

# EXACTLY SOLVABLE MODEL OF CORRELATED METAL-INSULATOR TRANSITION

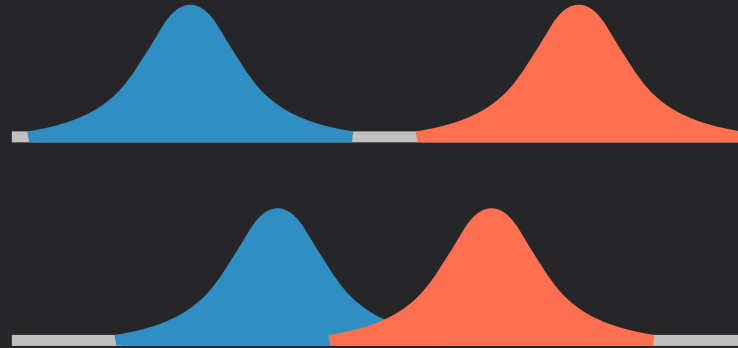
## Insights on Non-Fermi Liquid and Mott Insulator

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EPQM Seminar



# In A Nutshell

- An **exactly solvable** model that displays correlation-driven transition from a **non-Fermi liquid** to a **Mott insulator**.
- Analyse the non-Fermi liquid in this **controlled setting** to understand its features.
- Study a generalisation of this model to obtain **Fermi arcs**

Where Do Mott Insulators  
and Non-Fermi Liquids Fit in  
the “Standard Model”?

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# Mott Insulators Are Different

Half-filled system is **metallic**  
in absence of interactions.



Dispersion away from band  
edge is **non-interacting**.

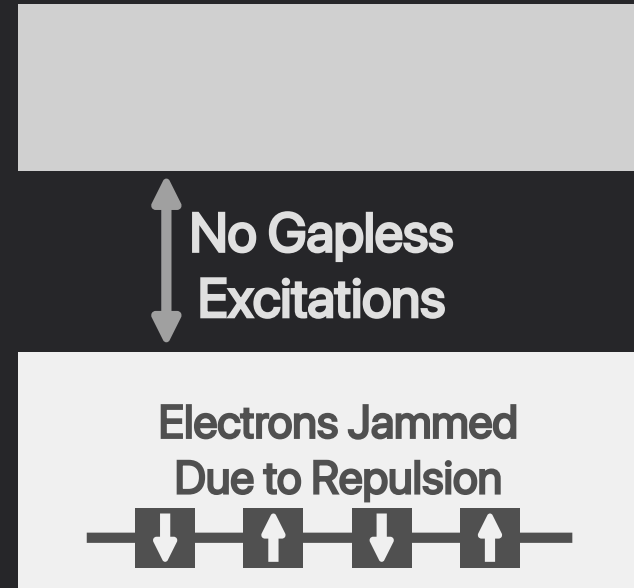
# Mott Insulators Are Different

Half-filled system is **metallic** in absence of interactions.



Dispersion away from band edge is **non-interacting**.

Add strong **interactions** – Mott Insulator!

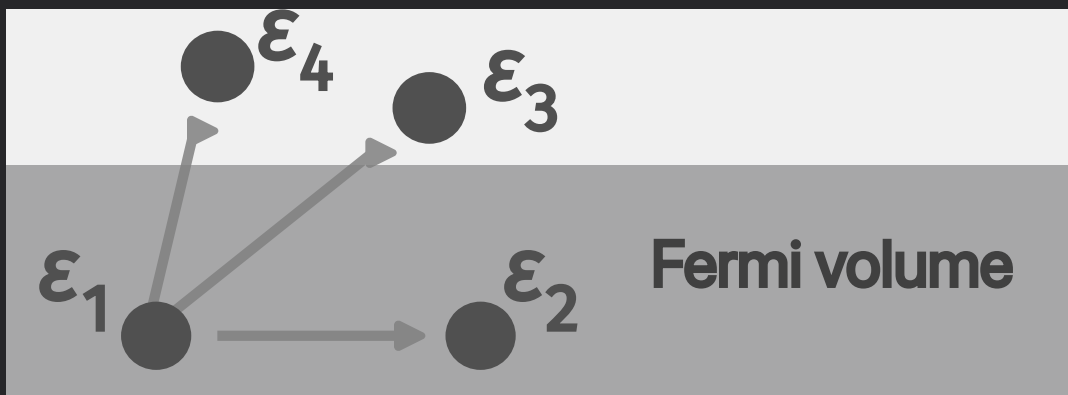


Gap opens inside the band:  $\begin{pmatrix} \varepsilon & U \\ U & \varepsilon \end{pmatrix} \rightarrow \varepsilon \pm U$

# Non-Fermi Liquids Are Different

## Landau Fermi Liquid Theory (Postulates)

- Theory describing how **metals arise** in interacting systems
- Lack of scattering **phase space** at low-energies
- Fermi surface and low-lying electronic excitations survive (**quasiparticles**).



$$\begin{aligned}\Gamma &\sim \int d\epsilon_4 d\epsilon_3 d\epsilon_2 \delta(\epsilon - \epsilon_2 - \epsilon_3 - \epsilon_4) \\ &\sim \epsilon^2 \\ \rho &\sim T^2\end{aligned}$$

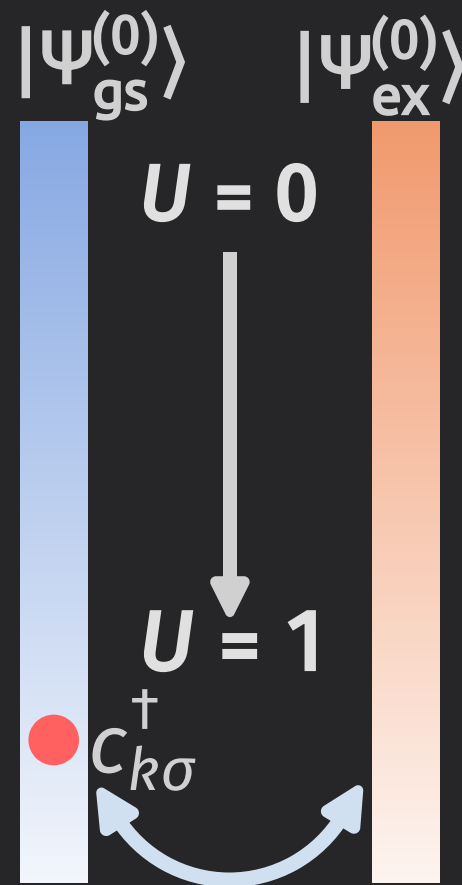
# Non-Fermi Liquids Are Different

## Landau Fermi Liquid Theory (Quantification)

- **Self-energy**  $\Sigma \sim i\omega^2$ . Quantifies **decay** rate. Vanishes very fast as  $\omega \rightarrow 0$ : essential for **quasiparticle** picture
- **Quasiparticle residue**: how similar are the true excitations to 1-particle excitations

$$Z = \langle \Psi_{\text{ex}} | c_{k\sigma}^\dagger | \Psi_{\text{gs}} \rangle$$

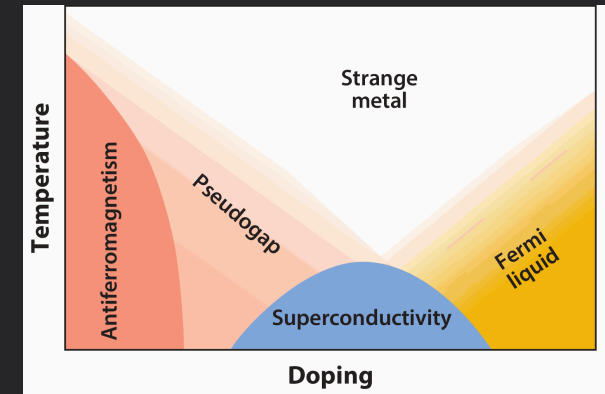
- $Z = \left(1 - \frac{\partial(\text{Re } \Sigma)}{\partial \omega}\right)^{-1}$ . Must be **non-zero** for Landau Fermi liquid.



