Tutorial 2

AID-521: Mathematics for Data Science

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Qn. 1.

Suppose that a random system of police patrol is devised so that a patrol officer may visit a given beat location $Y = 0, 1, 2, 3, ...$ times per half-hour period, with each location being visited an average of once per time period. Assume that Y possesses, approximately, a Poisson probability distribution.

(a) Calculate the probability that the patrol officer will miss a given location during a half-hour period.

(b) What is the probability that it will be visited once? Twice?

(c) What is the probability that it will be visited at least once?

Ans/Sol.

Qn. 2.

Arrivals of customers at a checkout counter follow a Poisson distribution. It is known that, during a given 30-minute period, one customer arrived at the counter.

(a) Find the probability that the customer arrived during the last 5 minutes of the 30-minute period.

Ans/Sol.

Qn. 3.

A local supermarket has three checkout counters. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random, independently of the other. Let Y_1 denote the number of customers who choose counter 1 and Y_2, the number who select counter 2.

(a) Find the joint probability function of Y_1 and Y_2.

(b) Find the values of joint cdf $F(y_1, y_2)$ at $(-1, 2), (1.5, 2)$, and $(5, 7)$.

Ans/Sol.

Qn. 4.

Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers. Let Y_1 denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week. Let Y_2 denote the proportion of the capacity of the bulk tank that is sold during the week. Because Y_1 and Y_2 are both proportions, both variables take on values between 0 and 1. Further, the amount sold, y_2, cannot exceed the amount available, y_1. Suppose that the joint density function for Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the probability that less than one-half of the tank will be stocked and more than one-quarter of the tank will be sold.

(b) Find the covariance between the amount in stock Y_1 and amount of sales Y_2.

Ans/Sol.

Qn. 5.

From a group of three Republicans, two Democrats, and one independent, a committee of two people is to be randomly selected. Let Y_1 denote the number of Republicans and Y_2 denote the number of Democrats on the committee.

(a) Find the joint probability function of Y_1 and Y_2.

(b) Find the marginal probability function of Y_1.

(c) Find the conditional distribution of Y_1 given that $Y_2 = 1$. That is, given that one of the two people on the committee is a Democrat, find the conditional distribution for the number of Republicans selected for the committee.

(d) Is the number of Republicans in the sample independent of the number of Democrats? (Is Y_1 independent of Y_2?)

Ans/Sol.

Qn. 6.

A soft-drink machine has a random amount Y_2 in supply at the beginning of a given day and dispenses a random amount Y_1 during the day (with measurements in gallons). It is not resupplied during the day, and hence $Y_1 \le Y_2$. It has been observed that Y_1 and Y_2 have a joint density given by

$$f(y_1, y_2) = \begin{cases} 1/2, & 0 \le y_1 \le y_2 \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

That is, the points (y_1, y_2) are uniformly distributed over the triangle with the given boundaries.

(a) Find the conditional density of Y_1 given $Y_2 = y_2$.

(b) Evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 1.5 gallons at the start of the day.

(c) Find the conditional expectation of the amount of sales, Y_1, given that $Y_2 = 1.5$.

Ans/Sol.

Qn. 7.

$$f(y_1, y_2) = \begin{cases} 6y_1 y_2^2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$g(y_1, y_2) = \begin{cases} 2, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that Y_1 and Y_2 are independent if their joint distribution is given by $f(y_1, y_2)$.

(b) Show that Y_1 and Y_2 are dependent if their joint distribution is given by $g(y_1, y_2)$.

Ans/Sol.

Qn. 8.

A process for producing an industrial chemical yields a product containing two types of impurities. For a specified sample from this process, let Y_1 denote the proportion of impurities in the sample and let Y_2 denote the proportion of type I impurities among all impurities found. Suppose that the joint distribution of Y_1 and Y_2 can be modeled by the following probability density function:

$$f(y_1, y_2) = \begin{cases} 2(1 - y_1), & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the expected value of the proportion of type I impurities in the sample.

Ans/Sol.

Qn. 9.

