Tutorial 2

AID-521: Mathematics for Data Science

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Qn. 1.

Suppose that a random system of police patrol is devised so that a patrol officer may visit a given beat location $Y = 0, 1, 2, 3, ...$ times per half-hour period, with each location being visited an average of once per time period. Assume that Y possesses, approximately, a Poisson probability distribution.

(a) Calculate the probability that the patrol officer will miss a given location during a half-hour period.

(b) What is the probability that it will be visited once? Twice?

(c) What is the probability that it will be visited at least once?

Ans/Sol.

Qn. 2.

Arrivals of customers at a checkout counter follow a Poisson distribution. It is known that, during a given 30-minute period, one customer arrived at the counter.

(a) Find the probability that the customer arrived during the last 5 minutes of the 30-minute period.

Ans/Sol.

Qn. 3.

A local supermarket has three checkout counters. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random, independently of the other. Let Y_1 denote the number of customers who choose counter 1 and Y_2, the number who select counter 2.

(a) Find the joint probability function of Y1 and Y2.

(b) Find the values of joint cdf $F(y_1, y_2)$ at $(-1, 2), (1.5, 2)$, and $(5, 7)$.

Ans/Sol.

Qn. 4.

Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers. Let Y_1 denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week. Let Y_2 denote the proportion of the capacity of the bulk tank that is sold during the week. Because Y_1 and Y_2 are both proportions, both variables take on values between 0 and 1. Further, the amount sold, y_2, cannot exceed the amount available, y_1. Suppose that the joint density function for Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the probability that less than one-half of the tank will be stocked and more than one-quarter of the tank will be sold.

(b) Find the covariance between the amount in stock Y_1 and amount of sales Y_2.

Ans/Sol.

Qn. 5.

From a group of three Republicans, two Democrats, and one independent, a committee of two people is to be randomly selected. Let Y_1 denote the number of Republicans and Y_2 denote the number of Democrats on the committee.

(a) Find the joint probability function of Y_1 and Y_2.

(b) Find the marginal probability function of Y_1.

(c) Find the conditional distribution of Y_1 given that $Y_2 = 1$. That is, given that one of the two people on the committee is a Democrat, find the conditional distribution for the number of Republicans selected for the committee.

(d) Is the number of Republicans in the sample independent of the number of Democrats? (Is Y_1 independent of Y_2?)

Ans/Sol.

Qn. 6.

A soft-drink machine has a random amount Y_2 in supply at the beginning of a given day and dispenses a random amount Y_1 during the day (with measurements in gallons). It is not resupplied during the day, and hence $Y_1 \le Y_2$. It has been observed that Y_1 and Y_2 have a joint density given by

$$f(y_1, y_2) = \begin{cases} 1/2, & 0 \le y_1 \le y_2 \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

That is, the points (y_1, y_2) are uniformly distributed over the triangle with the given boundaries.

(a) Find the conditional density of Y_1 given $Y_2 = y_2$.

(b) Evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 1.5 gallons at the start of the day.

(c) Find the conditional expectation of the amount of sales, Y_1, given that $Y_2 = 1.5$.

Ans/Sol.

Qn. 7.

$$f(y_1, y_2) \;=\; \begin{cases} 6y_1y_2^2, & 0 \le y_1 \le 1,\; 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$g(y_1, y_2) \;=\; \begin{cases} 2, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that Y_1 and Y_2 are independent if their joint distribution is given by $f(y_1, y_2)$.

(b) Show that Y_1 and Y_2 are dependent if their joint distribution is given by $g(y_1, y_2)$.

Ans/Sol.

Qn. 8.

A process for producing an industrial chemical yields a product containing two types of impurities. For a specified sample from this process, let Y_1 denote the proportion of impurities in the sample and let Y_2 denote the proportion of type I impurities among all impurities found. Suppose that the joint distribution of Y_1 and Y_2 can be modeled by the following probability density function:

$$f(y_1, y_2) \;=\; \begin{cases} 2(1 - y_1), & 0 \le y_1 \le 1,\; 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the expected value of the proportion of type I impurities in the sample.

Ans/Sol.

Qn. 9.

Let Y_1 and Y_2 denote random variables. Then prove that

$$E[Y_1] \;=\; E[\; E[\; Y1|Y2 \;]\;],$$

where on the right-hand side the inside expectation is with respect to the conditional distribution of Y_1 given Y_2 and the outside expectation is with respect to the distribution of Y_2.

Ans/Sol.

Qn. 10.

A quality control plan for an assembly line involves sampling $n = 10$ finished items per day and counting Y, the number of defectives. If p denotes the probability of observing a defective, then Y has a binomial distribution, assuming that a large number of items are produced by the line. But p varies from day to day and is assumed to have a uniform distribution on the interval from 0 to 1/4.

(a) Find the expected value of Y .

Ans/Sol.

Qn. 11.

Let $Y_1, ..., Y_n$ be independent random variables with $E[Y_i] = \mu$ and $Var(Y_i) = \sigma^2$. (These variables may denote the outcomes of n independent trials of an experiment.) Define

$$\bar{Y} \;=\; \frac{1}{n}\sum_{i=1}^{n} Y_i$$

(a) Show that $E[\bar{Y}] = \mu$.

(b) Show that $Var(\bar{Y}) = \sigma^2/n$.

Qn. 12.

The number of defectives Y in a sample of $n = 10$ items selected from a manufacturing process follows a binomial probability distribution. An estimator of the fraction defective in the lot is the random variable $\hat{p} = Y/n$.

(a) Find the expected value of $\hat{p}$.

(b) Find the variance of $\hat{p}$.

Ans/Sol.