Multi-channel Kondo model URG

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CONTENTS

I.   INTRODUCTION OF THE HAMILTONIAN AND ITS ZERO MODE

The isotropic multi-channel Kondo model [1–13] is described by the Hamiltonian

$$H = \sum_{k,\alpha,\gamma} \epsilon_k^\gamma \hat{n}_{k\alpha}^\gamma + J \sum_{kk',\gamma} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} c_{k\alpha}^{\gamma\,\dagger} c_{k'\alpha'}^\gamma \ . \tag{1}$$

In the above equation, k, k' sum over the momentum states, α, α' sum over the spin indices and γ sums over the various channels. $\vec{S}_d, \vec{s}$ are the impurity and conduction bath spin vectors. The presence of a common spin-exchange coupling J for all the channels is what makes the problem isotropic with respect to the channels.

The zero mode of this Hamiltonian is obtained by keeping only the central component of the Fourier transform of the kinetic energy part:

$$\epsilon(\vec{r}) \simeq \delta_{\vec{r},0}\epsilon(\vec{r}=0) \implies \epsilon_{\vec{k}} = \sum_{\vec{r}} \epsilon(\vec{r})e^{i\vec{k}\cdot\vec{r}} = \epsilon(\vec{r}=0) = \sum_q \epsilon_{\vec{q}} \tag{2}$$

Assuming the dispersion already includes the chemical potential μ, we have

$$\sum_q \epsilon_{\vec{q}} = \rho\Delta\epsilon \sum_{\epsilon=\epsilon_F-D}^{\epsilon_F+D} (\epsilon - \mu) = \rho\Delta\epsilon\,(\epsilon_F - \mu) \tag{3}$$

If we set the chemical potential at the Fermi surface, this summation vanishes.

$$\epsilon_{\vec{k}} = \rho\Delta\epsilon\,(\epsilon_F - \mu) = 0 \tag{4}$$

and the zero mode Hamiltonian is then just the spin-exchange part of the Hamiltonian:

$$H_{\text{zero mode}} = J \sum_{kk',\gamma} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} c_{k\alpha}^{\gamma\,\dagger} c_{k'\alpha'}^\gamma \ . \tag{5}$$

We can make one more identification: the sum over the spins in momentum space amount to a local interaction at the zeroth sites of the indicates channels: $\sum_k c_{k\alpha}^{\gamma\,\dagger} = c_{\vec{r}=0,\alpha}^{\gamma\,\dagger}$, This suggests we shoud define spin operators for the zeroth sites of the channels:

$$\sum_{kk'} s_{\alpha\alpha'}^a c_{k\alpha}^{\gamma\,\dagger} c_{k'\alpha'}^\gamma = s_{\alpha\alpha'}^a c_{0\alpha}^{\gamma\,\dagger} c_{0\alpha'}^\gamma = s_\gamma^a \tag{6}$$

In the terms of the local spin operators $\vec{s}_\gamma = (s_\gamma^x, s_\gamma^y, s_\gamma^z)$, we can define the total spin of the zero mode of the bath: $\vec{s}_{\text{tot}} = \sum_\gamma \vec{s}_\gamma$ and the zero mode Hamiltonian then takes the simpler form of a star graph Hamiltonian:

$$H_{\text{zero mode}} = J\vec{S}_d \cdot \vec{s}_{\text{tot}} \ . \tag{7}$$

This star graph is shown schematically in fig. 1.
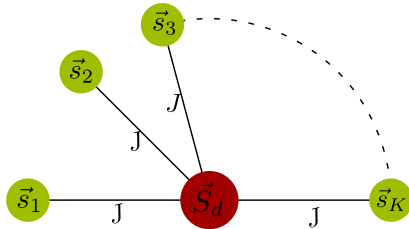

FIG. 1. Zero mode of the $K-$channel Kondo model, represented as a "star graph". The central red node denotes the impurity spin $\vec{S}_d$, while the outer nodes denote the spins $\vec{s}_\gamma$ of the zeroth sites of the channels $\gamma = 1, 2, \ldots, K$. The straight lines represent the spin-exchange interaction between the central node and the outer nodes mediated by the coupling J.

II. SYMMETRIES OF THE STAR GRAPH

This star graph commutes with a number of operators, revealing various symmetries of the problem. The first of such operators is s_{tot}^2. Henceforth we will drop the label zero mode. To see the commuation, note that s_{tot}^2 can be written as $s_{\text{tot}}^2 = \sum_{i=x,y,z} \left(s_{\text{tot}}^i\right)^2$. This then gives

$$\left[s_{\text{tot}}^2, H\right] = \left[\sum_{i=x,y,z} {s_{\text{tot}}^i}^2, J \sum_{i=x,y,z} S_d^i s_{\text{tot}}^i\right] = \sum_{i,j} J S_d^i \left\{s_{\text{tot}}^i, \left[s_{\text{tot}}^i, s_{\text{tot}}^j\right]\right\} = \sum_{i,j} J S_d^i \left\{s_{\text{tot}}^i, i\epsilon^{ijk} s_{\text{tot}}^k\right\} = 0 \qquad (8)$$

The total spin angular momentum operator $S^2 = \left(\vec{S}_d + \vec{s}_{\text{tot}}\right)^2$ also commutes with the Hamiltonian. To see this, note that we can rewrite the operator as

$$S^2 = \left(\vec{S}_d + \vec{s}_{\text{tot}}\right)^2 = \frac{3}{4} + s_{\text{tot}}^2 + \frac{2}{J} H \qquad (9)$$

Since we already know that s_{tot} commutes with H, we have that $\left[S^2, H\right] = 0$. The $z-$component of this total spin operator also commutes with the Hamiltonian:

$$[S^z, H] = \left[S^z, J\left(\frac{1}{2}S^2 - \frac{3}{8} - \frac{1}{2}s_{\text{tot}}^2\right)\right] = \frac{J}{2}\left([S^z, S^2] - [s_{\text{tot}}^z, s_{\text{tot}}^2]\right) = 0 \qquad (10)$$

One final operator that we will look at is the spin parity operator $\pi^a = \sigma_d^a \otimes_{i=1}^K \sigma_i^a$.

$$[H, \pi^a] = J\sum_{b,j}\left[S_d^b s_j^b, \sigma_d^a \otimes_{i=1}^K \sigma_i^a\right] = \sum_{bj}\left(S_d^b \sigma_d^a s_j^b \sigma_j^a - \sigma_d^a S_d^b \sigma_j^a s_j^b\right) \otimes_{i<j} \sigma_i^a \otimes_{i>j} \sigma_i^a = 0 \qquad (11)$$

III. EIGENVALUES AND EIGENSTATES OF THE STAR GRAPH

The star graph can be diagonalized. Since the Hamiltonian commutes with s_{tot}^2, it is already block-diagonal in the quantum number s_{tot}. Let us represent the quantum number of s_{tot}^z by m. For a particular s_{tot}, m can take values from the set $[-s_{\text{tot}}, s_{\text{tot}}]$. The spin S_d^z can also take values $\pm\frac{1}{2}$. From now on, we will assume we are in the subspace of a particular $s_{\text{tot}} = M$, so we will ignore that quantum number and write the kets simply as $|S_d^z, m\rangle$. So, the notation $|\uparrow, -1\rangle$ means the state with $S_d^z = \frac{1}{2}$ and $m = -1$. We will now show that even inside the block of $2 \times s_{\text{tot}}$ (or $2 \times s_{\text{tot}} + 1$, depending on whether it is odd or even) defined by a particular value of s_{tot}, the Hamiltonian actually separates into decoupled 2×2 blocks. To see why, first note that the terminal states $|\downarrow, -M\rangle$ and $|\uparrow, M\rangle$ are already eigenstates, because they cannot scatter (the impurity can only flip down, and this would require the bath to flip up, but s_{tot}^z is already at its maximum value M). The other $2M - 2$ states can be organized into 2×2 blocks formed by the states $|\uparrow, m\rangle$ and $|\downarrow, m+1\rangle$ for $m \in [-M, M-1]$. The fact that this block does not interact with the other blocks can be easily verified: if there was some other state which when acted upon by the Hamiltonian gave a non-zero projection on $|\uparrow, m\rangle$, it would have to come from $S_d^z = \downarrow$, and this would mean the bath spin would have had to flip down. This means the bath spin in that state would have to be $m + 1$, and that is precisely the other state in the block.

Defining $x_m^M = M(M+1) - m(m+1)$, the 2×2 blocks can be written as

$$H_m = \begin{pmatrix} \frac{Jm}{2} & \frac{J}{2}\sqrt{x_m^M} \\ \frac{J}{2}\sqrt{x_m^M} & -\frac{J}{2}(m+1) \end{pmatrix}, m \in [-M, M-1] \qquad (12)$$

The eigenvalues are

$$\lambda_\pm^M = -\frac{J}{4} \pm \frac{J}{2}\left(M + \frac{1}{2}\right) \qquad (13)$$

The eigenvalues of the terminal states are $\frac{JM}{2}$. The set of eigenvalues $\lambda_-^{M_{\max}}$ form the ground state subspace $\{|\Psi_{m,-}^M\rangle\}$ of degeneracy $2M_{\max} = K$. This common $K-$fold degenerate eigenvalue is $-\frac{J}{2}(M+1)$. The normalized eigenstates of the 2×2 blocks for each value of m are given by

$$|\Psi_{m,-}^M\rangle = \frac{1}{\sqrt{(M+m+1)(1+2M)}}\left[-\sqrt{x_m^M}|\uparrow, M, m\rangle + (M+m+1)|\downarrow, M, m+1\rangle\right], \quad m \in [-M, M-1] \qquad (14)$$

$$|\Psi_{m,+}^M\rangle = \frac{1}{\sqrt{(M-m)(M-m+1)}} \left[\sqrt{x_m^M} \, |\uparrow, M, m\rangle + (M-m) \, |\downarrow, M, m+1\rangle \right], \qquad m \in [-M, M-1] \quad (15)$$

Note that the index m that we are using to label the eigenvalues is actually related to a conserved quantity; the total spin operator along z, S^z. That is, the eigenstates $|\Psi_{m,\pm}^M\rangle$ have $S^z = S_d^z + s_{\text{tot}}^z = m + \frac{1}{2}$.

IV.   IMPORTANCE OF THE STAR GRAPH IN DETERMINING THE PHYSICS OF THE PROBLEM

Methods like CFT, Bethe ansatz, bosonization and NRG [3–5, 11, 14, 15] have shown that *neither the weak coupling nor the strong coupling Hamiltonians are stable fixed points of the multi-channel Kondo models.* Instead, the RG flows converge to an intermediate coupling fixed point. The RG flows *preserve the degeneracy of the ground state* [14, 16, 17], because the Hamiltonian commutes with the total magnetization operator $S^z = S_d^z + s_{\text{tot}}^z$ [15]. For example, the two-channel and the three-channel Kondo models have ground state degeneracies of 2 and 3 respectively, throughout the RG flow. Analytic and numerical studies also showed that *the two and three-channel models enjoy a weak-strong duality* [16, 17]: the model at strong coupling, when projected onto the ground state basis, can also be thought of as a Kondo model at weak coupling, but possibly with a modified central spin. For example, in the two channel Kondo model, the model at strong coupling can be mapped onto the same model with a small coupling parameter, but with the central Kondo spin-half impurity replaced by a new, composite spin-half object formed by the doubly-degenerate ground state of the strong coupling model [16]. The strong and weak coupling models are formally identical and the model is self-dual. The case of the three-channel model is different [17] because there, the central impurity gets replaced by a new *spin-1* impurity formed by the triply-degenerate ground state. Compared to the self-duality of the two-channel model, this case corresponds to a cross duality between a spin-1/2 and a spin-1 model. We will now show that these qualitative features can be captured very simply by the star graph.

At weak coupling $J \to 0^+$, the central spin is weakly coupled to the outer spins and can freely flip like a local moment. *This makes it prone to screening because of the $s^\pm$ terms in the star graph.* At strong coupling $J \to \infty^-$, the outer spin-half objects tightly bind with the central spin-half object to form a single spin. The remaining states in the conduction bath channels form new zero modes and create a new outer ring of spins that interact with the new central spin, *leading to another multi-channel Kondo problem with a new and small exchange coupling which is RG relevant, so the strong coupling fixed point $J \to \infty^-$ is unstable.* This shows that it can be argued, even from the zero mode, that the two limits of the model cannot be stable fixed points, and the actual fixed point has to lie somewhere in between.

Just like the full Hamiltonian, the star graph conserves the total spin S^z, and this leads to a $K-$fold degeneracy in the ground state of the $K-$channel zero mode. Since S^z commutes with the star graph, the ground states are eigenstates of S^z with distinct eigenvalues $\frac{K+1}{2}, \frac{K-1}{2}, \ldots, \frac{-K-1}{2}$. *This feature - the equality of energy eigenvalue for distinct values of S^z - persists for all values of $J \in [0^+, \infty^-]$.* This is because, the gap between the degenerate subspace and the next excited level is J or $J/2$, depending on whether K is even or odd, and the gap is always non-zero as long as J lies in the regime of RG flow. In other words, as the exchange coupling grows(shrinks) from 0^+ (∞^-) towards the fixed point value J^*, the degeneracy remains protected at K.

The fact that the strong coupling fixed point can be seen as a new zero mode problem with a modifed central spin reveals the duality mentioned before. At strong coupling $J \gg 1$, we can obtain a new star graph by defining a new spin-$\left(\frac{K-1}{2}\right)$ object from the $K-$fold degenerate ground state manifold and letting this interact with the new outer spins formed by the remaining sites of the conduction channels. This new interaction is generated by the minimal hopping that exists between the zeroth site and the first sites of each of the conduction bath channels, and is characterised by the small superexchange coupling $J' \sim t^2/J$. *The conclusion is that a large J model with a central spin-half object can be transformed into a small J' model with a central spin-$\left(\frac{K-1}{2}\right)$ object.* This is shown schematically in Fig. 2. The former is RG irrelevant while the latter is relevant, both flowing towards the stable fixed point $J = J' = J^*$. For $K = 2$, the central spin for the J'-problem is of size $\frac{K-1}{2} = 1/2$, which is the same as the J-problem, so the models are self-dual, as mentioned previously.

V.   PARITY OPERATORS AND THE IMPURITY MAGNETIZATION

We can also use the eigenvalues of π^z to label the eigenstates. We have the eigenvalue relation $\pi^z |S^z\rangle = (-1)^{S^z - \frac{1}{2}} |S^z\rangle$. Applying π^z on these eigenstates gives

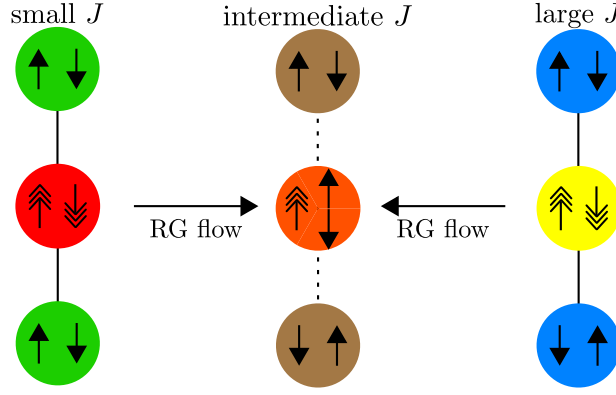$$\pi^z |\Psi_{m,-}^M\rangle = (-1)^m |\Psi_{m,-}^M\rangle \qquad (16)$$

FIG. 2. Duality of the RG flows as seen in the star graph Hamiltonian. The red and green circles represent the impurity and zeroth site spins respectively. At large J, the red circle binds with the green circles to form an effective spin $\frac{K-1}{2}$ object (yellow) that interacts with the remaining spin of the conduction bath (blue circles). The fixed point consists of the red (purple) and green (yellow) circles binding to form a doubly degenerate ground state (orange) that interacts with the zero mode of the remaining sites (brown) through the hopping.

This shows that the operator π^z splits the $K-$fold degenerate ground state manifold into two sub-manifolds of π^z parity ± 1. The odd values of m form the negative parity sector $\pi^z = -1$ while the even values form the positive parity sector $\pi^z = 1$.

