UPSC PHYSICS PYQ SOLUTION

Mechanics - Part 1

Contents

1 With an appropriate diagram, show that in the Rutherford scattering, the orbit of the particle is a hyperbola. Obtain an expression for impact parameter.

Introduction: We consider a positively charged particle of charge ze, mass m, and initial speed v_0 (kinetic energy $E = \frac{1}{2}mv_0^2$) approaching a heavy nucleus of charge Ze from infinity. The nucleus is assumed fixed due to its large mass ($M \gg m$). The particle is deflected by the repulsive Coulomb force, following a hyperbolic trajectory. The impact parameter p is the perpendicular distance from the nucleus to the particle's initial straight-line path. This derivation computes p in terms of the scattering angle ϕ.

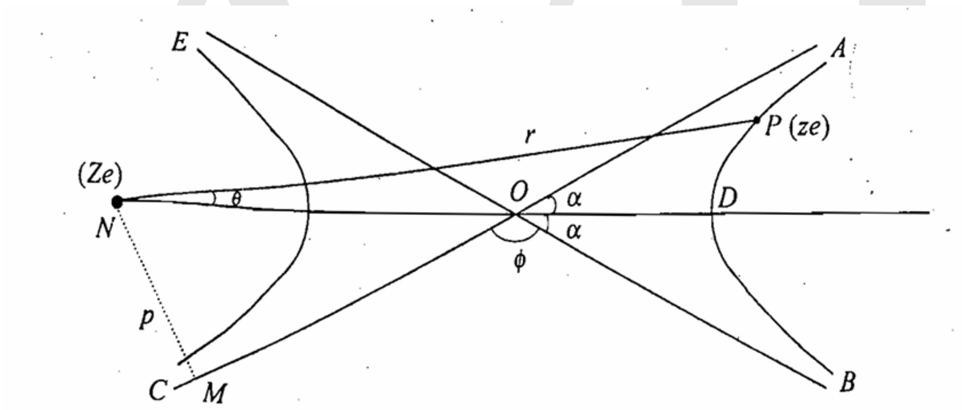
Solution: The repulsive Coulomb force and potential are:

$$F(r) = \frac{zZe^2}{4\pi\varepsilon_0 r^2}, \quad U(r) = \frac{zZe^2}{4\pi\varepsilon_0 r}. \tag{1}$$

The force is central, so angular momentum J and total energy E are conserved:

$$J = mv_0 p, \quad E = \frac{1}{2}mv_0^2, \tag{2}$$

where p is the impact parameter, and v_0 is the initial speed at infinity (where $U(r) \to 0$).



For a central force, the trajectory is described in polar coordinates (r, θ). The angular momentum is:

$$J = m\,r^2\,\dot{\theta}.$$

The effective potential for radial motion is:

$$U_{\text{eff}}(r) = U(r) + \frac{J^2}{2\,m\,r^2} = \frac{zZe^2}{4\pi\varepsilon_0\,r} + \frac{J^2}{2\,m\,r^2}.$$

The total energy is:

$$E = \frac{1}{2}m\,\dot{r}^2 + U_{\text{eff}}(r).$$

To find the orbit, we use the substitution $u = \frac{1}{r}$, so $r = \frac{1}{u}$, and:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\,\frac{d\theta}{dt} = -\frac{1}{u^2}\,\frac{du}{d\theta}\,\frac{J}{m\,r^2} = -\frac{J}{m}\,\frac{du}{d\theta}.$$

The radial equation becomes:

$$E = \frac{1}{2}\, m\Big(-\tfrac{J}{m}\,\tfrac{du}{d\theta}\Big)^2 \;+\; \frac{zZe^2}{4\pi\varepsilon_0}\,u \;+\; \frac{J^2}{2\,m}\,u^2,$$

which can be rearranged to

$$\Big(\tfrac{du}{d\theta}\Big)^2 = \frac{2\,m\,E}{J^2} \;-\; u^2 \;-\; \frac{2\,m\,k}{J^2}\,u, \qquad k = \frac{zZe^2}{4\pi\varepsilon_0}.$$

Differentiating once more with respect to θ yields the differential equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{m\,k}{J^2}.$$

The general solution is:

$$u(\theta) = \frac{m\,k}{J^2}\,(1 + e\,\cos\theta), \qquad \text{or} \quad \frac{\ell}{r} = 1 + e\,\cos\theta,$$

where

$$\ell = \frac{J^2}{m\,k}, \qquad e = \sqrt{1 + \frac{2\,E\,\ell}{k}}.$$

Since $E > 0$ and the potential is repulsive, $e > 1$, indicating a hyperbolic orbit.

The impact parameter p relates to the orbit's geometry. Using

$$J = m\,v_0\,p, \qquad E = \frac{1}{2}\,m\,v_0^2,$$

we find

$$\ell = \frac{J^2}{m\,k} = \frac{(m\,v_0\,p)^2}{m\cdot\frac{zZe^2}{4\pi\varepsilon_0}} = \frac{m\,v_0^2\,p^2}{\frac{zZe^2}{4\pi\varepsilon_0}} = \frac{2\,E\,p^2}{k} = \frac{8\,\pi\,\varepsilon_0\,E\,p^2}{z\,Z\,e^2}.$$

The orbit's asymptotes occur when $r \to \infty$, i.e. $u \to 0$, which gives:

$$1 + e\,\cos\alpha = 0 \implies \cos\alpha = -\frac{1}{e}.$$

Because the hyperbola is symmetric about the periapsis direction, one finds

$$\alpha = \frac{\pi - \phi}{2}.$$

Hence

$$\cos\Big(\tfrac{\pi}{2} - \tfrac{\phi}{2}\Big) = -\frac{1}{e} \implies \sin\Big(\tfrac{\phi}{2}\Big) = \frac{1}{e}.$$

