MCA Sem. I Core

Mathematical Techniques for Computer Applications (MCAC 103) L 1

Review of Pobability Theory

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Outline

Uncertainities in the World

1. Weather and and natural events
2. Transport services
3. Getting hit by a disease or an epidemic
4. Political party winning an election
5. Topics asked in examination
6. Score in the examination

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What about Computer Systems?

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What about Computer Systems?

1. Time taken to install a software
2. Time taken to print a document
3. Component in a computer system will work after t units of time
4. Uncertainities in Operating System/ Database System/ Communication Systems/ Decision Support Systems

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Mathematics of probability are the same in either case.

Examples

- **Probability(Rain on Jan 1)**
- **Probability (Car turns left at intersection)**
- **Probability(Getting admission in MSc DUCS)**
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- There is 60 % chance of getting oil here.
- There is 30 %chance of rain today
- There is 80 % chance that the disease will not re-occur

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2. Combinatorial reasoning
   - Permutations of n items - number of sequences of n items ($n!$)
   - Permutations of k out of n items - number of sequences of k items ($\frac{n!}{(n-k)!}$)
   - Combination of k out of n items - number of combinations (subsets) of k items ($\binom{n}{k}$)

Quick Reference

- Different ways of arranging n different things $\longrightarrow n!$
- Number of ways to arrange n things in a circle $\longrightarrow (n-1)!$
- Number of ways to arrange n things, of which r are identical and rest are different $\longrightarrow \frac{n!}{r!}$
- Number of ways to arrange n arranged things and m arranged things $\longrightarrow n! * m!$
- Number of ways to select r things from n things $\longrightarrow \binom{n}{r}$
- Number of ways to arrange n things taken r at a time $\longrightarrow^n P_r$

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- A test consists of 15 multiple choice questions, each permitting a choice of 4 alternatives. In how many ways can a student check- off answers to these questions?

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Examples: Combinatorial Reasoning - II

The number of permutations of $n(= n_1 + n_2 + \ldots, n_k)$ objects of which, n_1 are of one kind, n_2 are of other kind, ... n_k are of k^{th} kind $= \frac{n!}{n_1! * n_2! * \ldots * n_k!}$

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- In a graph containg n vertices, how many edges can be drawn?
- How many strings can be formed from the set of n characters?
- How many strings of size upto k can be formed from the set of n characters?

Basic Terminology

Experiment: **A procedure that can be (possibly) infinitely repeated and has a well-defined set of possible outcomes, each outcome depends on chance**
Sample Space: **Set of all possible (exhaustive) distinct and mutually exclusive outcomes of an experiment is the sample space, denoted by Ω or S**
Event: **An outcome of the experiment, which is of current interest and is subset of S. It is denoted by E**

Experiment	Sample space	Example Event
Does the car have air bags?	{Y, N}	Car w/o airbags
Rolling two dice and note sum	{2, 3, ... 12}	Even sum
KMs run per liter of fuel	$\{x \| 4 \leq x \leq 20\}$	$>$ 10 KM
Toss a coin till H appears	{H, TH, TTH .... }	Less than 4 heads
Count tosses till H appears	{0, 1, 2, 3 .... }	Less than 4 tosses
Favourite subject in BSc	{S1, S2, ... Sk}	Programming

Enumerate the Sample space and Events

1. A trainee takes 5 shots at a target and we care only whether each shot is a hit or a miss.
   Events: E1 - trainee will miss all five, E2 - trainee will hit all five, E3 - trainee will miss atleast 3, E4 - trainee will hit atmost 3
2. Electing 3 executives out of 20
   Events E1 - person X is not elected, E2 - two persons A and B are elected
3. Rolling a die until 4 appears
   Events: E1 - 4 appears on the 3rd roll, E2 - 4 appears after 3rd roll
4. A group of 5 boys and 10 girls is lined up in random order
   Events E1 - person in the 4th position is a boy, E2 - person in the 12th position is girl, E3 - particular boy is in the 3rd position

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Set Theory and Venn diagrams come handy

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1. AXIOM 1: $0 \leq P(E) \leq 1$
   Probability of an event lies between 0 and 1 (both inclusive)
2. AXIOM 2: $P(S) = 1$
   Probability of sample space is 1
3. AXIOM 3: For any sequence of mutually exclusive events $E_1, E_2, \ldots,\ P(\bigcup_{i=1}^{n} E_i) = \Sigma_{i=1}^{n} P(E_i)$, for $n = 1, 2, \ldots, \infty$
   Probability of occurance of atleast one of the mutually exclusive events is the sum of their respective probabilities

Simple Probability Questions

1. When we flip a balance coined 5 times what is the probability of getting exactly 3 heads? More than 3 heads?
2. A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find the probability that a number greater than 3 occurs on the single role of the die.
3. For a given experiment, the sample space is infinite, consiting of outcomes {O1, O2, ...}. Verify that $P(Oi) = (\frac{1}{2})^i$ is a probability measure.
4. Five cards are drawn from a deck of 52 playing cards. What is the probability of getting three cards of a kind and a pair.

Useful thoerems of Probability

1. If A and A' are complementary events in the same sample space then $P(A) = 1 - P(A')$
2. If A and B are events in a sample space S and $A \subseteq B$ then $P(A) \leq P(B)$
3. If A and B are events in a sample space S then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. Generalizing the result above for n events

$$\boxed{\begin{aligned} P\left(\bigcup_{j=1}^{n} A_j\right) &= \sum_{j=1}^{n} P(A_j) - \sum_{i=1}^{n} \sum_{\substack{j=2 \\ i<j}}^{n} P(A_i A_j) + \sum_{i=1}^{n} \sum_{\substack{j=2 \\ i<j<k}}^{n} \sum_{k=3}^{n} P(A_i A_j A_k) \\ &\quad - \cdots + (-1)^{n-1} P(A_1 A_2 \ldots A_n), \end{aligned}}$$

$\boxed{\text{Odds of an event E (will occur) - } \frac{p}{(1-p)}, p \neq 0, 1}$

Practice Exercises

1. Express the event "A or B but not both" as venn diagram and set theoretic language.
2. Show that $P(A \cap B) \leq P(A) + P(B)$ and $P(A \cap B) \geq P(A) + P(B) - 1$
3. Prove by induction that for events $E_1, E_2, \ldots,\ P(\bigcup_{i=1}^{n} E_i) \leq \Sigma_{i=1}^{n} P(E_i)$
4. Two cards are randomly drawn from a deck of 52 playing cards. find the probability that both cards will be greater than 3 and less than 8.
5. 4 candidates are seeking a vacancy on a school board. If A is twice as likely to be elected as B, and B and C are given about the same chance of being elected, while C is twice as likely to be elected as D, then what are the probabilities that C will win? A will not win?