It is interesting to calculate the action of the parity operator π^x on these eigenstates. We can write that operator as

$$\pi^x = (S_d^+ + S_d^-) \otimes_{i=1}^K \left(s_i^+ + s_i^- \right) \tag{17}$$

We can write the eigenstates of s_{tot}^z in terms of the eigenstates of s_i^z using their Clebsch-Gordon coefficients:

$$|M, m\rangle = \sum_{\substack{\{m_i\} \\ \sum_i m_i = m}} |m_1, m_2, ..., m_K\rangle \, C_{\{m_i\}, m, M} \tag{18}$$

where $C_{\{m_i\}, m, M} = \langle m_1, m_2, ..., m_K | m, M \rangle$ are the Clebsch-Gordon coefficients. Applying the parity operator on the state gives

$$\pi^x |S_d^z, M, m\rangle = \sum_{\substack{\{m_i\} \\ \sum_i m_i = m}} C_{\{m_i\}, m, M} (S_d^+ + S_d^-)|S_d^z\rangle \otimes_{i=1}^K \left(s_i^+ + s_i^- \right) |m_i\rangle = \sum_{\substack{\{m_i\} \\ \sum_i m_i = m}} C_{\{m_i\}, m, M} |-S_d^z\rangle \otimes_{i=1}^K |-m_i\rangle \tag{19}$$

$$= \sum_{\substack{\{m_i\} \\ \sum_i m_i = -m}} C_{\{-m_i\}, m, M} |-S_d^z\rangle \otimes_{i=1}^K |m_i\rangle \tag{20}$$

Using the identity $\langle M_1 m_1, M_2 m_2, \dots | m, M \rangle = (-1)^{\sum_i M_i - M} \langle M_1, -m_1; M_2 - m_2; \dots | -m, M \rangle$, we have $C_{\{-m_i\}, m, M} = C_{\{m_i\}, -m, M}$. Substituting this, we get

$$\pi^x |S_d^z, M, m\rangle = \sum_{\substack{\{m_i\} \\ \sum_i m_i = -m}} C_{\{m_i\}, -m, M} |-S_d^z\rangle \otimes_{i=1}^K |m_i\rangle = |-S_d^z, M, -m\rangle \tag{21}$$

The action of π^x on the eigenstates can now be determined:

$$\pi^x |\Psi_{m,-}^M\rangle = -\sqrt{x_m^M} |\downarrow, M, -m\rangle + (M + m + 1) |\uparrow, M, -m-1\rangle = -|\Psi_{-m-1,-}^M\rangle \tag{22}$$

It can be seen that the action of the operator π^x is to interpolate between two ground states of opposite π^z eigenvaluesand, more specifically, opposite S^z values

$$\pi^x |S^z\rangle \to |-S^z\rangle \tag{23}$$

We can show that the impurity magnetization along the $x-$direction in some specific ground states can be related to matrix elements of the string operator π^x. The most general ground state is obtained by taking a general linear combination of all the normalized ground states:

$$|g_m^n\rangle = \frac{1}{\sqrt{2}} \left[|\Psi_{m,-}^M\rangle + e^{i\theta} |\Psi_{m+n,-1}^M\rangle \right], m \in [-M, M-2], n \in [1, M-m-1] \tag{24}$$

The expectation value of the impurity spin along x in this state is given by

$$(m_d^x)_m^n = \langle g_m^n|\sigma_d^x|g_m^n\rangle \tag{25}$$

The action of σ_d^x is as follows:

$$\sigma_d^x |\Psi_{m,-}^M\rangle = \frac{1}{\sqrt{(M+m+1)(1+2M)}} \left[-\sqrt{x_m^M} |\downarrow, M, m\rangle + (M+m+1) |\uparrow, M, m+1\rangle \right] \tag{26}$$

which means that the only $|\Psi_{m+n,-}^M\rangle$ with $n > 0$ that give a non-zero inner product with $\sigma_d^x |\Psi_{m,-}^M\rangle$ is $n = 1$. This observation then gives

$$(m_d^x)_m^n = \delta_{n,1} \langle g_m^1|\sigma_d^x|g_m^1\rangle = \delta_{n,1} \left[e^{i\theta} \langle \Psi_{m+1,-}^M|\sigma_d^x|\Psi_{m,-}^M\rangle + \text{h.c.} \right] = \delta_{n,1} \left[e^{i\theta} \langle \Psi_{m+1,-}^M|\sigma_d^x\pi^x|\Psi_{-m-1,-}^M\rangle + \text{h.c.} \right] \tag{27}$$

We thus see that the impurity magnetization $(m_d^x)_m^n$ along x for the state $|g_m^n\rangle$ is determined by the matrix element of the string operator $\sigma_d^x\pi^x$.
For $m = -M$, we get

$$\langle \Psi_{-M+1,-}^M|\sigma_d^x\pi^x|\Psi_{M-1,-}^M\rangle = \frac{\sqrt{2M-1}}{2M+1} \tag{28}$$

The ground state manifold is defined by $M = K/2$. Using this, we can write

$$(m_d^x)_{-M}^1 = \left[\langle \Psi_{1-M,-}^M|\sigma_d^x\pi^x|\Psi_{M-1,-}^M\rangle + \text{h.c.} \right] = \frac{2\sqrt{2M-1}}{2M+1} = \frac{2\sqrt{K-1}}{K+1} \tag{29}$$

We can similarly calculate the magnetization along the $z-$direction. The most general ground state is

$$|g\rangle = \sum_{m=-M}^{M-1} C_m |\Psi_{m,-}^M\rangle, \qquad m \in [-M, M-1], \qquad \sum_m |C_m|^2 = 1 \tag{30}$$

The expectation value of σ_z^d will be non-zero only in the diagonal terms: $\langle \Psi_m^M|\sigma_d^z|\Psi_{m'}^M\rangle = \delta_{mm'} \langle \Psi_m^M|\sigma_d^z|\Psi_m^M\rangle$. Using this, we get

$$\langle g|\sigma_d^z|g\rangle = \sum_{m=-M}^{M-1} |C_m|^2 \frac{\left[x_m^M - (M+m+1)^2 \right]}{(M+m+1)(1+2M)} \tag{31}$$

For the simpler case when $C_m = \delta_{m,\pm(M-\frac{1}{2})-\frac{1}{2}}$, we get

$$\langle g_\pm|\sigma_d^z|g_\pm\rangle = \pm\frac{2M-1}{2M+1} = \pm\frac{K-1}{K+1} \tag{32}$$

VI. DERIVATION OF URG EQUATIONS FOR THE MULTI-CHANNEL KONDO MODEL

A. URG procedure

The URG method [18–23] involves applying unitary transformations to decouple high energy degrees of freedom. The renormalization at step j is given by

$$\Delta H_j = \left(c^\dagger T \frac{1}{\hat{\omega} - H_D} T^\dagger c + T^\dagger c \frac{1}{\hat{\omega} - H_D} c^\dagger T \right), \tag{33}$$

where $c^\dagger T$ represents the off-diagonal part of the Hamiltonian with respect to the fluctuations of the electron we are decoupling and H_D is the diagonal part with respect to the same electron. Assuming we are decoupling a particular electron $q\beta$, we have

$$c^\dagger T = J \sum_{\substack{|k|<\Lambda_j,\\ \alpha=\uparrow,\downarrow}} \vec{S}_d \cdot \vec{s}_{\beta\alpha} c^\dagger_{q\beta} c_{k\alpha}, \ H_D = \epsilon_q \tau_{q\beta} + J S_d^z s_q^z \tag{34}$$

Usually we treat the $\hat{\omega}$ as number(s) and study the renormalization in the couplings as functions of the quantum fluctuation scales. Each value of the fluctuation scale defines an eigendirection of $\hat{\omega}$. We have then essentially traded off the complexity in the non-commutation of the diagonal and off-diagonal terms for all the directions in the manifold of $\hat{\omega}$.

Here we will do something different. We will redefine the $\hat{\omega}$ by pulling out the off-diagonal term from it: $\hat{\omega} \to \hat{\omega} - H_X$, and then study the renormalization at various orders by expanding the denominator in powers of H_X. Such a redefinition essentially amounts to a rotation of the eigendirections of $\hat{\omega}$. This is done in order to extract some information out of $\hat{\omega}$, specifically the dependence of the RG equations on the channel number $K = \sum_\gamma$. This dependence is in principle present even if we do not do such a redefinition and expansion, in the various directions and values of ω, because those values encode the non-perturbative information regarding scattering at all loops. However, it is difficult to read off this information directly. This step of redefinition followed by expansion is being done with the sole aim of exposing such information.

The expansion we are talking about is

$$\eta = \frac{1}{\hat{\omega} - H_D} T^\dagger c = \frac{1}{\omega' - H_D - H_X} T^\dagger c \simeq \frac{1}{\omega' - H_D} T^\dagger c + \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c \tag{35}$$

where $H_X = J \sum_{k,k'<\Lambda_j,\alpha,\alpha'} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} c^\dagger_{k\alpha} c_{k'\alpha'}$ is scattering between the entangled electrons. With this change, the second and third order renormalizations will take the form

$$\Delta H_j^{(2)} = c^\dagger T \frac{1}{\omega' - H_D} T^\dagger c + T^\dagger c \frac{1}{\omega - H_D} c^\dagger T \tag{36}$$

$$\Delta H_j^{(3)} = c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c + T^\dagger c \frac{1}{\omega - H_D} H_X \frac{1}{\omega - H_D} c^\dagger T \tag{37}$$

We will use the identity

$$S_d^a S_d^z S_d^b = \left(\frac{1}{4}\delta^{az} + \frac{i}{2}\sum_c \epsilon^{azc} S_d^c \right) S_d^b = \left(\frac{1}{4}\delta^{az} S_d^b + \frac{i}{8}\epsilon^{azb} - \frac{1}{4}\sum_{c_1,c} \epsilon^{azc_1}\epsilon^{c_1 bc} S_d^c \right) = \frac{1}{4}\left(\delta^{az} S_d^b - \delta^{ab} S_d^z + \delta^{bz} S_d^a \right) \tag{38}$$

B. Leading order renormalization

$$\Delta H_j^{(2)} = \underbrace{c^\dagger T \frac{1}{\omega' - H_D} T^\dagger c}_{\text{first term}} + \underbrace{T^\dagger c \frac{1}{\omega - H_D} c^\dagger T}_{\text{second term}} \tag{39}$$

This renormalization is identical to that in the single channel. There is no additional physics due to the presence of multiple channels at this order. It is shown in appendix A1.

C. Next-to-leading order renormalization

$$\Delta H_j^{(3)} = \underbrace{c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c}_{\text{first term}} + \underbrace{T^\dagger c \frac{1}{\omega - H_D} H_X \frac{1}{\omega - H_D} c^\dagger T}_{\text{second term}} \tag{40}$$

A general term of this summation has three sets of spin operators coming from $c^\dagger T$, H_X and $T^\dagger c$. If we had expressed the spin operators in terms of $S^z, S^\pm$, most of the terms would have atleast one S^+ or S^-, and by the same argument as in the single-channel case, the denominator will have anti-parallel spins and the Ising term will be negative, leading to the form: $\omega - D/2 - \epsilon_k/2 + J/4$. ϵ_k is the energy of the other electron that will be summed over. The only term that does not have even one $S^\pm$ is the one with three S^z. We can show that this term will also have the same denominator. An instance of this term (in shorthand) is

$$S_d^z c_{q\uparrow}^\dagger \frac{1}{\omega - D/2 - \epsilon_k/2 + J/2 S_d^z} S_d^z \frac{1}{\omega - D/2 - \epsilon_k/2 + J/2 S_d^z} S_d^z c_{q\uparrow} \tag{41}$$

$$\tag{42}$$

This can be split into up and down configurations of the impurity spin using the decomposition $S_d^z = \frac{1}{2}\left(\frac{1}{2} + S_d^z\right) - \frac{1}{2}\left(\frac{1}{2} - S_d^z\right)$. These configurations will have different quantum fluctuation scales ω, ω':

$$\frac{1}{2} S_d^z c_{q\uparrow}^\dagger \left[\frac{\left(\frac{1}{2} + S_d^z\right)}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} - \frac{\left(\frac{1}{2} - S_d^z\right)}{\left(\omega' - D/2 - \epsilon_k/2 - J/4\right)^2} \right] S_d^z c_{q\uparrow} \tag{43}$$

If we now use poor man's scaling values to relate the two ωs, we get $\omega' - \omega = J/2$. Substituting this will make both the denominators identical: $\omega - D/2 - \epsilon_k/2 + J/4$. This means that the denominator for all non-zero terms that renormalize the Hamiltonian is $\omega - D/2 - \epsilon_k/2 + J/4$.

1. Calculation of first term

$$c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = J^3 \sum_{\substack{q,k,k_1,k_2, \\ \beta,\alpha,\alpha_1,\alpha_2, \\ l_1,l_2,a,b,c}} \frac{c_{q\beta,l_1}^\dagger c_{k\alpha,l_1} S_d^a s_{\beta\alpha}^a S_d^b s_{\alpha_1\alpha_2}^b c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} c_{k\alpha,l_1}^\dagger c_{q\beta,l_1} S_d^c s_{\alpha\beta}^c}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} \tag{44}$$

q sums over the momenta being decoupled. k, k_1, k_2 sum over the momenta not being decoupled. $\beta, \alpha, \alpha_1, \alpha_2$ sum over the spin indices. l_1, l_2 sum over the channels. We will start simplifying this equation by summing over q. $c_{q\beta}^\dagger$ and $c_{q\beta}$ can be easily combined to form $\hat{n}_{q\beta}$, because they anti-commute with the other momenta. The sum gives $\sum_q \hat{n}_{q\beta l_1} = n(D)$. This gives

$$c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = J^3 n(D) \sum_{\substack{k,k_1,k_2, \\ \beta,\alpha,\alpha_1,\alpha_2, \\ l_1,l_2,a,b,c}} \frac{c_{k\alpha,l_1} S_d^a s_{\beta\alpha}^a S_d^b s_{\alpha_1\alpha_2}^b c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} c_{k\alpha,l_1}^\dagger S_d^c s_{\alpha\beta}^c}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} \tag{45}$$

The operators $c_{k\alpha}^\dagger$ and its conjugate can be brought together without any change of sign because there will be an even number of flips. The sum over k gives

$$\sum_k \frac{1 - \hat{n}_{k\alpha l_1}}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} = \rho \int \frac{d\epsilon\left[1 - \hat{n}(\epsilon)_{\alpha l_1}\right]}{\left(\omega - D/2 - \epsilon/2 + J/4\right)^2} = \rho \int_0^{D-2(\omega+J/4)} \frac{d\epsilon}{\left(\omega - D/2 - \epsilon/2 + J/4\right)^2} \tag{46}$$

The integration limits include only the positive energies, because of the $1-\hat{n}$ operator; the upper limit of the integration is chosen so as to make the denominator double, because this preserves the symmetry of the denominator. Performing the integration gives

$$\sum_k \frac{1 - \hat{n}_{k\alpha l_1}}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} = -\frac{1}{2} \frac{\rho}{\omega - D/2 + J/4} \tag{47}$$

The sum over the channel index l_1 produces a factor of K. $K = \sum_{l_1}$ is the total number of conduction bath channels. The entire expression is now

$$c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = -\frac{1}{2} \frac{J^3 n(D)\rho K}{\omega - D/2 + J/4} \sum_{\substack{\beta,\alpha,\alpha_1,\alpha_2, \\ a,b,c}} S_d^a s_{\beta\alpha}^a S_d^b s_{\alpha_1\alpha_2}^b S_d^c s_{\alpha\beta}^c \sum_{k_1,k_2,l_2} c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} \tag{48}$$

We now need to simplify the spin products. The sum over α, β can be carried out immediately: $\sum_{\alpha,\beta} s^a_{\beta\alpha} s^c_{\alpha\beta} = $
$\text{Trace}\,(s^a s^c) = \frac{1}{2}\delta^{ac}$. Substituting this gives

$$c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = -\frac{1}{2} \frac{J^3 n(D)\rho K}{\omega - D/2 + J/4} \frac{1}{2} \sum_{\alpha_1,\alpha_2,a,b} S^a_d S^b_d S^a_d \sum_{k_1,k_2,l_2} c^\dagger_{k_1\alpha_1,l_2} c_{k_2\alpha_2,l_2} s^b_{\alpha_1\alpha_2} \tag{49}$$

The spin product can now be carried out:

$$\sum_a S^a_d S^b_d S^a_d = \sum_a S^a_d \left[\frac{1}{4}\delta^{ab} + \frac{i}{2}\sum_c \epsilon^{bac} S^c_d \right] = \frac{1}{4}S^b_d - \frac{1}{4}\sum_{ace} \epsilon^{bac}\epsilon^{ace} S^e_d = \frac{1}{4}S^b_d - \frac{1}{4}S^b_d \sum_{ac} \epsilon^{bac}\epsilon^{acb} \tag{50}$$

$$= -\frac{1}{4}S^b_d \tag{51}$$

The renormalization becomes

$$\Delta H_1 = c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = -\frac{1}{2} \frac{J^3 n(D)\rho K}{\omega - D/2 + J/4} \frac{1}{2}\left(-\frac{1}{4}\right) \sum_{k_1,k_2,\alpha_1,\alpha_2,b,l_2} S^b_d s^b_{\alpha_1\alpha_2} c^\dagger_{k_1\alpha_1,l_2} c_{k_2\alpha_2,l_2} \tag{52}$$

$$= \frac{1}{16} \frac{J^3 n(D)\rho K}{\omega - D/2 + J/4} \sum_{k_1,k_2,\alpha_1,\alpha_2,l_2} \vec{S}_d \cdot \vec{s}_{\alpha_1\alpha_2} c^\dagger_{k_1\alpha_1,l_2} c_{k_2\alpha_2,l_2} \tag{53}$$

2. Calculation of second term

Like the single-channel case, the renormalization coming from the hole excitations is exactly the Hermitian conjugate of that in the particle sector. And since the renormalization ΔH_1 is Hermitian, we have $\Delta H_0 = \Delta H_1$

D. Total renormalization $\Delta H^{(3)}$

The total renormalization is twice that in the particle sector.

$$\Delta H^{(3)} = \frac{1}{8} \frac{J^3 n(D)\rho K}{\omega - D/2 + J/4} \sum_{k_1,k_2,\alpha_1,\alpha_2} \vec{S}_d \cdot \vec{s}_{\alpha_1\alpha_2} c^\dagger_{k_1\alpha_1,l_2} c_{k_2\alpha_2,l_2} \tag{54}$$

Combining with $\Delta H^{(2)}$ and replacing $n(D) = \rho|\delta D|$, we get

$$\frac{\Delta J}{|\Delta D|} = -\frac{J^2\rho}{\omega - D/2 + J/4} + \frac{1}{8}\frac{J^3\rho^2 K}{\omega - D/2 + J/4} = -\frac{J^2\rho}{\omega - D/2 + J/4}\left[1 - \frac{1}{8}J\rho K\right] \tag{55}$$

We choose $\omega = -D/2$ to get a clearer idea of what the equations say.

$$\frac{\Delta J}{|\Delta D|} = \frac{J^2\rho}{D - J/4}\left[1 - \frac{1}{8}J\rho K\right] \tag{56}$$

Quantities with zero in the subscript will denote their values in the bare Hamiltonian. Using $\delta D = -|\delta D|$, we can write the continuum form of the equation:

$$\frac{dJ}{dD} = \frac{J^2\rho}{D - J/4}\left(\frac{1}{8}J\rho K - 1\right) \tag{57}$$

For $D \gg J$, we can ignore the J in the denominator, and the equation reduces to the one-loop poor man's scaling form

$$\frac{dJ}{dD} \simeq \frac{J^2\rho}{D}\left(\frac{1}{8}J\rho K - 1\right) \tag{58}$$

This equation has a stable fixed point at $J^* = \frac{8}{\rho K}$. The value of the fixed point agrees with the poor man's scaling calculation in [24], but differs from the value of $\frac{2}{\rho K}$ obtained in other references [25, 26].
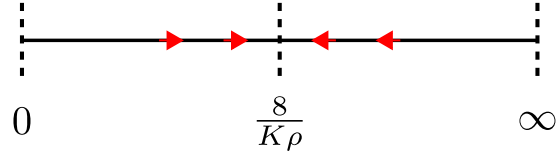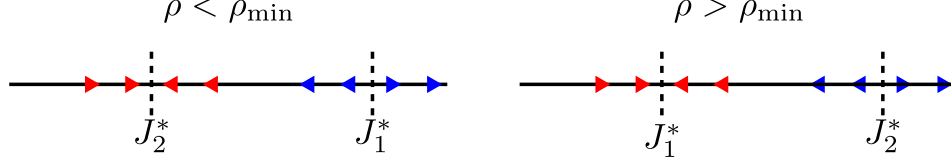
FIG. 3. Attractive finite J fixed point of poor man scaling RG equation

$\rho < \rho_{\min}$ $\rho > \rho_{\min}$

FIG. 4.

