Unit - V

(2018) (2021)

Mathematical Induction :- Let $S(n)$ be a statement that involves positive integer $n = 1, 2, 3, \ldots$. Then $S(n)$ is true for all positive integer n provided that

① $S(1)$ is true.

② $S(k+1)$ is true whenever $S(k)$ is true.

So, there are 3 steps of proof using the principle of mathematical induction.

Step 1 :- (Inductive base) Verify that $S(1)$ is true.

Step 2 :- (Inductive hypothesis) Assume that $S(k)$ is true for an arbitrary value of k.

Step 3 :- (Inductive step) Verify that $S(k+1)$ is true on basis of the inductive hypothesis.

(2021)

⑦(a) State Mathematical Induction. Using the Mathematical Induction, show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad n \geq 1.$$

Solⁿ :- Let $S(n)$ be the given statement.

① Inductive base :- For $n = 1$, we have

$$1^2 = \frac{1(1+1)(2+1)}{6} = \frac{\cancel{6}}{\cancel{6}} = 1$$

So, $S(1)$ is true.

② Inductive hypothesis :- Assuming that $S(k)$ is true, i.e.

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{—①}$$

③ Inductive step :- We wish to show the truth of $S(k+1)$, i.e.

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Now, $1^2+2^2+3^2+\cdots+(k+1)^2 = (1^2+2^2+3^2+\cdots+k^2)+(k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1) + k$$

$$= (2k+1)\left[\frac{k(k+1)}{6}+1\right]$$

$$= (2k+1)\left[\frac{k(k+1)+6}{6}\right]$$

$$= (2k+1)\left[\frac{k^2+k+6}{6}\right]$$

$$= (k+1)\left[\frac{k(2k+1)}{6}+1\right]+k$$

$$= (k+1)\left[\frac{2k^2+k+6}{6}\right]+k$$

$$= \cancel{(k+1)}\frac{(k+1)(2k^2+k+6)+6k}{6}$$

$$= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \quad \{\text{using eq}^n\ 1\}$$

$$= (k+1)\left[\frac{k(2k+1)}{6}+(k+1)\right]$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$= (k+1)\left[\frac{2k^2+7k+6}{6}\right]$$

$$= (k+1)\left[\frac{2k^2+4k+3k+6}{6}\right]$$

$$\Rightarrow 1^2+2^2+3^2+\cdots = (k+1)\left[\frac{2k(k+2)+3(k+2)}{6}\right]$$

$$+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Thus $S(k+1)$ is true whenever $S(k)$ is true. Therefore, by the principle of mathematical induction, $S(n)$ is true for all positive integer n.

Hence proved.

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EX41:- Show that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

by mathematical induction.

Sol^n:- Let the given statement be denoted by $S(n)$.

① **Inductive base:-** For $n=1$, we have

$$\frac{1}{1\cdot 2}=\frac{1}{1+1}$$

$$\Rightarrow \frac{1}{2}=\frac{1}{2}$$

Hence $S(1)$ is true.

② **Inductive hypothesis:-** Assume that $S(k)$ is true,

i.e.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad \text{—(1)}$$

③ **Inductive step:-** We wish to show that the statement is true for $n=k+1$, i.e.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}.$$

Now, $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}$

$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \{\text{using eq}^n\ (1)\}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}$$

$$\Rightarrow \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Thus $S(k+1)$ is true whenever $S(k)$ is true.
Therefore, by the principle of mathematical induction, $S(n)$ is true for all positive integer n.
Hence proved.

2020

2(a) Prove by Mathematical Induction that: $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.

Solⁿ:- Let $S(n)$ be the given statement.

① Inductive base :- For $n = 1$, we have

$$6^{1+2} + 7^{2+1} = 6^3 + 7^3 = 216 + 343 = 559 = 43 \times 13$$

which is divisible by 43.
So, $S(1)$ is true.

② Inductive hypothesis :- Assume that $S(k)$ is true, i.e.

$$6^{k+2} + 7^{2k+1} = 43 \times m, \quad (1)$$ for some positive integer m

③ Inductive step :- We wish to show ~~that~~ the truth of $S(k+1)$, i.e.

divisible

None

$6^{(k+1)+2} + 7^{2(k+1)+1} = 6^{k+3} + 7^{2k+3}$ is divisible by 43.