# Non-Fermi Liquids Are Different

## Violations Of Landau Fermi Liquid Theory

- **Tomonaga-Luttinger Liquid**: Interacting electrons in 1D → spin-charge separation!
- **Overscreened** fixed points in Kondo models → fractional entropy, diverging  $\chi$ ,  $C_v$
- **Strange Metal**: Normal state of unconventional SCs in Cu oxides, heavy fermions, pnictides



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Custers et al. (2003); Doiron-Leyraud et al. (2009); Emery & Kivelson (1992); Haldane (1981); Keimer et al. (2015)



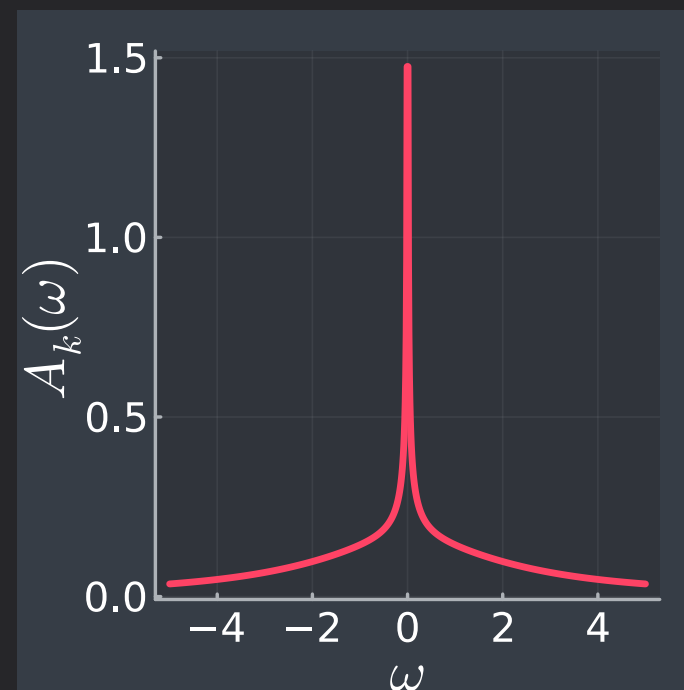
# Non-Fermi Liquids Are Different

## The Marginal Fermi Liquid

- **Phenomenological** explanation of normal state of cuprates:  $\Sigma \sim \omega \log(|\omega|) - i\pi |\omega|$
- Quasiparticle residue **vanishes** at Fermi surface

$$Z^{-1} = 1 - \frac{\partial(\text{Re } \Sigma)}{\partial \omega} \sim -\log(\omega) \rightarrow \infty$$

- Not accessible through **perturbative** corrections of Landau Fermi liquid



# Non-Fermi Liquids Are Different

## Main Takeaways

- Landau Fermi Liquid theory requires interacting eigenstates to be **adiabatically connected** to non-interacting eigenstates
- Non-Fermi liquids involve **vanishing quasiparticle residue**, signalling that the states are in fact not adiabatically connected.
- This typically means **non-perturbative** approaches are required to deal with such phases.
- The qualitatively different nature of excitations means that LFL and NFL correspond to **distinct fixed points** in the RG sense.

# An Exactly Solvable Model

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# The Hatsugai-Kohmoto Model

Consider long-ranged interaction in real-space.

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i,\sigma}^\dagger c_{j,\sigma} + \frac{U}{L^d} \sum_{i_1, i_2, r} c_{i_1+r, \uparrow}^\dagger c_{i_2-r, \downarrow}^\dagger c_{i_2, \downarrow} c_{i_1, \uparrow}$$



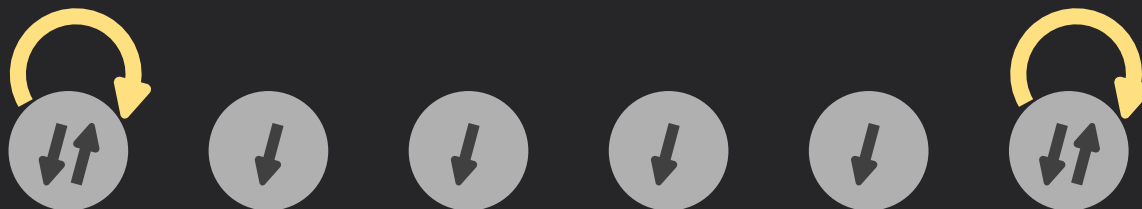
Switch to momentum space, Hamiltonian **becomes local!**

$$c_r^\dagger \sim \sum_k e^{-ikr} c_k^\dagger; \quad H = \sum_{\vec{k}} H_{\vec{k}}; \quad H_{\vec{k}} = \varepsilon_{\vec{k}} \sum_{\sigma} n_{\vec{k}, \sigma} + U n_{\vec{k} \uparrow} n_{\vec{k} \downarrow}$$

# The Hatsugai-Kohmoto Model

Contrast with the Hubbard interaction.

$$H_{\text{int}} \sim \sum_i n_{i,\uparrow} n_{i,\downarrow} = \sum_{k_1, k_2, q} c_{k_1+q, \uparrow}^\dagger c_{k_2-q, \downarrow}^\dagger c_{k_2, \downarrow} c_{k_1, \uparrow}$$



- local in real-space, highly **non-local** in  $k$  -space
- HK model is  $q = 0, k_1 = k_2$  (**zero mode!**) component of the Hubbard
- HKM is easier to solve than Hubbard (KE and PE **do not commute**)

# Spectrum

$$H = \sum_{\vec{k}} H_{\vec{k}}; \quad H_{\vec{k}} = \varepsilon_{\vec{k}} \sum_{\sigma} n_{\vec{k},\sigma} + U n_{\vec{k}\uparrow} n_{\vec{k}\downarrow}$$

Each  $H_k$  can be diagonalised.

