Week 6

Lecture 1

Singular Value Decomposition

1. Recap: Spectral Theorem

Let $A \in R^{n \times n}$ be real symmetric.
Then:

1. All eigenvalues of A are real.
2. A is orthogonally diagonalizable.

There exists orthogonal Q such that

$$Q^T A Q = \Lambda$$

where

- Λ is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_n$.
- Columns of Q are orthonormal eigenvectors $x_1, \ldots, x_n$.
- $Q^T Q = I$.

Equivalently,

$$A = Q \Lambda Q^T$$

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2. Motivation for SVD

Eigen decomposition requires:

- Square matrix
- Symmetric structure
  However:
  Any real $m \times n$ matrix can be decomposed, even if rectangular.
  This is the Singular Value Decomposition.

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3. Singular Value Decomposition

Theorem

For any real matrix $A \in R^{m \times n}$,

$$A = Q_1 \Sigma Q_2^T$$

where:

- $Q_1 \in R^{m \times m}$ is orthogonal
- $Q_2 \in R^{n \times n}$ is orthogonal
- $\Sigma \in R^{m \times n}$ has form

  $$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

  where

  $$D = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$$

  with $\sigma_i > 0$.

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4. Proof of SVD

Step 1: Consider $A^T A$

Let

$$A \in R^{m \times n}$$

Then

$$A^T A \in R^{n \times n}$$

and $A^T A$ is symmetric.
By spectral theorem:
There exists orthonormal eigenvectors $x_1, \ldots, x_n$ such that

$$A^T A x_i = \lambda_i x_i$$

with

$$x_i^T x_j = 0 \text{ if } i \neq j$$

$$\|x_i\| = 1$$

Step 2: Non-negativity of Eigenvalues

Compute

$$x_i^T A^T A x_i = \lambda_i x_i^T x_i$$

Since $\|x_i\| = 1$,

$$x_i^T A^T A x_i = \lambda_i$$

But also

$$x_i^T A^T A x_i = (Ax_i)^T (Ax_i) = \|Ax_i\|^2$$

Thus

$$\lambda_i = \|Ax_i\|^2 \geq 0$$

Hence eigenvalues of $A^T A$ are non-negative.

Step 3: Order Eigenvalues

Arrange:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$$

$$\lambda_{r+1} = \cdots = \lambda_n = 0$$

Define singular values:

$$\sigma_i = \sqrt{\lambda_i}$$

for $i = 1, \ldots, r$.

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5. Construct Left Singular Vectors

Define

$$y_i = \frac{1}{\sigma_i} A x_i$$

for $i = 1, \ldots, r$.

Step 5.1: Norm of y_i

$$\|y_i\| = \frac{1}{\sigma_i}\|Ax_i\|$$

But

$$\|Ax_i\|^2 = \lambda_i$$

Thus

$$\|y_i\| = \frac{\sqrt{\lambda_i}}{\sigma_i} = 1$$

Step 5.2: Orthogonality

For $i \neq j$:

$$y_i^T y_j = \frac{1}{\sigma_i \sigma_j}(Ax_i)^T(Ax_j)$$

$$= \frac{1}{\sigma_i \sigma_j} x_i^T A^T A x_j$$

$$= \frac{\lambda_j}{\sigma_i \sigma_j} x_i^T x_j$$

Since $x_i^T x_j = 0$,

$$y_i^T y_j = 0$$

Thus $y_1, \ldots, y_r$ are orthonormal.

Step 6: Extend to Basis

Extend $\{y_1, \ldots, y_r\}$ to orthonormal basis of R^m:

$$\{y_1, \ldots, y_m\}$$

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6. Define Orthogonal Matrices

Define

$$Q_1 = \{y_1, \ldots, y_m\}$$

$$Q_2 = \{x_1, \ldots, x_n\}$$

Both are orthogonal.

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7. Compute Σ

Compute

$$Q_1^T A Q_2$$

Entry in position i, j:

$$y_i^T A x_j$$

Case 1: $j \le r$

Since

$$A x_j = \sigma_j y_j$$

we get

$$y_i^T A x_j = \sigma_j y_i^T y_j$$

Thus

$$y_i^T A x_j = \begin{cases} \sigma_j & i = j \\ 0 & i \neq j \end{cases}$$

Case 2: $j > r$

Since $\lambda_j = 0$,

$$A x_j = 0$$

Hence

$$y_i^T A x_j = 0$$

Final Form

Therefore

$$Q_1^T A Q_2 = \Sigma$$

where

$$\Sigma = \begin{pmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{pmatrix}$$

Thus

$$A = Q_1 \Sigma Q_2^T$$

This is the Singular Value Decomposition.

