# URG analysis of the extended Kondo model with d-wave interaction

Abhirup Mukherjee, Siddhartha Lal

## I. HAMILTONIAN

We consider an impurity spin $\vec{S}_d$ interacting with a two-dimensional tight-binding conduction bath through the d-wave channel :

$$J\vec{S}_d \cdot \vec{S}_f, \quad \vec{S}_f = \frac{1}{2}\sum_{\sigma,\sigma'} \vec{\tau}_{\sigma,\sigma'} f^\dagger_\sigma f_{\sigma'} . \tag{1}$$

where $\vec{\tau}$ is the vector of sigma matrices. The spin $\vec{S}_f$ is constructed in terms of the d-wave electron $f_\sigma = \frac{1}{2}\left(c^\dagger_{L,\sigma} + c^\dagger_{R,\sigma} - c^\dagger_{U\sigma} - c^\dagger_{D\sigma}\right)$, where $L, R, U$ and $D$ indicate electrons at the positions $(x,y) = (-a,0), (a,0), (0,a)$ and $(0,-a)$ respectively, $a$ being the lattice spacing of the conduction bath lattice. We also consider local bath correlations in the $d-$wave channel:

$$-\frac{W}{2}\left(f^\dagger_\uparrow f_\uparrow - f^\dagger_\downarrow f_\downarrow\right)^2 . \tag{2}$$

In order to facilitate a momentum-space decoupling RG scheme, we Fourier transform the $d-$wave operator to momentum space:

$$f_\sigma = \frac{1}{2}\left(c^\dagger_{L,\sigma} + c^\dagger_{R,\sigma} - c^\dagger_{U\sigma} - c^\dagger_{D\sigma}\right) = \frac{1}{2}\sum_{\vec{k}} c^\dagger_{\vec{k},\sigma}\left[e^{-ik^x a} + e^{ik^x a} - e^{-ik^y a} - e^{ik^y a}\right] = \sum_{\vec{k}}\left[\cos\left(ak^x\right) - \cos\left(ak^y\right)\right] c^\dagger_{\vec{k},\sigma} . \tag{3}$$

The Kondo interaction term takes the following form in momentum space:

$$J\vec{S}_d \cdot \vec{S}_f = \frac{1}{2}J\sum_{\vec{k}_1,\vec{k}_2,\sigma,\sigma'} \vec{S}_d \cdot \vec{\tau}_{\sigma,\sigma'} c^\dagger_{\vec{k}_1\sigma} c_{\vec{k}_2,\sigma'} \prod_{i=1,2}\left[\cos\left(ak^x_i\right) - \cos\left(ak^y_i\right)\right] . \tag{4}$$

The local correlation term can be similarly written in momentum space:

$$-\frac{W}{2}\left(f^\dagger_\uparrow f_\uparrow - f^\dagger_\downarrow f_\downarrow\right)^2 = -\frac{W}{2}\sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\sigma}\left(c^\dagger_{\vec{k}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{k}_4,\sigma} - c^\dagger_{\vec{k}_1\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_3\bar{\sigma}} c_{\vec{k}_4,\bar{\sigma}}\right)\prod_{i=1,2,3,4}\left[\cos\left(ak^x_i\right) - \cos\left(ak^y_i\right)\right] . \tag{5}$$

Combining all the terms, the Hamiltonian can be formally written as

$$H = \sum_{\vec{k},\sigma}\varepsilon_{\vec{k}} c^\dagger_{\vec{k},\sigma} c_{\vec{k},\sigma} + \frac{1}{2}\sum_{\vec{k}_1,\vec{k}_2,\sigma,\sigma'} J_{\vec{k}_1,\vec{k}_2}\vec{S}_d \cdot \vec{\tau}_{\sigma,\sigma'} c^\dagger_{\vec{k}_1\sigma} c_{\vec{k}_2,\sigma'} - \frac{1}{2}\sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\sigma,\sigma'} W_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{k}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_3,\sigma'} c_{\vec{k}_4,\sigma'} . \tag{6}$$

with

$$\begin{aligned}
\varepsilon_{\vec{k}} &= -2t\left[\cos\left(ak^x\right) + \cos\left(ak^y\right)\right], \\
J_{\vec{k}_1,\vec{k}_2} &= J\prod_{i=1,2}\left[\cos\left(ak^x_i\right) - \cos\left(ak^y_i\right)\right], \\
W_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4} &= W\prod_{i=1,2,3,4}\left[\cos\left(ak^x_i\right) - \cos\left(ak^y_i\right)\right].
\end{aligned} \tag{7}$$

The coupling $W_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4}$ has certain symmetries. It is independent of the sequence of momentum indices. It remains invariant if any number of the momenta undergo rotation by an integer multiple of $\pi/2$. It also remains invariant if an even number of momenta are reflected about the nodal point in the corresponding quadrant.