For D not so large, the denominator also comes into play, and Eq. 56 holds. We get the possibility of two fixed points - one from the numerator and the other from the numerator.

The numerator and denominator fixed points, J_1^* and J_2^* respectively, are given by

$$J_1^* = \frac{8}{K\rho}, \qquad\qquad\qquad D^* = \frac{J_2^*}{4} \qquad (59)$$

For a given K, the position of J_1^* will be governed by ρ. In general, for each bare bandwidth D_0, there exists a minimal ρ, $\rho_{\min}(D_0)$, above which the the lower fixed point is the one from the numerator. That is, for $\rho > \rho_{\min}$, if we start scaling from small J_0, it grows until it hits J_1^* which acts as the attractive fixed point, and J_2^* lies at a higher value and acts as the repulsive fixed point. For $\rho < \rho_{\min}$, J will grow and hit J_2^* instead, and $J_1^* > J_2^*$ now becomes the repulsive fixed point.

$$\rho_{\min} = \text{minimum} \left\{ \rho, \text{ such that } \frac{8}{K\rho} < 4D^*(\rho) \right\} \qquad (60)$$

The RG flows towards the attractive fixed point J_1^* is shown in fig. 5.

$$K = 2, D_0 = 10, \rho = 1$$
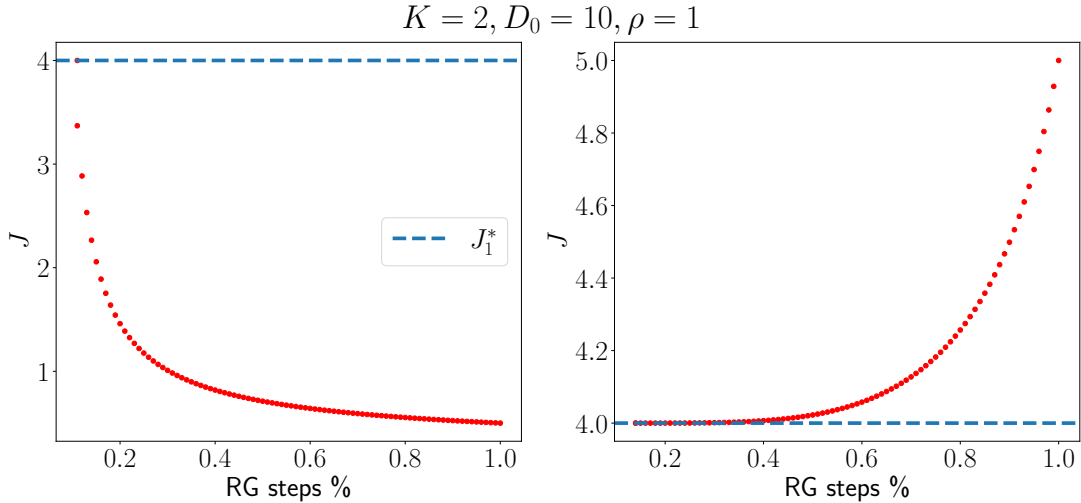
FIG. 5. Attractive flows towards J_1^*

This behaviour is shown schematically in fig. 4. In fig. 6, we plot $\rho_{\min}$ against the bare bandwidth. For large D_0, it essentially shrinks to zero, and the numerator becomes the first fixed point for essentially all ρ.

If we assume we are at a sufficiently large D_0 and $\rho > \rho_{\min}$, the lower fixed point is J_1^*. As shown in fig. 6, we have $J_1^* \ll D_0$. If we start with J_0 in the neighborhood of J_1^*, we can use $J_1^* \ll D_0$ to ignore J in the denominator and
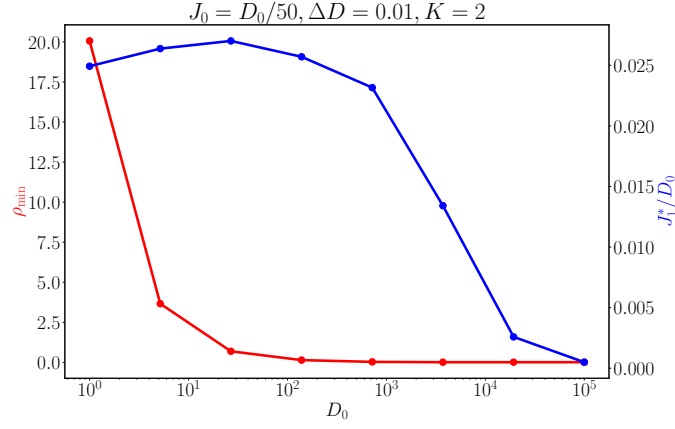
FIG. 6. Red curve shows variation of $\rho_{\min}$ against D_0. It vanishes at large D_0. Blue curve shows variation of the ratio J_1^*/D_0 with D_0. That shrinks as well, showing that the fixed point J_1^* remains finite in the thermodynamic limit, and the distance between J_1^* and J_2^* keeps growing.

the RG equation reduces to the poor man's scaling form Eq. 58. The denominator fixed point has effectively moved off to infinity. That this is true can also be argued from the single-channel Kondo model URG results. There, we saw that when the bandwidth is scaled to larger values, the strong coupling fixed point was stable at successively larger values of J^*. Since the denominator fixed point is identical in structure in both problems, its reasonable that the same thing will happen here.

VII. EFFECT OF ANISOTROPY ON RG FLOWS

For a general anisotropic multi-channel Kondo, the Hamiltonian is

$$H = \sum_{k,\alpha,\gamma} \epsilon_k^\gamma \hat{n}_{k\alpha}^\gamma + \sum_{kk',\gamma} J_\gamma \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} c_{k\alpha}^{\gamma\,\dagger} c_{k'\alpha'}^\gamma . \tag{61}$$

Let us consider the specific case where $K-1$ channels have the same coupling $J_1 = J_2 = ... = J_{K-1} = J_+$ and the remaining channel has a different coupling $J_K = J_-$. The RG equations for such a model are

$$\frac{\Delta J_+}{|\Delta D|} = \frac{J_+^2 \rho}{D - J_+/4} - \frac{\rho^2 J_+}{8} \left[\frac{(K-1)J_+^2}{D - J_+/4} + \frac{J_-^2}{D - J_-/4} \right] \tag{62}$$

$$\frac{\Delta J_-}{|\Delta D|} = \frac{J_-^2 \rho}{D - J_-/4} - \frac{\rho^2 J_-}{8} \left[\frac{(K-1)J_+^2}{D - J_+/4} + \frac{J_-^2}{D - J_-/4} \right] \tag{63}$$

Setting $J_+ = J_-$ leads to the critical fixed point at $J_+^* = J_-^* = J_* = \frac{8}{K\rho}$. We now perturb around this fixed point by defining new variables $j_\pm = J_\pm - J_*$. We also assume that $D - J_\pm/4 \simeq D - J_*/4$. The RG equations then take the form

$$\frac{\Delta j_+}{|\Delta D|} = \frac{\rho J_+}{D - J_*/4} \left[J_+ - \frac{\rho}{8} \left[(K-1)J_+^2 + J_-^2 \right] \right] \tag{64}$$

$$= \frac{\rho J_+}{D - J_*/4} \left[j_+ + J_* - \frac{1}{KJ_*} \left[(K-1)(J_* + j_+)^2 + (J_* + j_-)^2 \right] \right] \tag{65}$$

$$= \frac{\rho J_+}{KJ_*(D - J_*/4)} \left[KJ_*(j_+ + J_*) - (K-1)(J_* + j_+)^2 - (J_* + j_-)^2 \right] \tag{66}$$

$$= \frac{\rho J_+}{KJ_*(D - J_*/4)} \left[K\left(J_* j_+ + J_*^2\right) - (K-1)\left(J_*^2 + j_+^2 + 2J_* j_+\right) - \left(J_*^2 + j_-^2 + 2J_* j_-\right) \right] \tag{67}$$

$$= \frac{\rho J_+}{KJ_*(D - J_*/4)} \left[KJ_* j_+ - (K-1)\left(j_+^2 + 2J_* j_+\right) - \left(j_-^2 + 2J_* j_-\right) \right] \tag{68}$$

$$= \frac{\rho J_+}{KJ_*(D - J_*/4)} \left[-(K-2)J_* j_+ - (K-1)j_+^2 - j_-^2 - 2J_* j_- \right] \tag{69}$$

$$\frac{\Delta j_-}{|\Delta D|} = \frac{J_-\rho}{D - J_*/4} \left[J_- - \frac{\rho}{8} \left[(K-1)J_+^2 + J_-^2 \right] \right] \tag{70}$$

$$= \frac{J_-\rho}{KJ_*(D - J_*/4)} \left[K\left(J_* j_- + J_*^2\right) - (K-1)\left(J_*^2 + j_+^2 + 2J_* j_+\right) - \left(J_*^2 + j_-^2 + 2J_* j_-\right) \right] \tag{71}$$

$$= \frac{J_-\rho}{KJ_*(D - J_*/4)} \left[(K-2)J_* j_- - j_-^2 - (K-1)j_+^2 - 2(K-1)J_* j_+ \right] \tag{72}$$

$$\tag{73}$$

We will first look at the special case of $K = 2$, the two channel Kondo model. The equations simplify to

$$\frac{\Delta j_\pm}{|\Delta D|} = \frac{J_\pm \rho}{KJ_*(D - J_*/4)} \left[-\left(j_+^2 + j_-^2\right) - 2J_* j_\mp \right] \tag{74}$$

$$\tag{75}$$

For $j_- < 0, j_+ > 0$, we have $\Delta j_- < 0$. The coupling J_- therefore becomes irrelevant. For small j_+, we have $j_+^2 < 2J_*|j_-|$ and $\Delta j_+ > 0$. This means that the isotropic fixed point is repulsive under anisotropy [1]. The coupling j_+ being relevant means we have a single-channel Kondo problem. We already know the non-perturbative URG equation for the single-channel Kondo problem:

$$\frac{\Delta j_+}{|\Delta D|} = \frac{J_+^2 \rho}{D - J_+/4} \ , \tag{76}$$

and it leads to the strong coupling fixed point

We now look at the general K channel case. Let us first look at the regime $j_- < 0, j_+ > 0$. In this regime, we have $\Delta j_- < 0$, which means j_- will flow to larger negative values until it reaches $j_- = -J_*$ such that $J_- = J_* + j_- = 0$. j_+ is, on the other hand, relevant for small values of $j_\pm$. It will continue to grow until the numerator of Δj_+ vanishes. This condition is given by

$$(K-2)J_* j_+ + (K-1)j_+^2 + j_-^2 + 2J_* j_- = 0 \tag{77}$$

Substituting $j_- = -J_*$ gives

$$(K-1)j_+^2 + (K-2)J_* j_+ - J_*^2 = 0 \tag{78}$$

Solving for j_+ gives

$$j_{+,*} = \frac{-J_*(K-2) \pm \sqrt{(K-2)^2 J_*^2 + 4(K-1)J_*^2}}{2(K-1)} = \frac{J_*}{2(K-1)} \left[-(K-2) \pm K \right] = \frac{J_*}{K-1} \tag{79}$$

At the final step, we chose the positive solution, because j_+ is relevant in this regime. The new fixed point value of J_+ is therefore

$$J_+ = J_* + \frac{J_*}{K-1} = \frac{\frac{8}{K\rho} K}{K-1} = \frac{8}{(K-1)\rho} \tag{80}$$

In other words, the K channel fixed point flows to the $K - 1$ channel fixed point.

In the opposite regime $j_- > 0, j_+ < 0$, Δj_+ is negative. It has been checked numerically that J_+ ultimately flows to zero in this regime (fig. 7), and J_- remains relevant. Since there is no numerator fixed point in the relevant coupling J_- and because all other couplings are irrelevant, the equation for J_- is replaced by the single-channel Kondo coupling URG equation, and the low-energy physics is then of strong coupling..

VIII. DERIVATION OF URG EQUATIONS FOR THE MULTI-CHANNEL SPIN-S IMPURITY KONDO MODEL

The leading order renormalization is the same as the single-channel model, and is shown in the appendix. Following that calculation, the Hamiltonian we will work with is

$$H = \sum_{k\sigma} \epsilon_k \tau_{k\sigma} + \sum_{m=-S}^{S} \sum_{\substack{kl, \\ \sigma=\uparrow,\downarrow}} J_m^\sigma |m\rangle \langle m| c_{k\sigma}^\dagger c_{l\sigma} + \sum_{kl} \sum_{m=-S}^{S-1} J_m^t \left(|m+1\rangle \langle m| s_{kl}^- + \text{h.c.} \right) \tag{81}$$
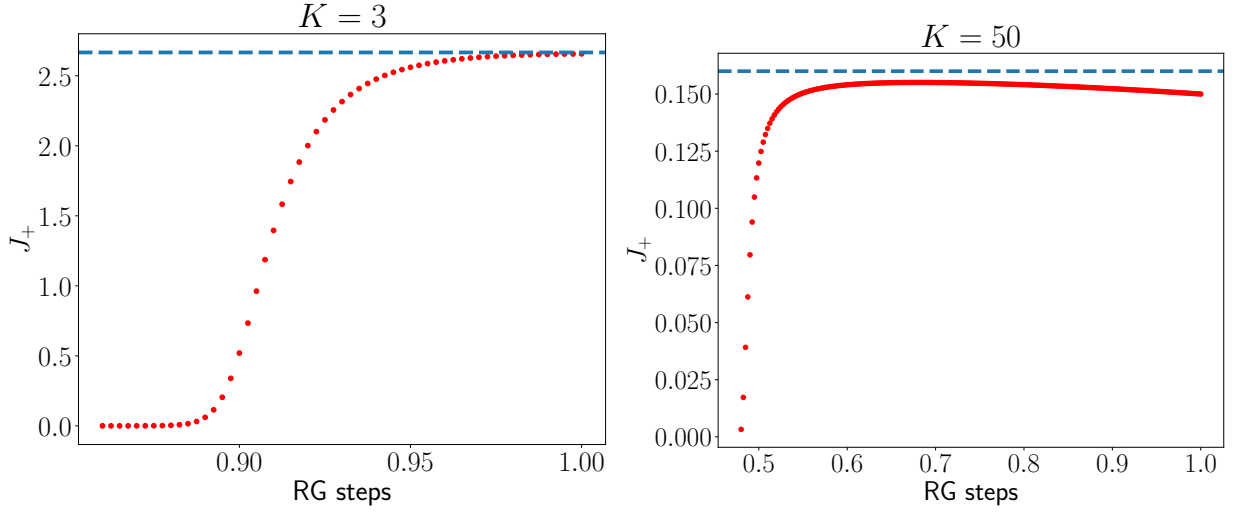
FIG. 7. Flow of the coupling J_+ when $j_+ < 0$ for two values of K

where $\{|m\rangle\}$ are the eigenstates of S_d^z, $J_m^\sigma = \frac{1}{2}\sigma m J$ in the UV Hamiltonian, and $J_m^t = J\frac{1}{2}\sqrt{S(S+1) - m(m+1)}$ is the coupling that connects $|m\rangle$ and $|m+1\rangle$.

The next order renormalization is of the form

$$\Delta H_j^{(3)} = \underbrace{c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c}_{\text{first term (particle sector)}} + \underbrace{T^\dagger c \frac{1}{\omega - H_D} H_X \frac{1}{\omega - H_D} c^\dagger T}_{\text{second term (hole sector)}} \tag{82}$$

Before we start, we will make another simplification. A general term of ΔH will be of the form

$$\sum_{q,k,l_1} c_{q\beta,l_1}^\dagger c_{k\alpha,l_1} |m_1\rangle \langle m_2| \frac{1}{\omega - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{\beta J}{4} S_d^z} |m_2\rangle c_{k_1\sigma_1,l_2}^\dagger c_{k_2\sigma_2,l_2} \langle m_3| \frac{1}{\omega - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{\beta J}{4} S_d^z} |m_3\rangle \langle m_4| c_{k\alpha,l_1}^\dagger c_{q\beta,l_1} \tag{83}$$

We have not bothered to write all the summations and the couplings correctly, because we will only simplify the denominator here. Evaluating the inner products gives

$$|m_1\rangle \langle m_4| \sum_k c_{q\beta}^\dagger c_{k\alpha} \frac{1}{\omega_{m_2,\beta} - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{\beta J m_2}{4}} c_{k_1\sigma_1,l_2}^\dagger c_{k_2\sigma_2,l_2} \frac{1}{\omega_{m_3,\beta} - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{\beta J m_3}{4}} c_{k\alpha}^\dagger c_{q\beta} \tag{84}$$

To compare with the spin-$\frac{1}{2}$ RG equations, we will transform the general spin-S ω to the spin-$\frac{1}{2}$ ω, using $\omega_{m,\sigma} \to \omega - \frac{J}{2}\left(m\sigma - \frac{1}{2}\right)$.

$$\hat{n}_{q\beta} |m_1\rangle \langle m_4| c_{k_1\sigma_1,l_2}^\dagger c_{k_2\sigma_2,l_2} \sum_k (1 - \hat{n}_{k\alpha}) \left(\frac{1}{\omega - \frac{D}{2} - \frac{\epsilon_k}{2} + \frac{J}{4}}\right)^2 \tag{85}$$

We can set $\sum_q \hat{n}_{q\beta} = n(D)$. Performing the sums over k and l_1 in the same fashion as previously gives

$$-\frac{1}{2} |m_1\rangle \langle m_4| c_{k_1\sigma_1,l_2}^\dagger c_{k_2\sigma_2,l_2} \frac{\rho n(D) K}{\omega - \frac{D}{2} + \frac{J}{4}} \tag{86}$$

Reinstating the complete summation and the couplings gives

$$-\frac{1}{2} \frac{\rho n(D) K}{\omega - \frac{D}{2} + \frac{J}{4}} \sum_{m_1,m_4} \sum_{k_1,k_2,l_2,\sigma_1\sigma_2} \lambda_1\lambda_2\lambda_3 |m_1\rangle \langle m_4| c_{k_1\sigma_1,l_2}^\dagger c_{k_2\sigma_2,l_2} \tag{87}$$

There is no sum over m_2 and m_3 because they are constrained by m_1 and m_4 respectively. λ_i represent the couplings present at the three interaction vertices.

