QHE in Torus Geometry

Notes on symmetries

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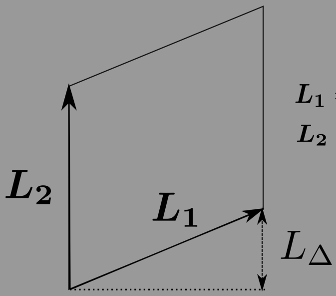
Introduction I

Working in torus geometry is equivalent to imposing periodic boundary conditions on a parallelogram unit cell along both its lattice vectors. Say we have following parallelogram :



$$\boldsymbol{L_1} = L_x\hat{x} + L_\triangle\hat{y}$$
$$\boldsymbol{L_2} = L_y\hat{y}$$

Figure: Unit cell

Introduction II

Then we demand that the physical observables of the system should not change if any particle(s) is(are) translated by a vector

$$\boldsymbol{L_{m,n}} = m\boldsymbol{L_1} + n\boldsymbol{L_2} \quad m, n \in \mathbb{Z}$$

Because such translation brings back the particle in the same position because of imposed pbc. So this is the first symmetry we have on torus geometry and we will see its implications on a qhe system next.

Magnetic Translation Operator I

In the absense of magnetic field, the generator of translation is canonical momenta. That is

$$\begin{aligned} T(\vec{a}) &= \exp\left(\frac{\iota}{\hbar}\vec{a}\cdot\boldsymbol{P}\right) \qquad (1) \\ \implies \quad T(\vec{a})\psi(\vec{x}) &= \psi(\vec{x}+\vec{a}) \end{aligned}$$

Magnetic Translation Operator II

The quantum Hall Hamiltonian looks like

$$H = \sum_i \frac{{\boldsymbol{\pi}_i}^2}{2m} + \frac{1}{2} \sum_{i,j} V(\vec{r_i} - \vec{r_j}) \tag{2}$$

with $V(\vec{r}_{i,j} + \boldsymbol{L_{m,n}}) = V(\vec{r}_{i,j})$ on torus, where $\boldsymbol{\pi}$'s are the physical momenta given as

$$\boldsymbol{\pi} = \boldsymbol{p} + |e|\boldsymbol{A}(\boldsymbol{r})$$

where $\boldsymbol{A}(\boldsymbol{r})$ is the magnetic potential and -ve charge of the electron is accounted for.

Magnetic Translation Operator III

The interest in translation operator, T, is as an accompanying symmetry of H in torus, with a possible reduction of H in smaller sectors corresponding to q-numbers of T. But here is a problem : even though, we have canonical momentum commuting with each other

$$[p_x, p_y] = 0$$

But physical moments do not commute

$$[\pi_x, \pi_y] \neq 0$$

Implying, $[H, \boldsymbol{p}] \neq 0$. Thus translations generated with $\boldsymbol{p}$ or $\boldsymbol{\pi}$ will not commute with H.

Magnetic Translation Operator IV

There is a way to remove the non-commuting part from the $\boldsymbol{\pi}$ operators as follows

$$\begin{aligned} \boldsymbol{K} &= \boldsymbol{\pi} - \frac{\hbar}{l^2}(\hat{z} \times \boldsymbol{r}) \\ \text{or} \quad \boldsymbol{K} &= \boldsymbol{\pi} - eB(\hat{z} \times \boldsymbol{r}) \end{aligned} \tag{3}$$

such that

$$\begin{aligned} & [\boldsymbol{K}, \boldsymbol{\pi}] = 0 \\ \Longrightarrow \quad & [\boldsymbol{K}, H] = 0 \end{aligned}$$

Magnetic Translation Operator V

If we make a modified "translation" operator using $\boldsymbol{K}$, as follows

$$T(\vec{a}) = \exp\left(\frac{\iota}{\hbar}\vec{a} \cdot \boldsymbol{K}\right) \tag{4}$$

which commutes with H, how do we know its action brings the right translation? Else why we are calling it a translation operator at all?

Magnetic Translation Operator VI

We know

$$\boldsymbol{K} = \boldsymbol{p} + e\boldsymbol{A}(\boldsymbol{r}) - eB(\hat{z} \times \boldsymbol{r}) \tag{5}$$

References