Introduction to Statistical Machine Learning

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(Many figures from C. M. Bishop, "Pattern Recognition and Machine Learning")

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Outlines

Part VIII

Linear Classification 2

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Three Models for Decision Problems

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In increasing order of complexity

- Find a discriminant function $f(\mathbf{x})$ which maps each input directly onto a class label.
- Discriminative Models
  1. Solve the inference problem of determining the posterior class probabilities $p(\mathcal{C}_k \mid \mathbf{x})$.
  2. Use decision theory to assign each new $\mathbf{x}$ to one of the classes.
- Generative Models
  1. Solve the inference problem of determining the class-conditional probabilities $p(\mathbf{x} \mid \mathcal{C}_k)$.
  2. Also, infer the prior class probabilities $p(\mathcal{C}_k)$.
  3. Use Bayes' theorem to find the posterior $p(\mathcal{C}_k \mid \mathbf{x})$.
  4. Alternatively, model the joint distribution $p(\mathbf{x}, \mathcal{C}_k)$ directly.
  5. Use decision theory to assign each new $\mathbf{x}$ to one of the classes.

Probabilistic Generative Models

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- Generative approach: model class-conditional densities $p(\mathbf{x} \,|\, \mathcal{C}_k)$ and priors $p(\mathcal{C}_k)$ to calculate the posterior probability for class $\mathcal{C}_1$

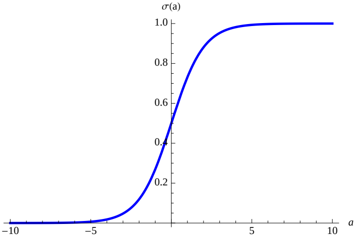
$$\begin{aligned}
p(\mathcal{C}_1 \,|\, \mathbf{x}) &= \frac{p(\mathbf{x} \,|\, \mathcal{C}_1) p(\mathcal{C}_1)}{p(\mathbf{x} \,|\, \mathcal{C}_1) p(\mathcal{C}_1) + p(\mathbf{x} \,|\, \mathcal{C}_2) p(\mathcal{C}_2)} \\
&= \frac{1}{1 + \exp(-a(\mathbf{x}))} = \sigma(a(\mathbf{x}))
\end{aligned}$$

where a and the logistic sigmoid function $\sigma(a)$ are given by

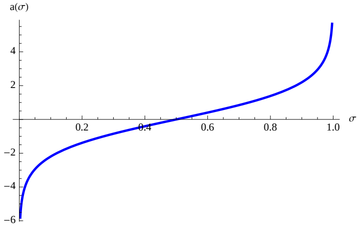
$$\begin{aligned}
a(\mathbf{x}) &= \ln \frac{p(\mathbf{x} \,|\, \mathcal{C}_1)\, p(\mathcal{C}_1)}{p(\mathbf{x} \,|\, \mathcal{C}_2)\, p(\mathcal{C}_2)} = \ln \frac{p(\mathbf{x}, \mathcal{C}_1)}{p(\mathbf{x}, \mathcal{C}_2)} \\
\sigma(a) &= \frac{1}{1 + \exp(-a)}.
\end{aligned}$$

Logistic Sigmoid

- The logistic sigmoid function $\sigma(a) = \frac{1}{1+\exp(-a)}$
- "squashing function' because it maps the real axis into a finite interval $(0, 1)$
- $\sigma(-a) = 1 - \sigma(a)$
- Derivative $\frac{d}{da}\sigma(a) = \sigma(a)\,\sigma(-a) = \sigma(a)\,(1 - \sigma(a))$
- Inverse is called logit function $a(\sigma) = \ln\left(\frac{\sigma}{1-\sigma}\right)$



Logistic Sigmoid $\sigma(a)$



Logit $a(\sigma)$

Probabilistic Generative Models - Multiclass

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- The normalised exponential is given by

$$p(\mathcal{C}_k \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \mathcal{C}_k)\, p(\mathcal{C}_k)}{\sum_j p(\mathbf{x} \mid \mathcal{C}_j)\, p(\mathcal{C}_j)} = \frac{\exp(a_k)}{\sum_j \exp(a_j)}$$

  where

$$a_k = \ln(p(\mathbf{x} \mid \mathcal{C}_k)\, p(\mathcal{C}_k)).$$

- Also called softmax function as it is a smoothed version of the $\max$ function.
- Example: If $a_k \gg a_j$ for all $j \neq k$, then $p(\mathcal{C}_k \mid \mathbf{x}) \simeq 1$, and $p(\mathcal{C}_j \mid \mathbf{x}) \simeq 0$.