The various terms in each sector can be represented as $a|b|c$, where $a, b, c \in \{z, +, -\}$ and represent the operator for the conduction electrons in the three stages. We will only calculate the terms in the particle sector, the ones in hole sector will be equal to these because of particle-hole symmetry.

a. $z|z|z$

$$-\frac{1}{2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}}\sum_{\beta}\sum_{m}\sum_{k_1,k_2,l_2,\sigma}\left(J_m^{\beta}\right)^2 J_m^{\sigma}\left|m\right\rangle\left\langle m\right| c_{k_1\sigma,l_2}^{\dagger}c_{k_2\sigma,l_2} \tag{88}$$

b. $-|z|+$

$$-\frac{1}{2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}}\sum_{m}\sum_{k_1,k_2,l_2,\sigma}\left(J_m^{t}\right)^2 J_m^{\sigma}\left|m+1\right\rangle\left\langle m+1\right| c_{k_1\sigma,l_2}^{\dagger}c_{k_2\sigma,l_2} \tag{89}$$

c. $+|z|-$

$$-\frac{1}{2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}}\sum_{m}\sum_{k_1,k_2,l_2,\sigma}\left(J_m^{t}\right)^2 J_{m+1}^{\sigma}\left|m\right\rangle\left\langle m\right| c_{k_1\sigma,l_2}^{\dagger}c_{k_2\sigma,l_2} \tag{90}$$

d. $+|+|-$

$$-\frac{1}{2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}}\sum_{m}\sum_{k_1,k_2,l_2} J_m^{t}\left(J_{m+1}^{t}\right)^2\left|m\right\rangle\left\langle m+1\right| c_{k_1\uparrow,l_2}^{\dagger}c_{k_2\downarrow,l_2} \tag{91}$$

e. $+|-|-$

$$-\frac{1}{2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}}\sum_{m}\sum_{k_1,k_2,l_2} J_m^{t}\left(J_{m+1}^{t}\right)^2\left|m+1\right\rangle\left\langle m\right| c_{k_1\downarrow,l_2}^{\dagger}c_{k_2\uparrow,l_2} \tag{92}$$

f. $-|-|+$

$$-\frac{1}{2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}}\sum_{m}\sum_{k_1,k_2,l_2}\left(J_m^{t}\right)^2 J_{m+1}^{t}\left|m+2\right\rangle\left\langle m+1\right| c_{k_1\downarrow,l_2}^{\dagger}c_{k_2\uparrow,l_2} \tag{93}$$

g. $-|+|+$

$$-\frac{1}{2}\frac{\rho n(D)K}{\omega - \frac{D}{2} + \frac{J}{4}}\sum_{m}\sum_{k_1,k_2,l_2}\left(J_m^{t}\right)^2 J_{m+1}^{t}\left|m+1\right\rangle\left\langle m+2\right| c_{k_1\uparrow,l_2}^{\dagger}c_{k_2\downarrow,l_2} \tag{94}$$

IX. MFL AND ORTHOGONALITY CATASTROPHE IN LOW-ENERGY EXCITATIONS OF 2-CKM

At the stable fixed point $J^* = J_1^* = \frac{8}{K\rho}$, the ground states of the Hamiltonian are those of the star graph model, with a degeneracy of K. We now specialise to the two-channel Kondo model. To find the low-energy excitations on top of this ground state manifold, one can insert a tight-binding nearest-neighbour hopping between the zeroth site (the one that holds the impurity) and the first site (site that's nearest to the zeroth site) as a perturbation and calculate the diagonal and off-diagonal terms generated by this perturbation. It is found that when we trace out the impurity, we are left only with real space off-diagonal terms:

$$V_{\text{eff}} = \frac{2t^2}{J^*}\left[\left(\sigma_{0,1}^{z}\right)^2 s_{0,2}^{+} + \left(\sigma_{0,2}^{z}\right)^2 s_{0,1}^{+}\right]\left(s_{1,1}^{-} + s_{1,2}^{-}\right) + \text{h.c.} \tag{95}$$

where $\sigma_{0,l}^{z} = \hat{n}_{0\uparrow,l} - \hat{n}_{0\downarrow,l}, s^{+} = c_{0\uparrow,l}^{\dagger}c_{0\downarrow,l}$ and $s^{-} = (s^{+})^{\dagger}$. The notation $0\sigma, l$ has the site index $i = 0, 1, 2, \ldots$ as the first label, the spin index $\sigma = \uparrow, \downarrow$ as the second label and the channel index $l = 1, 2$ as the third index.

These are the terms that are generated because of the presence of the impurity. Such a non-Fermi liquid (NFL) contribution to the effective Hamiltonian and the absence of any Fermi-liquid term should be contrasted with the local Fermi liquid excitations induced by the singlet ground state of the single-channel Kondo model at fourth order. We wish to point out that such NFL terms were also obtained by Coleman, et al. [12] in terms of Majorana fermions at the zeroth site and the first site. They then went on to compute a single-particle self-energy renormalization coming from this NFL term that matches the phenomenological [27] and microscopic forms of the marginal Fermi liquid

self-energy [18, 20]. We take a different route in order to calculate the self-energy contribution coming from Eq. 95, one that involves [20].

In [18], it was found that the normal phase of the Mott insulator was a marginal Fermi liquid in momentum space, defined by the interaction term

$$H_{\mathrm{MFL}} = \sum_{k,k',k'',\sigma} R\hat{n}_k\hat{n}_{k'\overline{\sigma}}\left(1 - \hat{n}_{k''\sigma}\right) \tag{96}$$

We wish to look for such a term in the effective Hamiltonian. For this, we will perform a perturbative treatment of the hopping at strong coupling $J \to \infty$ where the perturbative coupling t^2/J is arbitrarily small and again obtain Eq. 95. Such a change from the strong coupling model with parameter J to a weak-coupling model with parameter t^2/J amounts to a duality transformation [16, 17]. It can be shown that the duality transformation leads to an identical multi-channel Kondo model [16] (self-duality), which implies we can have identical RG flows, and our transformation simply extracts the NFL piece from the dual model. The self-duality also implies that the critical intermediate-coupling fixed point is unique and can be reached from either of the models.

Since we are looking for a 2 particle-1 hole term in $k-$space, we will keep only those terms that can produce such a term:

$$V_{\mathrm{eff}} = \frac{2t^2}{J} \sum_{l=1,2} \left(\sum_{\sigma} \hat{n}_{0\sigma,l}\right) s_{0,\bar{l}}^+ s_{1,\bar{l}}^- + \text{h.c.} \tag{97}$$

where $\bar{l} = 3 - l$ is the channel index complementary to l. To compare with Eq. 96, we will fourier transform this effective Hamiltonian (after inserting the kinetic energy of the bath) into $k-$space.

$$H_{\mathrm{eff}} = \sum_{k\sigma,l} \epsilon_{k,l}\hat{n}_{k\sigma,l} + \sum_{\sigma,\{k_i,k_i'\},l} \frac{2t^2}{J} e^{i(k_1-k_1')a} c_{k\sigma,l}^\dagger c_{k'\sigma,l} c_{k_2\uparrow,\bar{l}}^\dagger c_{k_2'\downarrow,\bar{l}} c_{k_1\downarrow,\bar{l}}^\dagger c_{k_1'\uparrow,\bar{l}} + \text{h.c.} \tag{98}$$

This form of the Hamiltonian is very similar to the three-particle interaction term in Appendix B of [20]. The channel indices in Eq. 98 can be mapped to the normal directions in [20]. As specualted earlier, the three-particle interaction in Eq. 98 has a diagonal component which can be obtained by setting $k = k'$, $k_1 = k_2'$ and $k_2 = k_1'$:

$$H_{\mathrm{eff,MFL}} = \sum_{\sigma,k,k_1,k_2,l} \frac{2t^2}{J} e^{i(k_1-k_2)a}\hat{n}_{k\sigma,l}\hat{n}_{k_2\uparrow,\bar{l}}\left(1 - \hat{n}_{k_1\downarrow,\bar{l}}\right) + \text{h.c.} = \sum_{\sigma,k,k_1,k_2,l} \frac{4t^2}{J}\cos a\left(k_1 - k_2\right)\hat{n}_{k\sigma,l}\hat{n}_{k_2\uparrow,\bar{l}}\left(1 - \hat{n}_{k_1\downarrow,\bar{l}}\right) \tag{99}$$

The largest term is obtained by choosing $k_1 = k_2 = k'$:

$$H_{\mathrm{eff,MFL}}^* = \frac{4t^2}{J} \sum_{\sigma,k,k',l} \hat{n}_{k\sigma,l}\hat{n}_{k'\uparrow,\bar{l}}\left(1 - \hat{n}_{k'\downarrow,\bar{l}}\right) \tag{100}$$

Following [20], one can follow the RG evolution of the dual coupling $R_j = \frac{4t^2}{J_j}$ at the jth RG step, in the form of the URG equation

$$\Delta R_j = \frac{R_j^2}{\bar{\omega} - \epsilon_j/2 + R_j/8} \tag{101}$$

In the RG equation, ϵ_j represents the energy of the jth isoenergetic shell. It is seen from the RG equation that R is relevant in the range of $\bar{\omega} = \epsilon_j - \omega$ that has been used throughout ($\omega < \epsilon_j/2$), leading to a fixed-point at $R_*/8 = \bar{\omega} - \epsilon_*/2$). The relevance of R is expected because the strong coupling J is irrelevant and $R \sim 1/J$.

The renormalization in R leads to a renormalization in the single-particle self-energy [20]. The $k-$space-averaged self-energy renormalization is

$$\Delta\Sigma_l(\omega) = \rho R_*^2 \int_0^{\epsilon_*} \frac{d\epsilon}{\bar{\omega} - \epsilon_j/2 + R_*/8} \tag{102}$$

leading to a $k-$space-averaged self-energy for each channel (eq. 33 of [20])

$$\Sigma_l(\omega_l) \sim \omega_l \ln|\omega_l| \tag{103}$$

where ω_l is the fluctuation scale for a particular channel. The self-energy also provides the quasiparticle residue for each channel[20]:

$$Z_l(\omega_l) = (2 - \ln \omega_l)^{-1} \tag{104}$$

As the energy scales $\omega_l \to 0$, the Z_l vanishes, implying that the ground state and lowest-lying excitations, in the presence of the NFL terms, are not adiabatically connected to the Fermi gas. This is the orthogonality catastrophe in the two-channel Kondo problem, and it is brought about by the presence of the terms in Eq. 100. Such terms were absent in the similar perturbative expansion of the single-channel Kondo model, because there was no multiply-degenerate ground state manifold that allowed scattering. This line of argument shows that the extra degeneracy of the ground state subspace that comes about when one upgrades from the single-channel Kondo model to the multi-channel Kondo models is at the heart of the NFL behaviour, and the orthogonality catastrophe should be a general feature of all multi-channel Kondo models, even though we have demonstrated this only for the two-channel case.

The fact that the marginal Fermi liquid term in eq. 100 involves all momenta means it is localised in real space. This can also be understood from the fact that the parent NFL term in eq. 95 is also local. A local NFL term with a self-energy of the form in eq. 103 was also obtained in the $\sigma - \tau$ model by Coleman et al. [12]. The common features show the universality between the two-channel Kondo and the $\sigma - \tau$ models.

X. EFFECTIVE HAMILTONIAN FOR LOW-ENERGY EXCITATIONS: $k-$SPACE

The fixed point Hamiltonian is

$$H^* = H_0 + J^* \vec{S}_d \cdot \vec{s}_{\text{tot}} \tag{105}$$

where $H_0 = \sum_{k,l,\sigma} \epsilon_{k,l} \hat{n}_{k\sigma,l}$ and $\vec{s}_{\text{tot}} = \sum_l \vec{s}_l = \sum_{kk'\alpha\alpha',l} \vec{\sigma}_{\alpha\alpha'} c^\dagger_{k\alpha,l} c_{k'\alpha',l}$. l sums over the channels. Henceforth we will drop the $*$. Obtaining the effective Hamiltonian involves obtaining the low energy excitations on top of this fixed point Hamiltonian. The large-energy excitations are ones that involve spin flips. This guides the separation of the Hamiltonian into a diagonal and an off-diagonal piece:

$$H = H_d + V = \underbrace{H_0 + J S_d^z s_{\text{tot}}^z}_{H_d} + \underbrace{\frac{J}{2} S_d^+ s_{\text{tot}}^- + \text{h.c.}}_{V + V^\dagger} \tag{106}$$

We define V as the interaction term that decreases s_{tot}^z by 1: $V |s_{\text{tot}}^z\rangle \to |s_{\text{tot}}^z - 1\rangle$. Similarly, we define $V^\dagger |s_{\text{tot}}^z\rangle \to |s_{\text{tot}}^z + 1\rangle$. The effective Hamiltonian that has the states $|S_d^z, s_{\text{tot}}, s_{\text{tot}}^z\rangle$ as eigenstates are

$$H_{\text{eff}} = H_d + V \frac{1}{E_{\text{gs}} - H_d} V = \sum_{k,l,\sigma} \epsilon_{k,l} \hat{n}_{k\sigma,l} + J S_d^z s_{\text{tot}}^z + \frac{J}{2} S_d^+ s_{\text{tot}}^- \frac{1}{E_{\text{gs}} - J S_d^z s_{\text{tot}}^z - H_0} \frac{J}{2} S_d^- s_{\text{tot}}^+ \tag{107}$$

$$+ \frac{J}{2} S_d^- s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} - J S_d^z s_{\text{tot}}^z - H_0} \frac{J}{2} S_d^+ s_{\text{tot}}^- \tag{108}$$

This is obtained from the Schrodinger equation for the ground state. If we expand the ground state in terms of $|S_d^z, s_{\text{tot}}, s_{\text{tot}}^z\rangle$, we have

$$|\Psi_{\text{gs}}\rangle = \sum_{S_d^z, s_{\text{tot}}, s_{\text{tot}}^z} C_{S_d^z, s_{\text{tot}}, s_{\text{tot}}^z} |S_d^z, s_{\text{tot}}, s_{\text{tot}}^z\rangle \tag{109}$$

The Schrodinger equation for the ground state can be written as

$$E_{\text{gs}} |\Psi_{\text{gs}}\rangle = H |\Psi_{\text{gs}}\rangle = (H_d + V) |\Psi_{\text{gs}}\rangle \implies (E_{\text{gs}} - H_d) \sum C_{S_d^z, s_{\text{tot}}, s_{\text{tot}}^z} |S_d^z, s_{\text{tot}}, s_{\text{tot}}^z\rangle = V \sum C_{S_d^z, s_{\text{tot}}, s_{\text{tot}}^z} |S_d^z, s_{\text{tot}}, s_{\text{tot}}^z\rangle \tag{110}$$

Since V only changes $S_d^z \to -S_d^z$ and $s_{\text{tot}}^z \to s_{\text{tot}}^z \pm 1$, we can simplify the equation into individual smaller equations. Let us take the case of two-channel, where the possible states are $s_{\text{tot}}, s_{\text{tot}}^z = (0,0), (1,-1), (1,0), (1,1)$. The individual equations for this model are

$$(E_{\text{gs}} - H_d) |S_d^z, 0, 0\rangle = (E_{\text{gs}} - H_d) \left|-\frac{1}{2}, 1, -1\right\rangle = (E_{\text{gs}} - H_d) \left|\frac{1}{2}, 1, 1\right\rangle = 0 \tag{111}$$

$$(E_{\text{gs}} - H_d)\, C_{\frac{1}{2},1,-1}\, |\tfrac{1}{2}, 1, -1\rangle = V C_{-\frac{1}{2},1,0}\, |-\tfrac{1}{2}, 1, 0\rangle \tag{112}$$

$$(E_{\text{gs}} - H_d)\, C_{-\frac{1}{2},1,0}\, |-\tfrac{1}{2}, 1, 0\rangle = V^\dagger C_{\frac{1}{2},1,-1}\, |\tfrac{1}{2}, 1, -1\rangle \tag{113}$$

$$(E_{\text{gs}} - H_d)\, C_{\frac{1}{2},1,0}\, |\tfrac{1}{2}, 1, 0\rangle = V C_{-\frac{1}{2},1,1}\, |-\tfrac{1}{2}, 1, 1\rangle \tag{114}$$

$$(E_{\text{gs}} - H_d)\, C_{-\frac{1}{2},1,1}\, |-\tfrac{1}{2}, 1, 1\rangle = V^\dagger C_{\frac{1}{2},1,0}\, |\tfrac{1}{2}, 1, 0\rangle \tag{115}$$

From eqs. 112 and 115, we can write

$$C_{\frac{1}{2},1,-1}\, |\tfrac{1}{2}, 1, -1\rangle = C_{-\frac{1}{2},1,0} \frac{1}{E_{\text{gs}} - H_d} V\, |-\tfrac{1}{2}, 1, 0\rangle, \qquad C_{-\frac{1}{2},1,1}\, |-\tfrac{1}{2}, 1, 1\rangle = C_{\frac{1}{2},1,0} \frac{1}{E_{\text{gs}} - H_d} V^\dagger\, |\tfrac{1}{2}, 1, 0\rangle \tag{116}$$

Substituting these into eqs. 114 and 113 gives

$$E_{\text{gs}}\, |\tfrac{1}{2}, 1, 0\rangle = \left(H_d + V \frac{1}{E_{\text{gs}} - H_d} V^\dagger \right) |\tfrac{1}{2}, 1, 0\rangle \tag{117}$$

$$E_{\text{gs}}\, |-\tfrac{1}{2}, 1, 0\rangle = \left(H_d + V^\dagger \frac{1}{E_{\text{gs}} - H_d} V \right) |-\tfrac{1}{2}, 1, 0\rangle \tag{118}$$

$$\tag{119}$$

These equations represent the Schrodinger equation for the states $|S_d^z, 1, 0\rangle$, and the right hand sides therefore give the effective Hamiltonians for those states. If we combine the states into a single subspace $|1, 0\rangle = \left\{ |\tfrac{1}{2}, 1, 0\rangle, |-\tfrac{1}{2}, 1, 0\rangle \right\}$, the effective Hamiltonian for this composite subspace becomes the sum of the two parts:

$$H_{\text{eff}}\, |1, 0\rangle \langle 1, 0| = \left(H_d + V G_0 V^\dagger + V^\dagger G_0 V \right) |1, 0\rangle \tag{120}$$

where $G_0 = (E_{\text{gs}} - H_d)^{-1}$. If we expand the subspace as $|1, 0\rangle = |\tfrac{1}{2}, 1, 0\rangle + |-\tfrac{1}{2}, 1, 0\rangle$, we recover eqs. 117. Solving similarly for the other states gives

$$H_{\text{eff}}\, |1, 1\rangle \langle 1, 1| = \left(H_d + V^\dagger G_0 V \right) |1, 1\rangle \tag{121}$$

$$H_{\text{eff}}\, |1, -1\rangle \langle 1, -1| = \left(H_d + V G_0 V^\dagger \right) |1, -1\rangle \tag{122}$$

One important conclusion that comes out of these calculations is that if the ground state is degenerate, the effective Hamiltonians is independent of which ground state we choose to start from in Eq. 109. This is because the only difference in the various degenerate ground states is in the coefficients $C_{S_d^z, s_{tot}, s_{tot}^z}$. Since the final effective Hamiltonians are independent of these coefficients, they will be the same irrespective of which ground state we start with.

