AID-521 Mathematics for Data Science

Module: Probability | Lecture: 2

CONDITIONING AFFECTS PROBABILITIES

conditional probability, independence, total probability, bayes’ theorem

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}; \quad P(B) > 0$$

P (Going to school | Falling ill) = P (Going to school) ?

Conditional Probability - Properties

- P(A|B) is a probability function. It satisfies the axiomatic definition of probability.
- Multiplication law:
  $P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$

What are the domain and range of P(A|B)?

Conditional Probability

Try yourself

A fruit basket contains 25 apples and oranges, of which 20 are apples.

If two fruits are randomly picked in sequence, what is the probability that both the fruits are apples?

(Is it $\frac{20}{25} \cdot \frac{19}{24}$?)

Independence

When are two events independent? Basically, the occurence of one should not depend on the other.

Two events A and B, with $P(A), P(B) \neq 0$, are independent when $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

Two events are independent *if and only if* $P(A \cap B) = P(A) \cdot P(B)$.

Total Probability Rule

Suppose $S = A_1 \cup A_2 \cup \cdots \cup A_n$, i.e., the sample space S is composed of n events $A_1, ..., A_n$.

Suppose $P(A_i) > 0$ for $i \in \{1, ..., n\}$, and $A_i \cap A_j = \phi$ for $i \neq j$.

Then, the following holds for any event B.

$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)$$

Total Probability

Try yourself

During an epidemic in a town, 40% of its inhabitants became sick. Of any 100 sick persons, 10 will need to be admitted to an emergency ward.

What is the probability that a randomly chosen person from this town will be admitted to an emergency ward?

Total Probability

Try yourself – solution

A: The event that a person is sick
B: The event that a person is admitted to emergency ward

Its given that $P(A) = 0.4$ and $P(B|A) = 0.1$
It is obvious that $P(B|A^c) = 0$

A and A^c are disjoint events.

Using total probability rule, we have $P(B)$
$= P(B|A)P(A) + P(B|A^c)P(A^c)$
$= 0.1 \times 0.4 + 0 \times (1 - P(A))$
$= 0.04$

Bayes' Theorem

- → Combines Conditional Probability and Total Probability Rule
- → Has many important implications, and is widely applied in many situations

Bayes' Theorem

Assume $S = A_1 \cup A_2 \cup \cdots \cup A_n$, where
$P(A_i) > 0$ for $i \in \{1, 2, ..., n\}$ and
$A_i \cap A_j = \phi$ for $i \neq j$.

Then for any event B with $P(B) > 0$, we have

$$P(A_j|B) = \frac{P(A_j) \cdot P(B|A_j)}{\sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)}$$

Try to prove the above theorem yourself.

Bayes' Theorem

→ Events $A_1, A_2, ..., A_n$ are called states

→ $P(A_i), i \in \{1, 2, ..., n\}$ are called a priori probabilities (or simply priors) of respective events A_i

→ For a given event A_j, $P(A_j|B)$ is called the posterior probability of A_j

Bayes' Theorem

Try yourself

Suppose a statistics class contains 70% male and 30% female students.

It is known that in a test, 5% of males and 10% of females got an ”A” grade.

If one student from this class is randomly selected and observed to have an ”A” grade, what is the probability that this is a male student?

Something worth noting

It is already given that if a person is male, his chance of having a ”A” grade (5%) is less than if he were female (10%). So we randomly pick a person in the class and found that the person has ”A” grade, it is quite intuitive that the person is more likely to be female, isn’t it?

Bayes' Theorem

Try yourself – solution

Denote the events of selecting male and female by M and F respectively.
Denote A as the event that a selected student has "A" grade.

Its given that $P(M) = 0.7, P(F) = 0.3, P(A|M) = 0.05$, and $P(A|F) = 0.1$

$$P(A) \; = \; P(A|M)P(M) + P(A|F)P(F) \; = \; ... = 0.065$$

$$P(M|A) = \frac{P(A|M) \cdot P(M)}{P(A)} = \frac{0.05 \times 0.7}{0.065} = \frac{7}{13}$$

Something worth noting

NO! This is because the probability of selecting a male or female person is not equal – since 70% of the class is male. So, even if males are less likely than females to get "A" grades, their population (70% of the class) is large enough that even there are many males with "A" grades.