Probabil. Generative Model - Continuous Input

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- Assume class-conditional probabilities are Gaussian, all classes share the same covariance. What can we say about the posterior probabilities?

$$
\begin{aligned}
p(\mathbf{x} \mid \mathcal{C}_k) &= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu}_k)^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}_k) \right\} \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2} \mathbf{x}^T \boldsymbol{\Sigma}^{-1} \mathbf{x} \right\} \\
&\quad \times \exp\left\{ \boldsymbol{\mu}_k^T \boldsymbol{\Sigma}^{-1} \mathbf{x} - \frac{1}{2} \boldsymbol{\mu}_k^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_k \right\}
\end{aligned}
$$

where we separated the quadratic term in $\mathbf{x}$ and the linear term.

Probabil. Generative Model - Continuous Input

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- For two classes
$$p(\mathcal{C}_1 \,|\, \mathbf{x}) = \sigma(a(\mathbf{x}))$$
- and $a(\mathbf{x})$ is
$$\begin{aligned} a(\mathbf{x}) &= \ln \frac{p(\mathbf{x} \,|\, \mathcal{C}_1)\, p(\mathcal{C}_1)}{p(\mathbf{x} \,|\, \mathcal{C}_2)\, p(\mathcal{C}_2)} \\ &= \ln \frac{\exp\left\{\boldsymbol{\mu}_1^T \boldsymbol{\Sigma}^{-1} \mathbf{x} - \frac{1}{2} \boldsymbol{\mu}_1^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_1\right\}}{\exp\left\{\boldsymbol{\mu}_2^T \boldsymbol{\Sigma}^{-1} \mathbf{x} - \frac{1}{2} \boldsymbol{\mu}_2^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_2\right\}} + \ln \frac{p(\mathcal{C}_1)}{p(\mathcal{C}_2)} \end{aligned}$$
- Therefore
$$p(\mathcal{C}_1 \,|\, \mathbf{x}) = \sigma(\mathbf{w}^T \mathbf{x} + w_0)$$

  where

$$\begin{aligned} \mathbf{w} &= \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \\ w_0 &= -\frac{1}{2} \boldsymbol{\mu}_1^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_1 + \frac{1}{2} \boldsymbol{\mu}_2^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_2 + \ln \frac{p(\mathcal{C}_1)}{p(\mathcal{C}_2)} \end{aligned}$$

Probabil. Generative Model - Continuous Input

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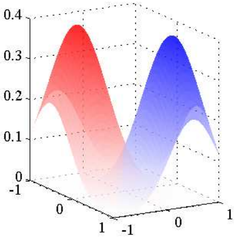
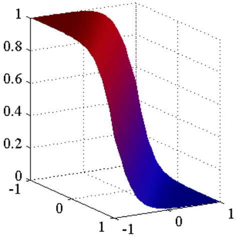
Class-conditional densities for two classes (left). Posterior probability $p(\mathcal{C}_1 \mid \mathbf{x})$ (right). Note the logistic sigmoid of a linear function of $\mathbf{x}$.





