# Resource estimation in quantum metrology

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## 1 Introduction

Quantum metrology investigates the means of achieving exceptionally high-resolution and sensitive measurements of physical parameters by leveraging quantum theory to describe the underlying systems. This discipline particularly emphasizes the utilization of quantum entanglement and quantum squeezing. Its objective is to advance measurement techniques to achieve superior precision compared to classical methods. When measuring a physical quantity, it's crucial to assess the accuracy of our measurements, typically through metrics like Mean Square Error (MSE). However, directly computing the exact MSE is often challenging because it relies on knowing the true value of the quantity being measured. Surprisingly, we can establish lower bounds on the MSE, ensuring that for any given experiment, the error cannot fall below a certain calculated threshold. Moreover, we can identify conditions where this lower bound is achieved, effectively minimizing the MSE. In parameter estimation theory, we delve into the probability distributions associated with observed data given the physical quantity's value. As these distributions shift across the probability space with changes in the physical quantity, our ability to detect these variations depends on the magnitude of these shifts. When distributions undergo significant changes, our measurements become more sensitive to alterations in the physical quantity compared to scenarios where distributions remain relatively stable. This establishes a crucial connection between measurement sensitivity, reflected in small MSE values, and the choice of distance functions within the probability space. Our objective is to identify a distance measure that quantitatively links to MSE, thus providing a metric in the probability space known as the Fisher information. We will examine essential properties of the Fisher information.

Expanding into quantum mechanics necessitates considering density operators instead of classical probability distributions. These operators, residing in a linear vector space, allow the construction of distance measures connecting them to minimal MSE in physical experiments. However, due to the richer structure of density operators, including non-commutative operators, the corresponding distance measures become more intricate. Depending on the chosen inner product in the density operator space, various distance measures can be crafted, each with nuanced physical interpretations.

The direct extension of the Fisher information, known as the quantum Fisher information, serves as a well-explored quantity that sets a limit on the achievable precision using resources available through quantum mechanics such as quantum entanglement, non-linearity, quantum squeezing etc. In this report we will explain Shot-noise sensitivity limit which is achieved with classical parameter estimation protocols and then we will explain Heisenberg sensitivity limit which uses resources like entanglement to improve sensitivity. Finally we will compare the resource estimation for both protocols as calculated by Microsoft resource estimator.

## 2 Estimation theory

In this section, we provide a concise overview of classical and quantum estimation theory[1]. We introduce fundamental concepts such as the expectation value and variance, which quantify the disparity between measured and true values. We establish a set of generic lower bounds that restrict the variance of parameter estimates, ultimately leading to the renowned Cramér-Rao bound. For a more comprehensive introduction to classical estimation theory, readers may refer to Kay (1993)[2]. Various aspects of estimation theory become apparent when we contemplate the following heuristic reasoning[3]: When we perform a measurement of an observable $M$, whose outcomes are contingent on a parameter $\theta$, we can attribute an estimate and an error to $\theta$.

$$\delta\theta = \frac{\Delta M}{|\partial_\theta \langle M \rangle|}, \tag{1}$$

where $\partial_\theta = \partial/\partial\theta$, and $\Delta M = \left[\left\langle M^2 \right\rangle - \langle M \rangle^2\right]^{\frac{1}{2}}$ is the standard deviation in the measurement outcomes. The denominator $|\partial_\theta \langle M \rangle|$ can be viewed as a local correction in the units of $\delta\theta$. If the metrology scheme is done with the measurement of $R$ resources, then the variance and error in $\theta$ is given by

$$\Delta M = \left[\sum_i^R \left\langle M_i^2 \right\rangle - \langle M_i \rangle^2\right]^{\frac{1}{2}} \text{ and } \delta\theta = \frac{\Delta M}{\left|\sum_i^R \partial_\theta \langle M_i \rangle\right|}. \tag{2}$$

It should be noted that the summation runs only over the resources that are measured, its because inference can only be drawn from measured systems.

### 2.1 Shot-noise sensitivity limit

The shot-noise limit arises due to the inherent quantum fluctuations in the detection of particles or photons. It sets a lower bound on the uncertainty in measurements, dictating that even under ideal conditions, the precision of measurements cannot surpass a certain threshold determined by the statistical nature of particle or photon detection. We perform the simulation by starting with states $|0\rangle$ and rotating it to equal superposition with Hadamard gate, followed by parameterization with $U(\theta) = e^{i\theta\sigma_z}$. The state is then rotated back with Hadamard gate to have parameter information in measurement statistics. The system is then measured in $Z$-basis($M = \sigma_z$) and same protocol is followed with every qubits as shown in Figure 1.

