UPSC PHYSICS PYQ SOLUTION
Quantum Mechanics - Part 4

Contents

31 Normalized wave function of a particle is given:

$$\psi(x) = N \exp\left(-\frac{x^2}{2a^2} + ikx\right).$$

Find the expectation value of position.

Introduction:
The expectation value of position $\langle x \rangle$ for a given wavefunction $\psi(x)$ is a fundamental concept in quantum mechanics. It provides the average position of a particle described by that wavefunction, giving insight into where the particle is likely to be found upon measurement. Mathematically, the expectation value of position is calculated as:

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 \, dx$$

where $|\psi(x)|^2$ represents the probability density function of the particle's position.

Solution:

Given the normalized wavefunction:

$$\psi(x) = N \exp\left(-\frac{x^2}{2a^2} + ikx\right)$$

The expectation value of position is given by:

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 \, dx$$

First, find $|\psi(x)|^2$:

$$|\psi(x)|^2 = \left|N \exp\left(-\frac{x^2}{2a^2} + ikx\right)\right|^2 = N^2 \exp\left(-\frac{x^2}{a^2}\right)$$

Now, calculate the expectation value:

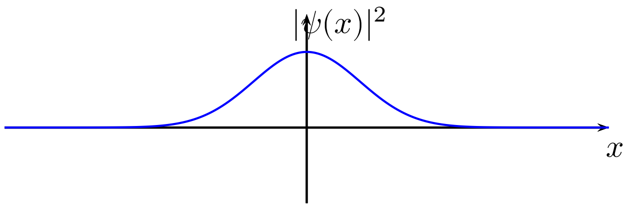
$$\langle x \rangle = N^2 \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{a^2}\right) \, dx$$

Notice that the integrand $x \exp\left(-\frac{x^2}{a^2}\right)$ is an odd function, and the limits are symmetric about zero. Therefore, the integral evaluates to zero:

$$\langle x \rangle = 0$$

Conclusion:

Below is a graph of the probability density function $|\psi(x)|^2$:



The expectation value of the position for the given wavefunction is zero. Physically, this means that the average position of the particle is centered at the origin. This result is consistent with the symmetry of the probability density function, which is evenly distributed about the origin. In other words, there is no preferred direction or location for the particle, indicating that it is equally likely to be found on either side of the origin.

32 Write the time-independent Schrödinger equation for a bouncing ball.

Introduction:
In quantum mechanics, a bouncing ball can be modeled as a particle subject to a gravitational potential. The potential energy increases linearly with height, similar to the classical potential energy function in a gravitational field.

Solution:

For a bouncing ball, the potential energy $V(z)$ is given by:

$$V(z) = mgz$$

where: - m is the mass of the ball, - g is the acceleration due to gravity, - z is the height above the ground.

The time-independent Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(z)}{dz^2} + V(z)\psi(z) = E\psi(z)$$

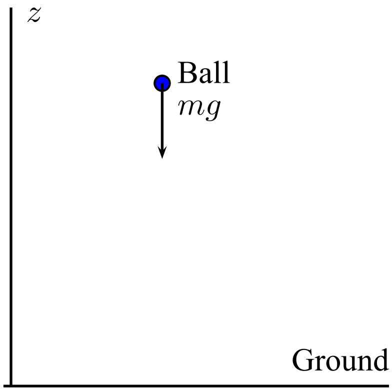
Substituting the potential $V(z) = mgz$, we get:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(z)}{dz^2} + mgz\psi(z) = E\psi(z)$$

Rewriting, we have:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(z)}{dz^2} + mgz\psi(z) = E\psi(z)$$

This is the time-independent Schrödinger equation for a particle in a linear potential, representing a bouncing ball in a gravitational field.



Conclusion:
The time-independent Schrödinger equation for a bouncing ball subject to a gravitational potential is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(z)}{dz^2} + mgz\psi(z) = E\psi(z)$$

This equation models the quantum behavior of a particle under the influence of gravity, providing insight into the quantized energy levels and wavefunctions of a bouncing ball in a gravitational field.

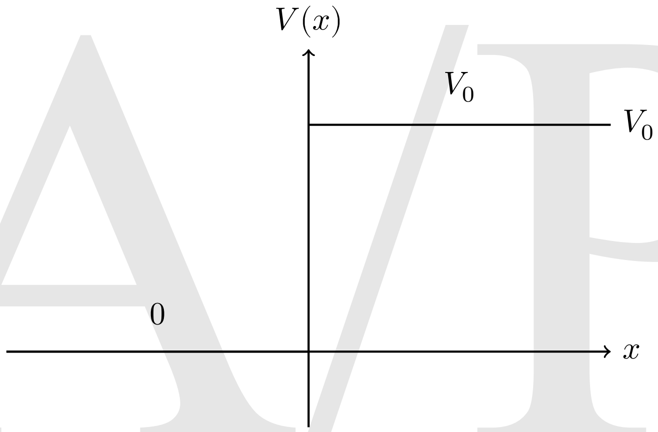
33 Solve the Schrödinger equation for a step potential and calculate the transmission and reflection coefficients for the case when the kinetic energy of the particle E_0 is greater than the potential energy V (i.e., $E_0 > V$).