Thus

$$\csc\Big(\tfrac{\phi}{2}\Big) = e.$$

Squaring both sides,

$$\csc^2\Big(\tfrac{\phi}{2}\Big) = e^2 \;=\; 1 + \cot^2\Big(\tfrac{\phi}{2}\Big) \;=\; 1 + \Big[\tfrac{2\,E\,p\,(4\pi\varepsilon_0)}{z\,Z\,e^2}\Big]^2,$$

where we have used $e^2 = 1 + \frac{2\,E\,\ell}{k}$ and $\ell = \frac{2\,E\,p^2}{k}$, $k = \frac{z\,Z\,e^2}{4\pi\varepsilon_0}$. Hence

$$\cot\left(\frac{\phi}{2}\right) = \frac{2\,E\,p\,(4\pi\varepsilon_0)}{z\,Z\,e^2}.$$

Solving for p, we obtain:

$$\boxed{p = \frac{z\,Z\,e^2}{2\,E\,(4\pi\varepsilon_0)}\,\cot\left(\frac{\phi}{2}\right).}$$

Conclusion: The impact parameter determines the scattering cross-section, which is crucial for experiments such as Rutherford's gold-foil experiment. A smaller p corresponds to a larger scattering angle ϕ, indicating a closer approach to the nucleus.

2 Prove that as a result of an elastic collision of two particles under the non-relativistic regime with equal masses, the scattering angle will be 90°. Illustrate your answer with a vector diagram.

Introduction:

We are to prove that in a non-relativistic, elastic collision between two particles of equal mass, where one is initially at rest, the angle between their velocities after collision is 90°. This scenario is common in atomic and nuclear physics. Assumptions:

1. Both particles have the same mass m.
2. The collision is elastic: kinetic energy and momentum are conserved.
3. The second particle is initially at rest.
4. The scattering is analyzed in the laboratory frame.

Solution:

Let the incoming particle (mass m) have an initial velocity $\vec{v}_0$. The target particle (also of mass m) is initially at rest.

Let the velocities after collision be:

- $\vec{v}_1$ for the incoming particle
- $\vec{v}_2$ for the initially stationary particle

Conservation of Momentum:

Total momentum before collision:

$$\vec{p}_{\text{initial}} = m\vec{v}_0$$

Total momentum after collision:

$$\vec{p}_{\text{final}} = m\vec{v}_1 + m\vec{v}_2$$

So:

$$\vec{v}_0 = \vec{v}_1 + \vec{v}_2 \quad (1)$$

Conservation of Kinetic Energy:

Total kinetic energy before collision:

$$K_{\text{initial}} = \frac{1}{2}mv_0^2$$

Total kinetic energy after collision:

$$K_{\text{final}} = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

So:

$$v_0^2 = v_1^2 + v_2^2 \quad (2)$$

From equation (1), square both sides:

$$v_0^2 = |\vec{v}_1 + \vec{v}_2|^2 = v_1^2 + v_2^2 + 2\vec{v}_1 \cdot \vec{v}_2$$

Substitute from equation (2) into this:

$$v_1^2 + v_2^2 = v_1^2 + v_2^2 + 2\vec{v}_1 \cdot \vec{v}_2 \Rightarrow 2\vec{v}_1 \cdot \vec{v}_2 = 0$$

Therefore,

$$\vec{v}_1 \cdot \vec{v}_2 = 0$$

This implies that $\vec{v}_1$ and $\vec{v}_2$ are perpendicular. Hence, the angle between the velocities of the two particles after collision is 90°.

Diagram:

This vector diagram shows $\vec{v}_0$ along the initial direction. After collision, $\vec{v}_1$ and $\vec{v}_2$ are at right angles to each other, completing the vector triangle $\vec{v}_1 + \vec{v}_2 = \vec{v}_0$.

Conclusion:

In an elastic, non-relativistic collision between two equal masses where one is initially at rest, the two particles scatter at 90° to each other. This result follows directly from conservation of momentum and kinetic energy.

3 If the forces acting on a particle are conservative, show that the total energy of the particle which is the sum of the kinetic and potential energies is conserved.

Introduction:

We are to show that if a particle is subjected only to conservative forces, then its total mechanical energy—defined as the sum of its kinetic energy (K) and potential energy (U)—remains constant over time. A conservative force $\vec{F}$ satisfies $\vec{F} = -\nabla U$, where U is the potential energy function. The principle of energy conservation in this context stems directly from Newton's second law and the definition of conservative forces.

Solution:

Let the mass of the particle be m, its position vector be $\vec{r}(t)$, and velocity be $\vec{v} = \frac{d\vec{r}}{dt}$.

The kinetic energy is given by:

$$K = \frac{1}{2}mv^2$$

The time derivative of kinetic energy is:

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot m\frac{d\vec{v}}{dt}$$

From Newtonś second law:

$$m\frac{d\vec{v}}{dt} = \vec{F}$$

Therefore:

$$\frac{dK}{dt} = \vec{v}\cdot\vec{F}$$

If the force $\vec{F}$ is conservative, it can be written as:

$$\vec{F} = -\nabla U$$

Then:

$$\frac{dK}{dt} = \vec{v}\cdot(-\nabla U) = -\vec{v}\cdot\nabla U$$

The time derivative of the potential energy U is:

$$\frac{dU}{dt} = \nabla U\cdot\frac{d\vec{r}}{dt} = \nabla U\cdot\vec{v}$$

Therefore:

$$\frac{dK}{dt} = -\frac{dU}{dt}$$

Add both sides:

$$\frac{dK}{dt} + \frac{dU}{dt} = 0 \Rightarrow \frac{d}{dt}(K+U) = 0$$

This implies that the total energy $E = K + U$ is constant in time.

Conclusion:

We have shown that for a particle under the influence of conservative forces, the rate of change of kinetic energy is the negative of the rate of change of potential energy. Therefore, the total mechanical energy $E = K + U$ remains constant. This is the mathematical statement of the conservation of energy in conservative systems.

