Convex Optimization Methods for Computing Channel Capacity

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The Communication Problem, Formally

The Problem Statement:

- The source possess M distinct messages, one of which it wishes to communicate with the destination.
- The noisy channel takes in one of the N input-symbol (say i) and produces one of the M output symbol with probability distribution $\mathbf{Q}_i$ *independently* of everything else.

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Maximum Achievable Rate

Over all encoding and decoding schemes, what is the maximum achievable rate, for arbitrarily small probability of error ?

$$\max \liminf \frac{\log M}{n} \tag{1}$$

s.t.

$$\mathbb{P}_n(M \neq \hat{M}) \searrow 0 \tag{2}$$

Where it all started - Shannon (1948)

34 *The Mathematical Theory of Communication*

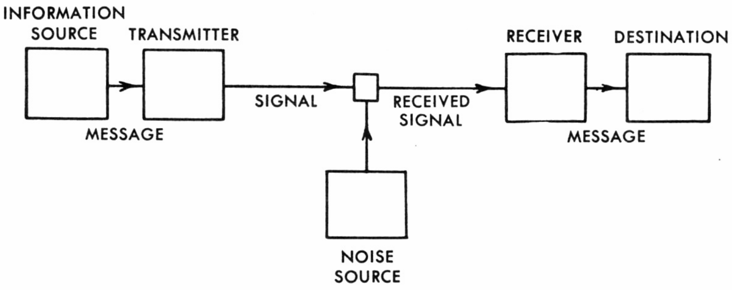
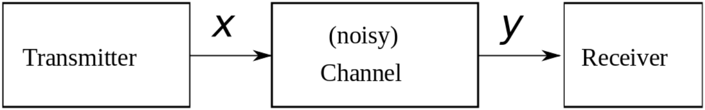


Fig. 1. — Schematic diagram of a general communication system.

The Fundamental Limit: Channel Capacity



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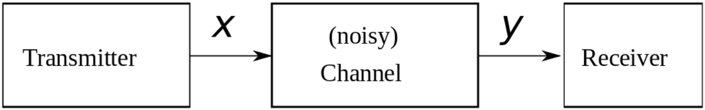

Theorem: Shannon 1948

For every channel matrix $\mathbf{Q}$, maximum achievable rate is given by

$$C = \max_{\mathbf{p}_X} I(X;Y) \tag{3}$$

Where $I(X;Y)$ denotes the *mutual information* between the random variables X and Y.

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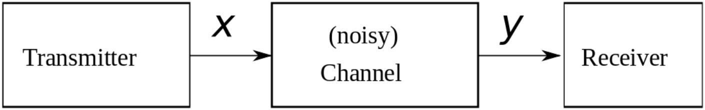

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Objective of this talk

Solve the optimization problem 3.

Review of some useful functionals

- For two PMF **p** and **q** with the same support, the K-L divergence between **p** and **q** is given by,

$$D(\mathbf{p}||\mathbf{q}) = \sum_{x \in X} p_x \log \frac{p_x}{q_x}$$

Property:

$$D(\mathbf{p}||\mathbf{q}) \geq 0 \tag{4}$$

With equality iff $\mathbf{p} = \mathbf{q}$.

- **Mutual Information**

$$I(X;Y) = I(\mathbf{p}, \mathbf{Q}) = \sum_{i=1}^{N} p_i \Big(\sum_{j=1}^{M} Q_{ij} \log Q_{ij} \Big) - \sum_{j=1}^{M} q_j \log q_j \tag{5}$$

Where,

$$\mathbf{q} = \mathbf{p}\mathbf{Q} \tag{6}$$

The PMF **q** is known as the output distribution.

Some Properties of mutual information $I(X;Y) = I(\mathbf{p}, \mathbf{Q})$

Lemma

$I(X;Y) \equiv I(\mathbf{p}, \mathbf{Q})$ *is concave in the variable* $\mathbf{p}$.

Thus the problem 3 corresponds to maximizing a differentiable concave function over the probability simplex.

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- All *off-the-shelf* constrained convex optimization methods are applicable.
- Slow in practice as they do not take into account the structure of the problem.