Introduction:
The step potential is a fundamental problem in quantum mechanics that illustrates the behavior of a particle encountering a sudden change in potential energy. This problem is essential for understanding phenomena such as quantum tunneling and reflection.

Consider a particle encountering a step potential:

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } x \geq 0 \end{cases}$$

Below is a diagram illustrating the step potential:



Solution:

Consider a particle encountering a step potential:

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } x \geq 0 \end{cases}$$

The Schrödinger equation in regions where $V(x)$ is constant is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

For $x < 0$ (Region I), where $V(x) = 0$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E_0\psi(x)$$

The general solution is:

$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}$$

where:

$$k_1 = \sqrt{\frac{2mE_0}{\hbar^2}}$$

For $x \geq 0$ (Region II), where $V(x) = V_0$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E_0\psi(x)$$

This simplifies to:

$$\frac{d^2\psi(x)}{dx^2} = k_2^2\psi(x)$$

where:

$$k_2 = \sqrt{\frac{2m(E_0 - V_0)}{\hbar^2}}$$

The general solution is:

$$\psi_{II}(x) = Ce^{ik_2x}$$

Since we consider the particle coming from the left and moving to the right, there will be no wave traveling to the left in Region II ($D = 0$):

$$\psi_{II}(x) = Ce^{ik_2x}$$

Boundary Conditions:

At $x = 0$, the wavefunctions and their first derivatives must be continuous:

$$\psi_I(0) = \psi_{II}(0)$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}$$

Applying these conditions:

1. Continuity of wavefunction:

$$A + B = C$$

2. Continuity of derivative:

$$ik_1A - ik_1B = ik_2C$$

Solving these equations for A, B, and C:

From the first equation:

$$C = A + B$$

Substituting into the second equation:

$$ik_1A - ik_1B = ik_2(A + B)$$

Rearranging:

$$k_1A - k_1B = k_2A + k_2B$$

$$(k_1 - k_2)A = (k_1 + k_2)B$$

$$\frac{A}{B} = \frac{k_1 + k_2}{k_1 - k_2}$$

Therefore, the reflection coefficient R is:

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2$$

The transmission coefficient T is given by:

$$T = \left|\frac{C}{A}\right|^2 = \left|\frac{2k_1}{k_1 + k_2}\right|^2$$

Conclusion:
For a particle encountering a step potential with $E_0 > V_0$, the transmission and reflection coefficients are given by:

$$R = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2, \quad T = \left|\frac{2k_1}{k_1 + k_2}\right|^2$$

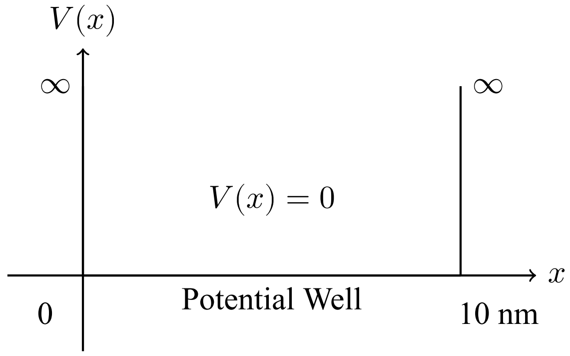
These coefficients describe the probability of the particle being reflected or transmitted at the potential step.

An application of the step potential is seen in the behavior of electrons in semiconductor devices, where they encounter potential barriers at junctions, leading to phenomena like tunneling and reflection that are crucial for the operation of diodes and transistors.

34 Calculate the lowest energy of an electron confined to move in a 1-dimensional potential well of width 10 nm.

Introduction:
The problem of an electron confined in a one-dimensional potential well, also known as a ”particle in a box,” demonstrates the concept of quantized energy levels in quantum mechanics.

Below is a diagram illustrating the one-dimensional potential well:



Solution:

For an electron in a one-dimensional box of width $L = 10$ nm $= 10 \times 10^{-9}$ m, the energy levels are given by:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

The normalized wave function is:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

The lowest energy corresponds to the ground state ($n = 1$):

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2}$$

Substitute the values: - Planck’s constant $\hbar = 1.0545718 \times 10^{-34}$ J · s - Electron mass $m = 9.10938356 \times 10^{-31}$ kg - Width $L = 10 \times 10^{-9}$ m

Calculating:

$$E_1 = \frac{\pi^2(1.0545718 \times 10^{-34})^2}{2(9.10938356 \times 10^{-31})(10 \times 10^{-9})^2}$$

$$E_1 \approx 6.024 \times 10^{-20} \text{ J}$$

To convert this energy into electronvolts (eV):

$$1 \text{ eV} = 1.60218 \times 10^{-19} \text{ J}$$

$$E_1 \approx \frac{6.024 \times 10^{-20}}{1.60218 \times 10^{-19}} \text{ eV}$$

$$E_1 \approx 0.376 \text{ eV}$$

Conclusion:
The lowest energy of an electron confined in a one-dimensional potential well of width 10 nm is approximately 0.376 eV.