To calculate these effective Hamiltonians, we will calculate the individual terms. We can easily simplify the S_d^z in the denominator of G_0, because $S_d^\pm \frac{1}{A + B S_d^z} = S_d^\pm \frac{1}{A \mp \frac{1}{2} B}$:

$$V G_0 V^\dagger = \frac{J^2}{4} s_{\text{tot}}^- \frac{\frac{1}{2} + S_d^z}{E_{\text{gs}} + \frac{J}{2} s_{\text{tot}}^z - H_0} s_{\text{tot}}^+ \tag{123}$$

$$V^\dagger G_0 V = \frac{J^2}{4} s_{\text{tot}}^+ \frac{\frac{1}{2} - S_d^z}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z - H_0} s_{\text{tot}}^- \tag{124}$$

Since H_0 does not commute with the spin operators, we will need to expand the denominator to make sense of this Hamiltonian.

$$V G_0 V^\dagger = s_{\text{tot}}^- \frac{1}{E_{\text{gs}} + \frac{J}{2} s_{\text{tot}}^z} \left[1 + \frac{1}{E_{\text{gs}} + \frac{J}{2} s_{\text{tot}}^z} H_0 + \frac{1}{E_{\text{gs}} + \frac{J}{2} s_{\text{tot}}^z} H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2} s_{\text{tot}}^z} H_0 + \ldots \right] s_{\text{tot}}^+ \tag{125}$$

$$V^\dagger G_0 V = s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} \left[1 + \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} H_0 + \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} H_0 \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} H_0 + \ldots \right] s_{\text{tot}}^- \tag{126}$$

This is an expansion in H_0^n / J^{n+1}, $n = 0, 1, 2, \ldots$. The $n = 0$ terms give

$$s_{\text{tot}}^- \frac{1}{E_{\text{gs}} + \frac{J}{2} s_{\text{tot}}^z} s_{\text{tot}}^+ = s_{\text{tot}}^- s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} + \frac{J}{2} (s_{\text{tot}}^z + 1)} \tag{127}$$

$$s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} - \frac{J}{2}s_{\text{tot}}^z} s_{\text{tot}}^- = s_{\text{tot}}^+ s_{\text{tot}}^- \frac{1}{E_{\text{gs}} - \frac{J}{2}\left(s_{\text{tot}}^z - 1\right)} \tag{128}$$

One of the $n = 1$ terms gives

$$s_{\text{tot}}^- \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} H_0 s_{\text{tot}}^+ = \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)}\right)^2 s_{\text{tot}}^- H_0 s_{\text{tot}}^+ \tag{129}$$

Next we calculate the commutator:

$$\left[s_{\text{tot}}^+, H_0\right] = X_{1,\text{tot}}^\dagger = \sum_l X_{1,l}^\dagger = \sum_{kk',l} \left(\epsilon_k - \epsilon_{k'}\right) c_{k'\uparrow}^\dagger c_{k\downarrow} \tag{130}$$

where $X_{n,l} \equiv \sum_{k,k'} \left(\epsilon_k - \epsilon_{k'}\right)^n c_{k\downarrow}^\dagger c_{k'\uparrow}$. Substituting this commutator gives

$$s_{\text{tot}}^- \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} H_0 s_{\text{tot}}^+ = \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)}\right)^2 \left(s_{\text{tot}}^- s_{\text{tot}}^+ H_0 - s_{\text{tot}}^- X_{1,\text{tot}}^\dagger\right) \tag{131}$$

The other $n = 1$ term gives

$$s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} - \frac{J}{2}s_{\text{tot}}^z} \frac{1}{E_{\text{gs}} - \frac{J}{2}s_{\text{tot}}^z} H_0 s_{\text{tot}}^- = \left(\frac{1}{E_{\text{gs}} - \frac{J}{2}\left(s_{\text{tot}}^z - 1\right)}\right)^2 \left(s_{\text{tot}}^+ s_{\text{tot}}^- H_0 + s_{\text{tot}}^+ X_{1,\text{tot}}\right) \tag{132}$$

One of the $n = 2$ terms gives

$$s_{\text{tot}}^- \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} H_0 s_{\text{tot}}^+ \tag{133}$$

$$= \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)}\right)^2 s_{\text{tot}}^- H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} H_0 s_{\text{tot}}^+ \tag{134}$$

$$= \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)}\right)^2 s_{\text{tot}}^- H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} \left(s_{\text{tot}}^+ H_0 - X_{1,\text{tot}}^\dagger\right) \tag{135}$$

$$= \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)}\right)^2 \left[s_{\text{tot}}^- H_0 s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)} H_0 - s_{\text{tot}}^- H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} X_{1,\text{tot}}^\dagger\right] \tag{136}$$

$$= \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)}\right)^2 \left[s_{\text{tot}}^- \left(s_{\text{tot}}^+ H_0 - X_{1,\text{tot}}^\dagger\right) \frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)} H_0 - s_{\text{tot}}^- H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} X_{1,\text{tot}}^\dagger\right] \tag{137}$$

$$= \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)}\right)^2 \left[s_{\text{tot}}^- s_{\text{tot}}^+ H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)} H_0 - s_{\text{tot}}^- \left(X_{1,\text{tot}}^\dagger \frac{1}{E_{\text{gs}} + \frac{J}{2}\left(s_{\text{tot}}^z + 1\right)} H_0 + H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} X_{1,\text{tot}}^\dagger\right)\right] \tag{138}$$

$$\tag{139}$$

At this point, we need the commutator between H_0 and s_{tot}^z:

$$\left[H_0, s_{\text{tot}}^z\right] = Z_{1,\text{tot}} = \sum_l Z_{1,l} = \sum_{k,k',l} \left(\epsilon_k - \epsilon_{k'}\right) \frac{1}{2} \left(c_{k\uparrow,l}^\dagger c_{k'\uparrow,l} - c_{k\downarrow,l}^\dagger c_{k'\downarrow,l}\right) \tag{140}$$

This gives the relation

$$H_0(a + bs_{\text{tot}}^z) = (a + bs_{\text{tot}}^z)H_0 + Z_{1,\text{tot}} \implies \frac{1}{a + bs_{\text{tot}}^z} H_0 = H_0 \frac{1}{a + bs_{\text{tot}}^z} + \frac{1}{a + bs_{\text{tot}}^z} Z_{1,\text{tot}} \frac{1}{a + bs_{\text{tot}}^z} \tag{141}$$

Using this, we get

$$s_{\text{tot}}^- \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}s_{\text{tot}}^z} H_0 s_{\text{tot}}^+ \tag{142}$$

$$= \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)}\right)^2 \left[s_{\text{tot}}^- s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)} H_0 H_0 - s_{\text{tot}}^- s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)} Z_{1,\text{tot}} \frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)} H_0 \right. \tag{143}$$

$$\left. - s_{\text{tot}}^- \left(X_{1,\text{tot}}^\dagger \frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)} H_0 + H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2} s_{\text{tot}}^z} X_{1,\text{tot}}^\dagger\right)\right] \tag{144}$$

$$= \left(\frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)}\right)^2 s_{\text{tot}}^- s_{\text{tot}}^+ \left[\frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)} H_0 H_0 - \frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)} Z_{1,\text{tot}} H_0 \frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)}\right] \tag{145}$$

At the last step, we dropped the three-particle scattering terms.

The other $n = 2$ term gives

$$s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} H_0 \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} H_0 s_{\text{tot}}^- \tag{146}$$

$$= \left(\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)}\right)^2 s_{\text{tot}}^+ H_0 \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} H_0 s_{\text{tot}}^- \tag{147}$$

$$= \left(\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)}\right)^2 s_{\text{tot}}^+ H_0 \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} \left(s_{\text{tot}}^- H_0 + X_{1,\text{tot}}\right) \tag{148}$$

$$= \left(\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)}\right)^2 \left[s_{\text{tot}}^+ H_0 s_{\text{tot}}^- \frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} H_0 + s_{\text{tot}}^+ H_0 \frac{1}{E_{\text{gs}} - \frac{J}{2} s_{\text{tot}}^z} X_{1,\text{tot}}\right] \tag{149}$$

$$= \left(\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)}\right)^2 s_{\text{tot}}^+ \left(s_{\text{tot}}^- H_0 + X_{1,\text{tot}}\right) \frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} H_0 \tag{150}$$

$$= \left(\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)}\right)^2 \left[s_{\text{tot}}^+ s_{\text{tot}}^- H_0 \frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} H_0\right] \tag{151}$$

$$= \left(\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)}\right)^2 s_{\text{tot}}^+ s_{\text{tot}}^- \left[\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} H_0^2 - \frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} Z_{1,\text{tot}} \frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} H_0\right] \tag{152}$$

$$= \left(\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)}\right)^2 s_{\text{tot}}^+ s_{\text{tot}}^- \left[H_0^2 \frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} - \frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} Z_{1,\text{tot}} H_0 \frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)}\right] \tag{153}$$

If we look at the effective Hamiltonian for a subspace $|s_{\text{tot}}, s_{\text{tot}}^z\rangle$, we can replace the following operators with scalars:

$$\frac{1}{E_{\text{gs}} + \frac{J}{2}(s_{\text{tot}}^z + 1)} = \gamma_{s_{\text{tot}}}^{s_{\text{tot}}^z} \tag{154}$$

$$\frac{1}{E_{\text{gs}} - \frac{J}{2}(s_{\text{tot}}^z - 1)} = \gamma_{s_{\text{tot}}}^{-s_{\text{tot}}^z}, \tag{155}$$

$$s_{\text{tot}}^- s_{\text{tot}}^+ = s_{\text{tot}}(s_{\text{tot}} + 1) - s_{\text{tot}}^z(s_{\text{tot}}^z + 1) = \chi_{s_{\text{tot}}}^{s_{\text{tot}}^z} \tag{156}$$

$$s_{\text{tot}}^+ s_{\text{tot}}^- = s_{\text{tot}}(s_{\text{tot}} + 1) - s_{\text{tot}}^z(s_{\text{tot}}^z - 1) = \chi_{s_{\text{tot}}}^{-s_{\text{tot}}^z} \tag{157}$$

They are not all independent: $\left(\gamma_{s_{\text{tot}}}^{s_{\text{tot}}^z}\right)^{-1} + \left(\gamma_{s_{\text{tot}}}^{-s_{\text{tot}}^z}\right)^{-1} = E_{\text{gs}} + J, \chi_{s_{\text{tot}}}^{s_{\text{tot}}^z} - \chi_{s_{\text{tot}}}^{-s_{\text{tot}}^z} = -2s_{\text{tot}}^z$. For the two-channel problem, we have four possible states in total: $(s_{\text{tot}}, s_{\text{tot}}^z) = \{(0,0), (1,-1), (1,0), (1,1)\}$. The states in the Eq. 111 cannot be acted on by V or $V^\dagger$, so the effective Hamiltonian for these states will consist of only the diagonal part:

$$H_{\text{eff}} = H_0 + \frac{J}{2} s_{\text{tot}}^z \tag{158}$$

The factors γ and χ for the other states are

$$s_{\text{tot}}, s_{\text{tot}}^z = \qquad\qquad (1, -1), \qquad\qquad\qquad (1, 0), \qquad\qquad (1, 1) \tag{159}$$

$$\gamma_{s_{\text{tot}}}^{s_{\text{tot}}^z} = \qquad\qquad \frac{1}{E_{\text{gs}}}, \qquad\qquad \frac{1}{E_{\text{gs}} + J/2}, \qquad\qquad -- \tag{160}$$

$$\chi_{s_{\text{tot}}}^{s_{\text{tot}}^z} = \qquad\qquad 2, \qquad\qquad 2, \qquad\qquad -- \tag{161}$$

$$\gamma_{s_{\text{tot}}}^{-s_{\text{tot}}^z} = \qquad\qquad --, \qquad\qquad \frac{1}{E_{\text{gs}} + J/2}, \qquad\qquad \frac{1}{E_{\text{gs}}} \tag{162}$$

$$\chi_{s_{\text{tot}}}^{-s_{\text{tot}}^z} = \qquad\qquad --, \qquad\qquad 2, \qquad\qquad 2 \tag{163}$$

$$\tag{164}$$

The effective Hamiltonians for these states are:

$$H_{\text{eff}}^{1,1} = H_0 + J S_d^z + \frac{J^2}{4} \frac{2}{E_{\text{gs}}} \left[1 + \frac{H_0}{E_{\text{gs}}} + \frac{s_{\text{tot}}^+ X_{1,\text{tot}}}{2 E_{\text{gs}}} + \frac{H_0^2}{E_{\text{gs}}^2} - \frac{Z_{1,\text{tot}} H_0}{E_{\text{gs}}^3} \right] \left(\frac{1}{2} - S_d^z \right) \tag{165}$$

$$H_{\text{eff}}^{1,-1} = H_0 - J S_d^z + \frac{J^2}{4} \frac{2}{E_{\text{gs}}} \left[1 + \frac{H_0}{E_{\text{gs}}} - \frac{s_{\text{tot}}^- X_{1,\text{tot}}^\dagger}{2 E_{\text{gs}}} + \frac{H_0^2}{E_{\text{gs}}^2} - \frac{Z_{1,\text{tot}} H_0}{E_{\text{gs}}^3} \right] \left(\frac{1}{2} + S_d^z \right) \tag{166}$$

$$H_{\text{eff}}^{1,0} = H_0 + \frac{J^2}{2\left(E_{\text{gs}} + J/2\right)} \left[1 + \frac{H_0 + (1/2 + S_d^z) s_{\text{tot}}^+ X_{1,\text{tot}} - (1/2 - S_d^z) s_{\text{tot}}^- X_{1,\text{tot}}^\dagger}{2\left(E_{\text{gs}} + J/2\right)} + \frac{H_0^2}{\left(E_{\text{gs}} + J/2\right)^2} - \frac{Z_{1,\text{tot}} H_0}{\left(E_{\text{gs}} + J/2\right)^3} \right] \tag{167}$$

$$\tag{168}$$

The terms have the following meanings:

$$H_0 = \sum_{k,\sigma,l} \epsilon_k^l \hat{n}_{k,\sigma,l} \tag{169}$$

$$H_0^2 = \sum_{k_1,k_2,l_1,l_2,\sigma_1,\sigma_2} \epsilon_k^{l_1} \epsilon_{k_2}^{l_2} \hat{n}_{k_1,\sigma_1,l_1} \hat{n}_{k_2,\sigma_2,l_2} \tag{170}$$

$$s_{\text{tot}}^+ X_{1,\text{tot}} = \sum_{k_1,q_1,k_2,q_2,l_1,l_2} c_{k_1,\uparrow,l_1}^\dagger c_{q_1,\downarrow,l_1} \left(\epsilon_{k_2}^{l_2} - \epsilon_{q_2}^{l_2} \right) c_{k_2,\downarrow,l_2}^\dagger c_{q_2,\uparrow,l_2} \tag{171}$$

$$s_{\text{tot}}^- X_{1,\text{tot}}^\dagger = \sum_{k_1,q_1,k_2,q_2,l_1,l_2} c_{k_1,\downarrow,l_1}^\dagger c_{q_1,\uparrow,l_1} \left(\epsilon_{k_2}^{l_2} - \epsilon_{q_2}^{l_2} \right) c_{q_2,\uparrow,l_2}^\dagger c_{k_2,\downarrow,l_2} \tag{172}$$

$$Z_{1,\text{tot}} H_0 = \sum_{k_1,q_1,k_2,\sigma_2,l_1,l_2} \left(\epsilon_{k_1}^{l_1} - \epsilon_{q_1}^{l_1} \right) \epsilon_{k_2}^{l_2} \left(c_{k_1,\uparrow,l_1}^\dagger c_{q_1,\uparrow,l_1} - c_{k_1,\downarrow,l_1}^\dagger c_{q_1,\downarrow,l_1} \right) \hat{n}_{k_2,\sigma_2,l_2} \tag{173}$$

$$\tag{174}$$

XI.   IMPURITY SUSCEPTIBILITY FROM ZERO-MODE FIXED POINT HAMILTONIAN

The zero-mode approximation of the fixed point Hamiltonian is a star graph Hamiltonian:

$$H = J^* \vec{S}_d \cdot \vec{s}_{\text{tot}} \tag{175}$$

where $\vec{s}_{\text{tot}} = \sum_l \vec{s}_l$ is the total spin operator for all the channels. We insert a magnetic field that acts only on the impurity and then attempt to diagonalize the Hamiltonian.

