AID-521 Mathematics for Data Science

Module: Probability | Lecture: 6

FUNCTIONS OF RANDOM VARIABLES

method of distribution functions, transformation method, method of mgf*

What will we see here?

- $\rightarrow$ Given a r.v. X and a function $g(\cdot)$ on X, we already know how to calculate $E[g(X)]$
- $\rightarrow$ How can we calculate the probability function of $g(X)$?
- $\rightarrow$ Three widely used methods: (1) method of distribution function, (2) method of transformations, (3) method of moment-generating functions

Method of Distribution Function

X is a random variable with with known pdf $f_X(x)$.

Y is a function of X.

$$F_Y(y) \;=\; P(Y \leq y) \;=\; P(X \in \{Y \leq y\})$$

$$f_Y(y) \;=\; \frac{dF_Y(y)}{dy}$$

Method of Distribution Function

Try yourself
(For sol., see Ram & Tsokos, Example 3.4.5)

$X_1, X_2, \ldots, X_n$ are continuous i.i.d. r.v.s with pdf $f(x)$ and cdf $F(x)$. Find the pdfs of

$$Y_m = \min(X_1, X_2, \ldots, X_n) \text{ and } Y_M = \max(X_1, X_2, \ldots, X_n).$$

Method of Distribution Function

Try yourself
(For sol., see Ram & Tsokos, Example 3.4.1)

Find the pdf of X^2 where X is the standard normal r.v.

Method of Distribution Function

Try yourself

Y has the density function

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the pdf of $U = 3Y - 1$.

Answer

$$f_U(u) = \begin{cases} \frac{2}{9}(u+1), & -1 \leq u < 2, \\ 0, & \text{elsewhere} \end{cases}$$

Transformation Method

X is a univariate r.v. with with pdf $f_X(x)$ and cdf F_X.

$Y = g(X)$, such that g is differentiable and g^{-1} exists.
Since g^{-1} exists, $g(\cdot)$ is <u>either increasing or decreasing</u>.

$$F_Y(y) = P(g(X) \leq y)$$

$$= \begin{cases} P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)), & \text{if } g(\cdot) \text{ is increasing,} \\ P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y)), & \text{if } g(\cdot) \text{ is decreasing,} \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

Transformation Method

Probability Integral Transformation

X is a continuous r.v. with pdf f and cdf F. Let $Y = F(X)$.

$$\begin{aligned} P(Y \le y) &= P(F(X) \le y) = P(X \le F^{-1}(y)) \\ &= \int_{-\infty}^{F^{-1}(y)} f_X(x)\, dx = F_X(x)\Big|_{-\infty}^{F^{-1}(y)} = y \end{aligned}$$

$$f(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 1, & 0 < y < 1, \\ 0, & \text{otherwise} \end{cases}$$

Y has a uniform probability distribution between 0 and 1.

$F(X)$ takes values between 0 and 1

Transformation Method

Try yourself

Y has the density function

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the pdf of $U = 3Y - 1$.

Answer

$$f_U(u) = \begin{cases} \frac{2}{9}(u+1), & -1 \leq u < 2, \\ 0, & \text{elsewhere} \end{cases}$$

Transformation Method - Multivariate

(X, Y) have the joint pdf $f(x, y)$.

$g_1(X, Y) = U$ and $g_2(X, Y) = V$ are mappings from (X, Y) to (U, V) s.t. they are one-one and onto.

Hence $\exists$ inverse functions h_1, h_2 s.t.
$x = h_1^{-1}(u, v)$, $y = h_2^{-1}(u, v)$.

The Jacobian of the transformation is defined as

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

The joint pdf of U and V is given by

$$f(u, v) = f(h_1^{-1}(u, v),\ h_2^{-1}(u, v)) \cdot |J|$$

Transformation Method - Multivariate

Try yourself

Let X and Y be independent r.v.s with common pdf

$$f(x) = e^{-x}, \quad x > 0.$$

Find the joint pdf of

$$U = \frac{X}{X+Y}, \quad \text{and} \quad V = X + Y$$

Transformation Method - Multivariate

Try yourself

The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & 0 \leq y_1,\ 0 \leq y_2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the *joint* density of $U = Y_1 + Y_2$

Method of Moment-Generating Functions

X and Y are r.v.s.

If moment-generating functions for X and Y

$\rightarrow$ exist, and

$\rightarrow$ are equal,

then X and Y have the same probability distribution.

Method of Moment-Generating Functions

Try yourself

Find the pdf of X^2 where X is the standard normal r.v.