## Excercise 5.5.2

1. Let $\mathcal{F} = \sigma(\theta, Z_1, Z_2, \ldots) = \sigma(\theta, Y_1, Y_2, \ldots)$, $\mathcal{F}_\infty = \sigma(Y_1, Y_2, \ldots)$, and $\mu = E(Z_i)$

2. Let $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i = \frac{1}{n}\sum_{i=1}^n Z_i + \theta$

3. Each $\bar{Y}_n$ is $\mathcal{F}_\infty$ measurable and hence $Y = \liminf \bar{Y}_n - \mu$ is $\mathcal{F}_\infty$ measurable

4. By SLLN, there exists $\Omega_0 \in \mathcal{F}$ such that $P(\Omega_0) = 1$ and $Y = \theta$ on $\Omega_0$

5. $EY_+ = EY_+ I_{\Omega_0} + EY_+ I_{\Omega_0^c} \leq E\theta_+ + \infty P(\Omega_0^c) < \infty$ (using the standard measure theoretic convention $\infty.0 = 0$ ). Similarly, $EY_- < \infty$ and hence $Y$ is integrable

6. For any $A \in \mathcal{F}_\infty$,

$$\begin{aligned}
\int_A \theta dP &= \int_{A\cap\Omega_0} \theta dP + \int_{A\cap\Omega_0^c} \theta dP \\
&= \int_{A\cap\Omega_0} Y dP + \int_{A\cap\Omega_0^c} \theta dP \\
&= \int_A Y dP + \int_{A\cap\Omega_0^c} (\theta - Y) dP \\
&= \int_A Y dP
\end{aligned}$$

   where the second term vanishes as both $\theta$ and $Y$ are integrable and $P(\Omega_0^c \cap A) = 0$

7. By uniqueness of conditional expectation, as $Y \in \mathcal{F}_\infty$, $Y = E(\theta|\mathcal{F}_\infty)$ on some $\Omega_1$ where $P(\Omega_1) = 1$

8. $\theta = E(\theta|\mathcal{F})$ on $\Omega_0 \cap \Omega_1$.