## II. RG SCHEME

At any given step $j$ of the RG procedure, we decouple the states $\{\vec{q}\}$ on the isoenergetic surface of energy $\varepsilon_j$. The diagonal Hamiltonian $H_D$ for this step consists of all terms that do not change the occupancy of the states $\{\vec{q}\}$:

$$H_D^{(j)} = \varepsilon_j \sum_{q,\sigma} \tau_{q,\sigma} + \frac{1}{2}\sum_{\vec{q}} J_{\vec{q},\vec{q}} S_d^z \left(\hat{n}_{\vec{q},\uparrow} - \hat{n}_{\vec{q},\downarrow}\right) - \frac{1}{2}\sum_{\vec{q}_1} W_{\vec{q}_1,\vec{q}_1,\vec{q}_1,\vec{q}_1} \left(\hat{n}_{\vec{q}_1,\uparrow} - \hat{n}_{\vec{q}_1,\downarrow}\right)^2 \ . \tag{8}$$

where $\tau = \hat{n} - 1/2$. The three terms, respectively, are the kinetic energy of the momentum states on the isoenergetic shell that we are decoupling, the Ising interaction energy between the impurity spin and the spins formed by these momentum states and, finally, the local correlation energy associated with these states arising from the $W$ term.

The off-diagonal part of the Hamiltonian on the other hand leads to scattering in the states $\{\vec{q}\}$:

$$\begin{aligned} H_X^{(j)} = & \underbrace{\sum_{\vec{k},\vec{q},\sigma,\sigma'} J_{\vec{k},\vec{q}} \vec{S}_d \cdot \vec{\tau}_{\sigma,\sigma'} \left[c^\dagger_{\vec{q}\sigma} c_{\vec{k},\sigma} + \text{h.c.}\right]}_{T_1^\dagger + T_1} \\ & \underbrace{-\frac{1}{2} \sum_{\vec{q}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\sigma,\sigma'} W_{\vec{q}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4} \sigma\sigma' \left[c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_3,\sigma'} c_{\vec{k}_4,\sigma'} + \text{h.c.}\right]}_{T_2^\dagger + T_2} \\ & \underbrace{-\frac{1}{2} \sum_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4,\sigma,\sigma'} W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4} \sigma\sigma' \left[c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{q}_2,\sigma'} c_{\vec{k}_4,\sigma'} + \text{h.c.}\right]}_{T_3^\dagger + T_3} \\ & \underbrace{-\frac{1}{2} \sum_{\vec{q}_1,\vec{k}_2,\vec{k}_3,\vec{q}_2,\sigma,\sigma'} W_{\vec{q}_1,\vec{k}_2,\vec{k}_3,\vec{q}_2} \sigma\sigma' c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_3,\sigma'} c_{\vec{q}_2,\sigma'}}_{T_4} \\ & \underbrace{-\frac{1}{2} \sum_{\vec{q}_1,\vec{k}_2,\vec{k}_3,\vec{q}_2,\sigma,\sigma'} W_{\vec{q}_1,\vec{k}_2,\vec{k}_3,\vec{q}_2} \sigma\sigma' c^\dagger_{\vec{k}_3,\sigma'} c_{\vec{q}_2,\sigma'} c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma}}_{T_5} \\ & \underbrace{-\frac{1}{2} \sum_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{q}_3,\sigma,\sigma'} W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{q}_3} \sigma\sigma' \left[c^\dagger_{\vec{q}_2,\sigma'} c_{\vec{q}_3,\sigma'} c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} + \text{h.c.}\right]}_{T_6^\dagger + T_6} \ , \\ & \underbrace{+ \sum_{\vec{k},\vec{q},\sigma,\sigma'} J_{\vec{q}',\vec{q}} \vec{S}_d \cdot \vec{\tau}_{\sigma,\sigma'} \left[c^\dagger_{\vec{q}\sigma} c_{\vec{q}',\sigma}\right]}_{T_7} \ . \end{aligned} \tag{9}$$

The first term $T_1$ is an impurity-mediated scattering between the states $\vec{q}$ at energy $\varepsilon_j$ and the states $\vec{k}$ at energies below $\varepsilon_j$. Terms $T_2$ through $T_6$ involve two-particle scattering between these momentum states involving an increasing number of participating states from the isoenergetic shell $\varepsilon_j$, through the Hubbard-like local term $W$. The renormalisation of the Hamiltonian is constructed from the general expression

$$\Delta H^{(j)} = H_X \frac{1}{\omega - H_D} H_X \ . \tag{10}$$

## III. RENORMALISATION OF THE BATH CORRELATION TERM $W$

In order to lead to a renormalisation of the $W-$term, there must be a total of four uncontracted momentum indices $k_i$ and two contracted indices $q_1, q_2$. The following combinations of scattering processes are compatible: (i) $T_2^\dagger G T_6 + T_6^\dagger G T_2$, (ii) $T_3^\dagger G T_3 + T_3 G T_3^\dagger$, (iii) $T_4 G T_4$, (iv) $T_5 G T_5$ and (v) $T_4 G T_5 + T_5 G T_4$.

