot}^z \to s_{\rm tot}^z \pm 1$, we can simplify the equation into individual smaller equations. For the two-channel model, the possible states are $(s_{\rm tot}, s_{\rm tot}^z) = (0,0), (1,-1), (1,0), (1,1)$. The individual equations for these states are

$$E_{\rm gs} |\frac{1}{2}, 1, 0\rangle = \left(H_d + V \frac{1}{E_{\rm gs} - H_d} V^\dagger\right) |\frac{1}{2}, 1, 0\rangle \tag{37}$$

$$E_{\rm gs} |-\frac{1}{2}, 1, 0\rangle = \left(H_d + V^\dagger \frac{1}{E_{\rm gs} - H_d} V\right) |-\frac{1}{2}, 1, 0\rangle \tag{38}$$

$$\tag{39}$$

These equations represent the Schrodinger equation for the states $|S_d^z, 1, 0\rangle$, and the right hand sides therefore give the effective Hamiltonians for those states. If we combine the states into a single subspace $|1,0\rangle = \left\{|\frac{1}{2}, 1, 0\rangle, |-\frac{1}{2}, 1, 0\rangle\right\}$, the effective Hamiltonian for this composite subspace becomes the sum of the two parts:

$$H_{\rm eff}^{1,0} |1,0\rangle \langle 1,0| = \left(H_d + V G_0 V^\dagger + V^\dagger G_0 V\right) |1,0\rangle \tag{40}$$

where $G_0 = (E_{\rm gs} - H_d)^{-1}$. If we expand the subspace as $|1,0\rangle = |\frac{1}{2}, 1, 0\rangle + |-\frac{1}{2}, 1, 0\rangle$, we recover eqs. 37. Solving similarly for the other states gives

$$H_{\rm eff}^{1,1} |1,1\rangle \langle 1,1| = \left(H_d + V^\dagger G_0 V\right) |1,1\rangle \tag{41}$$

$$H_{\rm eff}^{1,-1} |1,-1\rangle \langle 1,-1| = \left(H_d + V G_0 V^\dagger\right) |1,-1\rangle \tag{42}$$

To calculate these effective Hamiltonians, we will calculate the individual terms. We can easily simplify the S_d^z in the denominator of G_0, because $S_d^\pm \frac{1}{A + B S_d^z} = S_d^\pm \frac{1}{A \mp \frac{1}{2} B}$:

$$V G_0 V^\dagger = \frac{J^2}{4} s_{\rm tot}^- \frac{\frac{1}{2} + S_d^z}{E_{\rm gs} + \frac{J}{2} s_{\rm tot}^z - H_0} s_{\rm tot}^+ \tag{43}$$

$$V^\dagger G_0 V = \frac{J^2}{4} s_{\rm tot}^+ \frac{\frac{1}{2} - S_d^z}{E_{\rm gs} - \frac{J}{2} s_{\rm tot}^z - H_0} s_{\rm tot}^- \tag{44}$$

Since H_0 does not commute with the spin operators, we will need to expand the denominator to make sense of this Hamiltonian.

$$V G_0 V^\dagger = s_{\rm tot}^- \frac{1}{E_{\rm gs} + \frac{J}{2} s_{\rm tot}^z} \left[1 + \frac{1}{E_{\rm gs} + \frac{J}{2} s_{\rm tot}^z} H_0 + \frac{1}{E_{\rm gs} + \frac{J}{2} s_{\rm tot}^z} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s_{\rm tot}^z} H_0 + \ldots\right] s_{\rm tot}^+ \tag{45}$$

$$V^\dagger G_0 V = s_{\rm tot}^+ \frac{1}{E_{\rm gs} - \frac{J}{2} s_{\rm tot}^z} \left[1 + \frac{1}{E_{\rm gs} - \frac{J}{2} s_{\rm tot}^z} H_0 + \frac{1}{E_{\rm gs} - \frac{J}{2} s_{\rm tot}^z} H_0 \frac{1}{E_{\rm gs} - \frac{J}{2} s_{\rm tot}^z} H_0 + \ldots\right] s_{\rm tot}^- \tag{46}$$

