

# Generating Hubbard Model Solutions from Anderson Impurity Model Solutions

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## 1 Introduction

This is an attempt to obtain various quantities like Greens functions, self-energies, spectral functions and (if possible) energies and wavefunctions of the Hubbard model, using a cluster-bath approach. The cluster-bath system is taken to be a single-impurity Anderson model with a correlated bath. The correlation will be brought about in two ways: a self-energy  $\Sigma(k, \omega)$  of the bath, and a double occupancy repulsion cost  $U_b$ . The Hubbard and the correlated single-impurity Anderson models are defined using the Hamiltonians

$$H_{\text{hubb}} = -t^H \sum_{\sigma, \langle i, j \rangle} \left( c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.} \right) + U^H \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - \mu^H \sum_{i\sigma} \hat{n}_{i\sigma} \quad (1.1)$$

$$H_{\text{siam}} = \sum_{k\sigma} [\epsilon_k + \Sigma(k, \omega)] \hat{n}_{k\sigma} + \epsilon_d^A \sum_{\sigma} \hat{n}_{d\sigma} + U^A \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} + U_b \sum_{kk'} \hat{n}_k \hat{n}_{k'} - t^A \sum_{k\sigma} \left( c_{d\sigma}^\dagger c_{k\sigma} + \text{h.c.} \right) \quad (1.2)$$

Broadly speaking, the method involves first solving the SIAM using a unitary renormalisation group approach, to get the low energy effective theory, and then combining the low energy Hamiltonians in a symmetrized fashion to get the Hamiltonian for the Hubbard model lattice. It is reminiscent of dynamical mean-field theory (DMFT) - both involve an impurity-solver that solves an auxiliary system. The difference, however, lies in the following points:

- While DMFT primarily works with Greens functions and self-energies, this method involves Hamiltonians. The impurity-solver in DMFT provides an impurity Greens function (which is then equated with the local Greens function of the bath), while the impurity-solver in this method actually provides a low energy Hamiltonian.
- The final step of DMFT is the self-consistency equation, where the impurity and bath-local quantities are set equal. This ensures all sites, along with the impurity site, have the same self-energy, something which is required on grounds of translational invariance. The present method, however, brings about the translational invariance in a different way. It symmetrizes the Hamiltonians itself, such that all quantities then derived from the Hamiltonian are then guaranteed to have the symmetry.

The meaning of each of these statements will become clearer when we describe the method in more detail.

## 2 Philosophy of the method

The method is closely tied to the auxiliary system approach described in [1]. We can view the full Hamiltonian as a sum of two component Hamiltonians  $H_1, H_2$  connected via the interaction term  $H_{12}$ .

$$H = \begin{pmatrix} H_1 & H_{12} \\ H_{12}^* & H_2 \end{pmatrix} = H_1 |1\rangle \langle 1| + H_2 |2\rangle \langle 2| + H_{12} |1\rangle \langle 2| + H_{12}^* |2\rangle \langle 1| \quad (2.1)$$

where  $|1(2)\rangle$  actually represents a sum over all basis kets of the subsystem 1(2). As an example, we can split the the Hubbard model Hamiltonian between a particular site  $i = p$  and the rest of the lattice as follows:

$$\begin{aligned} H_{\text{hubb}} = & \underbrace{U^H \hat{n}_{p\uparrow} \hat{n}_{p\downarrow} - \mu^H \sum_{\sigma} \hat{n}_{p\sigma}}_{H_1} \\ & + \underbrace{U^H \sum_{i \neq p} \hat{n}_{i\uparrow} \hat{n}_{p\downarrow} - \mu^H \sum_{i \neq p, \sigma} \hat{n}_{i\sigma} - t^H \sum_{\substack{\sigma, \langle i, j \rangle \\ i \neq p \neq j}} (c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.})}_{H_2} \\ & - \underbrace{t^H \sum_{\substack{\sigma, \\ i \in \text{N.N. of } p}} (c_{i\sigma}^\dagger c_{p\sigma} + \text{h.c.})}_{H_{12} + H_{12}^*} \end{aligned} \quad (2.2)$$

The Greens function of the full Hamiltonian can also be split in a similar fashion:

$$G(\omega) = \begin{pmatrix} G_1 & G_{12} \\ G_{12}^* & G_2 \end{pmatrix} \quad (2.3)$$

The subsystem 1 is usually taken to be the "smaller system", and consequently, subsystem 2 represents the "bath". The smaller system is typically chosen such that its eigenstates are known exactly. Progress is then made by choosing a simpler version of the bath  $H_2$  and a simpler form also for its coupling  $H_{12}$  with the smaller system. This combination of the smaller system and the simpler bath is then called the *auxiliary system*. A typical auxiliary system for the Hubbard model would be the SIAM, where the impurity represents an arbitrary site  $p$  of the lattice, the bath represents the rest of the lattice sites and the hybridisation term between the impurity and the bath represents the coupling term  $H_{12}$ . Such a construction is shown in fig. 1. *It should be noted that any reasonable choice of the cluster and bath would break the translational symmetry of the full model. To allow computing quantities, one would need to make the bath (which is a much larger system) simpler than the cluster (which is a single site). This distinction breaks the translational symmetry of the Hubbard model. For eg., if one chooses eq. 1.2 as the auxiliary system, then the fact that the impurity has an onsite correlation while the bath only has a global capacitive cost ( $\sim U_b N^2$ ) means we have broken the symmetry between the cluster and the bath.*

The algorithm of DMFT then involves starting with some local self-energy of the bath,  $\Sigma(\omega)$ , and using an impurity solver to calculate the impurity Greens function in the presence of this self-energy. This impurity Greens function is then used to calculate the impurity self-energy  $\Sigma_d(\omega)$ , and the self-energy of the bath is then set equal to this impurity self-energy:  $\Sigma(\omega) = \Sigma_d(\omega)$ , because we expect, on grounds of the lattice symmetry, that the impurity is the same as any other site in the bath. This is said to be the self-consistency step, because the bath self-energy is completely determined only at the end. With this

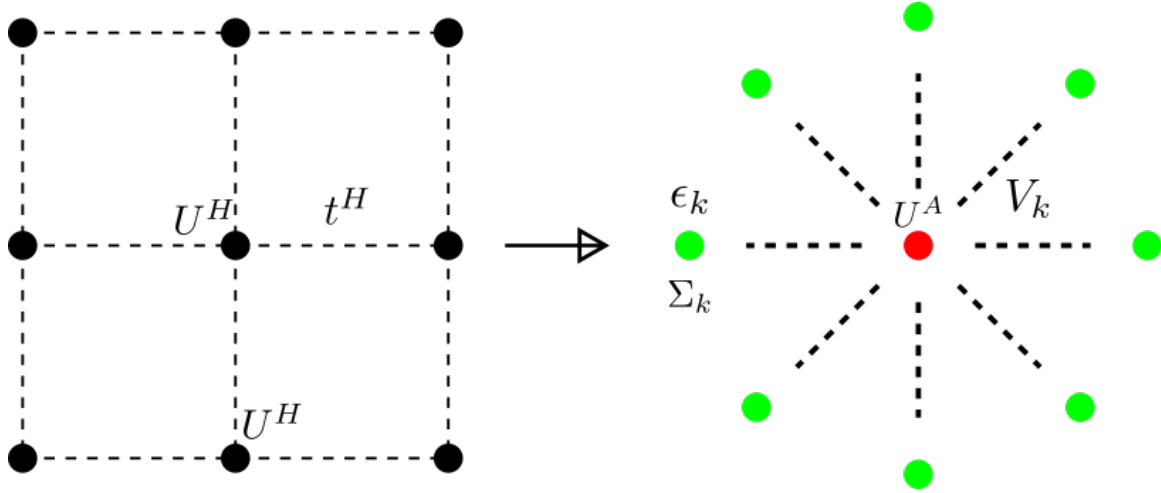


Figure 1: *Left*: Full Hubbard model lattice with onsite repulsion  $U^H$  on all sites and hopping between nearest neighbour sites with strength  $t^H$ . *Right*: Extraction of the auxiliary (cluster+bath) system from the full lattice. The central site on left becomes the impurity site (red) on the right (with an onsite repulsion  $\epsilon_d$ ), while the rest of the  $N - 1$  sites on the left form a conduction bath (green circles) (with dispersion  $\epsilon_k$  and correlation modelled by the self-energy  $\Sigma_k(\omega)$ ) that hybridizes with the impurity through the coupling  $V$ .

updated bath self-energy, one then repeats the entire process until there is no further change in the bath self-energy at the self-consistency step.

The present method intends to calculate the quantities in a different fashion. We start with a SIAM (with a correlated bath having a non-trivial self-energy), and solve it using the unitary renormalisation group approach to get to a fixed-point Hamiltonian. The fixed point Hamiltonian will in general involve the impurity site (with renormalised parameters  $\epsilon_d^*, U^*$ ) interacting with a smaller number of momentum states. Assuming the impurity-bath couplings are much larger than the dispersion of the bath, we can approximate the conduction bath part by a zero-mode of the kinetic energy part. The zero mode is defined as  $c_0 = \frac{1}{\sqrt{N}} \sum_k c_k$ , so it is just the zeroth site. What this means is that all the momentum states will then collapse to a single site (which we call the zero mode site, and represents the origin of the lattice). We will also pick out the zeroth site part of the onsite correlation part, and replace that with a single correlation coupling  $U_z^A$ . Such a model, shown in fig. 2 is exactly solvable.

$$H_{\text{corr}}^A = -t^A \sum_{\sigma} \left( c_{d\sigma}^\dagger c_{z\sigma} + \text{h.c.} \right) + U^A \tau_{d\uparrow} \tau_{d\downarrow} + U_z^A \tau_{z\uparrow} \tau_{z\downarrow} \quad (2.4)$$

The motivation for extracting the zero mode is that it leads to a very simple effective Hamiltonian (the asymmetric Anderson molecule) without losing much of the low energy physics.