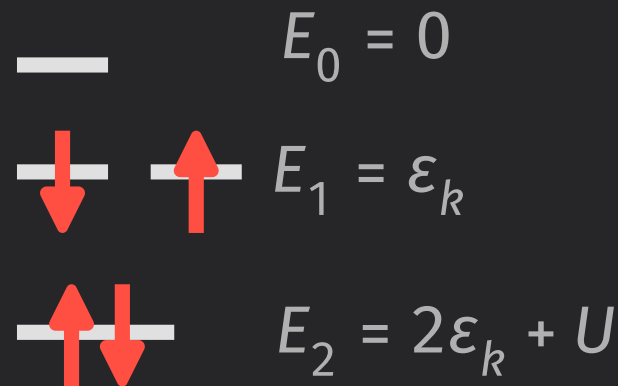
$$|0\rangle : E = 0, \quad |\sigma\rangle : E = \varepsilon_k, \quad |2\rangle : E = 2\varepsilon_k + U$$

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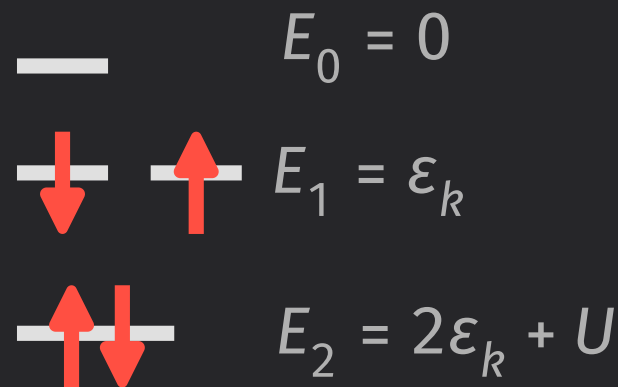


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## Case Of Half-Filling

$$E(\mu) = E - \mu n_k, \quad \mu = \frac{U}{2}$$

$$E_0 = 0, \quad E_1 = \varepsilon_k - \frac{U}{2}, \quad E_2 = 2\varepsilon_k$$

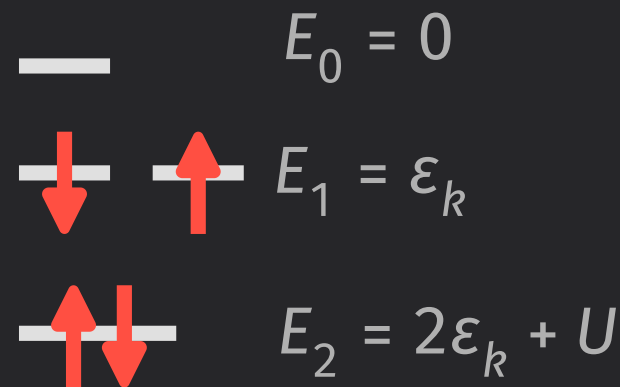


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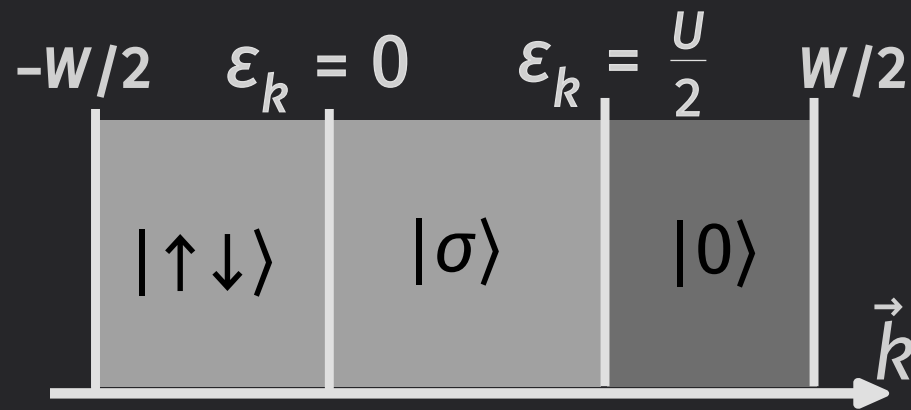
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# Introduction to Greens Functions

Nature of **excitations** can be studied through Greens function

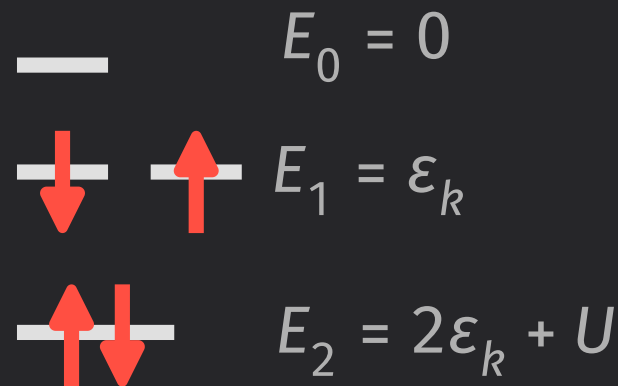
$$G_\nu(t) = -i\theta(t)\langle\{c_\nu(t), c_\nu^\dagger\}\rangle$$

- Non-interacting system:  $G_k(\omega + i\eta) = \frac{1}{\omega + i\eta - \epsilon_k}$
- **Poles** of Greens function  $\rightarrow$  one-particle excitations
- **Zeroes** of Greens function  $\rightarrow$  destruction of one-particle excitations

# Exact Single-Particle Greens Function of the HKM

Greens function can be calculated as

$$G_{k\sigma}(\omega) = \frac{P_e(k\sigma)}{\omega - E_x} + \frac{P_h(k\sigma)}{\omega + E_x}$$



# Exact Single-Particle Greens Function of the HKM

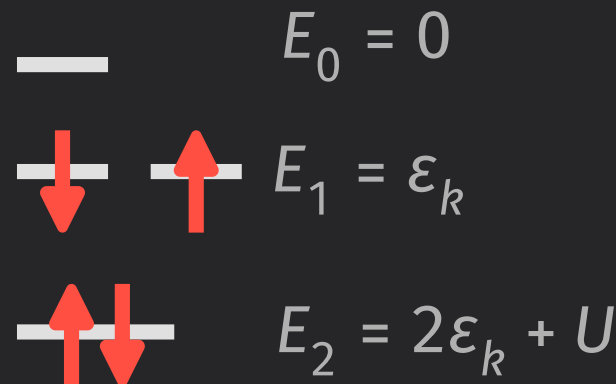
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**If opposite spin is unoccupied**

- Particle addition:  $E_x = \varepsilon_k$
- Particle removal:  $E_x = -\varepsilon_k$

$$G \rightarrow \frac{1 - \langle n_{k\bar{\sigma}} \rangle}{\omega - \varepsilon_k}$$



$$|0\rangle \rightarrow |\sigma\rangle$$

$$|\sigma\rangle \rightarrow |0\rangle$$

# Exact Single-Particle Greens Function of the HKM

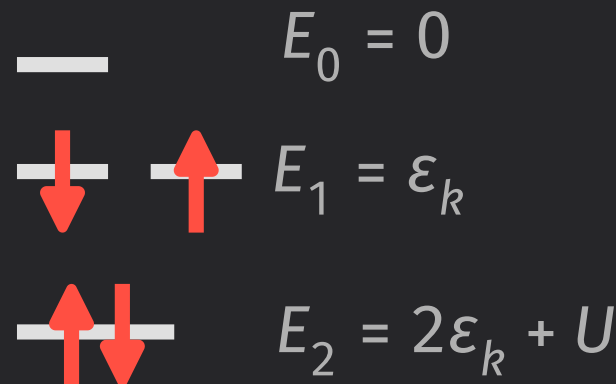
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**If opposite spin is occupied**

- Particle addition:  $E_x = \varepsilon_k + U$
- Particle removal:  $E_x = -\varepsilon_k - U$

$$G \rightarrow \frac{\langle n_{k\bar{\sigma}} \rangle}{\omega - \varepsilon_k - U}$$