35 Using Schrödinger Equation to Obtain Eigen-functions and Eigenvalues for a 1-Dimensional Harmonic Oscillator. Sketch the profiles of eigenfunc ons for first three energy states.

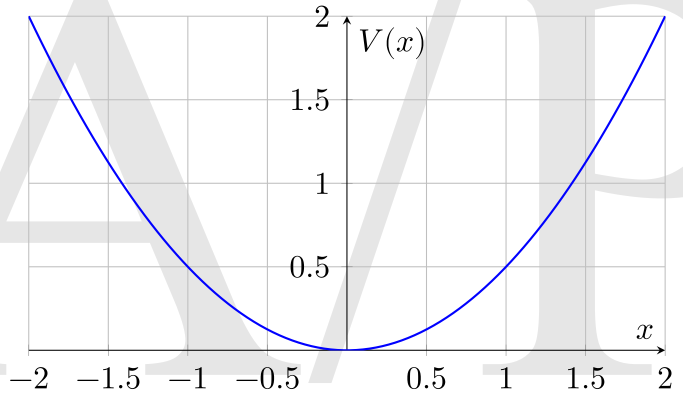
Introduction:
The quantum harmonic oscillator is a fundamental model in quantum mechanics that describes a particle subject to a restoring force proportional to its displacement from equilibrium. This is represented by the potential energy function:

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

where m is the mass of the particle and ω is the angular frequency of the oscillator. This potential is quadratic in x, making it an ideal system to illustrate quantized energy levels and wavefunctions.

Below is a graph illustrating the potential $V(x)$ of a harmonic oscillator:

Potential of the Harmonic Oscillator



Solution:

The force acting on a particle executing linear harmonic oscillation is given by Hooke's law:

$$F = -kx$$

where x represents the displacement from the equilibrium position, and k is the force constant. This linear relationship indicates that the force is always directed towards the equilibrium position and its magnitude increases linearly with the displacement.

The corresponding potential energy function, $V(x)$, associated with this force is quadratic and is expressed as:

$$V(x) = \frac{1}{2}kx^2$$

In terms of the mass m of the particle and the angular frequency ω, where $\omega = \sqrt{\frac{k}{m}}$, the potential energy can be rewritten as:

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

The time-independent Schrödinger equation for a particle of mass m in this potential is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x)$$

Simplifying, we obtain:

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2 x^2\right)\psi(x) = 0$$

To simplify this equation, we introduce the dimensionless eigenvalue λ and the dimensionless variable ξ:

$$\lambda = \frac{2E}{\hbar\omega}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x$$

Substituting these into the Schrödinger equation transforms it into:

$$\frac{d^2\psi(\xi)}{d\xi^2} + \left(\lambda - \xi^2\right)\psi(\xi) = 0$$

This differential equation is known as the Hermite equation. The solutions to this equation are the Hermite polynomials $H_n(\xi)$. The eigenfunctions of the harmonic oscillator are thus given by:

$$\psi_n(\xi) = A_n H_n(\xi) e^{-\xi^2/2}$$

where A_n is the normalization constant. These polynomials satisfy the orthogonality condition and are well-suited to describe the quantum states of the harmonic oscillator.

To solve the Hermite equation, we assume a power series solution:

$$\psi(\xi) = e^{-\xi^2/2}\sum_{n=0}^{\infty} a_n \xi^n$$

Substituting this series into the differential equation and matching coefficients for each power of ξ, we derive a recurrence relation for the coefficients a_n:

First, we compute the derivatives:

$$\begin{aligned}
\frac{d\psi(\xi)}{d\xi} &= e^{-\xi^2/2}\left(\sum_{n=0}^{\infty} a_n n\xi^{n-1} - \xi\sum_{n=0}^{\infty} a_n\xi^n\right) \\
&= e^{-\xi^2/2}\left(\sum_{n=1}^{\infty} a_n n\xi^{n-1} - \xi\sum_{n=0}^{\infty} a_n\xi^n\right) \\
&= e^{-\xi^2/2}\left(\sum_{n=1}^{\infty} a_n n\xi^{n-1} - \sum_{n=0}^{\infty} a_n\xi^{n+1}\right)
\end{aligned}$$

