14 Mar 22

Cofactor Expansion method for finding $|A|$

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ & & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{pmatrix}$$

1. Recursive method of finding Det of $n\times n$ matrix
2. Recursion stops when 1×1 matrix is obtained

If $A = (a)$, $|A| = a$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad |A| = a_{11}a_{22} - a_{12}a_{22}$$

Minor ij — matrix A_{ij} $n-1 \times n-1$ — obtained from A after removing i^{th} row and j^{th} column.

Cofactor ij $= (-1)^{i+j}\,\text{Det}(A_{ij}) = C_{ij}$

$|A| = \text{Det}(A) = \sum a_{ij} C_{ij}$

sign of the cofactor is imp -

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ \vdots & & & \vdots & \end{bmatrix}$$

Choose a column or Row arbitrarily.

Find $|A|$ for $A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 7 & 0 \\ 10 & 14 & 0 \end{pmatrix}$

Properties : Consider $A = [A_1 \; A_2 \cdot \; A_n]$ Row vector of Col. vector

a) If a col A_j is represented as sum of two vectors

$A_j' + A_j''$

ie $A = [A_1 \; A_2 \cdot \; A_j' + A_j'' \; \cdots \; A_n]$

Then $|A| = |(A_1 \; A_2 \; A_j' \cdot A_n)| + |(A_1 A_2 \cdots A_j'' \; A_n)|$

b). If x is a scalar, then

$$|(A_1 \; A_2 \; xA_j \; A_n)| = x\,|(A_1 \; A_2 \cdots A_n)|$$

c) If two rows/col. are equal then det is 0

d). If I is a unit matrix $|I| = 1$

e) If scalar multiple of a row (col) is added to another, determinant does not change

f. Swapping of rows or col. multiplies the determinant by (-1)

Proofs are simple

Complexity of Cofactor method.?

Cofactor method is efficent if there are several zero entries

Use elementary Row/Col operations when there are fewer zeros.

Use a combination.

Application of Determinant

(1) Inverse $A^{-1} = \frac{1}{|A|} \frac{\text{Adjoint}(A)}{\text{Adjugate}}$

$$\text{Adjoint}(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & & \\ C_{n1} & C_{n2} & & C_{nn} \end{pmatrix}^T$$

This is an expensive way of computing A^{-1}

If $|A| = 0$, A^{-1} doesnot exist.

(2) Solving system of LE.

If $|A| \neq 0$, then apply Cramers rule to find x_i's.

$$x_j = \frac{A_j}{|A|}$$ where A_j is the matrix obtained by replacing Col A_j by B.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -3 \\ -3 & 4 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix}$$

Vector space V is a set of objects, which can be added and multiplied by numbers, in such a way that

① If $v_1, v_2 \in V$ then $v_1 + v_2 \in V$.

② If $v_1 \in V$ & c is a number, $cv_1 \in V$.

And following properties are satisfied

1. Given the elements u, v, w of V, we have $(u + v) + w = u + (v + w)$.
2. There is an element of V, denoted by 0, such that for all elements u of V, $0 + u = u = u + 0$
3. For all elements u of V, the element $(-1)u$ is such that $u + (-1)u = 0$,
4. For all elements u, v of V, we have $u + v = v + u$.
5. If c is a number, then $c(u + v) = cu + cv$.
6. If a, b are two numbers, then $(a + b)v = av + bv$, and $(ab)v = a(bv)$.
7. For all elements u of V, $1.u = u$ (1 is a number)

Vectorspace Model for Document Representation

D_1: —
D_2 —
D_3 —
⋮
D_n —

Documents

$$\begin{array}{c|cccc|} & D_1 & D_2 & \cdots & D_n \\ \hline t_1 & & & & \\ t_2 & & & & \\ \text{terms} \ \vdots & & \boxed{a_{ij}} \rightarrow \text{weight of term } t_i \text{ in doc } D_j & & \\ t_m & & & & \\ \hline \end{array}$$

$A_{m \times n}$ — term-Doc Matrix

weight of term t_i in doc D_j

a_{ij} can be term frequency (tf) or a funtion of tf.