This is an expansion in $H_0^n / J^{n+1}, n = 0, 1, 2, \ldots$. Expanding up to $n = 2$ and keeping at most two particle interaction terms, the effective Hamiltonians for these states are:

$$H_{\rm eff}^{1,1} = H_0 + J S_d^z + \frac{J^2}{4} \frac{2}{E_{\rm gs}} \left[1 + \frac{H_0}{E_{\rm gs}} + \frac{s_{\rm tot}^+ X_{1,\rm tot}}{2 E_{\rm gs}} + \frac{H_0^2}{E_{\rm gs}^2} - \frac{Z_{1,\rm tot} H_0}{E_{\rm gs}^3}\right] \left(\frac{1}{2} - S_d^z\right) \tag{47}$$

$$H_{\rm eff}^{1,-1} = H_0 - J S_d^z + \frac{J^2}{4} \frac{2}{E_{\rm gs}} \left[1 + \frac{H_0}{E_{\rm gs}} - \frac{s_{\rm tot}^- X_{1,\rm tot}^\dagger}{2 E_{\rm gs}} + \frac{H_0^2}{E_{\rm gs}^2} - \frac{Z_{1,\rm tot} H_0}{E_{\rm gs}^3}\right] \left(\frac{1}{2} + S_d^z\right) \tag{48}$$

$$H_{\rm eff}^{1,0} = H_0 + \frac{J^2}{2\left(E_{\rm gs} + \frac{J}{2}\right)} \left[1 + \frac{H_0 + \left(\frac{1}{2} + S_d^z\right) s_{\rm tot}^+ X_{1,\rm tot} - \left(\frac{1}{2} - S_d^z\right) s_{\rm tot}^- X_{1,\rm tot}^\dagger}{2\left(E_{\rm gs} + \frac{J}{2}\right)} + \frac{H_0^2}{\left(E_{\rm gs} + \frac{J}{2}\right)^2} - \frac{Z_{1,\rm tot} H_0}{\left(E_{\rm gs} + \frac{J}{2}\right)^3}\right] \tag{49}$$

We employed the definitions $X_{n,\text{tot}} \equiv \sum_l \sum_{k,k'} (\epsilon_k - \epsilon_{k'})^n c^\dagger_{k\downarrow} c_{k'\uparrow}, X_{n,l}$ and $Z_{1,\text{tot}} \equiv \sum_{k,k',l} (\epsilon_k - \epsilon_{k'}) \frac{1}{2} \left(c^\dagger_{k\uparrow,l} c_{k'\uparrow,l} - c^\dagger_{k\downarrow,l} c_{k'\downarrow,l} \right)$. Focusing on the effective Hamiltonian for $(1, 0)$, we see lots of non-Fermi liquid terms of the form $s^+_{\text{tot}} X_{1,\text{tot}}, s^-_{\text{tot}} X^\dagger_{1,\text{tot}}, Z_{1,\text{tot}} H_0$. These arise because of the degenerate manifold and the increased availability of states in the Hilbert space for scattering, as compared to the unique singlet ground state of the single-channel Kondo model.

B. Low temperature thermodynamic behaviour

In order to understnad this non-Fermi liquid we calculate some thermodyanmic quantities like impurity specific heat and suseptibility. First we calcualte the self energy of this NFL hamiltonian. In the real space one can get a diagonal piece from the above equation by using the Fermionic anticommutation relations.

$$H^{off,(2)}_{eff}|_{diag} = -(16t^2/3)[(S^z_1)^2 + (S^z_2)^2] \tag{50}$$

We can find the corresponding momentum space Hamiltonian by doing the fouries transforamtion to the Hamiltonian $H^{off,(2)}_{eff}|_{diag}$ as.

$$H^{off,(2)}_{eff}|_{diag} = -\frac{4t^2}{3}\frac{1}{N}\Bigg[\sum_{k,\sigma} n_{k\sigma}(1 - \frac{1}{N}\sum_{k_2} n_{k_2,-\sigma}) + \sum_{k,\sigma} \tilde{n}_{k\sigma}(1 - \frac{1}{N}\sum_{k_2} \tilde{n}_{k_2,-\sigma})\Bigg] \tag{51}$$

The above relation leads to the correction to the kinetic energy, which is

$$\bar{\epsilon}_k - \epsilon_k = \Sigma_k = -\frac{4t^2}{3N^2}\left(1 - \frac{N}{e^{(\epsilon_k - \mu)/k_B T} + 1}\right) \tag{52}$$

Using the self-energy we calculate the impurity speciftic heat defined as $C_{imp} = C(J^*) - C(0)$ which is defined as

$$C_{imp} = \sum_{\Lambda,\sigma} \beta \left[\frac{(\bar{\epsilon}_\Lambda)^2 e^{\beta \bar{\epsilon}_\Lambda}}{(e^{\beta \bar{\epsilon}_\Lambda} + 1)^2} - \frac{(\epsilon_\Lambda)^2 e^{\beta \epsilon_\Lambda}}{(e^{\beta \epsilon_\Lambda} + 1)^2} \right] \tag{53}$$

