10 Mar 22

# System of Linear Equations

$$
\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\ a_{1n}x_n &= b_1 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\ a_{mn}x_n &= b_1
\end{aligned}\right\}
$$

System of $m$ linear equations in $n$ variables.

Many real world systems can be represented

$AX = B$ — Compact form

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Homogenous — if vector $B$ is $0$

Non-homogenous — otherwise

Trivial solution — if $X = 0$ $\{x_1, \ldots x_n$ all are $0\}$

Non-trivial solution, otherwise (at least one $x_i \neq 0$)

Consider homogenous System of Linear Eq.

$$a_{11}x_1 + a_{12} \cdots + a_{1n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n = 0$$

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{\text{coefficient Matrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} A_1 X &= 0 \\ A_2 X &= 0 \\ &\vdots \\ A_n X &= 0 \end{aligned}\right\} \text{Geometrically?}$$

$$E_1 = (1\ 0\ 0 \cdots 0)$$

$$E_2 = (0\ 1\ 0\ 0 \cdots 0)$$

$$\vdots$$

$$E_n = (0\ 0\ 0 \cdots 1)$$

Solution?

If $n > m$: At least one nontrivial solution. Why? Intuition & Proof

To solve ① gradually eliminate variables

② When left with $n-m+1$ variables, assign arbitrary values to all but 1.

③ Work backwards and find values of eliminated variables

If $n = m$ or $n > m$, $\left\langle \begin{array}{l} \text{unique sol.} \\ \text{no solution} \end{array} \right.$

$$3x - 2y + z + 2w = 0$$
$$x + y - z - w = 0$$
$$2x + 2y + 3z = 0$$

Reduce till you get $4 - 3 + 1 = 2$ variables

And find solution(s).

(1) If $n > m$, infinite solutions of the system.

(2) If $X$ and $Y$ are solutions, then $X+Y$ is also a solution, $cX$ is also a solution.

(3) If $X$ is a solution, then it is also the solution of $c_1A_1 + c_2A_2 \cdots + c_nA_n = 0$ (linear combination)

# Non-Homogenous System of linear Eq.

$$AX = B$$

$$\begin{bmatrix} a_{11}x_1 & \cdots & a_{1n}x_n \\ \vdots & & \\ a_{m1}x_1 & \cdots & a_{mn}x_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

# Augmented Matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & & \\ a_{n1} & \cdots & a_{nm} & b_n \end{bmatrix}$$

$n \times \overline{m+1}$

Apply elementary Row and Col. operation to reduce to Row Echlon form (Gauss Elimination)

$$3x - 2y + z + 2w = 1$$
$$x + y - z - w = -2$$
$$2x - y + 3z = 4$$

Coefficient Matrix

$$AX = B$$

$$\left[\begin{array}{cccc|c} 3 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & -1 & -2 \\ 2 & -1 & 3 & 0 & 4 \end{array}\right]$$

Augmented

After applying a sequence of operations,

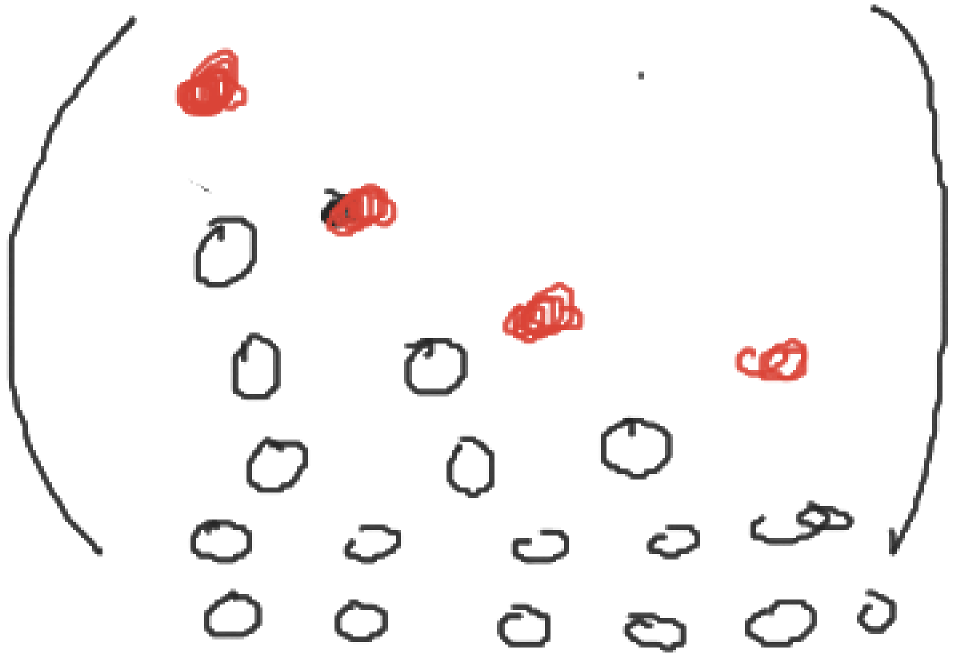
$$
\begin{aligned}
x + y - z - w &= -2 \\
15y - 12z - 15w &= -21 \\
12z - 5w &= 19.
\end{aligned}
$$