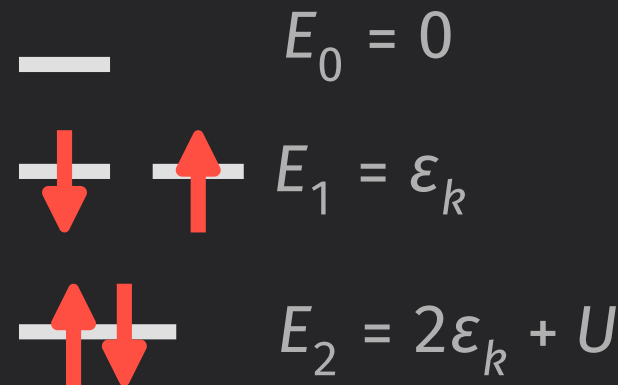
$$|-\sigma\rangle \rightarrow |\sigma, -\sigma\rangle$$

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# Exact Single-Particle Greens Function of the HKM

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$$G_{k\sigma}(\omega) = \frac{P_e(k\sigma)}{\omega - E_x} + \frac{P_h(k\sigma)}{\omega + E_x}$$



## Total Greens Function

$$G_{k\sigma} = \frac{1 - \langle n_{k\bar{\sigma}} \rangle}{\omega - \epsilon_k} + \frac{\langle n_{k\bar{\sigma}} \rangle}{\omega - \epsilon_k - U}$$

$(\epsilon_k \rightarrow \epsilon_k - \mu)$

# Correlated Metal-Insulator Transition

The Case Of **Half-Filling**:  $2\mu = U, \langle n_{k\sigma} \rangle = \frac{1}{2}$

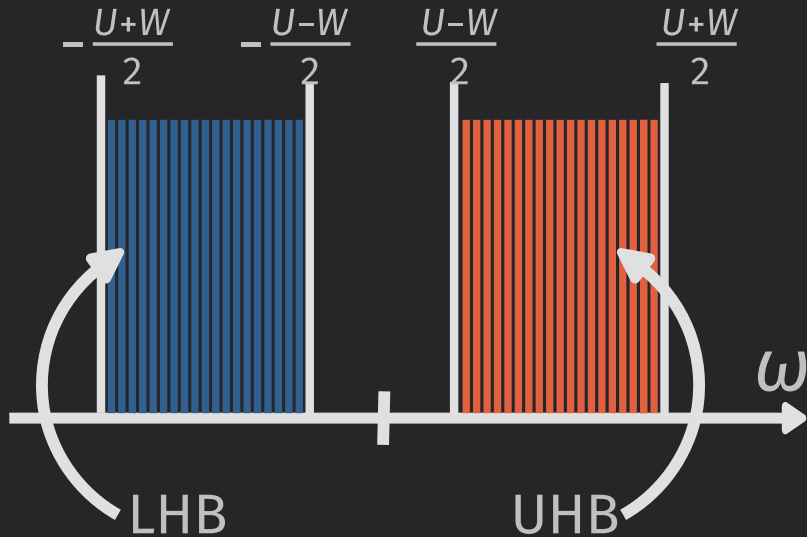
$$G_{k\sigma} = \frac{1}{2} \left[ (\omega - \varepsilon_k + U/2)^{-1} + (\omega - \varepsilon_k - U/2)^{-1} \right]$$

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$U > W$  (**Mott Insulator**)



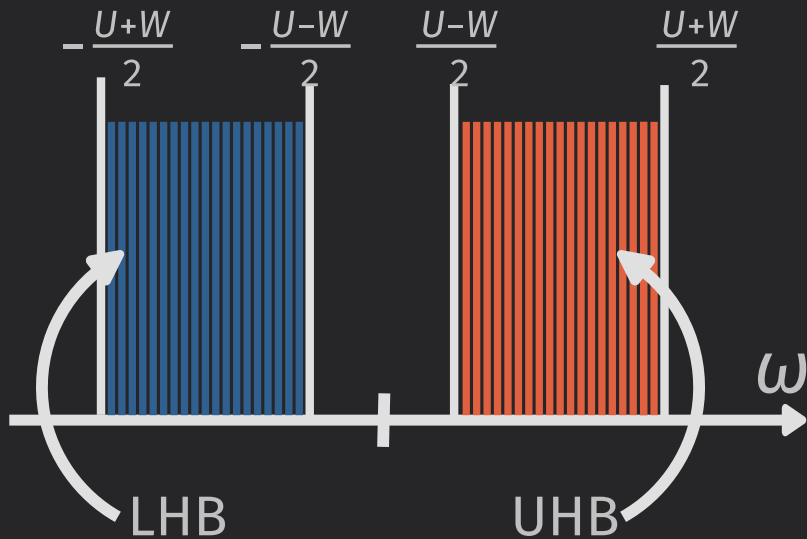


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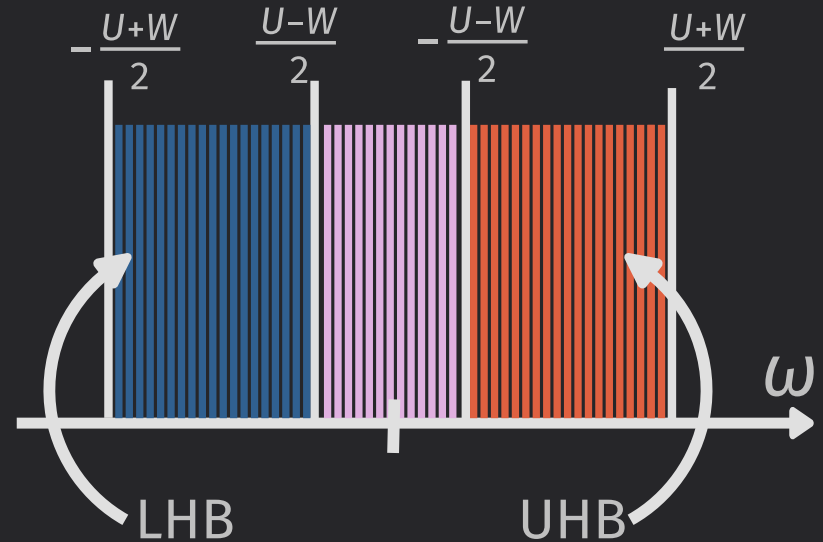
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$U > W$  (**Mott Insulator**)



$U < W$  (Metal)