We will then create the the entire Hubbard Hamiltonian by combining the zero-mode Hamiltonians according to a prescription which is guided by the symmetries of the problem. The impurity electron will be read off as an arbitrary site,  $p$ , of the lattice, while the zero-mode site becomes the site that is nearest to it. We will first demonstrate this explicitly for the Hubbard dimer. Then we will show how we can create a full Hubbard Hamiltonian by joining Anderson molecule Hamiltonians. At this point, we will assume that we have a Hubbard model in mind that has motivated a correlated SIAM as the auxiliary system, and we have performed renormalisation group analysis on this auxiliary system to get down to an effective Hamiltonian. We will also assume that the parameters of the auxiliary system have been chosen such that in the effective Hamiltonian, the impurity and zero mode have the same onsite repulsion:  $U^A = U_z^A$  (this is required for translational invariance). What this all means is that the RG analysis of the auxiliary model

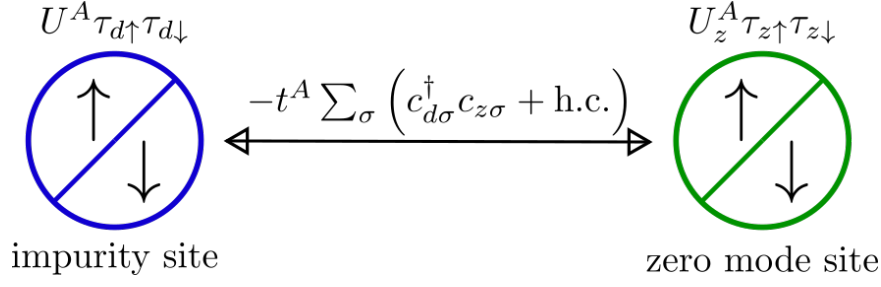


Figure 2: Correlated asymmetric Anderson molecule schematic version. It consists of two sites, impurity(blue) and zero mode(green). The impurity site has onsite repulsion, and there is inter-site hopping.

(appropriately chosen to model the bare Hubbard Hamiltonian) has led to a stable fixed point theory effective Hamiltonian described by a Hubbard dimer

$$H^D(U^D, t^D) \equiv -t^D \sum_{\sigma} (c_{0\sigma}^\dagger c_{1\sigma} + \text{h.c.}) + U^D (\tau_{0\uparrow} \tau_{0\downarrow} + \tau_{1\uparrow} \tau_{1\downarrow}) \quad (2.5)$$

This Hubbard dimer is shown in fig. 3.

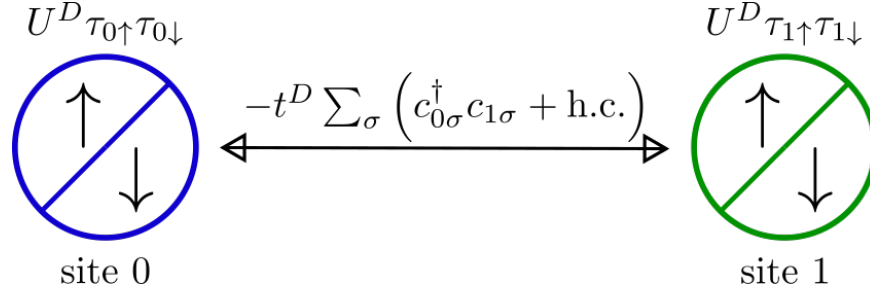


Figure 3: Hubbard dimer schematic version. It again consists of two sites, like the Anderson molecule, but now both sites have onsite repulsion, and there is again inter-site hopping.

The next step is to restore the lost translational invariance of the full system. We intend to do this by recreating a full Hubbard model,  $\tilde{H}$ , through the iterative real-space translation of the Hubbard dimer effective Hamiltonian obtained above. We will describe this process in detail below. However, we first specify the two approximations made along the entire journey from the original Hubbard model to the reconstructed one:

- We have replaced the full Hubbard model by an auxiliary system described by the SIAM Hamiltonian in eq. 1.2. The accuracy of this assumption is determined by the choice of the SIAM parameters, particularly the self-energy and repulsion of the bath. As discussed before, the very choice of the cluster and bath spoils the translational invariance of the parent model.
- We then perform a unitary RG on  $H^A$ . This leads to a fixed-point Hamiltonian  $\mathcal{U}_A H^A \mathcal{U}_A^\dagger$ . At this point, we extract a zero mode of this Hamiltonian to obtain the Hubbard dimer Hamiltonian:  $H^D = Z [\mathcal{U}_A H^A \mathcal{U}_A^\dagger] Z^{-1}$ , where  $Z$  defines the projection operator that extracts the zero mode from the effective stable fixed Hamiltonian  $\mathcal{U}_A H^A \mathcal{U}_A^\dagger$ . This constitutes the second approximation we make. This approximation further reduces the symmetry, because we have dropped all other lattice sites and kept just the zeroth lattice site (i.e., the site to which the impurity is coupled through single electron hybridisation).

We now tile the real-space lattice with this Hubbard dimer Hamiltonian to restore translational invariance (shown in a later section), and obtain a new Hubbard Hamiltonian,  $\tilde{H} = T [H^D] T^{-1}$ , where  $T$  denotes

the operator that performs the set of iterative real-space translations, and enables the dimer Hamiltonian to span the target real-space lattice. We quote the final form of  $\tilde{H}$  here to explain what the tiling means, but the explanation is given in a later section.

$$\tilde{H} = -\frac{2t^D}{Nw} \sum_{\sigma \langle ij \rangle} \left( c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.} \right) + \frac{2U^D}{N} \sum_i \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} = \frac{2}{Nw} \sum_{\langle ij \rangle} H^D(i, j, t^D, U^D) \quad (2.6)$$

What that means is that we have placed the Hubbard dimer at all nearest neighbour pairs to reconstruct a new Hubbard model. If we assume that the tiling mostly rectifies the explicit symmetry-breaking made while choosing the auxiliary system as well as taking the zero mode, we can write

$$\tilde{H} = \mathcal{U} H^H \mathcal{U}^{-1} = T Z \left[ \mathcal{U}_A H^A \mathcal{U}_A^\dagger \right] Z^{-1} T^{-1}, \quad (2.7)$$

where  $\mathcal{U} = T Z \mathcal{U}_A$  is some transformation that is either a unitary, or, at the very least, a similarity transformation that maps from the original to the reconstructed Hubbard Hamiltonian. The existence of  $\mathcal{U}$  is contingent on how good the approximations are, and strictly speaking needs to be verified.

### 3 Solution of the Hubbard dimer using the Anderson molecule

This section tries to see how far we can go if we just work with the Anderson molecule as the smallest unit of tiling. We will attempt to reproduce the entire spectrum of a Hubbard dimer by creating a new Hamiltonian made up purely of Anderson molecules. This will guide us in deciding how to generalize the "tiling method" for a general  $N$ -site Hubbard model, as well as give indications as to whether we need a different smallest unit for tiling.

The Hubbard dimer and Anderson molecules (zero-mode) are defined by the following respective Hamiltonians:

$$\begin{aligned} H^H &= -t^H \sum_{\sigma} \left( c_{1\sigma}^\dagger c_{2\sigma} + \text{h.c.} \right) + U^H \sum_{i=1,2} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - \mu^H \sum_{\sigma, i=1,2} \hat{n}_{i\sigma} \\ H^A &= -t^A \sum_{\sigma} \left( c_{d\sigma}^\dagger c_{z\sigma} + \text{h.c.} \right) + \epsilon_d^A \sum_{\sigma} \hat{n}_{d\sigma} + U^A \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \end{aligned} \quad (3.1)$$

In the first Hamiltonian, the indices  $i = 1, 2$  refer to the two lattice sites that constitute the dimer. In the second Hamiltonian, the subscript  $d$  indicates the impurity site, while the subscript  $z$  indicates the zero-mode site. First, we will assume that the Hubbard dimer is at half-filling ( $\frac{1}{2}U^H = \mu^H$ ):

$$\begin{aligned} H^H &= -t^H \sum_{\sigma} \left( c_{1\sigma}^\dagger c_{2\sigma} + \text{h.c.} \right) + U^H \sum_{i=1,2} \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} + \left( \frac{1}{2}U^H - \mu^H \right) \sum_{\sigma, i=1,2} \hat{n}_{i\sigma} + \text{constant} \\ &= -t^H \sum_{\sigma} \left( c_{1\sigma}^\dagger c_{2\sigma} + \text{h.c.} \right) + U^H \sum_{i=1,2} \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} \end{aligned} \quad (3.2)$$

Since the Hubbard Hamiltonian is at half-filling, we will also place the impurity at half-filling by setting  $\epsilon_d^A = -\frac{1}{2}U^A$ :

$$\begin{aligned} H^A &= -t^A \sum_{\sigma} \left( c_{d\sigma}^\dagger c_{z\sigma} + \text{h.c.} \right) + \left( \epsilon_d^A + \frac{1}{2}U^A \right) \sum_{\sigma} \hat{\tau}_{d\sigma} + U^A \hat{\tau}_{d\uparrow} \hat{\tau}_{d\downarrow} + \text{constant} \\ &= -t^A \sum_{\sigma} \left( c_{d\sigma}^\dagger c_{z\sigma} + \text{h.c.} \right) + U^A \hat{\tau}_{d\uparrow} \hat{\tau}_{d\downarrow} \end{aligned} \quad (3.3)$$

