

Multi-channel Kondo model URG

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1 Introduction

$$H = \sum_{k,\sigma,\gamma} \epsilon_k^\gamma \hat{n}_{k\sigma}^\gamma + J \sum_{kk',\gamma} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} c_{k\alpha}^\gamma c_{k'\alpha'}^\gamma \quad (1)$$

$$\Delta H = \tau \left(c^\dagger T \frac{1}{\hat{\omega} - H_D} T^\dagger c + T^\dagger c \frac{1}{\hat{\omega} - H_D} c^\dagger T \right) \quad (2)$$

2 Leading order renormalization

2.1 Second term

$$\begin{aligned} & T^\dagger c \frac{1}{\hat{\omega} - H_D} c^\dagger T \\ &= J^2 \sum_{q\beta kk'\alpha\alpha'} c_{k'\alpha'}^\dagger c_{q\beta} \vec{S}_d \cdot \vec{s}_{\alpha'\beta} \frac{1}{\omega - \epsilon_q \tau_{q\beta} - JS_d^z s_q^z} c_{q\beta}^\dagger c_{k\alpha} \vec{S}_d \cdot \vec{s}_{\beta\alpha} \\ &= J^2 \sum_{q\beta kk'\alpha\alpha',a,b} c_{k'\alpha'}^\dagger c_{q\beta} S_d^a s_{\alpha'\beta}^a \frac{1}{\omega - \frac{1}{2}D - JS_d^z s_q^z} c_{q\beta}^\dagger c_{k\alpha} S_d^b s_{\beta\alpha}^b \quad \left[\tau = \frac{1}{2}, \epsilon_q = D \right] \\ &= J^2 \sum_{q\beta kk'\alpha\alpha',a,b} c_{k'\alpha'}^\dagger c_{q\beta} S_d^a s_{\alpha'\beta}^a \frac{\omega - \frac{1}{2}D + JS_d^z s_q^z}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} c_{q\beta}^\dagger c_{k\alpha} S_d^b s_{\beta\alpha}^b \\ &= J^2 \sum_{q\beta kk'\alpha\alpha',a,b} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a s_{\alpha'\beta}^a \frac{\left(\omega - \frac{1}{2}D\right)(1 - \hat{n}_{q\beta}) + JS_d^z c_{q\beta} s_q^z c_{q\beta}^\dagger}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b s_{\beta\alpha}^b \quad (3) \\ &= J^2 \sum_{q\beta kk'\alpha\alpha',a,b} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a s_{\alpha'\beta}^a \frac{\left(\omega - \frac{1}{2}D\right)(1 - \hat{n}_{q\beta}) + JS_d^z \left(s_q^z + \frac{\beta}{2}\right)(1 - \hat{n}_{q\beta})}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b s_{\beta\alpha}^b \quad \left[[c_{q\beta}, s_q^z] = \frac{\beta}{2} c_{q\beta} \right] \\ &= J^2 |\delta D| \sum_{q\beta kk'\alpha\alpha',a,b} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a s_{\alpha'\beta}^a s_{\beta\alpha}^b \frac{\left(\omega - \frac{1}{2}D\right) + JS_d^z \beta}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b \quad \left[s_q^z = \frac{\beta}{2} \right] \\ &= \frac{J^2 \left(\omega - \frac{1}{2}D\right) |\delta D|}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} \sum_{kk'\alpha\alpha',a,b} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a S_d^b \left(s^a s^b\right)_{\alpha'\alpha} \quad [\text{term 1}] \\ &+ \frac{J^3}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} |\delta D| \sum_{\beta kk'\alpha\alpha',a,b} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a S_d^z S_d^b s_{\alpha'\beta}^a s_{\beta\alpha}^b \quad [\text{term 2}] \end{aligned}$$

We will now simplify the terms individually. In term 1, only the configurations $a \neq b$ can lead to a non-trivial impurity spin operator and hence contribute to renormalization. For $a \neq b$, we have $S_d^a S_d^b =$

