Multi-channel Kondo model URG

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I. INTRODUCTION OF THE HAMILTONIAN AND ITS ZERO MODE

The isotropic multi-channel Kondo model [1–13] is described by the Hamiltonian

$$H = \sum_{k,\alpha,\gamma} \epsilon_k^\gamma \hat{n}_{k\alpha}^\gamma + J \sum_{kk',\gamma} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} {c_{k\alpha}^\gamma}^\dagger c_{k'\alpha'}^\gamma \ . \tag{1}$$

In the above equation, k, k' sum over the momentum states, α, α' sum over the spin indices and γ sums over the various channels. $\vec{S}_d, \vec{s}$ are the impurity and conduction bath spin vectors. The presence of a common spin-exchange coupling J for all the channels is what makes the problem isotropic with respect to the channels.

The zero mode of this Hamiltonian is obtained by keeping only the central component of the Fourier transform of the kinetic energy part:

$$\epsilon(\vec{r}) \simeq \delta_{\vec{r},0}\epsilon(\vec{r}=0) \implies \epsilon_{\vec{k}} = \sum_{\vec{r}} \epsilon(\vec{r}) e^{i\vec{k}\cdot\vec{r}} = \epsilon(\vec{r}=0) = \sum_{q} \epsilon_{\vec{q}} \tag{2}$$

Assuming the dispersion already includes the chemical potential μ, we have

$$\sum_{q} \epsilon_{\vec{q}} = \rho\Delta\epsilon \sum_{\epsilon=\epsilon_F-D}^{\epsilon_F+D} (\epsilon-\mu) = \rho\Delta\epsilon\,(\epsilon_F-\mu) \tag{3}$$

If we set the chemical potential at the Fermi surface, this summation vanishes.

$$\epsilon_{\vec{k}} = \rho\Delta\epsilon\,(\epsilon_F-\mu) = 0 \tag{4}$$

and the zero mode Hamiltonian is then just the spin-exchange part of the Hamiltonian:

$$H_{\text{zero mode}} = J \sum_{kk',\gamma} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} {c^{\gamma}_{k\alpha}}^{\dagger} c^{\gamma}_{k'\alpha'} \ . \tag{5}$$

We can make one more identification: the sum over the spins in momentum space amount to a local interaction at the zeroth sites of the indicates channels: $\sum_k {c^{\gamma}_{k\alpha}}^{\dagger} = {c^{\gamma}_{\vec{r}=0,\alpha}}^{\dagger}$, This suggests we shoud define spin operators for the zeroth sites of the channels:

$$\sum_{kk'} s^a_{\alpha\alpha'} {c^{\gamma}_{k\alpha}}^{\dagger} c^{\gamma}_{k'\alpha'} = s^a_{\alpha\alpha'} {c^{\gamma}_{0\alpha}}^{\dagger} c^{\gamma}_{0\alpha'} = s^a_{\gamma} \tag{6}$$

In the terms of the local spin operators $\vec{s}_\gamma = (s^x_\gamma, s^y_\gamma, s^z_\gamma)$, we can define the total spin of the zero mode of the bath: $\vec{s}_{\text{tot}} = \sum_\gamma \vec{s}_\gamma$ and the zero mode Hamiltonian then takes the simpler form

$$H_{\text{zero mode}} = J\vec{S}_d \cdot \vec{s}_{\text{tot}} \tag{7}$$

II. SYMMETRIES OF THE HAMILTONIAN

This zero mode Hamiltonian commutes with a number of operators, revealing various symmetries of the problem. The first such operators is s^2_{tot}. Henceforth we will drop the label zero mode. To see the commuation, note that s^2_{tot} can be written as $s^2_{\text{tot}} = \sum_{i=x,y,z} \left(s^i_{\text{tot}}\right)^2$. This then gives

$$\left[s^2_{\text{tot}}, H\right] = \left[\sum_{i=x,y,z} {s^i_{\text{tot}}}^2, J\sum_{i=x,y,z} S^i_d s^i_{\text{tot}}\right] = \sum_{i,j} JS^i_d \left\{s^i_{\text{tot}}, \left[s^i_{\text{tot}}, s^j_{\text{tot}}\right]\right\} = \sum_{i,j} JS^i_d \left\{s^i_{\text{tot}}, i\epsilon^{ijk} s^k_{\text{tot}}\right\} = 0 \tag{8}$$

The total spin angular momentum operator $S^2 = \left(\vec{S}_d + \vec{s}_{\text{tot}}\right)^2$ also commutes with the Hamiltonian. To see this, note that we can rewrite the operator as

$$S^2 = \left(\vec{S}_d + \vec{s}_{\text{tot}}\right)^2 = \frac{3}{4} + s_{\text{tot}}\left(s_{\text{tot}}+1\right) + \frac{2}{J}H \tag{9}$$

Since we already know that s_{tot} commutes with H, we have that $\left[S^2, H\right] = 0$. The $z-$component of this total spin operator also commutes with the Hamiltonian:

$$[S^z, H] = \left[S^z, J\left(\frac{1}{2}S^2 - \frac{3}{8} - \frac{1}{2}s^2_{\text{tot}}\right)\right] = \frac{J}{2}\left(\left[S^z, S^2\right] - \left[s^z_{\text{tot}}, s^2_{\text{tot}}\right]\right) = 0 \tag{10}$$

One final operator that we will look at is the spin parity operator $\pi^a = \sigma^a_d \otimes_{i=1}^K \sigma^a_i$.

$$[H, \pi^a] = J\sum_{b,j} \left[S^b_d s^b_j, \sigma^a_d \otimes_{i=1}^K \sigma^a_i\right] = \sum_{bj} \left(S^b_d \sigma^a_d s^b_j \sigma^a_j - \sigma^a_d S^b_d \sigma^a_j s^b_j\right) \otimes_{i<j} \sigma^a_i \otimes_{i>j} \sigma^a_i = 0 \tag{11}$$

III. EIGENVALUES AND EIGENSTATES

The zero mode Hamiltonian can be diagonalized. Since the Hamiltonian commutes with $s^2_{\rm tot}$, it is already block-diagonal in the quantum number $s_{\rm tot}$. Let us represent the quantum number of $s^z_{\rm tot}$ by m. For a particular $s_{\rm tot}$, m can take values from the set $[-s_{\rm tot}, s_{\rm tot}]$. The spin S^z_d can also take values $\pm\frac{1}{2}$. From now on, we will assume we are in the subspace of a particular $s_{\rm tot} = M$, so we will ignore that quantum number and write the kets simply as $|S^z_d, m\rangle$. So, the notation $|\uparrow, -1\rangle$ means the state with $S^z_d = \frac{1}{2}$ and $m = -1$. We will now show that even inside the block of $2 \times s_{\rm tot}$ (or $2 \times s_{\rm tot} + 1$, depending on whether it is odd or even) defined by a particular value of $s_{\rm tot}$, the Hamiltonian actually separates into decoupled 2×2 blocks. To see why, first note that the terminal states $|\downarrow, -M\rangle$ and $|\uparrow, M\rangle$ are already eigenstates, because they cannot scatter (the impurity can only flip down, and this would require the bath to flip up, but $s^z_{\rm tot}$ is already at its maximum value M). The other $2M - 2$ states can be organized into 2×2 blocks formed by the states $|\uparrow, m\rangle$ and $|\downarrow, m + 1\rangle$ for $m \in [-M, M - 1]$. The fact that this block does not interact with the other blocks can be easily verified: if there was some other state which when acted upon by the Hamiltonian gave a non-zero projection on $|\uparrow, m\rangle$, it would have to come from $S^z_d =\downarrow$, and this would mean the bath spin would have had to flip down. This means the bath spin in that state would have to be $m + 1$, and that is precisely the other state in the block.

Defining $x^M_m = M(M + 1) - m(m + 1)$, the 2×2 blocks can be written as

$$H_m = \begin{pmatrix} \frac{Jm}{2} & \frac{J}{2}\sqrt{x^M_m} \\ \frac{J}{2}\sqrt{x^M_m} & -\frac{J}{2}(m+1) \end{pmatrix}, m \in [-M, M-1] \tag{12}$$

The eigenvalues are

$$\lambda^M_\pm = -\frac{J}{4} \pm \frac{J}{2}\left(M + \frac{1}{2}\right) \tag{13}$$

The eigenvalues of the terminal states are $\frac{JM}{2}$. The set of eigenvalues $\lambda^{M_{\rm max}}_-$ form the ground state subspace $\left\{|\Psi^M_{m,-}\rangle\right\}$ of degeneracy $2M_{\rm max} = K$. This common $K-$fold degenerate eigenvalue is $-\frac{J}{2}(M + 1)$. The normalized eigenstates of the 2×2 blocks for each value of m are given by

$$|\Psi^M_{m,-}\rangle = \frac{1}{\sqrt{(M+m+1)(1+2M)}}\left[-\sqrt{x^M_m}\,|\uparrow, M, m\rangle + (M + m + 1)\,|\downarrow, M, m + 1\rangle\right], \quad m \in [-M, M-1] \tag{14}$$

$$|\Psi^M_{m,+}\rangle = \frac{1}{\sqrt{(M-m)(M-m+1)}}\left[\sqrt{x^M_m}\,|\uparrow, M, m\rangle + (M - m)\,|\downarrow, M, m + 1\rangle\right], \quad m \in [-M, M-1] \tag{15}$$

Note that the index m that we are using to label tje eigenvalues is actually related to a conserved quantity; the total spin operator along z, S^z. That is, the eigenstates $|\Psi^M_{m,\pm}\rangle$ have $S^z = S^z_d + s^z_{\rm tot} = m + \frac{1}{2}$.

IV. PARITY OPERATORS AND THE IMPURITY MAGNETIZATION

We can also use the eigenvalues of π^z to label the eigenstates. We have the eigenvalue relation $\pi^z\,|S^z\rangle = (-1)^{S^z - \frac{1}{2}}\,|S^z\rangle$. Applying π^z on these eigenstates gives

$$\pi^z\,|\Psi^M_{m,-}\rangle = (-1)^m\,|\Psi^M_{m,-}\rangle \tag{16}$$

This shows that the operator π^z splits the $K-$fold degenerate ground state manifold into two sub-manifolds of π^z parity ± 1. The odd values of m form the negative parity sector $\pi^z = -1$ while the even values form the positive parity sector $\pi^z = 1$.

It is interesting to calculate the action of the parity operator π^x on these eigenstates. We can write that operator as

$$\pi^x = (S^+_d + S^-_d) \otimes^K_{i=1} \left(s^+_i + s^-_i\right) \tag{17}$$

We can write the eigenstates of $s^z_{\rm tot}$ in terms of the eigenstates of s^z_i using their Clebsch-Gordon coefficients:

$$|M, m\rangle = \sum_{\substack{\{m_i\} \\ \sum_i m_i = m}} |m_1, m_2, ..., m_K\rangle\, C_{\{m_i\},m,M} \tag{18}$$

where $C_{\{m_i\},m,M} = \langle m_1, m_2, ..., m_K | m, M\rangle$ are the Clebsch-Gordon coefficients. Applying the parity operator on the state gives

$$\pi^x |S_d^z, M, m\rangle = \sum_{\substack{\{m_i\} \\ \sum_i m_i = m}} C_{\{m_i\},m,M} (S_d^+ + S_d^-) |S_d^z\rangle \otimes_{i=1}^K \left(s_i^+ + s_i^-\right) |m_i\rangle = \sum_{\substack{\{m_i\} \\ \sum_i m_i = m}} C_{\{m_i\},m,M} |-S_d^z\rangle \otimes_{i=1}^K |-m_i\rangle \tag{19}$$

$$= \sum_{\substack{\{m_i\} \\ \sum_i m_i = -m}} C_{\{-m_i\},m,M} |-S_d^z\rangle \otimes_{i=1}^K |m_i\rangle \tag{20}$$

Using the identity $\langle M_1 m_1, M_2 m_2, \dots | m, M\rangle = (-1)^{\sum_i M_i - M} \langle M_1, -m_1; M_2 - m_2; \dots | -m, M\rangle$, we have $C_{\{-m_i\},m,M} = C_{\{m_i\},-m,M}$. Substituting this, we get

$$\pi^x |S_d^z, M, m\rangle = \sum_{\substack{\{m_i\} \\ \sum_i m_i = -m}} C_{\{m_i\},-m,M} |-S_d^z\rangle \otimes_{i=1}^K |m_i\rangle = |-S_d^z, M, -m\rangle \tag{21}$$

The action of π^x on the eigenstates can now be determined:

$$\pi^x |\Psi_{m,-}^M\rangle = -\sqrt{x_m^M} |\downarrow, M, -m\rangle + (M + m + 1) |\uparrow, M, -m - 1\rangle = - |\Psi_{-m-1,-}^M\rangle \tag{22}$$

It can be seen that the action of the operator π^x is to interpolate between two ground states of opposite π^z eigenvaluesand, more specifically, opposite S^z values

$$\pi^x |S^z\rangle \to |-S^z\rangle \tag{23}$$

We can show that the impurity magnetization along the $x-$direction in some specific ground states can be related to matrix elements of the string operator π^x. The most general ground state is obtained by taking a general linear combination of all the normalized ground states:

$$|g_m^n\rangle = \frac{1}{\sqrt{2}} \left[|\Psi_{m,-}^M\rangle + e^{i\theta} |\Psi_{m+n,-1}^M\rangle\right], m \in [-M, M-2], n \in [1, M-m-1] \tag{24}$$

The expectation value of the impurity spin along x in this state is given by

$$(m_d^x)_m^n = \langle g_m^n | \sigma_d^x | g_m^n\rangle \tag{25}$$

The action of σ_d^x is as follows:

$$\sigma_d^x |\Psi_{m,-}^M\rangle = \frac{1}{\sqrt{(M+m+1)(1+2M)}} \left[-\sqrt{x_m^M} |\downarrow, M, m\rangle + (M + m + 1) |\uparrow, M, m + 1\rangle\right] \tag{26}$$

which means that the only $|\Psi_{m+n,-}^M\rangle$ with $n > 0$ that give a non-zero inner product with $\sigma_d^x |\Psi_{m,-}^M\rangle$ is $n = 1$. This observation then gives

$$(m_d^x)_m^n = \delta_{n,1} \langle g_m^1 | \sigma_d^x | g_m^1\rangle = \delta_{n,1} \left[e^{i\theta} \langle \Psi_{m+1,-}^M | \sigma_d^x | \Psi_{m,-}^M\rangle + \text{h.c.}\right] = \delta_{n,1} \left[e^{i\theta} \langle \Psi_{m+1,-}^M | \sigma_d^x \pi^x | \Psi_{-m-1,-}^M\rangle + \text{h.c.}\right] \tag{27}$$

We thus see that the impurity magnetization $(m_d^x)_m^n$ along x for the state $|g_m^n\rangle$ is determined by the matrix element of the string operator $\sigma_d^x \pi^x$.