Now, $6^{k+3} + 7^{2k+3} = 6^{k+3} + 7^{2k+1+2}$

$= 6^{k+2} \cdot 6 + 7^{2k+1} \cdot 7^2$

$= 6^{k+3} \cdot 6 + 6 \cdot 7^{2k+1} + 43 \cdot 7^{2k+1}$

$= 6(6^{k+2} + 7^{2k+1}) + 43(7^{2k+1})$

$= 6 \times (43 \times m) + 43(7^{2k+1})$ {using eqn ①}

$= 43(6m + 7^{2k+1})$

which is divisible by 43.

Thus $S(k+1)$ is true whenever $S(k)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for all positive integer n.

Hence proved.

(2018)

③ (b) State the Principle of Mathematical Induction. And show that $8^n - 3^n$ is divisible by 5 for $n \geq 1$.

Sol^n:- Let $S(n)$ be the given statement.

① Inductive base :- For $n = 1$, we have

$8^1 - 3^1 = 8 - 3 = 5$

which is divisible by 5.

So, $S(1)$ is true.

② Inductive hypothesis :- Assume that $S(k)$ is true, i.e.

$8^k - 3^k = 5 \times m$ —① , for some $m \geq 1$

③ Inductive step :- We wish to show the truth of $S(k+1)$, i.e.

$8^{k+1} - 3^{k+1}$ is divisible by 5.

Now, $8^{k+1} - 3^{k+1}$ $= 8^k \cdot 8 - 3^k \cdot 3$

$= 8^k \cdot 8 - 3^k \cdot 8 + 3^k \cdot 5$

$= 8(8^k - 3^k) + 5(3^k)$

$= 8(5 \times m) + 5(3^k)$ {using eq^n 1}

$= 5(8m + 3^k)$

which is divisible by 5.

Thus $S(k+1)$ is true whenever $S(k)$ is true. Therefore, by the principle of mathematical induction, $S(n)$ is true for $n \geq 1$.

Hence proved.

(2022) 7 (b) Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for each positive integer n.

Sol^n:- Let $S(n)$ be the given statement.

1 Inductive base :- For $n = 1$, we have

$1^3 + 2 \times 1 = 1 + 2 = 3$

which is divisible by 3.

So, $S(1)$ is true.

2 Inductive hypothesis :- Assume that $S(k)$ is true, i.e

$k^3 + 2k = 3 \times m$ — (1), for some positive integer m

3 Inductive step :- We wish to show the truth of $S(k+1)$, i.e.

$(k+1)^3 + 2(k+1)$ is divisible by 3.

Now, $(k+1)^3 + 2(k+1) = k^3 + 1 + 3k(k+1) + 2k + 2$

$\{\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)\}$

$= (k^3 + 2k) + 3 + 3k^2 + 3k$

$= 3 \times m + 3(1 + k^2 + k)$ {using eq^n ①}

$= 3(m + 1 + k^2 + k)$

which is divisible by 3.

Thus $S(k+1)$ is true whenever $S(k)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for all positive integer n.

Hence proved.

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③(a) Prove by mathematical induction: $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$.

Sol^n:- Let $S(n)$ be the given statement.

① Inductive base :- For $n = 2$, we have

$2^4 - 4 \times 2^2 = 16 - 4 \times 4$

$= \cancel{16-16}$

$= 0$

which is divisible by 3.

So, $S(2)$ is true.

② Inductive hypothesis :- Assume that $S(k)$ is true, i.e.

$k^4 - 4k^2 = 3 \times m$ — ①, for some $m \geq 2$

③ Inductive step :- We wish to show the truth of $S(k+1)$, i.e.

$(k+1)^4 - 4(k+1)^2$ is divisible by 3.