### A. Correlated scattering involve one electron on the shell $\varepsilon_j$

The first term is of the form

$$T_6^\dagger G T_2 = \sum \sigma_1\sigma_1' W_{\vec{k}_3,\vec{k}_4,\vec{k}_1,\vec{q}_1} c^\dagger_{\vec{k}_3,\sigma_1'} c_{\vec{k}_4,\sigma_1'} c^\dagger_{\vec{k}_1,\sigma_1} c_{\vec{q}_1,\sigma_1} \frac{1}{\omega - H_D} W_{\vec{q}_1,\vec{q}_2,\vec{q}_2,\vec{k}_2} \Big( -\hat{n}_{\vec{q}_2,\bar{\sigma}_1} c^\dagger_{\vec{q}_1,\sigma_1} c_{\vec{k}_2,\sigma_1} + \hat{n}_{\vec{q}_2,\sigma_1} c^\dagger_{\vec{q}_1,\sigma_1} c_{\vec{k}_2,\sigma_1} + c^\dagger_{\vec{q}_1,\sigma_1} \left(1 - \hat{n}_{\vec{q}_2,\bar{\sigma}_1}\right) c_{\vec{k}_2,\sigma_1} \Big) . \tag{11}$$

The change in occupancy of the state $\vec{q}_1\sigma_1$ from 1 to 0 leads to an excited state energy $H_D = \varepsilon(q_1)\tau_{q_1\sigma_1} - \frac{1}{2}W_{\vec{q}_1}\left(\hat{n}_{\vec{q}_1,\uparrow} - \hat{n}_{\vec{q}_1,\uparrow}\right)^2 = -\frac{1}{2}\varepsilon(q_1) - \frac{1}{2}W_{\vec{q}_1}$, where $W_{\vec{q}_1}$ is a shorthand for $W_{\vec{q}_1,\vec{q}_1,\vec{q}_1,\vec{q}_1}$. We consider the first two terms for now. The number operators $\hat{n}_{\vec{q}_2,\sigma_1}, \hat{n}_{\vec{q}_2,\bar{\sigma}_1}$ project the initial state to that in which $\vec{q}_2$ is occupied, henceforth referred to as the particle sector (PS). The operator $c_{\vec{q}_1,\sigma_1}$ can be combined with its Hermitian conjugate on the other side to give another number operator, leading to another projection. Since the two terms are otherwise identical, their opposite signs lead to them cancelling each other. The remaining term involves the hole projection operator $1 - \hat{n}_{\vec{q}_2,\sigma_1}$ which projects onto the set of initial states in which the momentum state $\vec{q}_2$ is unoccupied, henceforth referred to as the hole sector (HS). Evaluating this term in the same way leads to

$$T_2^\dagger G T_6 = -\sum_{\{\vec{k}_i\},\sigma_1\sigma_1'} c^\dagger_{\vec{k}_3,\sigma_1'} c_{\vec{k}_4,\sigma_1'} c^\dagger_{\vec{k}_1,\sigma_1} c_{\vec{k}_2,\sigma_1} \sum_{\substack{\vec{q}_1\in\text{PS},\\ \vec{q}_2\in\text{HS}}} \frac{W_{\vec{q}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4} W_{\vec{q}_1,\vec{q}_2,\vec{q}_2,\vec{k}_1}}{\omega + \frac{1}{2}\varepsilon(q_1) + \frac{1}{2}W_{\vec{q}_1}} . \tag{12}$$

The particle-hole transformed term $T_6^\dagger G T_2$ can be evaluated in the same way:

$$T_6^\dagger G T_2 = \sum W_{\vec{q}_1,\vec{q}_2,\vec{q}_2,\vec{k}_2} \Big( -c^\dagger_{\vec{q}_1,\sigma_1} c_{\vec{k}_2,\sigma_1} \hat{n}_{\vec{q}_2,\bar{\sigma}_1} + c^\dagger_{\vec{q}_1,\sigma_1} c_{\vec{k}_2,\sigma_1} \hat{n}_{\vec{q}_2,\sigma_1} + c^\dagger_{\vec{q}_1,\sigma_1} \left(1 - \hat{n}_{\vec{q}_2,\bar{\sigma}_1}\right) c_{\vec{k}_2,\sigma_1} \Big) \frac{1}{\omega - H_D} \times \sigma_1\sigma_1' W_{\vec{k}_3,\vec{k}_4,\vec{k}_1,\vec{q}_1} c^\dagger_{\vec{k}_3,\sigma_1'} c_{\vec{k}_4,\sigma_1'} c^\dagger_{\vec{k}_1,\sigma_1} c_{\vec{q}_1,\sigma_1} . \tag{13}$$

For such a scattering process, the excited energy is given by $H_D = \varepsilon(q_1)\tau_{q_1\sigma_1} - \frac{1}{2}W_{\vec{q}_1}\left(\hat{n}_{\vec{q}_1,\uparrow} - \hat{n}_{\vec{q}_1,\uparrow}\right)^2 = \frac{1}{2}\varepsilon(q_1) - \frac{1}{2}W_{\vec{q}_1}$. The change of the sign in front of $\varepsilon(\vec{q}_1)$ arises from the fact that in this process, the state $\vec{q}_1$ is occupied in the intermediate excited state, owing to the $c^\dagger_{\vec{q}_1,\sigma_1}$ operator to the right of the Greens function. Cancelling the first two terms in eq. 13 (just as in the previous process) and evaluating the last term gives

$$T_6^\dagger G T_2 = \sum_{\{\vec{k}_i\},\sigma_1\sigma_1'} c^\dagger_{\vec{k}_3,\sigma_1'} c_{\vec{k}_4,\sigma_1'} c^\dagger_{\vec{k}_1,\sigma_1} c_{\vec{k}_2,\sigma_1} \sum_{\vec{q}_1,\vec{q}_2\in\text{HS}} \frac{W_{\vec{k}_3,\vec{k}_4,\vec{k}_1,\vec{q}_1} W_{\vec{q}_1,\vec{q}_2,\vec{q}_2,\vec{k}_2}}{\omega - \frac{1}{2}\varepsilon(q_1) + \frac{1}{2}W_{\vec{q}_1}} . \tag{14}$$