For $m = -M$, we get

$$\langle \Psi_{-M+1,-}^M | \sigma_d^x \pi^x | \Psi_{M-1,-}^M\rangle = \frac{\sqrt{2M-1}}{2M+1} \tag{28}$$

The ground state manifold is defined by $M = K/2$. Using this, we can write

$$(m_d^x)_{-M}^1 = \left[\langle \Psi_{1-M,-}^M | \sigma_d^x \pi^x | \Psi_{M-1,-}^M\rangle + \text{h.c.}\right] = \frac{2\sqrt{2M-1}}{2M+1} = \frac{2\sqrt{K-1}}{K+1} \tag{29}$$

We can similarly calculate the magnetization along the $z-$direction. The most general ground state is

$$|g\rangle = \sum_{m=-M}^{M-1} C_m \, |\Psi^M_{m,-}\rangle \, , \qquad\qquad m \in [-M, M-1] \, , \qquad\qquad \sum_m |C_m|^2 = 1 \tag{30}$$

The expectation value of σ^d_z will be non-zero only in the diagonal terms: $\langle \Psi^M_m | \sigma^z_d | \Psi^M_{m'} \rangle = \delta_{mm'} \langle \Psi^M_m | \sigma^z_d | \Psi^M_m \rangle$. Using this, we get

$$\langle g | \sigma^z_d | g \rangle = \sum_{m=-M}^{M-1} |C_m|^2 \frac{\left[x^M_m - (M+m+1)^2 \right]}{(M+m+1)(1+2M)} \tag{31}$$

For the simpler case when $C_m = \delta_{m,\pm(M-\frac{1}{2})-\frac{1}{2}}$, we get

$$\langle g_\pm | \sigma^z_d | g_\pm \rangle = \pm \frac{2M-1}{2M+1} = \pm \frac{K-1}{K+1} \tag{32}$$

V. DERIVATION OF URG EQUATIONS FOR THE MULTI-CHANNEL KONDO MODEL

A. URG procedure

The URG method [14–19] involves applying unitary transformations to decouple high energy degrees of freedom. The renormalization at step j is given by

$$\Delta H_j = \left(c^\dagger T \frac{1}{\hat{\omega} - H_D} T^\dagger c + T^\dagger c \frac{1}{\hat{\omega} - H_D} c^\dagger T \right) \, , \tag{33}$$

where $c^\dagger T$ represents the off-diagonal part of the Hamiltonian with respect to the fluctuations of the electron we are decoupling and H_D is the diagonal part with respect to the same electron. Assuming we are decoupling a particular electron $q\beta$, we have

$$c^\dagger T = J \sum_{k<\Lambda_j, \alpha} \vec{S}_d \cdot \vec{s}_{\beta\alpha} c^\dagger_{q\beta} c_{k\alpha}, \qquad\qquad H_D = \epsilon_q \tau_{q\beta} + J S^z_d s^z_q \tag{34}$$

Usually we treat the $\hat{\omega}$ as number(s) and study the renormalization in the couplings as functions of the quantum fluctuation scales. Each value of the fluctuation scale defines an eigendirection of $\hat{\omega}$. We have then essentially traded off the complexity in the non-commutation of the diagonal and off-diagonal terms for all the directions in the manifold of $\hat{\omega}$.

Here we will do something different. We will redefine the $\hat{\omega}$ by pulling out the off-diagonal term from it: $\hat{\omega} \to \hat{\omega} - H_X$, and then study the renormalization at various orders by expanding the denominator in powers of H_X. Such a redefinition essentially amounts to a rotation of the eigendirections of $\hat{\omega}$. This is done in order to extract some information out of $\hat{\omega}$, specifically the dependence of the RG equations on the channel number $K = \sum_\gamma$. This dependence is in principle present even if we do not do such a redefinition and expansion, in the various directions and values of ω, because those values encode the non-perturbative information regarding scattering at all loops. However, it is difficult to read off this information directly. This step of redefinition followed by expansion is being done with the sole aim of exposing such information.

The expansion we are talking about is

$$\eta = \frac{1}{\hat{\omega} - H_D} T^\dagger c = \frac{1}{\omega' - H_D - H_X} T^\dagger c \simeq \frac{1}{\omega' - H_D} T^\dagger c + \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c \tag{35}$$

where $H_X = J \sum_{k,k'<\Lambda_j,\alpha,\alpha'} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} c^\dagger_{k\alpha} c_{k'\alpha'}$ is scattering between the entangled electrons. With this change, the second and third order renormalizations will take the form

$$\Delta H^{(2)}_j = c^\dagger T \frac{1}{\omega' - H_D} T^\dagger c + T^\dagger c \frac{1}{\omega - H_D} c^\dagger T \tag{36}$$

$$\Delta H^{(3)}_j = c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c + T^\dagger c \frac{1}{\omega - H_D} H_X \frac{1}{\omega - H_D} c^\dagger T \tag{37}$$

We will use the identity

$$S_d^a S_d^z S_d^b = \left(\frac{1}{4}\delta^{az} + \frac{i}{2}\sum_c \epsilon^{azc} S_d^c\right) S_d^b = \left(\frac{1}{4}\delta^{az} S_d^b + \frac{i}{8}\epsilon^{azb} - \frac{1}{4}\sum_{c_1,c} \epsilon^{azc_1}\epsilon^{c_1bc} S_d^c\right) = \frac{1}{4}\left(\delta^{az} S_d^b - \delta^{ab} S_d^z + \delta^{bz} S_d^a\right) \quad (38)$$

B. Leading order renormalization

$$\Delta H_j^{(2)} = \underbrace{c^\dagger T \frac{1}{\omega' - H_D} T^\dagger c}_{\text{first term}} + \underbrace{T^\dagger c \frac{1}{\omega - H_D} c^\dagger T}_{\text{second term}} \quad (39)$$

This renormalization is identical to that in the single channel. There is no additional physics due to the presence of multiple channels at this order. It is shown in appendix A.

C. Next-to-leading order renormalization

$$\Delta H_j^{(3)} = \underbrace{c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c}_{\text{first term}} + \underbrace{T^\dagger c \frac{1}{\omega - H_D} H_X \frac{1}{\omega - H_D} c^\dagger T}_{\text{second term}} \quad (40)$$

A general term of this summation has three sets of spin operators coming from $c^\dagger T, H_X$ and $T^\dagger c$. If we had expressed the spin operators in terms of $S^z, S^\pm$, most of the terms would have atleast one S^+ or S^-, and by the same argument as in the single-channel case, the denominator will have anti-parallel spins and the Ising term will be negative, leading to the form: $\omega - D/2 - \epsilon_k/2 + J/4$. ϵ_k is the energy of the other electron that will be summed over. The only term that does not have even one $S^\pm$ is the one with three S^z. We can show that this term will also have the same denominator. An instance of this term (in shorthand) is

$$S_d^z c_{q\uparrow}^\dagger \frac{1}{\omega - D/2 - \epsilon_k/2 + J/2 S_d^z} S_d^z \frac{1}{\omega - D/2 - \epsilon_k/2 + J/2 S_d^z} S_d^z c_{q\uparrow} \quad (41)$$

$$(42)$$

This can be split into up and down configurations of the impurity spin using the decomposition $S_d^z = \frac{1}{2}\left(\frac{1}{2} + S_d^z\right) - \frac{1}{2}\left(\frac{1}{2} - S_d^z\right)$. These configurations will have different quantum fluctuation scales ω, ω':

$$\frac{1}{2} S_d^z c_{q\uparrow}^\dagger \left[\frac{\left(\frac{1}{2} + S_d^z\right)}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} - \frac{\left(\frac{1}{2} - S_d^z\right)}{\left(\omega' - D/2 - \epsilon_k/2 - J/4\right)^2}\right] S_d^z c_{q\uparrow} \quad (43)$$

If we now use poor man's scaling values to relate the two ωs, we get $\omega' - \omega = J/2$. Substituting this will make both the denominators identical: $\omega - D/2 - \epsilon_k/2 + J/4$. This means that the denominator for all non-zero terms that renormalize the Hamiltonian is $\omega - D/2 - \epsilon_k/2 + J/4$.

1. *Calculation of first term*

$$c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = J^3 \sum_{\substack{q,k,k_1,k_2,\\ \beta,\alpha,\alpha_1,\alpha_2,\\ l_1,l_2,a,b,c}} \frac{c_{q\beta,l_1}^\dagger c_{k\alpha,l_1} S_d^a s_{\beta\alpha}^a S_d^b s_{\alpha_1\alpha_2}^b c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} c_{k\alpha,l_1}^\dagger c_{q\beta,l_1} S_d^c s_{\alpha\beta}^c}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} \quad (44)$$

q sums over the momenta being decoupled. k, k_1, k_2 sum over the momenta not being decoupled. $\beta, \alpha, \alpha_1, \alpha_2$ sum over the spin indices. l_1, l_2 sum over the channels. We will start simplifying this equation by summing over q. $c_{q\beta}^\dagger$

and $c_{q\beta}$ can be easily combined to form $\hat{n}_{q\beta}$, because they anti-commute with the other momenta. The sum gives $\sum_q \hat{n}_{q\beta l_1} = n(D)$. This gives

$$c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = J^3 n(D) \sum_{\substack{k,k_1,k_2,\\ \beta,\alpha,\alpha_1,\alpha_2,\\ l_1,l_2,a,b,c}} \frac{c_{k\alpha,l_1} S_d^a s_{\beta\alpha}^a S_d^b s_{\alpha_1\alpha_2}^b c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} c_{k\alpha,l_1}^\dagger S_d^c s_{\alpha\beta}^c}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} \tag{45}$$

The operators $c_{k\alpha}^\dagger$ and its conjugate can be brought together without any change of sign because there will be an even number of flips. The sum over k gives

$$\sum_k \frac{1 - \hat{n}_{k\alpha l_1}}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} = \rho \int \frac{d\epsilon \left[1 - \hat{n}(\epsilon)_{\alpha l_1}\right]}{\left(\omega - D/2 - \epsilon/2 + J/4\right)^2} = \rho \int_0^{D-2(\omega+J/4)} \frac{d\epsilon}{\left(\omega - D/2 - \epsilon/2 + J/4\right)^2} \tag{46}$$

The integration limits include only the positive energies, because of the $1-\hat{n}$ operator; the upper limit of the integration is chosen so as to make the denominator double, because this preserves the symmetry of the denominator. Performing the integration gives

$$\sum_k \frac{1 - \hat{n}_{k\alpha l_1}}{\left(\omega - D/2 - \epsilon_k/2 + J/4\right)^2} = -\frac{1}{2} \frac{\rho}{\omega - D/2 + J/4} \tag{47}$$

The sum over the channel index l_1 produces a factor of K. $K = \sum_{l_1}$ is the total number of conduction bath channels. The entire expression is now

$$c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = -\frac{1}{2} \frac{J^3 n(D) \rho K}{\omega - D/2 + J/4} \sum_{\substack{\beta,\alpha,\alpha_1,\alpha_2,\\ a,b,c}} S_d^a s_{\beta\alpha}^a S_d^b s_{\alpha_1\alpha_2}^b S_d^c s_{\alpha\beta}^c \sum_{k_1,k_2,l_2} c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} \tag{48}$$

We now need to simplify the spin products. The sum over α, β can be carried out immediately: $\sum_{\alpha,\beta} s_{\beta\alpha}^a s_{\alpha\beta}^c = \text{Trace}\left(s^a s^c\right) = \frac{1}{2}\delta^{ac}$. Substituting this gives

$$c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = -\frac{1}{2} \frac{J^3 n(D) \rho K}{\omega - D/2 + J/4} \frac{1}{2} \sum_{\alpha_1,\alpha_2,a,b} S_d^a S_d^b S_d^a \sum_{k_1,k_2,l_2} c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} s_{\alpha_1\alpha_2}^b \tag{49}$$

The spin product can now be carried out:

$$\sum_a S_d^a S_d^b S_d^a = \sum_a S_d^a \left[\frac{1}{4}\delta^{ab} + \frac{i}{2}\sum_c \epsilon^{bac} S_d^c\right] = \frac{1}{4} S_d^b - \frac{1}{4}\sum_{ace} \epsilon^{bac}\epsilon^{ace} S_d^e = \frac{1}{4} S_d^b - \frac{1}{4} S_d^b \sum_{ac} \epsilon^{bac}\epsilon^{acb} \tag{50}$$

$$= -\frac{1}{4} S_d^b \tag{51}$$

The renormalization becomes

$$\Delta H_1 = c^\dagger T \frac{1}{\omega' - H_D} H_X \frac{1}{\omega' - H_D} T^\dagger c = -\frac{1}{2} \frac{J^3 n(D) \rho K}{\omega - D/2 + J/4} \frac{1}{2} \left(-\frac{1}{4}\right) \sum_{k_1,k_2,\alpha_1,\alpha_2,b,l_2} S_d^b s_{\alpha_1\alpha_2}^b c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} \tag{52}$$

$$= \frac{1}{16} \frac{J^3 n(D) \rho K}{\omega - D/2 + J/4} \sum_{k_1,k_2,\alpha_1,\alpha_2,l_2} \vec{S_d} \cdot \vec{s}_{\alpha_1\alpha_2} c_{k_1\alpha_1,l_2}^\dagger c_{k_2\alpha_2,l_2} \tag{53}$$

2. Calculation of second term

Like the single-channel case, the renormalization coming from the hole excitations is exactly the Hermitian conjugate of that in the particle sector. And since the renormalization ΔH_1 is Hermitian, we have $\Delta H_0 = \Delta H_1$

D. Total renormalization $\Delta H^{(3)}$

The total renormalization is twice that in the particle sector.

$$\Delta H^{(3)} = \frac{1}{8}\frac{J^3 n(D)\rho K}{\omega - D/2 + J/4} \sum_{k_1,k_2,\alpha_1,\alpha_2} \vec{S}_d \cdot \vec{s}_{\alpha_1\alpha_2} c^\dagger_{k_1\alpha_1,l_2} c_{k_2\alpha_2,l_2} \tag{54}$$

Combining with $\Delta H^{(2)}$ and replacing $n(D) = \rho|\delta D|$, we get

$$\frac{\Delta J}{|\Delta D|} = -\frac{J^2\rho}{\omega - D/2 + J/4} + \frac{1}{8}\frac{J^3\rho^2 K}{\omega - D/2 + J/4} = -\frac{J^2\rho}{\omega - D/2 + J/4}\left[1 - \frac{1}{8}J\rho K\right] \tag{55}$$

We choose $\omega = -D/2$ to get a clearer idea of what the equations say.

$$\frac{\Delta J}{|\Delta D|} = \frac{J^2\rho}{D - J/4}\left[1 - \frac{1}{8}J\rho K\right] \tag{56}$$

Quantities with zero in the subscript will denote their values in the bare Hamiltonian. Using $\delta D = -|\delta D|$, we can write the continuum form of the equation:

$$\frac{\mathrm{d}J}{\mathrm{d}D} = \frac{J^2\rho}{D - J/4}\left(\frac{1}{8}J\rho K - 1\right) \tag{57}$$

For $D \gg J$, we can ignore the J in the denominator, and the equation reduces to the one-loop poor man's scaling form

$$\frac{\mathrm{d}J}{\mathrm{d}D} \simeq \frac{J^2\rho}{D}\left(\frac{1}{8}J\rho K - 1\right) \tag{58}$$

This equation has a stable fixed point at $J^* = \frac{8}{\rho K}$. The value of the fixed point agrees with the poor man's scaling calculation in [20], but differs from the value of $\frac{2}{\rho K}$ obtained in other references [21, 22] [**MOST PROBABLY THERE ARE OTHER REFERENCES**].