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8. Structural Interpretation

From

$$A = Q_1 \Sigma Q_2^T$$

we obtain

$$AA^T = Q_1 \Sigma \Sigma^T Q_1^T$$

Thus columns of Q_1 are eigenvectors of AA^T.
Similarly,

$$A^T A = Q_2 \Sigma^T \Sigma Q_2^T$$

Thus columns of Q_2 are eigenvectors of $A^T A$.

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9. Key Observations

1. SVD exists for every real $m \times n$ matrix.
2. Singular values are square roots of eigenvalues of $A^T A$.
3. Q_2 contains eigenvectors of $A^T A$.
4. Q_1 contains eigenvectors of AA^T.
5. No symmetry or squareness assumption required.

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Lecture 2

Example of Singular Value Decomposition

Problem Setup

Given the matrix

$$A = \begin{pmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{pmatrix}$$

Goal: Find the Singular Value Decomposition of A.

Step 1: Check Diagonalizability

Eigenvalues of A

Solve $\det A - \lambda I = 0$.
Eigenvalue:

$$\lambda = \sqrt{2}$$

with algebraic multiplicity 2.
Compute

$$A - \sqrt{2}I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Solving

$$A - \sqrt{2}I \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

gives

$$x_2 = 0$$

Thus eigenvectors are multiples of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Only one linearly independent eigenvector.

Conclusion

A is not diagonalizable.
Therefore proceed with SVD.

Step 2: Compute $A^T A$

$$A^T = \begin{pmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{pmatrix}$$

Then

$$A^T A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}$$

This is real symmetric.

Step 3: Eigenvalues of $A^T A$

Solve

$$\det \begin{pmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 3-\lambda \end{pmatrix} = 0$$

This gives

$$\lambda_1 = 4, \quad \lambda_2 = 1$$

Step 4: Singular Values

Singular values are

$$\sigma_i = \sqrt{\lambda_i}$$

Thus

$$\sigma_1 = 2, \quad \sigma_2 = 1$$

So

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Step 5: Eigenvectors of $A^T A$

For $\lambda_1 = 4$

Solve

$$A^T A - 4I = \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$$

Eigenvector:

$$\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

Normalize:
Length is $\sqrt{3}$.

$$x_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

For $\lambda_2 = 1$

Solve

$$A^T A - I = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$$

Eigenvector:

$$\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$$

Normalize:
Length is $\sqrt{3}$.

$$x_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$$

Step 6: Construct Q_2

$$Q_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$$

Columns are orthonormal eigenvectors of $A^T A$.

Step 7: Compute Left Singular Vectors

Using

$$\sigma_i y_i = A x_i$$

Compute y_1

$$A x_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2\sqrt{2} \\ 2 \end{pmatrix}$$

Divide by $\sigma_1 = 2$:

$$y_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

Compute y_2

$$A x_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$

Divide by $\sigma_2 = 1$:

$$y_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$

Normalize:

$$y_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

Step 8: Construct Q_1

$$Q_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}$$

Columns are orthonormal.

Final SVD

$$A = Q_1 \Sigma Q_2^T$$

Where

$$Q_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Q_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$$

Important Observations

1. A is not diagonalizable.
2. $A^T A$ is symmetric and diagonalizable.
3. Singular values are square roots of eigenvalues of $A^T A$.
4. Columns of Q_2 are eigenvectors of $A^T A$.
5. Columns of Q_1 are eigenvectors of AA^T.
6. SVD exists even when A is not diagonalizable.

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Lecture 3

Positive Definiteness and Quadratic Forms

1. Motivation via Quadratic Functions

Consider the quadratic function

$$f(x, y) = 2x^2 + 4xy + y^2$$

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2. Stationary Point

A stationary point occurs when first derivatives vanish.
Compute partial derivatives:

$$\frac{\partial f}{\partial x} = 4x + 4y$$

$$\frac{\partial f}{\partial y} = 4x + 2y$$

At $(0, 0)$:

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

Thus $(0, 0)$ is a stationary point.

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3. Nature of Stationary Point

Second derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

The origin is a minimum for this function.

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4. General Quadratic Form in Two Variables

Consider

$$f(x, y) = ax^2 + 2bxy + cy^2$$

Observation:
Every quadratic function has a stationary point at $(0, 0)$.

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5. Definition: Positive Definite Function

A function f is positive definite if

$$f(0,0) = 0$$

and

$$f(x,y) > 0 \quad \text{for all } (x,y) \neq (0,0)$$

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6. Necessary Conditions

Evaluate at $(1,0)$:

$$f(1,0) = a$$

If f is positive definite:

$$a > 0$$

Evaluate at $(0,1)$:

$$f(0,1) = c$$

Thus:

$$c > 0$$

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7. Completing the Square

Rewrite

$$ax^2 + 2bxy + cy^2$$

as

$$a\left(x + \frac{b}{a}y\right)^2 + \left(c - \frac{b^2}{a}\right)y^2$$

For positive definiteness:

1. $a > 0$
2. $c - \frac{b^2}{a} > 0$

Equivalent to:

$$ac - b^2 > 0$$

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8. Necessary and Sufficient Condition

The quadratic form

$$f(x, y) = ax^2 + 2bxy + cy^2$$

is positive definite if and only if

$$a > 0$$

and

$$ac - b^2 > 0$$

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9. Other Cases

If

$$ac = b^2$$

then f is positive semidefinite.
If

$$ac < b^2$$

then $(0, 0)$ is a saddle point.