We describe the celebrated Blahut-Arimoto Algorithm for solving the problem.

- we need to obtain a variational characterization of the mutual information $I(X;Y)$.

A Variational Characterization of $I(X;Y) = I(\mathbf{p}, \mathbf{Q})$

For a set of conditional input distributions $\Phi = \{\phi(\cdot|j), j \in \mathcal{Y}\}$ indexed by the output symbol j, define the functional

$$\tilde{I}(\mathbf{p}, \mathbf{Q}; \phi) = \sum_{i=1}^{N} \sum_{j=1}^{M} p_i Q_{ij} \log \frac{\phi(i|j)}{p_i}$$

Proposition: For a fixed $\mathbf{Q}$ $\tilde{I}(\mathbf{p}, \mathbf{Q}; \phi)$ is concave individually in $\mathbf{p}$ and ϕ.

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Theorem

For any matrix of conditional probabilities ϕ, we have

$$\max_{\phi} \tilde{I}(\mathbf{p}, \mathbf{Q}; \phi) = I(\mathbf{p}, \mathbf{Q}) \tag{7}$$

where maxima is achieved for $\phi(i|j) = \phi^*(i|j) = p_i \frac{Q_{ij}}{\sum_{i=1}^{N} p_i Q_{ij}}$.

Reformulation of the Optimization Problem

With the help from the previous theorem we can reformulate the original optimization problem OPT as follows

Capacity Reformulation

$$C = \max_{\mathbf{p}} \max_{\phi} \tilde{I}(\mathbf{p}, \mathbf{Q}; \phi) \tag{8}$$

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- This is attractive in this case as there are closed form solutions for both the optimization problems.
- Concave character of $\tilde{I}(\mathbf{p}, \mathbf{Q}; \phi)$ guarantees that the method converges to optima.

Iterative Algorithm for solving OPT

Blahut-Arimoto Algorithm for Channel Capacity

Step 1: Initialize $\mathbf{p}^{(1)}$ to the uniform distribution over $\mathcal{X}$, i.e. $p_i^{(1)} = \frac{1}{|\mathcal{X}|}$ for all $i \in \mathcal{X}$. Set t to 1.
Step 2: Find $\phi^{(t+1)}$ as follows:

$$\phi^{(t+1)}(i|j) = \frac{p_i^{(t)} Q_{ij}}{\sum_k p_k^{(t)} Q_{kj}}, \quad \forall i,j \tag{9}$$

Step 3: Update $\mathbf{p}^{(t+1)}$ as follows:

$$p_i^{(t+1)} = \frac{r_i^{(t+1)}}{\sum_{k \in \mathcal{X}} r_k^{(t+1)}} \tag{10}$$

Where,

$$r_i^{(t+1)} = \exp\left(\sum_j Q_{ij} \log \phi^{(t+1)}(i|j)\right) \tag{11}$$

Step 4: Set $t \leftarrow t+1$ and goto Step 2.

Convergence Rates and Improvements

Theorem

The BA algorithm has a convergence rate $\Theta(\frac{1}{t})$.

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- By plugging-in the solution ϕ^* can re-write the BA iteration as follows

$$\mathbf{p}^{t+1} = \arg\max_{\mathbf{p}} \left(\sum_{i=1}^{N} p_i D(\mathbf{Q}_i || \mathbf{q}^t) - D(\mathbf{p} || \mathbf{p}^t) \right)$$

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- Using the idea of appropriately emphasizing/attenuating the penalty term via a weighting factor γ_t, we try the following iteration instead

$$\mathbf{p}^{t+1} = \arg\max_{\mathbf{p}} \left(\sum_{i=1}^{N} p_i D(\mathbf{Q}_i || \mathbf{q}^t) - \gamma_t D(\mathbf{p} || \mathbf{p}^t) \right)$$

Proximal Reformulations Contd.