4 Discuss the problem of scattering of a charged particle by a Coulomb field. Hence, obtain an expression for Rutherford scattering cross-section. What is the importance of the above expression?

Introduction: The scattering of a beam of charged particles (e.g. protons or α-particles) by a heavy, positively charged nucleus is a central problem in both classical and quantum mechanics. Although the full treatment is quantum-mechanical, the classical analysis already reveals the key geometric and kinematic features of the process. In the classical picture, an incident particle of charge $+ze$ and mass m approaches a fixed nucleus of charge $+Ze$ with initial speed v_0 and straight-line trajectory at large distance. As it draws near, it experiences a repulsive Coulomb force, its path deviates from a straight line, and after passing by the nucleus it emerges again along a straight line in a new direction. The angle between the incident and outgoing trajectories is called the *scattering angle* ϕ. By analyzing the relationship between the impact parameter p and the scattering angle ϕ, one derives the *differential scattering cross-section* $d\sigma/d\Omega$, which for a purely Coulomb (inverse-square) force yields Rutherford's famous formula.

Scattering cross-section (general definitions): Let a uniform beam of particles move toward the scattering center (the nucleus) with flux (or intensity) I_0, defined as the number of particles crossing unit area per unit time, normal to the beam direction. We assume all incident particles have the same mass m and same kinetic energy $E = \frac{1}{2}\, m\, v_0^2$. Denote by $d\sigma$ the area element in the plane perpendicular to the beam (the "impact-parameter plane") such that any particle whose initial straight-line path falls within $d\sigma$ is scattered. Let $d\Omega$ be the solid angle about a particular scattering direction (i.e. angle ϕ measured from the original beam axis) into which these particles are deflected.

If $I(\Omega)$ is the number of particles scattered per unit time into the infinitesimal solid angle $d\Omega$ about direction Ω, then

$$I_0\, d\sigma \;=\; I(\Omega)\, d\Omega \quad\Longrightarrow\quad \frac{d\sigma}{d\Omega} \;=\; \frac{I(\Omega)}{I_0} \tag{3}$$

is called the *differential scattering cross-section* $\sigma(\Omega)$. In other words,

$$\sigma(\Omega) \;=\; \frac{d\sigma}{d\Omega} \;=\; \frac{\text{(number scattered per unit time into } d\Omega)}{\text{(incident flux } I_0)}.$$

The *total cross-section* is then

$$\sigma_{\text{total}} \;=\; \int_{\Omega} \sigma(\Omega)\, d\Omega \;=\; \int_{\Omega} \frac{d\sigma}{d\Omega}\, d\Omega\,.$$

Scattering angle ϕ and impact parameter p: By definition, the *scattering angle* ϕ is the angle between the incident direction and the outgoing direction of the particle after interaction. The *impact parameter* p is the perpendicular distance from the center of force (nucleus) to the initial, undeviated straight-line trajectory of the particle. Because the Coulomb force is central, the problem has axial symmetry about the beam axis (XX'), and all particles with the same p are scattered by the same angle ϕ.

In spherical coordinates about the beam axis, the solid angle element is

$$d\Omega \;=\; 2\pi\, \sin\phi\, d\phi\,,$$

since there is full 2π azimuthal symmetry and $\phi \in [0, \pi]$ is the polar (scattering) angle.

On the other hand, particles in the incident beam whose impact parameters lie between p and $p + dp$ occupy an annular ring of area

$$d\sigma \;=\; 2\pi\, p\, dp \quad \text{in the impact-parameter plane.}$$

All those particles with impact in $[p, p + dp]$ are scattered into polar angles in $[\phi, \phi + d\phi]$. Hence, the number of incident particles per unit time in that annular ring is

$$I_0\, d\sigma \;=\; I_0\, (2\pi\, p\, dp)\,.$$

At the same time, the number of particles scattered per unit time into the corresponding solid angle $d\Omega = 2\pi\, \sin\phi\, d\phi$ is

$$I(\Omega)\, d\Omega \;=\; I(\Omega)\, (2\pi\, \sin\phi\, d\phi).$$

Since each ring of impact parameter p corresponds one-to-one with the scattering angle ϕ, we set

$$I_0\, (2\pi\, p\, dp) \;=\; I(\Omega)\, (2\pi\, \sin\phi\, d\phi).$$

But from (3) we have $I(\Omega) = I_0\, \sigma(\phi)$ (here $\sigma(\phi) = \sigma(\Omega)$ with Ω determined by ϕ). Therefore,

$$I_0\, (2\pi\, p\, dp) \;=\; I_0\, \sigma(\phi)\, (2\pi\, \sin\phi\, d\phi) \;\Longrightarrow\; 2\pi\, p\, dp \;=\; 2\pi\, \sigma(\phi)\, \sin\phi\, d\phi.$$

Solving for $\sigma(\phi)$ yields

$$\sigma(\phi) \;=\; \frac{p}{\sin\phi}\, \left|\frac{dp}{d\phi}\right|. \tag{4}$$

A minus sign sometimes appears if one writes $dp/d\phi$ as negative (since p decreases as ϕ increases), but the absolute-value form above is most common. Equation (4) is the general result for *any* central potential: once you know the functional relation $p(\phi)$, you substitute into (4) to obtain the differential cross-section $\sigma(\phi)$.