# Non-Fermi Liquid Signatures In Metallic Phase

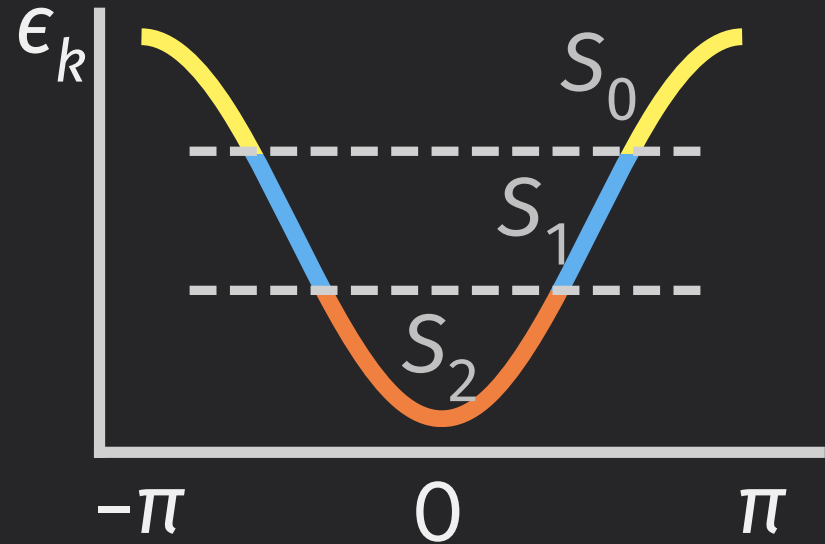
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# Signature I: Composite Gapless Excitations

$$G_{k\sigma} = \frac{1}{2} \left[ (\omega - \varepsilon_k + U/2)^{-1} + (\omega - \varepsilon_k - U/2)^{-1} \right]$$

Momentum states classified into three groups:

- $S_2 : \varepsilon_k < -U/2$ :  
**Both poles** below  $\omega = 0$ :  $\langle n_k \rangle = 2$
- $S_1 : -U/2 < \varepsilon_k < U/2$ :  
**One pole** below  $\omega = 0$ :  $\langle n_k \rangle = 1$
- $S_0 : \varepsilon_k > U/2$ :  
**No pole** below  $\omega = 0$ :  $\langle n_k \rangle = 0$



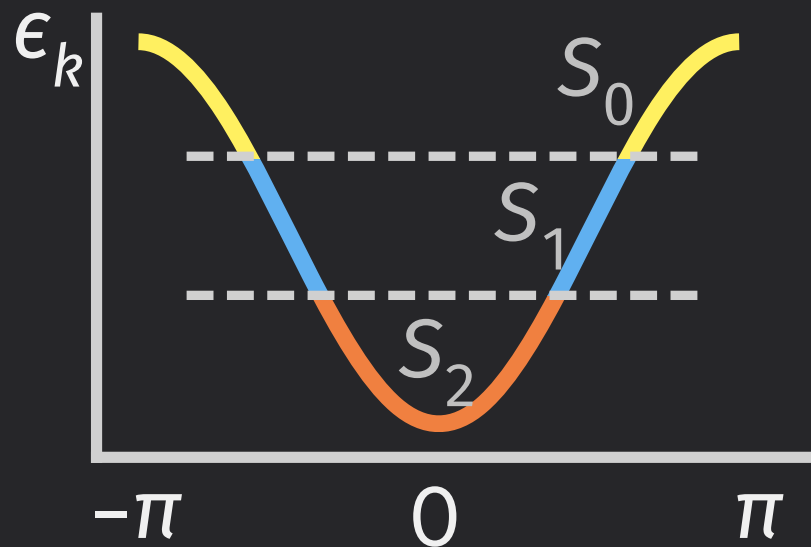
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Momentum states classified into three groups:

Ground state is a **mixed** state.

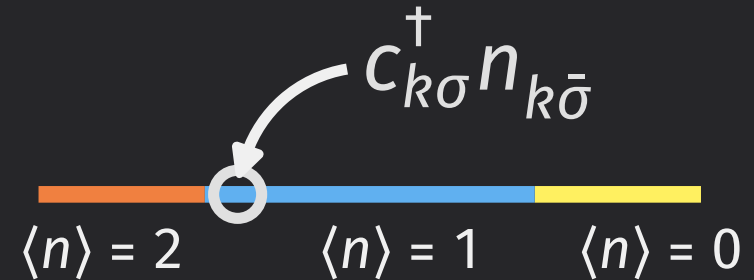
- $k$  –states in  $S_2$  are doubly-occupied.
- $k$  –states in  $S_1$  are half-filled.
- $2^{N_1}$ -fold degenerate.



# Signature I: Composite Gapless Excitations

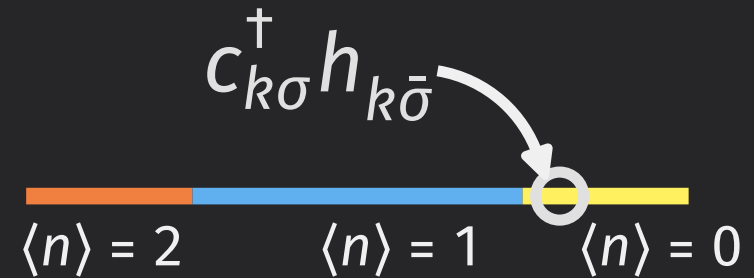
## Near $S_2 - S_1$ boundary ( $\varepsilon_k = -U/2$ )

- Excitation operator:  $c_{k\sigma}^\dagger n_{k\bar{\sigma}}$
- Excitation energy is  $\varepsilon_k + \frac{U}{2} \rightarrow 0^+$



## Near $S_1 - S_0$ boundary ( $\varepsilon_k = U/2$ )

- Excitation operator:  $c_{k\sigma}^\dagger (1 - n_{k\bar{\sigma}})$
- Excitation energy is  $\varepsilon_k - \frac{U}{2} \rightarrow 0^+$



**Projectors are needed** because the other excitations are gapped.

# Signature I: Composite Gapless Excitations

**Near  $S_2 - S_1$  boundary**

$$c_{k\sigma}^\dagger n_{k\bar{\sigma}}$$

**Near  $S_1 - S_0$  boundary**

$$c_{k\sigma}^\dagger (1 - n_{k\bar{\sigma}})$$

## Excitations are Non-Fermi Liquid In Nature!

- Strong correlations lead to **composite** (hole/double) excitations
- Excitations are **not electronic** (do not satisfy  $\{.\}$  relations)
- **Breakdown** of quasiparticle picture, and hence of Fermi liquid theory

## Signature II: Divergence of Self-Energy

We have at 1/2-filling:

$$G_{k\sigma} = \frac{1}{\omega - \varepsilon_k + \frac{U^2/4}{\omega - \varepsilon_k}}.$$

**Self-energy** is  $\Sigma(\omega) = \frac{U^2/4}{\omega - \varepsilon_k}$

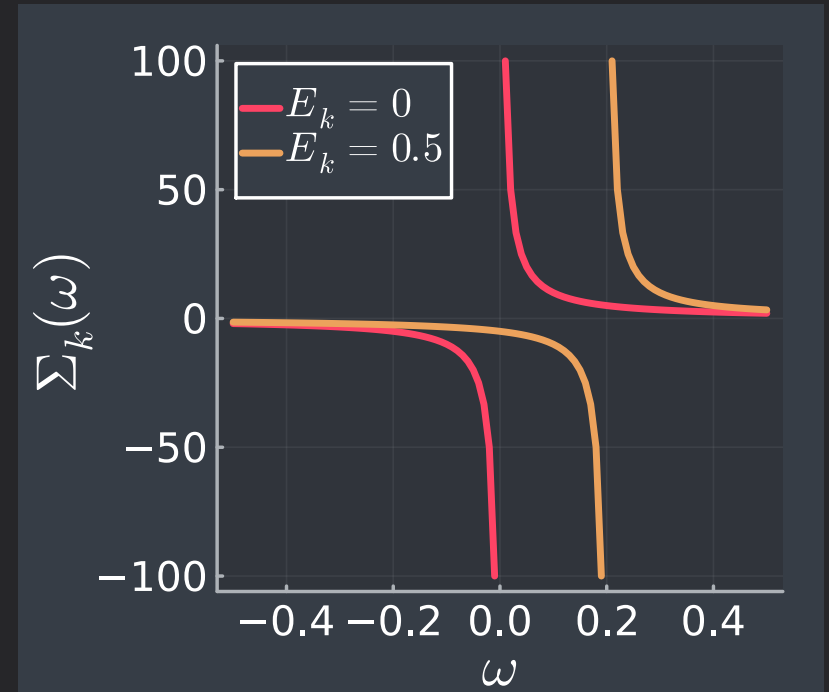
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- **Diverges** along  $\varepsilon_k = 0$  as  $\omega \rightarrow 0$
- **Violates** Fermi Liquid Theory
- Leads to zeros of Greens function
- Death of Landau **quasiparticles**