The first step is to recreate the Hubbard dimer Hamiltonian eq. 3.2 using the Anderson molecule Hamiltonian eq. 3.3:

$$\begin{aligned}
H^H &= -t^H \sum_{\sigma} \left( c_{1\sigma}^{\dagger} c_{2\sigma} + \text{h.c.} \right) + U^H \sum_{i=1,2} \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} \\
&= \frac{1}{2} \left[ -t^H \sum_{\sigma} \left( c_{1\sigma}^{\dagger} c_{2\sigma} + \text{h.c.} \right) + t^H \sum_{\sigma} \left( c_{2\sigma}^{\dagger} c_{1\sigma} + \text{h.c.} \right) \right] + \frac{1}{2} 2U^H \sum_{i=1,2} \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} \\
&= \frac{1}{2} \left[ -t^A \sum_{\sigma} \left( c_{d\sigma}^{\dagger} c_{z\sigma} + \text{h.c.} \right) \Big|_{\substack{z \rightarrow 2, d \rightarrow 1 \\ t^A \rightarrow t^H}} + t^A \sum_{\sigma} \left( c_{2\sigma}^{\dagger} c_{1\sigma} + \text{h.c.} \right) \Big|_{\substack{d \rightarrow 2, z \rightarrow 1 \\ t^A \rightarrow t^H}} \right] \\
&\quad + \frac{1}{2} \left( U^A \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} \Big|_{\substack{d \rightarrow 1 \\ U^A \rightarrow 2U^H}} + U^A \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} \Big|_{\substack{d \rightarrow 2 \\ U^A \rightarrow 2U^H}} \right) \\
&= \frac{1}{2} \left[ H^A \left( t^A \rightarrow t^H, U^A \rightarrow 2U^H, d \rightarrow 1, z \rightarrow 2 \right) + H^A \left( t^A \rightarrow t^H, U^A \rightarrow 2U^H, d \rightarrow 2, z \rightarrow 1 \right) \right]
\end{aligned} \tag{3.4}$$

The conclusion we can draw from this is that the Hubbard dimer Hamiltonian can be obtained from the Anderson dimer Hamiltonian in the following fashion:

- The essential idea is that we have to create a local Hubbard Hamiltonian for each site of the Hubbard lattice by replacing the impurity label  $d$  in the Anderson dimer with the label of the particular site. So if there are two sites, we will get two local Hamiltonians obtained by replacing  $d$  with 1 and 2 respectively. For each local Hamiltonian, the zero-mode label  $z$  is replaced by the site that is nearest to the one that  $d$  is being replaced by. So, if  $d \rightarrow 1(2)$ , then  $z \rightarrow 2(1)$ .
- This, however, is not the only change that we must make, in order to get the local Hamiltonian for a particular site. Along with  $d$  and  $z$ , we must also make the transformations  $t^A \rightarrow t^H, U^A \rightarrow 2U^H$ .
- Finally, once we have the local Hamiltonians for sites 1 and 2, we average them to get the total Hubbard Hamiltonian.

Note that we expect most of these "rules" to be specific for the dimer, and there will be generalizations to most of them for a general  $N$ -site Hubbard Hamiltonian.

The wavefunctions for the  $N = 2$  sector can also be connected through these transformations. Since both the Hamiltonians are analytically solvable, we can write down their groundstate wavefunctions [2]:

$$\begin{aligned}
|\Psi_{\text{GS}}^H\rangle &= a_1(U^H, t^H) \frac{1}{\sqrt{2}} (|\uparrow_1, \downarrow_2\rangle - |\downarrow_1, \uparrow_2\rangle) - a_2(U^H, t^H) \sqrt{2} (|\uparrow_1 \downarrow_1, \rangle - |\uparrow_2 \downarrow_2\rangle) \\
|\Psi_{\text{GS}}^A\rangle &= a_1\left(\frac{1}{2}U^A, t^A\right) \frac{1}{\sqrt{2}} (|\uparrow_d, \downarrow_z\rangle - |\downarrow_d, \uparrow_z\rangle) - a_2\left(\frac{1}{2}U^A, t^A\right) \sqrt{2} (|\uparrow_d \downarrow_d, \rangle - |\uparrow_z \downarrow_z\rangle) \\
E_{\text{GS}}^H &= -\frac{1}{2}\Delta(U^H, t^H), E_{\text{GS}}^A = -\frac{1}{2}\Delta\left(\frac{1}{2}U^A, t^A\right)
\end{aligned} \tag{3.5}$$

where

$$a_1(U, t) \equiv \frac{4t}{\sqrt{2\Delta(U, t) (\Delta(U, t) - U)}}, \quad a_2(U, t) \equiv \sqrt{\frac{\Delta(U, t) - U}{2\Delta(U, t)}}, \quad \Delta(U, t) \equiv \sqrt{U^2 + 16t^2} \tag{3.6}$$

$a_1, a_2$  satisfy  $a_1(-U, t) = -a_2(U, t)$  and  $a_1(U, t)a_2(U, t) = \frac{2t}{\Delta(U, t)}$ . From the forms of the wavefunctions and eigenenergies, we can immediately write down

$$|\Psi_{\text{GS}}^H\rangle = \frac{1}{2} \left[ |\Psi_{\text{GS}}^A\rangle (t^A, U^A \rightarrow t^H, 2U^H, d \rightarrow 1, z \rightarrow 2) + |\Psi_{\text{GS}}^A\rangle (t^A, U^A \rightarrow t^H, 2U^H, d \rightarrow 2, z \rightarrow 1) \right] \quad (3.7)$$

$$E_{\text{GS}}^H = E_{\text{GS}}^A (t^A, U^A \rightarrow t^H, 2U^H)$$

This shows that the rules laid out before work for the Hamiltonians, as well as the wavefunctions and energy eigenvalues of the  $N = 2, 0, 4$  sector. These sectors specifically work because it is only in these sectors can we ensure that  $n_d = n_z$ , which is required for the Hubbard Hamiltonian because  $n_1 = n_2$ . In the other sectors ( $N = 1, 3$ ), the impurity site and the zero-mode sites have to be singly-occupied in some part, and since the impurity site incurs a single-occupation cost of  $-\frac{U^H}{2}$  which is not borne by the zero-mode site, there is an intrinsic dissimilarity between the two sites of the Anderson molecule in this regime. This dissimilarity does not exist for the Hubbard model, so we cannot hope to connect the two models in this regime. Going forward, we will switch to using Hubbard dimers as the smallest tiling unit for a general Hubbard model.

## 4 Creating General $N$ -site Hubbard Hamiltonian from Hubbard dimers

We will follow the strategy outlined in the previous section. For concreteness, we will consider a lattice of  $N$  lattice sites and  $w$  nearest neighbours for each site. Note that a uniform number of nearest neighbours means that there is perfect translational invariance on the lattice, which means there cannot be any edge sites. This is achieved by applying periodic boundary conditions on the edges of the lattice. A square 2d lattice is thus placed on a 2-torus.

For each site  $i$  out of the  $N$  sites (having nearest neighbours  $\{i_j, j \in [1, w]\}$ ), we will create  $w$  local Hamiltonians  $\{H_{i,i_j}^H, i \in [1, N], j \in [1, w]\}$  out of the Hubbard dimer Hamiltonians, by setting the impurity index  $d$  to  $i$  and the zero-mode index  $z$  to  $i_j$ . We will also need to suitably transform the Hamiltonian parameters  $U^A, t^A$ , but we will figure those out as we go along. All  $N$  sites will then produce  $Nw$  local Hamiltonians in total, and the general Hubbard Hamiltonian should be the average of these local Hamiltonians.

The local Hamiltonians look like

$$H_{i,i_j}^D = -t^D \sum_{\sigma} \left( c_{i\sigma}^{\dagger} c_{i_j\sigma} + \text{h.c.} \right) + U^D \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} + U^D \hat{\tau}_{i_j\uparrow} \hat{\tau}_{i_j\downarrow} \quad (4.1)$$

We haven't transformed the dimer parameters  $U^D, t^D$  to the Hubbard parameters yet. The sum of all local Hamiltonians for a particular site  $i$  then gives the total Hamiltonian for that site:

$$H_i^D = \sum_{j \in [1, w]} H_{i,i_j}^D = \sum_{\substack{\sigma \\ j \in \text{N.N. of } i}} -t^D \left( c_{i\sigma}^{\dagger} c_{j\sigma} + \text{h.c.} \right) + w U^D \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} + U^D \sum_{j \in \text{N.N. of } i} \hat{\tau}_{j\uparrow} \hat{\tau}_{j\downarrow} \quad (4.2)$$

If we now sum over all the lattice sites and take the average of all the Hamiltonians ( $Nw$  in number), we get

$$\begin{aligned}
\tilde{H} &= \frac{1}{Nw} \sum_{i=1}^N H_i^D \\
&= \frac{1}{Nw} \left[ \sum_{\sigma, i=1}^N \sum_{j \in \text{NN of } i} -t^D \left( c_{\sigma}^{\dagger} c_{j\sigma} + \text{h.c.} \right) + wU^D \sum_{i=1}^N \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} + U^D \sum_{i=1}^N \sum_{j \in \text{N.N. of } i} \hat{\tau}_{j\uparrow} \hat{\tau}_{j\downarrow} \right] \\
&= \frac{1}{Nw} \left[ -t^D 2 \sum_{\sigma, \langle ij \rangle} \left( c_{\sigma}^{\dagger} c_{j\sigma} + \text{h.c.} \right) + 2U^D \sum_{\langle ij \rangle} \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} + 2U^D \sum_{\langle ij \rangle} \hat{\tau}_{j\uparrow} \hat{\tau}_{j\downarrow} \right] \\
&= \frac{2}{Nw} \sum_{\langle ij \rangle} \left[ -t^D \sum_{\sigma} \left( c_{\sigma}^{\dagger} c_{j\sigma} + \text{h.c.} \right) + U^D \left( \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} + \hat{\tau}_{j\uparrow} \hat{\tau}_{j\downarrow} \right) \right] \\
&= \frac{2}{Nw} \sum_{\langle ij \rangle} H^D(i, j, t^D, U^D)
\end{aligned} \tag{4.3}$$