Coulomb scattering: determination of $p(\phi)$: In the special case of a repulsive Coulomb potential

$$V(r) \;=\; \frac{Z\, z\, e^2}{4\pi\, \varepsilon_0\, r},$$

one must integrate the orbit equation or use conservation laws to find p as a function of ϕ. A standard derivation (by solving the orbitś differential equation in polar form) shows that the trajectory is a hyperbola whose eccentricity e satisfies

$$e \;=\; \frac{1}{\sin\!\left(\frac{\phi}{2}\right)} \quad\Longrightarrow\quad \cot\!\left(\frac{\phi}{2}\right) \;=\; \sqrt{e^2 - 1} \;=\; \frac{2\, E\, p\, (4\pi\, \varepsilon_0)}{Z\, z\, e^2},$$

where $E = \frac{1}{2}\, m\, v_0^2$ is the incident kinetic energy. Hence one finds

$$p(\phi) \;=\; \frac{Z\, z\, e^2}{4\pi\, \varepsilon_0\, 2\, E}\; \cot\!\left(\frac{\phi}{2}\right) \;=\; \frac{Z\, z\, e^2}{8\pi\, \varepsilon_0\, E}\; \cot\!\left(\frac{\phi}{2}\right).$$

Differentiating with respect to ϕ gives

$$\frac{dp}{d\phi} = \frac{Z\,z\,e^2}{8\pi\,\varepsilon_0\,E} \cdot \frac{d}{d\phi}\Big[\cot\big(\tfrac{\phi}{2}\big)\Big] = \frac{Z\,z\,e^2}{8\pi\,\varepsilon_0\,E}\left(-\tfrac{1}{2}\right)\csc^2\!\Big(\tfrac{\phi}{2}\Big) = -\,\frac{Z\,z\,e^2}{16\pi\,\varepsilon_0\,E}\,\csc^2\!\Big(\tfrac{\phi}{2}\Big).$$

Taking the absolute value,

$$\left|\frac{dp}{d\phi}\right| = \frac{Z\,z\,e^2}{16\pi\,\varepsilon_0\,E}\,\csc^2\!\Big(\tfrac{\phi}{2}\Big).$$

Substitute $p(\phi)$ and $|dp/d\phi|$ into the general formula (4):

$$\sigma(\phi) = \frac{p(\phi)}{\sin\phi}\left|\frac{dp}{d\phi}\right| = \frac{1}{\sin\phi}\left[\tfrac{Z\,z\,e^2}{8\pi\,\varepsilon_0\,E}\,\cot\big(\tfrac{\phi}{2}\big)\right] \times \left[\tfrac{Z\,z\,e^2}{16\pi\,\varepsilon_0\,E}\,\csc^2\big(\tfrac{\phi}{2}\big)\right].$$

Use the identity $\sin\phi = 2\,\sin\big(\frac{\phi}{2}\big)\cos\big(\frac{\phi}{2}\big)$ and $\cot\big(\frac{\phi}{2}\big) = \cos\big(\frac{\phi}{2}\big)/\sin\big(\frac{\phi}{2}\big)$ to simplify:

$$\frac{\cot(\frac{\phi}{2})}{\sin\phi} = \frac{\cos(\frac{\phi}{2})}{\sin(\frac{\phi}{2})}\cdot\frac{1}{2\,\sin(\frac{\phi}{2})\,\cos(\frac{\phi}{2})} = \frac{1}{2\,\sin^2(\frac{\phi}{2})\,\sin(\frac{\phi}{2})\,\cos(\frac{\phi}{2})} = \frac{1}{2\,\sin^3(\frac{\phi}{2})\,\cos(\frac{\phi}{2})}.$$

Multiplying by $\csc^2(\frac{\phi}{2}) = 1/\sin^2(\frac{\phi}{2})$ yields

$$\frac{\cot(\frac{\phi}{2})\,\csc^2(\frac{\phi}{2})}{\sin\phi} = \frac{1}{2\,\sin^5(\frac{\phi}{2})\,\cos(\frac{\phi}{2})}.$$

Thus

$$\sigma(\phi) = \left(\tfrac{Z\,z\,e^2}{8\pi\,\varepsilon_0\,E}\right)\left(\tfrac{Z\,z\,e^2}{16\pi\,\varepsilon_0\,E}\right) \times \frac{1}{2\,\sin^5(\frac{\phi}{2})\,\cos(\frac{\phi}{2})} = \frac{(Z\,z\,e^2)^2}{256\,\pi^2\,\varepsilon_0^2\,E^2}\,\frac{1}{\sin^4(\frac{\phi}{2})}.$$

Equivalently, one writes the famous *Rutherford differential scattering cross-section* as

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{(Z\,z\,e^2)^2}{256\,\pi^2\,\varepsilon_0^2\,E^2}\,\frac{1}{\sin^4(\frac{\phi}{2})} = \left(\frac{Z\,z\,e^2}{16\,\pi\,\varepsilon_0\,E}\right)^2\frac{1}{\sin^4(\frac{\phi}{2})}.} \tag{5}$$

Importance of the Rutherford formula:

1. *Evidence for a compact nuclear charge*. Rutherford's $\sin^{-4}(\phi/2)$ law agreed precisely with the α-particle scattering experiments of 1911. The observed angular distribution could only be explained if (i) the positive charge of the atom is concentrated in a very small nucleus, and (ii) the interatomic potential at distances $\sim 10^{-14}$–10^{-12} m is purely Coulombic $\propto 1/r$. This discovery overturned the "plum-pudding" model and established the nuclear model of the atom.

2. *Dependence on charges and energy*. Equation (5) shows

$$\frac{d\sigma}{d\Omega} \propto (Z\,z)^2\,\frac{1}{E^2}\,\frac{1}{\sin^4(\phi/2)},$$

so by varying the projectile energy E or comparing targets of different nuclear charge Z, one can extract information about nuclear charge and size.

3. *Forward-peaked scattering*. Because $\sin^{-4}(\phi/2)$ diverges as $\phi \to 0$, most particles are scattered at very small angles ("halo" of forward scattering). In practice, detectors must be positioned at small ϕ to measure the bulk of scattering events.

4. *Benchmark for more complex interactions*. The Rutherford formula is the classical benchmark for pure Coulomb scattering. Deviations at small angles indicate atomic-electron screening; deviations at large angles indicate finite nuclear size or non-Coulomb forces.