Now, $(k+1)^4 - 4(k+1)^2 = (k+1)^2\left[(k+1)^2 - 4\right]$

$$= (k^2+1+2k)\left[k^2+1+2k-4\right]$$
$$= (k^2+1+2k)(k^2+2k-3)$$
$$= k^4 + \underline{k^2} + \underline{2k^3} + \underline{2k^3} + 2k + \underline{4k^2} - \underline{3k^2} - 3 - 6k$$
$$= k^4 + 4k^3 + 2k^2 - 4k - 3$$
$$= (k^4 - 4k^2) + 6k^2 - 3 + 4k^3 - 4k$$
$$= 3\times m + 3(2k^2-1) + 4k(k^2-1) \quad \{\text{using eq}^n\ (1)\}$$
$$= 3\left[m + 2k^2 - 1 + \frac{4k}{3}(k^2-1)\right]$$

$$\cancel{= (k^2+4+4k)\left[k^2+4+4k-4\right]}$$
$$\cancel{= (k^2+4+4k)(k^2+4k)}$$
$$\cancel{= k^4 + 4k^2 + 4k^3 + 4k^3 + 16k + 16k^2}$$
$$\cancel{= k^4 + 8k^3 + 20k^2 + 16k}$$
$$\cancel{= k^4 + 8k^3 + 24k^2 + 16k}$$
$$\cancel{= (k^4 - 4k^2) + 8k^3 + 24k^2 + 16k}$$
$$\cancel{= 3\times m + 3\left(\frac{8}{3}k^3 + 8k^2 + \frac{16}{3}k\right) \quad \{\text{using eq}^n\ (1)\}}$$
$$\cancel{= 3\left(m + \frac{8}{3}k^3 + 8k^2 + \frac{16}{3}k\right)}$$

which is divisible by 3.

Thus $S(k+1)$ is true whenever $S(k)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for all $n \geq 2$.

Hence proved.

<u>Recurrence Relation</u> :- A recurrence relation for the sequence $\{S_n\}$ is an equation that relates S_n in terms of one or more of the previous terms of the sequence, $S_0, S_1, \ldots, S_{n-1}$ for all integer $n \geq n_0$, where n_0 is a non negative integer. The specification of the values of S_n, $n < n_0$ is called the initial conditions of a recurrence relation.

For example, the recurrence relation of the sequence

$$S = \{5, 8, 11, 14, 17, \ldots\} \text{ is}$$

$$S_n = S_{n-1} + 3, \quad n \geq 2$$

with initial condition $S_1 = 5$.

EX1:-(a) Find the first four terms of the following recurrence relation

$$a_k = 2a_{k-1} + k, \text{ for all integers } k \geq 2, a_1 = 1.$$

Solⁿ:- $a_1 = 1$

$$a_2 = 2a_1 + 2 = 2\times1 + 2 = 2+2 = 4$$

$$a_3 = 2a_2 + 3 = 2\times4 + 3 = 8+3 = 11$$

$$a_4 = 2a_3 + 4 = 2\times11 + 4 = 22+4 = 26. \quad \text{Ans}$$

Homogeneous recurrence relation :- The basic approach for solving homogeneous relation is to look for solution of the form $a_n = r^n$.
Now, $a_n = r^n$ is a solutions of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \text{ iff}$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

Dividing both sides by r^{n-k}, we get

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$$

$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

which is called the characteristic eqⁿ of the recurrence relation. The solutions of this eqⁿ are called the characteristic roots of the recurrence relation. A characteristic eqⁿ of the kth degree has k characteristic roots.

4.16 Distinct roots :- If the characteristic eqⁿ has distinct roots $r_1, r_2, \ldots, r_k$, then the general form of the solutions for homogeneous eqⁿ is

$$a_n = b_1 r_1^n + b_2 r_2^n + \cancel{b_3 r_3^n} + \cdots + b_k r_k^n,$$

where $b_1, b_2, \ldots, b_k$ are constants which may be taken to satisfy any initial conditions.

EX7:- Solve $a_n = a_{n-1} + 2a_{n-2}$, $n \geq 2$ with the initial conditions $a_0 = 0$, $a_1 = 1$.