There we used the identity

$$\sum_{i=1}^N \sum_{j \in \text{NN of } i} = 2 \sum_{\langle ij \rangle} \tag{4.4}$$

The sum  $\langle ij \rangle$  is over all nearest-neighbour pairs. Thus,  $\tilde{H}$  can be written as

$$\tilde{H} = -\frac{2t^D}{Nw} \sum_{\sigma, \langle ij \rangle} \left( c_{\sigma}^{\dagger} c_{j\sigma} + \text{h.c.} \right) + \frac{2U^D}{N} \sum_i \hat{\tau}_{i\uparrow} \hat{\tau}_{i\downarrow} = \frac{2}{Nw} \sum_{\langle ij \rangle} H^D(i, j, t^D, U^D) \tag{4.5}$$

The conclusion is that on tiling the Hubbard dimer Hamiltonians into all the nearest neighbour pairs, we end up with a new Hubbard model Hamiltonian with "renormalised parameters" given by

$$\tilde{t} = \frac{2t^D}{Nw} \quad , \quad \tilde{U} = \frac{2U^D}{N} . \tag{4.6}$$

It is thus apparent that translating the Hubbard dimers throughout the lattice has restored translational invariance, and generated correlations on all sites.

The claim made in a previous section was that this Hamiltonian is very close to a similarity transformed Hubbard Hamiltonian:

$$\tilde{H} = \frac{2}{Nw} \sum_{\langle ij \rangle} H^D(i, j, t^D, U^D) \equiv \mathcal{U} H^H \mathcal{U}^{-1} , \tag{4.7}$$

where  $\mathcal{U}$  is in general distinct from the unitary transformation that completes the URG starting from the bare cluster+bath Anderson model up to the fixed-point Hamiltonian. As shown earlier,  $U$  transformation that involves three steps: the URG transformations  $\mathcal{U}_A$ , the extracting of the zero mode  $Z$  and the tiling procedure outline in this section,  $T$

$$\mathcal{U} = T Z \mathcal{U}_A . \tag{4.8}$$



## 5 Formal expressions for single particle Greens functions and other related many-body quantites

### 5.1 Expressing matrix elements of the inverse single particle Greens function in terms of Hubbard dimer counterparts

If the groundstate wavefunction of  $H^H$  is  $|0\rangle$ , then the groundstate wavefunction of  $\tilde{H}$  is  $|\tilde{0}\rangle = \mathcal{U}|0\rangle$ . The advantage of  $\tilde{H}$  being connected to  $H^H$  through unitary (or atleast similarity) transformations is that they will have the same single-particle Greens functions

$$\begin{aligned} (G_H)_{ij}^\sigma(t) &\equiv -i\theta(t) \langle 0 | \left\{ c_{i\sigma}, c_{j\sigma}^\dagger \right\} | 0 \rangle = -i\theta(t) \langle 0 | \mathcal{U}^{-1} \left\{ \mathcal{U} c_{i\sigma} \mathcal{U}^{-1} \mathcal{U} c_{j\sigma}^\dagger \mathcal{U}^{-1} + \mathcal{U} c_{j\sigma}^\dagger \mathcal{U}^{-1} \mathcal{U} c_{i\sigma} \mathcal{U}^{-1} \right\} \mathcal{U} | 0 \rangle \\ &= -i\theta(t) \langle \tilde{0} | \left\{ \tilde{c}_{i\sigma} \tilde{c}_{j\sigma}^\dagger + \tilde{c}_{j\sigma}^\dagger \tilde{c}_{i\sigma} \right\} | \tilde{0} \rangle \\ &= \left( \tilde{G} \right)_{i\tilde{\sigma} j\tilde{\sigma}}(t) . \end{aligned} \quad (5.1)$$

This says that the Greens function  $G_H$  of the bare Hubbard model  $H^H$  for excitations  $|j\sigma\rangle \equiv c_{j\sigma}^\dagger |0\rangle$  on top of its ground state  $|0\rangle$  is exactly equal to the Greens function  $\tilde{G}$  of the renormalised Hamiltonian  $\tilde{H}$  for renormalised excitations  $|\tilde{j}\sigma\rangle \equiv \tilde{c}_{j\sigma}^\dagger |\tilde{0}\rangle \equiv \left( \mathcal{U} c_i^\dagger \mathcal{U}^{-1} \right) \mathcal{U} |0\rangle = \mathcal{U} |j\sigma\rangle$  on top of its renormalised ground state  $|\tilde{0}\rangle = \mathcal{U} |0\rangle$ . This means,

$$\langle i\sigma | G_H | j\sigma \rangle = \langle \tilde{i}\sigma | \tilde{G} | \tilde{j}\sigma \rangle \quad (5.2)$$

This same unitary connection also allows us to equate the inverse Greens functions:

$$\left[ (G_H)^{-1} \right]_{ij}^\sigma = \langle i\sigma | \omega - H^H | j\sigma \rangle = \langle i\sigma | \mathcal{U}^{-1} \left( \omega - \mathcal{U} H^H \mathcal{U}^{-1} \right) \mathcal{U} | j\sigma \rangle = \langle \tilde{i}\sigma | \left( \omega - \tilde{H} \right) | \tilde{j}\sigma \rangle = \left[ \left( \tilde{G} \right)^{-1} \right]_{\tilde{i}, \tilde{j}}^\sigma \quad (5.3)$$

We will now see how we can use the Hubbard dimer expansion to get simple expressions for the inverse Greens functions of  $\tilde{H}$ , and hence  $H^H$ .

Equation 4.5 can be rewritten in terms of inverse Greens functions. However, before proceeding, we should note that even though eq. 4.5 used the indices  $i, j$ , the correct indices are actually  $\tilde{i}, \tilde{j}$ , in light of the fact that operators get renormalised as  $c_i \rightarrow \tilde{c}_i \equiv \mathcal{U} c_i$ . With this in mind, we can write

$$\omega - \left( \tilde{G} \right)^{-1}(\omega) = \frac{2}{Nw} \sum_{\langle \tilde{i}, \tilde{j} \rangle} \left[ \omega - G_D^{-1}(\omega, \tilde{i}, \tilde{j}) \right] \quad (5.4)$$

where  $\tilde{G}$  and  $G_D$  are the Greens functions for the new (symmetrized) Hubbard model and the Hubbard dimer respectively:

$$\tilde{G}(\omega) = \frac{1}{\omega - \tilde{H}}, \quad G_D(\omega) = \frac{1}{\omega - H^D} \quad (5.5)$$

Since the  $\omega$  on the RHS of eq. 5.4 is independent of the summation indices, they can be pulled out along with a factor. The factor is just the total number of nearest neighbour pairs, which is  $\frac{Nw}{2}$ . This allows it to cancel the  $\omega$  on the LHS. The equation then simplifies to

$$\left( \tilde{G} \right)^{-1}(\omega) = \frac{2}{Nw} \sum_{\langle \tilde{i}, \tilde{j} \rangle} G_D^{-1}(\omega, \tilde{i}, \tilde{j}) \quad (5.6)$$

where  $G_D^{-1}(\tilde{i}, \tilde{j}, t^D, U^D)$  is the Greens function (matrix) of the Hubbard dimer Hamiltonian with  $\tilde{i}, \tilde{j}$  as the two sites.

We will now calculate the single-particle site-diagonal and site-off-diagonal Greens functions. First lets consider the diagonal Greens function at  $\tilde{i}^{\text{th}}$  site. We will extract this using the definition

$$\left[ \tilde{G}(\omega)^{-1} \right]_{\tilde{i}\tilde{i}}^{\sigma} = \langle \tilde{i}\sigma | \tilde{G}(\omega)^{-1} | \tilde{i}\sigma \rangle = \langle i\sigma | G_H(\omega)^{-1} | i\sigma \rangle = \left[ G_H(\omega)^{-1} \right]_{ii}^{\sigma} \quad (5.7)$$

This means that we will take the matrix element of eq. 5.6 between the excitations above the exact ground state of  $\tilde{H}$ , which is  $|\tilde{i}\rangle$ . The only terms that will contribute on the RHS of eq. 5.6 are those that have  $\tilde{i}$  in one of the indices. There will be  $w$  such terms, because  $\tilde{i}$  has  $w$  nearest neighbours. Thus, the right hand side will be a sum of  $w$  terms, each term being the inverse Greens function of a Hubbard dimer. Each term will be a real space local Greens function, and because of translational invariance, it will be the same on both sites. So we will represent the real space local inverse Greens function as  $\left[ G_D^{-1}(\omega) \right]_{00}^{\sigma}$ . The entire thing is thus  $w$  times this local Greens function.

$$\left( G_H^{-1}(\omega) \right)_{ii}^{\sigma} = \frac{2}{Nw} \left[ G_D^{-1}(\omega) \right]_{00}^{\sigma} \times w = \frac{2}{N} \left[ G_D^{-1}(\omega) \right]_{00}^{\sigma} \equiv g_0 \quad (5.8)$$

