Multi-channel Kondo model URG

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1 Introduction

$$H = \sum_{k,\sigma,\gamma} \epsilon_k^\gamma \hat{n}_{k\sigma}^\gamma + J \sum_{kk',\gamma} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} {c_{k\alpha}^\gamma}^\dagger c_{k'\alpha'}^\gamma \tag{1}$$

$$\Delta H = \left[c^\dagger T, \eta\right] = \left(c^\dagger T \frac{1}{\hat{\omega} - H_D} T^\dagger c - T^\dagger c \frac{1}{\hat{\omega} - H_D} c^\dagger T\right) \tag{2}$$

2 Leading order renormalization

2.1 Second term

$$\begin{aligned}
&T^\dagger c \frac{1}{\hat{\omega} - H_D} c^\dagger T \\
&= J^2 \sum_{q\beta kk'\alpha\alpha'} c^\dagger_{k'\alpha'} c_{q\beta} \vec{S}_d \cdot \vec{s}_{\alpha'\beta} \frac{1}{\omega - \epsilon_q \tau_{q\beta} - J S_d^z s_q^z} c^\dagger_{q\beta} c_{k\alpha} \vec{S}_d \cdot \vec{s}_{\beta\alpha} \\
&= J^2 \sum_{q\beta kk'\alpha\alpha',a,b} c^\dagger_{k'\alpha'} c_{q\beta} S_d^a s^a_{\alpha'\beta} \frac{1}{\omega - \frac{1}{2}D - J S_d^z s_q^z} c^\dagger_{q\beta} c_{k\alpha} S_d^b s^b_{\beta\alpha} && \left[\tau = \frac{1}{2}, \epsilon_q = D\right] \\
&= J^2 \sum_{q\beta kk'\alpha\alpha',a,b} c^\dagger_{k'\alpha'} c_{q\beta} S_d^a s^a_{\alpha'\beta} \frac{\omega - \frac{1}{2}D + J S_d^z s_q^z}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} c^\dagger_{q\beta} c_{k\alpha} S_d^b s^b_{\beta\alpha} \\
&= J^2 \sum_{q\beta kk'\alpha\alpha',a,b} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a s^a_{\alpha'\beta} \frac{\left(\omega - \frac{1}{2}D\right)\left(1 - \hat{n}_{q\beta}\right) + J S_d^z c_{q\beta} s_q^z c^\dagger_{q\beta}}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b s^b_{\beta\alpha} \\
&= J^2 \sum_{q\beta kk'\alpha\alpha',a,b} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a s^a_{\alpha'\beta} \frac{\left(\omega - \frac{1}{2}D\right)\left(1 - \hat{n}_{q\beta}\right) + J S_d^z \left(s_q^z + \frac{\beta}{2}\right)\left(1 - \hat{n}_{q\beta}\right)}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b s^b_{\beta\alpha} && \left[\left[c_{q\beta}, s_q^z\right] = \frac{\beta}{2} c_{q\beta}\right] \\
&= J^2 |\delta D| \sum_{q\beta kk'\alpha\alpha',a,b} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a s^a_{\alpha'\beta} s^b_{\beta\alpha} \frac{\left(\omega - \frac{1}{2}D\right) + J S_d^z \beta}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b && \left[s_q^z = \frac{\beta}{2}\right] \\
&= \frac{J^2 \left(\omega - \frac{1}{2}D\right)|\delta D|}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} \sum_{kk'\alpha\alpha',a,b} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a S_d^b \left(s^a s^b\right)_{\alpha'\alpha} && \text{[term 1]} \\
&+ \frac{J^3}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} |\delta D| \sum_{\beta kk'\alpha\alpha',a,b} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a S_d^z S_d^b s^a_{\alpha'\beta} s^b_{\beta\alpha} \beta && \text{[term 2]}
\end{aligned} \tag{3}$$

We will now simplify the terms individually. In term 1, only the configurations $a \neq b$ can lead to a non-trivial impurity spin operator and hence contribute to renormalization. For $a \neq b$, we have $S_d^a S_d^b =$

$\frac{i}{2}\sum_c \epsilon^{abc} S_d^c$. Therefore,

$$\sum_{a,b} S_d^a S_d^b \left(s^a s^b\right)_{\alpha'\alpha} = -\frac{1}{4}\sum_{c,e} S_d^c \left(s^e\right)_{\alpha'\alpha} \sum_{a,b} \epsilon^{abc}\epsilon^{abe} = -\frac{1}{4}\sum_{c,e} S_d^c \left(s^e\right)_{\alpha'\alpha} 2\delta_{ce} = -\frac{1}{2}\vec{S}_d \cdot \vec{s}_{\alpha'\alpha} \tag{4}$$