General Case - K Classes, Shared Covariance

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- Use the normalised exponential

$$p(\mathcal{C}_k \,|\, \mathbf{x}) = \frac{p(\mathbf{x} \,|\, \mathcal{C}_k) p(\mathcal{C}_k)}{\sum_j p(\mathbf{x} \,|\, \mathcal{C}_j) p(\mathcal{C}_j)} = \frac{\exp(a_k)}{\sum_j \exp(a_j)}$$

where

$$a_k = \ln \left(p(\mathbf{x} \,|\, \mathcal{C}_k) p(\mathcal{C}_k) \right).$$

- to get a linear function of $\mathbf{x}$

$$a_k(\mathbf{x}) = \mathbf{w}_k^T \mathbf{x} + w_{k0}.$$

where

$$\mathbf{w}_k = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_k$$

$$w_{k0} = -\frac{1}{2} \boldsymbol{\mu}_k^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_k + p(\mathcal{C}_k).$$

General Case - K Classes, Different Covariance

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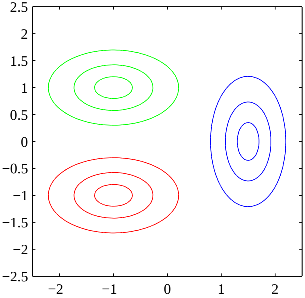
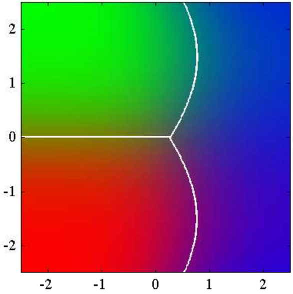
- If each class-conditional probability has a different covariance, the quadratic terms $-\frac{1}{2}\mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x}$ do not longer cancel each other out.
- We get a quadratic discriminant.





Maximum Likelihood Solution

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- Given the functional form of the class-conditional densities $p(\mathbf{x} \mid \mathcal{C}_k)$, can we determine the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?

Maximum Likelihood Solution

- Given the functional form of the class-conditional densities $p(\mathbf{x} \,|\, \mathcal{C}_k)$, can we determine the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?
- Not without data ;-)
- Given also a data set $(\mathbf{x}_n, t_n)$ for $n = 1, \dots, N$. (Using the coding scheme where $t_n = 1$ corresponds to class $\mathcal{C}_1$ and $t_n = 0$ denotes class $\mathcal{C}_2$.
- Assume the class-conditional densities to be Gaussian with the same covariance, but different mean.
- Denote the prior probability $p(\mathcal{C}_1) = \pi$, and therefore $p(\mathcal{C}_2) = 1 - \pi$.
- Then

$$
\begin{aligned}
p(\mathbf{x}_n, \mathcal{C}_1) &= p(\mathcal{C}_1) p(\mathbf{x}_n \,|\, \mathcal{C}_1) = \pi \, \mathcal{N}(\mathbf{x}_n \,|\, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}) \\
p(\mathbf{x}_n, \mathcal{C}_2) &= p(\mathcal{C}_2) p(\mathbf{x}_n \,|\, \mathcal{C}_2) = (1 - \pi) \, \mathcal{N}(\mathbf{x}_n \,|\, \boldsymbol{\mu}_2, \boldsymbol{\Sigma})
\end{aligned}
$$

Maximum Likelihood Solution

- Thus the likelihood for the whole data set $\mathbf{X}$ and $\mathbf{t}$ is given by

$$p(\mathbf{t}, \mathbf{X} \mid \pi, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}) = \prod_{n=1}^{N} [\pi \, \mathcal{N}(\mathbf{x}_n \mid \boldsymbol{\mu}_1, \boldsymbol{\Sigma})]^{t_n} \times$$
$$[(1-\pi) \, \mathcal{N}(\mathbf{x}_n \mid \boldsymbol{\mu}_2, \boldsymbol{\Sigma})]^{1-t_n}$$

- Maximise the log likelihood
- The term depending on π is

$$\sum_{n=1}^{N} (t_n \ln \pi + (1-t_n) \ln(1-\pi))$$

- which is maximal for

$$\pi = \frac{1}{N} \sum_{n=1}^{N} t_n = \frac{N_1}{N} = \frac{N_1}{N_1 + N_2}$$

where N_1 is the number of data points in class $\mathcal{C}_1$.

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Maximum Likelihood Solution

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- Similarly, we can maximise the log likelihood (and thereby the likelihood $p(\mathbf{t}, \mathbf{X} \,|\, \pi, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma})$) depending on the mean $\boldsymbol{\mu}_1$ or $\boldsymbol{\mu}_2$, and get

$$\boldsymbol{\mu}_1 = \frac{1}{N_1} \sum_{n=1}^{N} t_n \, \mathbf{x}_n$$

$$\boldsymbol{\mu}_2 = \frac{1}{N_2} \sum_{n=1}^{N} (1 - t_n) \, \mathbf{x}_n$$

- For each class, this are the means of all input vectors assigned to this class.