$\boxed{\text{One Free Parameter}}$

Row echlon form.

1. All zero rows are at the bottom

2. For two successive non zero rows the non-zero entry in the second row starts atleast one position further to the right than the first row



# Geometric Interpretation of infinite, unique and no solution

(1) One eq in two variables leads to infinite solutions

(2) Two equations in two variable lead to unique solution

(3) Two equations in two variables may sometimes lead to no solution

Similar visualizations can be done in 3-D.

How to recognize solutions in Row Echlon Form (Non-Homo)

(1) Zero row at the bottom of matrix (Infinite)

(2) Every col other than last has leading coefficient (Unique)

(3) Zero row has non-zero element in last column (In consistent)

Row operations can be represented by matrix operations

1. Let $E$ be a matrix obtained from $I_{n\times n}$ by exchanging $R_i$ with $R_j$. Then $EA$ causes similar change in $A_{n\times n}$

2. Let $E$ be a matrix obtained from $I$ by multiplying $r^{th}$ row by $c$ and adding to $s^{th}$ row. Then $EA$ causes similar change in $A_{n\times n}$

Reduced Row Echlon Form

1 Leading coefficients are 1

2. Only non zero element in the col.

$$AX = B$$

$$X = A^{-1} B$$

Implicitly multiplying $\underline{A^{-1}B}$

Finding $A^{-1}$

$$\overset{I}{\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}} \overset{A}{\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}}$$

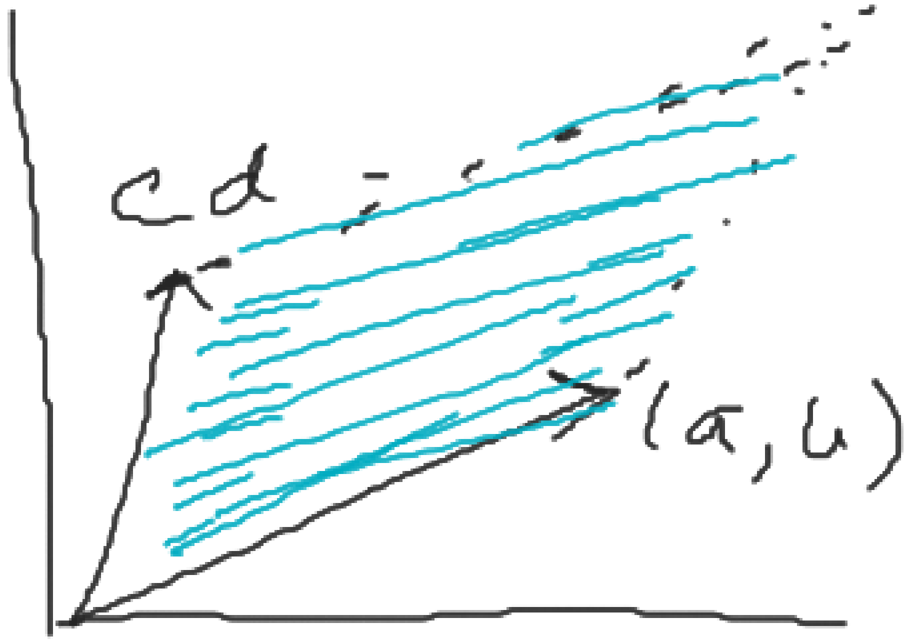
Apply elementary row operations on both matrices (Row Equivalence)

$$\underset{A^{-1}}{\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}} \underset{I}{\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}}$$

Can also find $A^{-1}$ using determinant method

Determinent → vol of the region enclosed by the vectors

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$




$\begin{bmatrix} M \end{bmatrix} \begin{bmatrix} V \end{bmatrix}$ → scaling factor is $|M|$.

Let $B = 2A$. What happens to scaling factor $|B|$?

Each dimension is doubled.

Sign of the determinant determines orientation.

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

http://mathinsight.org/determinant_geometric_properties