Here we are intersted in the low temperature behavior of this impurity specific-heat. Using the self-energy obtained above we calculate the impurity susecptibility for $t = 0.1$ and $\alpha = 1$. The above Fig.4 shows that at low temperature follows logarithmic behavior for the single channel case which is in agreement with the results known in the literature [CITE].

$$\frac{C_{imp}}{T} \propto \log T \tag{54}$$

Apart from this impurity specific heat one can calculate the magnetic susceptibilty. For that we do the exact diagonalizaton of the NFL Hamiltonian eq.(**??**) and get

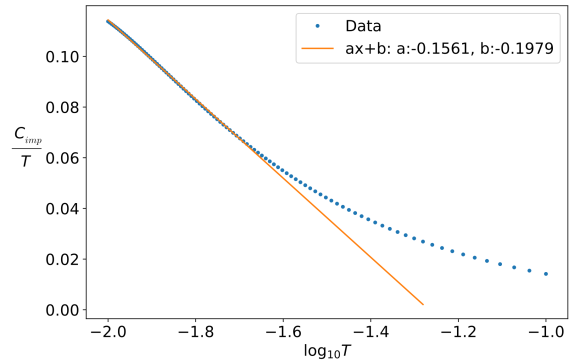

FIG. 4. This figure shows the variation of the impurity specific heat with the temperature.

the entire eigenspectrum. On all those different state we can calculate teh expectation value of the magnetization squared $\langle (S^z)^2 \rangle$. Using these expectation values we can calculate the suseptibility by using the formula

$$\chi = \beta \left[\frac{\sum e^{-\beta \bar{\epsilon}_\Lambda} \langle \bar{S}^{z2} \rangle}{\sum e^{-\beta \bar{\epsilon}_\Lambda}} - \frac{\sum e^{-\beta \epsilon_\Lambda} \langle S^{z2} \rangle}{\sum e^{-\beta \epsilon_\Lambda}} \right] \tag{55}$$

We numerically compute the χ from the definition above and study it's dependence on temperature. The above

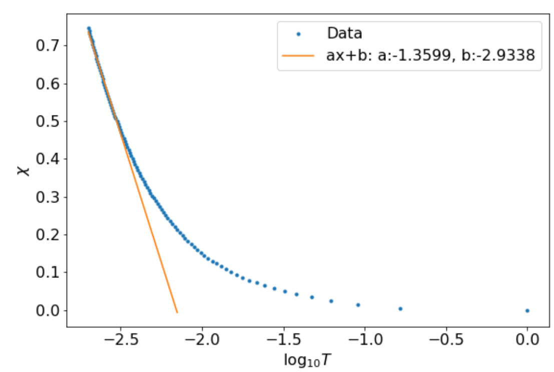

FIG. 5. Suseptibility

figure shows that at low temperture the suseptibility follows the logarithmic behavior.

$$\chi(T) \propto \log T \tag{56}$$

As expected both the C_{imp}/T and χ follows the logarithmic behavior which is known in the literature for the two channel Kondo problem. From the slopes we can calculate the Wilson ratio $C_{imp}/T\chi$, we find that in our case where we have taken upto two nearest neighbors to each zeroth sides of two channels the Wilson ratio is ≈ 8.3 which is greater than the actual value. For a smaller system with only one nearest neighbor to the zeroth site of both the channels we find the Wilson number if even higher shows that in a larger system the Wilson ratio approaches a smaller value.

C. Three channel LEH

Similar to the two channel case, we here calculate the low energy effective Hamiltonain for three channel Kondo problem by introducing the real space hoppting on top of the zero mode three-channel stargraph model. This zero mode three channel stargraph model has three fold degenerate ground states with total $2^4 = 16$ states in the eigen spectrum. The three degenerate ground states are given as

$$\begin{aligned}|\alpha_{-1}\rangle &= c|1000\rangle - b(|0100\rangle + |0010\rangle + |0001\rangle)\\ |\alpha_{+1}\rangle &= b(|1110\rangle + |1101\rangle + |1011\rangle) - c|0111\rangle\\ |\alpha_0\rangle &= -a(|1100\rangle + |1010\rangle + |1001\rangle)\\ &\quad + a(|0110\rangle + |0101\rangle + |0011\rangle)\end{aligned} \tag{57}$$