Sol^n:- The given recurrence relation is

$$a_n = a_{n-1} + 2a_{n-2}$$

$$\Rightarrow a_n - a_{n-1} - 2a_{n-2} = 0 \quad \text{—(1)}$$

is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ is a solution of eq^n (1).

The characteristic eq^n is

$$r^2 - r - 2 = 0$$

$$\Rightarrow r^2 - 2r + r - 2 = 0$$

$$\Rightarrow r(r-2) + 1(r-2) = 0$$

$$\Rightarrow (r-2)(r+1) = 0$$

$$\Rightarrow r = 2, -1 \text{ distinct real roots.}$$

So the general sol^n is

$$a_n = b_1 (2)^n + b_2 (-1)^n \quad \text{—(2)}$$

Again $a_0 = 0 \Rightarrow 0 = b_1 + b_2$ —(3)

And $a_1 = 1 \Rightarrow 1 = 2b_1 - b_2$ —(4)

$$1 = 3b_1$$

$$\Rightarrow \boxed{b_1 = \frac{1}{3}}$$

From eq^n (3)

$$0 = \frac{1}{3} + b_2$$

$$\Rightarrow \boxed{b_2 = -\frac{1}{3}}$$

Hence the explicit sol^n is given by

$$\boxed{a_n = \frac{1}{3}(2)^n - \frac{1}{3}(-1)^n}.$$

Multiple roots :- If the characteristic eq^n of a homogeneous recurrence relation is $(r-2)^3 = 0$, then $r = 2$ is a required root of multiplicity 3. Then the general sol^n is

$$a_n = (b_1 + n b_2 + n^2 b_3) 2^n$$

In general, if r is a root of the characteristic eq^n of mth order of a given recurrence relation with multiplicity m, then the general form of the solution is

$$a_n = (b_1 + n b_2 + n^2 b_3 + \cdots + n^{m-1} b_m) r^n$$

where $b_1, b_2, b_3, \cdots, b_m$ are constants which may be taken to satisfy any initial conditions.

Mixed roots :- A combination of distinct and multiple roots is also possible, i.e. some roots of a characteristic eq^n are distinct and some roots are equal. If the characteristic eq^n of a homogeneous recurrence relation of 5th order is $(r-2)(r-4)(r-3)^3 = 0$, then

$$r = 2, 4, 3, 3, 3.$$

Then the general sol^n is

$$a_n = b_1 2^n + b_2 4^n + (b_3 + n b_4 + n^2 b_5) 3^n$$

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7 (a) Solve the following recurrence relation: $a_n - 4a_{n-1} + 4a_{n-2} = 0$ with initial conditions $a_0 = 1$ and $a_1 = 6$.

Sol^n:- The given recurrence relation is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0 \quad \text{--- (1)}$$

Let $a_n = r^n$ is a sol^n of eq^n (1).

Then the characteristic eq^n is

$$r^2 - 4r + 4 = 0$$

$$\Rightarrow (r-2)^2 = 0$$

$\Rightarrow r = 2, 2$

Thus the general solⁿ is

$$a_n = (b_1 + n b_2) 2^n \quad \text{--- (2)}$$

Again $a_0 = 1 \Rightarrow \boxed{1 = b_1}$

And $a_1 = 6 \Rightarrow 6 = (b_1 + b_2) 2$

$$\Rightarrow \frac{\cancel{6}^{\,3}}{\cancel{2}} = b_1 + b_2$$

$$\Rightarrow 3 = 1 + b_2$$

$$\Rightarrow 3 - 1 = b_2$$

$$\Rightarrow \boxed{b_2 = 2}$$

Hence, the required solⁿ is

$$\boxed{a_n = (1 + 2n) 2^n} \quad \text{. Ans}$$

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(7)(a) Solve the following Recurrence Relation:
$a_n = 4(a_{n-1} - a_{n-2})$ with initial conditions $a_0 = a_1 = 1$.

Solⁿ:- The given recurrence relation can be written as

$$a_n = 4a_{n-1} - 4a_{n-2}$$

$$\Rightarrow a_n - 4a_{n-1} + 4a_{n-2} = 0 \quad \text{--- (1)}$$

Let $a_n = r^n$ is a solⁿ of eqⁿ (1).