## Signature II: Divergence of Self-Energy

**How Does A Diverging Self-Energy Leave The System Metallic?**

## Signature II: Divergence of Self-Energy

### How Does A Diverging Self-Energy Leave The System Metallic?

Greens functions for composite excitations do not have self-energy!

$$d_{k\sigma}^\dagger = c_{k\sigma}^\dagger n_{k\sigma}, \quad G_d = \frac{1}{\omega - \varepsilon_k - \frac{U}{2}}$$

$$h_{k\sigma}^\dagger = c_{k\sigma}^\dagger (1 - n_{k\sigma}), \quad G_h = \frac{1}{\omega - \varepsilon_k + \frac{U}{2}}$$

These can therefore propagate with **long lifetimes**.

Is This A Realistic Model of  
Interacting Electrons?

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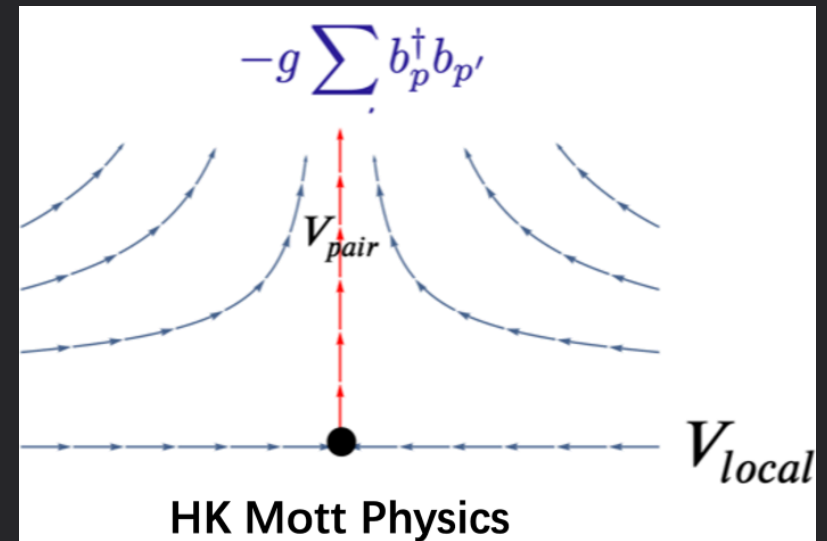
# Stability Under Small Perturbations

$$S = S_{\text{FL}} + U \int c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger c_{k\downarrow} c_{k\uparrow} + \\ u \int c_{k_1+q,\uparrow}^\dagger c_{k_2-q,\downarrow}^\dagger c_{k_2,\downarrow} c_{k_1,\uparrow}$$

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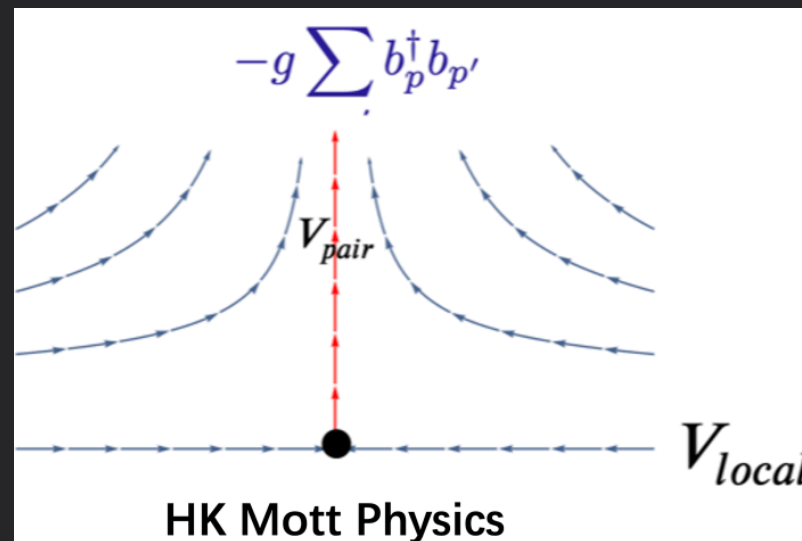
- HK interaction strongly **relevant** under RG flow
- **Stable** under local interaction terms
- Displays superconducting instability



# Stability Under Small Perturbations

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- HK interaction strongly **relevant** under RG flow
- **Stable** under local interaction terms
- Displays superconducting instability



**Key Point:** HK interaction is the **source of Mottness** in Hubbard model.

# Link With The 2D Hubbard Model

Alternative form of HK interaction:

$$U \sum_k n_{k\uparrow} n_{k\downarrow} \sim -J \sum_k \mathbf{s}_k \cdot \mathbf{s}_k$$

## Baskaran Model

$$\sum_k \varepsilon_k n_{k\sigma} + \left( \sum_k \mathbf{s}_k \right)^2 - J \sum_k \mathbf{s}_k \cdot \mathbf{s}_k$$

- Spin-charge separation of NFL
- Gapped spinon excitations

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- Spin-charge separation of NFL
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- Similar model emerges from 2D Hubbard Model, **Mott liquid**  
$$- \sum_k \mathbf{S}_k \cdot \mathbf{S}_{-k} + \sum_k \mathbf{C}_k \cdot \mathbf{C}_{-k}$$

$$S^z = n_{k\uparrow} - n_{k^T\uparrow}, C^z = n_{k\uparrow} + n_{k^R\uparrow} - 1$$