where $a = 0.4082482904638638$, $b = 0.2886751345948$, $c = 0.8660254037844386$ **(PLEASE CHANGE THIS!)**. Here the state is represented by $|n_d, n_1, n_2, n_3\rangle$, where $n_i = 1/0$ represents the spin configuration $S_i^z = \pm 1/2$ respectively. Next we use degenerate perturbation theory to get the LEH which contains diagonal and off-diagonal terms. In this case we get non-zero contribution from all the three ground states $|J^z = -1\rangle$, $|J^z = 0\rangle$ and $|J^z = 1\rangle$. We get the diagonal contribution to the LEH in the second order to be

$$H^{(2)}_{eff,diag} = -\frac{7.2J^2}{\alpha}\hat{I} \tag{58}$$

The contribution associater to different ground states $|J^z = -1\rangle$, $|J^z = 0\rangle$ and $|J^z = 1\rangle$ is given respectively as $-\frac{2.4J^2}{\alpha}\hat{I} + \hat{\mathcal{F}}$, $-\frac{2.4J^2}{\alpha}\hat{I}$, $-\frac{2.4J^2}{\alpha}\hat{I} - \hat{\mathcal{F}}$, where $\hat{\mathcal{F}}$ is a function of diagonal number oprators of sites nearest neighbor to the zeroth site of each channels. The off-diagonal terms in the LEH is appearing due to the scattering between pair of degenerate states (α_0, α_{+1}) and (α_0, α_{-1}), there is no contribution in the second order coming from the scattering between $(\alpha_{-1}, \alpha_{+1})$. The effective low energy Hamiltonian in the second ordere is given as

$$H^{(2)}_{eff,off-diag} = \sum_{\substack{(ijk)=\\(123),(231),(312)}} \left[c_{i\uparrow}c^{\dagger}_{i\downarrow}\left(-2ab\left\{\Sigma_{jk} + c^{\dagger}_{j\uparrow}c_{j\downarrow}c_{k\uparrow}c^{\downarrow}_{k\downarrow}\right\} - ac(\Omega_{jk} + \tilde{\Omega}_{jk})\right) + \text{ h.c.}\right] \otimes \hat{\Xi}_l \tag{59}$$

where different operators are defined as

$$\begin{aligned}\Sigma_{i,j} &= n_{i\uparrow}(1-n_{i\downarrow})(1-n_{j\uparrow})n_{j\downarrow} + (1-n_{i\uparrow})n_{i\downarrow}n_{j\uparrow}(1-n_{j\downarrow})\\ \Omega_{i,j} &= 4S_i^z S_j^z n_{i\uparrow}n_{j\uparrow}\\ \tilde{\Omega}_{i,j} &= 4S_i^z S_j^z (1-n_{i\uparrow})(1-n_{j\uparrow})\\ \hat{\Xi}_l &= (c^{\dagger}_{l_1\uparrow}c_{l_1\downarrow} + c^{\dagger}_{l_2\uparrow}c_{l_2\downarrow} + c^{\dagger}_{l_3\uparrow}c_{l_3\downarrow} + \text{h.c.})\end{aligned} \tag{60}$$

IV. HAMILTONIAN RG OF SPIN-S IMPURITY MCK MODEL

The Hamiltonian for the channel-isotropic MCK model is given in eq. **??**. As mentioned in the same section, the Hamiltonian $H_{(j-1)}$ of the $(j-1)^{\text{th}}$ RG step is obtained from the Hamiltonian $H_{(j)}$ of the preceding RG step by applying a unitary transformation $U_{(j)}$: $H_{(j-1)} = U_{(j)}H_{(j)}U^{\dagger}_{(j)}$. The unitary transformation is obtained in terms of the fermionic operator $\eta_{(j)}$:

$$U_{(j)} = \frac{1}{\sqrt{2}}\left(1 + \eta_{(j)} - \eta^{\dagger}_{(j)}\right) , \tag{61}$$

$$\hat{\omega}_{(j)} = H_{(j-1)} - H^i_{(j)},\ T_{(j)} \equiv \text{Tr}\left(H_{(j)}c_j\right) , \tag{62}$$

$$\eta^{\dagger}_{(j)} = \frac{1}{\hat{\omega}_{(j)} - \text{Tr}\left(H_{(j)}\hat{n}_j\right)}c^{\dagger}_j T_{(j)} . \tag{63}$$

The operator $\hat{\omega}_{(j)}$ encodes the quantum fluctuation scales arising from the interplay of the kinetic energy terms and the interaction terms of the Hamiltonian.