Then the characteristic eqⁿ is

$$r^2 - 4r + 4 = 0$$

$$\Rightarrow (r-2)^2 = 0$$

$$\Rightarrow r = 2, 2$$

Thus the general solⁿ is

$$a_n = (b_1 + n b_2) 2^n \quad \text{--- (2)}$$

Again $a_0 = 1 \Rightarrow \boxed{1 = b_1}$

And $a_1 = 1 \Rightarrow 1 = (b_1 + b_2) 2$

$$\Rightarrow 1 = (1 + b_2) 2$$

$$\Rightarrow \frac{1}{2} = 1 + b_2$$

$$\Rightarrow \frac{1}{2} - 1 = b_2$$

$$\Rightarrow \boxed{b_2 = -\frac{1}{2}}$$

So, the required solⁿ is

$$\boxed{a_n = \left(1 - \frac{1}{2} n\right) 2^n}.$$ Ans

EX10:- Solve the recurrence relation

$$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0.$$

Solⁿ:- Let $a_n = r^n$ be a solⁿ of the given recurrence relation.

The characteristic eqⁿ is

$$r^3 - 8r^2 + 21r - 18 = 0$$

$$\Rightarrow (r-2)(r^2 - 6r + 9) = 0$$

$$\Rightarrow (r-2)(r-3)^2 = 0$$

$$\Rightarrow r = 2, 3, 3$$

$8 - 32 + 42 - 18 = \cancel{56} - \cancel{56} = 0$

$$\begin{array}{r} r^2 - 6r + 9 \\ r-2 \overline{) r^3 - 8r^2 + 21r - 18} \\ \underline{r^3 - 2r^2} \\ -6r^2 + 21r \\ \underline{-6r^2 + 12r} \\ 9r - 18 \\ \underline{9r - 18} \\ 0 \end{array}$$

Thus the general solⁿ is

$$\boxed{a_n = b_1 (2)^n + (b_2 + n b_3) 3^n}.$$

(2022)

1 (i) Solve the recurrence relation: $a_n - 3a_{n-1} + 2a_{n-2} = 0$.

Solⁿ:- Let $a_n = r^n$ be a solⁿ of the given recurrence relation.

The characteristic eq^n is

$$r^2 - 3r + 2 = 0$$
$$\Rightarrow r^2 - 2r - r + 2 = 0$$
$$\Rightarrow r(r-2) - 1(r-2) = 0$$
$$\Rightarrow (r-2)(r-1) = 0$$
$$\Rightarrow r = 1, 2 \text{ distinct real roots.}$$

Thus the general sol^n is

$$\boxed{a_n = b_1(1)^n + b_2(2)^n}.$$

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1 (b) Solve the recurrence relation:

$$a_n - 5a_{n-1} + 6a_{n-2} = 0.$$

Sol^n:- Let $a_n = r^n$ be a sol^n of the given recurrence relation.

The characteristic eq^n is

$$r^2 - 5r + 6 = 0$$
$$\Rightarrow r^2 - 3r - 2r + 6 = 0$$
$$\Rightarrow r(r-3) - 2(r-3) = 0$$
$$\Rightarrow (r-3)(r-2) = 0$$
$$\Rightarrow r = 2, 3 \text{ distinct real roots.}$$

Thus the general sol^n is

$$\boxed{a_n = b_1(2)^n + b_2(3)^n}.$$

Non Homogeneous recurrence relation:- A second order non homogeneous linear recurrence relation with constant coefficients is of the form

$$a_n + 5a_{n-1} + 6a_{n-2} = f(n).$$

Case I:- When $f(n) = $ constant

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(7) (b) Solve the following recurrence relation:

$a_n + 6a_{n-1} + 9a_{n-2} = 3.$

Given that: $a_0 = 0$ and $a_1 = 1$.

Solⁿ:- The associated homogeneous recurrence relation is

$$a_n + 6a_{n-1} + 9a_{n-2} = 0 \quad \text{--- (1)}$$

Let $a_n = r^n$ be a solⁿ of eqⁿ (1).