Maximum Likelihood Solution

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- Finally, the log likelihood $\ln p(\mathbf{t}, \mathbf{X} \mid \pi, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma})$ can be maximised for the covariance $\boldsymbol{\Sigma}$ resulting in

$$\boldsymbol{\Sigma} = \frac{N_1}{N}\mathbf{S}_1 + \frac{N_2}{N}\mathbf{S}_2$$

$$\mathbf{S}_k = \frac{1}{N_k}\sum_{n \in \mathcal{C}_k}(\mathbf{x}_n - \boldsymbol{\mu}_k)(\mathbf{x}_n - \boldsymbol{\mu}_k)^T$$

Discrete Features - Naive Bayes

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- Assume the input space consists of discrete features, in the simplest case $x_i \in \{0, 1\}$.
- For a D-dimensional input space, a general distribution would be represented by a table with 2^D entries.
- Together with the normalisation constraint, this are $2^D - 1$ independent variables.
- Grows exponentially with the number of features.
- The Naive Bayes assumption is that all features conditioned on the class $\mathcal{C}_k$ are independent of each other.

$$p(\mathbf{x} \mid \mathcal{C}_k) = \prod_{i=1}^{D} \mu_{k_i}^{x_i} (1 - \mu_{k_i})^{1-x_i}$$

Discrete Features - Naive Bayes

- With the naive Bayes

$$p(\mathbf{x} \mid \mathcal{C}_k) = \prod_{i=1}^{D} \mu_{k_i}^{x_i} (1 - \mu_{k_i})^{1-x_i}$$

- we can then again find the factors a_k in the normalised exponential

$$p(\mathcal{C}_k \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \mathcal{C}_k) p(\mathcal{C}_k)}{\sum_j p(\mathbf{x} \mid \mathcal{C}_j) p(\mathcal{C}_j)} = \frac{\exp(a_k)}{\sum_j \exp(a_j)}$$

- as a linear function of the x_i

$$a_k(\mathbf{x}) = \sum_{i=1}^{D} \{x_i \ln \mu_{k_i} + (1 - x_i) \ln(1 - \mu_{k_i})\} + \ln p(\mathcal{C}_k).$$

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Three Models for Decision Problems

In increasing order of complexity

- Find a discriminant function $f(\mathbf{x})$ which maps each input directly onto a class label.
- Discriminative Models
  1. Solve the inference problem of determining the posterior class probabilities $p(\mathcal{C}_k \mid \mathbf{x})$.
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  2. Also, infer the prior class probabilities $p(\mathcal{C}_k)$.
  3. Use Bayes' theorem to find the posterior $p(\mathcal{C}_k \mid \mathbf{x})$.
  4. Alternatively, model the joint distribution $p(\mathbf{x}, \mathcal{C}_k)$ directly.
  5. Use decision theory to assign each new $\mathbf{x}$ to one of the classes.

Probabilistic Discriminative Models

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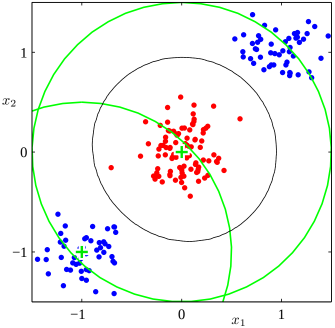
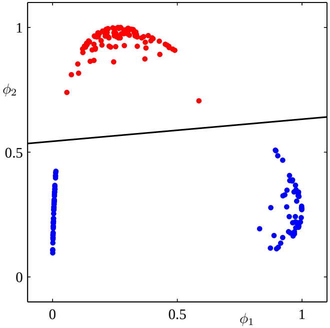
- Maximise a likelihood function defined through the conditional distribution $p(\mathcal{C}_k \mid \mathbf{x})$ directly.
- Discriminative training
- Typically fewer parameters to be determined.
- As we learn the posteriror $p(\mathcal{C}_k \mid \mathbf{x})$ directly, prediction may be better than with a generative model where the class-conditional density assumptions $p(\mathbf{x} \mid \mathcal{C}_k)$ poorly approximate the true distributions.
- But: discriminative models can not create synthetic data, as $p(\mathbf{x})$ is not modelled.