The URG equation for the single-channel Kondo model [**?**] shows a stable strong coupling fixed point. Ferromagnetic interactions are irrelevant. Strictly speaking, that RG equation already encodes, in principle, the multi-channel behaviour, through a modified $\hat{\omega}$. To extract this information, we consider the strong coupling fixed-point $J \gg D$ as a fixed point and analyze its stability from the star graph perspective. For the exactly-screened case, the star graph decouples from the conduction bath, leaving behind a local Fermi liquid interaction on the first site. Similarly, in the under-screened regime, the ground state is composed of states where the impurity spin is only partially screened by the conduction channels. If a particular configuration of the bath-impurity system has the total conduction bath spin down, the impurity will have a residual up spin. This induces a ferromagnetic super-exchange coupling that is irrelevant under RG, so this fixed point is stable as well.

We now come to the over-screened case, where there is a residual spin on the conduction channel site. The neighbouring electrons will now hop in with spins opposite to that of the impurity, so an antiferromagnetic interaction will be induced, and such an interaction is relevant under the RG. This shows that the over-screened regime can-not have a stable strong coupling fixed point, and we need to search for an intermediate coupling fixed point. We therefore need the generator of the unitary transformation that incorporates third order scattering scatterings explicitly. We should take account of all possible processes that render the set of states $\{|\hat{n}_{q\beta}=1\rangle, |\hat{n}_{q\beta}=0\rangle\}$ diagonal. The higher order generator itself has two scattering processes, such that the entire renormalisation term $c^\dagger_{q\beta} T\eta$ has in total three coherent processes. The complete generator upto third order can be written as

$$\eta = \frac{1}{\hat{\omega}-H_D}T^\dagger c \simeq \frac{1}{\omega'-H_D}T^\dagger c + \frac{1}{\omega'-H_D}H_X\frac{1}{\omega'-H_D}T^\dagger c + \frac{1}{\omega'-H_D}T^\dagger c\frac{1}{\omega'-H_D}H_X \tag{64}$$

where $H_X = J\sum_{k,k'<\Lambda_j,\alpha,\alpha'}\vec{S}_d\cdot\vec{s}_{\alpha\alpha'}c^\dagger_{k\alpha}c_{k'\alpha'}$ is scattering between the entangled electrons. There are two third order terms in the above equation corresponding to the two possible sequences in which the processes can occur while keeping the total renormalisation $c^\dagger_{q\beta}T\eta$ diagonal in $q\beta$. The second order processes remain unchanged. The total renormalisation takes the form:

$$\Delta H_{(j)} = \underbrace{c^\dagger T\frac{1}{\omega'-H_D}T^\dagger c + \left(c^\dagger \leftrightarrow c\right)}_{\Delta H^{(2)}_{(j)}} + \underbrace{c^\dagger T\frac{1}{\omega'-H_D}H_X\frac{1}{\omega'-H_D}T^\dagger c + c^\dagger T\frac{1}{\omega'-H_D}T^\dagger c\frac{1}{\omega'-H_D}H_X + \left(c^\dagger \leftrightarrow c\right)}_{\Delta H^{(3)}_{(j)}} \tag{65}$$

$\Delta H^{(2)}_{(j)}$ and $\Delta H^{(3)}_{(j)}$ are the renormalisation arising from the the second and third order processes respectively.

It is easier to see the RG flow of the couplings if we write the Hamiltonian in terms of the eigenstates of S_d^z. These eigenstates are defined by $S_d^z|m_d\rangle = m_d|m_d\rangle, m_d\in[-S,S]$. In terms of these eigenstates, the Hamiltonian becomes

$$\mathcal{H} = \sum_{k\sigma}\epsilon_k\tau_{k\sigma} + \sum_{m_d=-S}^{S}\sum_{\substack{kl,\\ \sigma=\uparrow,\downarrow}}J^\sigma_{m_d}|m_d\rangle\langle m_d|c^\dagger_{k\sigma}c_{l\sigma} + \sum_{kl}\sum_{m_d=-S}^{S-1}J^t_{m_d}\left(|m_d+1\rangle\langle m_d|s^-_{kl} + \text{h.c.}\right) \tag{66}$$

where k,l sum over the momentum states, σ sums over the spin indices, $J^\sigma_m = \frac{1}{2}\sigma mJ$ in the UV Hamiltonian, and $J^t_m = J\frac{1}{2}\sqrt{S(S+1)-m(m+1)}$ is the coupling that connects $|m\rangle$ and $|m+1\rangle$. We first calculate $\Delta H^{(2)}_{(j)}$. There will be two types of processes - those processes that start from an occupied state (particle sector) and those that start from a vacant state (hole sector). Due to particle hole symmetry of the Hamiltonian, they will be equal to each other and we will only calculate the particle sector contribution.