The characteristic eqⁿ is

$$r^2 + 6r + 9 = 0$$

$$\Rightarrow (r+3)^2 = 0$$

$$\Rightarrow r = -3, -3$$

So, the solⁿ of eqⁿ (1) is

$$a_n^{(h)} = (b_1 + n b_2)(-3)^n$$

To find the particular solⁿ of the given eqⁿ, let $a_n = A$.

Substituting in the given eqⁿ

$$A + 6A + 9A = 3$$

$$\Rightarrow 16A = 3$$

$$\Rightarrow A = \frac{3}{16}$$

$\therefore$ Particular solⁿ $a_n^{(p)} = \frac{3}{16}$

Thus ~~Hence~~ the general solⁿ is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\Rightarrow a_n = (b_1 + n b_2)(-3)^n + \frac{3}{16} \quad \text{--- (2)}$$

Again $a_0 = 0 \Rightarrow 0 = b_1 + \frac{3}{16}$

$$\Rightarrow \boxed{b_1 = -\frac{3}{16}}$$

And $a_1 = 1 \Rightarrow 1 = (b_1 + b_2)(-3) + \frac{3}{16}$

$$\Rightarrow 1 - \frac{3}{16} = (b_1 + b_2)(-3)$$

$$\Rightarrow \frac{13}{16} = (b_1 + b_2)(-3)$$

$$\Rightarrow \frac{13}{16 \times (-3)} = b_1 + b_2$$

$$\Rightarrow -\frac{13}{48} = -\frac{3}{16} + b_2$$

$$\Rightarrow -\frac{13}{48} + \frac{3}{16} = b_2$$

$$\Rightarrow \frac{-13+9}{48} = b_2$$

$$\Rightarrow \frac{-\cancel{4}^{\,1}}{\cancel{48}_{\,12}} = b_2$$

$$\Rightarrow \boxed{b_2 = -\frac{1}{12}}$$

Hence, the required solⁿ is

$$\boxed{a_n = \left(-\frac{3}{16} - \frac{1}{12}n\right)(-3)^n + \frac{3}{16}}.$$

Case II :- When $f(n) = a^n$, where a is a root of characteristic eqⁿ.

(42) (2022)

(2)(a) Solve the recurrence $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$.

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7 (b) Use generating functions to solve the recurrence relation,

$a_n - 9a_{n-1} + 20a_{n-2} = 0$ where $a_0 = -3$ and $a_1 = -10$.

Solⁿ:- Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$, where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 2 to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n - 9\sum_{n=2}^{\infty} a_{n-1}x^n + 20\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\Rightarrow [G(x) - a_0 - a_1 x] - 9x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 20x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0$$

$$\Rightarrow [G(x) - a_0 - a_1 x] - 9x[G(x) - a_0] + 20x^2 G(x) = 0,$$

here $G(x) = \sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ is the generating function for the sequence $\{a_{n-1}\}$ and $G(x) = \sum_{n=2}^{\infty} a_{n-2}x^{n-2}$ is the generating function for the sequence $\{a_{n-2}\}$.

$$\Rightarrow G(x)[1 - 9x + 20x^2] = a_0 + a_1 x - 9a_0 x$$

$$\Rightarrow G(x) = \frac{a_0 + a_1 x - 9a_0 x}{1 - 9x + 20x^2}$$

$$= \frac{-3 - 10x + 27x}{1 - 9x + 20x^2} \quad \{\because a_0 = -3 \text{ and } a_1 = -10\}$$

$$= \frac{-3 + 17x}{20x^2 - 9x + 1}$$

$$= \frac{-3 + 17x}{20x^2 - 5x - 4x + 1}$$

$$= \frac{-3+17x}{5x(4x-1)-1(4x-1)}$$

$$= \frac{-3+17x}{(4x-1)(5x-1)}$$

$$= \frac{-3+17x}{(1-4x)(1-5x)}$$

$$\Rightarrow G(x) = -\frac{5}{1-4x} + \frac{2}{1-5x}$$

{By using Partial fraction}

$$\therefore \sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 5^n x^n - 5\sum_{n=0}^{\infty} 4^n x^n$$