The sequence $\{\gamma_t\}$ is chosen so that we have strict improvement of Capacity estimate at every iteration. Define the *maximum KLD-induced eigenvalue* of $\mathbf{Q}$ as

$$\lambda_{KL}^2(\mathbf{Q}) = \sup_{\mathbf{p}\neq\mathbf{p}'} \frac{D(\mathbf{pQ}||\mathbf{p}'\mathbf{Q})}{D(\mathbf{p}||\mathbf{p}')}$$

It can be shown that $0 \leq \lambda_{KL}^2(\mathbf{Q}) \leq 1$.

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The capacity estimates improves at every iteration if we take $\gamma_t \geq \lambda^2_{KL}(\mathbf{Q})$.

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Lemma

The capacity estimates improves at every iteration if we take $\gamma_t \geq \lambda^2_{KL}(\mathbf{Q})$.

- However $\lambda^2_{KL}(\mathbf{Q})$ might be difficult to estimate.
- A step-size $\gamma_t = \frac{D(\mathbf{p}^{(t)}\mathbf{Q}||\mathbf{p}^{(t-1)}\mathbf{Q})}{D(\mathbf{p}^{(t)}||\mathbf{p}^{(t-1)})}$ is found to work well in practice.
- Convergence rate boosted by at least a factor of γ_∞^{-1}.

Accelerated BA Algorithm

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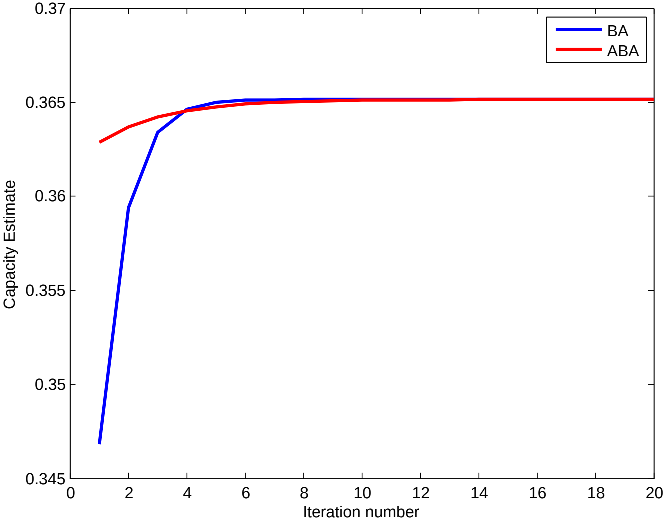
Step 1: Initialize $\mathbf{p}^{(1)}$ to the uniform distribution over $\mathcal{X}$, i.e. $p_i^{(1)} = \frac{1}{|\mathcal{X}|}$ for all $i \in \mathcal{X}$. Set t to 1.

Step 2: Repeat until convergence:

$$\mathbf{q}^{(t)} = \mathbf{p}^{(t)}\mathbf{Q} \tag{12}$$

$$p_i^{(t+1)} = p_i^{(t)} \frac{\exp\left(\gamma_t^{-1} D(\mathbf{Q}_i || \mathbf{q}^{(t)})\right)}{\sum_k p_k^{(t)} \exp\left(\gamma_t^{-1} D(\mathbf{Q}_k || \mathbf{q}^{(t)})\right)} \quad , \forall i \in \mathcal{X} \tag{13}$$

Numerical Simulation



Dual Approach

Finally we take the Lagrange dual of the problem OPT. By straight-forward calculations, it turns out to be the following Geometric Program

$$\min_{\mathbf{z}} \sum_{j=1}^{M} z_j$$

Subject to,

$$\begin{aligned} \prod_{j=1}^{M} z_j^{P_{ij}} &\geq \exp\big(-H(\mathbf{Q}_i)\big), \quad i = 1, 2, \ldots, N \\ \mathbf{z} &\geq \mathbf{0} \end{aligned}$$

- The above GP is useful for deriving outer bounds on capacity.

Conclusion and References

- We have discussed both classical and accelarated Blahut-Arimoto Algorithm for computing Channel capacity of a discrete memoryless channel.
- We have discussed their convergence properties and connection with proximal algorithms

References:

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- G. Matz and P. Duhamel, Information geometric formulation and interpretation of accelerated blahut-arimoto-type algorithms,
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