Original Input versus Feature Space

- Used direct input $\mathbf{x}$ until now.
- All classification algorithms work also if we first apply a fixed nonlinear transformation of the inputs using a vector of basis functions $\phi(\mathbf{x})$.
- Example: Use two Gaussian basis functions centered at the green crosses in the input space.





Original Input versus Feature Space

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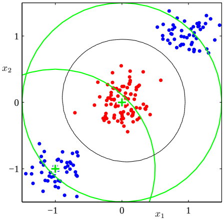
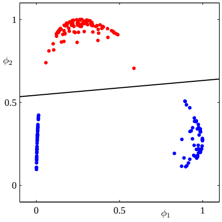
- Linear decision boundaries in the feature space correspond to nonlinear decision boundaries in the input space.
- Classes which are NOT linearly separable in the input space can become linearly separable in the feature space.
- BUT: If classes overlap in input space, they will also overlap in feature space.
- Nonlinear features $\phi(\mathbf{x})$ can not remove the overlap; but they may increase it !





Original Input versus Feature Space

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- Fixed basis functions do not adapt to the data and therefore have important limitations (see discussion in Linear Regression).
- Understanding of more advanced algorithms becomes easier if we introduce the feature space now and use it instead of the original input space.
- Some applications use fixed features successfully by avoiding the limitations.
- We will therefore use ϕ instead of $\mathbf{x}$ from now on.

Logistic Regression is Classification

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- Two classes where the posterior of class $\mathcal{C}_1$ is a logistic sigmoid $\sigma()$ acting on a linear function of the feature vector ϕ
$$p(\mathcal{C}_1 \mid \phi) = y(\phi) = \sigma(\mathbf{w}^T \phi)$$
- $p(\mathcal{C}_2 \mid \phi) = 1 - p(\mathcal{C}_1 \mid \phi)$
- Model dimension is equal to dimension of the feature space M.
- Compare this to fitting two Gaussians

$$\underbrace{2M}_{\text{means}} + \underbrace{M(M+1)/2}_{\text{shared covariance}} = M(M+5)/2$$

- For larger M, the logistic regression model has a clear advantage.

Logistic Regression is Classification

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- Determine the parameter via maximum likelihood for data $(\boldsymbol{\phi}_n, t_n)$, $n = 1, \dots, N$, where $\boldsymbol{\phi}_n = \boldsymbol{\phi}(\mathbf{x}_n)$. The class membership is coded as $t_n \in \{0, 1\}$.
- Likelihood function

$$p(\mathbf{t} \,|\, \mathbf{w}) = \prod_{n=1}^{N} y_n^{t_n} (1 - y_n)^{1-t_n}$$

  where $y_n = p(\mathcal{C}_1 \,|\, \boldsymbol{\phi}_n)$.
- Error function : negative log likelihood resulting in the cross-entropy error function

$$E(\mathbf{w}) = -\ln p(\mathbf{t} \,|\, \mathbf{w}) = -\sum_{n=1}^{N} \{t_n \ln y_n + (1 - t_n) \ln(1 - y_n)\}$$

Logistic Regression is Classification

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- Error function (cross-entropy error)

$$E(\mathbf{w}) = -\sum_{n=1}^{N} \{t_n \ln y_n + (1 - t_n) \ln(1 - y_n)\}$$

- $y_n = p(\mathcal{C}_1 \mid \phi_n) = \sigma(\mathbf{w}^T \phi_n)$
- Gradient of the error function (using $\frac{d\sigma}{da} = \sigma(1 - \sigma)$)

$$\nabla E(\mathbf{w}) = \sum_{n=1}^{N} (y_n - t_n)\phi_n$$

- gradient does not contain any sigmoid function
- for each data point error is product of deviation $y_n - t_n$ and basis function ϕ_n.
- BUT : maximum likelihood solution can exhibit over-fitting even for many data points; should use regularised error or MAP then.