We now assume that the Brillouin zone of the lattice in which our conduction bath is embedded is symmetrical about the Fermi surface. This essentially amounts to working at particle-hole symmetry, by setting the chemical potential of the bath to zero. This symmetry leads to two consequences:

- If this is the case, the states in the particle sector will reside on the isoenergetic shell of energy $-|\varepsilon_j|$, while those in the hole sector will reside at energy $|\varepsilon_j|$, at equal distances from the Fermi surface (which lies at zero energy). This ensures that in the denominators of the RG equation, we have the simplification

$$\varepsilon(\vec{q}_1)\Big|_{\text{PS}} = -\varepsilon(\vec{q}_2)\Big|_{\text{HS}} = -|\varepsilon_j| . \tag{15}$$

- We also consider the symmetry of of the coupling $W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}$ to reflections about the nodal point in the same quadrant of the Brillouin zone. For any momentum $\vec{q}_1$ in the hole sector, we can find a corresponding point $\tilde{\vec{q}}_1$ in the particle sector by reflecting about the nodal point. This corresponds to the transformation $q_x \to \pi - q_x, q_y \to \pi - q_y$, leading to a sign change of the factor $[\cos(aq_1^x) - \cos(aq_1^y)]$. This leaves the product $W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}$ unchanged. The diagonal coupling $W_{\vec{q}_1}$ also remains unchanged by themselves, since it involves a product of four copies of the factor.

These two features ensure that the inner summations over $\vec{q}_1, \vec{q}_2$ are identical in eqs. 12 and 14, leading to the vanishing of the total renormalisation $T_6^\dagger G T_2 + T_2^\dagger G T_6$.

### B. Scattering across the Fermi surface involving two electrons on the shell $\varepsilon_j$

We now consider the second term $T_3^\dagger G T_3$:

$$
\begin{aligned}
T_3^\dagger G T_3 &= \frac{1}{4}\sum W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{q}_2,\sigma'} c_{\vec{k}_4,\sigma'} \frac{1}{\omega - H_D} W_{\vec{q}_1,\vec{k}'_2,\vec{q}_2,\vec{k}'_4}\sigma\sigma' c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{q}_2,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{q}_1,\sigma} \\
&= \frac{1}{4}\sum \hat{n}_{\vec{q}_1,\sigma}\hat{n}_{\vec{q}_2,\sigma'} c_{\vec{k}_2,\sigma} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{k}_1,\sigma'} c^\dagger_{\vec{k}_3,\sigma} \frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega + \frac{1}{2}\left[\varepsilon(q_1)+\varepsilon(q_2)\right] + \frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)} \\
&= \frac{1}{4}\sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{k}_2,\sigma} \sum_{\vec{q}_1,\vec{q}_2\in\mathrm{PS}} \frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega + \frac{1}{2}\left[\varepsilon(q_1)+\varepsilon(q_2)\right] + \frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\ ,
\end{aligned} \tag{16}
$$

where $\vec{q}_1,\vec{q}_2$ are summed over all momentum states in the isoenergetic shell and in the particle sector (PS) (states are occupied in the ground state).

The particle-hole transformed term is $T_3 G T_3^\dagger$:

$$
\begin{aligned}
T_3 G T_3^\dagger &= \frac{1}{4}\sum W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}\sigma\sigma' c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{q}_2,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{q}_1,\sigma} \frac{1}{\omega - H_D} W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{q}_2,\sigma'} c_{\vec{k}_4,\sigma'} \\
&= \frac{1}{4}\sum \left(1-\hat{n}_{\vec{q}_1,\sigma}\right)\left(1-\hat{n}_{\vec{q}_2,\sigma'}\right) c^\dagger_{\vec{k}_1,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{k}_2,\sigma} c_{\vec{k}_4,\sigma'} \frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega - \frac{1}{2}\left[\varepsilon(q_1)+\varepsilon(q_2)\right] + \frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)} \\
&= \frac{1}{4}\sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{k}_2,\sigma} \sum_{\vec{q}_1,\vec{q}_2\in\mathrm{HS}} \frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega - \frac{1}{2}\left[\varepsilon(q_1)+\varepsilon(q_2)\right] + \frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\ ,
\end{aligned} \tag{17}
$$

where the hole projectors $\left(1-\hat{n}_{\vec{q}_1,\sigma}\right)\left(1-\hat{n}_{\vec{q}_2,\sigma'}\right)$ force the momenta $\vec{q}_1,\vec{q}_2$ to extend over the states only in the hole sector (states that are unoccupied in the ground state). The change in the sign of $\left[\varepsilon(q_1)+\varepsilon(q_2)\right]$ in the denominator compared to the denominator in $T_3^\dagger G T_3$ is for the same reason.