Term 1 therefore simplifies to

$$-\frac{1}{2}\frac{J^2\left(\omega - \frac{1}{2}D\right)|\delta D|}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} \sum_{kk'\alpha\alpha',c} c^\dagger_{k'\alpha'} c_{k\alpha} \vec{S}_d \cdot \vec{s}_{\alpha'\alpha} \quad \text{[term 1]} \tag{5}$$

In term 2, we use the identity:

$$S^a S^z S^b = \frac{1}{4}\sum_c \left(\delta_{ac}\delta_{zb} - \delta_{ab}\delta_{zc}\right) S^c \tag{6}$$

Substituting this gives

$$\begin{aligned}
&\sum_{\beta,a,b} S_d^a S_d^z S_d^b s^a_{\alpha'\beta} s^b_{\beta\alpha}\beta = \frac{1}{4}\sum_{\beta,c}\beta\left(S_d^c s^c_{\alpha'\beta} s^z_{\beta\alpha} - S_d^z s^c_{\alpha'\beta} s^c_{\beta\alpha}\right) \\
&\implies \text{term 2} = \frac{J^3|\delta D|}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2}\frac{1}{4}\sum_{\beta kk'\alpha\alpha',c} c^\dagger_{k'\alpha'} c_{k\alpha}\beta s^c_{\alpha'\beta}\left(S_d^c s^z_{\beta\alpha} - S_d^z s^c_{\beta\alpha}\right) \\
&\qquad = \frac{J^3|\delta D|}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2}\frac{1}{4}\sum_{\beta kk'\alpha\alpha',c} c^\dagger_{k'\alpha'} c_{k\alpha}\left(\frac{1}{2}S_d^c s^c_{\alpha'\alpha} - \beta S_d^z s^c_{\beta\alpha} s^c_{\alpha'\beta}\right)
\end{aligned} \tag{7}$$

2.2 First term

$$\begin{aligned}
& c^\dagger T \frac{1}{\hat{\omega} - H_D} T^\dagger c \\
&= J^2 \sum_{q\beta kk'\alpha\alpha'} c^\dagger_{q\beta} c_{k\alpha} \vec{S}_d \cdot \vec{s}_{\beta\alpha} \frac{1}{\omega' - \epsilon_q \tau_{q\beta} - J S_d^z s_q^z} c^\dagger_{k'\alpha'} c_{q\beta} \vec{S}_d \cdot \vec{s}_{\alpha'\beta} \\
&= J^2 \sum_{q\beta kk'\alpha\alpha' ab} c^\dagger_{q\beta} c_{k\alpha} S_d^a s^a_{\beta\alpha} \frac{1}{\omega' - \frac{1}{2}D - J S_d^z s_q^z} c^\dagger_{k'\alpha'} c_{q\beta} S_d^b s^b_{\alpha'\beta} && \left[\tau = -\frac{1}{2}, \epsilon_q = -D\right] \\
&= J^2 \sum_{q\beta kk'\alpha\alpha' ab} c^\dagger_{q\beta} c_{k\alpha} S_d^a s^a_{\beta\alpha} \frac{\left(\omega' - \frac{1}{2}D\right) + J S_d^z s_q^z}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} c^\dagger_{k'\alpha'} c_{q\beta} S_d^b s^b_{\alpha'\beta} \\
&= J^2 \sum_{q\beta kk'\alpha\alpha' ab} c^\dagger_{q\beta} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a s^a_{\beta\alpha} \frac{-\left(\omega' - \frac{1}{2}D\right) - J S_d^z s_q^z}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} c_{q\beta} S_d^b s^b_{\alpha'\beta} && \left[c_k c^\dagger_{k'} \sim -c^\dagger_{k'} c_k\right] \\
&= J^2 \sum_{q\beta kk'\alpha\alpha' ab} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a s^a_{\beta\alpha} \frac{-\left(\omega' - \frac{1}{2}D\right)\hat{n}_{q\beta} - J S_d^z c^\dagger_{q\beta} s_q^z c_{q\beta}}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b s^b_{\alpha'\beta} \\
&= J^2 \sum_{q\beta kk'\alpha\alpha' ab} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a s^a_{\beta\alpha} \frac{-\left(\omega' - \frac{1}{2}D\right)\hat{n}_{q\beta} - J S_d^z \left(s_q^z - \frac{\beta}{2}\right)\hat{n}_{q\beta}}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b s^b_{\alpha'\beta} && \left[\left[c^\dagger_{q\beta}, s_q^z\right] = -\frac{\beta}{2}c^\dagger_{q\beta}\right] \\
&= J^2|\delta D| \sum_{\beta kk'\alpha\alpha' ab} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a s^a_{\beta\alpha} \frac{-\left(\omega' - \frac{1}{2}D\right) + J S_d^z \beta}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} S_d^b s^b_{\alpha'\beta} && \left[s_q^z = -\frac{1}{2}\beta\right] \\
&= -\frac{J^2\left(\omega' - \frac{1}{2}D\right)|\delta D|}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} \sum_{kk'\alpha\alpha' ab} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a S_d^b \left(s^b s^a\right)_{\alpha'\alpha} && \text{[term 3]} \\
&\quad + \frac{J^3|\delta D|}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} \sum_{q\beta kk'\alpha\alpha' ab} c^\dagger_{k'\alpha'} c_{k\alpha} S_d^a S_d^z S_d^b s^a_{\beta\alpha} s^b_{\alpha'\beta}\beta && \text{[term 4]}
\end{aligned} \tag{8}$$