In the particle sector, we have $(\hat{n}_{q\beta}=1)$, so we will work at a negative energy shell $\epsilon_q=-D$. The renormalisation can schematically represented as $H^I_0\frac{1}{\omega-H^D_{q\beta}}H^I_1$. Both H^I_0 and H^I_1 have all three operators $S^z_d, S^\pm_d$. We first consider specifically the case of spin-$\frac{1}{2}$ impurity. Those terms that have identical operators on both sides can be ignored because ${S_d^z}^2=$ constant and ${S^\pm}^2=0$. All the six terms that *will* renormalise the Hamiltonian have a spin flip operator on at least one side of the Greens function. This means that in the denominator of the Greens function, S^z_d and s^z_{qq} have to be anti-parallel in order to produce a non-zero result for that term. This means we can identically replace $S^z_d s^z_{qq} = -\frac{1}{4}$. Also, in the particle sector, the Greens function always has $c_{q\beta}$ in front of it, so $\epsilon_q\tau_{q\beta}=\frac{D}{2}$. The upshot of all this is that the denominator of all scattering processes for the spin-$\frac{1}{2}$ impurity Hamiltonian will be $\omega-\frac{D}{2}+\frac{J}{4}$.

We now come to the general case of spin-S impurity. The various terms that renormalise the Hamiltonian can be described in terms of the bath spin operators that come into them. For example, the term that has s^z on both sides of the intervening Greens function can be represented as $z|z$. There are 7 such terms: $z|z, \pm|\mp, z|\pm, \pm|z$. Each of these terms occur both in the particle and the hole sectors. We will demonstrate the calculation of two of these terms. The $z|z$ $+|-$ terms evaluate in the following manner.

$$z|z: \sum_{kk',m,\sigma}c^\dagger_{q\sigma}c_{k'\sigma}|m\rangle\langle m|\frac{{J^\sigma_m}^2}{\omega-\frac{D}{2}+\frac{J}{2}\sigma S^z_d}|m\rangle\langle m|c^\dagger_{k\sigma}c_{q\sigma} = -\sum_{kk',m,\sigma}n_{q\sigma}\frac{{J^\sigma_m}^2c^\dagger_{k\sigma}c_{k'\sigma}|m\rangle\langle m|}{\omega_{m,\sigma}-\frac{D}{2}+\frac{J}{2}\sigma m} \tag{67}$$

$$+|-: \sum_{kk',m}c^\dagger_{q\uparrow}c_{k'\downarrow}|m\rangle\langle m+1|\frac{{J^t_m}^2}{\omega-\frac{D}{2}+\frac{J}{2}S^z_d}|m+1\rangle\langle m|c^\dagger_{k\downarrow}c_{q\uparrow} = -n_{q\uparrow}\sum_{kk',m}\frac{{J^t_m}^2c^\dagger_{k\downarrow}c_{k'\downarrow}|m\rangle\langle m|}{\omega_{m+1,\uparrow}-\frac{D}{2}+\frac{J}{2}(m+1)} \tag{68}$$

We similarly compute the rest of the terms. We again define $\sum_q \hat{n}_{q\sigma} = n(D)$. To compare with the spin-$\frac{1}{2}$ RG equations, we will transform the general spin-S ω to the spin-$\frac{1}{2}$ ω, using $\omega_{m,\sigma} \to \omega - \frac{J}{2}\left(m\sigma - \frac{1}{2}\right)$.