We can obtain two additional terms by switching the sequence of operators on the right hand side of the propagator:

$$
\begin{aligned}
T_3^\dagger G \overline{T}_3 &= \frac{1}{4}\sum W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{q}_2,\sigma'} c_{\vec{k}_4,\sigma'} \frac{1}{\omega - H_D} W_{\vec{q}_1,\vec{k}'_2,\vec{q}_2,\vec{k}'_4}\sigma\sigma' c^\dagger_{\vec{k}_3,\sigma} c_{\vec{q}_1,\sigma} c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{q}_2,\sigma'} \\
&= \frac{1}{4}\sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{k}_2,\sigma} \sum_{\vec{q}_1,\vec{q}_2\in\mathrm{PS}} \frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega + \frac{1}{2}\left[\varepsilon(q_1)+\varepsilon(q_2)\right] + \frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\ ,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
T_3 G \overline{T}_3^\dagger &= \frac{1}{4}\sum W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}\sigma\sigma' c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{q}_2,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{q}_1,\sigma} \frac{1}{\omega - H_D} W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{q}_2,\sigma'} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} \\
&= \frac{1}{4}\sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{k}_2,\sigma} \sum_{\vec{q}_1,\vec{q}_2\in\mathrm{HS}} \frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega - \frac{1}{2}\left[\varepsilon(q_1)+\varepsilon(q_2)\right] + \frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\ ,
\end{aligned} \tag{19}
$$

### C. Forward and tangential scattering involving two electrons on the shell $\varepsilon_j$

The remaining sets of terms that we need to consider are:

$$
\begin{aligned}
T_4 G T_4 &= \frac{1}{4}\sum W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}\sigma\sigma' c^\dagger_{\vec{q}_1,\sigma} c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{q}_2,\sigma'} \frac{1}{\omega - H_D} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{q}_2,\sigma'} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{q}_1,\sigma} \\
&= \frac{1}{4}\sum \hat{n}_{\vec{q}_1,\sigma}\left(1-\hat{n}_{\vec{q}_2,\sigma'}\right) c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{k}_3,\sigma} \frac{W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}}{\omega + \frac{1}{2}\varepsilon\left(q_1\right) - \frac{1}{2}\varepsilon\left(q_2\right) + \frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)} \\
&= -\frac{1}{4}\sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_1,\sigma'} c_{\vec{k}_4,\sigma'} c^\dagger_{\vec{k}_3,\sigma} c_{\vec{k}_2,\sigma} \sum_{\substack{\vec{q}_1\in\mathrm{PS},\\ \vec{q}_2\in\mathrm{HS}}} \frac{W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2} W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}}{\omega + \frac{1}{2}\varepsilon\left(q_1\right) - \frac{1}{2}\varepsilon\left(q_2\right) + \frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\ .
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
T_5GT_5 &= \frac{1}{4}\sum W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}\sigma\sigma' c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{q}_2,\sigma'}c^\dagger_{\vec{q}_1,\sigma}c_{\vec{k}_2,\sigma}\frac{1}{\omega-H_D}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{k}_3,\sigma}c_{\vec{q}_1,\sigma}c^\dagger_{\vec{q}_2,\sigma'}c_{\vec{k}_4,\sigma'} \\
&= \frac{1}{4}\sum \hat{n}_{\vec{q}_1,\sigma}\left(1-\hat{n}_{\vec{q}_2,\sigma'}\right)c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{k}_2,\sigma}c^\dagger_{\vec{k}_3,\sigma}c_{\vec{k}_4,\sigma'}\frac{W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}}{\omega+\frac{1}{2}\varepsilon\left(q_1\right)-\frac{1}{2}\varepsilon\left(q_2\right)+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)} \\
&= -\frac{1}{4}\sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{k}_4,\sigma'}c^\dagger_{\vec{k}_3,\sigma}c_{\vec{k}_2,\sigma}\sum_{\substack{\vec{q}_1\in\text{PS},\\ \vec{q}_2\in\text{HS}}}\frac{W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}}{\omega+\frac{1}{2}\varepsilon\left(q_1\right)-\frac{1}{2}\varepsilon\left(q_2\right)+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\ .
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
T_4GT_5 &= \frac{1}{4}\sum W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}\sigma\sigma' c^\dagger_{\vec{q}_1,\sigma}c_{\vec{k}_2,\sigma}c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{q}_2,\sigma'}\frac{1}{\omega-H_D}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{k}_3,\sigma}c_{\vec{q}_1,\sigma}c^\dagger_{\vec{q}_2,\sigma'}c_{\vec{k}_4,\sigma'} \\
&= -\frac{1}{4}\sum \hat{n}_{\vec{q}_1,\sigma}\left(1-\hat{n}_{\vec{q}_2,\sigma'}\right)c_{\vec{k}_2,\sigma}c^\dagger_{\vec{k}_1,\sigma'}c^\dagger_{\vec{k}_3,\sigma}c_{\vec{k}_4,\sigma'}\frac{W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}}{\omega+\frac{1}{2}\varepsilon\left(q_1\right)-\frac{1}{2}\varepsilon\left(q_2\right)+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)} \\
&= -\frac{1}{4}\sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{k}_4,\sigma'}c^\dagger_{\vec{k}_3,\sigma}c_{\vec{k}_2,\sigma}\sum_{\substack{\vec{q}_1\in\text{PS},\\ \vec{q}_2\in\text{HS}}}\frac{W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}}{\omega+\frac{1}{2}\varepsilon\left(q_1\right)-\frac{1}{2}\varepsilon\left(q_2\right)+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\ .
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
T_5GT_4 &= \frac{1}{4}\sum W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}\sigma\sigma' c^\dagger_{\vec{k}_3,\sigma}c_{\vec{q}_2,\sigma}c^\dagger_{\vec{q}_1,\sigma'}c_{\vec{k}_4,\sigma'}\frac{1}{\omega-H_D}W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}\sigma\sigma' c^\dagger_{\vec{q}_2,\sigma}c_{\vec{k}_2,\sigma}c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{q}_1,\sigma'} \\
&= -\frac{1}{4}\sum \hat{n}_{\vec{q}_1,\sigma}\left(1-\hat{n}_{\vec{q}_2,\sigma'}\right)c^\dagger_{\vec{k}_3,\sigma}c_{\vec{k}_4,\sigma'}c_{\vec{k}_2,\sigma}c^\dagger_{\vec{k}_1,\sigma'}\frac{W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}}{\omega+\frac{1}{2}\varepsilon\left(q_1\right)-\frac{1}{2}\varepsilon\left(q_2\right)+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)} \\
&= -\frac{1}{4}\sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_3,\sigma}c_{\vec{k}_2,\sigma}c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{k}_4,\sigma'}\sum_{\substack{\vec{q}_1\in\text{PS},\\ \vec{q}_2\in\text{HS}}}\frac{W_{\vec{q}_1,\vec{k}_1,\vec{q}_2,\vec{k}_2}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_4}}{\omega+\frac{1}{2}\varepsilon\left(q_1\right)-\frac{1}{2}\varepsilon\left(q_2\right)+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\ .
\end{aligned}
\tag{23}
$$