term 3 can be made identical to term 1 using the relation: $s^b s^a = -s^b s^a$ for $a \neq b$. With this change, term 3 becomes

$$\text{term 3} = \frac{J^2\left(\omega' - \frac{1}{2}D\right)|\delta D|}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2}\sum_{kk'\alpha\alpha' ab} c^\dagger_{k'\alpha'}c_{k\alpha}S_d^a S_d^b\left(s^a s^b\right)_{\alpha'\alpha} = \frac{1}{2}\frac{J^2\left(\omega' - \frac{1}{2}D\right)|\delta D|}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2}\sum_{kk'\alpha\alpha',c} c^\dagger_{k'\alpha'}c_{k\alpha}\vec{S_d}\cdot\vec{s}_{\alpha'\alpha} \tag{9}$$

For term 4, we get

$$\begin{aligned}\sum_{\beta,a,b} S_d^a S_d^z S_d^b s^b_{\alpha'\beta}s^a_{\beta\alpha}\beta &= \frac{1}{4}\sum_{\beta,a,b,c}\beta\left(\delta_{ac}\delta_{zb} - \delta_{ab}\delta_{zc}\right)S^c s^b_{\alpha'\beta}s^a_{\beta\alpha} \\ &= \frac{1}{4}\sum_{\beta,c}\beta s^c_{\beta\alpha}\left(S_d^c s^z_{\alpha'\beta} - S_d^z s^c_{\alpha'\beta}\right)\end{aligned} \tag{10}$$

which gives

$$\text{term 4} = \frac{J^3|\delta D|}{\left(\omega' - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2}\frac{1}{4}\sum_{\beta kk'\alpha\alpha',c} c^\dagger_{k'\alpha'}c_{k\alpha}\left(\frac{1}{2}S_d^c s^c_{\alpha'\alpha} \ - \beta S_d^z s^c_{\beta\alpha}s^c_{\alpha'\beta}\right) \tag{11}$$

2.3 Total renormalization

From the formula for the renormalization ΔH, we write

$$\Delta H = \frac{1}{2}\left(\text{term 3} + \text{term 4} - \text{term 1} - \text{term 2}\right) \tag{12}$$

From the constraints of URG and particle-hole symmetry, we have the constraint $\omega + \omega' = H_d^0 + H_d^1$. H_d^0 is the diagonal part when the current node is unoccupied and H_d^1 is when its occupied. For our case, $H_d^0 + H_d^1 = \sum \epsilon_q \tau_q = D$, because both the hole and particle states have energy of $\frac{D}{2}$ (ϵ_q and τ flip sign together). We therefore have $(\omega - D/2) = -(\omega' - D/2)$. term 2 and term 4 cancel each other and term 3 becomes equal to term 1. The total renormalization at second order is therefore (relabelling ω' as ω)

$$\Delta H = \text{term 3} = \frac{1}{2}\frac{J^2\left(\omega - \frac{1}{2}D\right)|\delta D|}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2}\sum_{kk'\alpha\alpha',c} c^\dagger_{k'\alpha'}c_{k\alpha}\vec{S_d}\cdot\vec{s}_{\alpha'\alpha} \tag{13}$$

which gives

$$\Delta J = \frac{1}{2}\frac{J^2\left(\omega - \frac{1}{2}D\right)|\delta D|}{\left(\omega - \frac{1}{2}D\right)^2 - \frac{1}{16}J^2} \tag{14}$$

The choice of $\omega = D$ gives

$$\Delta J = \frac{J^2 D|\delta D|}{D^2 - \frac{1}{4}J^2} \tag{15}$$

For $J \ll D$, we get the one-loop PMS form.