Now we come to the off-diagonal Greens function for the nearest neighbour sites  $i$  and  $j$ . We will extract this using the definition

$$\left[ \tilde{G}(\omega)^{-1} \right]_{\tilde{i}\tilde{j}}^{\sigma} = \langle \tilde{i}\sigma | \tilde{G}(\omega)^{-1} | \tilde{j}\sigma \rangle = \langle i\sigma | G_H(\omega)^{-1} | j\sigma \rangle = \left[ G_H(\omega)^{-1} \right]_{ij}^{\sigma} \quad (5.9)$$

We will again take the matrix element of eq. 5.6 between the excitations above the exact ground state of  $\tilde{H}$ . This will receive contribution from only one nearest neighbour pair in the summation of the RHS of eq. 5.6, namely that of  $(i, j)$ . This will not be a diagonal Greens function. Instead, it involves two nearest neighbour sites. We call this Greens function  $\left[ G_D^{-1}(\omega) \right]_{01}^{\sigma}$ .

$$\left( G_H^{-1}(\omega) \right)_{ij}^{\sigma} = \frac{2}{Nw} \left[ G_D^{-1}(\omega) \right]_{01}^{\sigma} \equiv g_1 \quad (5.10)$$

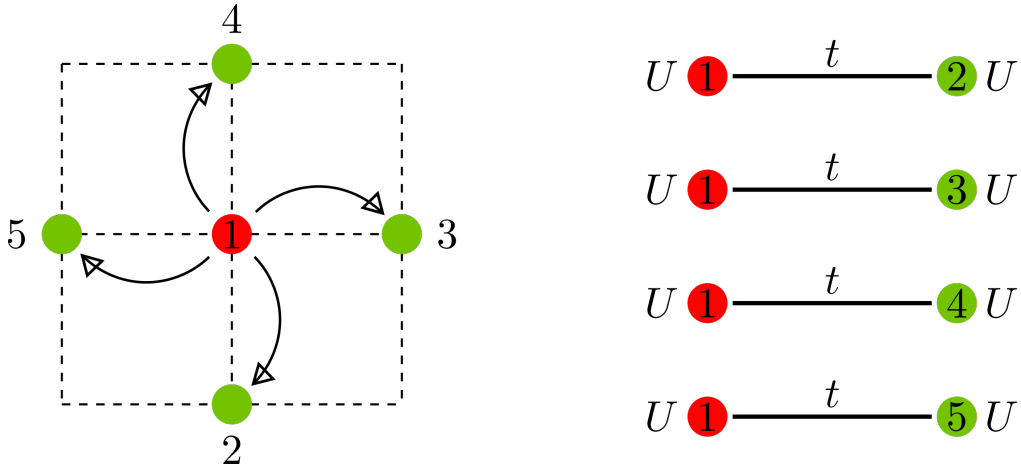


Figure 4: *Left:* Part of the lattice that is picked out by the Greens function on the LHS of eq. 5.8. *Right:* Hamiltonian whose Greens functions appear on the right hand side of same equation. The two submodels are identical.

## 5.2 Constructing full Greens function matrix from the inverse matrix via Fourier transformation to $k$ -space

Since the only real-space interaction is that between the nearest neighbours, there are no further real space Greens functions, and the matrix  $G^{-1}$  is tridiagonal in real space. Because of translational invariance, we know that  $(G_H^{-1})_{ii}$  and  $(G_H^{-1})_{ij}$  are independent of  $i, j$ . Furthermore, all expressions below are for a particular spin  $\sigma$ ; from the spin rotational symmetry of the Hubbard Hamiltonian, the same expressions are obtained for the opposite spin.

$$G_H^{-1}(\omega) = \begin{pmatrix} (G^{-1})_{00} & (G^{-1})_{01} & \dots & \dots & (G^{-1})_{0N} \\ (G^{-1})_{10} & (G^{-1})_{11} & (G^{-1})_{12} & \dots & \dots \\ \dots & (G^{-1})_{21} & (G^{-1})_{22} & (G^{-1})_{23} & \dots \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \end{pmatrix} = \begin{pmatrix} g_0 & g_1 & \dots & \dots & g_1 \\ g_1 & g_0 & g_1 & \dots & \dots \\ \dots & g_1 & g_0 & g_1 & \dots \\ & & \dots & & \\ & & \dots & & \end{pmatrix} \quad (5.11)$$

It can be written in the operator form as

$$G_H^{-1}(\omega) = g_0 \sum_{i=1}^N |i\rangle \langle i| + g_1 \sum_{\langle i,j \rangle} (|i\rangle \langle j| + \text{h.c.}) \quad (5.12)$$

We can Fourier transform the real space basis kets  $|i\rangle \equiv |\vec{r}_i\rangle$  into momentum space:

$$\begin{aligned} |i\rangle &= \frac{1}{\sqrt{N}} \sum_{\vec{k}=\vec{k}_1}^{\vec{k}_N} e^{i\vec{k} \cdot \vec{r}_i} |\vec{k}\rangle \Leftrightarrow |k\rangle = \frac{1}{\sqrt{N}} \sum_{\vec{i}} e^{-i\vec{k} \cdot \vec{r}_i} |\vec{r}_i\rangle \\ \sum_i |i\rangle \langle i| &= \frac{1}{N} \sum_i \sum_{\vec{k}, \vec{q}} e^{i(\vec{k}-\vec{q}) \cdot \vec{r}_i} |\vec{k}\rangle \langle \vec{q}| = \frac{1}{N} \sum_{\vec{k}, \vec{q}} N \delta(\vec{k} - \vec{q}) |\vec{k}\rangle \langle \vec{q}| = \sum_{\vec{k}=\vec{k}_1}^{\vec{k}_N} |\vec{k}\rangle \langle \vec{k}| \\ \sum_{\langle i,j \rangle} |i\rangle \langle j| &= \frac{1}{N} \sum_{\langle i,j \rangle} \sum_{\vec{k}, \vec{q}} e^{i(\vec{k} \cdot \vec{r}_i - \vec{q} \cdot \vec{r}_j)} |\vec{k}\rangle \langle \vec{q}| = \frac{1}{N} \frac{1}{2} \sum_{\hat{e}_i=\hat{e}_1}^{\hat{e}_D} \sum_{\vec{k}, \vec{q}} e^{-i\vec{q} \cdot a_i \hat{e}_i} |\vec{k}\rangle \langle \vec{q}| \sum_{i=1}^N e^{i(\vec{k}-\vec{q}) \cdot \vec{r}_i} \left[ \sum_{\langle i,j \rangle} = \frac{1}{2} \sum_i \sum_{j \in \text{NN of } i} \right] \\ &= \frac{1}{2} \sum_{\hat{e}_i=\hat{e}_1}^{\hat{e}_D} \sum_{\vec{q}} e^{-i\vec{q} \cdot \hat{e}_i} |\vec{q}\rangle \langle \vec{q}| = \frac{1}{2} \sum_{\vec{q}} \left( \sum_{i_1}^D e^{-ia_{i_1} q_{i_1}} \right) |\vec{q}\rangle \langle \vec{q}| \end{aligned}$$

$\hat{e}_i$  are the primitive lattice unit vectors for the general  $d$ -dimensional lattice.  $q_i$  are the projections of  $\vec{q}$  along those vectors.  $a_i$  are the lattice spacings along each primitive vector. Adding the Hermitian conjugate simply extracts the real part:

$$\sum_{\langle i,j \rangle} |i\rangle \langle j| + \text{h.c.} = \sum_{\vec{q}} \underbrace{\sum_{i_1}^D \cos(a_{i_1} q_{i_1})}_{\equiv \xi_{\vec{q}}} |\vec{q}\rangle \langle \vec{q}| \quad (5.13)$$

On a 2D square lattice, for instance, we obtain

$$\begin{aligned}\xi_{\vec{q}} &= 2(\cos q_x a_x + \cos q_y a_y) \equiv \frac{-\epsilon_{\vec{q}}}{t^H}, \\ \epsilon_{\vec{q}} &= -2t^H(\cos q_x a_x + \cos q_y a_y).\end{aligned}\tag{5.14}$$

The inverse Greens function thus becomes diagonal in momentum space

$$G_H^{-1}(\omega) = \sum_{\vec{k}=\vec{k}_1}^{\vec{k}_N} (g_0 + g_1 \xi_{\vec{k}}) |\vec{k}\rangle \langle \vec{k}| = \begin{pmatrix} g_0 + g_1 \xi_{\vec{k}_1} & & & \\ & g_0 + g_1 \xi_{\vec{k}_2} & & \\ & & g_0 + g_1 \xi_{\vec{k}_3} & \\ \dots & \dots & \dots & \dots \end{pmatrix}\tag{5.15}$$

The matrix entries are written in momentum space, and the matrix is diagonal in this basis.

### 5.3 Calculation of self energy matrix from the Dyson equation

Using Dyson's equation for the self-energy  $\Sigma = G_0^{-1} - G^{-1}$  (where the matrix  $G_{0,H}^{-1}(\omega) = \sum_{\vec{k}=\vec{k}_1}^{\vec{k}_N} (\omega - \epsilon_k) |\vec{k}\rangle \langle \vec{k}| = \sum_{\vec{k}=\vec{k}_1}^{\vec{k}_N} (\omega + t^H \xi_k) |\vec{k}\rangle \langle \vec{k}|$  is the inverse Greens function for the appropriate non-interacting tight-binding system), we obtain the self-energy matrix as

$$\begin{aligned}\Sigma_H(\omega) &= G_{0,H}^{-1} - G_H^{-1}, \\ &= \sum_{\vec{k}=\vec{k}_1}^{\vec{k}_N} (\omega - g_0 + (t^H - g_1) \xi_{\vec{k}}) |\vec{k}\rangle \langle \vec{k}|\end{aligned}\tag{5.16}$$

This is the full self energy matrix. One can get both real space and momentum space self energies by taking appropriate matrix elements. For instance, the self-energy in momentum space is

$$\Sigma_H(\vec{k}, \omega) \equiv \langle \vec{k} | \Sigma_H(\omega) | \vec{k} \rangle = \omega - g_0 + (t^H - g_1) \xi_{\vec{k}}.\tag{5.17}$$

Thus, we can use eqs.(5.14), (5.8) and (5.10) to obtain the full self-energy  $\Sigma(\vec{k}, \omega)$  for the Hubbard model on the 2D square lattice.