### D. Net renormalisation for the bath correlation term

Adding contributions from all these terms, the total renormalisation in the Hamiltonian in the context of $W$ comes out to be

$$
\begin{aligned}
\Delta H = \sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_3,\sigma}c_{\vec{k}_2,\sigma}c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{k}_4,\sigma'}\Bigg[&\frac{1}{2}\sum_{\vec{q}_1,\vec{q}_2\in\text{PS}}\frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega+\frac{1}{2}\left[\varepsilon(q_1)+\varepsilon(q_2)\right]+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)} \\
&+\frac{1}{2}\sum_{\vec{q}_1,\vec{q}_2\in\text{HS}}\frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega-\frac{1}{2}\left[\varepsilon(q_1)+\varepsilon(q_2)\right]+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)} \\
&-\sum_{\substack{\vec{q}_1\in\text{PS},\\ \vec{q}_2\in\text{HS}}}\frac{W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}}{\omega+\frac{1}{2}\left[\varepsilon(q_1)-\varepsilon(q_2)\right]+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\Bigg]
\end{aligned}
\tag{24}
$$

Following the arguments laid down below eq. 14, we know that the following relation holds:

$$
\varepsilon(\vec{q}_1)\Big|_{\text{PS}} = -\varepsilon(\vec{q}_2)\Big|_{\text{HS}} = -|\varepsilon_j|\ . \tag{25}
$$

Following the same arguments, we also know that the product coupling $W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1}$ and the diagonal couplings $W_{\vec{q}_1}$ and $W_{\vec{q}_2}$ remain unchanged if $\vec{q}_2$ is transformed between the particle and hole sectors. Using these properties, we get

$$
\begin{aligned}
\Delta H &= \sum_{\substack{1,2,3,4,\\ \sigma,\sigma'}} c^\dagger_{\vec{k}_3,\sigma}c_{\vec{k}_2,\sigma}c^\dagger_{\vec{k}_1,\sigma'}c_{\vec{k}_4,\sigma'}\sum_{\vec{q}_1,\vec{q}_2\in\text{HS}}W_{\vec{q}_1,\vec{k}_2,\vec{q}_2,\vec{k}_4}W_{\vec{q}_1,\vec{k}_3,\vec{q}_2,\vec{k}_1} \\
&\times\left[\frac{1/2}{\omega-|\varepsilon_j|+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}+\frac{1/2}{\omega-|\varepsilon_j|+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}-\frac{1}{\omega-|\varepsilon_j|+\frac{1}{2}\left(W_{\vec{q}_1}+W_{\vec{q}_2}\right)}\right] \\
&= 0\ .
\end{aligned}
\tag{26}
$$