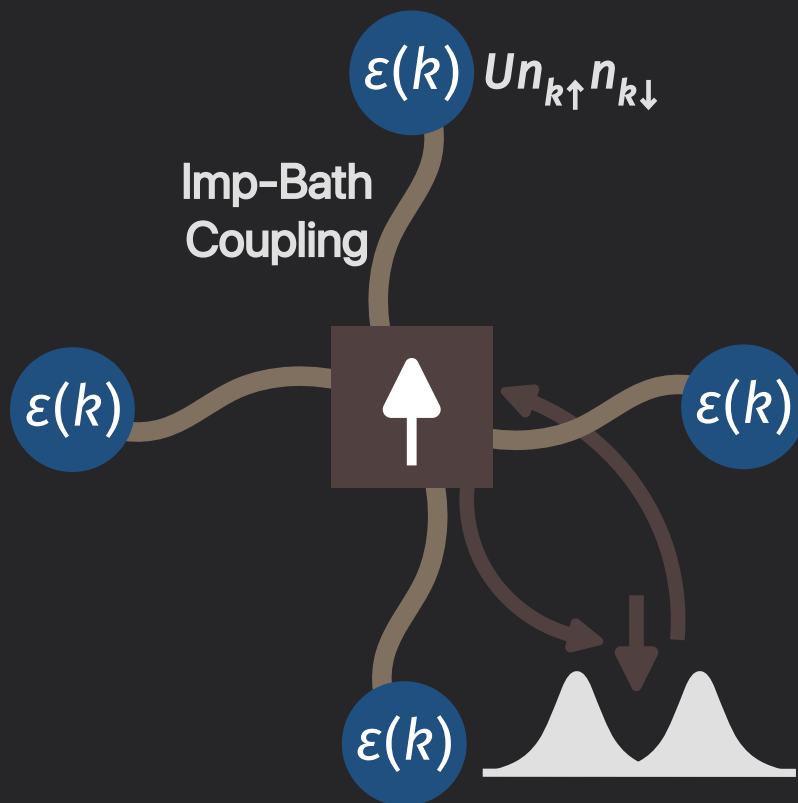
# Summary of Main Ideas

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# Avenues for Futher Investigation

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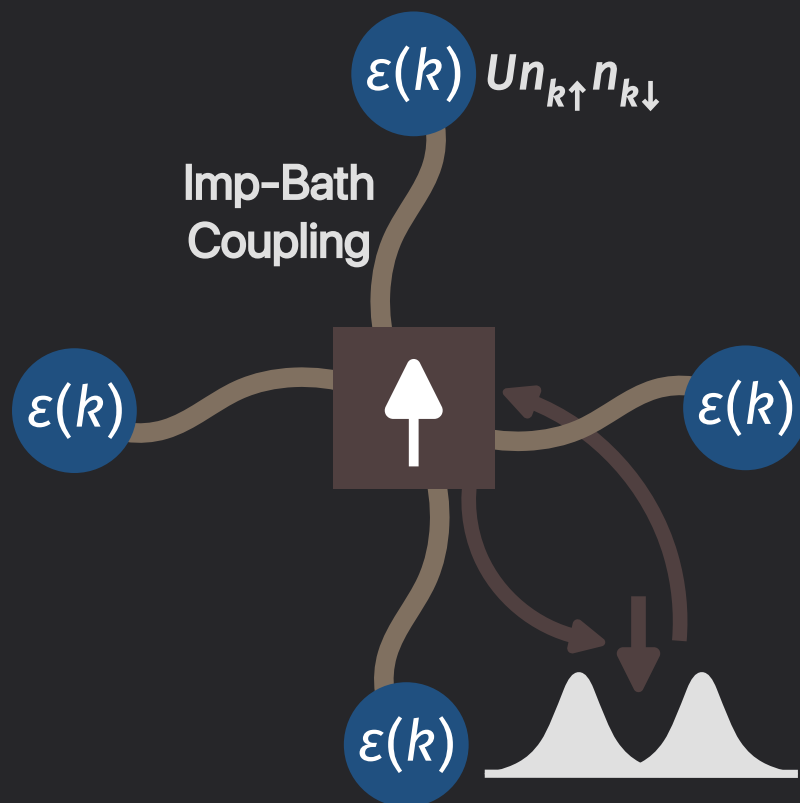
# Kondo Screening in Hatsugai-Kohmoto Model



Consider local moment  
hybridising with HK Model

$$H = H_{\text{Kondo}} + H_{\text{HKM}}$$

# Kondo Screening in Hatsugai-Kohmoto Model

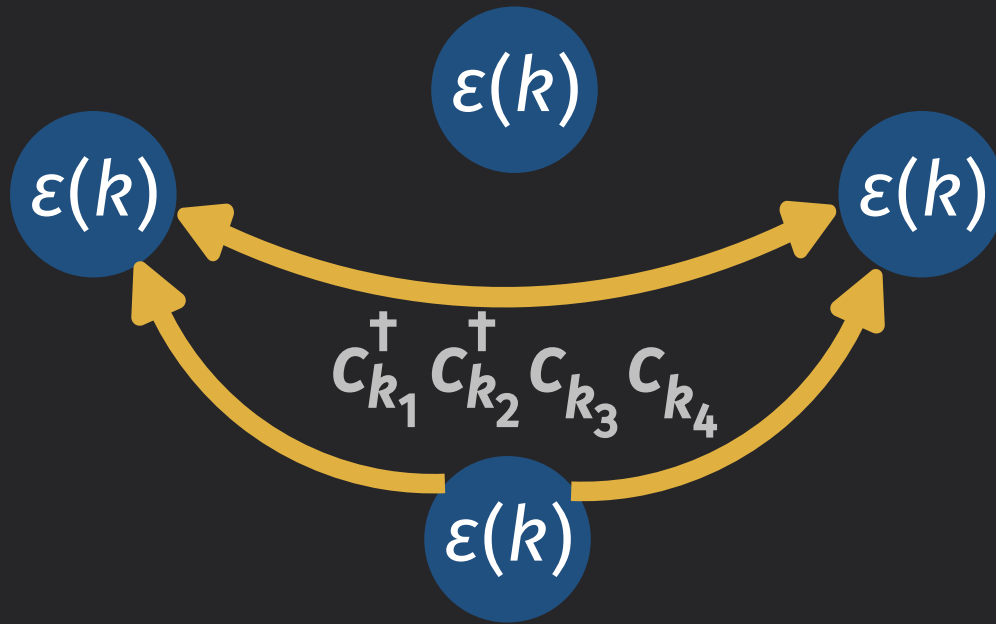


Consider local moment  
hybridising with HK Model

$$H = H_{\text{Kondo}} + H_{\text{HKM}}$$

How does **absence** of quasiparticles  
affect Kondo screening?

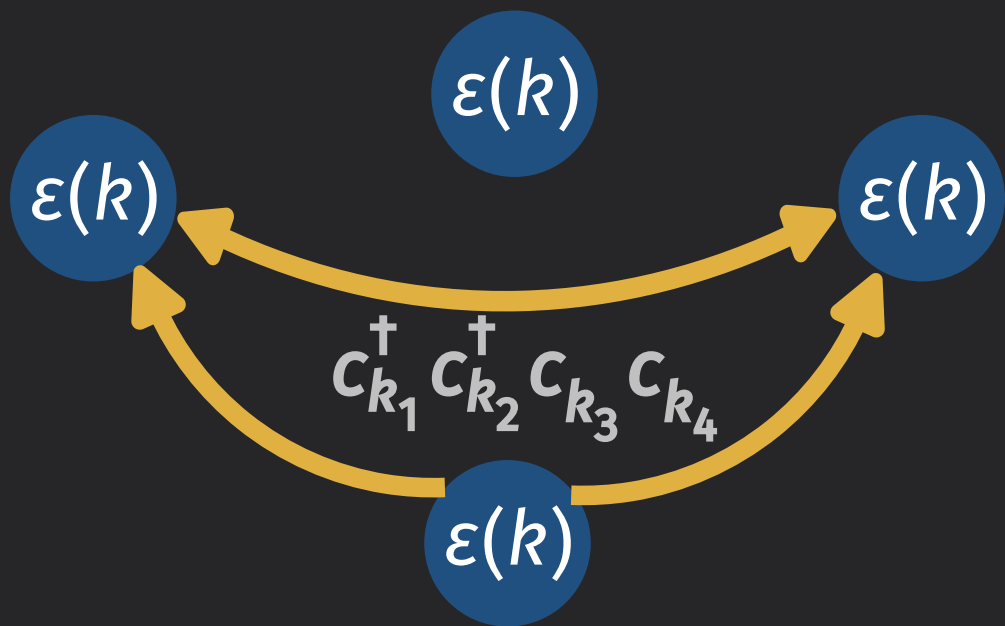
# Toy Model for Thermalisation and Many-Body Scars