5. *Foundation for quantum-mechanical scattering*. In quantum theory, the Rutherford result emerges in the Born approximation for high incident energies. Thus, Rutherford scattering underpins much of modern nuclear and particle-physics scattering analysis.

Conclusion: By analyzing a beam of charged particles scattered by a fixed nucleus via the Coulomb force, one finds a hyperbolic trajectory whose eccentricity satisfies $e = 1/\sin(\phi/2)$. Using the relation $p(\phi) = (Z\,z\,e^2/(8\pi\,\varepsilon_0\,E))\,\cot(\phi/2)$ and substituting into the general formula $\sigma(\phi) = (p/\sin\phi)\,|dp/d\phi|$, one obtains Rutherford's celebrated differential cross-section (equation (5)). Its perfect $\sin^{-4}(\phi/2)$ angular dependence and $(Z\,z)^2/E^2$ scaling provided the first direct evidence for a small, highly charged nucleus and set the stage for modern scattering theory.

5 Write down precisely the conservation theorems for energy, linear momentum, and angular momentum of a particle with their mathematical forms.

Introduction: In classical mechanics, the conservation theorems for energy, linear momentum, and angular momentum describe fundamental symmetries of nature and are pivotal in analyzing the motion of particles. These theorems are valid under specific conditions where corresponding external forces or torques are absent or conservative in nature. Below, we present each conservation law along with its precise mathematical expression.

Solution:

Conservation of Energy:

If the net force acting on a particle is conservative, the total mechanical energy (sum of kinetic and potential energies) of the particle remains constant over time.

Let T denote kinetic energy and U the potential energy of the particle. Then the total mechanical energy is

$$E = T + U.$$

The conservation law states:

$$\frac{dE}{dt} = \frac{d}{dt}(T + U) = 0.$$

Alternatively,

$$E = \text{constant}.$$

Conservation of Linear Momentum:

If the net external force acting on a particle is zero, the linear momentum $\vec{p}$ of the particle remains constant in time.

Let $\vec{p} = m\vec{v}$, where m is the mass and $\vec{v}$ is the velocity of the particle. Then,

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} = 0 \quad \Rightarrow \quad \vec{p} = \text{constant}.$$

Conservation of Angular Momentum:

If the net external torque acting on a particle about a fixed point (or axis) is zero, the angular momentum $\vec{L}$ of the particle about that point (or axis) remains constant.

Angular momentum of a particle with respect to point O is given by

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v},$$

where $\vec{r}$ is the position vector of the particle relative to point O.

The torque about point O is

$$\vec{\tau} = \frac{d\vec{L}}{dt}.$$

Therefore, if $\vec{\tau}_{\text{net}} = 0$, then

$$\frac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant}.$$

Conclusion:

The conservation theorems state that, in absence of non-conservative forces, external forces, or external torques respectively:

1. Total mechanical energy $(T + U)$ remains constant.
2. Linear momentum $(\vec{p})$ remains constant if $\vec{F}_{\text{net}} = 0$.
3. Angular momentum $(\vec{L})$ remains constant if $\vec{\tau}_{\text{net}} = 0$.

These principles are fundamental tools in solving a wide range of problems in mechanics and underlie the symmetries of physical laws.

6 Show that the differential scattering cross-section can be expressed as $\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|$, where s is the impact parameter and θ is the scattering angle.

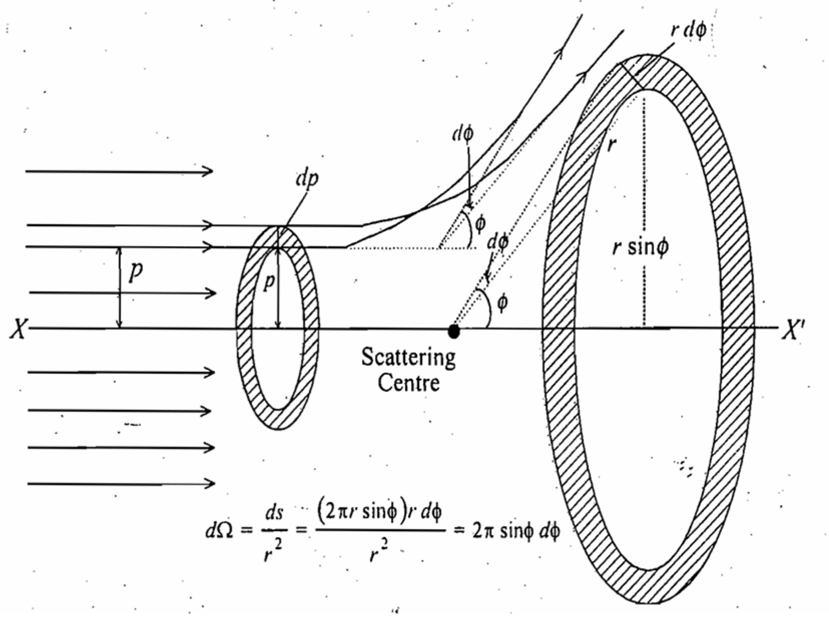
Introduction: In classical scattering theory, the differential scattering cross-section $\sigma(\theta)$ quantifies the likelihood that an incoming particle is scattered into a solid angle element $d\Omega$ centered around scattering angle θ. The relationship between the impact parameter s and the scattering angle θ helps us derive this expression. We assume an axially symmetric scattering center such that azimuthal symmetry around the beam axis holds.

Solution:

Consider an incident beam of particles with uniform flux. The number of particles scattered into a solid angle $d\Omega$ about θ equals the number of particles whose impact parameters lie between s and $s + ds$.