$$H(h) = J^* \vec{S}_d \cdot \vec{s}_{\text{tot}} + h S_d^z \tag{176}$$

The Hamiltonian commutes with s_{tot}^2:

$$\left[s_{\text{tot}}^2, H(h) \right] = \left[\sum_{i=x,y,z} s_{\text{tot}}^{i\ 2}, J^* \sum_{i=x,y,z} S_d^i s_{\text{tot}}^i \right] = \sum_{i,j} J^* S_d^i \left\{ s_{\text{tot}}^i, \left[s_{\text{tot}}^i, s_{\text{tot}}^j \right] \right\} = \sum_{i,j} J^* S_d^i \left\{ s_{\text{tot}}^i, i\epsilon^{ijk} s_{\text{tot}}^k \right\} = 0 \tag{177}$$

This means the Hamiltonian is already block-diagonal in the quantum number s_{tot}. Let us represent the quantum number of s_{tot}^z by m. For a particular s_{tot}, m can take values from the set $[-s_{\text{tot}}, s_{\text{tot}}]$. The spin S_d^z can also take

values $\pm\frac{1}{2}$. From now on, we will assume we are in the subspace of a particular $s_{\text{tot}} = M$, so we will ignore that quantum number and write the kets simply as $|S_d^z, m\rangle$. So, the notation $|\uparrow, -1\rangle$ means the state with $S_d^z = \frac{1}{2}$ and $m = -1$. We will now show that even inside the block of $2 \times s_{\text{tot}}$ (or $2 \times s_{\text{tot}} + 1$, depending on where it is odd or even) defined by a particular value of s_{tot}, the Hamiltonian actually separates into decoupled 2×2 blocks. To see why, first note that the terminal states $|\downarrow, -M\rangle$ and $|\uparrow, M\rangle$ are already eigenstates, because they cannot scatter (the impurity can only flip down, and this would require the bath to flip up, but s_{tot}^z is already at its maximum value M). The other $2M - 2$ states can be organized into 2×2 blocks formed by the states $|\uparrow, m\rangle$ and $|\downarrow, m+1\rangle$ for $m \in [-M, M-1]$. The fact that this block does not interact with the other blocks can be observation: if there was some other state which when acted upon by the Hamiltonian gave a non-zero projection on $|\uparrow, m\rangle$, it would have to come from $S_d^z = \downarrow$, and this would mean the bath spin would have had to flip down. This means the bath spin in that state would have to be $m + 1$, and that is precisely the other state in the block.

Defining $\epsilon_m^h = \frac{1}{2}(Jm + h)$ and $x_m^M = M(M+1) - m(m+1)$, the 2×2 blocks can be written as

$$H_m = \begin{pmatrix} \epsilon_m^h & \frac{J}{2}\sqrt{x_m^M} \\ \frac{J}{2}\sqrt{x_m^M} & -\left(\epsilon_m^h + J/2\right) \end{pmatrix} \tag{178}$$

The eigenvalues are

$$\lambda_{m,\pm}^{M,h} = \frac{1}{2}\left[-J/2 \pm \sqrt{J^2/4 + J^2 x_m^M + 4\epsilon_m^h\left(\epsilon_m^h + J/2\right)}\right] = -J/4 \pm \sqrt{J^2 x_m^M/4 + \alpha^2} \tag{179}$$

where $\alpha = \epsilon_m^h + J/4$. The eigenvalues of the terminal states are $\pm\epsilon_{\pm M}^h$. For $h = 0$, the ground state subspace is K−fold degenerate and is formed by the negative solutions of Eq. 179. This common K−fold degenerate eigenvalue is $-J(M+1)/2$. The full list of energy eigenvalues at a particular value of M is

$$JM/2 - h/2, \qquad\qquad |\downarrow, M, -M\rangle, \qquad\qquad m = -M \tag{180}$$

$$-J/4 \pm \sqrt{J^2 x_m^M/4 + \alpha^2}, \qquad \{|\uparrow, M, m\rangle, |\downarrow, M, m+1\rangle\}, \qquad m = -M, \ldots, M-1 \tag{181}$$

$$JM/2 + h/2, \qquad\qquad |\uparrow, M, M\rangle, \qquad\qquad m = M \tag{182}$$

$$\tag{183}$$

The eigenstates for each value of M, m are given by

$$\left(\pm\frac{1}{2}\sqrt{J^2 x_m^{M2} + 4\alpha^2} - \frac{J}{2}\sqrt{x_m^M}\right)|\uparrow, M, m\rangle + \left(\frac{J}{2}(m + \frac{1}{2}) + \frac{h}{2}\right)|\downarrow, M, m+1\rangle \tag{184}$$

The partition function is

$$Z(h) = \sum_{M=M_{\min}}^{M_{\max}}\left[\sum_{\substack{m=-M, \\ m\in\mathbb{Z}}}^{M-1}\left(e^{-\beta\lambda_{m,+}^{M,h}} + e^{-\beta\lambda_{m,-}^{M,h}}\right) + e^{-\beta\epsilon_M^h} + e^{\beta\epsilon_{-M}^h}\right] \tag{185}$$

$$= \sum_{M=M_{\min}}^{M_{\max}}\left[\sum_{\substack{m=-M, \\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4}\cosh\beta\sqrt{J^2 x_m^M/4 + \alpha^2} + 2e^{-\beta JM/2}\cosh\beta h/2\right] \tag{186}$$

where $M_{\max} = K/2$ for a K−channel Kondo model, and $M_{\min} = 0$ if K is even, otherwise $1/2$. This is yet not the complete partition function, because we have not accounted for the possibility that there multiple subspaces of M. For example, the $K = 3$ case states can be obtained by adding the third spin-half onto the states $S = 0, 1$. $S = 0$ gives $s_{\text{tot}} = 1/2$ and $S = 1$ gives $s_{\text{tot}} = 1/2, 3/2$. So, $s_{\text{tot}} = 1/2$ appears twice. These two subspaces are actually orthogonal, because the quantum numbers for the individual channels are different. We need to count the number of instances of a particular subspace $s_{\text{tot}} = M$. It turns out that this number is given by

$$r_M^K = {}^{K-1}C_{K/2-M} \tag{187}$$

which means the correct partition function is

$$Z(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K\left[\sum_{\substack{m=-M, \\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4}\cosh\beta\sqrt{J^2 x_m^M/4 + \alpha^2} + 2e^{-\beta JM/2}\cosh\beta h/2\right] \tag{188}$$

To calculate the impurity magnetic susceptibility, we will use the expression

$$\chi = \frac{1}{\beta} \lim_{h \to 0} \left[\frac{Z(h)''}{Z(h)} - \left(\frac{Z(h)'}{Z(h)} \right)^2 \right] \tag{189}$$

where the $\prime$ indicates derivative with respect to h. We will now calculate these derivatives. For that we will need

$$\frac{d\epsilon_m^h}{dh} = \frac{1}{2} = \frac{d\alpha}{dh} \tag{190}$$

$$\frac{d\lambda_{m,\pm}^{M,h}}{dh} = \pm \frac{\alpha}{\sqrt{J^2 + 4\alpha^2}} \tag{191}$$

We are now ready to compute the derivatives of Z:

$$\frac{dZ(h)}{dh} = \beta \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} e^{\beta J/4} \alpha \frac{\sinh \beta \sqrt{J^2 x_m^M/4 + \alpha^2}}{\sqrt{J^2 x_m^M/4 + \alpha^2}} + e^{-\beta JM/2} \sinh (\beta h/2) \right] \tag{192}$$

$$\frac{d^2 Z(h)}{dh^2} = \beta \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\frac{1}{2} \sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} \frac{e^{\beta J/4}}{\sqrt{J^2 x_m^M/4 + \alpha^2}} \left(\sinh \beta \sqrt{J^2 x_m^M/4 + \alpha^2} + \beta \alpha^2 \frac{\cosh \beta \sqrt{J^2 x_m^M/4 + \alpha^2}}{\sqrt{J^2 x_m^M/4 + \alpha^2}} \right. \right. \tag{193}$$

$$\left. \left. - \frac{\alpha^2 \sinh \beta \sqrt{J^2 x_m^M/4 + \alpha^2}}{J^2 x_m^M/4 + \alpha^2} \right) + e^{-\beta JM/2} \frac{\beta}{2} \cosh (\beta h/2) \right] \tag{194}$$

$$\tag{195}$$

We will now take the limit of $h \to 0$. Note that $\alpha(h \to 0) = \frac{J}{2}(m + 1/2)$ and hence

$$(J^2 x_m^M/4 + \alpha^2)(h \to 0) = \frac{J^2}{4} \left[M(M+1) - m(m+1) + (m+1/2)^2 \right] = \frac{J^2}{4} (M+1/2)^2 \tag{196}$$

For brevity, we define $\theta_M = \beta J(M+1/2)/2$ and $\Sigma_M = \sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} (m+1/2)^2$. It can be shown that this summation, for $M = M_{\max} = K/2$, evaluates to $\Sigma_{\max} = K(K+1)(K-1)/12$.

$$\lim_{h \to 0} Z(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} 2e^{\beta J/4} \cosh \beta \frac{J}{2} (M+1/2) + 2e^{-\beta JM/2} \right] = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[4M e^{\beta J/4} \cosh \theta_M + 2e^{-\beta JM/2} \right] \tag{197}$$

$$\lim_{h \to 0} \frac{dZ(h)}{dh} = \beta \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} e^{\beta J/4} (m+1/2) \frac{\sinh \beta \frac{J}{2} (M+1/2)}{(M+1/2)} \right] = 0 \tag{198}$$

$$\lim_{h \to 0} \frac{d^2 Z(h)}{dh^2} = \frac{\beta^2}{2} \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\frac{e^{\beta J/4}}{\theta_M} \left(2M \sinh \theta_M + \frac{\beta^2 J^2}{4} \left[\frac{\cosh \theta_M}{\theta_M} - \frac{\sinh \theta_M}{\theta_M^2} \right] \Sigma_M \right) + e^{-\beta JM/2} \right] \tag{199}$$

$$\tag{200}$$

These expressions have been used to compute the impurity susceptibility for various values of K in fig. 8.

Since the final expressions are formidable, we write down the expressions specifically for the single-channel and two-channel models. For single-channel, we have $M = \frac{1}{2}$ and $m = \pm\frac{1}{2}$. The terminal states are $S_d^z = -1/2, m = -1/2$ and $S_d^z = 1/2, m = 1/2$. There is therefore just one 2×2 block, and that is at $m = -1/2$.

$$\lim_{h \to 0} Z(h) = 2e^{\beta J/4} \cosh \beta \frac{J}{2} + 2e^{-\beta J/4} \tag{201}$$

$$\lim_{h \to 0} \frac{dZ(h)}{dh} = 0 \tag{202}$$

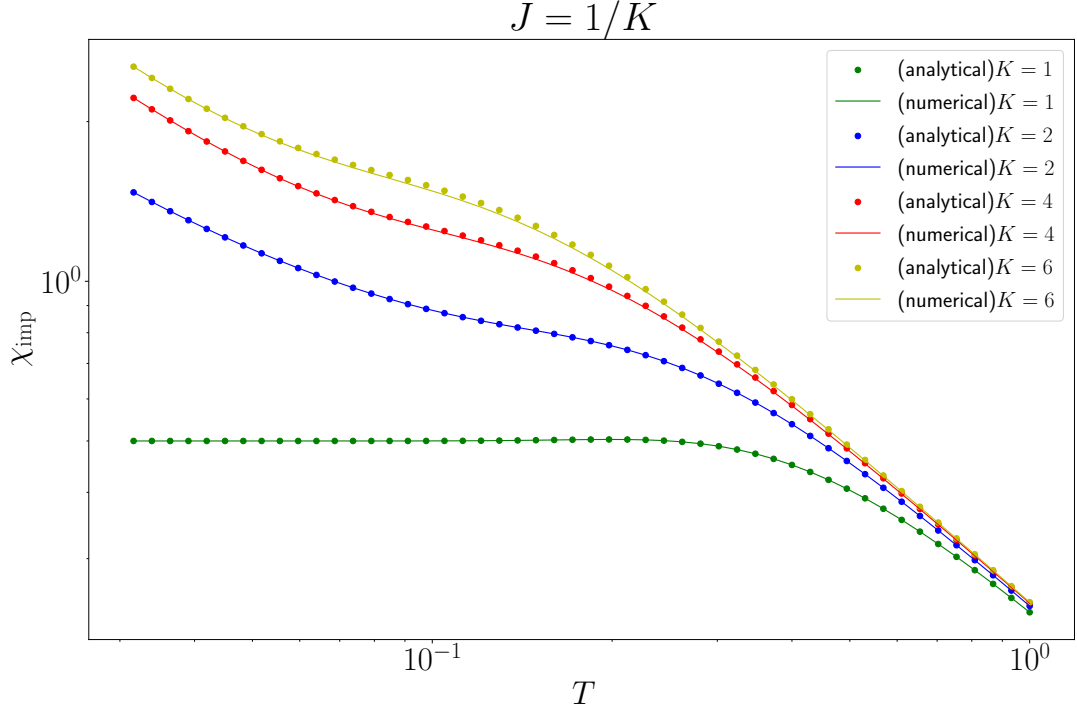FIG. 8. Impurity susceptibility for $K = 1, 2, 4, 6$, calculated numerically as well as using the analytical expressions.

$$\lim_{h \to 0} \frac{\mathrm{d}^2 Z(h)}{\mathrm{d}h^2} = \frac{\beta}{J} \left(e^{\beta J/4} \sinh \beta \frac{J}{2} + e^{-\beta J/4} \frac{J\beta}{2} \right) \tag{203}$$

$$\chi = \frac{1}{\beta} \lim_{h \to 0} \left[\frac{Z(h)''}{Z(h)} - \left(\frac{Z(h)'}{Z(h)} \right)^2 \right] = \frac{1}{J} \frac{\left(2e^{\beta J/2} \sinh \beta \frac{J}{2} + J\beta \right)}{4e^{\beta J/2} \cosh \beta \frac{J}{2} + 4} \tag{204}$$

This expression matches with the direct calculation of the susceptibility of the single-channel Kondo model.

At low temperature $\beta \to \infty$, only the highest value $M_{\max}$ will survive:

$$Z \to 2r_{M_{\max}}^K M_{\max} e^{\beta \frac{J}{2}(M_{\max}+1)} \tag{205}$$

$$Z'' \to r_{M_{\max}}^K \left(\frac{\beta}{2(M_{\max}+1/2)} \right)^2 e^{\beta \frac{J}{2}(M_{\max}+1)} \Sigma_{\max} \tag{206}$$

$$\chi \to \frac{\beta \Sigma_{\max}}{2M_{\max} (2M_{\max}+1)^2} = \frac{\beta K(K+1)(K-1)/12}{K(K+1)^2} = \frac{\beta(K-1)}{12(K+1)} \tag{207}$$

At high temperatures $\beta \to 0$, we get

$$Z \to \sum_{M=M_{\min}}^{M_{\max}} r_M^K [4M+2] \tag{208}$$

$$Z'' \to \frac{\beta^2}{2} \sum_{M=M_{\min}}^{M_{\max}} r_M^K [2M+1] \tag{209}$$

$$\chi \to 1/4 \tag{210}$$

XII.   BATH SUSCEPTIBILITY FROM ZERO-MODE FIXED POINT HAMILTONIAN

We insert a magnetic field that acts only on the bath and then attempt to diagonalize the Hamiltonian.

$$H(h) = J^* \vec{S}_d \cdot \vec{s}_{\mathrm{tot}} + h s_{\mathrm{tot}}^z \tag{211}$$

Defining $x_m^M = M(M+1) - m(m+1)$, the 2×2 blocks can be written as

$$H_m = \begin{pmatrix} m(J/2 + h) & J\sqrt{x_m^M}/2 \\ J\sqrt{x_m^M}/2 & (m+1)(h - J/2) \end{pmatrix} \tag{212}$$

The eigenvalues are

$$\lambda_{m,\pm}^{M,h} = -J/4 + (2m+1)h/2 \pm \frac{1}{2}\sqrt{J^2 x_m^M + [(2m+1)h - J/2]^2 - 4m(m+1)(h^2 - J^2/4)} \tag{213}$$

$$= -J/4 + (2m+1)h/2 \pm \frac{1}{2}\sqrt{J^2(M+1/2)^2 + h^2 - (2m+1)Jh} = -J/4 + (2m+1)h/2 \pm \phi_m^M \tag{214}$$

The eigenvalues of the terminal states are $JM/2 \pm hM$. The partition function is

$$Z(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} \left(e^{-\beta\lambda_{m,+}^{M,h}} + e^{-\beta\lambda_{m,-}^{M,h}} \right) + e^{-\beta JM/2} \left(e^{\beta hM} + e^{-\beta hM} \right) \right] \tag{215}$$

$$= \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} 2e^{\beta(J/4 - (m+1/2)h)} \cosh\beta\phi_m^M + 2e^{-\beta JM/2} \cosh\beta Mh \right] \tag{216}$$

We will now take the derivatives.