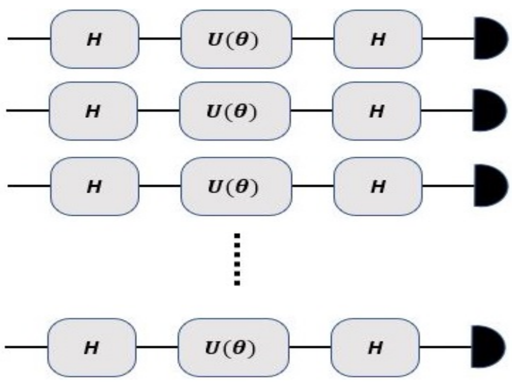


Figure 1: Experimental setup to obtain shot-noise sensitivity limit

The expectation value for $Z$-measurement for each qubit for this protocol will be given as $\langle M_i \rangle = \cos(\theta)$. Using Eqn(2) we can calculate the sensitivity as follows

$$\delta\theta = \frac{\left[\sum_i^R \langle M_i^2 \rangle - \langle M_i \rangle^2\right]^{\frac{1}{2}}}{\left|\sum_i^R \partial_\theta \langle M_i \rangle\right|} = \frac{\sqrt{R}\sin\theta}{|R\partial_\theta \cos\theta|} = \frac{1}{\sqrt{R}}. \tag{3}$$

The $\delta\theta$ decreases with $O(1/\sqrt{R})$, hence our sensitivity increases as $O(\sqrt{R})$. It should be noted that this metrology scheme do not uses any quantum phenomena and hence the result can be replicated with classical resources. In next section we will see what is the effect on sensitivity when we use quantum entanglement.

## 2.2 Heisenberg sensitivity limit

While classical metrology techniques have constraints, quantum metrology offers the potential to approach or surpass the shot-noise limit through the exploitation of quantum phenomena such as entanglement and squeezing. The Heisenberg limit originates from the Heisenberg uncertainty principle, a fundamental tenet of quantum mechanics. This principle states that the precision of simultaneous measurements of certain pairs of observables, such as position and momentum, is fundamentally constrained. In the context of quantum metrology, the Heisenberg limit manifests as an ultimate bound on the achievable precision in measurements of complementary observables. Remarkably, quantum metrology protocols can sometimes reach or even surpass the Heisenberg limit by harnessing quantum entanglement and employing sophisticated measurement techniques.

We perform the simulation by starting with states $|0\rangle$ and rotating only the last qubit to equal superposition state with Hadamard gate. We then use that qubit as control and entangle it with all qubits using CNOT gates followed by parameterization with $U(\theta) = (e^{i\theta\sigma_z})^{\otimes R}$. After phase accusation we do the phase-kickback and push information of phase acquired by all qubits to the last qubit. The state is then rotated with Hadamard gate to have parameter information in measurement statistics. The system is then measured in $Z$-basis($M = \sigma_z$) as shown in Figure 2.

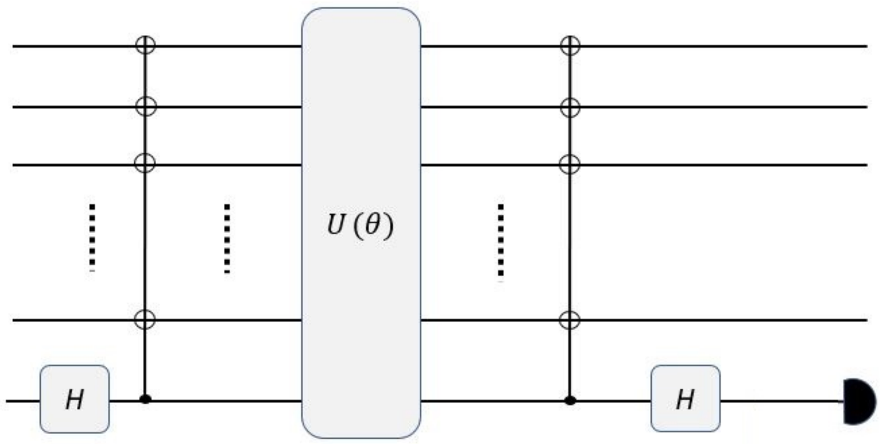


Figure 2: Experimental setup to obtain Heisenberg sensitivity limit

The expectation value for $Z$-measurement is given as $\langle M_i \rangle = \cos(R\theta)$ hence the sensitivity can be calculated as

$$\delta\theta = \frac{\left[\langle M_i^2 \rangle - \langle M_i \rangle^2\right]^{\frac{1}{2}}}{|\partial_\theta \langle M_i \rangle|} = \frac{\sin(R\theta)}{|\partial_\theta \cos(R\theta)|} = \frac{1}{R}. \tag{4}$$

The $\delta\theta$ decreases with $O(1/R)$, hence our sensitivity increases as $O(R)$. Hence using quantum entanglement between qubits one can get better sensitivity even after using same number of logical qubit resources as classical metrology.

## 3 Resource estimation

We simulate the experiment to obtain shot-noise sensitivity limit and Heisenberg sensitivity limit with 25 qubits each. To get the statistics correct we measured qubits for 100,000 times. The angle with which the state was rotated was taken as $\theta = 1.0$ Rad. The resource estimation was done using Microsoft resource estimator for various experimental setup (given by qubitParams) and results are shown as follows.