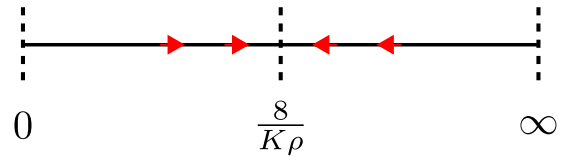

FIG. 1. Attractive finite J fixed point of poor man scaling RG equation

For D not so large, the denominator also comes into play, and eq.56 holds. We get the possibility of two fixed points - one from the numerator and the other from the numerator.

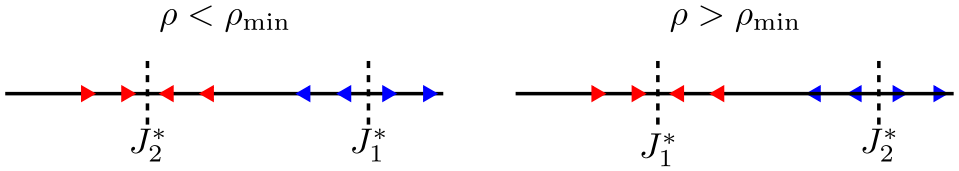

FIG. 2.

The numerator and denominator fixed points, J_1^* and J_2^* respectively, are given by

$$J_1^* = \frac{8}{K\rho}, \qquad D^* = \frac{J_2^*}{4} \tag{59}$$

For a given K, the position of J_1^* will be governed by ρ. In general, for each bare bandwidth D_0, there exists a minimal ρ, $\rho_{\min}(D_0)$, above which the the lower fixed point is the one from the numerator. That is, for $\rho > \rho_{\min}$, if

we start scaling from small J_0, it grows until it hits J_1^* which acts as the attractive fixed point, and J_2^* lies at a higher value and acts as the repulsive fixed point. For $\rho < \rho_{\min}$, J will grow and hit J_2^* instead, and $J_1^* > J_2^*$ now becomes the repulsive fixed point.

$$\rho_{\min} = \text{minimum} \left\{ \rho, \text{ such that } \frac{8}{K\rho} < 4D^*(\rho) \right\} \tag{60}$$

The RG flows towards the attractive fixed point J_1^* is shown in fig. 3.

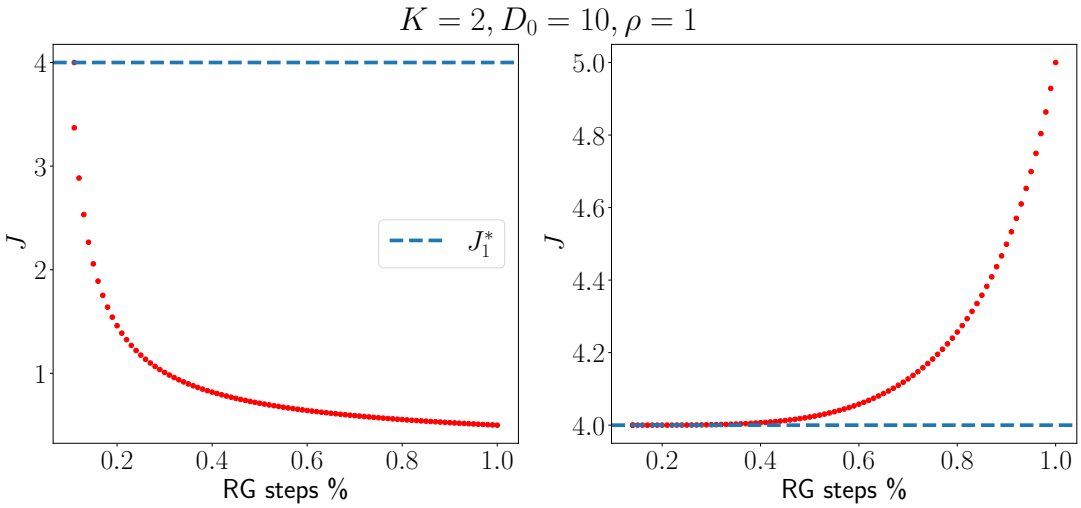

FIG. 3. Attractive flows towards J_1^*

This behaviour is shown schematically in fig. 2. In fig. 4, we plot $\rho_{\min}$ against the bare bandwidth. For large D_0,

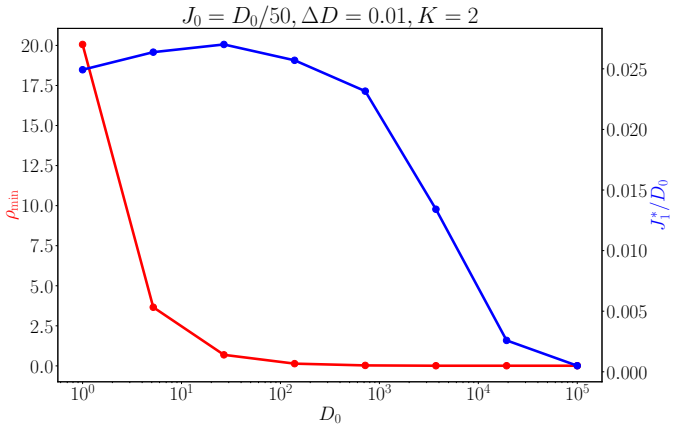

FIG. 4. Red curve shows variation of $\rho_{\min}$ against D_0. It vanishes at large D_0. Blue curve shows variation of the ratio J_1^*/D_0 with D_0. That shrinks as well, showing that the fixed point J_1^* remains finite in the thermodynamic limit, and the distance between J_1^* and J_2^* keeps growing.

it essentially shrinks to zero, and the numerator becomes the first fixed point for essentially all ρ.

If we assume we are at a sufficiently large D_0 and $\rho > \rho_{\min}$, the lower fixed point is J_1^*. As shown in fig. 4, we have $J_1^* \ll D_0$. If we start with J_0 in the neighborhood of J_1^*, we can use $J_1^* \ll D_0$ to ignore J in the denominator and the RG equation reduces to the poor man's scaling form eq. 58. The denominator fixed point has effectively moved off to infinity. That this is true can also be argued from the single-channel Kondo model URG results. There, we saw that when the bandwidth is scaled to larger values, the strong-coupling fixed point was stable at successively larger values of J^*. Since the denominator fixed point is identical in structure in both problems, its reasonable that the same thing will happen here.

VI. EFFECT OF ANISOTROPY ON RG FLOWS

For a general anisotropic multi-channel Kondo, the Hamiltonian is

$$H = \sum_{k,\alpha,\gamma} \epsilon_k^\gamma \hat{n}_{k\alpha}^\gamma + \sum_{kk',\gamma} J_\gamma \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} {c_{k\alpha}^\gamma}^\dagger c_{k'\alpha'}^\gamma \ . \tag{61}$$

Let us consider the specific case where $K-1$ channels have the same coupling $J_1 = J_2 = ... = J_{K-1} = J_+$ and the remaining channel has a different coupling $J_K = J_-$. The RG equations for such a model are

$$\frac{\Delta J_+}{|\Delta D|} = \frac{J_+^2 \rho}{D - J_+/4} - \frac{\rho^2 J_+}{8}\left[\frac{(K-1)J_+^2}{D-J_+/4} + \frac{J_-^2}{D-J_-/4}\right] \tag{62}$$

$$\frac{\Delta J_-}{|\Delta D|} = \frac{J_-^2 \rho}{D - J_-/4} - \frac{\rho^2 J_-}{8}\left[\frac{(K-1)J_+^2}{D-J_+/4} + \frac{J_-^2}{D-J_-/4}\right] \tag{63}$$

Setting $J_+ = J_-$ leads to the critical fixed point at $J_+^* = J_-^* = J_* = \frac{8}{K\rho}$. We now perturb around this fixed point by defining new variables $j_\pm = J_\pm - J_*$. We also assume that $D - J_\pm/4 \simeq D - J_*/4$. The RG equations then take the form

$$\frac{\Delta j_+}{|\Delta D|} = \frac{\rho J_+}{D - J_*/4}\left[J_+ - \frac{\rho}{8}\left[(K-1)J_+^2 + J_-^2\right]\right] \tag{64}$$

$$= \frac{\rho J_+}{D - J_*/4}\left[j_+ + J_* - \frac{1}{KJ_*}\left[(K-1)\left(J_* + j_+\right)^2 + \left(J_* + j_-\right)^2\right]\right] \tag{65}$$

$$= \frac{\rho J_+}{KJ_*\left(D - J_*/4\right)}\left[KJ_*\left(j_+ + J_*\right) - (K-1)\left(J_* + j_+\right)^2 - \left(J_* + j_-\right)^2\right] \tag{66}$$

$$= \frac{\rho J_+}{KJ_*\left(D - J_*/4\right)}\left[K\left(J_* j_+ + J_*^2\right) - (K-1)\left(J_*^2 + j_+^2 + 2J_* j_+\right) - \left(J_*^2 + j_-^2 + 2J_* j_-\right)\right] \tag{67}$$

$$= \frac{\rho J_+}{KJ_*\left(D - J_*/4\right)}\left[KJ_* j_+ - (K-1)\left(j_+^2 + 2J_* j_+\right) - \left(j_-^2 + 2J_* j_-\right)\right] \tag{68}$$

$$= \frac{\rho J_+}{KJ_*\left(D - J_*/4\right)}\left[-(K-2)J_* j_+ - (K-1)j_+^2 - j_-^2 - 2J_* j_-\right] \tag{69}$$

$$\frac{\Delta j_-}{|\Delta D|} = \frac{J_-\rho}{D - J_*/4}\left[J_- - \frac{\rho}{8}\left[(K-1)J_+^2 + J_-^2\right]\right] \tag{70}$$

$$= \frac{J_-\rho}{KJ_*\left(D - J_*/4\right)}\left[K\left(J_* j_- + J_*^2\right) - (K-1)\left(J_*^2 + j_+^2 + 2J_* j_+\right) - \left(J_*^2 + j_-^2 + 2J_* j_-\right)\right] \tag{71}$$

$$= \frac{J_-\rho}{KJ_*\left(D - J_*/4\right)}\left[(K-2)J_* j_- - j_-^2 - (K-1)j_+^2 - 2(K-1)J_* j_+\right] \tag{72}$$

$$\tag{73}$$

We will first look at the special case of $K = 2$, the two channel Kondo model. The equations simplify to

$$\frac{\Delta j_\pm}{|\Delta D|} = \frac{J_\pm \rho}{KJ_*\left(D - J_*/4\right)}\left[-\left(j_+^2 + j_-^2\right) - 2J_* j_\mp\right] \tag{74}$$

$$\tag{75}$$

For $j_- < 0, j_+ > 0$, we have $\Delta j_- < 0$. The coupling J_- therefore becomes irrelevant. For small j_+, we have $j_+^2 < 2J_*|j_-|$ and $\Delta j_+ > 0$. This means that the isotropic fixed point is repulsive under anisotropy [1]. The coupling j_+ being relevant means we have a single-channel Kondo problem. We already know the non-perturbative URG equation for the single-channel Kondo problem:

$$\frac{\Delta j_+}{|\Delta D|} = \frac{J_+^2 \rho}{D - J_+/4} \ , \tag{76}$$

and it leads to the strong-coupling fixed point

We now look at the general K channel case. Let us first look at the regime $j_- < 0, j_+ > 0$. In this regime, we have $\Delta j_- < 0$, which means j_- will flow to larger negative values until it reaches $j_- = -J_*$ such that $J_- = J_* + j_- = 0$.

j_+ is, on the other hand, relevant for small values of $j_\pm$. It will continue to grow until the numerator of Δj_+ vanishes. This condition is given by

$$(K-2)\,J_* j_+ + (K-1) j_+^2 + j_-^2 + 2 J_* j_- = 0 \tag{77}$$

Substituting $j_- = -J_*$ gives

$$(K-1) j_+^2 + (K-2)\,J_* j_+ - J_*^2 = 0 \tag{78}$$

Solving for j_+ gives

$$j_{+,*} = \frac{-J_*(K-2) \pm \sqrt{(K-2)^2 J_*^2 + 4(K-1)J_*^2}}{2(K-1)} = \frac{J_*}{2(K-1)}\left[-(K-2) \pm K\right] = \frac{J_*}{K-1} \tag{79}$$

At the final step, we chose the positive solution, because j_+ is relevant in this regime. The new fixed point value of J_+ is therefore

$$J_+ = J_* + \frac{J_*}{K-1} = \frac{\frac{8}{K\rho}K}{K-1} = \frac{8}{(K-1)\rho} \tag{80}$$

In other words, the K channel fixed point flows to the $K-1$ channel fixed point.

In the opposite regime $j_- > 0, j_+ < 0$, Δj_+ is negative. It has been checked numerically that J_+ ultimately flows to zero in this regime (fig. 5), and J_- remains relevant. Since there is no numerator fixed point in the relevant coupling J_- and because all other couplings are irrelevant, the equation for J_- is replaced by the single-channel Kondo coupling URG equation, and the low-energy physics is then of strong-coupling..