$$Z'(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} 2e^{\beta(J/4 - (m+1/2)h)} \left(-\beta(m+1/2)\cosh\beta\phi_m^M + \beta\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\sinh\beta\phi_m^M \right) + 2\beta M e^{-\beta JM/2}\sinh\beta Mh \right] \tag{217}$$

$$Z''(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} 2e^{\beta(J/4 - (m+1/2)h)} \left(\beta^2(m+1/2)^2\cosh\beta\phi_m^M - 2\beta^2(m+1/2)\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\sinh\beta\phi_m^M \right. \right. \tag{218}$$

$$\left. \left. + \beta\frac{\mathrm{d}^2\phi_m^M}{\mathrm{d}h^2}\sinh\beta\phi_m^M + \beta^2\left(\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\right)^2\cosh\beta\phi_m^M \right) + 2\beta^2 M^2 e^{-\beta JM/2}\cosh\beta Mh \right] \tag{219}$$

$$\tag{220}$$

In the limit of $h \to 0$, we have

$$Z(h \to 0) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} 2e^{\beta J/4}\cosh\beta\phi_m^M + 2e^{-\beta JM/2} \right] \tag{221}$$

$$Z'(h \to 0) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} 2e^{\beta J/4}\left(-\beta(m+1/2)\cosh\beta\phi_m^M + \beta\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\sinh\beta\phi_m^M \right) \right] \tag{222}$$

$$Z''(h \to 0) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} 2e^{\beta J/4}\left(\beta^2(m+1/2)^2\cosh\beta\phi_m^M - 2\beta^2(m+1/2)\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\sinh\beta\phi_m^M \right. \right. \tag{223}$$

$$\left. \left. + \beta\frac{\mathrm{d}^2\phi_m^M}{\mathrm{d}h^2}\sinh\beta\phi_m^M + \beta^2\left(\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\right)^2\cosh\beta\phi_m^M \right) + 2\beta^2 M^2 e^{-\beta JM/2} \right] \tag{224}$$

$$\tag{225}$$

The ϕ and the derivatives are actually at $h \to 0$. We are interested in the low temperature behaviour. In the limit of $h \to 0$, $\phi_m^M \to \phi^M = J(M + 1/2)/2$. We can also look at the behaviour of the derivative:

$$\lim_{h \to 0} \frac{\mathrm{d}\phi_m^M}{\mathrm{d}h} = -\frac{J(m+1/2)}{4\phi^M} \tag{226}$$

$$\lim_{h \to 0} \frac{d^2\phi_m^M}{dh^2} = \frac{1}{4\phi^M} - \frac{J^2(m+1/2)^2}{16\left(\phi^M\right)^3} \tag{227}$$

$$\tag{228}$$

Substituting these gives

$$Z(h \to 0) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} 2e^{\beta J/4} \cosh \beta \phi^M + 2e^{-\beta JM/2} \right] = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[4Me^{\beta J/4} \cosh \beta \phi^M + 2e^{-\beta JM/2} \right] \tag{229}$$

$$Z'(h \to 0) = -\beta \sum_{M=M_{\min}}^{M_{\max}} r_M^K 2e^{\beta J/4} \left(\cosh \beta \phi^M + \frac{J}{4\phi^M} \sinh \beta \phi^M \right) \sum_{\substack{m=-M, \\ m \in \mathbb{Z}}}^{M-1} (m+1/2) = 0 \tag{230}$$

$$Z''(h \to 0) = \beta^2 \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[2e^{\beta J/4} \left\{ \left(1 + \frac{J^2}{16\phi^{M2}} \right) \Sigma_M \cosh \beta \phi^M + \left(\frac{2M}{4\beta\phi^M} + \frac{J}{2\phi^M} \left(1 - \frac{J}{16\beta\phi^{M2}} \right) \Sigma_M \right) \sinh \beta \phi^M \right\} \tag{231}$$

$$+ 2M^2 e^{-\beta JM/2} \right] \tag{232}$$

$$\tag{233}$$

where $\Sigma_M = \sum_m (m+1/2)^2$.

If we take the limit of $\beta \to \infty$, the hyperbolic functions can be replaced by exponentials. The terms with $1/\beta$ in Z'' drop out. All negative exponentials will also drop out. Moreover, out of all the positve exponentials, only the largest exponent will survive. The largest value Φ of ϕ^M occurs at $M = M_{\max} = K/2$. This maximum value is $\Phi = J(K+1)/4$. We therefore have

$$Z(h \to 0) = r_{K/2}^K 4Me^{\beta J/4} \frac{1}{2} e^{\beta\Phi} \tag{234}$$

$$Z''(h \to 0) = \beta^2 r_{K/2}^K 2e^{\beta J/4} \frac{1}{2} e^{\beta\Phi} \left(1 + \frac{J}{4\Phi} \right)^2 \Sigma_{\max} \tag{235}$$

$$\tag{236}$$

The susceptibility at low temperatures becomes

$$\chi(T \to 0) = \frac{2\beta \left(1 + \frac{J}{4\Phi} \right)^2 \Sigma_M}{4M} = \frac{\beta(K-1)(K+2)^2}{12(K+1)} \tag{237}$$

XIII. NON-ANALYTICITY IN THE FREE ENERGY

For $K > 1$, the Gibbs free energy at $T = 0$ becomes non-analytic under insertion of a magnetic field on the impurity. The thermal free energy is given by

$$F(h) = -\frac{1}{\beta} \ln Z(h) = -\frac{1}{\beta} \ln \sum_{E_n} e^{-\beta E_n} \tag{238}$$

At $T \to 0$, only the most negative energy $E_{\min}$ survives. Assuming a d'_{gs}−fold degenerate ground state for $h \neq 0$, the zero temperature free energy becomes

$$F(h \neq 0, T \to 0) = -\frac{1}{\beta} \ln d'_{gs} e^{-\beta E_{\min}} = E_{\min} - k_B T \ln d'_{gs} \tag{239}$$

In the star graph Hamiltonian with K−channels and a global magnetic field on the impurity

$$H = J\vec{S}_d \cdot \vec{s}_{tot} + h \left(S_d^z + s_{tot}^z \right) \; , \tag{240}$$

the energy eigenvalues for a particular value of $s_{\text{tot}} = M$ are given by

$$(a) \quad \frac{J}{2}M \pm h\left(M + \frac{1}{2}\right) \tag{241}$$

$$(b) \quad -\frac{J}{4} \pm \frac{J}{2}\left(M + \frac{1}{2}\right) + h\left(m + \frac{1}{2}\right), m \in [-M, M-1] \tag{242}$$

For small h, the ground state will be $-\frac{J}{4} - \frac{J}{2}\left(M + \frac{1}{2}\right) + h\left(m_{\text{min}} + \frac{1}{2}\right)$ for a particular value m_{min} that minimizes this energy. This specific value will depend on the sign of h:

$$m_{\text{min}} = \begin{cases} -M, & h > 0 \\ M - 1, & h < 0 \end{cases} \tag{243}$$

which means $E_{\text{min}} = -\frac{J}{2}(M+1) - |h|\left(M - \frac{1}{2}\right)$. We also know, from Eq. 187, that $d'_{\text{gs}} = {}^{K-1}C_{K/2-K/2} = 1$. The free energy for a non-zero field is therefore

$$F(h \neq 0, T \to 0) = -\frac{J}{2}(M+1) - |h|\left(M - \frac{1}{2}\right) \tag{244}$$

The first derivative of the free energy with respect to the field gives

$$F'(h \neq 0, T \to 0) = -\text{sign}(h)\left(M - \frac{1}{2}\right) \tag{245}$$

There we used the result that the derivative of $|x|$ is $\text{sign}(x)$. If we now take h to zero from both directions, we get

$$F'(h \to 0^{\pm}, T \to 0) = \mp\left(M - \frac{1}{2}\right) = \mp\frac{1}{2}(K - 1) \tag{246}$$

The first derivative of the impurity free energy is therefore discontinuous as $h \to 0$; it goes to different values depending on the direction in which we take the limit. The only case where it is not analytic is when $K = 1$; then the derivative goes to zero from both directions. This non-analyticity has also been verified numerically.
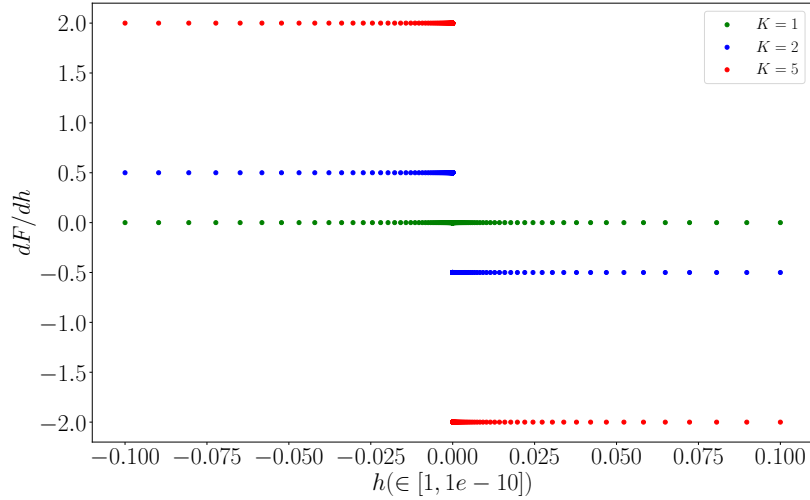

FIG. 9. Non-analytic free energy for $K > 1$ and analytic free energy for $K = 1$.

The non-analyticity for $K > 1$ occurs because the magnetic field is able to flip the ground state. For example, for $K = 2$, the states in question are $|M = 1, m = -1, 0\rangle$. For $h > 0$, the ground state occurs in the subspace $|S_d^z = 1/2, m = -1\rangle, |S_d^z = -1/2, m = 0\rangle$. If we now flip the magnetic field, the ground state subspace flips to $|S_d^z = 1/2, m = 0\rangle, |S_d^z = -1/2, m = 1\rangle$. Instead, if we look at the case of $K = 1$, the ground state is in the subspace of $|S_d^z = 1/2, m = -1/2\rangle, |S_d^z = -1/2, m = 1/2\rangle$, and since there is only this one subspace, the ground state is independent of the field. From this discussion, it is clear that the non-analyticity appears because there are multiple values of $m \in [-M, M-1]$ in the ground state manifold, which means that it is the ground state degeneracy that causes the non-analyticity.

XIV. THERMAL ENTROPY AT $T = 0$

Given the Helmholtz free energy $F = -k_B T \ln Z = -\frac{1}{\beta} \ln \sum_n e^{-\beta E_n}$, the thermal entropy at temperature T is given by

$$S(T) = -\frac{\partial F}{\partial T} = k_B \ln \sum_n e^{-\beta E_n} + \frac{\sum_n E_n e^{-\beta E_n}}{T \sum_n e^{-\beta E_n}} \tag{247}$$

In the limit of $T \to 0$, only the lowest energy state E_{gs} will survive. Assuming this state has a degeneracy d_{gs}, we have

$$S(T \to 0) = k_B \ln \left(d_{\mathrm{gs}} e^{-\beta E_{\mathrm{gs}}} \right) + \frac{d_{\mathrm{gs}} E_{\mathrm{gs}} e^{-\beta E_{\mathrm{gs}}}}{T d_{\mathrm{gs}} e^{-\beta E_{\mathrm{gs}}}} = k_B \ln d_{\mathrm{gs}} - \frac{1}{T} E_{\mathrm{gs}} + \frac{1}{T} E_{\mathrm{gs}} = k_B \ln d_{\mathrm{gs}} \tag{248}$$

This is a general result that holds for any system with a d_{gs}-fold degenerate ground state. For the star graph problem, we have a $K-$fold degenerate ground state, so the $T = 0$ entropy of the star graph Hamiltonian with K outer spins is $S(T \to 0) = k_B \ln K$. The impurity contribution to the total entropy is obtained by subtracting the entropy of the non-interacting bath. But we know that the entropy of a free Fermi gas vanishes at low temperatures. The impurity contribution is therefore

$$S_{\mathrm{imp}}(T \to 0) = k_B \ln K \tag{249}$$

We wish to point out that this value is different from the "standard" result

$$S = \ln 2 \cos \frac{\pi}{K + 2} \tag{250}$$

obtained from other methods like the Bethe ansatz (BA) and conformal field theory (CFT) calculations [2, 3, 6, 11, 28, 29]. The difference is not only in the value but also in the origin. Eq. 249 describes the thermal entropy coming purely from the quantum mechanical degeneracy of the star graph problem (which is the zero mode of the full multi-channel Kondo problem), while the result Eq. 250 presumably arises from the lowest-lying excitations on top of the zero mode arising from the conduction bath $k-$states. Another difference lies in the fact that while Eq. 249 holds true independent of system size, Eq. 250 is correct only in the thermodynamic limit and only if the limit of system size $L \to \infty$ is taken prior to taking the limit of temperature $T \to 0$ [2, 15, 30]. We note however that the entropy coming from the degeneracy is larger than that coming from the excitations (fig. 10), and consequently plays a very important role in the physics of the multi-channel Kondo problem, being responsible for the NFL physics and the critical behaviour near the fixed point.
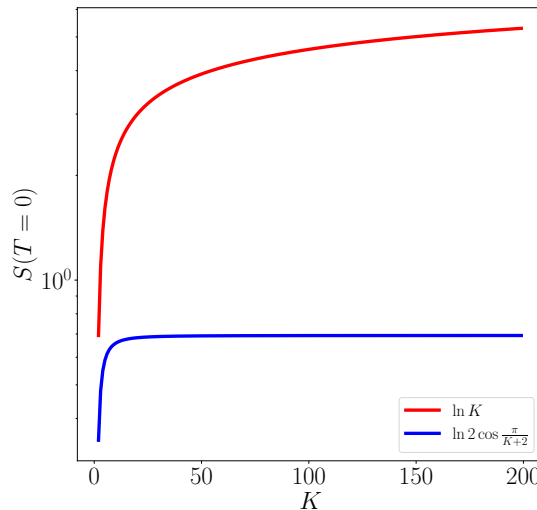

FIG. 10. Comparison of star graph zero temperature entropy with CFT/Bethe ansatz formula

Another related problem for which this result holds is the Ising model on a 1D chain. The partition function for that problem is $Z = \lambda_1^N + \lambda_2^N$, where $\lambda_{1,2} = e^{\beta J} \pm e^{-\beta J}$ are the eigenvalues of the $T-$matrix and N is the number of particles. As we come to $T \to 0$, we have $\lambda_{1,2} \simeq e^{\beta J}$ and $Z \simeq 2 e^{\beta N J}$, so $F = -k_B T \ln 2 - NJ$ and the entropy becomes $S = -\partial_T F = k_B \ln 2$. Note that the factor of 2 in Z is the degeneracy d_{gs} described above.

Appendix A: URG equations for the single-channel Kondo model

The single-channel SU(2) Kondo model consists of a spin-half impurity interacting with a band of itinerant conduction electrons through a spin-exchange coupling:

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k \tau_{k\sigma} + \sum_{kl} J^z S_d^z s_{kl}^z + \frac{1}{2} \sum_{kl} J^t \left(S_d^+ s_{kl}^- + S_d^- s_{kl}^+ \right) \tag{A1}$$

Here, $s_{kl}^z = \frac{1}{2} \left(c_{k\uparrow}^\dagger c_{l\uparrow} - c_{k\downarrow}^\dagger c_{l\downarrow} \right)$, $s_{kl}^- = c_{k\downarrow}^\dagger c_{l\uparrow}$ and $s_{kl}^+ = s_{lk}^{-\dagger}$. Also, $\tau = \hat{n} - \frac{1}{2}$. The indices k, l sum over the momentum states. $\vec{S}_d$ is the impurity spin operator with $S_d^z = \pm 1/2$.

The scheme is that we will disentangle an electron $q\beta$ from the Hamiltonian, q being the momentum and β the spin. The diagonal part of the Hamiltonian under this scheme is

$$H_{q\beta}^D = \epsilon_q \tau_{q\beta} + J^z S_d^z s_{qq}^z \tag{A2}$$

The off-diagonal parts at a particular RG step H_1^I and H_0^I, that start from particle and hole states respectively, are

$$H_1^I = \sum_{|k|<\Lambda,q} J^z S_d^z s_{kq}^z + \frac{1}{2} \sum_{|k|<\Lambda,q} J^t \left(S_d^+ s_{kq}^- + S_d^- s_{kq}^+ \right) \tag{A3}$$

$$H_0^I = \sum_{|k|<\Lambda,q} J^z S_d^z s_{qk}^z + \frac{1}{2} \sum_{|k|<\Lambda,q} J^t \left(S_d^+ s_{qk}^- + S_d^- s_{qk}^+ \right) \tag{A4}$$

H_1^I is the Hamiltonian term that scatters from the occupied configuration of q, H_0^I is the same from the unoccupied configuration. These are the terms that appear in the numerator.