### 5.4 Obtaining real space Greens functions from the full matrix

Since  $G_H^{-1}(\omega)$  is diagonal in this basis, we can invert the matrix trivially by replacing the diagonal elements with their reciprocals.

$$G_H(\omega) = \sum_{\vec{k}=\vec{k}_1}^{\vec{k}_N} (g_0 + g_1 \xi_{\vec{k}})^{-1} |\vec{k}\rangle \langle \vec{k}| = \begin{pmatrix} (g_0 + g_1 \xi_{\vec{k}_1})^{-1} & & & \\ & (g_0 + g_1 \xi_{\vec{k}_2})^{-1} & & \\ & & (g_0 + g_1 \xi_{\vec{k}_3})^{-1} & \\ \dots & \dots & \dots & \dots \end{pmatrix}\tag{5.18}$$

The diagonal elements of  $G_H$  in momentum space are simply  $G_H(\vec{k}, \omega)$

$$\begin{aligned}
G_H(\vec{k}, \omega) &= \left( g_0 + g_1 \xi_{\vec{k}} \right)^{-1} \\
&= \left\{ \frac{2}{N} \left[ G_D^{-1}(\omega) \right]_{00\sigma} + \frac{2}{Nw} \left[ G_D^{-1}(\omega) \right]_{01\sigma} \xi_{\vec{k}} \right\}^{-1} \\
&= \frac{N}{2} \left\{ \left[ G_D^{-1}(\omega) \right]_{00\sigma} + \frac{1}{w} \left[ G_D^{-1}(\omega) \right]_{01\sigma} \xi_{\vec{k}} \right\}^{-1}
\end{aligned} \tag{5.19}$$

The real-space Greens functions can now be obtained by taking matrix elements of the full Greens function matrix in eq. 5.18:

$$G_H(\vec{r}_i, \vec{r}_j, \omega) \equiv \langle \vec{r}_i | G_H(\omega) | \vec{r}_j \rangle = \sum_{\vec{k}} G_H(\vec{k}, \omega) \langle \vec{r}_i | \vec{k} \rangle \langle \vec{k} | \vec{r}_j \rangle \tag{5.20}$$

From the definition of the Fourier transforms mentioned previously, we can write

$$\langle \vec{r}_i | \vec{k} \rangle = \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{-\vec{k} \cdot \vec{r}} \langle \vec{r}_i | \vec{r} \rangle = \frac{1}{\sqrt{N}} e^{-\vec{k} \cdot \vec{r}_i} \tag{5.21}$$

Substituting this gives

$$G_H(\vec{r}_i, \vec{r}_j, \omega) = \frac{1}{N} \sum_{\vec{k}} G_H(\vec{k}, \omega) e^{\vec{k} \cdot (\vec{r}_j - \vec{r}_i)} \equiv G_H(\vec{r}_j - \vec{r}_i, \omega) \tag{5.22}$$

The final expression brings out the fact that the Greens function for excitations between  $\vec{r}_i$  and  $\vec{r}_j$  depends only on the relative position  $\vec{r}_j - \vec{r}_i$ , not on their absolute positions. This is expected on grounds of translational invariance. By choosing  $\vec{r}_i = \vec{r}_j$ , we can obtain the real space *local* Greens function:

$$(G_H)_{ii}(\omega) \equiv G_H(\vec{r}_j - \vec{r}_i = 0, \omega) = \frac{1}{N} \sum_{\vec{k}} G_H(\vec{k}, \omega) = \frac{1}{N} \sum_{\vec{k}} \left( g_0 + g_1 \xi_{\vec{k}} \right)^{-1} \tag{5.23}$$

We can now compute the  $k$ -space spectral function  $A_H(\vec{k}, \omega)$  and the real-space local spectral function  $A_H(\vec{r} = 0, \omega)$  as

$$\begin{aligned}
A_H(\vec{k}, \omega) &= -\frac{1}{\pi} \text{Im}(G_H(\vec{k}, \omega)) = -\frac{1}{\pi} \text{Im} \left[ (g_0 + g_1 \xi_{\vec{k}})^{-1} \right] \\
A_H(\vec{r} = 0, \omega) &= -\frac{1}{\pi} \text{Im}(G_H(\vec{r} = 0, \omega)) = -\frac{1}{N\pi} \text{Im} \left[ \sum_{\vec{k}} (g_0 + g_1 \xi_{\vec{k}})^{-1} \right] = -\frac{1}{N} \sum_{\vec{k}} A_H(\vec{k}, \omega)
\end{aligned} \tag{5.24}$$

We can again use eqs.(5.14), (5.8) and (5.10) to obtain the spectral functions  $A_H(\vec{k}, \omega)$  and  $A_H(\vec{r} = 0, \omega)$  for the Hubbard model on the 2D square lattice.

## 5.5 Obtaining $g_0$ and $g_1$

These two quantities are expressed in terms of matrix elements of the inverse Greens function  $G_D^{-1}$  of the Hubbard dimer. Since  $G_D$  is a  $2 \times 2$  matrix, it is easy to invert:

$$\begin{aligned}
G_D &= \begin{pmatrix} (G_D)_{00} & (G_D)_{01} \\ (G_D)_{01} & (G_D)_{00} \end{pmatrix} \\
\Rightarrow G_D^{-1} &= \frac{1}{\text{Det } G_D} \begin{pmatrix} (G_D)_{00} & -(G_D)_{01} \\ -(G_D)_{01} & (G_D)_{00} \end{pmatrix} \\
\Rightarrow g_0 &\equiv \frac{2}{N} (G_D^{-1})_{00} = \frac{2}{N} \frac{1}{\text{Det } G_D} (G_D)_{00}, \\
g_1 &\equiv \frac{2}{Nw} (G_D^{-1})_{00} = -\frac{2}{Nw} \frac{1}{\text{Det } G_D} (G_D)_{01}
\end{aligned} \tag{5.25}$$

Using the expressions for  $(G_D)_{00}$  and  $(G_D)_{01}$  shown in the appendix, we can obtain  $g_0$  and  $g_1$  very generally.

## 6 Analytic consistency checks

### 6.1 Single-particle Greens function and self energy for the Hubbard dimer

The simplest test involves calculating  $G_H(\vec{r}=0, \omega)$  for the Hubbard dimer ( $N=2, w=1$ ). The discrete set of momenta are  $\{\vec{k}_n\} = 0, \frac{\pi}{a}$ , so

$$\{\xi_{\vec{k}}\} = 1, -1 \tag{6.1}$$

Therefore, the Greens function at  $\vec{r}=0$  for the Hubbard dimer can be expressed as

$$G_H(\vec{r}=0, \omega) = \frac{1}{2} \left[ \frac{1}{g_0 + g_1} + \frac{1}{g_0 - g_1} \right] = \frac{g_0}{g_0^2 - g_1^2} \tag{6.2}$$

For the Hubbard dimer, we substitute  $N=2, w=1$  into the expressions for  $g_0$  and  $g_1$  to obtain

$$g_0 = \frac{1}{\text{Det } G_D} (G_D)_{00}, \quad g_1 = -\frac{1}{\text{Det } G_D} (G_D)_{01} \tag{6.3}$$

The Greens function at the zeroth site then becomes

$$G_H(\vec{r}=0, \omega) = \text{Det } G_D \frac{(G_D)_{00}}{(G_D)_{00}^2 - (G_D)_{01}^2} = \text{Det } G_D \frac{(G_D)_{00}}{\text{Det } G_D} = (G_D)_{00} \tag{6.4}$$

This is the expected result. Because the full Hubbard model in this case was simply the Hubbard dimer,  $G_H(\vec{r}=0, \omega)$  should just be  $(G_D)_{00}$ . Although this is a trivial example (we matched the Hubbard dimer with another Hubbard dimer), this atleast tells us that this algorithm for calculating the real space local Greens function from the inverse Greens function matrix elements is sensible.

## 6.2 On the Bethe lattice

Another test involves considering the case of infinite number of nearest neighbours  $w \rightarrow \infty$  (the coordination number, and effectively the dimensionality). Here, as has been argued in the DMFT literature, the correct scaling of the  $t^H$  hopping parameter is  $t^H \rightarrow t^H / \sqrt{w}$  such that the kinetic energy of the associated tight-binding lattice model is finite. This allows for the competition between the kinetic and potential terms of the Hamiltonian to drive a metal-insulator transition in the limit of  $w \rightarrow \infty$  as well. Further, it has been argued in the DMFT literature that the Greens function matrix of the Hubbard model on the Bethe lattice with  $w \rightarrow \infty$  becomes purely local (i.e., it has vanishing inter-site matrix elements). We can also see this from eqs.(5.8),(5.10) where  $(G^H)^{-1}$  matrix has no  $k$  dependence as  $(G^H)_{ij}^{-1} \propto 1/w \rightarrow 0$  as  $w \rightarrow \infty$ . Thus, in the self-energy

$$\begin{aligned} \Sigma_H(\vec{k}, \omega) &= \omega - g_0 + (t^H - g_1)\xi_{\vec{k}} \\ &\rightarrow \lim_{w \rightarrow \infty} \omega - g_0 + t^H \xi_{\vec{k}}, \end{aligned} \quad (6.5)$$

We can now see that even in the limit of  $w \rightarrow \infty$ , the competition between  $g_0$  (the on-site repulsion) and the hopping related kinetic energy ( $t^H \xi_{\vec{k}}$ ) can lead to a metal-insulator transition.