Let Y_1 and Y_2 denote random variables. Then prove that

$$E[Y_1] = E[\ E[\ Y_1|Y_2\ ]\ ],$$

where on the right-hand side the inside expectation is with respect to the conditional distribution of Y_1 given Y_2 and the outside expectation is with respect to the distribution of Y_2.

Ans/Sol.

Qn. 10.

A quality control plan for an assembly line involves sampling $n = 10$ finished items per day and counting Y, the number of defectives. If p denotes the probability of observing a defective, then Y has a binomial distribution, assuming that a large number of items are produced by the line. But p varies from day to day and is assumed to have a uniform distribution on the interval from 0 to 1/4.

(a) Find the expected value of Y .

Ans/Sol.

Qn. 11.

Let $Y_1, ..., Y_n$ be independent random variables with $E[Y_i] = \mu$ and $Var(Y_i) = \sigma^2$. (These variables may denote the outcomes of n independent trials of an experiment.) Define

$$\bar{Y} \quad = \quad \frac{1}{n}\sum_{i=1}^{n} Y_i$$

(a) Show that $E[\bar{Y}] \; = \; \mu$.

(b) Show that $Var(\bar{Y}) = \sigma^2/n$.

Qn. 12.

The number of defectives Y in a sample of $n = 10$ items selected from a manufacturing process follows a binomial probability distribution. An estimator of the fraction defective in the lot is the random variable $\hat{p} = Y/n$.

(a) Find the expected value of $\hat{p}$.

(b) Find the variance of $\hat{p}$.

Ans/Sol.

1) $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$; $x = 0, 1, 2 \ldots$

2) Uniform prob. for each 5 min block.

3) (a)

			Y_2	
		0	1	2
	0	~~[illegible]~~	..	
Y_1	1		..	
	2		..	

equally ~~likely~~

$S = \{(1,1), (1,2), (1,3), \ldots, (3,3)\}$

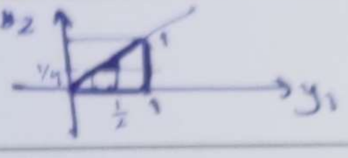
↳ equally likely ... $1/9$

Y_1 → 2, 1, 1, 1, 0, 0, 1, 0, 0

Y_2 → 0, 1, 0, 1, 2, 1, 0, 1, 0

			Y_2	
		0	1	2
	0	$1/9$		$1/9$
Y_1	1		$2/9$	0
	2	$1/9$	0	0

(b) $(0,0) + (0,1) + (0,2)$
$+ (1,0) + (1,1) + (1,2)$



4) (a) $P(0 \le Y_1 \le 0.5,\ Y_2 > 0.25)$

$$= \int_0^{0.5} \left(\int_{0.25}^{y_1} 3y_1 \, dy_2 \right) dy_1$$

$$= \int_{0.25}^{0.5} \int_{0.25}^{y_1} 3y_1 \, dy_2 \, dy_1$$

$$= \int_{0.25}^{0.5} 3y_1 \left(\int_{0.25}^{y_1} dy_2 \right) dy_1$$

$$= \int_{1/4}^{1/2} 3y_1 \left(y_1 - \frac{1}{4} \right) dy_1 = \dots$$

$$= 5/128$$

(b) $E(Y_1 Y_2) = \int_0^1 \int_0^{y_1} y_1 y_2 \, 3y_1 \, dy_2 \, dy_1$

$$= \dots = 3/10$$

$E[Y_1] = 3/4 \qquad E[Y_2] = 3/8$

$Cov(Y_1, Y_2) = \frac{3}{10} - \frac{3}{4} \cdot \frac{3}{8} = 0.02$

$\binom{4}{2} = \frac{4\times3\times\cancel{2!}}{2\times\cancel{2!}}$ $\quad \frac{6!}{2!\,4!} = \frac{6\times5}{2} = 15 \quad \frac{1!}{0!\,(1-0)!} = 1$

5) $3R,\ 2D,\ 1i \mid 6 \quad \gg 2$

(a)

		y_1 = 0	1	2
y_2	0	•	3/15	3/15
	1	2/15	6/15	•
	2	1/15	•	•

Combinatorics: so, no need for enlisting outcomes.

$$P(Y_1=1, Y_2=1) = \frac{\binom{3}{1}\binom{2}{1}}{\binom{6}{2}}$$ ← sample space size

(d) $p(y_1, y_2) = p(y_1)\cdot p(y_2) \quad \forall\, y_1, y_2$

$p(0,0) = 0$

$p_1(0) = \frac{\binom{3}{2}}{15} = \frac{3}{15}, \quad p_2(0) = 6/15$

Get a feeling!!

