MCA Sem. I Core

Mathematical Techniques for Computer Applications(MCAC 103) L 2

Conditional Pobability and Bayes' Theorem

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Outline

1. **Independent Events**
2. **Conditional Probability**
3. **Bayes' Theorem**

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3. **Time taken by a train to travel between stations A and B on two different days**

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$P(A_1A_2A_3) = 0$

$\boxed{\text{Events } A_1, A_2, A_3 \text{ are pairwise, but not mutually independent}}$

Applications in Reliability

1. There is a 1% probability for a hard drive to crash. Therefore, it has two backups, each having a 2% probability to crash, and all three components are mutually independent of each other. The stored information is lost only in an unfortunate situation when all three devices crash. What is the probability that the information is saved?

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2. Suppose that a shuttle's launch depends on three key devices that operate independently of each other and malfunction with probabilities 0.01, 0.02, and 0.02, respectively. If any of the key devices malfunctions, the launch will be postponed. Compute the probability for the shuttle to be launched on time, according to its schedule.

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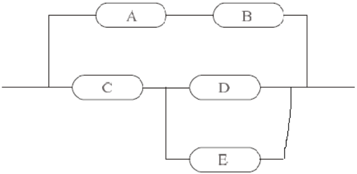

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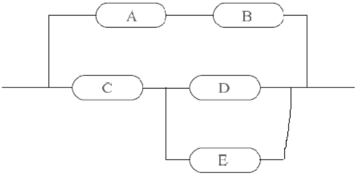

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Reliability of the system if each component is operable with probability 0.92 independently of the other components - (0.9868)
What is $((A \cap B) \cup (C \cap (D \cup E))))'$?

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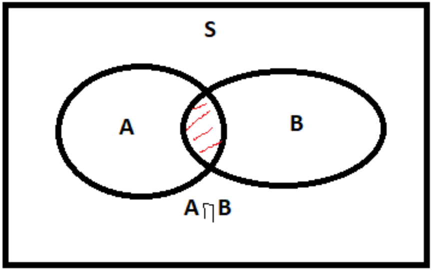
Useful when (i) initially partial information concerning the result of the experiment is available, (ii) some additional information is available



$P(A|B) = P(A \cap B)/P(B)$, where P(B) is not zero.

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Total Probability Theorem: relates the unconditional probability of an event A with its conditional probabilities

Consider a partition of the sample space S with mutually exclusive and exhaustive events $B_1, ..., B_k$. If these events also partition the event A, then $A = (A \cap B_1) \cup \ldots \cup (A \cap B_k)$ is also a union of k mutually exclusive events, and $P(A) = \Sigma_{i=1}^{k} P(A \cap B_i)$.

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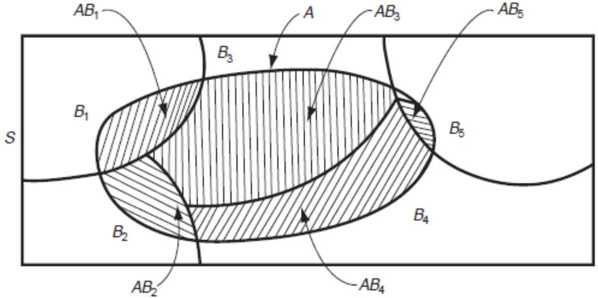
$P(A) = \Sigma_{i=1}^{k} P(A/B_i)P(B_i)$

Intuitive meaning is that P(A) is equal to a weighted average of $P(A|B_i)$, each term being weighted by the probability of the event on which it is conditioned.



Example Problems

1. In answering a question on a multiple-choice test a student either knows the answer or guesses. Let p be the probability that she knows the correct answer and $(1 - p)$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?
2. A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability 0.01, the test result will imply he has the disease.) If 0.5 percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?

Bayes' Theorem: Let A and B be two arbitrary events with $P(A) \neq 0$ and $P(B) \neq 0$. Then $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

Generalization: if $B_1, ..., B_k$ are exhaustive, mutually exclusive events is a sample space S and A is another event in the same sample space, then for event B_i

$P(B_i|A) = \frac{P(AB_i)}{P(B_i)}$

$P(B_i|A) = P(A|B_i) * P(B_i)/\Sigma_{j=1}^{k} P(A/B_j)P(B_j)$

Applications of Bayes' Theorem: Machine Learning

1. Naive Bayes Classifier: Find posterior probabilities of Classes
2. Bayesian Belief Networks : Reasoning with uncertain knowledge (formalizes subjective interpretations)

Evaluation of posteriori probability in terms of a priori information

A simple binary communication channel carries messages by using only two signals, say 0 and 1. We assume that, for a given binary channel, 40% of the time a 1 is transmitted; the probability that a transmitted 0 is correctly received is 0.90, and the probability that a transmitted 1 is correctly received is 0.95. Determine (a) the probability of a 1 being received, and (b) given a 1 is received, the probability that 1 was transmitted.

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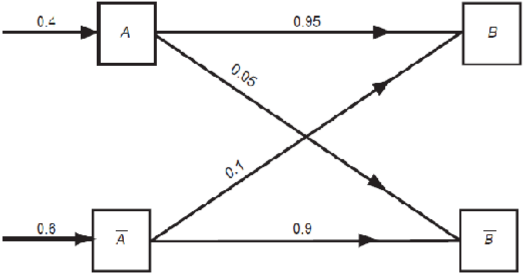

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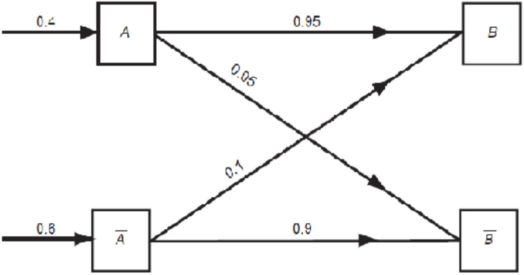

(0.44, 0.863)

An insurance company believes that people can be divided into two classes — those that are accident prone and those that are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a non-accident-prone person. If we assume that 30 percent of the population is accident prone, what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

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Practice Questions

1. A new computer program consists of two modules. The first module contains an error with probability 0.2. The second module is more complex; it has a probability of 0.4 to contain an error, independently of the first module. An error in the first module alone causes the program to crash with probability 0.5. For the second module, this probability is 0.8. If there are errors in both modules, the program crashes with probability 0.9. Suppose the program crashed.What is the probability of errors in both modules?

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