## 7 Calculating $g_0$ and $g_1$ in practice

The expressions for  $g_0$  and  $g_1$ , as mentioned above, involve taking matrix elements of the Hubbard dimer Hamiltonian against the exact wavefunctions of the full Hubbard model.

1. Since the full Hubbard model ground state wavefunctions are not readily available, one can, as the simplest approximation, assume the ground state has the form

$$|\tilde{0}\rangle = \sum_{\langle \tilde{i}\tilde{j} \rangle} |\Phi_{\tilde{i}\tilde{j}}^D\rangle \otimes |\Psi_{\tilde{i},\tilde{j}}\rangle \quad (7.1)$$

$|\Phi_{\tilde{i}\tilde{j}}^D\rangle$  would be the ground state of the Hubbard dimer with  $i, j$  creating the two sites of the dimer, while  $|\Psi_{\tilde{i},\tilde{j}}\rangle$  would be the wavefunction involving the rest of the sites. This of course assumes that the two sets  $(i, j)$  and  $(a, b, \dots, h, k, \dots)$  are not entangled, and is not true in general. With this assumption,  $g_0$  and  $g_1$  become related to the inverse Greens functions of the Hubbard dimer. Such a choice of the wavefunction is motivated by the fact that in the auxiliary system we chose, the bath had only a diagonal interaction; there was no off-diagonal two particle scattering term. It is further enforced when we extract the zero mode of the entire bath (i.e., we keep just the zeroth site). The zeroth site, along with the impurity, then forms the  $|\Phi\rangle$  part of the wavefunction. This will allow for analytic insight, but at the cost of accuracy as discussed above.

2. Another way of approaching the problem is to obtain the ground state wavefunction of a Hubbard model numerically for several values of  $U^H$  and  $t^H$ , and then computing the matrix elements of  $H^D$  against this wavefunction to obtain a family of  $(g_0, g_1)$ . This brings the numerical accuracy for a given finite-sized lattice realisation of the Hubbard model. However, this also means we lose any analytical insight into the structure of  $g_0$  and  $g_1$ .
3. The most promising approach of calculating  $g_0$  and  $g_1$  is by systematically improving the required ground state wavefunctions as follows. We obtain numerically the ground state wavefunction of a Hamiltonian that has not just the Hubbard dimer, but also a bath (with dispersion) that connects

to both the sites of the dimer:

$$H_{\text{bath}}^D(\tilde{i}, \tilde{j}) = \underbrace{U \left( \tau_{\tilde{i}\uparrow} \tau_{\tilde{i}\downarrow} + \tau_{\tilde{j}\uparrow} \tau_{\tilde{j}\downarrow} \right)}_{\text{Hubbard dimer}} - t \sum_{\sigma} \left( c_{i\sigma}^{\dagger} c_{j\sigma} + \text{h.c.} \right) + \underbrace{\sum_{k\sigma} \epsilon_{k\sigma} \tau_{k\sigma}}_{\text{bath}} - t \underbrace{\sum_{k\sigma} \left( c_{i\sigma}^{\dagger} c_{k\sigma} + c_{j\sigma}^{\dagger} c_{k\sigma} + \text{h.c.} \right)}_{\text{bath-dimer hybridisation}} \quad (7.2)$$

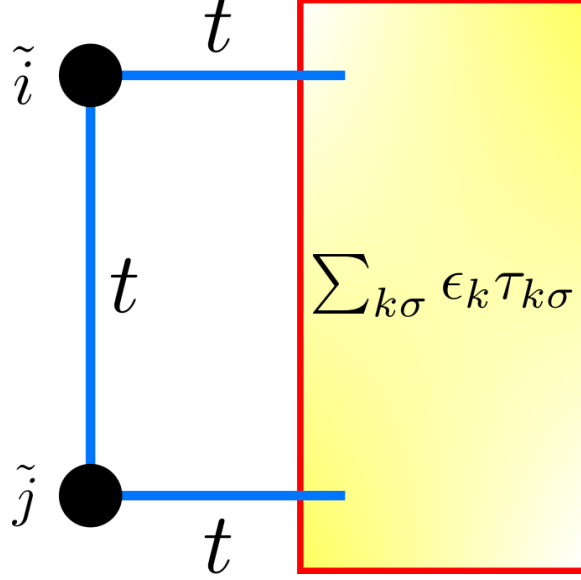


Figure 5: Hubbard dimer with dispersion that connects to both sites

By systematically increasing the number of momentum states in the bath dispersion, we can systematically improve the numerical computation of the ground state wavefunction. From here, we can compute matrix elements of  $H^D$ , and hence the functions  $g_0$  and  $g_1$ . The presence of the bath can be understood as follow. In the absence of a bath, i.e., we have just a Hubbard dimer, an electron can at most hop between the two sites and lead to the wavefunction in eq. 7.1. However, with a single particle hopping term connecting the dimer to a bath, an electron can also journey between the two sites of the bath via the bath. This leads to entanglement between the dimer's sites and those in the bath; this was clearly ignored in the ground state wavefunction eq. 7.1. The introduction of the bath dispersion offers the possibility that we can capture a site diagonal spectral function of the three peak form, because we have previously seen such a spectral function in the fixed point of the URG analysis of the SIAM (where the fixed point effective Hamiltonian was an Anderson molecule with dispersion). If this can be found, then a metal-insulator transition of the full Hubbard model can perhaps be captured by studying the spectral functions of the bath-coupled Hubbard dimer.

4. Another improvement can be made by introducing a self-energy  $\Sigma(k, \omega)$  into the bath dispersion. This introduces correlation within the bath. The Hamiltonian that we would need to solve would be the same as eq. 7.2, but with  $\epsilon_k$  replaced by  $\tilde{\epsilon}_k \equiv \epsilon_k + \Sigma(k, \omega)$ . The self-energy  $\Sigma(k, \omega)$  will have to be chosen depending on the phase we want to look at (e.g., metal, insulator etc.). With a  $\Sigma$  that is singular near the Fermi surface at  $\omega \rightarrow 0$ , the dispersion will become gapped and the phase will be insulating. In such a situation, there are no low-energy bath excitations within a given energy window proximate to its putative Fermi energy. This will localise the bath electrons, as well as confine all journeys between the two sites of the dimer to only the direct path. This is an indication of the localisation of electrons in the Hubbard model to holon-doublon excitations on nearest neighbour sites. In such a circumstance, eq. 7.1 will be a very good approximation to the



actual insulating ground state of the Hubbard model. On the other hand, if we use a self energy that vanishes near the Fermi surface ( $\Sigma(\omega) \sim \omega^2$ , i.e., a Fermi liquid bath), then we expect to end up in a metallic phase of the Hubbard model. This is simply due to the possibility of holons and doublons now dispersing throughout the lattice, and be concomitant with the presence of a pole in the single-particle Greens function  $G(k, \omega)$  of the Hubbard model. Introducing the self energy therefore gives us a larger variety of wavefunctions and features to work with. Further, this appears to be in line with the original proposals offered by Mott and Kohn for the Mott metal insulator transition as the localisation-delocalisation transition of holon-doublon pairs. With regards to the transition itself, the precise form of the bath  $\Sigma(k, \omega)$  (or spectral function) remains to be determined.

## 8 Future goals

1. Obtain the metal insulator transition of the 2D Hubbard model on the square lattice from our formalism. Compare with what is obtained from DMFT and its improvements. Also, determine the nature of the bath  $\Sigma(k, \omega)$  for the transition point.
2. While we have provided expressions for the single particle Greens functions of the  $N$ -site Hubbard model from their equivalent single particle Greens functions of the Hubbard dimer, we expect that eq.(5.6) holds quite generally for the two-particle Greens function sector as well. We will provide these expressions at a later point in time.
3. Study various other models of strong correlation with this approach: (i)

# Appendix: Spectra of Hubbard dimer and Anderson molecule

eigenstate	symbol	eigenvalue
$ 0, 0\rangle$	$ 0\rangle$	$\frac{U^H}{2}$
$\frac{1}{\sqrt{2}} ( \sigma, 0\rangle \pm  0, \sigma\rangle)$	$ 0\sigma_{\pm}\rangle$	$\mp t^H$
$ \sigma, \sigma\rangle$	$ \sigma\sigma\rangle$	$-\frac{U^H}{2}$
$\frac{1}{\sqrt{2}} ( \uparrow, \downarrow\rangle +  \downarrow, \uparrow\rangle)$	$ ST\rangle$	$-\frac{U^H}{2}$
$\frac{1}{\sqrt{2}} ( 2, 0\rangle -  0, 2\rangle)$	$ CS\rangle$	$\frac{U^H}{2}$
$a_1(U^H, t^H) \frac{1}{\sqrt{2}} ( \uparrow, \downarrow\rangle -  \downarrow, \uparrow\rangle) + a_2(U^H, t^H) \frac{1}{\sqrt{2}} ( 2, 0\rangle +  0, 2\rangle)$	$ -\rangle$	$-\frac{1}{2}\Delta(U^H, t^H)$
$-a_2(U^H, t^H) \frac{1}{\sqrt{2}} ( \uparrow, \downarrow\rangle -  \downarrow, \uparrow\rangle) + a_1(U^H, t^H) \frac{1}{\sqrt{2}} ( 2, 0\rangle +  0, 2\rangle)$	$ +\rangle$	$\frac{1}{2}\Delta(U^H, t^H)$
$\frac{1}{\sqrt{2}} ( \sigma, 2\rangle \pm  2, \sigma\rangle)$	$ 2\sigma_{\pm}\rangle$	$\pm t^H$
$ 2, 2\rangle$	$ 4\rangle$	$\frac{U^H}{2}$

