

## HW1

Due date: 04/05/2018

1. Durrett exercise 5.2.2
2. Complete the steps of Durrett example 5.2.4 (if you use the Borel-Cantelli lemma, explain why you can use it)
3. Durrett exercise 5.2.13
4. Let  $\tau$  be a stopping time with respect to a filtration  $\{\mathcal{F}_n\}$ . Show that
  - (a)  $\mathcal{F}_\tau = \{A \in \cup_n \mathcal{F}_n \mid A \cap \{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n\}$  is a  $\sigma$ -field
  - (b)  $\sigma$  is another stopping time such that  $\sigma \leq \tau$  a.s. then  $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$
  - (c) If  $X_n$  is adapted to  $\{\mathcal{F}_n\}$ , then  $X_\tau$  is  $\mathcal{F}_\tau$ -measurable
5. Let  $\{X_n\}$  be a martingale with  $\sup_n |X_n| < \infty$ . Show that there is a representation  $X_n = Y_n - Z_n$  where  $\{Y_n\}$  and  $\{Z_n\}$  are both non-negative martingales with  $\sup_n |Y_n| < \infty$  and  $\sup_n |Z_n| < \infty$ .
6. Let  $\{X_n\}$  and  $\{Y_n\}$  be positive integrable processes adapted to a filtration  $\{\mathcal{F}_n\}$  such that  $\sum_{n \geq 1} Y_n < \infty$  a.s. and  $E(X_{n+1} | \mathcal{F}_n) \leq (1 + Y_n)X_n + Y_n$ . Prove that  $X_n$  converges a.s. to a finite limit as  $n \rightarrow \infty$ .