$\frac{i}{2} \sum_c \epsilon^{abc} S_d^c$. Therefore,

$$\sum_{a,b} S_d^a S_d^b (s^a s^b)_{\alpha'\alpha} = -\frac{1}{4} \sum_{c,e} S_d^c (s^e)_{\alpha'\alpha} \sum_{a,b} \epsilon^{abc} \epsilon^{abe} = -\frac{1}{4} \sum_{c,e} S_d^c (s^e)_{\alpha'\alpha} 2\delta_{ce} = -\frac{1}{2} \vec{S}_d \cdot \vec{s}_{\alpha'\alpha} \quad (4)$$

Term 1 therefore simplifies to

$$-\frac{1}{2} \frac{J^2 (\omega - \frac{1}{2}D) |\delta D|}{(\omega - \frac{1}{2}D)^2 - \frac{1}{16}J^2} \sum_{kk'\alpha\alpha',c} c_{k'\alpha'}^\dagger c_{k\alpha} \vec{S}_d \cdot \vec{s}_{\alpha'\alpha} \quad [\text{term 1}] \quad (5)$$

In term 2, we use the identity:

$$S^a S^z S^b = \frac{1}{4} \sum_c (\delta_{ac} \delta_{zb} - \delta_{ab} \delta_{zc}) S^c \quad (6)$$

Substituting this gives

$$\begin{aligned} \sum_{\beta,a,b} S_d^a S_d^z S_d^b s_{\alpha'\beta}^a s_{\beta\alpha}^b \beta &= \frac{1}{4} \sum_{\beta,c} \beta (S_d^c s_{\alpha'\beta}^c s_{\beta\alpha}^z - S_d^z s_{\alpha'\beta}^c s_{\beta\alpha}^c) \\ \Rightarrow \text{term 2} &= \frac{J^3 |\delta D|}{(\omega - \frac{1}{2}D)^2 - \frac{1}{16}J^2} \frac{1}{4} \sum_{\beta kk'\alpha\alpha',c} c_{k'\alpha'}^\dagger c_{k\alpha} \beta s_{\alpha'\beta}^c (S_d^c s_{\beta\alpha}^z - S_d^z s_{\beta\alpha}^c) \\ &= \frac{J^3 |\delta D|}{(\omega - \frac{1}{2}D)^2 - \frac{1}{16}J^2} \frac{1}{4} \sum_{\beta kk'\alpha\alpha',c} c_{k'\alpha'}^\dagger c_{k\alpha} \left(\frac{1}{2} S_d^c s_{\alpha'\alpha}^c - \beta S_d^z s_{\beta\alpha}^c s_{\alpha'\beta}^c \right) \end{aligned} \quad (7)$$