Then,

$$\frac{d^2\psi(\xi)}{d\xi^2} = e^{-\xi^2/2}\left(\sum_{n=1}^{\infty} a_n n(n-1)\xi^{n-2} - 2\xi\sum_{n=1}^{\infty} a_n n\xi^{n-1} + \xi^2\sum_{n=0}^{\infty} a_n\xi^n\right)$$

$$= e^{-\xi^2/2}\left(\sum_{n=2}^{\infty} a_n n(n-1)\xi^{n-2} - 2\sum_{n=1}^{\infty} a_n n\xi^n + \sum_{n=0}^{\infty} a_n \xi^{n+2}\right)$$

Rewriting the Schrödinger equation:

$$e^{-\xi^2/2}\left(\sum_{n=2}^{\infty} a_n n(n-1)\xi^{n-2} - 2\sum_{n=1}^{\infty} a_n n\xi^n + \sum_{n=0}^{\infty} a_n \xi^{n+2}\right) + (\lambda - \xi^2)e^{-\xi^2/2}\sum_{n=0}^{\infty} a_n \xi^n = 0$$

Grouping terms by the power of ξ:

$$\sum_{n=2}^{\infty} a_n n(n-1)\xi^{n-2} - 2\sum_{n=1}^{\infty} a_n n\xi^n + \sum_{n=0}^{\infty} a_n \xi^{n+2} + \lambda\sum_{n=0}^{\infty} a_n \xi^n - \sum_{n=0}^{\infty} a_n \xi^{n+2} = 0$$

$$\sum_{n=0}^{\infty}\left(a_{n+2}(n+2)(n+1) - 2a_n n + \lambda a_n\right)\xi^n = 0$$

For the series to terminate, ensuring normalizable wavefunctions, λ must be an odd integer:

$$\lambda = 2n + 1$$

Thus, the quantized energy levels are given by:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

The corresponding normalized eigenfunctions are derived as follows:

The power series solution:

$$\psi(\xi) = e^{-\xi^2/2}\sum_{n=0}^{\infty} a_n \xi^n$$

Substituting into the Schrödinger equation:

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0$$

The Hermite polynomials $H_n(\xi)$ are defined as:

$$H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}\left(e^{-\xi^2}\right)$$

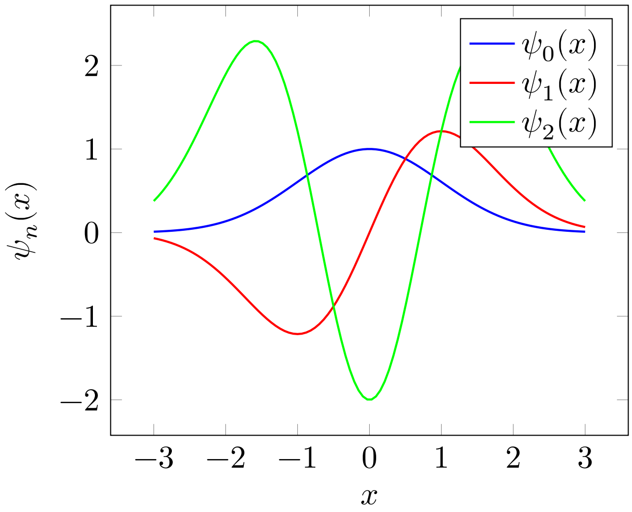
The normalized eigenfunctions are:

$$\psi_n(\xi) = \left(\frac{\alpha}{\sqrt{\pi}2^n n!}\right)^{1/2} H_n(\alpha\xi)e^{-\alpha^2\xi^2/2}$$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$.

Below are the plots of the first three eigenfunctions:

First Three Eigenfunctions of the Harmonic Oscillator



Conclusion

(i) The peculiar point is the ground state wave function of simple harmonic oscillator that is Gaussian in nature. This arises due to the unique boundary conditions of the system. SHO is the only system for which **equality of Heisenberg uncertainty principle** holds true (in ground state).

(ii) The derived **energy eigenvalues** $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ are quantized, meaning the system can only occupy specific energy levels. This indicates discrete energy states rather than a continuum.

(iii) **Applications**: Molecular Vibrations in Chemistry, where it explains the spectra observed in **infrared spectroscopy**. In quantum field theory, it serves as the basis for understanding **particle behavior in potential wells and for modeling quantized fields**.

36 Calculate the probability of transmission of an electron of 1.0 eV energy through a potential barrier of 4.0 eV and 0.1 nm width.

Introduction:. Here Quantum tunneling tunneling is taking place. Quantum tunneling occurs when particles pass through a barrier that they classically shouldn't be able to, due to their energy being lower than the potential of the barrier.