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10. Matrix Representation

Let

$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Then

$$f(x, y) = v^T A v$$

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11. General Quadratic Form

For $v = (x_1, \dots, x_n)^T$ and matrix $A = (a_{ij})$,

$$v^T A v = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Explicitly,

$$f(v) = a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + a_{nn}x_n^2$$

At $v = 0$:

$$f(v) = 0$$

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12. Examples

Example 1

$$f(x, y) = 2x^2 + 4xy + y^2$$

Here

$$a = 2, \quad b = 2, \quad c = 1$$

$$ac = 2, \quad b^2 = 4$$

Since

$$ac < b^2$$

Origin is a saddle point.

Example 2

$$f(x, y) = 2xy$$

Matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Saddle at origin.

Example 3

$$f(x_1, x_2, x_3) = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$$

Matrix form:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

At origin the function has a minimum.

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13. Connection to Linear Algebra

Quadratic form:

$$f(v) = v^T A v$$

Positive definiteness of f relates to properties of matrix A.
This leads to the definition of positive definite matrices.

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Lecture 4

Positive Definite Matrices

1. From Quadratic Forms to Matrices

Consider the quadratic form

$$f(x, y) = ax^2 + 2bxy + cy^2$$

This can be written as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Let

$$v = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Then

$$f(x, y) = v^T A v$$

Previously, we showed:

$$f(x, y) \text{ is positive definite}$$

if and only if

$$a > 0$$

and

$$ac - b^2 > 0$$

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2. Eigenvalue Interpretation in 2 by 2 Case

Let eigenvalues of A be λ_1, λ_2.
We have

$$\det A = ac - b^2 = \lambda_1 \lambda_2$$
$$\operatorname{trace} A = a + c = \lambda_1 + \lambda_2$$

If

$$a > 0$$

and

$$ac - b^2 > 0$$

then

$$\lambda_1 \lambda_2 > 0$$

and

$$\lambda_1 + \lambda_2 > 0$$

Thus

$$\lambda_1 > 0, \quad \lambda_2 > 0$$

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3. Definition: Positive Definite Matrix

Let A be a real symmetric $n \times n$ matrix.

Condition 1

A is positive definite if

$$v^T A v > 0$$

for all nonzero $v \in R^n$.

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4. Equivalent Characterization

Condition 2

All eigenvalues of A are strictly positive.

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5. Theorem

For a real symmetric matrix A, the following are equivalent:

1. $v^T A v > 0$ for all $v \neq 0$.
2. All eigenvalues of A are positive.

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6. Proof: Condition 1 Implies Condition 2

Assume

$$v^T A v > 0$$

for all $v \neq 0$.
Let

$$Ax = \lambda x$$

be an eigenpair with $x \neq 0$.
Then

$$x^T A x = x^T \lambda x$$
$$= \lambda x^T x$$

Since $x^T A x > 0$ and $x^T x > 0$,

$$\lambda > 0$$

Thus all eigenvalues are positive.

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7. Proof: Condition 2 Implies Condition 1

Assume all eigenvalues satisfy

$$\lambda_i > 0$$

Since A is real symmetric, by spectral theorem there exists an orthonormal basis of eigenvectors

$$x_1, \ldots, x_n$$

Any vector $x \in R^n$ can be written as

$$x = c_1 x_1 + \cdots + c_n x_n$$

Then

$$Ax = c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n$$

Now compute

$$x^T Ax$$

$$= (c_1 x_1 + \cdots + c_n x_n)^T (c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n)$$

Using orthonormality:

$$x_i^T x_j = 0 \text{ if } i \neq j$$

$$x_i^T x_i = 1$$

Thus

$$x^T Ax = c_1^2 \lambda_1 + \cdots + c_n^2 \lambda_n$$

Since each $\lambda_i > 0$ and not all c_i are zero for $x \neq 0$,

$$x^T Ax > 0$$

Thus Condition 1 holds.

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8. Final Definition

A real symmetric matrix A is positive definite if

$$v^T Av > 0$$

for all $v \neq 0$.
Equivalently,
A is positive definite if and only if all eigenvalues of A are strictly positive.

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9. Observations

1. Positive definiteness requires symmetry.
2. Quadratic form positivity and eigenvalue positivity are equivalent.
3. In the 2 by 2 case, positivity reduces to

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a > 0, \quad ac - b^2 > 0
$$

4. Spectral theorem is essential in proving equivalence.

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