MCA Sem. I Core

Mathematical Techniques for Computer Applications(MCAC 103) L 5

Expectation and Variance of Random Variables

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Outline

1. Expectation
2. Moments
3. Statistical Inequalities
4. Covariance
5. Conditional Expectation

Expectation of a random variable X is its mean (weighted average)

Discrete case - RV X

$$\mu = E[X] = \Sigma_x x P(x)$$

Continuous case - RV X

$$\mu = E[X] = \int_x x f(x) dx$$

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Practice Exercise - Consider RV X

1. If $P(0) = P(1) = \frac{1}{2}$, then $E[X] = \frac{1}{2}$
2. If $P(0) = \frac{1}{4}, P(1) = \frac{3}{4}$, then $E[X] = \frac{3}{4}$
3. If X is the outcome when we roll a fair die, then $E[X] = \frac{7}{2}$

Practice Exercise

1. Find the expected value of a discrete random variable X having the probability distribution

$$f(x) = \frac{|x-2|}{7} \text{ for } x = -1, 0, 1, 3$$

(1/7)

2. Find the expected value of a continuous random variable X having the probability density

$$f(x) = \begin{cases} \frac{1}{8}(x+1) & \text{for } 2 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

(37/12)

3. If the probability density is given by

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ 1/2 & \text{for } 1 < x \leq 2 \\ (3-x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

find $E[X]$.

Expectation of function of X - $g(x)$

Discrete case - RV X

$$E[g(x)] = \Sigma_x g(x)P(x)$$

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Other Properties of E[X]

1. $E[c] = c$, where c is a constant
2. $E[cX + b] = cE[X] + b$, where c, b are constants
3. $E[X + Y] = E[X] + E[Y]$
4. If $c_i, c_2, \ldots, c_n$ are constants, then $E[\Sigma_{i=1}^n c_i g_i(X)] = [\Sigma_{i=1}^n c_i E[g_i(X)]$

Practice Exercise

1. Find $E[g(X)]$, where $g(X) = X^2 - 5X + 3$, for Q3 on slide 4.

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1. Find $E[g(X)]$, where $g(X) = X^2 - 5X + 3$, for Q3 on slide 4. (-11/6)
2. If X is the number obtained after rolling a dice, find $E[2X^2 + 1]$ (94/3)

The r^{th} moment about the origin of the random variable X, denoted by μ_r' is expected value of X^r

$$\mu_r' = E[X^r] = \Sigma_x x^r P(x)$$

when X is discrete

$$\mu_r' = E[X^r] = \int_x x^r f(x) dx$$

When X is continuous rv

The r^{th} moment about the origin of the random variable X, denoted by μ'_r is expected value of X^r

$$\mu'_r = E[X^r] = \Sigma_x x^r P(x)$$

when X is discrete

$$\mu'_r = E[X^r] = \int_x x^r f(x) dx$$

When X is continuous rv

The r^{th} moment about mean of the random variable X, denoted by μ_r is expected value of $(X-\mu)^r$

$$\mu_r = E[(X-\mu)^r] = \Sigma_x (x-\mu)^r P(x)$$

when X is discrete

$$\mu_r = E[(X-\mu)^r] = \int_x (x-\mu)^r f(x) dx$$

When X is continuous rv

Note that ....

1. μ'_1 is mean of the distribution of X (denoted by μ)
2. μ_2 is the variance of the distribution denoted by σ^2
3. Positive square root of μ_2 is the sandard deviation denoted by σ
4. $\sigma^2 = \mu'_2 - \mu^2$
5. μ_3 describes skewness
6. If X has the variance σ^2, then $Var(aX + b) = a^2\sigma^2$

Practice Exercises

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3. A group of n cars enters an intersection from the south. Through prior observations, it is estimated that each car has the probability p of turning east, probability q of turning west, and probability r of going straight on $(p + q + r = 1)$. Assume that drivers behave independently and let X be the number of cars turning west. Find E[X].

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4. The waiting time X (in minutes) of a customer waiting to be served at a ticket counter has the density function

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find $E[X]$.