Vocabulary - V - Set of unique words (terms).

Each Col is a vector representation of the doc.

Each Doc is a point in m dimensional space.
Vector

Let Q be the query. Represent the query as Vector Q in same space, then $\cos(Q, D_i)$ gives similarity between D_i & Q.

D_1 : day(4) warm(2) bright (1) sunny (2)

D_2 : night(2) warm (2), cool(1)

D_3 : day (3) night(1) temp(2)

Q : day(1) cool(1) night(2) warm(1)

$$
\begin{array}{c|c}
 & \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\
\begin{array}{c} \text{day} \\ \text{warm} \\ \text{bright} \\ \text{sunny} \\ \text{night} \\ \text{cool} \\ \text{temp} \end{array} & A = \begin{bmatrix} 4 & 0 & 3 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
\qquad
\begin{array}{c} Q \\ 1 \\ 1 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{array}
$$

Find $\cos(Q D_1)$

$\cos(Q D_2)$

$\cos(Q D_3)$

Rank in order of relevance.

Linear Combination

Let V be an arbitrary vectorspace and let $v_1 \cdots v_n$ be the elements of V. Let $x_1 \cdots x_n$ be numbers then the expression

$$v = x_1 v_1 + x_2 v_2 + \cdots x_n v_n$$

v is a linear combination of $v_1 \cdots v_n$ 2

$x_1 \; x_2 \cdots x_n$ are coefficients.

System of Linear Eq.

$$AX = B$$

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} = B$$

$$A x_1 + A_2 x_2 + \cdots + A_n x_n = B$$

Let $\{V_1 \cdots V_n\}$ be a set of vectors in R^n.
$\{V_1 \cdots V_n\}$ are linearly dependent if for real no. $a_1, a_2 \cdots a_n$, following holds

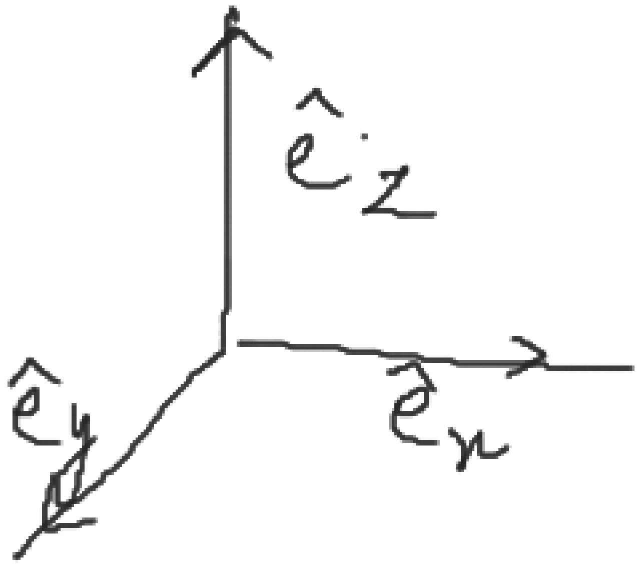
$$a_1 V_1 + a_2 V_2 + \cdots a_n V_n = 0 \quad \text{and } \forall i, a_i \neq 0$$

If such numbers do not exist, the collection is a set of linearly independent vectors.

Consider $\hat{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Basis vectors – orthogonal, all 3 dim vectors can be generated by linear combination of these 3 vectors.

$x\hat{e}_x + y\hat{e}_y + z\hat{e}_z = 0$, Do $x, y\ z$ exist?



Check $\begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}-1\\2\end{pmatrix}$, are they linearly indep.

$$a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}-1\\2\end{pmatrix} = 0$$, do a_1 & a_2 exists?

$$a_1 - a_2 = 0$$
$$a_1 + 2a_2 = 0$$

System of LE

trivial sol $(a_1 = a_2 = 0)$

hence linearly indep.