The ring of impact parameters between s and $s + ds$ corresponds to an area element:

$$dA = 2\pi s\, ds.$$



On the other hand, the number of particles scattered into solid angle $d\Omega = 2\pi \sin\theta\, d\theta$ is given by:

$$dN = I \cdot \sigma(\theta)\, d\Omega,$$

where I is the incident flux.

Equating the two expressions for the number of scattered particles:

$$I \cdot 2\pi s\, ds = I \cdot \sigma(\theta) \cdot 2\pi \sin\theta\, d\theta.$$

Canceling I and 2π from both sides:

$$s\, ds = \sigma(\theta) \sin\theta\, d\theta.$$

Solving for $\sigma(\theta)$:

$$\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|.$$

We use the absolute value since the function $s(\theta)$ may be decreasing with θ, and cross-section is physically non-negative.

Conclusion:

The differential scattering cross-section for a central force field is given by

$$\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|,$$

where s is the impact parameter and θ is the scattering angle. This result connects the spatial distribution of incident particles with the angular distribution of scattered particles.

7 (i) The distance between the centres of the carbon and oxygen atoms in the carbon monoxide (CO) gas molecule is 1.130×10^{-10} m. Locate the centre of mass of the molecule relative to the carbon atom.

Introduction:

We are given the internuclear distance between the carbon and oxygen atoms in a CO molecule as $d = 1.130 \times 10^{-10}$ m. The objective is to calculate the position of the center of mass (COM) of the molecule relative to the carbon atom, assuming a one-dimensional configuration along the molecular axis. Let m_C and m_O denote the atomic masses of carbon and oxygen, respectively.

Solution:

Assume the carbon atom is at position $x = 0$ and the oxygen atom is at position $x = d$. The center of mass of a two-particle system is given by:

$$x_{\text{COM}} = \frac{m_C \cdot 0 + m_O \cdot d}{m_C + m_O} = \frac{m_O \cdot d}{m_C + m_O}.$$

Substituting the approximate atomic masses in unified atomic mass units:

$$m_C = 12\,\text{u}, \quad m_O = 16\,\text{u},$$

we obtain:

$$x_{\text{COM}} = \frac{16}{12+16} \cdot 1.130 \times 10^{-10}\,\text{m} = \frac{16}{28} \cdot 1.130 \times 10^{-10}\,\text{m}.$$

Simplifying:

$$x_{\text{COM}} = \frac{4}{7} \cdot 1.130 \times 10^{-10}\,\text{m} = 0.5714 \cdot 1.130 \times 10^{-10}\,\text{m}.$$

$$x_{\text{COM}} \approx 6.448 \times 10^{-11}\,\text{m}.$$

Conclusion:

The center of mass of the CO molecule lies approximately 6.448×10^{-11} m from the carbon atom toward the oxygen atom along the molecular axis.

(ii) Find the centre of mass of a homogeneous semicircular plate of radius a.

Introduction: We are asked to determine the position of the center of mass (COM) of a homogeneous semicircular plate of radius a. Since the plate is homogeneous, its mass distribution is uniform. Due to symmetry, the center of mass must lie along the vertical axis passing through the center of the circle (i.e., the y-axis). Therefore, we only need to compute the y-coordinate of the center of mass. The x-coordinate will be zero by symmetry.

Solution: Let us place the semicircular plate in the xy-plane such that its flat edge lies along the x-axis, and the curved part lies in the upper half-plane ($y \geq 0$). The equation of the semicircle is:

$$x^2 + y^2 = a^2, \quad y \geq 0$$

The center of mass coordinates for a two-dimensional plate are given by:

$$\bar{x} = \frac{1}{A} \iint_{\text{plate}} x \, dA, \quad \bar{y} = \frac{1}{A} \iint_{\text{plate}} y \, dA$$

Since the plate is symmetric about the y-axis, $\bar{x} = 0$. We now compute $\bar{y}$.

We switch to polar coordinates: Let

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dA = r \, dr \, d\theta$$

For a semicircle of radius a in the upper half-plane, the limits are:

$$0 \leq r \leq a, \quad 0 \leq \theta \leq \pi$$

The total area of the semicircular plate is:

$$A = \frac{1}{2}\pi a^2$$

Now, compute the y-coordinate of the center of mass:

$$\bar{y} = \frac{1}{A} \iint_{\text{plate}} y \, dA = \frac{1}{A} \int_0^\pi \int_0^a (r\sin\theta) \cdot r \, dr \, d\theta$$

Simplify the integrand:

$$\bar{y} = \frac{1}{A} \int_0^\pi \sin\theta \int_0^a r^2 \, dr \, d\theta$$

Compute the inner integral:

$$\int_0^a r^2 \, dr = \left[\frac{r^3}{3}\right]_0^a = \frac{a^3}{3}$$

Now compute the outer integral:

$$\int_0^{\pi} \sin\theta \, d\theta = [-\cos\theta]_0^{\pi} = -\cos\pi + \cos 0 = 2$$

Putting it all together:

$$\bar{y} = \frac{1}{A} \cdot \left(2 \cdot \frac{a^3}{3}\right) = \frac{2a^3}{3A}$$

Recall $A = \frac{1}{2}\pi a^2$, so:

$$\bar{y} = \frac{2a^3}{3 \cdot \frac{1}{2}\pi a^2} = \frac{4a}{3\pi}$$

Conclusion: The center of mass of a homogeneous semicircular plate of radius a lies on the vertical axis of symmetry at a height of $\boxed{\frac{4a}{3\pi}}$ above the flat edge. Hence, the coordinates of the center of mass are $\boxed{(0, \frac{4a}{3\pi})}$.

8 A diatomic molecule can be considered to be made up of two masses m_1 and m_2 separated by a fixed distance r. Derive a formula for the distance of centre of mass, C, from mass m_1. Also show that the moment of inertia about an axis through C and perpendicular to r is μr^2 where $\mu = \frac{m_1 m_2}{m_1+m_2}$.