Solution:

The transmission probability T for a particle with energy E encountering a potential barrier V of width a is given by:

$$T = \exp\left(-2\kappa a\right)$$

where:

$$\kappa = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

Given:

- Energy $E = 1.0$ eV
- Potential $V = 4.0$ eV
- Width $a = 0.1$ nm $= 0.1 \times 10^{-9}$ m
- Electron mass $m = 9.10938356 \times 10^{-31}$ kg
- Planck's constant $\hbar = 1.0545718 \times 10^{-34}$ J $\cdot$ s
- 1 eV $= 1.60218 \times 10^{-19}$ J

Calculate κ:

$$\kappa = \sqrt{\frac{2 \times 9.10938356 \times 10^{-31} \times (4.0 - 1.0) \times 1.60218 \times 10^{-19}}{(1.0545718 \times 10^{-34})^2}}$$

$$\kappa \approx 1.14 \times 10^{10}\ \text{m}^{-1}$$

Calculate the transmission probability:

$$T = \exp\left(-2 \times 1.14 \times 10^{10} \times 0.1 \times 10^{-9}\right)$$

$$T = \exp\left(-2 \times 1.14\right)$$

$$T \approx \exp(-2.28)$$

$$T \approx 0.102$$

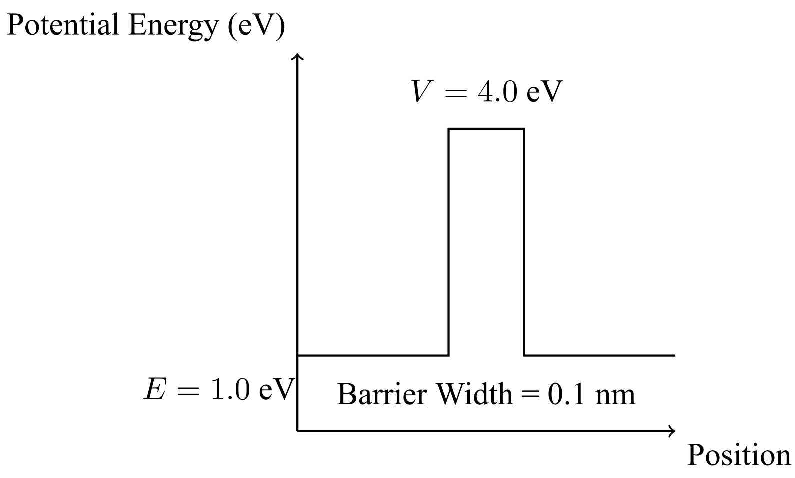
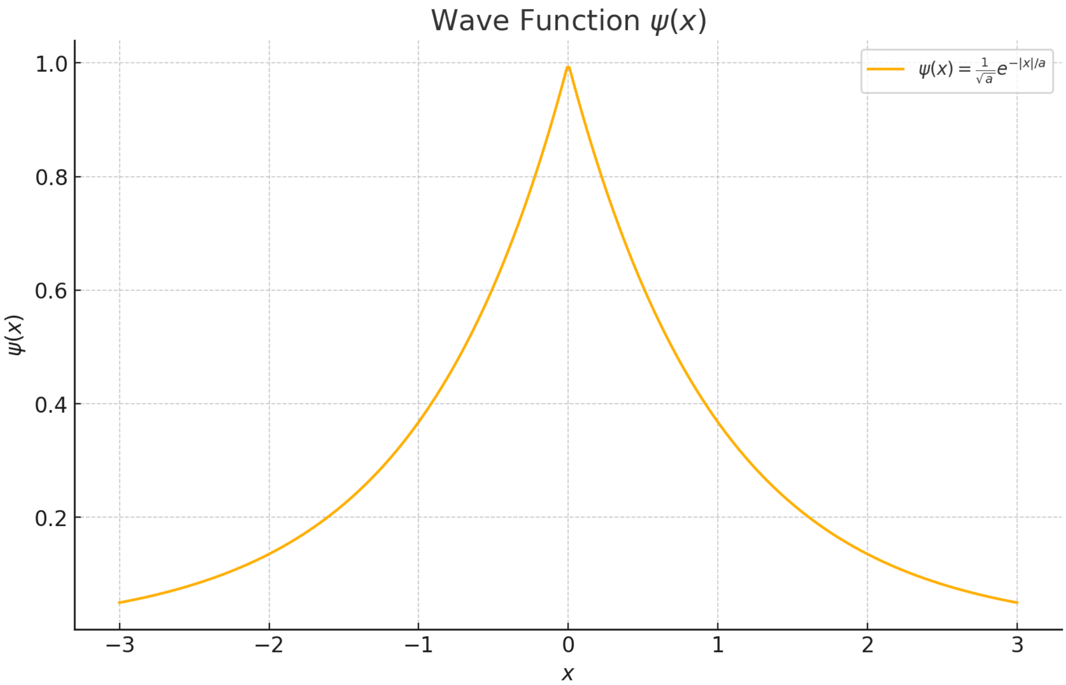


Figure 1: Potential Barrier Diagram

Conclusion: The probability of transmission of an electron with 1.0 eV energy through a potential barrier of 4.0 eV and 0.1 nm width is **approximately 0.102**, illustrating the quantum tunneling effect. Quantum tunneling is significant in various applications such as in the operation of **tunnel diodes** and the process of nuclear fusion in stars. This phenomenon also underpins the functionality of **scanning tunneling microscopes**, which can image surfaces at the atomic level.