1. Particle sector

The particle sector involves integrating out those states which are occupied ($\hat{n}_{q\beta} = 1$). We will work at an energy shell $\epsilon_q = -D$. The renormalization is

$$H_0^I \frac{1}{\omega - H_{q\beta}^D} H_1^I \tag{A5}$$

Both H_0^I and H_1^I have all three operators $S_d^z, S_d^\pm$. We call S_d^z the spin-keep term and the others spin-flip terms. The entire product will thus have $3 \times 3 = 9$ terms. Not all terms however renormalize the Hamiltonian. Those terms that have identical operators on both sides can be ignored because $S_d^{z2} = $ constant and $S^{\pm 2} = 0$. The other six terms will renormalize the Hamiltonian. This brings in one more simplification: all the six terms that *will* renormalize the Hamiltonian have a spin flip operator on at least one side of the Greens function. This means that in the denominator of the Greens function, S_d^z and s_{qq}^z have to be anti-parallel in order to produce a non-zero result for that term. This means we can identically replace $S_d^z s_{qq}^z = -\frac{1}{4}$. Also, in the particle sector, the Greens function always has $c_{q\beta}$ in front of it, so $\epsilon_q \tau_{q\beta} = D/2$. Substituting all this, we get

$$\frac{1}{\omega - D/2 + J/4} \sum_{|k,k'|<\Lambda,q} \left[\frac{1}{2} J^z J^t \left(S_d^z S_d^+ s_{qk'}^z s_{kq}^- + S_d^z S_d^- s_{qk'}^z s_{kq}^+ \right) + \frac{1}{2} J^t J^z \left(S_d^+ S_d^z s_{qk'}^- s_{kq}^z + S_d^- S_d^z s_{qk'}^+ s_{kq}^z \right) \right. \tag{A6}$$

$$\left. + \frac{1}{4} J^{t2} \left(S_d^- S_d^+ s_{qk'}^+ s_{kq}^- + S_d^+ S_d^- s_{qk'}^- s_{kq}^+ \right) \right] \tag{A7}$$

We now simplify the products and keep only terms diagonal in q. For example: $s_{qk'}^z s_{kq}^+ = \frac{1}{2} \hat{n}_{q\downarrow} s_{kk'}^+$ and $s_{qk'}^z s_{kq}^- = -\frac{1}{2} \hat{n}_{q\uparrow} s_{kk'}^-$. The renormalization becomes

$$\frac{1}{\omega - D/2 + J/4} \sum_{|k,k'|<\Lambda,q} \left[\frac{1}{4} J^z J^t \left(-\frac{1}{2} S_d^+ \hat{n}_q s_{kk'}^- - \frac{1}{4} S_d^- \hat{n}_q s_{kk'}^z \right) - \frac{1}{4} J^{t2} S_d^z \left(-\hat{n}_{q\uparrow} c_{k\downarrow}^\dagger c_{k'\downarrow} + \hat{n}_{q\downarrow} c_{k\uparrow}^\dagger c_{k'\uparrow} \right) \right] \tag{A8}$$

We now replace $\sum_q \hat{n}_{q\sigma} = n(D)$. The renormalization due to excitations coming from the particle sector is

$$\Delta H_1 = -\frac{1}{2}\frac{n(D)}{\omega - D/2 + J/4} \sum_{|k,k'|<\Lambda} \left[J^z J^t \frac{1}{2}\left(S_d^+ s_{kk'}^- + S_d^- s_{kk'}^z\right) + J^{t2} S_d^z s_{kk'}^z \right] \tag{A9}$$

The renormalization in the couplings coming from the particle sector is therefore,

$$\Delta J^z = -\frac{1}{2}\frac{J^{t2} n(D)}{\omega - D/2 + J/4}, \qquad\qquad \Delta J^t = -\frac{1}{2}\frac{J^z J^t n(D)}{\omega - D/2 + J/4} \tag{A10}$$

2. Hole sector

The hole sector involves integrating out those states which are vacant ($\hat{n}_{q\beta} = 1$). We will work at an energy shell $\epsilon_q = D$. The renormalization is

$$H_1^I \frac{1}{\omega - H_{q\beta}^D} H_0^I \tag{A11}$$

The same considerations as those in the particle sector apply here, and the denominator becomes $\omega - D/2 + J/4$, while the numerator is $H_1^I H_0^I$. Since this is just the Hermitian conjugate of the particle sector form, we do not need to calculate this separately, because the renormalization here will be $\Delta H_0 = \Delta H_1^\dagger = \Delta H_1$.

3. Scaling equations

Since the renormalization in the hole sector is equal to that in the particle sector, the total renormalization is simply twice that in the particle sector (eqs. A10):

$$\Delta J^z = -\frac{J^{t2} n(D)}{\omega - D/2 + J/4}, \qquad\qquad \Delta J^t = -\frac{J^z J^t n(D)}{\omega - D/2 + J/4} \tag{A12}$$

If we set $J_z = J_t = J$, we have an SU(2)-symmetric Kondo model $J\vec{S}_d \cdot \vec{s}$.

$$\Delta J = -\frac{J^2 n(D)}{\omega - D/2 + \frac{1}{4}J} \tag{A13}$$

To recover the one-loop form, we can replace ω with the bare value $-D/2$ and ignore the J in the denominator (small J).

$$\Delta J \approx \frac{J^2 n(D)}{D} \tag{A14}$$

$$\Delta J^{(2)} = -\frac{J^2 n(D)}{\omega - D/2 + J/4} \tag{A15}$$

For $\omega < D/2$, we get the flow towards the strong coupling fixed point. That is, there appears a stable fixed point at $J^* = 4|\omega - D/2|$ for all bare $J > 0$. We also get a decay towards the local moment fixed point $J^* = 0$ for $J < 0$. For $\omega = -D/2$ and $J \ll D$, we get the one-loop PMS form.

$$\Delta J^{(2)} = \frac{J^2 n(D)}{D - J/4} \simeq \frac{J^2 n(D)}{D} \tag{A16}$$

Appendix B: URG equations for the single-channel Kondo model with spin-S impurity

The Hamiltonian is the same as eq. A1, but the impurity is now a spin-S object, such that $S_d^z \in [-S, -S+1, \ldots, S-1, S]$.

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k \tau_{k\sigma} + \sum_{kl} J S_d^z s_{kl}^z + \frac{1}{2} \sum_{kl} J \left(S_d^+ s_{kl}^- + \text{h.c.} \right) \tag{B1}$$

It is easier to see the RG flow of the couplings if we write the Hamiltonian in terms of the eigenstates of S_d^z. These eigenstates are defined by $S_d^z |m\rangle = m |m\rangle$, $m \in [-S, S]$. In terms of these eigenstates, the Hamiltonian becomes

$$
\begin{aligned}
\mathcal{H} &= \sum_{k\sigma} \epsilon_k \tau_{k\sigma} + \sum_{kl} J S_d^z \sum_m |m\rangle \langle m| s_{kl}^z + \frac{1}{2} \sum_{kl} J \left(S_d^+ s_{kl}^- + \text{h.c.} \right) \sum_m |m\rangle \langle m| \\
&= \sum_{k\sigma} \epsilon_k \tau_{k\sigma} + \sum_{m=-S}^{S} \sum_{\substack{kl, \\ \sigma=\uparrow,\downarrow}} J_m^\sigma |m\rangle \langle m| c_{k\sigma}^\dagger c_{l\sigma} + \sum_{kl} \sum_{m=-S}^{S-1} J_m^t \left(|m+1\rangle \langle m| s_{kl}^- + \text{h.c.} \right)
\end{aligned}
\tag{B2}
$$

where $J_m^\sigma = \frac{1}{2}\sigma m J$ in the UV Hamiltonian, and $J_m^t = J\frac{1}{2}\sqrt{S(S+1) - m(m+1)}$ is the coupling that connects $|m\rangle$ and $|m+1\rangle$.

The various terms that renormalize the Hamiltonian can be described in terms of the bath spin operators that come into them. For example, the term that has s^z on both sides of the intervening Greens function can be represented as $z|z$. There are 7 such terms: $z|z, \pm|\mp, z|\pm, \pm|z$. Each of these terms occur both in the particle and the hole sectors.

1. Particle sector

We will now calculate each of these terms for the particle sector. The particle sector consists of those processes that start from an occupied state ($\hat{n}_{q\beta} = 1$).

a. $z|z$

$$\sum_{kk',m,\sigma} c_{q\sigma}^\dagger c_{k'\sigma} |m\rangle \langle m| \frac{J_m^{\sigma\,2}}{\omega - D/2 + \frac{J}{2}\sigma S_d^z} |m\rangle \langle m| c_{k\sigma}^\dagger c_{q\sigma} = -\sum_{kk',m,\sigma} n_{q\sigma} \frac{J_m^{\sigma\,2} c_{k\sigma}^\dagger c_{k'\sigma} |m\rangle \langle m|}{\omega_{m,\sigma} - D/2 + \frac{J}{2}\sigma m} \tag{B3}$$

b. $+|-$

$$\sum_{kk',m} c_{q\uparrow}^\dagger c_{k'\downarrow} |m\rangle \langle m+1| \frac{J_m^{t\,2}}{\omega - D/2 + \frac{J}{2}S_d^z} |m+1\rangle \langle m| c_{k\downarrow}^\dagger c_{q\uparrow} = -n_{q\uparrow} \sum_{kk',m} \frac{J_m^{t\,2} c_{k\downarrow}^\dagger c_{k'\downarrow} |m\rangle \langle m|}{\omega_{m+1,\uparrow} - D/2 + \frac{J}{2}(m+1)} \tag{B4}$$

c. $-|+$

$$\sum_{kk',m} c_{q\downarrow}^\dagger c_{k'\uparrow} |m+1\rangle \langle m| \frac{J_m^{t\,2}}{\omega - D/2 - \frac{J}{2}S_d^z} |m\rangle \langle m+1| c_{k\uparrow}^\dagger c_{q\downarrow} = -n_{q\downarrow} \sum_{kk',m} \frac{J_m^{t\,2} c_{k\uparrow}^\dagger c_{k'\uparrow} |m+1\rangle \langle m+1|}{\omega_{m,\downarrow} - D/2 - \frac{J}{2}m} \tag{B5}$$

d. $z|+$

$$\sum_{kk',m} c_{q\downarrow}^\dagger c_{k'\downarrow} |m\rangle \langle m| \frac{J_m^\downarrow J_m^t}{\omega - D/2 - \frac{J}{2}S_d^z} |m\rangle \langle m+1| c_{k\uparrow}^\dagger c_{q\downarrow} = -n_{q\downarrow} \sum_{kk',m} \frac{J_m^\downarrow J_m^t c_{k\uparrow}^\dagger c_{k'\downarrow} |m\rangle \langle m+1|}{\omega_{m,\downarrow} - D/2 - \frac{J}{2}m} \tag{B6}$$

e. $+|z$

$$\sum_{kk',m} c_{q\uparrow}^\dagger c_{k'\downarrow} |m\rangle \langle m+1| \frac{J_m^t J_{m+1}^\uparrow}{\omega - D/2 + \frac{J}{2}S_d^z} |m+1\rangle \langle m+1| c_{k\uparrow}^\dagger c_{q\uparrow} = -n_{q\uparrow} \sum_{kk',m} \frac{J_{m+1}^\uparrow J_m^t c_{k\uparrow}^\dagger c_{k'\downarrow} |m\rangle \langle m+1|}{\omega_{m+1,\downarrow} - D/2 + \frac{J}{2}(m+1)} \tag{B7}$$

f. $z|-$

$$\sum_{kk',m} c_{q\uparrow}^\dagger c_{k'\uparrow} |m+1\rangle \langle m+1| \frac{J_{m+1}^\uparrow J_m^t}{\omega - D/2 + \frac{J}{2}S_d^z} |m+1\rangle \langle m| c_{k\downarrow}^\dagger c_{q\uparrow} = -n_{q\uparrow} \sum_{kk',m} \frac{J_{m+1}^\uparrow J_m^t c_{k\downarrow}^\dagger c_{k'\uparrow} |m+1\rangle \langle m|}{\omega_{m+1,\uparrow} - D/2 + \frac{J}{2}(m+1)} \tag{B8}$$

g. $-|z$

$$\sum_{kk',m} c^\dagger_{q\downarrow} c_{k'\uparrow} |m+1\rangle \langle m| \frac{J^\downarrow_m J^t_m}{\omega - D/2 - \frac{J}{2} S^z_d} |m\rangle \langle m| c^\dagger_{k\downarrow} c_{q\downarrow} = -n_{q\downarrow} \sum_{kk',m} \frac{J^\downarrow_m J^t_m c^\dagger_{k\downarrow} c_{k'\uparrow} |m+1\rangle \langle m|}{\omega_{m,\downarrow} - D/2 - \frac{J}{2}(m)} \tag{B9}$$

To compare with the spin-$\frac{1}{2}$ RG equations, we will transform the general spin-S ω to the spin-$\frac{1}{2}$ ω, using $\omega_{m,\sigma} \rightarrow \omega - \frac{J}{2}\left(m\sigma - \frac{1}{2}\right)$. The first four terms renormalize J^σ_m:

$$\Delta J^\sigma_m = -\frac{(J^\sigma_m)^2 + \left(J^t_{m-\frac{1+\sigma}{2}}\right)^2}{\omega - D/2 + J/4} \tag{B10}$$

Here, we have defined $J^t_m = 0$ for $|m| > S$. Two relations can be obtained from this RG equation, the RG equations for the sum and difference of the couplings. These sum and difference are defined by rewriting the Ising part of the Hamiltonian in the following fashion:

$$\sum_{m=-S}^{S} \sum_{\substack{kl, \\ \sigma=\uparrow,\downarrow}} J^\sigma_m |m\rangle \langle m| c^\dagger_{k\sigma} c_{l\sigma} = \sum_{m=-S}^{S} \sum_{\sigma=\uparrow,\downarrow} J^+_m |m\rangle \langle m| \sum_\sigma \hat{n}_{0\sigma} + \sum_{m=-S}^{S} J^-_m |m\rangle \langle m| 2s^z_0 \tag{B11}$$

where $2J^\pm_m = J^\uparrow_m \pm J^\downarrow_m$. The RG equation for the sum of the couplings is

$$\Delta J^+_m = -\frac{1}{2} \frac{\sum_\sigma (J^\sigma_m)^2 + \sum_\sigma \left(J^t_{m-\frac{1+\sigma}{2}}\right)^2}{\omega - D/2 + J/4} = -\frac{J^2}{4} \frac{S(S+1)}{\omega - D/2 + J/4} \tag{B12}$$

This is an $m-$independent piece, so it can be summed over to produce an impurity-independent potential scattering term:

$$\sum_{m=-S}^{S} \sum_{\sigma=\uparrow,\downarrow} J^+ |m\rangle \langle m| \sum_\sigma \hat{n}_{0\sigma} = J^+ \mathbb{I} \sum_\sigma \hat{n}_{0\sigma} \tag{B13}$$

Since this term does not have any Kondo physics, we drop this piece and say that $\Delta J^+ = 0$ and so $\Delta J^\uparrow_m = -\Delta J^\downarrow_m \implies J^\uparrow_m = -J^\downarrow_m$ throughout the RG flow.

The second is the RG equation for the difference of the couplings:

$$\Delta J^-_m = -\frac{1}{2} \frac{\left(J^t_{m-1}\right)^2 - (J^t_m)^2}{\omega - D/2 + J/4} = -\frac{1}{4} \frac{mJ^2}{\omega - D/2 + J/4} \tag{B14}$$

The usual J^z Kondo coupling is defined as $J^z = 2J^-_m/m$. Substituting this gives

$$\Delta J^z = -\frac{1}{2} \frac{J^2}{\omega - D/2 + J/4} \tag{B15}$$

The last four terms renormalize J^t_m:

$$\Delta J^t_m = -\frac{J^t_m \left(J^\downarrow_m + J^\uparrow_{m+1}\right)}{\omega - D/2 + J/4} = -\frac{1}{2} \frac{J^t_m J}{\omega - D/2 + J/4} \implies J^t = -\frac{1}{2} \frac{J^2}{\omega - D/2 + J/4} \tag{B16}$$

J^t is the transverse coupling. Since we started from an isotropic model ($J^z = J^t = J$), we obtain $\Delta J^z = \Delta J^t$.

2. Hole sector

The RG equation in this sector is identical to that in the hole sector, because the renormalization terms are Hermitian conjugate to this in the particle sector.

[1] Nozières, Ph. and Blandin, A., J. Phys. France **41**, 193 (1980).

[2] J. Gan, N. Andrei, and P. Coleman, Phys. Rev. Lett. **70**, 686 (1993).

[3] V. J. Emery and S. Kivelson, Phys. Rev. B **46**, 10812 (1992).

[4] J. Gan, **6**, 4547 (1994).

[5] A. M. Tsvelick and P. B. Wiegmann, Zeitschrift für Physik B Condensed Matter **54**, 201 (1984).

[6] A. M. Tsvelick and P. B. Wiegmann, Journal of Statistical Physics **38**, 125 (1985).

[7] O. Parcollet and A. Georges, Phys. Rev. Lett. **79**, 4665 (1997).

[8] T. Kimura and S. Ozaki, Journal of the Physical Society of Japan **86**, 084703 (2017), https://doi.org/10.7566/JPSJ.86.084703.

[9] D. Bensimon, A. Jerez, and M. Lavagna, Phys. Rev. B **73**, 224445 (2006).

[10] D. L. Cox and M. Jarrell, **8**, 9825 (1996).

[11] I. Affleck and A. W. Ludwig, Nuclear Physics B **360**, 641 (1991).

[12] P. Coleman, L. B. Ioffe, and A. M. Tsvelik, Phys. Rev. B **52**, 6611 (1995).

[13] I. Affleck and A. W. Ludwig, Physical Review B **48**, 7297 (1993).

[14] H. B. Pang and D. L. Cox, Phys. Rev. B **44**, 9454 (1991).

[15] J. von Delft, G. Zaránd, and M. Fabrizio, Phys. Rev. Lett. **81**, 196 (1998).

[16] C. Kolf and J. Kroha, Phys. Rev. B **75**, 045129 (2007).

[17] R. Žitko and M. Fabrizio, Phys. Rev. B **95**, 085121 (2017).

[18] A. Mukherjee and S. Lal, Nuclear Physics B **960**, 115170 (2020).

[19] A. Mukherjee and S. Lal, Nuclear Physics B **960**, 115163 (2020).

[20] A. Mukherjee and S. Lal, New Journal of Physics **22**, 063007 (2020).

[21] A. Mukherjee and S. Lal, New Journal of Physics **22**, 063008 (2020).

[22] S. Patra and S. Lal, Phys. Rev. B **104**, 144514 (2021).

[23] S. Pal, A. Mukherjee, and S. Lal, **21**, 023019 (2019).

[24] V. Tripathi, *Landau Fermi Liquids and Beyond* (CRC Press, 2018).

[25] Y. Kuramoto, The European Physical Journal B - Condensed Matter and Complex Systems **5**, 457 (1998).

[26] E. Kogan, Journal of Physics Communications **2**, 085001 (2018).

[27] C. Varma, Z. Nussinov, and W. Van Saarloos, Physics Reports **361**, 267 (2002).

[28] N. Andrei and C. Destri, Phys. Rev. Lett. **52**, 364 (1984).

[29] P. Sacramento and P. Schlottmann, Physics Letters A **142**, 245 (1989).

[30] A. V. Rozhkov, International Journal of Modern Physics B **12**, 3457 (1998), https://doi.org/10.1142/S0217979298002805.