## IV. RENORMALISATION OF THE KONDO SCATTERING TERM $J$

We take a closer look at the Kondo scattering terms $T_1$ and $T_7$ in $H_X$:

$$\begin{aligned} T_1^\dagger + T_7 =& \underbrace{\frac{1}{2}\sum_{\vec{k},\vec{q},\sigma} J_{\vec{k},\vec{q}} S_d^z \sigma c^\dagger_{\vec{q},\sigma} c_{\vec{k},\sigma}}_{T^\dagger_{1,z}} + \underbrace{\frac{1}{2}\sum_{\vec{k},\vec{q}} J_{\vec{k},\vec{q}} S_d^+ c^\dagger_{\vec{q},\downarrow} c_{\vec{k},\uparrow}}_{T^\dagger_{1,+-}} + \underbrace{\frac{1}{2}\sum_{\vec{k},\vec{q}} S_d^- c^\dagger_{\vec{q},\uparrow} c_{\vec{k},\downarrow}}_{T^\dagger_{1,-+}} \\ & \underbrace{\frac{1}{2}\sum_{\vec{q}',\vec{q},\sigma} J_{\vec{q}',\vec{q}} S_d^z \sigma c^\dagger_{\vec{q},\sigma} c_{\vec{q}',\sigma}}_{T^\dagger_{7,z}} + \underbrace{\frac{1}{2}\sum_{\vec{q}',\vec{q}} J_{\vec{q}',\vec{q}} S_d^+ c^\dagger_{\vec{q},\downarrow} c_{\vec{q}',\uparrow}}_{T^\dagger_{7,+-}} + \underbrace{\frac{1}{2}\sum_{\vec{q}',\vec{q}} S_d^- c^\dagger_{\vec{q},\uparrow} c_{\vec{q}',\downarrow}}_{T^\dagger_{7,-+}} \end{aligned} \tag{27}$$

We note that scattering processes involving the pairs $\left(T^\dagger_{1,\pm\mp}, T_{1,\pm\mp}\right)$, $(T_{1,z}, T_6)$ and $(T_{7,z}, T_6)$ will renormalise the $S_d^z$ term, while those involving the pairs $(T_{1,z}, T_{1,\pm\mp})$ and $(T_{1,\pm\mp}, T_6)$ will renormalise the $S_d^\pm$ terms.

We first consider the renormalisation to the $S_d^z$ term, arising purely from the Kondo terms:

$$\sum_{\sigma=\pm} T^\dagger_{1,\sigma\bar{\sigma}} G T_{1,\sigma\bar{\sigma}} = \sum_{\sigma=\pm} \frac{1}{4} \sum_{\vec{k}_2,\vec{q}} J_{\vec{k}_2,\vec{q}} S_d^\sigma c^\dagger_{\vec{q},\bar{\sigma}} c_{\vec{k}_2,\sigma} \frac{1}{\omega - H_D} \sum_{\vec{k}_1,\vec{q}} J_{\vec{k}_1,\vec{q}} S_d^{\bar{\sigma}} c^\dagger_{\vec{k}_1,\sigma} c_{\vec{q},\bar{\sigma}} \ , \tag{28}$$

where $c_{k,+(-)} \equiv c_{k,\uparrow(\downarrow)}$. The excitation energy for such processes is given by $H_D = |\varepsilon_j| - J_{\vec{q}}/4 - W_{\vec{q}}/2$, due to the fact that the impurity spin flip and the spin flip of the conduction bath state $\vec{q}$ occurs in an anti-parallel fashion. Substituting this, performing the usual contraction and projection of the state $\vec{q}$ and carrying out the spin manipulation $S_d^\sigma S_d^{\bar{\sigma}} = \frac{1}{2} + \sigma S_d^z$ results in the expression

$$\begin{aligned} \sum_{\sigma=\pm} T^\dagger_{1,\sigma\bar{\sigma}} G T_{1,\sigma\bar{\sigma}} &= \sum_{\sigma=\pm} \frac{1}{4} \sum_{\vec{k}_2,\vec{k}_1} \left(\frac{1}{2} + \sigma S_d^z\right) c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_1,\sigma} \sum_{\vec{q}} \hat{n}_{\vec{q},\bar{\sigma}} \frac{J_{\vec{k}_2,\vec{q}} J_{\vec{k}_1,\vec{q}}}{\omega - |\varepsilon_j| + J_{\vec{q}}/4 + W_{\vec{q}}/2} \\ &= -\frac{1}{4} \sum_{\vec{k}_2,\vec{k}_1} S_d^z \left(c^\dagger_{\vec{k}_1,\uparrow} c_{\vec{k}_2,\uparrow} - c^\dagger_{\vec{k}_1,\downarrow} c_{\vec{k}_2,\downarrow}\right) \sum_{\vec{q}\in \mathrm{PS}} \frac{J_{\vec{k}_2,\vec{q}} J_{\vec{k}_1,\vec{q}}}{\omega - |\varepsilon_j| + J_{\vec{q}}/4 + W_{\vec{q}}/2} + \Big[\text{pot. scatt. terms}\Big] \ . \end{aligned} \tag{29}$$