37 The wave function of a particle is given as $\psi(x) = \frac{1}{\sqrt{a}} e^{-|x|/a}$. Find the probability of locating the particle in the range $-a \le x \le a$.

Introduction: The wave function $\psi(x)$ provides the probability amplitude for finding a particle at position x. The probability of locating the particle in a specific range is given by the integral of the square of the wave function over that range.



The general expression for the probability P of finding the particle in the range $x_1 \le x \le x_2$ is given by:

$$P = \int_{x_1}^{x_2} |\psi(x)|^2 \, dx$$

Solution: The probability P of finding the particle in the range $-a \le x \le a$ is given by:

$$P = \int_{-a}^{a} |\psi(x)|^2 \, dx$$

Given the wave function:

$$\psi(x) = \frac{1}{\sqrt{a}} e^{-|x|/a}$$

The square of the wave function is:

$$|\psi(x)|^2 = \left(\frac{1}{\sqrt{a}} e^{-|x|/a} \right)^2 = \frac{1}{a} e^{-2|x|/a}$$

Thus, the probability is:

$$P = \int_{-a}^{a} \frac{1}{a} e^{-2|x|/a} \, dx$$

Since the wave function is symmetric about $x = 0$, we can simplify the integral:

$$P = 2\int_0^a \frac{1}{a}e^{-2x/a}\,dx$$

Evaluating the integral:

$$P = 2\left[-\frac{1}{2}e^{-2x/a}\right]_0^a$$

$$P = 2\left[-\frac{1}{2}e^{-2a/a} + \frac{1}{2}\right]$$

$$P = 2\left[-\frac{1}{2}e^{-2} + \frac{1}{2}\right]$$

$$P = 1 - e^{-2}$$

Conclusion: The probability of locating the particle in the range $-a \leq x \leq a$ is $1 - e^{-2}$. This result illustrates how the wave function's exponential decay affects the probability distribution within a finite range.

38 Calculate the zero-point energy of a system consisting of a mass of 10^{-3} kg connected to a fixed point by a spring which is stretched by 10^{-2} m by a force of 10^{-1} N. The system is constrained to move only in one direction.

Introduction:

Zero-point energy is the lowest possible energy that a quantum mechanical physical system may have. It is the energy of the ground state of the system. In the case of a harmonic oscillator, the zero-point energy is $\frac{1}{2}\hbar\omega$, where ω is the angular frequency of the oscillator.

Solution:

First, we need to determine the spring constant k using Hooke's Law:

$$F = kx$$

Given:

- Force, $F = 10^{-1}$ N
- Displacement, $x = 10^{-2}$ m

We solve for k:

$$k = \frac{F}{x} = \frac{10^{-1}}{10^{-2}} = 10\,\text{N/m}$$

Next, we find the angular frequency ω of the system:

$$\omega = \sqrt{\frac{k}{m}}$$

Given:

- Mass, $m = 10^{-3}$ kg
- Spring constant, $k = 10$ N/m

We solve for ω:

$$\omega = \sqrt{\frac{10}{10^{-3}}} = \sqrt{10^4} = 100\,\text{rad/s}$$

The zero-point energy E_0 of a quantum harmonic oscillator is given by:

$$E_0 = \frac{1}{2}\hbar\omega$$

Using the reduced Planck constant $\hbar \approx 1.054 \times 10^{-34}$ Js, we get:

$$E_0 = \frac{1}{2} \times 1.054 \times 10^{-34} \times 100 = 5.27 \times 10^{-33}\,\text{J}$$

Conclusion:

The zero-point energy of the system is 5.27×10^{-33} J. This energy represents the lowest energy state of the harmonic oscillator system, even at absolute zero temperature. This concept is illustrates the inherent energy present in all quantum systems due to the Heisenberg uncertainty principle.

39 The general wave function of harmonic oscillator (one-dimensional) are of the form

$$u_n(x) = \sum_{k=0}^{\infty} a_k y^k e^{-y^2/2}$$

with $y = \sqrt{\frac{m\omega}{\hbar}}x$, and coefficients a_k are determined by recurrence relations

$$a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)}a_k$$

Corresponding energy levels are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

Discuss the parity of these wave functions. What happens, if the potential for $x \leq 0$ is infinite (half harmonic oscillator)?

Introduction:

The general wave function of a one-dimensional harmonic oscillator is given by a series solution involving Hermite polynomials. The parity of a wave function refers to its behavior under spatial inversion, $x \to -x$.