The renormalisation in J^σ_m is

$$\Delta J^\sigma_m = -n(D)\frac{\left(J^\sigma_m\right)^2 + \left(J^t_{m-\frac{1+\sigma}{2}}\right)^2}{\omega - \frac{D}{2} + \frac{J}{4}} \,. \tag{69}$$

Here, we have defined $J^t_m = 0$ for $|m| > S$. Two relations can be obtained from this RG equation, the RG equations for the sum and difference of the couplings: $J^\pm_m = \frac{1}{2}\left(J^\uparrow_m \pm J^\downarrow_m\right)$. The RG equation for the sum of the couplings is

$$\begin{aligned}\Delta J^+_m &= -n(D)\frac{\sum_\sigma \left(J^\sigma_m\right)^2 + \sum_\sigma \left(J^t_{m-\frac{1+\sigma}{2}}\right)^2}{2\left(\omega - \frac{D}{2} + \frac{J}{4}\right)} \\ &= -n(D)\frac{J^2}{4}\frac{S(S+1)}{\omega - \frac{D}{2} + \frac{J}{4}}\end{aligned} \tag{70}$$

This is an $m-$independent piece, so it can be summed over to produce an impurity-independent potential scattering term, which we ignore.

The second is the RG equation for the difference of the couplings:

$$\Delta J^-_m = -n(D)\frac{1}{2}\frac{\left(J^t_{m-1}\right)^2 - \left(J^t_m\right)^2}{\omega - \frac{D}{2} + \frac{J}{4}} = -\frac{1}{4}\frac{n(D)mJ^2}{\omega - \frac{D}{2} + \frac{J}{4}} \tag{71}$$

The usual J Kondo coupling is recovered through $J = 2J^-_m/m$. Substituting this gives

$$\Delta_{\text{p sector}} J = -\frac{1}{2}n(D)\frac{J^2}{\omega - \frac{D}{2} + \frac{J}{4}} \tag{72}$$

We can also obtain the RG equation for J from the transverse renormalisation:

$$\Delta J^t_m = -\frac{n(D)J^t_m\left(J^\downarrow_m + J^\uparrow_{m+1}\right)}{\omega - \frac{D}{2} + \frac{J}{4}} = -\frac{1}{2}\frac{n(D)J^t_m J}{\omega - \frac{D}{2} + \frac{J}{4}} \tag{73}$$

Since $J^t_m \propto J$, we have

$$\Delta J_{\text{p sector}} = -\frac{1}{2}\frac{n(D)J^2}{\omega - \frac{D}{2} + \frac{J}{4}} \tag{74}$$

The total renormalisation from both particle and hole sectors, at this order, is

$$\Delta J^{(2)} = -\frac{n(D)J^2}{\omega - \frac{D}{2} + \frac{J}{4}} \tag{75}$$

We now come to the third order renormalisation. Following eq. 65, the next order renormalisation is

$$\begin{aligned}\Delta H^{(3)}_j &= c^\dagger T\frac{1}{\omega' - H_D}H_X\frac{1}{\omega' - H_D}T^\dagger c \\ &+ c^\dagger T\frac{1}{\omega' - H_D}T^\dagger c\frac{1}{\omega' - H_D}H_X\end{aligned} \tag{76}$$

The first term will be of the form

$$\sum_{q,k,l_1} c^\dagger_{q\beta,l_1}c_{k\alpha,l_1}\left|m_1\right\rangle\left\langle m_2\right|\frac{1}{\omega - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{\beta J}{4}S^z_d}\left|m_2\right\rangle c^\dagger_{k_1\sigma_1,l_2}c_{k_2\sigma_2,l_2}\left\langle m_3\right|\frac{1}{\omega - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{\beta J}{4}S^z_d}\left|m_3\right\rangle\left\langle m_4\right|c^\dagger_{k\alpha,l_1}c_{q\beta,l_1} \tag{77}$$

We have not bothered to write all the summations and the couplings correctly, because we will only simplify the denominator here. Evaluating the inner products gives

$$\sum_{qk,l_1}\frac{\left|m_1\right\rangle\left\langle m_4\right|c^\dagger_{q\beta}c_{k\alpha}c^\dagger_{k_1\sigma_1,l_2}}{\omega_{m_2,\beta} - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{\beta J m_2}{4}}\frac{c_{k_2\sigma_2,l_2}c^\dagger_{k\alpha}c_{q\beta}}{\omega_{m_3,\beta} - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{\beta J m_3}{4}} \tag{78}$$

We again use $\omega_{m,\sigma} \to \omega - \frac{J}{2}\left(m\sigma - \frac{1}{2}\right)$.

$$\left|m_1\right\rangle\left\langle m_4\right|c^\dagger_{k_1\sigma_1,l_2}c_{k_2\sigma_2,l_2}\sum_{qk,l_1}\frac{\hat{n}_{q\beta}\left(1 - \hat{n}_{k\alpha}\right)}{\left(\omega - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{J}{4}\right)^2} \tag{79}$$

