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"NUMERICAL INTERPO- -LATION WITH UNEQUAL INTERVALS"

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"Numerical Interpolation With Unequal Intervals"

Introduction :-

Interpolation is a fundamental technique in numerical analysis used to estimate unknown values of a function based on known data points. It is particularly useful in cases where data is obtained from experiments or observations and an explicit mathematical function is unavailable.

When the given data points have equal intervals, simpler interpolation methods such as Newton's Forword or Backword Interpolation can be applied. However, in many real-world scenarios, data points are spaced, we focus at irregular intervals, necessitating more advanced interpolation techniques.

In this report, we focus on two widely used interpolation methods for unequal intervals:

(i) Lagrange's Interpolation Formula

(ii) Newton's Divided Difference Interpolation

Both methods construct polynomial approximations for the given data points and are essentials in scientific computing, engineering applications and data analysis.

Derivation / Formulation :-

(i) Lagrange's Interpolation Formula :-

Let $y = f(x)$ be a function defined in the interval $[a, b]$ and is only known on a set of $(n+1)$ distinct arguments, $x_0, x_1, x_2 \dots x_{r-1}, x_r, x_{r+1} \dots x_{n-1}, x_n$ in general, are not equally spaced, in the interval of definition of $f(x)$. Let the corresponding values of $f(x)$ on the set of arguments x_j $(j = 0, 1, \dots n)$ are: $y_0 = f(x_0)$, $y_1 = f(x_1)$ $y_2 = f(x_2)$, $\dots$ $y_r = f(x_r) \dots$, $y_n = f(x_n)$ i.e. $y_i = f(x_i)$ $(i = 0, 1, 2 \dots n)$.

Now our object is to find a polynomial $L(x)$ of degree not greater than n and such that $L(x)$ replaces $f(x)$ on the set of interpolation points x_j $(j = 0, 1, 2 \dots n)$.

$$L(x_j) = f(x_j) = y_j \quad (j = 0, 1, 2 \dots n) \qquad \text{.......} (1)$$

Let us set,

$$w(x) = (x - x_0)(x - x_1)(x - x_2) \dots\dots$$

$(x-x_{r-1})\ (x-x_r)\ (x-x_{r+1})\ \cdots$

$(x-x_{n-1})\ (x-x_n) \quad \cdots\cdots \quad (2)$

Differentiating (2) w.r.t x, we get,

$$\omega'(x) = (x-x_1)(x-x_2)\cdots(x-x_{r-1})(x-x_r)(x-x_{r+1}) \cdots (x-x_{n-1})(x-x_n)$$

$$+ (x-x_0)(x-x_2)\cdots(x-x_{r-1})(x-x_r)(x-x_{r+1}) \cdots\cdots (x-x_{n-1})(x-x_n)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots +$$

$$(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{r-1})(x-x_{r+1})\cdots \cdots(x-x_{n-1})(x-x_n)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots +$$

$$(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{r-1})(x-x_r)(x-x_{r+1})\cdots(x-x_n)$$

$$+ (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{r-1})(x-x_r)(x-x_{r+1})\cdots(x-x_{n-1})$$

$$\therefore \omega'(x_r) = (x_r-x_0)(x_r-x_1)(x_r-x_2)\cdots\cdots(x_r-x_{r-1})(x_r-x_{r+1})\cdots\cdots(x_r-x_{n-1})(x_r-x_n) \quad \cdots\cdots \quad (3)$$

Where $x_j\ (j=0,1,2,\cdots,n)$ are the interpolating points.

Let us consider a polynomial $\omega_r(x)$ of degree n given by,

$$\omega_r(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{r-1})(x-x_{r+1})\cdots\cdots(x-x_{n-1})(x-x_n)}{(x_r-x_0)(x_r-x_1)(x_r-x_2)\cdots(x_r-x_{r-1})(x_r-x_{r+1})\cdots\cdots(x_r-x_{n-1})(x_r-x_n)} \quad \cdots\cdots \quad (4)$$

$$= \frac{w(x)}{(x-x_r)\, w'(x_r)} \quad \cdots\cdots (5)$$

Thus,

$w_r(x_j)$

$$= \frac{(x_j-x_0)(x_j-x_1)(x_j-x_2)\cdots(x_j-x_{r-1})(x_j-x_{r+1})\cdots(x_j-x_{n-1})(x_j-x_n)}{(x_r-x_0)(x_r-x_1)(x_r-x_2)\cdots(x_r-x_{r-1})(x_r-x_{r+1})\cdots(x_r-x_{n-1})(x_r-x_n)}$$

$$\left.\begin{aligned} &= 0 \quad \text{when } r \neq j \\ &= 1 \quad \text{when } r = j \end{aligned}\right\} \quad \cdots\cdots (6)$$

Now, we consider a polynomial of degree n, given by

$$L(x) = \sum_{r=0}^{n} w_r(x)\, f(x_r) \quad \cdots\cdots (7)$$

as $w_r(x)$ is a polynomial of degree n.

Now,

$$L(x_j) = \sum_{r=0}^{n} w_r(x_j) f(x_r)$$

$$= w_j(x_j)\, f(x_j)$$

$$= f(x_j) \quad \cdots\cdots (8)$$

So, $L(x)$ satisfies the conditions (1) and $L(x)$ is indeed the interpolation polynomial which is unique. The polynomial $L(x)$ defined by equality (7) i.e.,

$$L(x) = \sum_{r=0}^{n} w_r(x) f(x_r) = \sum_{r=0}^{n} \frac{w(x) f(x_r)}{(x-x_r)(w'(x_r))}$$

$$= w(x) \sum_{r=0}^{n} \frac{f(x_r)}{(x-x_r) w'(x_r)} \quad \cdots\cdots (9)$$

is called Lagrange's Interpolation Formula and the error in Lagrange's Interpolation Formula is

$$R_{n+1} = (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)\frac{f^{n+1}(\xi)}{(n+1)!}$$

$$= w(x) \frac{f^{n+1}(\xi)}{(n+1)!} \quad (x_0 < \xi < x_n)$$

The functions

$$w_r(x) = \frac{w(x)}{(x-x_r) w'(x_r)}, \quad (r = 0,1,2,\cdots,n) \quad \cdots\cdots (10)$$

are called the Lagrangian Functions.

