AID-521 Mathematics for Data Science

Module: Probability | Lecture: 5

MULTIVARIATE (OR JOINT) PROB. DISTRIBUTIONS

bivariate distributions, marginal and conditional distributions, independent r.v.s, conditional expectation, covariance & correlation

Why multivariate?

→ Till now, we have seen univariate r.v.s

→ In real scenarios, two r.v.s may be related to each other. Hence it is important to model the joint outcomes of the joint underlying processes.

→ Bivariate distributions forms a nice starting point to dive into multivariate distributions.

Joint Distribution of Two R.V.s

X and Y are two random variables.

If both r.v.s are discrete, the joint probability mass function (joint pmf) of X and Y is

$$f(x,y) \;=\; P(X = x, Y = y)$$

If both r.v.s are continuous, $f(x,y)$ is called the joint probability density function (joint pdf) of X and Y iff

$$P(a \leq X \leq b,\; c \leq Y \leq d) \;=\; \int_a^b \int_c^d f(x,y)\, dx\, dy$$

Necessary Conditions

- $\rightarrow$ $f(x, y) \geq 0 \; \forall \, x, y$

- $\rightarrow$ $\sum_{x,y} f(x, y) = 1$, when X and Y are discrete

- $\rightarrow$ $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$,
  when X and Y are continuous

Marginal Distribution

Let $f(x, y)$ be the joint probability function of r.v.s X and Y.

The marginal probability distribution of X is denoted as $f_X(x)$ and is defined by

$$f_X(x) = \begin{cases} \sum_{\forall \text{ values } y} f(x, y), & \text{if } X \text{ and } Y \text{ are discrete,} \\ \\ \int_{-\infty}^{\infty} f(x, y)\, dy, & \text{if } X \text{ and } Y \text{ are continuous.} \end{cases}$$

Conditional Probability Distribution

Earlier we had seen $P(A|B) = \frac{P(A \cap B)}{P(B)}$ for events A and B, which are subsets of S.

Now, we are dealing with random variables.

Let $f(x, y)$ be the joint distribution of r.v.s X and Y.

The conditional probability distribution of X given Y is

$$f(x|Y = y) = \begin{cases} \dfrac{P(X{=}x, Y{=}y)}{\sum\limits_{\forall \text{ values } x} P(X{=}x, Y{=}y)}, & \text{if } X, Y \text{ are discrete,} \\[2ex] \dfrac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\, dx}, & \text{if } X, Y \text{ are continuous.} \end{cases}$$

Independent Random Variables

Earlier we had seen $P(A \cap B) = P(A) \cdot P(B)$ when events A and B are independent. Remember? Now, we are dealing with random variables.

Let X and Y be two r.v.s with joint probability (mass/density) function $f(x, y)$.

Then, X and Y are independent r.v.s iff the joint probability function is the product of the marginals, i.e.,

$$f(x, y) \quad = \quad f_X(x) \cdot f_Y(y) \;\; \forall \, x, y.$$

y

		-2	0	1	4	Sum
	-1	0.2	0.1	0.0	0.2	0.5
x	3	0.1	0.2	0.1	0.0	0.4
	5	0.1	0.0	0.0	0.0	0.1
	Sum	0.4	0.3	0.1	0.2	1.0

Joint distribution $f(x, y)$,
Marginal distribution $f_X(x)$,
Conditional distribution $f(x|Y = y)$,
Independence

Try yourself
(For sol., see Ram & Tsokos, Example 3.3.3)

$$f(x,y) = \begin{cases} 3x, & 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

→ Find $P(X \leq \frac{1}{2},\ \frac{1}{4} < Y < \frac{3}{4})$.

→ Find the marginals $f_X(x)$ and $f_Y(y)$.

→ Calculate the conditional probability $f(x|Y = \frac{1}{2})$.

→ Are X and Y independent random variables?

Expected Value & Properties

Let $f(x, y)$ be the joint probability function of X and Y.

The expected value of $g(X, Y)$ is

$$E[\,g(X,Y)\,] \;=\; \begin{cases} \sum\limits_{x,y} g(x,y)\cdot f(x,y), & \text{if } X,\, Y \text{ are discrete,} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\cdot f(x,y)\,dx\,dy, & \text{if } X,\, Y \text{ are continuous.} \end{cases}$$

Important Properties:-

→ $E[aX + bY] \;=\; a\cdot E[X] \;+\; b\cdot E[Y]$

→ If X and Y are independent, then $E[X\cdot Y] = E[X]\cdot E[Y]$

Conditional Expectation & Properties

Let X and Y be jointly distributed as $f(x, y)$, and $g(\cdot)$ be a function of x.

Given $Y = y$, the conditional expectation of $g(x)$ is a function of y defined as

$$E[\, g(X) \mid Y = y \,] \;=\; \begin{cases} \sum_x g(x, y) \cdot f(x|y), & \text{if } X, Y \text{ are discrete,} \\ \int_{-\infty}^{\infty} g(x, y) \cdot f(x|y)\, dx, & \text{if } X, Y \text{ are continuous.} \end{cases}$$

Important Properties:-

$\rightarrow$ $E[X] \;=\; E[\, E[X|Y] \,]$

$\rightarrow$ $Var(X) \;=\; E[\, Var(\, X|Y \,) \,] \;+\; Var(\, E[\, X|Y \,] \,)$

Covariance & Correlation

X and Y are two random variables, such that

$E[X] = \mu_X,\ E[Y] = \mu_Y$, and $Var(X) = \sigma_X^2,\ Var(Y) = \sigma_Y^2$.

The covariance between X and Y is defined as

$$\sigma_{XY} \ = \ Cov(X,Y) \ = \ E[\ (X-\mu_X)\cdot(Y-\mu_Y)\ ] = E[XY] - \mu_X \cdot \mu_Y.$$

The correlation between X and Y is defined as

$$\rho(X,Y) = \rho = = \frac{Cov(X,Y)}{\sqrt{\sigma_X^2 \cdot \sigma_Y^2}}$$

$\rightarrow$ A measure of linear relationship between X and Y

Covariance & Correlation -- Properties

→ $-1 \leq \rho \leq 1$

→ If X and Y are independent, then $\rho = 0$. (Try to prove)
The converse is not necessarily true.

→ $Cov(X, X) \ = \ Var(X)$

→ $Cov(\ a_1 X + b_1,\ a_2 Y + b_2\ ) \ = \ a_1 \cdot a_2 \cdot Cov(\ X, Y\ )$

→ $Var(aX + bY) \ = \ a^2 \cdot Var(X) \ + \ b^2 \cdot Var(Y) \ + \ 2ab \cdot Cov(X, Y)$