The particle-hole exchanged partner is

$$\begin{aligned} \sum_{\sigma=\pm} T_{1,\sigma\bar{\sigma}} G T^\dagger_{1,\sigma\bar{\sigma}} &= \sum_{\sigma=\pm} \frac{1}{4} \sum_{\vec{k}_1,\vec{q}} J_{\vec{k}_1,\vec{q}} S_d^{\bar{\sigma}} c^\dagger_{\vec{k}_1,\sigma} c_{\vec{q},\bar{\sigma}} \frac{1}{\omega - H_D} \sum_{\vec{k}_2,\vec{q}} J_{\vec{k}_2,\vec{q}} S_d^\sigma c^\dagger_{\vec{q},\bar{\sigma}} c_{\vec{k}_2,\sigma} \\ &= \sum_{\sigma=\pm} \frac{1}{4} \sum_{\vec{k}_2,\vec{k}_1} \left(\frac{1}{2} + \bar{\sigma} S_d^z\right) c^\dagger_{\vec{k}_1,\sigma} c_{\vec{k}_2,\sigma} \sum_{\vec{q}} (1 - \hat{n}_{\vec{q},\bar{\sigma}}) \frac{J_{\vec{k}_2,\vec{q}} J_{\vec{k}_1,\vec{q}}}{\omega - |\varepsilon_j| + J_{\vec{q}}/4 + W_{\vec{q}}/2} \\ &= -\frac{1}{4} \sum_{\vec{k}_2,\vec{k}_1,\sigma} S_d^z \sigma c^\dagger_{\vec{k}_1,\sigma} c_{\vec{k},\sigma} \sum_{\vec{q}\in \mathrm{HS}} \frac{J_{\vec{k}_2,\vec{q}} J_{\vec{k}_1,\vec{q}}}{\omega - |\varepsilon_j| + J_{\vec{q}}/4 + W_{\vec{q}}/2} + \Big[\text{pot. scatt. terms}\Big] \ . \end{aligned} \tag{30}$$

Next, we consider scattering processes involving $T_6$:

$$T^\dagger_{1,z} G T_6 = \frac{1}{4} \sum_{\vec{k}_1,\vec{k}_2,\vec{q},\sigma} \sigma J_{\vec{k}_2,\vec{q}} S_d^z c^\dagger_{\vec{q},\sigma} c_{\vec{k}_2,\sigma} \frac{1}{\omega - H_D} W_{\vec{q},\vec{q},\vec{q},\vec{k}_1} \left[-c^\dagger_{\vec{k}_1,\sigma} c_{\vec{q},\sigma} \hat{n}_{\vec{q},\bar{\sigma}} + c^\dagger_{\vec{k}_1,\sigma} c_{\vec{q},\sigma} \hat{n}_{\vec{q},\sigma} + c^\dagger_{\vec{k}_1,\sigma} (1 - \hat{n}_{-\vec{q},\sigma}) c_{\vec{q},\sigma}\right] \ . \tag{31}$$

For the same reasons as those outlined above eq. 12, the first two terms on the right-hand side of the Greens function cancel each other. The remaining term evaluates to

$$T^\dagger_{1,z} G T_6 = \frac{1}{4} \sum_{\vec{k}_1,\vec{k}_2,\sigma} \sigma S_d^z c_{\vec{k}_2,\sigma} c^\dagger_{\vec{k}_1,\sigma} \sum_{\vec{q}\in \mathrm{PS}} \frac{J_{\vec{k}_2,\vec{q}} W_{\vec{q},\vec{q},\vec{q},\vec{k}_1}}{\omega - |\varepsilon_j| + J_{\vec{q}}/4 + W_{\vec{q}}/2} = -\frac{1}{4} \sum_{\vec{k}_1,\vec{k}_2,\sigma} S_d^z \sigma c^\dagger_{\vec{k}_1,\sigma} c_{\vec{k}_2,\sigma} \sum_{\vec{q}\in \mathrm{PS}} \frac{J_{\vec{k}_2,\vec{q}} W_{\vec{q},\vec{q},\vec{q},\vec{k}_1}}{\omega - |\varepsilon_j| + J_{\vec{q}}/4 + W_{\vec{q}}/2} \ . \tag{32}$$

Another scattering process is obtained by switching the operators $T_1$ and $T_6$:

$$\begin{aligned} T_6^\dagger G T_{1,z} &= \frac{1}{4} \sum_{\vec{k}_1,\vec{k}_2,\vec{q},\sigma} \left[ -c^\dagger_{\vec{q},\sigma_1} c_{\vec{k}_2,\sigma_1} \hat{n}_{-\vec{q},\bar{\sigma}_1} + c^\dagger_{\vec{q},\sigma_1} c_{\vec{k}_2,\sigma_1} \hat{n}_{-\vec{q},\sigma_1} + c^\dagger_{\vec{q},\sigma_1} \left(1 - \hat{n}_{-\vec{q},\sigma_1}\right) c_{\vec{k}_2,\sigma_1} \right] \frac{W_{\vec{q},\vec{q},\vec{q},\vec{k}_2} J_{\vec{k}_1,\vec{q}}}{\omega - H_D} \sigma S_d^z c^\dagger_{\vec{k}_1,\sigma} c_{\vec{q},\sigma} \\ &= -\frac{1}{4} \sum_{\vec{k}_1,\vec{k}_2,\sigma} S_d^z \sigma c^\dagger_{\vec{k}_1,\sigma} c_{\vec{k}_2,\sigma} \sum_{\vec{q}\in \mathrm{PS}} \frac{J_{\vec{k}_1,\vec{q}} W_{\vec{q},\vec{q},\vec{q},\vec{k}_2}}{\omega - |\varepsilon_j| + J_{\vec{q}}/4 + W_{\vec{q}}/2} \end{aligned} \tag{33}$$

One can also construct