(ii) Newton's Divided Differences Interpolation:-

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labor of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employee what are called, "Divided Differences". Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2) \dots$ be given points, then the first divided difference for the arguments x_0, x_1 is defined by the relation $[x_0, x_1]$ or $\underset{x_1}{\Delta} y_0 = \frac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2]$ or, $\underset{x_2}{\Delta} y_0 = \frac{y_2 - y_1}{x_2 - x_1}$ and

$[x_2, x_3]$ or $\underset{x_3}{\Delta} y_0 = \frac{y_3 - y_2}{x_3 - x_2}$

The third divided difference for x_0, x_1, x_2, x_3 is defined as

$[x_0, x_1, x_2, x_3]$ or $\underset{x_1, x_2, x_3}{\Delta^3} y_0 = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_3]}{x_2 - x_0}$

The second divided difference for x_0, x_1, x_2 is defined as,

$[x_0, x_1, x_2]$ or $\underset{x_1, x_2}{\Delta^2} = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$

Let $y_0, y_1, \dots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, x_2, \dots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

So, that $y = y_0 + (x - x_0)[x, x_0]$

Again $[x, x_0, x_1] = \dfrac{[x, x_0] - [x_0, x_1]}{x - x_1}$

which gives $[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \cdots (2)$$

Also, $[x, x_0, x_1, x_2] = \dfrac{[x, x_0, x_1] - [x, x_0, x_2]}{x - x_2}$

Which gives $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1) \cdots (x - x_n)[x, x_0, x_1, \ldots x_n]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_0, x_1, x_2] + \cdots \quad (3)$$

which is called Newton's general interpolation formula with divided differences.

Associated Theorems and Their Proofs :-

(a) Uniqueness of Interpolation Polynomial :-

Theorem:

Given $n+1$ distinct data points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n), there exists a unique polynomial $P(x)$ of degree at most n that interpolates the data.

Proof:

Assume there are two polynomials $P(x)$ and $Q(x)$ of degree at most n that interpolate the same $n+1$ data points. Consider the polynomial $R(x) = P(x) - Q(x)$. Since both $P(x)$ and $Q(x)$ pass through the data points, $R(x)$ has $n+1$ roots at $x_0, x_1, \ldots, x_n$. However, a non-zero polynomial of degree at most n can have at most n roots. Therefore, $R(x)$ must be the zero polynomial, implying $P(x) = Q(x)$. This proves the uniqueness of the interpolating polynomial.

(b) Error Analysis in Interpolation :-

Theorem:

Let $P(x)$ be the interpolating polynomial of degree n for a function $f(x)$ at the points

$x_0, x_1, \ldots, x_n$. The interpolation error at any point x is given by,

$$E(x) = f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)} \prod_{i=0}^{n} (x - x_i)$$

where ξ lies in the interval containing $x_0, x_1, \ldots, x_n$ and x.

Proof :

The proof ~~lie~~ relies on the Rolle's theorem and the properties of derivatives. By constructing an auxiliary function and applying the mean value theorem, the error term can be derived. The detailed proof is beyond the scope of this report but is available in advanced numerical analysis textbooks.

Applications :-

(i) Scientific Data Analysis : Interpolation is used to estimate values between experimental data points, enabling the analysis of trends and pattern.

(ii) Computer Graphics : Lagrange's and Newton's methods are used to generate smooth curves and surfaces in 3D modeling and animation.

(iii) Financial Modeling : Interpolation is applied to estimate missing financial data, such as stock prices or interest rates, at specific times.

(iv) Geostatistics : Interpolation is used to estimate values at unmeasured locations based on sampled data, such as in mineral exploration or environment monitoring.

(v) Signal Processing : Interpolation is used to reconstruct signals from irregularly sampled data points.

Example :-

(1) Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Evalute $f(9)$, using Lagrange's formula

Solution :

(1) Here $x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$ and $y_0 = 150$, $y_1 = 392$, $Y_2 = 1452$, $Y_3 = 2366$ and $Y_4 = 5202$

Putting $x=9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} + \frac{2366}{3} + \frac{578}{5}$$

$$= 810$$

② Determine $f(x)$ as a polynomial in x for the following data:

x	-4	-1	0	2	5
$y=f(x)$	1245	33	5	9	1335

Solution:

The divided differences table is,

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
				13	
2	9		88		
		442			
5	1335		.		

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + \ldots$$

$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94)$$
$$+ (x+4)(x+1)(x-0)(-14) + (x+4)(x+1)x(x-2)(3)$$

$$= 3x^4 - 5x^2 + 6x^2 - 14x + 5$$

References :-

The following references have been really helpful while preparing this report -

(i) "An Introduction to Numerical Analysis" - Kendall E. Atkinson.

(ii) "Applied Numerical Analysis" - Curtis F. Gerald & Patrick O. Wheatley.

(iii) "Numerical Methods for Scientists and Engineering" - R. W. Hamming.

(iv) "Numerical Analysis" - Richard L. Burden & J. Douglas Faires.

(v) "Numerical Methods for Engineers" - Steven C. Chapra & Raymond P. Canale.