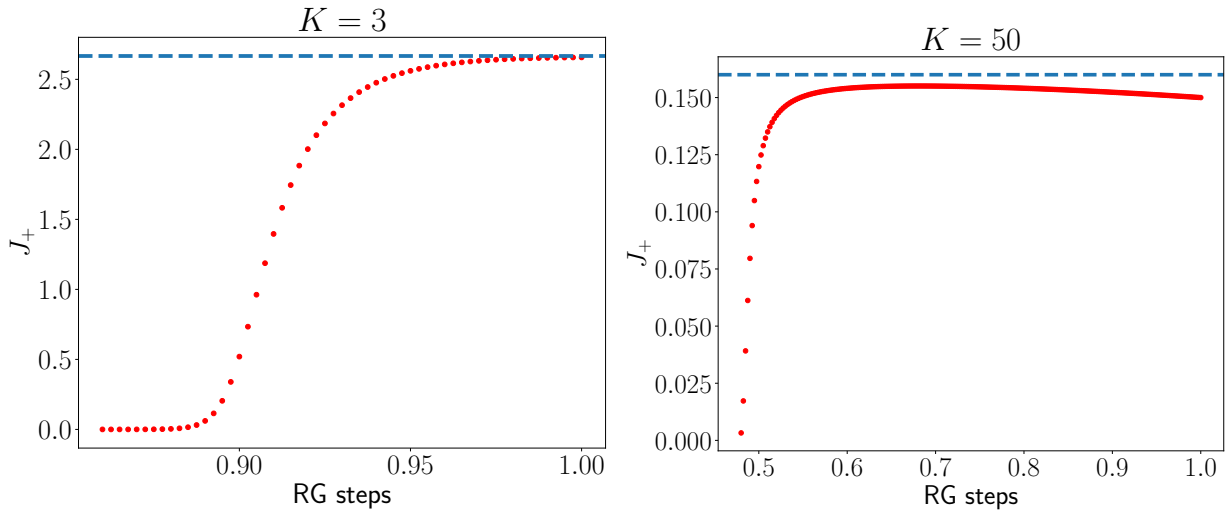

FIG. 5. Flow of the coupling J_+ when $j_+ < 0$ for two values of K

VII. DUALITY OF THE HAMILTONIAN RG FLOW

The infrared multi-channel Kondo Hamiltonian is given by

$$H = \sum_{k,\alpha,\gamma} \epsilon_k^\gamma \hat{n}_{k\alpha}^\gamma + J_< \sum_{kk',\gamma} \vec{S}_d \cdot \vec{s}_{\alpha\alpha'} {c_{k\alpha}^\gamma}^\dagger c_{k'\alpha'}^\gamma \ . \tag{81}$$

where $J_<$ indicates that it parametrizes the weak-coupling Hamiltonian. *Assuming one can also write a strong-coupling Hamiltonian that is identical to the weak-coupling one, the model can then be said to possess S-duality.* This has been demonstrated for the two-channel Kondo [23] by defining fermionic and bosonic operators F_σ and B from the ground states and excited states respectively of the zero-mode Hamiltonian eq. 7, then rewriting the Hamiltonian eq. 1 in

terms of these fermions and bosons by resolving the Hamiltonian in the eigenbasis of the zero mode Hamiltonian, and finally by decoupling the interaction between the bosons and the fermions through a Schrieffer-Wolff transformation (SWT) and projecting on to the fermionic states. The last step is what makes it a strong-coupling Hamiltonian: the SWT and the subsequent projection is perturbative in powers of a new coupling $J_> = t^2/J_<$ and makes sense only in the strong-coupling limit $J_< \gg t$. In that sense, the SWT removes the interaction between the ground state manifold and the excited levels. The result of these three steps is another two-channel Kondo model, but now parametrized by the strong-coupling coupling $J_>$ [23].

If a general $K-$channel Kondo model possesses S-duality under a transformation $J_> = \frac{\gamma D_0^2}{J_<} = \frac{\gamma 16 t^2}{J_<}$, $D_0 = 4t$ being the bare cut-off of the problem, we can write Hamiltonians $H(J_<)$ and $H(J_>)$ that are identical in form. This means that the two Hamiltonians will have identical properties. An immediate and obvious conclusion of this is that both the Hamiltonians will have identical RG flows:

$$\Delta J_< = \beta(J_<), \quad \Delta J_> = \beta(J_>) \tag{82}$$

where β represents the RG beta function. From URG calculations, we know that there is a unique attractive fixed point along the RG flow. This means that we must have $J_<^* = J_>^* = J^*$, and that $\beta(J_>^*) = 0$. From eq. 57, we get

$$J_>^* = \frac{8}{K\rho} = J_<^* \implies \frac{\gamma 16 t^2}{J_<^*} = J_<^* \implies \gamma = \left(\frac{J_<^*}{4t}\right)^2 = \left(\frac{2}{K\rho t}\right)^2 \tag{83}$$

This, then, is the transformation that makes the critical fixed point self-dual.

VIII. MFL AND ORTHOGONALITY CATASTROPHE IN LOW-ENERGY EXCITATIONS OF 2-CKM

At the stable fixed point $J^* = J_1^* = \frac{8}{K\rho}$, the ground states of the Hamiltonian are those of the star graph model, with a degeneracy of K. We now specialise to the two-channel Kondo model. To find the low-energy excitations on top of this ground state manifold, one can insert a tight-binding nearest-neighbour hopping between the zeroth site (the one that holds the impurity) and the first site (site that's nearest to the zeroth site) as a perturbation and calculate the diagonal and off-diagonal terms generated by this perturbation. It is found that when we trace out the impurity, we are left only with real space off-diagonal terms:

$$V_{\text{eff}} = \frac{2t^2}{J^*}\left[\left(\sigma_{0,1}^z\right)^2 s_{0,2}^+ + \left(\sigma_{0,2}^z\right)^2 s_{0,1}^+\right]\left(s_{1,1}^- + s_{1,2}^-\right) + \text{h.c.} \tag{84}$$

where $\sigma_{0,l}^z = \hat{n}_{0\uparrow,l} - \hat{n}_{0\downarrow,l}, s^+ = c_{0\uparrow,l}^\dagger c_{0\downarrow,l}$ and $s^- = (s^+)^\dagger$. The notation $0\sigma, l$ has the site index $i = 0, 1, 2, \ldots$ as the first label, the spin index $\sigma = \uparrow, \downarrow$ as the second label and the channel index $l = 1, 2$ as the third index.

These are the terms that are generated because of the presence of the impurity. Such a non-Fermi liquid contribution to the effective Hamiltonian and the absence of any Fermi-liquid term should be contrasted with the local Fermi liquid excitations induced by the singlet ground state of the single-channel Kondo model at fourth order. We wish to point out that such non-Fermi liquid terms were also obtained by Coleman, et al. [12] in terms of Majorana fermions at the zeroth site and the first site. They then went on to compute a single-particle self-energy renormalization coming from this non-Fermi liquid term that matches the phenomenological [24] and microscopic forms of the marginal Fermi liquid self-energy [14, 16]. We take a different route in order to calculate the self-energy contribution coming from eq. 84, one that involves [16].

In [14], it was found that the normal phase of the Mott insulator was a marginal Fermi liquid in momentum space, defined by the interaction term

$$H_{\text{MFL}} = \sum_{k,k',k'',\sigma} R\hat{n}_k \hat{n}_{k'\overline{\sigma}}\left(1 - \hat{n}_{k''\sigma}\right) \tag{85}$$

We wish to look for such a term in the effective Hamiltonian. For this, we will perform a perturbative treatment of the hopping at strong-coupling $J \to \infty$ where the perturbative coupling t^2/J is arbitrarily small and again obtain eq. 84. Sicne we are looking for a 2 particle-1 hole term in $k-$space, we will keep only those terms that can produce such a term:

$$V_{\text{eff}} = \frac{2t^2}{J}\sum_{l=1,2}\left(\sum_\sigma \hat{n}_{0\sigma,l}\right) s_{0,\bar{l}}^+ s_{1,\bar{l}}^- + \text{h.c.} \tag{86}$$

where $\bar{l} = 3 - l$. To compare with eq. 85, we will fourier transform this effective Hamiltonian (after inserting the kinetic energy of the bath) into $k-$space.

$$H_{\text{eff}} = \sum_{k\sigma,l} \epsilon_{k,l} \hat{n}_{k\sigma,l} + \sum_{\sigma,\{k_i,k_i'\},l} \frac{2t^2}{J} e^{i(k_1-k_1')a} c^\dagger_{k\sigma,l} c_{k'\sigma,l} c^\dagger_{k_2\uparrow,\bar{l}} c_{k_2'\downarrow,\bar{l}} c^\dagger_{k_1\downarrow,\bar{l}} c_{k_1'\uparrow,\bar{l}} + \text{h.c.} \tag{87}$$

This form of the Hamiltonian is very similar to the three-particle interaction term in Appendix B of [16]. The channel indices in eq. 87 can be mapped to the normal directions in [16]. Eq. 87 has a diagonal component.

IX. EFFECTIVE HAMILTONIAN FOR LOW-ENERGY EXCITATIONS: $k-$SPACE

The fixed point Hamiltonian is

$$H^* = H_0 + J^* \vec{S}_d \cdot \vec{s_{\text{tot}}} \tag{88}$$

where $H_0 = \sum_{k,l,\sigma} \epsilon_{k,l} \hat{n}_{k\sigma,l}$ and $\vec{s}_{\text{tot}} = \sum_l \vec{s}_l = \sum_{kk'\alpha\alpha',l} \vec{\sigma}_{\alpha\alpha'} c^\dagger_{k\alpha,l} c_{k'\alpha',l}$. l sums over the channels. Henceforth we will drop the $*$. Obtaining the effective Hamiltonian involves obtaining the low energy excitations on top of this fixed point Hamiltonian. The large-energy excitations are ones that involve spin flips. This guides the separation of the Hamiltonian into a diagonal and an off-diagonal piece:

$$H = H_d + V = \underbrace{H_0 + J S_d^z s_{\text{tot}}^z}_{H_d} + \underbrace{\frac{J}{2} S_d^+ s_{\text{tot}}^- + \text{h.c.}}_{V+V^\dagger} \tag{89}$$

We define V as the interaction term that decreases s^z_{tot} by 1: $V|s^z_{\text{tot}}\rangle \to |s^z_{\text{tot}} - 1\rangle$. Similarly, we define $V^\dagger |s^z_{\text{tot}}\rangle \to |s^z_{\text{tot}} + 1\rangle$. The effective Hamiltonian that has the states $|S^z_d, s_{\text{tot}}, s^z_{\text{tot}}\rangle$ as eigenstates are

$$H_{\text{eff}} = H_d + V \frac{1}{E_{\text{gs}} - H_d} V = \sum_{k,l,\sigma} \epsilon_{k,l} \hat{n}_{k\sigma,l} + J S_d^z s_{\text{tot}}^z + \frac{J}{2} S_d^+ s_{\text{tot}}^- \frac{1}{E_{\text{gs}} - J S_d^z s_{\text{tot}}^z - H_0} \frac{J}{2} S_d^- s_{\text{tot}}^+ \tag{90}$$

$$+ \frac{J}{2} S_d^- s_{\text{tot}}^+ \frac{1}{E_{\text{gs}} - J S_d^z s_{\text{tot}}^z - H_0} \frac{J}{2} S_d^+ s_{\text{tot}}^- \tag{91}$$

This is obtained from the Schrodinger equation for the ground state. If we expand the ground state in terms of $|S^z_d, s_{\text{tot}}, s^z_{\text{tot}}\rangle$, we have

$$|\Psi_{\text{gs}}\rangle = \sum_{S^z_d, s_{\text{tot}}, s^z_{\text{tot}}} C_{S^z_d, s_{\text{tot}}, s^z_{\text{tot}}} |S^z_d, s_{\text{tot}}, s^z_{\text{tot}}\rangle \tag{92}$$

The Schrodinger equation for the ground state can be written as

$$E_{\text{gs}} |\Psi_{\text{gs}}\rangle = H |\Psi_{\text{gs}}\rangle = (H_d + V) |\Psi_{\text{gs}}\rangle \implies (E_{\text{gs}} - H_d) \sum C_{S^z_d, s_{\text{tot}}, s^z_{\text{tot}}} |S^z_d, s_{\text{tot}}, s^z_{\text{tot}}\rangle = V \sum C_{S^z_d, s_{\text{tot}}, s^z_{\text{tot}}} |S^z_d, s_{\text{tot}}, s^z_{\text{tot}}\rangle \tag{93}$$

Since V only changes $S^z_d \to -S^z_d$ and $s^z_{\text{tot}} \to s^z_{\text{tot}} \pm 1$, we can simplify the equation into individual smaller equations. Let us take the case of two-channel, where the possible states are $s_{\text{tot}}, s^z_{\text{tot}} = (0,0), (1,-1), (1,0), (1,1)$. The individual equations for this model are

$$(E_{\text{gs}} - H_d) |S^z_d, 0, 0\rangle = (E_{\text{gs}} - H_d) |-\frac{1}{2}, 1, -1\rangle = (E_{\text{gs}} - H_d) |\frac{1}{2}, 1, 1\rangle = 0 \tag{94}$$

$$(E_{\text{gs}} - H_d) C_{\frac{1}{2},1,-1} |\frac{1}{2}, 1, -1\rangle = V C_{-\frac{1}{2},1,0} |-\frac{1}{2}, 1, 0\rangle \tag{95}$$

$$(E_{\text{gs}} - H_d) C_{-\frac{1}{2},1,0} |-\frac{1}{2}, 1, 0\rangle = V^\dagger C_{\frac{1}{2},1,-1} |\frac{1}{2}, 1, -1\rangle \tag{96}$$

$$(E_{\text{gs}} - H_d) C_{\frac{1}{2},1,0} |\frac{1}{2}, 1, 0\rangle = V C_{-\frac{1}{2},1,1} |-\frac{1}{2}, 1, 1\rangle \tag{97}$$

$$(E_{\text{gs}} - H_d) C_{-\frac{1}{2},1,1} |-\frac{1}{2}, 1, 1\rangle = V^\dagger C_{\frac{1}{2},1,0} |\frac{1}{2}, 1, 0\rangle \tag{98}$$

From eqs. 95 and 98, we can write

$$C_{\frac{1}{2},1,-1}\,|\frac{1}{2},1,-1\rangle = C_{-\frac{1}{2},1,0}\frac{1}{E_{\rm gs}-H_d}V\,|-\frac{1}{2},1,0\rangle\,,\qquad C_{-\frac{1}{2},1,1}\,|-\frac{1}{2},1,1\rangle = C_{\frac{1}{2},1,0}\frac{1}{E_{\rm gs}-H_d}V^\dagger\,|\frac{1}{2},1,0\rangle \tag{99}$$

Substituting these into eqs. 97 and 96 gives

$$E_{\rm gs}\,|\frac{1}{2},1,0\rangle = \left(H_d + V\frac{1}{E_{\rm gs}-H_d}V^\dagger\right)|\frac{1}{2},1,0\rangle \tag{100}$$

$$E_{\rm gs}\,|-\frac{1}{2},1,0\rangle = \left(H_d + V^\dagger\frac{1}{E_{\rm gs}-H_d}V\right)|-\frac{1}{2},1,0\rangle \tag{101}$$

$$\tag{102}$$

These equations represent the Schrodinger equation for the states $|S_d^z,1,0\rangle$, and the right hand sides therefore give the effective Hamiltonians for those states. If we combine the states into a single subspace $|1,0\rangle = \left\{|\frac{1}{2},1,0\rangle\,,|-\frac{1}{2},1,0\rangle\right\}$, the effective Hamiltonian for this composite subspace becomes the sum of the two parts:

$$H_{\rm eff}\,|1,0\rangle\,\langle 1,0| = \left(H_d + VG_0V^\dagger + V^\dagger G_0 V\right)|1,0\rangle \tag{103}$$

where $G_0 = (E_{\rm gs}-H_d)^{-1}$. If we expand the subspace as $|1,0\rangle = |\frac{1}{2},1,0\rangle + |-\frac{1}{2},1,0\rangle$, we recover eqs. 100. Solving similarly for the other states gives

$$H_{\rm eff}\,|1,1\rangle\,\langle 1,1| = \left(H_d + V^\dagger G_0 V\right)|1,1\rangle \tag{104}$$

$$H_{\rm eff}\,|1,-1\rangle\,\langle 1,-1| = \left(H_d + VG_0V^\dagger\right)|1,-1\rangle \tag{105}$$

One important conclusion that comes out of these calculations is that if the ground state is degenerate, the effective Hamiltonians is independent of which ground state we choose to start from in eq. 92. This is because the only difference in the various degenerate ground states is in the coefficients $C_{S_d^z,s_{tot},s_{tot}^z}$. Since the final effective Hamiltonians are independent of these coefficients, they will be the same irrespective of which ground state we start with.