Solution:

1. **Wave Function and Recurrence Relation:**

The wave function $u_n(x)$ is expressed as a series involving the Hermite polynomials $H_n(y)$:

$$u_n(x) = H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

The coefficients a_k in the series are determined by the recurrence relation:

$$a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)}a_k$$

2. **Parity of Wave Functions:**

The wave functions $u_n(x)$ for the harmonic oscillator have definite parity:

$$u_n(-x) = (-1)^n u_n(x)$$

This means:

- For even n: $u_n(x)$ is an even function.
- For odd n: $u_n(x)$ is an odd function.

This behavior is a result of the properties of the Hermite polynomials, which alternate in parity.

3. **Half Harmonic Oscillator**:

If the potential is infinite for $x \leq 0$, the wave function must vanish at $x = 0$:

$$u_n(0) = 0$$

For the half harmonic oscillator, this condition is satisfied only by the odd-parity solutions:

- Only wave functions with odd n are valid.
- These wave functions naturally vanish at $x = 0$, satisfying the boundary condition.

Conclusion:

The wave functions of a harmonic oscillator exhibit definite parity, with even n corresponding to even functions and odd n corresponding to odd functions. For a half harmonic oscillator, where the potential is infinite for $x \leq 0$, only the odd-parity wave functions are valid, as they meet the boundary condition $u_n(0) = 0$. This restriction reduces the number of allowed energy levels and changes the overall behavior of the system.

40 Which of the following functions is/are acceptable solution(s) of the Schrödinger equation?

- $\psi(x) = Ae^{-ikx} + Be^{ikx}$
- $\psi(x) = Ae^{-kx} + Be^{kx}$
- $\psi(x) = A\sin 3kx + B\cos 5kx$
- $\psi(x) = A\sin 3kx + B\sin 5kx$
- $\psi(x) = A\tan kx$

Introduction:

For a wave function to be an acceptable solution to the Schrödinger equation, it must satisfy the following conditions:

- **Normalizability:** The wave function must be square-integrable over all space, meaning that the integral of $|\psi(x)|^2$ over all space must be finite.
- **Continuity:** The wave function must be continuous and have continuous first derivatives.
- **Boundary Conditions:** The wave function must satisfy the boundary conditions of the physical system.
- **Eigenfunction:** The wave function must be an eigenfunction of the Hamiltonian operator.

Solution:

1. **Function (i):** $\psi(x) = Ae^{-ikx} + Be^{ikx}$

This is a general solution for the free particle Schrödinger equation. It represents a superposition of plane waves traveling in opposite directions and is a valid solution. These are solutions to the time-independent Schrödinger equation in a region where the potential $V(x) = 0$:

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

Hence, $\psi(x) = Ae^{-ikx} + Be^{ikx}$ is an acceptable solution.

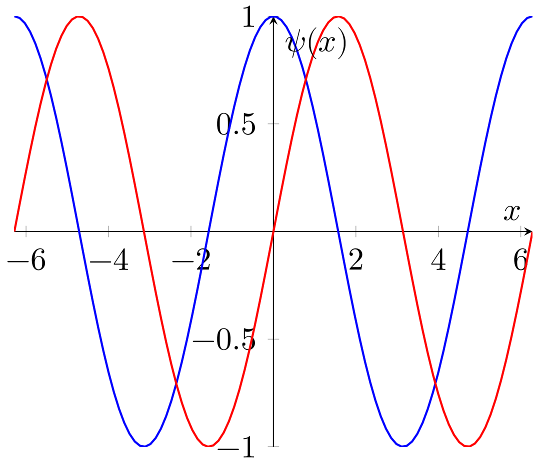


Figure 2: Plot of $\psi(x) = Ae^{-ikx} + Be^{ikx}$

2. **Function (ii):** $\psi(x) = Ae^{-kx} + Be^{kx}$

This form is typically a solution to the Schrödinger equation with an imaginary wave number $k = i\kappa$, which can represent a decaying or growing exponential. For bound states, normalizability requires that only the decaying exponent is present. In general, for regions with a potential step or barrier, $\psi(x) = Ae^{-kx} + Be^{kx}$ is valid but must be handled with boundary conditions to ensure physicality.

