Impurity entanglement entropy of 2-channel fixed point

The zero-mode Hamiltonian at the renormalisation group fixed point is a star graph with the impurity as the central node and the zeroth sites of the two conduction channels as the outer nodes: $H_0 = J^* \vec{S}_d \cdot \sum_l \vec{S}_{0,(\alpha)}$, where α (and Greek letters in genral) denote the channel index. The ground-states of this $2-$channel star graph are

$$|\sigma/2\rangle = \frac{\sigma}{\sqrt{6}} \left(2 \left|\sigma\bar{\sigma}\sigma\right\rangle - \left|\bar{\sigma}\sigma\sigma\right\rangle - \left|\sigma\sigma\bar{\sigma}\right\rangle\right), \sigma = \pm 1 \tag{1}$$

where three confifurations inside the ket are those of the impurity spin, the $\alpha = 1$ channel zeroth site and the $\alpha = 2$ channel zeroth site respectively. In the full Hamiltonian, there is also the hopping term: $H_{\text{int}} = -\frac{t}{N} \sum_{k,\alpha} \left(c^\dagger_{0\sigma,(\alpha)} c_{k\sigma,(\alpha)} + \text{h.c.}\right)$. Introducing the $k-$space states into H_0 leads to the following ground-states, classified by $S^z_{\text{tot}} = S^z_d + \sum_l S^z_{0,l} + \sum_k \sum_l S^z_{k,l}$ and the filling ν of $k-$space electrons:

$$S^z_{\text{tot}} = 0 : \begin{cases} \nu = 1 : \left|\sigma/2\right\rangle \left|h_{\sigma,(\alpha)}\right\rangle, \\ \nu = 3 : \left|\sigma/2\right\rangle \left|e_{\bar{\sigma},(\alpha)}\right\rangle, \end{cases} \quad \sigma = \pm 1, \alpha = 1, 2 \tag{2}$$

$$S^z_{\text{tot}} = \frac{\sigma}{2} : \nu = 2 : \left|\sigma/2\right\rangle \left|\phi\right\rangle, \sigma = \pm 1 \tag{3}$$

$$S^z_{\text{tot}} = \sigma : \begin{cases} \nu = 3 : \left|\sigma/2\right\rangle \left|e_{\sigma,(\alpha)}\right\rangle, \\ \nu = 1 : \left|\sigma/2\right\rangle \left|h_{\bar{\sigma},(\alpha)}\right\rangle, \end{cases} \quad \sigma = \pm 1, \alpha = 1, 2 \tag{4}$$

where $|\phi\rangle$ is the filled Fermi sea, and $\left|e_{\sigma,(\alpha)}\right\rangle$ and $\left|\sigma,(\alpha)\right\rangle$ are gapless particle and hole excitations: $\left|e_{\sigma,(\alpha)}\right\rangle = \sum_{k \in FS} c^\dagger_{k\sigma,(\alpha)} \left|\phi\right\rangle, \left|h_{\sigma,(\alpha)}\right\rangle = \sum_{k \in FS} c_{k\sigma,(\alpha)} \left|\phi\right\rangle$. The filling ν_1 and ν_2 in $|\phi\rangle$ were set to unity, so that the filling ν of the $|e\rangle$ and $|h\rangle$ states are 3 and 1 respectively.

In order to obtain the impurity entanglement entropy, we will use the result that the impurity reduced density matrix can be written in terms of the impurity magnetisation (m^z_d):

$$\rho_{\text{imp}} = \frac{1}{2}\mathcal{I} + m^z_d \sigma^z \tag{5}$$

where $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix along z. The problem then boils down to calculating the magnetisation m^z_d. For that, we will first obtain the new ground-states in the presence of H_{int}. After that, we will insert a global magnetic field into the new ground-state subspace, which will favour the states with the most positive S^z_{tot}. The expectation value of S^z_d in these favoured ground-states, in the presence of a vanishing field, will be the impurity magnetisation. The fact that the subspace with highest S^z_{tot} will be favoured means we only need to solve H_{int} in the subspace of $S^z_{\text{tot}} = +1$. This is because the full Hamiltonian and H_{int} conserve S^z_{tot}. The pertinent ground-states are

$$S^z_{\text{tot}} = 1 : \begin{cases} \nu = 3 : \begin{cases} \nu_{(1)} = 2 : \left|1/2\right\rangle \left|e_{\uparrow,(1)}\right\rangle \\ \nu_{(2)} = 2 : \left|1/2\right\rangle \left|e_{\uparrow,(2)}\right\rangle \end{cases} \\ \nu = 1 : \begin{cases} \nu_{(1)} = 0 : \left|1/2\right\rangle \left|h_{\downarrow,(1)}\right\rangle \\ \nu_{(2)} = 0 : \left|1/2\right\rangle \left|h_{\downarrow,(2)}\right\rangle \end{cases} \end{cases} \tag{6}$$

Since H_{int} conserves the filling $\nu_{(\alpha)}$ of each channel individually, these four states are not mixed by the perturbation H_{int}, at any order. The impurity magnetisation then reduces to

$$m^z_d = \langle 1/2 | S^z_d | 1/2 \rangle = \frac{1}{6} \left(4 \times \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{3} \tag{7}$$

The impurity entanglement entropy can then be obtained from ρ_{imp}:

$$S_{\text{EE}}(d) = -\left(\frac{1}{2} + m^z_d\right) \ln \left(\frac{1}{2} + m^z_d\right) - \left(\frac{1}{2} - m^z_d\right) \ln \left(\frac{1}{2} - m^z_d\right) \simeq 0.65 \ln 2 \tag{8}$$