To calculate these effective Hamiltonians, we will calculate the individual terms. We can easily simplify the S_d^z in the denominator of G_0, because $S_d^\pm \frac{1}{A+BS_d^z} = S_d^\pm \frac{1}{A\mp\frac{1}{2}B}$:

$$VG_0V^\dagger = \frac{J^2}{4}s_{\rm tot}^-\frac{\frac{1}{2}+S_d^z}{E_{\rm gs}+\frac{J}{2}s_{\rm tot}^z - H_0}s_{\rm tot}^+ \tag{106}$$

$$V^\dagger G_0 V = \frac{J^2}{4}s_{\rm tot}^+\frac{\frac{1}{2}-S_d^z}{E_{\rm gs}-\frac{J}{2}s_{\rm tot}^z - H_0}s_{\rm tot}^- \tag{107}$$

Since H_0 does not commute with the spin operators, we will need to expand the denominator to make sense of this Hamiltonian.

$$VG_0V^\dagger = s_{\rm tot}^-\frac{1}{E_{\rm gs}+\frac{J}{2}s_{\rm tot}^z}\left[1+\frac{1}{E_{\rm gs}+\frac{J}{2}s_{\rm tot}^z}H_0 + \frac{1}{E_{\rm gs}+\frac{J}{2}s_{\rm tot}^z}H_0\frac{1}{E_{\rm gs}+\frac{J}{2}s_{\rm tot}^z}H_0 + \ldots\right]s_{\rm tot}^+ \tag{108}$$

$$V^\dagger G_0 V = s_{\rm tot}^+\frac{1}{E_{\rm gs}-\frac{J}{2}s_{\rm tot}^z}\left[1+\frac{1}{E_{\rm gs}-\frac{J}{2}s_{\rm tot}^z}H_0 + \frac{1}{E_{\rm gs}-\frac{J}{2}s_{\rm tot}^z}H_0\frac{1}{E_{\rm gs}-\frac{J}{2}s_{\rm tot}^z}H_0 + \ldots\right]s_{\rm tot}^- \tag{109}$$

This is an expansion in $H_0^n/J^{n+1}, n = 0,1,2,\ldots$. The $n=0$ terms give

$$s_{\rm tot}^-\frac{1}{E_{\rm gs}+\frac{J}{2}s_{\rm tot}^z}s_{\rm tot}^+ = s_{\rm tot}^-s_{\rm tot}^+\frac{1}{E_{\rm gs}+\frac{J}{2}\left(s_{\rm tot}^z+1\right)} \tag{110}$$

$$s_{\rm tot}^+\frac{1}{E_{\rm gs}-\frac{J}{2}s_{\rm tot}^z}s_{\rm tot}^- = s_{\rm tot}^+s_{\rm tot}^-\frac{1}{E_{\rm gs}-\frac{J}{2}\left(s_{\rm tot}^z-1\right)} \tag{111}$$

One of the $n=1$ terms gives

$$s_{\rm tot}^-\frac{1}{E_{\rm gs}+\frac{J}{2}s_{\rm tot}^z}\frac{1}{E_{\rm gs}+\frac{J}{2}s_{\rm tot}^z}H_0s_{\rm tot}^+ = \left(\frac{1}{E_{\rm gs}+\frac{J}{2}\left(s_{\rm tot}^z+1\right)}\right)^2 s_{\rm tot}^-H_0s_{\rm tot}^+ \tag{112}$$

Next we calculate the commutator:

$$\left[s^+_{\rm tot}, H_0\right] = X^\dagger_{1,\rm tot} = \sum_l X^\dagger_{1,l} = \sum_{kk',l} \left(\epsilon_k - \epsilon_{k'}\right) c^\dagger_{k'\uparrow} c_{k\downarrow} \tag{113}$$

where $X_{n,l} \equiv \sum_{k,k'} \left(\epsilon_k - \epsilon_{k'}\right)^n c^\dagger_{k\downarrow} c_{k'\uparrow}$. Substituting this commutator gives

$$s^-_{\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} H_0 s^+_{\rm tot} = \left(\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right)^2 \left(s^-_{\rm tot} s^+_{\rm tot} H_0 - s^-_{\rm tot} X^\dagger_{1,\rm tot}\right) \tag{114}$$

The other $n = 1$ term gives

$$s^+_{\rm tot} \frac{1}{E_{\rm gs} - \frac{J}{2} s^z_{\rm tot}} \frac{1}{E_{\rm gs} - \frac{J}{2} s^z_{\rm tot}} H_0 s^-_{\rm tot} = \left(\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right)^2 \left(s^+_{\rm tot} s^-_{\rm tot} H_0 + s^+_{\rm tot} X_{1,\rm tot}\right) \tag{115}$$

One of the $n = 2$ terms gives

$$s^-_{\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} H_0 s^+_{\rm tot} \tag{116}$$

$$= \left(\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right)^2 s^-_{\rm tot} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} H_0 s^+_{\rm tot} \tag{117}$$

$$= \left(\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right)^2 s^-_{\rm tot} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} \left(s^+_{\rm tot} H_0 - X^\dagger_{1,\rm tot}\right) \tag{118}$$

$$= \left(\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right)^2 \left[s^-_{\rm tot} H_0 s^+_{\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} H_0 - s^-_{\rm tot} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} X^\dagger_{1,\rm tot}\right] \tag{119}$$

$$= \left(\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right)^2 \left[s^-_{\rm tot} \left(s^+_{\rm tot} H_0 - X^\dagger_{1,\rm tot}\right) \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} H_0 - s^-_{\rm tot} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} X^\dagger_{1,\rm tot}\right] \tag{120}$$

$$= \left(\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right)^2 \left[s^-_{\rm tot} s^+_{\rm tot} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} H_0 - s^-_{\rm tot} \left(X^\dagger_{1,\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} H_0 + H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} X^\dagger_{1,\rm tot}\right)\right] \tag{121}$$

$$\tag{122}$$

At this point, we need the commutator between H_0 and $s^z_{\rm tot}$:

$$\left[H_0, s^z_{\rm tot}\right] = Z_{1,\rm tot} = \sum_l Z_{1,l} = \sum_{k,k',l} \left(\epsilon_k - \epsilon_{k'}\right) \frac{1}{2} \left(c^\dagger_{k\uparrow,l} c_{k'\uparrow,l} - c^\dagger_{k\downarrow,l} c_{k'\downarrow,l}\right) \tag{123}$$

This gives the relation

$$H_0(a + b s^z_{\rm tot}) = (a + b s^z_{\rm tot}) H_0 + Z_{1,\rm tot} \implies \frac{1}{a + b s^z_{\rm tot}} H_0 = H_0 \frac{1}{a + b s^z_{\rm tot}} + \frac{1}{a + b s^z_{\rm tot}} Z_{1,\rm tot} \frac{1}{a + b s^z_{\rm tot}} \tag{124}$$

Using this, we get

$$s^-_{\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} H_0 s^+_{\rm tot} \tag{125}$$

$$= \left(\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right)^2 \left[s^-_{\rm tot} s^+_{\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} H_0 H_0 - s^-_{\rm tot} s^+_{\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} Z_{1,\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} H_0 \right. \tag{126}$$

$$\left. - s^-_{\rm tot} \left(X^\dagger_{1,\rm tot} \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} H_0 + H_0 \frac{1}{E_{\rm gs} + \frac{J}{2} s^z_{\rm tot}} X^\dagger_{1,\rm tot}\right)\right] \tag{127}$$

$$= \left(\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right)^2 s^-_{\rm tot} s^+_{\rm tot} \left[\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} H_0 H_0 - \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} Z_{1,\rm tot} H_0 \frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)}\right] \quad (128)$$

At the last step, we dropped the three-particle scattering terms.

The other $n = 2$ term gives

$$s^+_{\rm tot} \frac{1}{E_{\rm gs} - \frac{J}{2} s^z_{\rm tot}} \frac{1}{E_{\rm gs} - \frac{J}{2} s^z_{\rm tot}} H_0 \frac{1}{E_{\rm gs} - \frac{J}{2} s^z_{\rm tot}} H_0 s^-_{\rm tot} \quad (129)$$

$$= \left(\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right)^2 s^+_{\rm tot} H_0 \frac{1}{E_{\rm gs} - \frac{J}{2} s^z_{\rm tot}} H_0 s^-_{\rm tot} \quad (130)$$

$$= \left(\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right)^2 s^+_{\rm tot} H_0 \frac{1}{E_{\rm gs} - \frac{J}{2} s^z_{\rm tot}} \left(s^-_{\rm tot} H_0 + X_{1,\rm tot}\right) \quad (131)$$

$$= \left(\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right)^2 \left[s^+_{\rm tot} H_0 s^-_{\rm tot} \frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} H_0 + s^+_{\rm tot} H_0 \frac{1}{E_{\rm gs} - \frac{J}{2} s^z_{\rm tot}} X_{1,\rm tot}\right] \quad (132)$$

$$= \left(\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right)^2 s^+_{\rm tot} \left(s^-_{\rm tot} H_0 + X_{1,\rm tot}\right) \frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} H_0 \quad (133)$$

$$= \left(\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right)^2 \left[s^+_{\rm tot} s^-_{\rm tot} H_0 \frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} H_0\right] \quad (134)$$

$$= \left(\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right)^2 s^+_{\rm tot} s^-_{\rm tot} \left[\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} H_0^2 - \frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} Z_{1,\rm tot} \frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} H_0\right] \quad (135)$$

$$= \left(\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right)^2 s^+_{\rm tot} s^-_{\rm tot} \left[H_0^2 \frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} - \frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} Z_{1,\rm tot} H_0 \frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)}\right] \quad (136)$$

If we look at the effective Hamiltonian for a subspace $|s_{\rm tot}, s^z_{\rm tot}\rangle$, we can replace the following operators with scalars:

$$\frac{1}{E_{\rm gs} + \frac{J}{2}\left(s^z_{\rm tot}+1\right)} = \gamma^{s^z_{\rm tot}}_{s_{\rm tot}} \quad (137)$$

$$\frac{1}{E_{\rm gs} - \frac{J}{2}\left(s^z_{\rm tot}-1\right)} = \gamma^{-s^z_{\rm tot}}_{s_{\rm tot}}, \quad (138)$$

$$s^-_{\rm tot} s^+_{\rm tot} = s_{\rm tot}\left(s_{\rm tot}+1\right) - s^z_{\rm tot}\left(s^z_{\rm tot}+1\right) = \chi^{s^z_{\rm tot}}_{s_{\rm tot}} \quad (139)$$

$$s^+_{\rm tot} s^-_{\rm tot} = s_{\rm tot}\left(s_{\rm tot}+1\right) - s^z_{\rm tot}\left(s^z_{\rm tot}-1\right) = \chi^{-s^z_{\rm tot}}_{s_{\rm tot}} \quad (140)$$

They are not all independent: $\left(\gamma^{s^z_{\rm tot}}_{s_{\rm tot}}\right)^{-1} + \left(\gamma^{-s^z_{\rm tot}}_{s_{\rm tot}}\right)^{-1} = E_{\rm gs} + J, \chi^{s^z_{\rm tot}}_{s_{\rm tot}} - \chi^{-s^z_{\rm tot}}_{s_{\rm tot}} = -2s^z_{\rm tot}$. For the two-channel problem, we have four possible states in total: $(s_{\rm tot}, s^z_{\rm tot}) = \{(0,0),(1,-1),(1,0),(1,1)\}$. The states in the eq. 94 cannot be acted on by V or $V^\dagger$, so the effective Hamiltonian for these states will consist of only the diagonal part:

$$H_{\rm eff} = H_0 + \frac{J}{2} s^z_{\rm tot} \quad (141)$$

The factors γ and χ for the other states are

$$s_{\rm tot}, s^z_{\rm tot} = \quad (1,-1), \quad (1,0), \quad (1,1) \quad (142)$$

$$\gamma^{s^z_{\rm tot}}_{s_{\rm tot}} = \quad \frac{1}{E_{\rm gs}}, \quad \frac{1}{E_{\rm gs} + J/2}, \quad -- \quad (143)$$

$$\chi^{s^z_{\rm tot}}_{s_{\rm tot}} = \quad 2, \quad 2, \quad -- \quad (144)$$

$$\gamma^{-s^z_{\rm tot}}_{s_{\rm tot}} = \quad --, \quad \frac{1}{E_{\rm gs} + J/2}, \quad \frac{1}{E_{\rm gs}} \quad (145)$$

$$\chi_{s_{\rm tot}}^{-s_{\rm tot}^z} = \quad --, \quad 2, \quad 2 \tag{146}$$

$$\tag{147}$$

The effective Hamiltonians for these states are:

$$H_{\rm eff}^{1,1} = H_0 + JS_d^z + \frac{J^2}{4}\frac{2}{E_{\rm gs}}\left[1 + \frac{H_0}{E_{\rm gs}} + \frac{s_{\rm tot}^+ X_{1,\rm tot}}{2E_{\rm gs}} + \frac{H_0^2}{E_{\rm gs}^2} - \frac{Z_{1,\rm tot}H_0}{E_{\rm gs}^3}\right]\left(\frac{1}{2} - S_d^z\right) \tag{148}$$