Consider HK Model perturbed by  
Hubbard interaction

$$H = H_{\text{HKM}} + P_\nu H_{\text{Hub}} P_\nu$$

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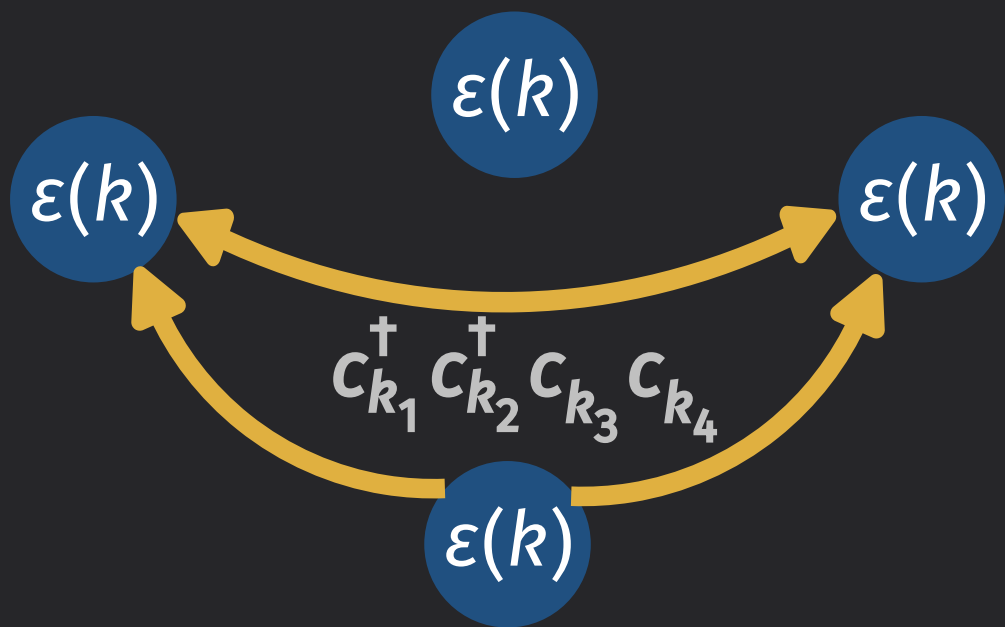


Consider HK Model perturbed by  
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$$H = H_{\text{HKM}} + P_\nu H_{\text{Hub}} P_\nu$$

- $H_{\text{Hub}}$  allows **thermalisation**  
of  $k$ -states

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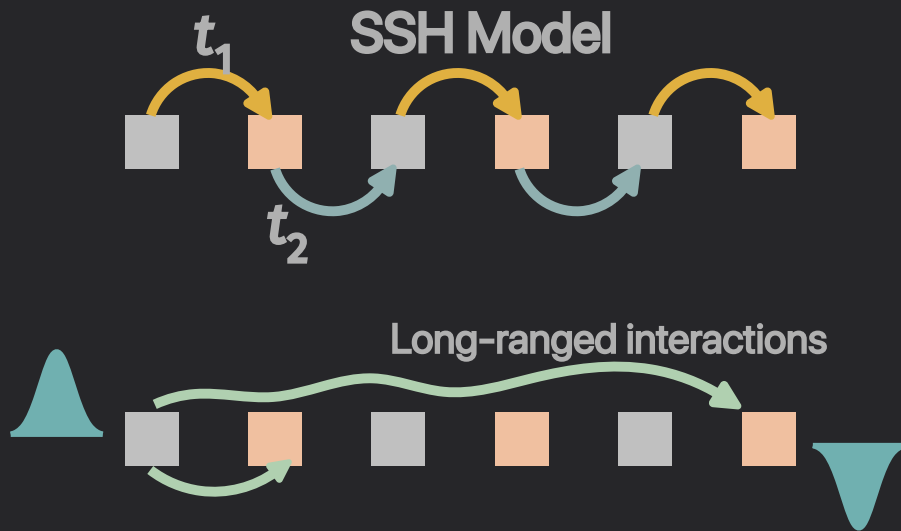


Consider HK Model perturbed by  
Hubbard interaction

$$H = H_{\text{HKM}} + P_\nu H_{\text{Hub}} P_\nu$$

- $H_{\text{Hub}}$  allows **thermalisation** of  $k$ -states
- $P_\nu$  will preserve certain sectors.  
**Scars?**

# Effect of Hatsugai-Kohmoto Interaction on Edge Modes of SSH Chain



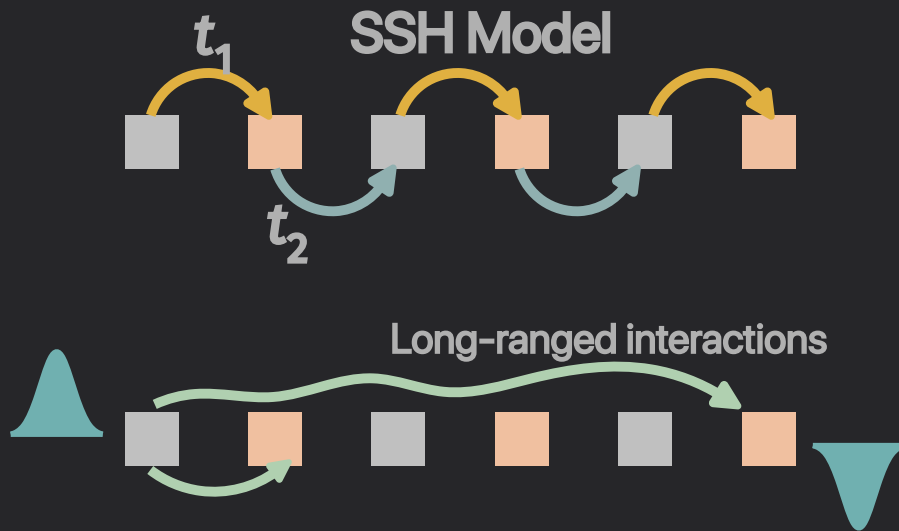
Add long-ranged HKM interactions to SSH chain.

$$H_{\text{SSH}} = -t_1 c_1^\dagger c_2 - t_2 c_2^\dagger c_3 + \dots$$

$$H = H_{\text{HKM}} + H_{\text{SSH}}$$



# Effect of Hatsugai-Kohmoto Interaction on Edge Modes of SSH Chain



Add long-ranged HKM interactions to SSH chain.

$$H_{\text{SSH}} = -t_1 c_1^\dagger c_2 - t_2 c_2^\dagger c_3 + \dots$$

$$H = H_{\text{HKM}} + H_{\text{SSH}}$$

- Fate of topological **edge modes**?

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