2.2 First term

$$\begin{aligned} c^\dagger T \frac{1}{\hat{\omega} - H_D} T^\dagger c &= J^2 \sum_{q\beta kk'\alpha\alpha'} c_{q\beta}^\dagger c_{k\alpha} \vec{S}_d \cdot \vec{s}_{\beta\alpha} \frac{1}{\omega' - \epsilon_q \tau_{q\beta} - JS_d^z s_q^z} c_{k'\alpha'}^\dagger c_{q\beta} \vec{S}_d \cdot \vec{s}_{\alpha'\beta} \\ &= J^2 \sum_{q\beta kk'\alpha\alpha'ab} c_{q\beta}^\dagger c_{k\alpha} S_d^a s_{\beta\alpha}^a \frac{1}{\omega' - \frac{1}{2}D - JS_d^z s_q^z} c_{k'\alpha'}^\dagger c_{q\beta} S_d^b s_{\alpha'\beta}^b \quad \left[\tau = -\frac{1}{2}, \epsilon_q = -D \right] \\ &= J^2 \sum_{q\beta kk'\alpha\alpha'ab} c_{q\beta}^\dagger c_{k\alpha} S_d^a s_{\beta\alpha}^a \frac{(\omega' - \frac{1}{2}D) + JS_d^z s_q^z}{(\omega' - \frac{1}{2}D)^2 - \frac{1}{16}J^2} c_{k'\alpha'}^\dagger c_{q\beta} S_d^b s_{\alpha'\beta}^b \\ &= J^2 \sum_{q\beta kk'\alpha\alpha'ab} c_{q\beta}^\dagger c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a s_{\beta\alpha}^a \frac{-(\omega' - \frac{1}{2}D) - JS_d^z s_q^z}{(\omega' - \frac{1}{2}D)^2 - \frac{1}{16}J^2} c_{q\beta} S_d^b s_{\alpha'\beta}^b \quad [c_k c_{k'}^\dagger \sim -c_{k'}^\dagger c_k] \\ &= J^2 \sum_{q\beta kk'\alpha\alpha'ab} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a s_{\beta\alpha}^a \frac{-(\omega' - \frac{1}{2}D) \hat{n}_{q\beta} - JS_d^z c_{q\beta}^\dagger s_q^z c_{q\beta}}{(\omega' - \frac{1}{2}D)^2 - \frac{1}{16}J^2} S_d^b s_{\alpha'\beta}^b \quad (8) \\ &= J^2 \sum_{q\beta kk'\alpha\alpha'ab} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a s_{\beta\alpha}^a \frac{-(\omega' - \frac{1}{2}D) \hat{n}_{q\beta} - JS_d^z (s_q^z - \frac{\beta}{2}) \hat{n}_{q\beta}}{(\omega' - \frac{1}{2}D)^2 - \frac{1}{16}J^2} S_d^b s_{\alpha'\beta}^b \quad \left[[c_{q\beta}^\dagger, s_q^z] = -\frac{\beta}{2} c_{q\beta}^\dagger \right] \\ &= J^2 |\delta D| \sum_{\beta kk'\alpha\alpha'ab} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a s_{\beta\alpha}^a \frac{-(\omega' - \frac{1}{2}D) + JS_d^z \beta}{(\omega' - \frac{1}{2}D)^2 - \frac{1}{16}J^2} S_d^b s_{\alpha'\beta}^b \quad \left[s_q^z = -\frac{1}{2} \beta \right] \\ &= -\frac{J^2 (\omega' - \frac{1}{2}D) |\delta D|}{(\omega' - \frac{1}{2}D)^2 - \frac{1}{16}J^2} \sum_{kk'\alpha\alpha'ab} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a S_d^b (s^b s^a)_{\alpha'\alpha} \quad [\text{term 3}] \\ &\quad + \frac{J^3 |\delta D|}{(\omega' - \frac{1}{2}D)^2 - \frac{1}{16}J^2} \sum_{q\beta kk'\alpha\alpha'ab} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a S_d^z S_d^b s_{\beta\alpha}^a s_{\alpha'\beta}^b \beta \quad [\text{term 4}] \end{aligned}$$

term 3 can be made identical to term 1 using the relation: $s^b s^a = -s^b s^a$ for $a \neq b$. With this change, term 3 becomes

$$\text{term 3} = \frac{J^2 \left(\omega' - \frac{1}{2}D\right) |\delta D|}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} \sum_{kk'\alpha\alpha'ab} c_{k'\alpha'}^\dagger c_{k\alpha} S_d^a S_d^b (s^a s^b)_{\alpha'\alpha} = \text{term 1} \quad (9)$$

For term 4, we get

$$\begin{aligned} \sum_{\beta,a,b} S_d^a S_d^z S_d^b s_{\alpha'\beta}^b s_{\beta\alpha}^a \beta &= \frac{1}{4} \sum_{\beta,a,b,c} \beta (\delta_{ac} \delta_{zb} - \delta_{ab} \delta_{zc}) S^c s_{\alpha'\beta}^b s_{\beta\alpha}^a \\ &= \frac{1}{4} \sum_{\beta,c} \beta s_{\beta\alpha}^c (S_d^c s_{\alpha'\beta}^z - S_d^z s_{\alpha'\beta}^c) \end{aligned} \quad (10)$$

which gives

$$\text{term 4} = \frac{J^3 |\delta D|}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} \frac{1}{4} \sum_{\beta k k' \alpha \alpha', c} c_{k'\alpha'}^\dagger c_{k\alpha} \left(\frac{1}{2} S_d^c s_{\alpha'\alpha}^c - \beta S_d^z s_{\beta\alpha}^c s_{\alpha'\beta}^c \right) \quad (11)$$