Introduction: We are given a diatomic molecule composed of two point masses, m_1 and m_2, separated by a fixed distance r. We are to derive:

1. The distance of the center of mass (denoted C) from the mass m_1.
2. The moment of inertia about an axis perpendicular to the line joining the masses and passing through the center of mass.

We assume the system lies along the x-axis, with m_1 at $x = 0$ and m_2 at $x = r$.

Solution:

Let the distance of the center of mass from m_1 be x_C. By the definition of center of mass for point masses:

$$x_C = \frac{m_1 \cdot 0 + m_2 \cdot r}{m_1 + m_2} = \frac{m_2 r}{m_1 + m_2}$$

So, the center of mass is located at a distance

$$\boxed{x_C = \frac{m_2 r}{m_1 + m_2}}$$

from mass m_1.

Next, we compute the moment of inertia about an axis through the center of mass and perpendicular to the line joining the masses.

Let I be this moment of inertia. The perpendicular distances of the two masses from the center of mass are:

1. For m_1: $x_C = \frac{m_2 r}{m_1 + m_2}$
2. For m_2: $r - x_C = r - \frac{m_2 r}{m_1 + m_2} = \frac{m_1 r}{m_1 + m_2}$

Hence, the moment of inertia is:

$$I = m_1 x_C^2 + m_2 (r - x_C)^2 = m_1 \left(\frac{m_2 r}{m_1 + m_2} \right)^2 + m_2 \left(\frac{m_1 r}{m_1 + m_2} \right)^2$$

Factor out r^2:

$$I = r^2 \left[m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 \right]$$

Simplify the expression inside the brackets:

$$I = r^2 \cdot \frac{1}{(m_1+m_2)^2}\left[m_1 m_2^2 + m_2 m_1^2\right] = r^2 \cdot \frac{m_1 m_2 (m_1+m_2)}{(m_1+m_2)^2}$$

Cancel one factor of $(m_1 + m_2)$:

$$I = r^2 \cdot \frac{m_1 m_2}{m_1+m_2}$$

Thus, the moment of inertia about the center of mass is:

$$\boxed{I = \mu r^2 \quad \text{where} \quad \mu = \frac{m_1 m_2}{m_1+m_2}}$$

Conclusion: The distance of the center of mass from mass m_1 is $\frac{m_2 r}{m_1+m_2}$. The moment of inertia about an axis perpendicular to the line joining the two masses and passing through the center of mass is μr^2, where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass.

9 A ball moving with a speed of 9 m/s strikes an identical stationary ball such that after the collision the direction of each ball makes an angle 30° with the original line of motion. Find the speed of the balls after the collision. Is the kinetic energy conserved in this collision?

Introduction: We are given that a moving ball with speed $v = 9\,\text{m/s}$ collides with an identical stationary ball. After the collision, both balls move at an angle of 30° to the original direction of motion of the moving ball.

We are to determine:

1. The speed of each ball after the collision.
2. Whether kinetic energy is conserved.

Assumptions:

1. The balls are identical, so their masses are equal (let mass be m).
2. The motion occurs in a frictionless, two-dimensional plane.

Solution:

Let both balls have mass m. Let the initial velocity of the first ball be $\vec{v}_1 = 9\hat{i}$. The second ball is initially stationary, so its initial velocity is $\vec{v}_2 = 0$. After the collision, let the speeds of the two balls be v_1' and v_2'. The problem states that "the direction of each ball makes an angle 30° with the original line of motion." For momentum conservation in the y-direction, one ball must go at +30° and the other at −30° with respect to the initial line of motion (the x-axis). Given that the balls are identical and the problem implies a symmetric outcome, it is reasonable to assume $v_1' = v_2' = v'$.

Initial momentum:
In x-direction: $P_{x,\text{initial}} = mv_1 + mv_2 = m \cdot 9 + m \cdot 0 = 9m$

In y-direction: $P_{y,\text{initial}} = 0$

After the collision, let the velocity of the first ball be $\vec{v'}_1$ at an angle of +30° with the x-axis, and the velocity of the second ball be $\vec{v'}_2$ at an angle of −30° with the x-axis. We assume $v_1' = v_2' = v'$.

Final momentum components:
x-direction:

$$P_{x,\text{final}} = mv' \cos(30^\circ) + mv' \cos(-30^\circ) = mv' \cos(30^\circ) + mv' \cos(30^\circ) = 2mv' \cos(30^\circ)$$

y-direction:

$$P_{y,\text{final}} = mv' \sin(30^\circ) + mv' \sin(-30^\circ) = mv' \sin(30^\circ) - mv' \sin(30^\circ) = 0$$

The y-component of momentum is conserved.

By conservation of linear momentum in the x-direction:

$$P_{x,\text{initial}} = P_{x,\text{final}}$$

$$9m = 2mv' \cos(30^\circ)$$

Cancel m from both sides:

$$9 = 2v' \cos(30^\circ)$$

Substitute the value of $\cos(30^\circ) = \frac{\sqrt{3}}{2}$:

$$9 = 2v' \cdot \frac{\sqrt{3}}{2}$$

$$9 = v'\sqrt{3}$$

Solving for v':

$$v' = \frac{9}{\sqrt{3}} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}\,\text{m/s}$$

So, the speed of each ball after the collision is $\boxed{3\sqrt{3}\,\text{m/s}}$.

Now we check for conservation of kinetic energy.