6) <a> use = Y_1 $\leq$ tot = Y_2

$f(y_1 | Y_2 = y_2) = \cancel{\int_{-\infty}^{y_2} f(y_1, y_2)}$

$= \dfrac{f(y_1, y_2)}{\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1} = \dfrac{1/2}{\begin{cases} 1/2\, y_2 & ; 0 \leq y_2 \leq 2 \\ 0 & ; \text{else} \end{cases}} = 1/y_2$

$[0 < y_2 \leq 2]$

<b> $P(Y_1 \leq 1/2 \mid Y_2 = 1.5)$

$= \int_0^{1/2} f(y_1 \mid y_2 = 1.5)\, dy_1$

$= \int_0^{1/2} \left(\frac{1}{y_2} \mid y_2 = 1.5\right) dy_1 = \int_0^{1/2} \frac{1}{1.5}\, dy_1$

$= \frac{1}{2} \cdot \frac{1}{1.5} = \frac{1}{3}$

<c> $E[Y_1 \mid Y_2 = 1.5]$

$= \int_0^{1.5 = y_2} y_1 f(y_1 \mid y_2 = 1.5)\, dy_1$

$= \int_0^{y_2} \frac{y_1}{y_2}\, dy_1 = \frac{1}{y_2} * \frac{y_2^2}{2} = \frac{y_2}{2}$

$= 0.75$

7) (a) $f_{Y_1}(y_1) = \int_0^1 6y_1 y_2^2 \, dy_2$ $\quad 0 < y_1 \le 1$

$= 2y_1$ $\quad$ else 0

$f_{Y_2}(y_2) = \int_0^1 6y_1 y_2^2 \, dy_1$

$= 3y_2^2$

(b) $f_{Y_1}(y_1)$ $\quad 0 \le y_1 \le 1$

$= \begin{cases} \int_0^{y_1} 2\, dy_2 = 2y_1 \\ 0 & \text{else} \end{cases}$

$f_{Y_2}(y_2) = \begin{cases} \int_{y_2}^1 2\, dy_1 = 2(1-y_2) \\ 0 \end{cases}$

$\therefore f(y_1, y_2) \neq f_{Y_1}(\cdot)\, f_{Y_2}(\cdot)$

8) Prop^n of Type I impurities in the entire sample $= Y_1 \cdot Y_2$

$\therefore E[Y_1 Y_2] = \int_0^1 \int_0^1 y_1 y_2 \, 2(1-y_1) \, dy_2 \, dy_1$

$= 1/6.$

11) Do yourself ... just apply definit^n.

9) Thm ✓ [SOL]

10) (*) $E[Y_1] = \underset{(Y_2)}{E}\Big[\underset{(Y_1)}{E[Y_1 | Y_2]} \Big]$

■ $E[Y]$, given p, $= np$
(binomial dist^n)

$$E[Y] = E\Big[E[Y|p]\Big] = E(np)$$

$$= n \cdot E(p) = n \cdot \left(\frac{0 + 1/4}{2}\right)$$

↳ uniform dist^n

$$= n/8$$

$$= \frac{10}{8} \quad (\text{as } n = 10)$$

$$= 1.25$$

↳ Long Run expected # defect / day

12) $E(\hat{p}) = E\left(\frac{Y}{n}\right)$

$= \frac{1}{n} E(Y)$ { $\because$ n is constant

$= \frac{1}{n} \times np$ { $\because$ Exp. value of binomial $= np$.

$= p$

$Var(\hat{p}) = Var\left(\frac{Y}{n}\right)$

$= \left(\frac{1}{n}\right)^2 Var(Y)$ { $\because$ n is constant (use property of variance)

$= \frac{1}{n^2} \cdot np(1-p)$

$= \frac{p(1-p)}{n}$.