In [25]:
```
result = qsharp.estimate(
    "Shotnoise("+str(no_of_qubits)+","+str(angle)+","+str(no_shots)+")",
    [
        {
        "qubitParams": { "name": "qubit_gate_ns_e3" }},
        {
        "qubitParams": { "name": "qubit_gate_ns_e4" }},
        {
        "qubitParams": { "name": "qubit_gate_us_e3" }},
        {
        "qubitParams": { "name": "qubit_gate_us_e4" }},
        {
        "qubitParams": { "name": "qubit_maj_ns_e4" },
        "qecScheme": { "name": "floquet_code" }},
        {
        "qubitParams": { "name": "qubit_maj_ns_e6" },
        "qecScheme": { "name": "floquet_code" }},
    ]
)
result[:]
```

Out[25]:

▼ Physical resource estimates

| Item | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Runtime | 1 mins | 32 secs | 1 days | 13 hours | 24 secs | 11 secs |
| rQOPS | 7.86M | 15.00M | 5.24k | 10.00k | 20.00M | 44.00M |
| Physical qubits | 4.09M | 770.53k | 1.88M | 514.49k | 7.38M | 312.31k |

▶ Resource estimates breakdown

▶ Logical qubit parameters

▶ T factory parameters

▼ Pre-layout logical resources

| Item | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Logical qubits (pre-layout) | 25 | 25 | 25 | 25 | 25 | 25 |
| T gates | 0 | 0 | 0 | 0 | 0 | 0 |
| Rotation gates | 2.50M | 2.50M | 2.50M | 2.50M | 2.50M | 2.50M |
| Rotation depth | 100.00k | 100.00k | 100.00k | 100.00k | 100.00k | 100.00k |
| CCZ gates | 0 | 0 | 0 | 0 | 0 | 0 |
| CCIX gates | 0 | 0 | 0 | 0 | 0 | 0 |
| Measurement operations | 2.50M | 2.50M | 2.50M | 2.50M | 2.50M | 2.50M |

In [26]:
```
result = qsharp.estimate(
    "Heisenberg("+str(no_of_qubits)+","+str(angle)+","+str(no_shots)+")",
    [
        {
        "qubitParams": { "name": "qubit_gate_ns_e3" }},
        {
        "qubitParams": { "name": "qubit_gate_ns_e4" }},
        {
        "qubitParams": { "name": "qubit_gate_us_e3" }},
        {
        "qubitParams": { "name": "qubit_gate_us_e4" }},
        {
        "qubitParams": { "name": "qubit_maj_ns_e4" },
        "qecScheme": { "name": "floquet_code" }},
        {
        "qubitParams": { "name": "qubit_maj_ns_e6" },
        "qecScheme": { "name": "floquet_code" }},
    ]
)
result[:]
```

Out[26]: ▼ Physical resource estimates

| Item | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Runtime | 41 secs | 22 secs | 17 hours | 9 hours | 16 secs | 7 secs |
| rQOPS | 7.86M | 15.00M | 5.24k | 10.00k | 20.00M | 44.00M |
| Physical qubits | 6.06M | 1.14M | 2.76M | 756.49k | 10.97M | 462.79k |

▶ Resource estimates breakdown
▶ Logical qubit parameters
▶ T factory parameters
▼ Pre-layout logical resources

| Item | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Logical qubits (pre-layout) | 25 | 25 | 25 | 25 | 25 | 25 |
| T gates | 0 | 0 | 0 | 0 | 0 | 0 |
| Rotation gates | 2.50M | 2.50M | 2.50M | 2.50M | 2.50M | 2.50M |
| Rotation depth | 100.00k | 100.00k | 100.00k | 100.00k | 100.00k | 100.00k |
| CCZ gates | 0 | 0 | 0 | 0 | 0 | 0 |
| CCIX gates | 0 | 0 | 0 | 0 | 0 | 0 |
| Measurement operations | 100.03k | 100.03k | 100.03k | 100.03k | 100.03k | 100.03k |

We observe that in realistic scenario the experiment to obtain Heisenberg sensitivity limit will take roughly 30% more physical qubits than its classical counterpart (shot-noise limit). The additional physical qubits are required because the experiment to obtain Heisenberg sensitivity limit uses entanglement and typically creating and maintaining entanglement is a resource-intensive process. We also saw that Heisenberg sensitivity limit experiment takes roughly 25% to 33% less time than the experiment for shot-noise limit, this is because there are less qubits to measure and reset in the experiment for Heisenberg limit.

## References


[1] Jasminder S Sidhu and Pieter Kok. Geometric perspective on quantum parameter estimation. *AVS Quantum Science*, 2(1), 2020.

[2] Monson H Hayes. *Statistical digital signal processing and modeling*. John Wiley & Sons, 1996.

[3] Géza Tóth and Dénes Petz. Extremal properties of the variance and the quantum fisher information. *Physical Review A*, 87(3):032324, 2013.