{$\because$ General term $= a^n$
$\therefore$ Generating function,
$G(x) = \frac{1}{1-ax}$}

Hence $a_n = 2\cdot 5^n - 5\cdot 4^n$ which is the required sol^n. Ans

7(a) Determine the numeric function corresponding to the following Generating function:

$$A(z) = \frac{7z^2}{(1-2z)(1+3z)}.$$

Sol^n :- It is given that

$$A(z) = \frac{7z^2}{(1-2z)(1+3z)}$$

$$= 7\left[\frac{z^2}{1-2z+3z-6z^2}\right]$$

$$= 7\left[\frac{z^2}{-6z^2+z+1}\right]$$

$$= -7\left[\frac{z^2}{6z^2-z-1}\right]$$

$$= -7\left[\frac{1}{6} + \frac{\frac{z}{6}+\frac{1}{6}}{6z^2-z-1}\right]$$

$$= -7\left[\frac{1}{6} + \frac{1}{6}\left(\frac{z+1}{6z^2-z-1}\right)\right]$$

$$= -\frac{7}{6} - \frac{7}{6}\left(\frac{z+1}{6z^2-z-1}\right)$$

$$= -\frac{7}{6} + \frac{7}{6}\left(\frac{z+1}{-6z^2+z+1}\right)$$

$$= -\frac{7}{6} + \frac{7}{6}\left[\frac{z+1}{(1-2z)(1+3z)}\right]$$

$$= -\frac{7}{6} + \frac{7}{6}\left[\frac{3}{5}\cdot\frac{1}{1-2z} + \frac{2}{5}\cdot\frac{1}{1+3z}\right]$$

{By using Partial fraction}

$$\Rightarrow A(z) = -\frac{7}{6} + \frac{7}{10}\cdot\frac{1}{1-2z} + \frac{7}{15}\cdot\frac{1}{1+3z}$$

$$\therefore \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty}\left(-\frac{7}{6}\right)z^n + \frac{7}{10}\sum_{n=0}^{\infty}(2)^n z^n + \frac{7}{15}\sum_{n=0}^{\infty}(-3)^n z^n$$

We know that if General term $= a^n$, then Generating function $G(x) = \frac{1}{1-ax}$

Hence $a_n = -\frac{7}{6} + \frac{7}{10}(2)^n + \frac{7}{15}(-3)^n$

Thus, the numeric function corresponding to the generating function $A(z)$ is:

$$\boxed{a_n = -\frac{7}{6} + \frac{7}{10}(2)^n + \frac{7}{15}(-3)^n}$$

(2021) (3)

(2)(a)(ii) In how many ways, can 7 boys and 5 girls be seated in a row, so that no two girls may sit together?

Sol^n:- Since there is no restriction on boys, first of all we fix the positions of 7 boys. Their positions are indicated as

$$\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times B_7 \times$$

Now, 7 boys can be arranged in $\lfloor 7$ ways.

Now, if 5 girls sit at places (including the two ends) indicated by $\times$, then no two of the 5 girls will sit together.

Clearly, 5 girls can be seated in 8 places in ${}^8P_5$ ways.

Hence the required number of ways of seating 7 boys and 5 girls under the given condition $= {}^8P_5 \times \lfloor 7$.

(2021) (1)

(1)(ii) Find the number of handshakes (हाथ मिलाना) in party of 12 people, where each two of them shake hands with each other.

Sol^n:- When two people shake hands, it is counted as one handshake, not two. Therefore, the total number of handshakes is the same as number of ways of selecting 2 people from among 12 people. This can be done

$${}^{12}C_2 = \frac{\overset{6}{\cancel{12}} \times 11}{\cancel{2} \times 1} = 66 \text{ ways}.$$

(2019) (7)

(2) (a) Find the minimum number of students in a class to be sure that four out them are born in the same month.

Sol^n:- We consider each month as a pigeonhole, then $m = 12$ and we have to find the minimum number of students (pigeons) so that four out of them are born in the same month. Take

$(n-1)/m + 1 = 4$

$\Rightarrow (n-1)/m = 4-1$

$\Rightarrow (n-1)/12 = 3 \Rightarrow n-1 = 36 \Rightarrow n = 37$ which is the required minimum number of students. Ans

(2020) (2021) (2018) (5)

(1)(j) State the pigeonhole principle.