$$H_{\rm eff}^{1,-1} = H_0 - JS_d^z + \frac{J^2}{4}\frac{2}{E_{\rm gs}}\left[1 + \frac{H_0}{E_{\rm gs}} - \frac{s_{\rm tot}^- X_{1,\rm tot}^\dagger}{2E_{\rm gs}} + \frac{H_0^2}{E_{\rm gs}^2} - \frac{Z_{1,\rm tot}H_0}{E_{\rm gs}^3}\right]\left(\frac{1}{2} + S_d^z\right) \tag{149}$$

$$H_{\rm eff}^{1,0} = H_0 + \frac{J^2}{2\left(E_{\rm gs} + J/2\right)}\left[1 + \frac{H_0 + (1/2 + S_d^z)\, s_{\rm tot}^+ X_{1,\rm tot} - (1/2 - S_d^z)\, s_{\rm tot}^- X_{1,\rm tot}^\dagger}{2\left(E_{\rm gs} + J/2\right)} + \frac{H_0^2}{\left(E_{\rm gs} + J/2\right)^2} - \frac{Z_{1,\rm tot}H_0}{\left(E_{\rm gs} + J/2\right)^3}\right] \tag{150}$$

$$\tag{151}$$

The terms have the following meanings:

$$H_0 = \sum_{k,\sigma,l} \epsilon_k^l \hat{n}_{k,\sigma,l} \tag{152}$$

$$H_0^2 = \sum_{k_1,k_2,l_1,l_2,\sigma_1,\sigma_2} \epsilon_k^{l_1}\epsilon_{k_2}^{l_2}\hat{n}_{k_1,\sigma_1,l_1}\hat{n}_{k_2,\sigma_2,l_2} \tag{153}$$

$$s_{\rm tot}^+ X_{1,\rm tot} = \sum_{k_1,q_1,k_2,q_2,l_1,l_2} c_{k_1,\uparrow,l_1}^\dagger c_{q_1,\downarrow,l_1}\left(\epsilon_{k_2}^{l_2} - \epsilon_{q_2}^{l_2}\right) c_{k_2,\downarrow,l_2}^\dagger c_{q_2,\uparrow,l_2} \tag{154}$$

$$s_{\rm tot}^- X_{1,\rm tot}^\dagger = \sum_{k_1,q_1,k_2,q_2,l_1,l_2} c_{k_1,\downarrow,l_1}^\dagger c_{q_1,\uparrow,l_1}\left(\epsilon_{k_2}^{l_2} - \epsilon_{q_2}^{l_2}\right) c_{q_2,\uparrow,l_2}^\dagger c_{k_2,\downarrow,l_2} \tag{155}$$

$$Z_{1,\rm tot}H_0 = \sum_{k_1,q_1,k_2,\sigma_2,l_1,l_2} \left(\epsilon_{k_1}^{l_1} - \epsilon_{q_1}^{l_1}\right)\epsilon_{k_2}^{l_2}\left(c_{k_1,\uparrow,l_1}^\dagger c_{q_1,\uparrow,l_1} - c_{k_1,\downarrow,l_1}^\dagger c_{q_1,\downarrow,l_1}\right)\hat{n}_{k_2,\sigma_2,l_2} \tag{156}$$

$$\tag{157}$$

X. IMPURITY SUSCEPTIBILITY FROM ZERO-MODE FIXED POINT HAMILTONIAN

The zero-mode approximation of the fixed point Hamiltonian is a star graph Hamiltonian:

$$H = J^* \vec{S_d} \cdot \vec{s}_{\rm tot} \tag{158}$$

where $\vec{s}_{\rm tot} = \sum_l \vec{s}_l$ is the total spin operator for all the channels. We insert a magnetic field that acts only on the impurity and then attempt to diagonalize the Hamiltonian.

$$H(h) = J^* \vec{S_d} \cdot \vec{s}_{\rm tot} + hS_d^z \tag{159}$$

The Hamiltonian commutes with $s_{\rm tot}^2$:

$$\left[s_{\rm tot}^2, H(h)\right] = \left[\sum_{i=x,y,z} {s_{\rm tot}^i}^2, J^* \sum_{i=x,y,z} S_d^i s_{\rm tot}^i\right] = \sum_{i,j} J^* S_d^i \left\{s_{\rm tot}^i, \left[s_{\rm tot}^i, s_{\rm tot}^j\right]\right\} = \sum_{i,j} J^* S_d^i \left\{s_{\rm tot}^i, i\epsilon^{ijk} s_{\rm tot}^k\right\} = 0 \tag{160}$$

This means the Hamiltonian is already block-diagonal in the quantum number $s_{\rm tot}$. Let us represent the quantum number of $s_{\rm tot}^z$ by m. For a particular $s_{\rm tot}$, m can take values from the set $[-s_{\rm tot}, s_{\rm tot}]$. The spin S_d^z can also take values $\pm\frac{1}{2}$. From now on, we will assume we are in the subspace of a particular $s_{\rm tot} = M$, so we will ignore that quantum number and write the kets simply as $|S_d^z, m\rangle$. So, the notation $|\uparrow, -1\rangle$ means the state with $S_d^z = \frac{1}{2}$ and $m = -1$. We will now show that even inside the block of $2 \times s_{\rm tot}$ (or $2 \times s_{\rm tot} + 1$, depending on where it is odd or even) defined by a particular value of $s_{\rm tot}$, the Hamiltonian actually separates into decoupled 2×2 blocks. To see why, first note that the terminal states $|\downarrow, -M\rangle$ and $|\uparrow, M\rangle$ are already eigenstates, because they cannot scatter (the impurity can only flip down, and this would require the bath to flip up, but $s_{\rm tot}^z$ is already at its maximum value M). The other

$2M-2$ states can be organized into 2×2 blocks formed by the states $|\uparrow, m\rangle$ and $|\downarrow, m+1\rangle$ for $m \in [-M, M-1]$. The fact that this block does not interact with the other blocks can be observation: if there was some other state which when acted upon by the Hamiltonian gave a non-zero projection on $|\uparrow, m\rangle$, it would have to come from $S_d^z = \downarrow$, and this would mean the bath spin would have had to flip down. This means the bath spin in that state would have to be $m+1$, and that is precisely the other state in the block.

Defining $\epsilon_m^h = \frac{1}{2}(Jm+h)$ and $x_m^M = M(M+1) - m(m+1)$, the 2×2 blocks can be written as

$$H_m = \begin{pmatrix} \epsilon_m^h & \frac{J}{2}\sqrt{x_m^M} \\ \frac{J}{2}\sqrt{x_m^M} & -\left(\epsilon_m^h + J/2\right) \end{pmatrix} \tag{161}$$

The eigenvalues are

$$\lambda_{m,\pm}^{M,h} = \frac{1}{2}\left[-J/2 \pm \sqrt{J^2/4 + J^2 x_m^M + 4\epsilon_m^h\left(\epsilon_m^h + J/2\right)}\right] = -J/4 \pm \sqrt{J^2 x_m^M/4 + \alpha^2} \tag{162}$$

where $\alpha = \epsilon_m^h + J/4$. The eigenvalues of the terminal states are $\pm\epsilon_{\pm M}^h$. For $h=0$, the ground state subspace is $K-$fold degenerate and is formed by the negative solutions of eq. 162. This common $K-$fold degenerate eigenvalue is $-J(M+1)/2$. The full list of energy eigenvalues at a particular value of M is

$$JM/2 - h/2, \qquad |\downarrow, M, -M\rangle, \qquad m = -M \tag{163}$$

$$-J/4 \pm \sqrt{J^2 x_m^M/4 + \alpha^2}, \qquad \{|\uparrow, M, m\rangle, |\downarrow, M, m+1\rangle\}, \qquad m = -M, \ldots, M-1 \tag{164}$$

$$JM/2 + h/2, \qquad |\uparrow, M, M\rangle, \qquad m = M \tag{165}$$

$$\tag{166}$$

The eigenstates for each value of M, m are given by

$$\left(\pm\frac{1}{2}\sqrt{J^2 x_m^{M\,2} + 4\alpha^2} - \frac{J}{2}\sqrt{x_m^M}\right)|\uparrow, M, m\rangle + \left(\frac{J}{2}(m+\frac{1}{2}) + \frac{h}{2}\right)|\downarrow, M, m+1\rangle \tag{167}$$

The partition function is

$$Z(h) = \sum_{M=M_{\min}}^{M_{\max}}\left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1}\left(e^{-\beta\lambda_{m,+}^{M,h}} + e^{-\beta\lambda_{m,-}^{M,h}}\right) + e^{-\beta\epsilon_M^h} + e^{\beta\epsilon_{-M}^h}\right] \tag{168}$$

$$= \sum_{M=M_{\min}}^{M_{\max}}\left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4}\cosh\beta\sqrt{J^2 x_m^M/4 + \alpha^2} + 2e^{-\beta JM/2}\cosh\beta h/2\right] \tag{169}$$

where $M_{\max} = K/2$ for a $K-$channel Kondo model, and $M_{\min} = 0$ if K is even, otherwise $1/2$. This is yet not the complete partition function, because we have not accounted for the possibility that there multiple subspaces of M. For example, the $K=3$ case states can be obtained by adding the third spin-half onto the states $S = 0, 1$. $S=0$ gives $s_{\text{tot}} = 1/2$ and $S=1$ gives $s_{\text{tot}} = 1/2, 3/2$. So, $s_{\text{tot}} = 1/2$ appears twice. These two subspaces are actually orthogonal, because the quantum numbers for the individual channels are different. We need to count the number of instances of a particular subspace $s_{\text{tot}} = M$. It turns out that this number is given by

$$r_M^K = {}^{K-1}C_{K/2-M} \tag{170}$$

which means the correct partition function is

$$Z(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K\left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4}\cosh\beta\sqrt{J^2 x_m^M/4 + \alpha^2} + 2e^{-\beta JM/2}\cosh\beta h/2\right] \tag{171}$$

To calculate the impurity magnetic susceptibility, we will use the expression

$$\chi = \frac{1}{\beta}\lim_{h\to 0}\left[\frac{Z(h)''}{Z(h)} - \left(\frac{Z(h)'}{Z(h)}\right)^2\right] \tag{172}$$

where the $\prime$ indicates derivative with respect to h. We will now calculate these derivatives. For that we will need

$$\frac{\mathrm{d}\epsilon_m^h}{\mathrm{d}h} = \frac{1}{2} = \frac{\mathrm{d}\alpha}{\mathrm{d}h} \tag{173}$$

$$\frac{\mathrm{d}\lambda_{m,\pm}^{M,h}}{\mathrm{d}h} = \pm\frac{\alpha}{\sqrt{J^2+4\alpha^2}} \tag{174}$$

We are now ready to compute the derivatives of Z:

$$\frac{\mathrm{d}Z(h)}{\mathrm{d}h} = \beta \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} e^{\beta J/4}\alpha \frac{\sinh\beta\sqrt{J^2x_m^M/4+\alpha^2}}{\sqrt{J^2x_m^M/4+\alpha^2}} + e^{-\beta JM/2}\sinh(\beta h/2)\right] \tag{175}$$

$$\frac{\mathrm{d}^2Z(h)}{\mathrm{d}h^2} = \beta \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\frac{1}{2}\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} \frac{e^{\beta J/4}}{\sqrt{J^2x_m^M/4+\alpha^2}} \left(\sinh\beta\sqrt{J^2x_m^M/4+\alpha^2} + \beta\alpha^2 \frac{\cosh\beta\sqrt{J^2x_m^M/4+\alpha^2}}{\sqrt{J^2x_m^M/4+\alpha^2}} \right.\right. \tag{176}$$

$$\left.\left. -\frac{\alpha^2\sinh\beta\sqrt{J^2x_m^M/4+\alpha^2}}{J^2x_m^M/4+\alpha^2}\right) + e^{-\beta JM/2}\frac{\beta}{2}\cosh(\beta h/2)\right] \tag{177}$$

$$\tag{178}$$

We will now take the limit of $h\to 0$. Note that $\alpha(h\to 0) = \frac{J}{2}(m+1/2)$ and hence

$$(J^2x_m^M/4+\alpha^2)(h\to 0) = \frac{J^2}{4}\left[M(M+1)-m(m+1)+(m+1/2)^2\right] = \frac{J^2}{4}(M+1/2)^2 \tag{179}$$

For brevity, we define $\theta_M = \beta J(M+1/2)/2$ and $\Sigma_M = \sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1}(m+1/2)^2$. It can be shown that this summation, for $M = M_{\max} = K/2$, evaluates to $\Sigma_{\max} = K(K+1)(K-1)/12$.

$$\lim_{h\to 0} Z(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4}\cosh\beta\frac{J}{2}(M+1/2) + 2e^{-\beta JM/2}\right] = \sum_{M=M_{\min}}^{M_{\max}} r_M^K\left[4Me^{\beta J/4}\cosh\theta_M + 2e^{-\beta JM/2}\right] \tag{180}$$

$$\lim_{h\to 0}\frac{\mathrm{d}Z(h)}{\mathrm{d}h} = \beta\sum_{M=M_{\min}}^{M_{\max}} r_M^K\left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} e^{\beta J/4}(m+1/2)\frac{\sinh\beta\frac{J}{2}(M+1/2)}{(M+1/2)}\right] = 0 \tag{181}$$

$$\lim_{h\to 0}\frac{\mathrm{d}^2Z(h)}{\mathrm{d}h^2} = \frac{\beta^2}{2}\sum_{M=M_{\min}}^{M_{\max}} r_M^K\left[\frac{e^{\beta J/4}}{\theta_M}\left(2M\sinh\theta_M + \frac{\beta^2J^2}{4}\left[\frac{\cosh\theta_M}{\theta_M} - \frac{\sinh\theta_M}{\theta_M^2}\right]\Sigma_M\right) + e^{-\beta JM/2}\right] \tag{182}$$

$$\tag{183}$$

These expressions have been used to compute the impurity susceptibility for various values of K in fig. 6.