Initial kinetic energy (KE_{initial}):

$$KE_{\text{initial}} = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{1}{2}m(9)^2 + \frac{1}{2}m(0)^2 = \frac{81}{2}m = 40.5m$$

Final kinetic energy (KE_{final}): Since both balls have speed $v' = 3\sqrt{3}\,\text{m/s}$:

$$KE_{\text{final}} = \frac{1}{2}m(v')^2 + \frac{1}{2}m(v')^2 = 2 \cdot \frac{1}{2}m(3\sqrt{3})^2 = m(3^2 \cdot (\sqrt{3})^2) = m(9 \cdot 3) = 27m$$

Comparing KE_{initial} and KE_{final}:
$KE_{\text{initial}} = 40.5m$
$KE_{\text{final}} = 27m$

Since $KE_{\text{final}} \neq KE_{\text{initial}}$ ($27m \neq 40.5m$), the kinetic energy is NOT conserved in this collision. Therefore, the collision is inelastic.

Conclusion: After the collision, each ball moves with speed $\boxed{3\sqrt{3}\,\text{m/s}}$ at an angle of 30° to the original direction. The kinetic energy is **not conserved**, indicating that the collision is inelastic.

10 (i) If a particle of mass m is in a central force field $f(r)\hat{r}$, then show that its path must be a plane curve, where $\hat{r}$ is a unit vector in the direction of position vector $\vec{r}$.

Introduction: We are asked to show that the motion of a particle of mass m under the influence of a central force $\vec{F} = f(r)\hat{r}$ lies in a plane. Here, $f(r)$ is a scalar function depending only on the distance $r = |\vec{r}|$ from a fixed point (typically the origin), and $\hat{r}$ is the radial unit vector in the direction of the position vector $\vec{r}$. This is a general result in classical mechanics concerning central forces.

Solution:

Let $\vec{r}(t)$ be the position vector of the particle at time t, and let $\vec{F}$ be the force acting on the particle. Then Newton's second law gives:

$$m\ddot{\vec{r}} = \vec{F} = f(r)\hat{r}$$

The defining property of a central force is that it is directed along the radial direction $\hat{r}$, i.e., it has no component perpendicular to $\vec{r}$.

Let us define the angular momentum of the particle about the origin:

$$\vec{L} = \vec{r} \times m\vec{v} = m\vec{r} \times \dot{\vec{r}}$$

Differentiate $\vec{L}$ with respect to time:

$$\frac{d\vec{L}}{dt} = m\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = m(\dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}) = m\vec{r} \times \ddot{\vec{r}}$$

Since $\vec{F} = m\ddot{\vec{r}} = f(r)\hat{r}$ is parallel to $\vec{r}$, the cross product $\vec{r} \times \ddot{\vec{r}} = 0$.

Therefore:

$$\frac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant vector}$$

This shows that angular momentum is conserved in both magnitude and direction.

Since $\vec{L}$ is constant in direction, it defines a fixed plane perpendicular to $\vec{L}$ in which both $\vec{r}$ and $\vec{v}$ must lie for all time (because $\vec{r} \times \vec{v}$ remains parallel to $\vec{L}$).

Hence, the motion of the particle is confined to a fixed plane.

Conclusion: The angular momentum vector $\vec{L}$ of the particle remains constant in a central force field, implying that the motion is confined to a plane perpendicular to $\vec{L}$. Thus, the path of a particle under a central force $\vec{F} = f(r)\hat{r}$ must lie in a plane.

(ii) A block of mass m having negligible dimension is sliding freely in x-direction with velocity $\vec{v} = v\hat{i}$ as shown in the diagram. What is its angular momentum $\vec{L}_O$ about origin O and its angular momentum $\vec{L}_A$ about the point A on y-axis?

Introduction: We are given a point-like block of mass m moving with a velocity $\vec{v} = v\hat{i}$ along the x-axis.

We are to determine its angular momentum:

1. $\vec{L}_O$ about the origin O.
2. $\vec{L}_A$ about a point A located on the y-axis.

Assume the position of the particle at a particular instant is $\vec{r} = x\hat{i} + y\hat{j}$. The general definition of angular momentum of a particle about a point P is:

$$\vec{L}_P = \vec{r}_P \times \vec{p}$$

where $\vec{r}_P$ is the position vector of the particle relative to point P, and $\vec{p} = m\vec{v}$ is its linear momentum.

Solution:

Angular momentum about the origin O:

Let the position vector of the particle be:

$$\vec{r}_O = x\hat{i} + y\hat{j}$$

The velocity is $\vec{v} = v\hat{i}$, so the momentum is:

$$\vec{p} = mv\hat{i}$$

Now compute $\vec{L}_O$:

$$\vec{L}_O = \vec{r}_O \times \vec{p} = (x\hat{i} + y\hat{j}) \times (mv\hat{i}) = x\hat{i} \times mv\hat{i} + y\hat{j} \times mv\hat{i}$$

Recall cross product identities:

- $\hat{i} \times \hat{i} = 0$
- $\hat{j} \times \hat{i} = -\hat{k}$

Then:

$$\vec{L}_O = 0 + ymv(-\hat{k}) = -mvy\hat{k}$$

Angular momentum about point A on the y-axis:

Let point A have coordinates $(0, y_0)$. The position vector of the particle relative to point A is:

$$\vec{r}_A = \vec{r} - \vec{r}_A^{\,\text{(position)}} = (x\hat{i} + y\hat{j}) - (0\hat{i} + y_0\hat{j}) = x\hat{i} + (y - y_0)\hat{j}$$

Then compute:
$$\vec{L}_A = \vec{r}_A \times \vec{p} = (x\hat{i} + (y - y_0)\hat{j}) \times (mv\hat{i})$$

Again,

$$\vec{L}_A = x\hat{i} \times mv\hat{i} + (y - y_0)\hat{j} \times mv\hat{i} = 0 + (y - y_0)(-mv\hat{k}) = -mv(y - y_0)\hat{k}$$

Conclusion: The angular momenta of the block are:

- About origin O: $\boxed{\vec{L}_O = -mvy\,\hat{k}}$
- About point A on the y-axis at height y_0: $\boxed{\vec{L}_A = -mv(y - y_0)\,\hat{k}}$

In both cases, the angular momentum vector is directed along the z-axis.