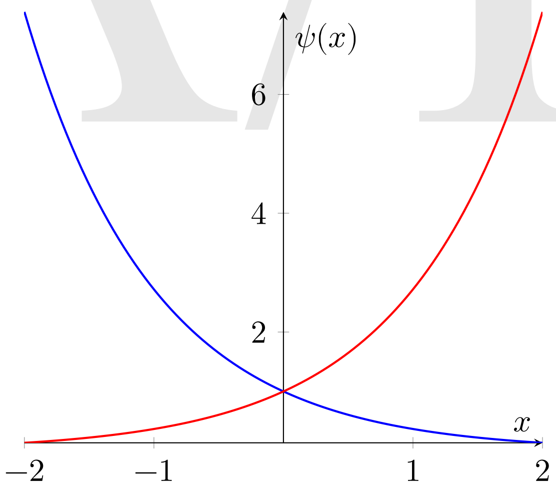


Figure 3: Plot of $\psi(x) = Ae^{-kx} + Be^{kx}$

3. **Function (iii):** $\psi(x) = A\sin 3kx + B\cos 5kx$

This function is not a solution to the Schrödinger equation for a single potential because it mixes different wave numbers. Each term would need to independently satisfy the Schrödinger equation, but having different wave numbers $3k$ and $5k$ implies different energies which cannot coexist in a single eigenstate.

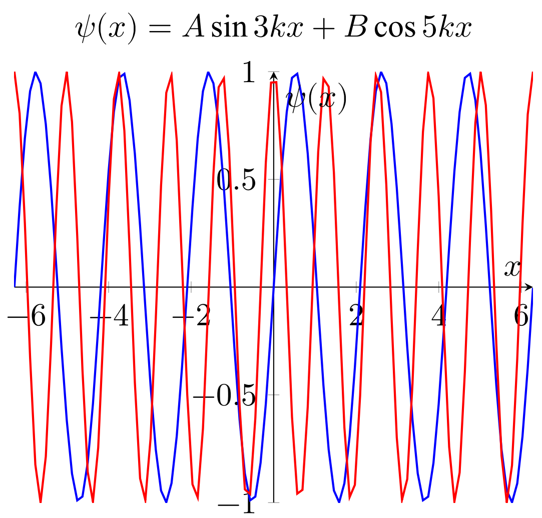


Figure 4: Plot of $\psi(x) = A\sin 3kx + B\cos 5kx$

4. **Function (iv):** $\psi(x) = A\sin 3kx + B\sin 5kx$

Similar to the previous case, this function mixes different wave numbers and therefore different energies. It cannot be a single eigenstate of the Schrödinger equation due to the presence of different k values, implying different energy eigenvalues.

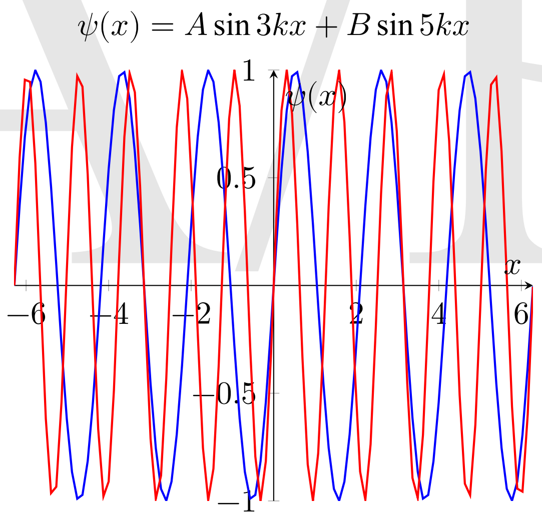


Figure 5: Plot of $\psi(x) = A\sin 3kx + B\sin 5kx$

5. **Function (v):** $\psi(x) = A\tan kx$

The tangent function $\tan kx$ is not acceptable as a wave function because it has singularities where $kx = \frac{\pi}{2}$, which makes it non-normalizable and unphysical for describing a quantum state.

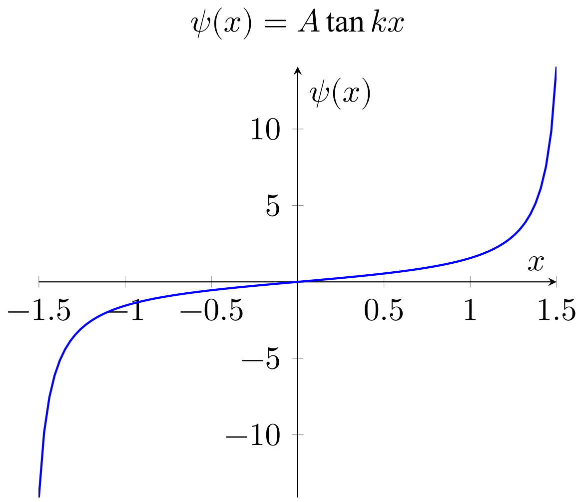


Figure 6: Plot of $\psi(x) = A\tan kx$

Conclusion:

Among the given functions, the acceptable solutions to the Schrödinger equation are:

- $\psi(x) = Ae^{-ikx} + Be^{ikx}$ (i)
- $\psi(x) = Ae^{-kx} + Be^{kx}$ (ii), provided it is appropriately normalized and satisfies boundary conditions.

Functions (iii), (iv), and (v) are not acceptable solutions due to mixing of different wave numbers (implying different energies) or non-normalizability.