Since the final expressions are formidable, we write down the expressions specifically for the single-channel and two-channel models. For single-channel, we have $M = \frac{1}{2}$ and $m = \pm\frac{1}{2}$. The terminal states are $S_d^z = -1/2, m = -1/2$ and $S_d^z = 1/2, m = 1/2$. There is therefore just one 2×2 block, and that is at $m = -1/2$.

$$\lim_{h\to 0} Z(h) = 2e^{\beta J/4}\cosh\beta\frac{J}{2} + 2e^{-\beta J/4} \tag{184}$$

$$\lim_{h\to 0}\frac{\mathrm{d}Z(h)}{\mathrm{d}h} = 0 \tag{185}$$

$$\lim_{h\to 0}\frac{\mathrm{d}^2Z(h)}{\mathrm{d}h^2} = \frac{\beta}{J}\left(e^{\beta J/4}\sinh\beta\frac{J}{2} + e^{-\beta J/4}\frac{J\beta}{2}\right) \tag{186}$$

$$\chi = \frac{1}{\beta}\lim_{h\to 0}\left[\frac{Z(h)''}{Z(h)} - \left(\frac{Z(h)'}{Z(h)}\right)^2\right] = \frac{1}{J}\frac{\left(2e^{\beta J/2}\sinh\beta\frac{J}{2} + J\beta\right)}{4e^{\beta J/2}\cosh\beta\frac{J}{2}+4} \tag{187}$$

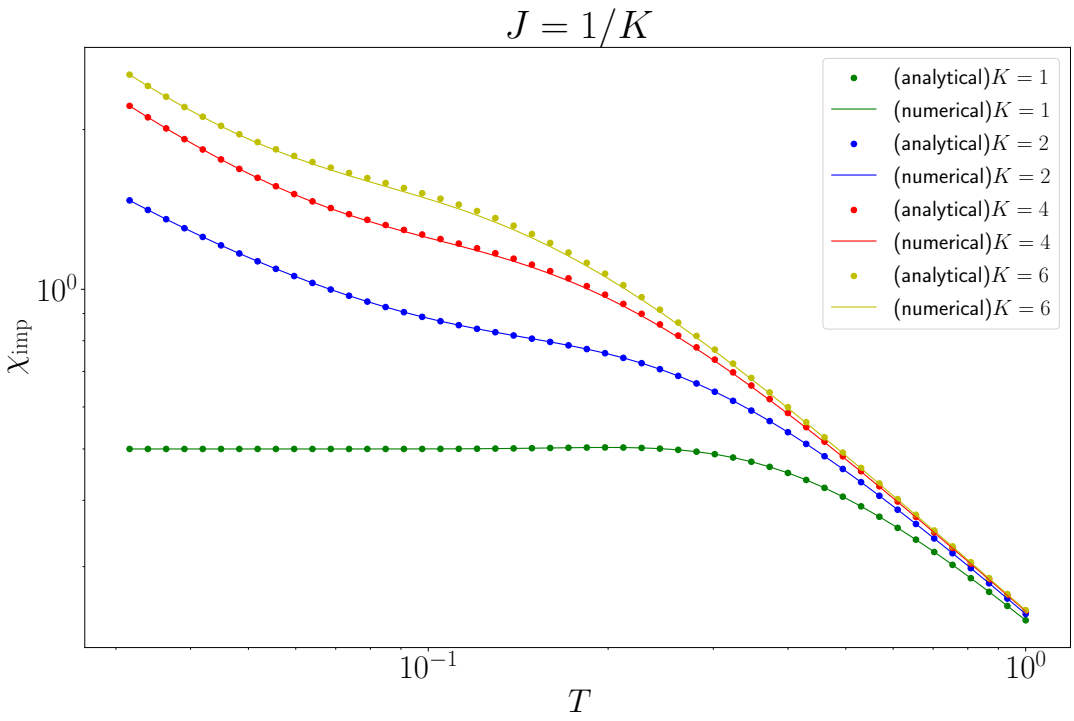

FIG. 6. Impurity susceptibility for $K = 1, 2, 4, 6$, calculated numerically as well as using the analytical expressions.

This expression matches with the direct calculation of the susceptibility of the single-channel Kondo model.

At low temperature $\beta \to \infty$, only the highest value $M_{\max}$ will survive:

$$Z \to 2r^K_{M_{\max}} M_{\max} e^{\beta \frac{J}{2}(M_{\max}+1)} \tag{188}$$

$$Z'' \to r^K_{M_{\max}} \left(\frac{\beta}{2(M_{\max}+1/2)}\right)^2 e^{\beta \frac{J}{2}(M_{\max}+1)} \Sigma_{\max} \tag{189}$$

$$\chi \to \frac{\beta \Sigma_{\max}}{2M_{\max}\left(2M_{\max}+1\right)^2} = \frac{\beta K(K+1)(K-1)/12}{K(K+1)^2} = \frac{\beta(K-1)}{12(K+1)} \tag{190}$$

At high temperatures $\beta \to 0$, we get

$$Z \to \sum_{M=M_{\min}}^{M_{\max}} r^K_M \left[4M+2\right] \tag{191}$$

$$Z'' \to \frac{\beta^2}{2} \sum_{M=M_{\min}}^{M_{\max}} r^K_M \left[2M+1\right] \tag{192}$$

$$\chi \to 1/4 \tag{193}$$

XI. BATH SUSCEPTIBILITY FROM ZERO-MODE FIXED POINT HAMILTONIAN

We insert a magnetic field that acts only on the bath and then attempt to diagonalize the Hamiltonian.

$$H(h) = J^* \vec{S}_d \cdot \vec{s}_{\rm tot} + h s^z_{\rm tot} \tag{194}$$

Defining $x^M_m = M(M+1) - m(m+1)$, the 2×2 blocks can be written as

$$H_m = \begin{pmatrix} m(J/2+h) & J\sqrt{x^M_m}/2 \\ J\sqrt{x^M_m}/2 & (m+1)(h-J/2) \end{pmatrix} \tag{195}$$

The eigenvalues are

$$\lambda_{m,\pm}^{M,h} = -J/4 + (2m+1)h/2 \pm \frac{1}{2}\sqrt{J^2 x_m^M + \left[(2m+1)h - J/2\right]^2 - 4m(m+1)(h^2 - J^2/4)} \tag{196}$$

$$= -J/4 + (2m+1)h/2 \pm \frac{1}{2}\sqrt{J^2(M+1/2)^2 + h^2 - (2m+1)Jh} = -J/4 + (2m+1)h/2 \pm \phi_m^M \tag{197}$$

The eigenvalues of the terminal states are $JM/2 \pm hM$. The partition function is

$$Z(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} \left(e^{-\beta\lambda_{m,+}^{M,h}} + e^{-\beta\lambda_{m,-}^{M,h}} \right) + e^{-\beta JM/2} \left(e^{\beta hM} + e^{-\beta hM} \right) \right] \tag{198}$$

$$= \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta(J/4-(m+1/2)h)} \cosh\beta\phi_m^M + 2e^{-\beta JM/2}\cosh\beta Mh \right] \tag{199}$$

We will now take the derivatives.

$$Z'(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta(J/4-(m+1/2)h)} \left(-\beta(m+1/2)\cosh\beta\phi_m^M + \beta\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\sinh\beta\phi_m^M \right) + 2\beta M e^{-\beta JM/2}\sinh\beta Mh \right] \tag{200}$$

$$Z''(h) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta(J/4-(m+1/2)h)} \left(\beta^2(m+1/2)^2\cosh\beta\phi_m^M - 2\beta^2(m+1/2)\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\sinh\beta\phi_m^M \right. \right. \tag{201}$$

$$\left. \left. + \beta\frac{\mathrm{d}^2\phi_m^M}{\mathrm{d}h^2}\sinh\beta\phi_m^M + \beta^2\left(\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\right)^2\cosh\beta\phi_m^M \right) + 2\beta^2 M^2 e^{-\beta JM/2}\cosh\beta Mh \right] \tag{202}$$

$$\tag{203}$$

In the limit of $h \to 0$, we have

$$Z(h\to 0) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4}\cosh\beta\phi_m^M + 2e^{-\beta JM/2} \right] \tag{204}$$

$$Z'(h\to 0) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4} \left(-\beta(m+1/2)\cosh\beta\phi_m^M + \beta\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\sinh\beta\phi_m^M \right) \right] \tag{205}$$

$$Z''(h\to 0) = \sum_{M=M_{\min}}^{M_{\max}} r_M^K \left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4} \left(\beta^2(m+1/2)^2\cosh\beta\phi_m^M - 2\beta^2(m+1/2)\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\sinh\beta\phi_m^M \right. \right. \tag{206}$$

$$\left. \left. + \beta\frac{\mathrm{d}^2\phi_m^M}{\mathrm{d}h^2}\sinh\beta\phi_m^M + \beta^2\left(\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h}\right)^2\cosh\beta\phi_m^M \right) + 2\beta^2 M^2 e^{-\beta JM/2} \right] \tag{207}$$

$$\tag{208}$$

The ϕ and the derivatives are actually at $h \to 0$. We are interested in the low temperature behaviour. In the limit of $h \to 0$, $\phi_m^M \to \phi^M = J(M+1/2)/2$. We can also look at the behaviour of the derivative:

$$\lim_{h\to 0}\frac{\mathrm{d}\phi_m^M}{\mathrm{d}h} = -\frac{J(m+1/2)}{4\phi^M} \tag{209}$$

$$\lim_{h\to 0}\frac{\mathrm{d}^2\phi_m^M}{\mathrm{d}h^2} = \frac{1}{4\phi^M} - \frac{J^2(m+1/2)^2}{16\left(\phi^M\right)^3} \tag{210}$$

(211)

Substituting these gives

$$Z(h\to 0)=\sum_{M=M_{\min}}^{M_{\max}} r_M^K\left[\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1} 2e^{\beta J/4}\cosh\beta\phi^M+2e^{-\beta JM/2}\right]=\sum_{M=M_{\min}}^{M_{\max}} r_M^K\left[4Me^{\beta J/4}\cosh\beta\phi^M+2e^{-\beta JM/2}\right] \tag{212}$$

$$Z'(h\to 0)=-\beta\sum_{M=M_{\min}}^{M_{\max}} r_M^K 2e^{\beta J/4}\left(\cosh\beta\phi^M+\frac{J}{4\phi^M}\sinh\beta\phi^M\right)\sum_{\substack{m=-M,\\ m\in\mathbb{Z}}}^{M-1}(m+1/2)=0 \tag{213}$$

$$Z''(h\to 0)=\beta^2\sum_{M=M_{\min}}^{M_{\max}} r_M^K\left[2e^{\beta J/4}\left\{\left(1+\frac{J^2}{16\phi^{M2}}\right)\Sigma_M\cosh\beta\phi^M+\left(\frac{2M}{4\beta\phi^M}+\frac{J}{2\phi^M}\left(1-\frac{J}{16\beta\phi^{M2}}\right)\Sigma_M\right)\sinh\beta\phi^M\right\}\right. \tag{214}$$

$$\left.+2M^2e^{-\beta JM/2}\right] \tag{215}$$

$$\tag{216}$$

where $\Sigma_M=\sum_m(m+1/2)^2$.

If we take the limit of $\beta\to\infty$, the hyperbolic functions can be replaced by exponentials. The terms with $1/\beta$ in Z'' drop out. All negative exponentials will also drop out. Moreover, out of all the positve exponentials, only the largest exponent will survive. The largest value Φ of ϕ^M occurs at $M=M_{\max}=K/2$. This maximum value is $\Phi=J(K+1)/4$. We therefore have

$$Z(h\to 0)=r_{K/2}^K 4Me^{\beta J/4}\frac{1}{2}e^{\beta\Phi} \tag{217}$$

$$Z''(h\to 0)=\beta^2 r_{K/2}^K 2e^{\beta J/4}\frac{1}{2}e^{\beta\Phi}\left(1+\frac{J}{4\Phi}\right)^2\Sigma_{\max} \tag{218}$$

$$\tag{219}$$

The susceptibility at low temperatures becomes

$$\chi(T\to 0)=\frac{2\beta\left(1+\frac{J}{4\Phi}\right)^2\Sigma_M}{4M}=\frac{\beta(K-1)(K+2)^2}{12(K+1)} \tag{220}$$

XII. NON-ANALYTICITY IN THE FREE ENERGY

For $K>1$, the Gibbs free energy at $T=0$ becomes non-analytic under insertion of a magnetic field on the impurity. The thermal free energy is given by

$$F(h)=-\frac{1}{\beta}\ln Z(h)=-\frac{1}{\beta}\ln\sum_{E_n}e^{-\beta E_n} \tag{221}$$

At $T\to 0$, only the most negative energy $E_{\min}$ survives. Assuming a $d'_{\rm gs}-$fold degenerate ground state for $h\neq 0$, the zero temperature free energy becomes

$$F(h\neq 0,T\to 0)=-\frac{1}{\beta}\ln d'_{\rm gs}e^{-\beta E_{\min}}=E_{\min}-k_BT\ln d'_{\rm gs} \tag{222}$$

In the star graph Hamiltonian with $K-$channels and a global magnetic field on the impurity

$$H=J\vec{S_d}\cdot\vec{s}_{\rm tot}+h\left(S_d^z+s_{\rm tot}^z\right)\ , \tag{223}$$

the energy eigenvalues for a particular value of $s_{\rm tot}=M$ are given by

$$(a)\quad \frac{J}{2}M\pm h\left(M+\frac{1}{2}\right) \tag{224}$$

$$(b) \quad -\frac{J}{4} \pm \frac{J}{2}\left(M+\frac{1}{2}\right) + h\left(m+\frac{1}{2}\right), m \in [-M, M-1] \tag{225}$$

For small h, the ground state will be $-\frac{J}{4} - \frac{J}{2}\left(M+\frac{1}{2}\right) + h\left(m_{\min}+\frac{1}{2}\right)$ for a particular value $m_{\min}$ that minimizes this energy. This specific value will depend on the sign of h:

$$m_{\min} = \begin{cases} -M, & h > 0 \\ M-1, & h < 0 \end{cases} \tag{226}$$

which means $E_{\min} = -\frac{J}{2}(M+1) - |h|\left(M-\frac{1}{2}\right)$. We also know, from eq. 170, that $d'_{\text{gs}} = {}^{K-1}C_{K/2-K/2} = 1$. The free energy for a non-zero field is therefore

$$F(h \neq 0, T \to 0) = -\frac{J}{2}(M+1) - |h|\left(M-\frac{1}{2}\right) \tag{227}$$

The first derivative of the free energy with respect to the field gives

$$F'(h \neq 0, T \to 0) = -\text{sign}(h)\left(M-\frac{1}{2}\right) \tag{228}$$

There we used the result that the derivative of $|x|$ is $\text{sign}(x)$. If we now take h to zero from both directions, we get

$$F'(h \to 0^{\pm}, T \to 0) = \mp\left(M-\frac{1}{2}\right) = \mp\frac{1}{2}(K-1) \tag{229}$$

The first derivative of the impurity free energy is therefore discontinuous as $h \to 0$; it goes to different values depending on the direction in which we take the limit. The only case where it is not analytic is when $K = 1$; then the derivative goes to zero from both directions. This non-analyticity has also been verified numerically.