Sol^n:- The Pigeon hole principle is sometime useful in counting methods.

If n pigeons (कबूतरों) are assigned to m pigeonholes (कबूतर के रहने का स्थान) then at least one pigeonhole contains two or more pigeons ($m<n$).

6

Extended Pigeonhole Principle :- If n pigeons are assigned to m pigeonholes and $n > m$, then one of the pigeonholes must contain at least $(n-1)/m + 1$ pigeons.

2019 4

7 (b) Find the number of possible ways in which the letters of the word COTTON can be arranged so that the two T's do not come together.

Solⁿ :- There are two T's, two O's and the rest two letters are different in the word COTTON.

Hence the number of arrangements of the letters without any restriction

$$= \frac{6!}{2!\,2!}$$

$$= \frac{6 \times 5 \times \cancel{4}^{2} \times 3 \times 2 \times 1}{\cancel{2} \times \cancel{2}}$$

$$= 180$$

Considering the two T's as one letter, the number of letters to be arranged is 5.

So, the number of arrangements in which both T come together $= \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times \cancel{2} \times 1}{\cancel{2}}$

$$= 60$$

Hence, the number of ways in which two T's do not come together $= 180 - 60$

$$= 120. \quad \underline{\text{Ans}}$$

2018 2

2 (a) Everybody in a room shakes hands with everybody else. The total number of handshakes is 66. How many people are there in the room?

Sol^n:- If there are n people in the room. Then there are $\frac{n(n-1)}{2}$ handshakes.

According to question,

$$\frac{n(n-1)}{2} = 66$$

$$\Rightarrow n^2 - n = 132$$

$$\Rightarrow n^2 - n - 132 = 0$$

$$\Rightarrow n^2 - 12n + 11n - 132 = 0$$

$$\Rightarrow n(n-12) + 11(n-12) = 0$$

$$\Rightarrow (n-12)(n+11) = 0$$

$$\Rightarrow n = 12, -11$$

Since the number of people cannot be negative, therefore

$$n = 12$$

$\therefore$ Total number of people in the room $= 12$. Ans

(2021)

2 (a)(i) State all PEANO's axioms.

(2018)

7 (a) Determine $S^2 a$ and $S^{-2} a$ for the following numeric functions:

$$a_r = \begin{cases} 2, & 0 \le r \le 3 \\ 2^{-r}, & r \geq 4 \end{cases}$$

Sol:- Here, $a_r = \begin{cases} 2, & 0 \le r \le 3 \\ 2^{-r}, & r \geq 4 \end{cases}$

We know that

$$b = S^i a : b_r = \begin{cases} 0, & 0 \le r \le i-1 \\ a_{r-i}, & r \geq i \end{cases}$$

$0 \le r \le 1$, 0

$r \geq 2$, a_{r-2}

$r = 2$, $a_0 = 2$

$r = 3$, $a_1 = 2$

$r = 4$, $a_2 = 2$

$r = 5$, $a_3 = 2$

$r = 6$, $a_4 = 2^{-r}$ $r = 7$, $a_5 = 2^{-r}$

$$\therefore\ b = S^2 a : b_r = \begin{cases} 0, & 0 \le r \le 1 \\ 2, & 2 \le r \le 5 \\ 2^{-r}, & r \geq 6 \end{cases}$$

Now, $b : S^{-i}a : b_x = \begin{cases} a_{x+i}, & x \geq 0 \end{cases}$

$\Rightarrow b_x = a_{x+i} = a_{x+2}$

$x = 0, \; a_2 = 2$

$x = 1, \; a_3 = 2$

$x = 2, \; a_4 = 2^{-x}$

$x = 3, \; a_5 = 2^{-x}$

$\therefore \; b = S^{-2}a : b_x = \begin{cases} 2, & 0 \leq x \leq 1 \\ 2^{-x}, & x \geq 2 \end{cases}$

Ans