Table 1: Spectrum of Hubbard dimer at half-filling

eigenstate	symbol	eigenvalue
$ 0, 0\rangle$	$ 0\rangle$	$\frac{U^A}{4}$
$a_1(U^A, t^A)  \sigma, 0\rangle + a_2(U^A, t^A)  0, \sigma\rangle$	$ 0\sigma_{-}\rangle$	$-\frac{1}{4}\Delta(U^A, t^A)$
$-a_2(U^A, t^A)  \sigma, 0\rangle + a_1(U^A, t^A)  0, \sigma\rangle$	$ 0\sigma_{+}\rangle$	$\frac{1}{4}\Delta(U^A, t^A)$
$ \sigma, \sigma\rangle$	$ \sigma\sigma\rangle$	$-\frac{U^A}{4}$
$\frac{1}{\sqrt{2}} ( \uparrow, \downarrow\rangle +  \downarrow, \uparrow\rangle)$	$ ST\rangle$	$-\frac{U^A}{4}$
$\frac{1}{\sqrt{2}} ( 2, 0\rangle -  0, 2\rangle)$	$ CS\rangle$	$\frac{U^A}{4}$
$a_1(\frac{1}{2}U^A, t^A) \frac{1}{\sqrt{2}} ( \uparrow, \downarrow\rangle -  \downarrow, \uparrow\rangle) + a_2(\frac{1}{2}U^A, t^A) \frac{1}{\sqrt{2}} ( 2, 0\rangle +  0, 2\rangle)$	$ -\rangle$	$-\frac{1}{2}\Delta(\frac{1}{2}U^A, t^A)$
$-a_2(\frac{1}{2}U^A, t^A) \frac{1}{\sqrt{2}} ( \uparrow, \downarrow\rangle -  \downarrow, \uparrow\rangle) + a_1(\frac{1}{2}U^A, t^A) \frac{1}{\sqrt{2}} ( 2, 0\rangle +  0, 2\rangle)$	$ +\rangle$	$\frac{1}{2}\Delta(\frac{1}{2}U^A, t^A)$
$a_1(U^A, t^A)  \sigma, 2\rangle - a_2(U^A, t^A)  2, \sigma\rangle$	$ 2\sigma_{-}\rangle$	$-\frac{1}{4}\Delta(U^A, t^A)$
$a_2(U^A, t^A)  \sigma, 2\rangle + a_1(U^A, t^A)  2, \sigma\rangle$	$ 2\sigma_{+}\rangle$	$\frac{1}{4}\Delta(U^A, t^A)$
$ 2, 2\rangle$	$ 4\rangle$	$\frac{U^A}{4}$

Table 2: Spectrum of Anderson molecule at particle-hole symmetry

## Appendix: Local Greens function for the Hubbard dimer

From the spectral representation, we have the following expression for the local Greens function for the Hubbard dimer at site 0:

$$G_{D,00}^\sigma(\omega) = \frac{1}{Z} \sum_{m,n} ||\langle m | c_{i\sigma} | n \rangle||^2 \left( e^{-\beta E_m} + e^{-\beta E_n} \right) \frac{1}{\omega + E_m - E_n} \quad (8.1)$$

$m, n$  sum over the exact eigenstates.  $E_m, E_n$  are the corresponding energies. We are interested in the  $T \rightarrow 0$  Greens function. In that limit, all exponentials except that for the ground state  $E_{gs}$  will die out. The exponential inside the summation will then cancel the exponential in the partition function.

$$\begin{aligned} G_{D,00}^\sigma(\omega, T \rightarrow 0) &= \sum_n \left[ ||\langle GS | c_{i\sigma} | n \rangle||^2 \frac{1}{\omega + E_{GS} - E_n} + ||\langle n | c_{i\sigma} | GS \rangle||^2 \frac{1}{\omega + E_n - E_{GS}} \right] \\ &= \sum_n \left[ ||\langle n | c_{i\sigma}^\dagger | GS \rangle||^2 \frac{1}{\omega + E_{GS} - E_n} + ||\langle n | c_{i\sigma} | GS \rangle||^2 \frac{1}{\omega + E_n - E_{GS}} \right] \end{aligned} \quad (8.2)$$

The ground state  $|GS\rangle$  is just the state  $|-\rangle$  in the table 1. We will choose to look at  $\sigma = \uparrow$ . Then,

$$\begin{aligned} c_{1\uparrow} |-\rangle &= \frac{a_1}{\sqrt{2}} |0, \downarrow\rangle + \frac{a_2}{\sqrt{2}} |\downarrow, 0\rangle \\ c_{1\uparrow}^\dagger |-\rangle &= -\frac{a_1}{\sqrt{2}} |2, \uparrow\rangle + \frac{a_2}{\sqrt{2}} |\uparrow, 2\rangle \end{aligned} \quad (8.3)$$

The set of states  $|n\rangle$  that give non-zero inner product  $|GS\rangle$  are therefore

$$\begin{aligned} \{|n\rangle\} &= |0 \downarrow_\pm\rangle \\ ||\langle n | c_{\uparrow\sigma} | GS \rangle||^2 &= \frac{1}{4} (a_2 \pm a_1)^2 = \frac{1}{4} (1 \pm 2a_1 a_2) \\ \{E_n\} &= \mp t \end{aligned} \quad (8.4)$$

for the second inner product, and

$$\begin{aligned} \{|n\rangle\} &= |2 \uparrow_\pm\rangle \\ ||\langle n | c_{\uparrow\sigma}^\dagger | GS \rangle||^2 &= \frac{1}{4} (a_2 \mp a_1)^2 = \frac{1}{4} (1 \mp 2a_1 a_2) \\ \{E_n\} &= \pm t \end{aligned} \quad (8.5)$$

for the first. The Greens function is therefore

$$G_{D,00}^\uparrow(\omega, T \rightarrow 0) = \left( \frac{1}{2} + \frac{2t}{\Delta} \right) \frac{\omega}{\omega^2 - \left( \frac{\Delta}{2} - t \right)^2} + \left( \frac{1}{2} - \frac{2t}{\Delta} \right) \frac{\omega}{\omega^2 - \left( \frac{\Delta}{2} + t \right)^2} = G_{D,00}^\downarrow(\omega, T \rightarrow 0). \quad (8.6)$$

In the atomic limit ( $t = 0$ ), the Greens function simplifies to

$$G_{D,00}^\uparrow(\omega, T \rightarrow 0) \Big|_{\text{atomic}} = \frac{\omega}{\omega^2 - \frac{1}{4}U^2} \quad (8.7)$$

In the atomic limit, the singly-occupied state has zero energy:

$$E_1(t = 0) = \langle 1, 0 | \left( U\tau_{0\uparrow}\tau_{0\downarrow} + U\tau_{1\uparrow}\tau_{1\downarrow} \right) | 1, 0 \rangle = 0 \quad (8.8)$$

We can write the atomic limit Greens function in terms of this energy and the self energy:

$$G_{D,00}^{\uparrow}(\omega, T \rightarrow 0) \Big|_{\text{atomic}} = \frac{1}{\omega - E_1(t=0) - \Sigma(t=0)} = \frac{1}{\omega - 0 - \frac{U^2}{4\omega}} \quad (8.9)$$

The self energy in the atomic limit can be read off as

$$\Sigma(t=0) = \frac{U^2}{4\omega} \quad (8.10)$$

The site local spectral function can also be calculated from the local Greens function:

$$\begin{aligned} A(0 \uparrow, \omega) &= -\frac{1}{\pi} \text{Im } G_{D,00}^{\uparrow}(\omega) \\ &= \left( \frac{1}{4} - \frac{t}{\Delta} \right) \left[ \delta(\omega - \frac{1}{2}\Delta - t) + \delta(\omega + \frac{1}{2}\Delta + t) \right] \\ &\quad + \left( \frac{1}{4} + \frac{t}{\Delta} \right) \left[ \delta(\omega - \frac{1}{2}\Delta + t) + \delta(\omega + \frac{1}{2}\Delta - t) \right] \\ &= A(0 \downarrow, \omega) . \end{aligned} \quad (8.11)$$

Finally, the inter-site Greens function for the Hubbard dimer is given by

$$G_{D,01}^{\uparrow}(\omega, T \rightarrow 0) = \left( \frac{1}{2} + \frac{2t}{\Delta} \right) \frac{t}{(\omega + \frac{\Delta}{2})^2 - t^2} + \left( \frac{1}{2} - \frac{2t}{\Delta} \right) \frac{t}{(\omega - \frac{\Delta}{2})^2 - t^2} = G_{D,01}^{\downarrow}(\omega, T \rightarrow 0) . \quad (8.12)$$

Once we have calculated the diagonal and off-diagonal matrix elements of the Greens function, we can compute the parameters  $g_0$  and  $g_1$ :

$$g_0 \equiv \frac{2}{N} (G_D^{-1})_{00} = \frac{2}{N} \frac{1}{\text{Det } G_D} (G_D)_{00}, g_1 \equiv \frac{2}{Nw} (G_D^{-1})_{00} = -\frac{2}{Nw} \frac{1}{\text{Det } G_D} (G_D)_{01} \quad (8.13)$$

## References

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