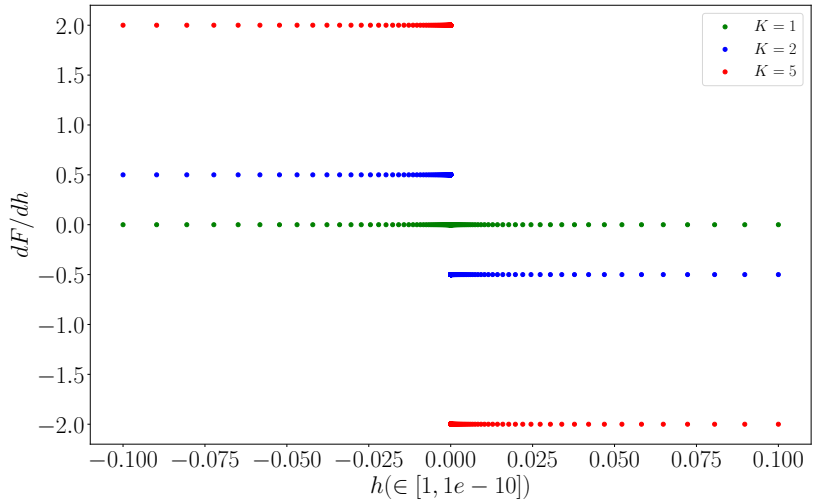

FIG. 7. Non-analytic free energy for $K > 1$ and analytic free energy for $K = 1$.

The non-analyticity for $K > 1$ occurs because the magnetic field is able to flip the ground state. For example, for $K = 2$, the states in question are $|M = 1, m = -1, 0\rangle$. For $h > 0$, the ground state occurs in the subspace $|S_d^z = 1/2, m = -1\rangle, |S_d^z = -1/2, m = 0\rangle$. If we now flip the magnetic field, the ground state subspace flips to $|S_d^z = 1/2, m = 0\rangle, |S_d^z = -1/2, m = 1\rangle$. Instead, if we look at the case of $K = 1$, the ground state is in the subspace of $|S_d^z = 1/2, m = -1/2\rangle, |S_d^z = -1/2, m = 1/2\rangle$, and since there is only this one subspace, the ground state is independent of the field. From this discussion, it is clear that the non-analyticity appears because there are multiple values of $m \in [-M, M-1]$ in the ground state manifold, which means that it is the ground state degeneracy that causes the non-analyticity.

XIII. THERMAL ENTROPY AT $T = 0$

Given the Helmholtz free energy $F = -k_B T \ln Z = -\frac{1}{\beta} \ln \sum_n e^{-\beta E_n}$, the thermal entropy at temperature T is given by

$$S(T) = -\frac{\partial F}{\partial T} = k_B \ln \sum_n e^{-\beta E_n} + \frac{\sum_n E_n e^{-\beta E_n}}{T \sum_n e^{-\beta E_n}} \tag{230}$$

In the limit of $T \to 0$, only the lowest energy state $E_{\rm gs}$ will survive. Assuming this state has a degeneracy $d_{\rm gs}$, we have

$$S(T \to 0) = k_B \ln \left(d_{\rm gs} e^{-\beta E_{\rm gs}} \right) + \frac{d_{\rm gs} E_{\rm gs} e^{-\beta E_{\rm gs}}}{T d_{\rm gs} e^{-\beta E_{\rm gs}}} = k_B \ln d_{\rm gs} - \frac{1}{T} E_{\rm gs} + \frac{1}{T} E_{\rm gs} = k_B \ln d_{\rm gs} \tag{231}$$

This is a general result that holds for any system with a $d_{\rm gs}$-fold degenerate ground state. For the star graph problem, we have a $K-$fold degenerate ground state, so the $T = 0$ entropy of the star graph Hamiltonian with K outer spins is $S(T \to 0) = k_B \ln K$. The impurity contribution to the total entropy is obtained by subtracting the entropy of the non-interacting bath. But we know that the entropy of a free Fermi gas vanishes at low temperatures. The impurity contribution is therefore

$$S_{\rm imp}(T \to 0) = k_B \ln K \tag{232}$$

We wish to point out that this value is different from the "standard" result

$$S = \ln 2 \cos \frac{\pi}{K+2} \tag{233}$$

obtained from other methods like the Bethe ansatz (BA) and conformal field theory (CFT) calculations [2, 3, 6, 11, 25, 26]. The difference is not only in the value but also in the origin. Eq. 232 describes the thermal entropy coming purely from the quantum mechanical degeneracy of the star graph problem (which is the zero mode of the full multi-channel Kondo problem), while the result eq. 233 presumably arises from the lowest-lying excitations on top of the zero mode arising from the conduction bath $k-$states. Another difference lies in the fact that while eq. 232 holds true independent of system size, eq. 233 is correct only in the thermodynamic limit and only if the limit of system size $L \to \infty$ is taken prior to taking the limit of temperature $T \to 0$ [2, 27, 28]. We note however that the entropy coming from the degeneracy is larger than that coming from the excitations (fig. 8), and consequently plays a very important role in the physics of the multi-channel Kondo problem, being responsible for the non-Fermi liquid physics and the critical behaviour near the fixed point.

Another related problem for which this result holds is the Ising model on a 1D chain. The partition function for that problem is $Z = \lambda_1^N + \lambda_2^N$, where $\lambda_{1,2} = e^{\beta J} \pm e^{-\beta J}$ are the eigenvalues of the $T-$matrix and N is the number of particles. As we come to $T \to 0$, we have $\lambda_{1,2} \simeq e^{\beta J}$ and $Z \simeq 2e^{\beta N J}$, so $F = -k_B T \ln 2 - NJ$ and the entropy becomes $S = -\partial_T F = k_B \ln 2$. Note that the factor of 2 in Z is the degeneracy $d_{\rm gs}$ described above.

Appendix A: URG analysis of the single-channel Kondo model

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k \tau_{k\sigma} + \sum_{k,l} J^z S_d^z s_{kl}^z + \frac{1}{2} \sum_{k,l} J^t \left(S_d^+ s_{kl}^- + S_d^- s_{kl}^+ \right) \tag{A1}$$

where $s_{kl}^z = \frac{1}{2} \left(c_{k\uparrow}^\dagger c_{l\uparrow} - c_{k\downarrow}^\dagger c_{l\downarrow} \right)$, $s_{kl}^- = c_{k\downarrow}^\dagger c_{l\uparrow}$ and $s_{kl}^+ = s_{lk}^{-\dagger}$. Also, $\tau = \hat{n} - \frac{1}{2}$. k, l sum over the momentum states. $\vec{S}_d$ is the impurity spin operator.

The scheme is that we will disentangle an electron $q\beta$ from the Hamiltonian, q being the momentum and β the spin. The diagonal part of the Hamiltonian under this scheme is

$$H_{q\beta}^D = \epsilon_q \tau_{q\beta} + J^z S_d^z s_{qq}^z \tag{A2}$$

The off-diagonal parts at a particular RG step H_1^I and H_0^I, that start from particle and hole states respectively, are

$$H_1^I = \sum_{|k|<\Lambda, q} J^z S_d^z s_{kq}^z + \frac{1}{2} \sum_{|k|<\Lambda, q} J^t \left(S_d^+ s_{kq}^- + S_d^- s_{kq}^+ \right) \tag{A3}$$

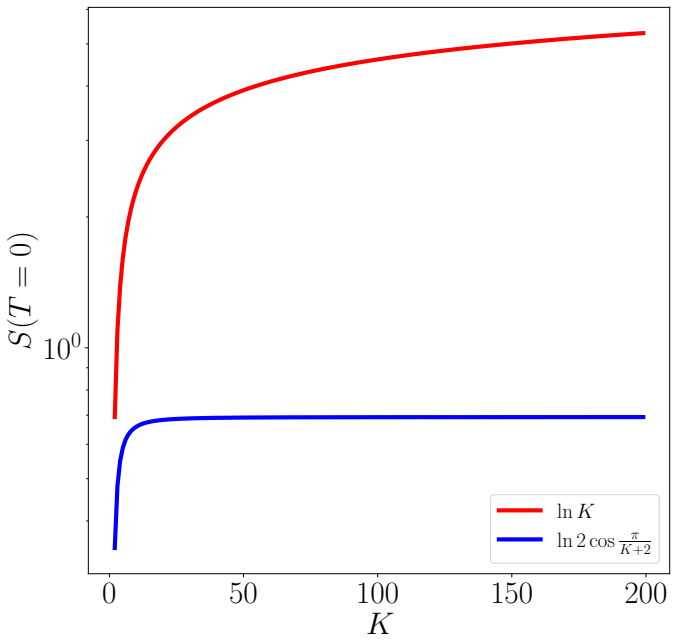

FIG. 8. Comparison of star graph zero temperature entropy with CFT/Bethe ansatz formula

$$H_0^I = \sum_{|k|<\Lambda,q} J^z S_d^z s_{qk}^z + \frac{1}{2}\sum_{|k|<\Lambda,q} J^t \left(S_d^+ s_{qk}^- + S_d^- s_{qk}^+\right) \tag{A4}$$

H_1^I is the Hamiltonian term that scatters from the occupied configuration of q, H_0^I is the same from the unoccupied configuration. These are the terms that appear in the numerator.

1. Particle sector

The particle sector involves integrating out those states which are occupied ($\hat{n}_{q\beta} = 1$). We will work at an energy shell $\epsilon_q = -D$. The renormalization is

$$H_0^I \frac{1}{\omega - H_{q\beta}^D} H_1^I \tag{A5}$$

Both H_0^I and H_1^I have all three operators $S_d^z, S_d^\pm$. We call S_d^z the spin-keep term and the others spin-flip terms. The entire product will thus have $3\times 3 = 9$ terms. Not all terms however renormalize the Hamiltonian. Those terms that have identical operators on both sides can be ignored because ${S_d^z}^2 =$ constant and ${S^\pm}^2 = 0$. The other six terms will renormalize the Hamiltonian. This brings in one more simplification: all the six terms that *will* renormalize the Hamiltonian have a spin flip operator on at least one side of the Greens function. This means that in the denominator of the Greens function, S_d^z and s_{qq}^z have to be anti-parallel in order to produce a non-zero result for that term. This means we can identically replace $S_d^z s_{qq}^z = -\frac{1}{4}$. Also, in the particle sector, the Greens function always has $c_{q\beta}$ in front of it, so $\epsilon_q \tau_{q\beta} = D/2$. Substituting all this, we get

$$\frac{1}{\omega - D/2 + J/4} \sum_{|k,k'|<\Lambda,q} \left[\frac{1}{2} J^z J^t \left(S_d^z S_d^+ s_{qk'}^z s_{kq}^- + S_d^z S_d^- s_{qk'}^z s_{kq}^+\right) + \frac{1}{2} J^t J^z \left(S_d^+ S_d^z s_{qk'}^- s_{kq}^z + S_d^- S_d^z s_{qk'}^+ s_{kq}^z\right)\right. \tag{A6}$$

$$\left. + \frac{1}{4} J^{t2} \left(S_d^- S_d^+ s_{qk'}^+ s_{kq}^- + S_d^+ S_d^- s_{qk'}^- s_{kq}^+\right)\right] \tag{A7}$$

We now simplify the products and keep only terms diagonal in q. For example: $s_{qk'}^z s_{kq}^+ = \frac{1}{2}\hat{n}_{q\downarrow} s_{kk'}^+$ and $s_{qk'}^z s_{kq}^- =$

$-\frac{1}{2}\hat{n}_{q\uparrow}s^-_{kk'}$. The renormalization becomes

$$\frac{1}{\omega - D/2 + J/4}\sum_{|k,k'|<\Lambda,q}\left[\frac{1}{4}J^zJ^t\left(-\frac{1}{2}S^+_d\hat{n}_q s^-_{kk'} - \frac{1}{4}S^-_d\hat{n}_q s^z_{kk'}\right) - \frac{1}{4}{J^t}^2 S^z_d\left(-\hat{n}_{q\uparrow}c^\dagger_{k\downarrow}c_{k'\downarrow} + \hat{n}_{q\downarrow}c^\dagger_{k\uparrow}c_{k'\uparrow}\right)\right] \tag{A8}$$

We now replace $\sum_q \hat{n}_{q\sigma} = n(D)$. The renormalization due to excitations coming from the particle sector is

$$\Delta H_1 = -\frac{1}{2}\frac{n(D)}{\omega - D/2 + J/4}\sum_{|k,k'|<\Lambda}\left[J^zJ^t\frac{1}{2}\left(S^+_d s^-_{kk'} + S^-_d s^z_{kk'}\right) + {J^t}^2 S^z_d s^z_{kk'}\right] \tag{A9}$$

The renormalization in the couplings coming from the particle sector is therefore,

$$\Delta J^z = -\frac{1}{2}\frac{{J^t}^2 n(D)}{\omega - D/2 + J/4}, \qquad \Delta J^t = -\frac{1}{2}\frac{J^zJ^t n(D)}{\omega - D/2 + J/4} \tag{A10}$$

2. Hole sector

The hole sector involves integrating out those states which are vacant ($\hat{n}_{q\beta} = 1$). We will work at an energy shell $\epsilon_q = D$. The renormalization is

$$H^I_1\frac{1}{\omega - H^D_{q\beta}}H^I_0 \tag{A11}$$

The same considerations as those in the particle sector apply here, and the denominator becomes $\omega - D/2 + J/4$, while the numerator is $H^I_1H^I_0$. Since this is just the Hermitian conjugate of the particle sector form, we do not need to calculate this separately, because the renormalization here will be $\Delta H_0 = \Delta H^\dagger_1 = \Delta H_1$.

3. Scaling equations

Since the renormalization in the hole sector is equal to that in the particle sector, the total renormalization is simply twice that in the particle sector (eqs. A10):

$$\Delta J^z = -\frac{{J^t}^2 n(D)}{\omega - D/2 + J/4}, \qquad \Delta J^t = -\frac{J^zJ^t n(D)}{\omega - D/2 + J/4} \tag{A12}$$

If we set $J_z = J_t = J$, we have an SU(2)-symmetric Kondo model $J\vec{S_d}\cdot\vec{s}$.

$$\Delta J = -\frac{J^2 n(D)}{\omega - D/2 + \frac{1}{4}J} \tag{A13}$$

To recover the one-loop form, we can replace ω with the bare value $-D/2$ and ignore the J in the denominator (small J).

$$\Delta J \approx \frac{J^2 n(D)}{D} \tag{A14}$$

$$\Delta J^{(2)} = -\frac{J^2 n(D)}{\omega - D/2 + J/4} \tag{A15}$$

For $\omega < D/2$, we get the flow towards the strong-coupling fixed point. That is, there appears a stable fixed point at $J^* = 4|\omega - D/2|$ for all bare $J > 0$. We also get a decay towards the local moment fixed point $J^* = 0$ for $J < 0$. For $\omega = -D/2$ and $J \ll D$, we get the one-loop PMS form.

$$\Delta J^{(2)} = \frac{J^2 n(D)}{D - J/4} \simeq \frac{J^2 n(D)}{D} \tag{A16}$$

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