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find $E[X]$. (1/2 minutes)

$$\boxed{\text{Markov Inequality: } P(X \geq a) \leq \frac{E(X)}{a}, \text{ for } X, a \geq 0}$$

1. Gives probability bound for extreme values of a rv i.e. $P(X \geq a)$, where $X \geq 0$
2. Useful when the exact distribution of X is unknown, but $E(X)$ is known
3. Intuitive Proof,

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3. Intuitive Proof,

$$E(X) = \Sigma_{x=0}^{x=\infty} xP(x) = \Sigma_{x=0}^{x=a} xP(x) + \Sigma_{x=a}^{x=\infty} xP(x)$$

$$E(X) \geq \Sigma_{x=a}^{x=\infty} xP(x) \geq a\Sigma_{x=a}^{x=\infty} P(x) = aP(X \geq a)$$

4. Loose bound - not useful sometimes

Alternate Proof of Markov Inequality

Define a rv Y as follows:

$$Y = \begin{cases} 0 & \textbf{for } x < a \\ a & \textbf{for } x \geq a \end{cases}$$

Thus $Y \leq X$
$E(Y) \leq E(X)$

$0 * P(y = 0) + a * P(y = a) \leq E(X)$

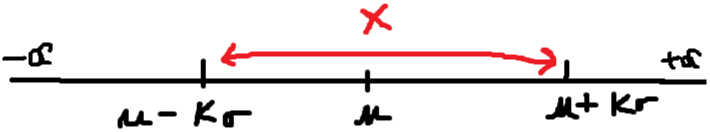
$a\Sigma_{x=a}^{x=\infty} P(x) \leq E(X)$

$a(P(X \geq a)) \leq E(X)$

$P(X \geq a) \leq \frac{E(X)}{a}$

Chebychev's Inequality: $P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$,

for rv X with mean and Var as μ, σ and constant $k \geq 0$



1. Gives lower bound of the probability that rv will take value with in k standard deviations
2. Useful when we do not have exact probability distribution for rv X

Chebychev's Inequality: $P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$,

for rv X with mean and Var as μ, σ and constant $k \geq 0$

Proof using Markov's Inequality

Since $(X - \mu)^2$ is a nonnegative random variable, we can apply Markov's inequali
(with $a = k^2$) to obtain

$$P\{(X - \mu)^2 \geq k^2\} \leq \frac{E[(X - \mu)^2]}{k^2}$$

But since $(X - \mu) \geq k^2$ if and only if $|X - \mu| \geq k$, Equation 4.9.1 is equivalent to

$$P\{|X - \mu| \geq k\} \leq \frac{E[(X - \mu)^2]}{k^2} = \frac{\sigma^2}{k^2}$$

and the proof is complete. □

Alternate Proof

$$\sigma^2 = E(X - E(X))^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

But, $(x-\mu)^2 \geq k^2\sigma^2$ for $x \leq (\mu - k\sigma)$ or $x \leq (\mu + k\sigma)$, we can write

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)dx$$

$$\frac{1}{k^2} \geq \int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx$$

$$\frac{1}{k^2} \geq P(|x-\mu| \geq k\sigma)$$

$$\frac{1}{k^2} \geq 1 - P(|x-\mu| < k\sigma)$$

$$P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Covariance of two random variab;e X and Y, denoted by $Cov(X, Y)$ is denoted by $Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$

If X and Y are independent random variables, the $E(XY) = E(X)E(Y)$

Properties of Covariance for any random variables X, Y, Z and constant k

1. Cov(X, X) = Var(X)
2. Cov(X, Y) = Cov(Y, X)
3. Cov(kX, Y) = kCov(X, Y)
4. Cov(X, Y+Z) = Cov(X, Y) + Cov(X, Z)

Practice Exercises

Suppose $X_1, X_2, \ldots, X_n$ are independently and identically distributed rvs with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Consider the sample mean $\bar{X} = \frac{1}{n}\Sigma_{i=1}^{n} X_i$ as a derived random variable. Then show that :

1. $E(\bar{X}) = \mu$
2. $Var(\bar{X}) = \sigma^2/n$
3. $Cov(\bar{X}, X_i - \bar{X}) = 0$

Conditional Expectation

1. **Conditional expectation of a rv is defined in terms of conditional probability distribution.**
2. **E(X|Y) is a function of is a rv Y, and hence rv itself.**

If X is a discrete rv and $p(x|y)$ is the conditional probablity distribution of X given Y=y at x, then the conditional expectation $E(X|Y=y)$ is

$\mu_{X|y} = \Sigma_x x * p(x|y)$

Correspondingly, conditional variance is defined as:

$\sigma^2_{X|y} = E[(X - \mu_{X|y})^2|y]$

$E(X^2|y) - \mu^2_{X|y}$