Check $\begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}1\\-1\\2\end{pmatrix} \begin{pmatrix}3\\1\\4\end{pmatrix}$, are they linearly indep.

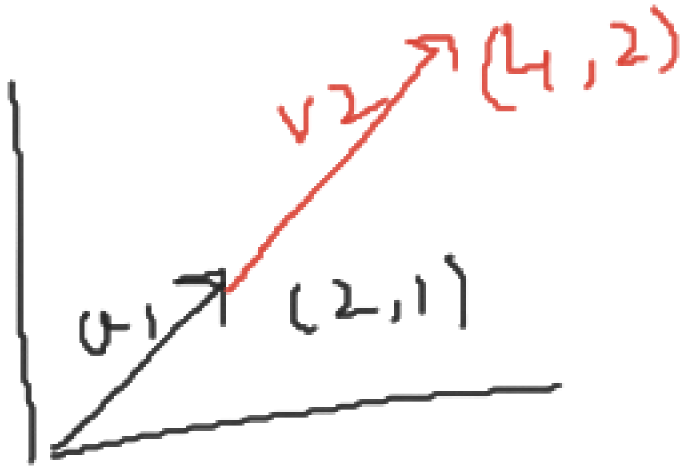
Construct a system of LE (homogenous) and solve.

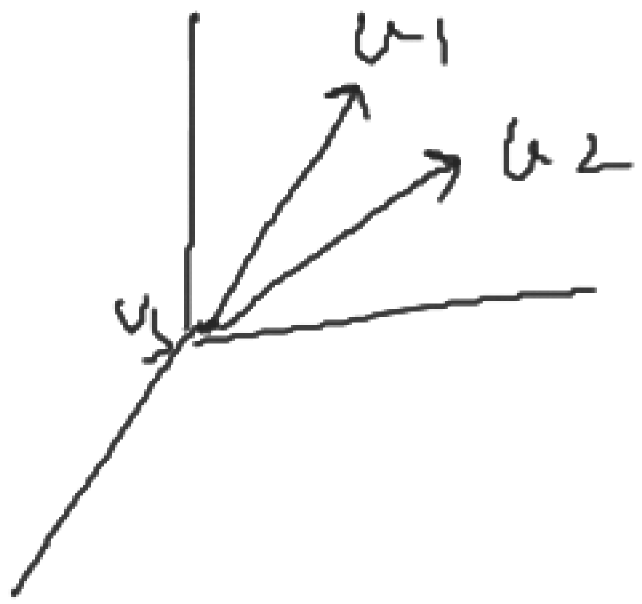
Check $\begin{pmatrix}1\\1\\-2\end{pmatrix}\begin{pmatrix}1\\-1\\2\end{pmatrix}\begin{pmatrix}3\\1\\4\end{pmatrix}$?

Set of linearly dependent vectors have $\boxed{\text{redundant info}}$.

If the set contains 0 vector, then there is linear dependency.

$\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix}$





$v_3 = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Rank gives the # of number of linearly independent rows/columns.

Row Rank - LI rows

Col Rank - LI columns

Row rank = Column rank = rank of the matrix.

In a system of LE, if # variables $(n) >$ # Eq (m)

We know that non trivial Sol exists

What if $n = m$?

Q: Do we have sufficient information?

Another interpretation of Rank

Order of largest order minor with non-zero determinent (non-singular matrix)

<u>Method</u> : Reduce it to RE form & count the number of non-zero rows/col.

$$\text{Rank}(A_{m\times n}) \leq \min(n, m)$$

$$\text{Rank}(A_{n\times n}) \leq n$$

If $\det(A_{n\times n}) \neq 0$, $\text{Rank}(A) = n$

$$\text{Rank}(AB) \leq \text{Min}(\text{Rank}(A), \text{Rank}(B))$$

$$\text{Rank}(A) = \text{Rank}(A^T)$$