We define $\sum_q \hat{n}_{q\beta} = n(D)$. Performing the sums over k and l_1 gives

$$-\frac{1}{2}\left|m_1\right\rangle\left\langle m_4\right|c^\dagger_{k_1\sigma_1,l_2}c_{k_2\sigma_2,l_2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}} \tag{80}$$

ρ is the density of states which we have taken to be constant. Reinstating the complete summation and the couplings gives

$$-\frac{1}{2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}}\sum_{\substack{m_1,m_4,k_1,k_2,\\ l_2,\sigma_1\sigma_2}}\lambda_1\lambda_2\lambda_3\left|m_1\right\rangle\left\langle m_4\right|c^\dagger_{k_1\sigma_1,l_2}c_{k_2\sigma_2,l_2} \tag{81}$$

There is no sum over m_2 and m_3 because they are constrained by m_1 and m_4 respectively. λ_i represent the couplings present at the three interaction vertices. $k_{1,2}$ sum over the momenta, $\sigma_{1,2}$ sum over the spin indices and l_2 sums over the channels.

The second term in eq. 76 can be evaluated in an almost identical fashion. The integral here be negative of the first term, because of an exchange in the scattering processes.

$$\frac{1}{2}\frac{\rho n(D)K}{\omega-\frac{D}{2}+\frac{J}{4}}\sum_{\substack{m_1,m_4,k_1,k_2,\\ l_2,\sigma_1\sigma_2}}\lambda_1\lambda_3\lambda_2\left|m_1\right\rangle\left\langle m_4\right|c^\dagger_{k_1\sigma_1,l_2}c_{k_2\sigma_2,l_2} \tag{82}$$

The first group of terms (those that appear in 81) in the particle sector can be represented as $a|b'|c$, where $a,b,c\in\{z,+,-\}$ and represent the operator for the conduction electrons in the three ceonnected processes. The $'$ on b indicates that it is the state of the electrons *not being decoupled*. The second group of terms (those that appear in 82) are therefore represented as $a|b|c'$, because in this group, the interaction H_X between the electrons that are not being decoupled occur at the very end. We will only calculate the terms in the particle sector, the ones in hole sector will be equal to these because of particle-hole symmetry. The full list of terms is: $z|z'|z \quad z|z|z' \quad -|z'|+ \quad -|+|z' \quad +|z'|- \quad +|-|z' \quad z|+'|z \quad z|-'|z \quad +|+'|- \quad -|+'|+ \quad z|z|+' \quad z|z|-' \quad +|-|+' \quad +|-|-' \quad -|+|+' \quad z|z'|z \quad z|z|z' \quad -|z'|+ \quad -|+|z' \quad +|z'|- \quad +|-|z' \quad z|+'|z \quad z|-'|z \quad +|+'|- \quad -|+'|+ \quad z|z|+' \quad z|z|-' \quad +|-|+' \quad +|-|-' \quad -|+|+'$.

The total renormalisation in J^σ_m is

$$\Delta J^\sigma_m=\frac{1}{2}\frac{\rho n(D)K}{\omega-\frac{D}{2}+\frac{J}{4}}\left[\left(J^t_{m-1}\right)^2J^\sigma_m+\left(J^t_m\right)^2J^\sigma_m-\left(J^t_{m-1}\right)^2J^\sigma_{m-1}-\left(J^t_m\right)^2J^\sigma_{m+1}\right]=\frac{1}{2}\frac{\rho n(D)K}{\omega-\frac{D}{2}+\frac{J}{4}}J^3m\sigma \tag{83}$$

Since we had defined $J^\sigma_m\equiv\frac{1}{2}Jm\sigma$, we have $\Delta J=\frac{2}{m\sigma}\Delta J^\sigma_m$, and we get $\Delta_{\text{p sector}}J=\frac{1}{4}\frac{\rho n(D)K}{\omega-\frac{D}{2}+\frac{J}{4}}J^3$. Combining with the hole sector renormalisation, we get

$$\Delta J^{(3)}=\frac{1}{2}\frac{\rho n(D)K}{\omega-\frac{D}{2}+\frac{J}{4}}J^3 \tag{84}$$

The total renormalisation in J after combining all orders is

$$\Delta J=-\frac{n(D)J^2}{\omega-\frac{D}{2}+\frac{J}{4}}+\frac{1}{2}\frac{\rho n(D)K}{\omega-\frac{D}{2}+\frac{J}{4}}J^3 \tag{85}$$