Laplace Approximation

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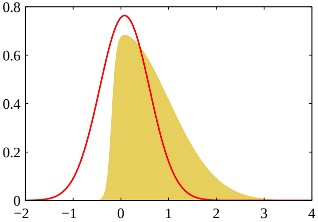
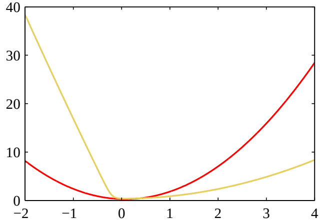
- Given a continous distribution $p(x)$ which is not Gaussian, can we approximate it by a Gaussian $q(x)$?
- Need to find a mode of $p(x)$. Try to find a Gaussian with the same mode.



Non-Gaussian (yellow) and Gaussian approximation (red).



Negative log of the Non-Gaussian (yellow) and Gaussian approx. (red).

Laplace Approximation

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- Assume $p(x)$ can be written as

$$p(z) = \frac{1}{Z} f(z)$$

with normalisation $Z = \int f(z) \, \mathrm{d}z$.

- Furthermore, assume Z is unknown !
- A mode of $p(z)$ is at a point z_0 where $p'(z_0) = 0$.
- Taylor expansion of $\ln f(z)$ at z_0

$$\ln f(z) \simeq \ln f(z_0) - \frac{1}{2} A (z - z_0)^2$$

where

$$A = -\frac{d^2}{dz^2} \ln f(z) \mid_{z=z_0}$$

Laplace Approximation

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- Exponentiating

$$\ln f(z) \simeq \ln f(z_0) - \frac{1}{2} A (z - z_0)^2$$

- we get

$$f(z) \simeq f(z_0) \exp\{-\frac{A}{2}(z - z_0)^2\}.$$

- And after normalisation we get the Laplace approximation

$$q(z) = \left(\frac{A}{2\pi}\right)^{1/2} \exp\{-\frac{A}{2}(z - z_0)^2\}.$$

- Only defined for precision $A > 0$ as only then $p(z)$ has a maximum.

Laplace Approximation - Vector Space

- Approximate p($\mathbf{z}$) for $z \in \mathbb{R}^M$

$$p(\mathbf{z}) = \frac{1}{Z} f(\mathbf{z}).$$

- we get the Taylor expansion

$$\ln f(\mathbf{z}) \simeq \ln f(\mathbf{z}_0) - \frac{1}{2}(\mathbf{z} - \mathbf{z}_0)^T \mathbf{A}(\mathbf{z} - \mathbf{z}_0)$$

- where the Hessian $\mathbf{A}$ is defined as

$$\mathbf{A} = -\nabla\nabla \ln f(\mathbf{z}) \mid_{\mathbf{z}=\mathbf{z}_0}.$$

- The Laplace approximation of $p(\mathbf{z})$ is then

$$q(\mathbf{z}) = \frac{|\mathbf{A}|^{1/2}}{(2\pi)^{M/2}} \exp\left\{-\frac{1}{2}(\mathbf{z} - \mathbf{z}_0)^T \mathbf{A}(\mathbf{z} - \mathbf{z}_0)\right\}$$
$$= \mathcal{N}(\mathbf{z} \,|\, \mathbf{z}_0, \mathbf{A}^{-1})$$

Bayesian Logistic Regression

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- Exact Bayesian inference for the logistic regression is intractable.
- Why? Need to normalise a product of prior probabilities and likelihoods which itself are a product of logistic sigmoid functions, one for each data point.
- Evaluation of the predictive distribution also intractable.
- Therefore we will use the Laplace approximation.

Bayesian Logistic Regression

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- Assume a Gaussian prior because we want a Gaussian posterior.

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w} \,|\, \mathbf{m}_0, \mathbf{S}_0)$$

  for fixed hyperparameter $\mathbf{m}_0$ and $\mathbf{S}_0$.
- Hyperparameters are parameters of a prior distribution. In contrast to the model parameters $\mathbf{w}$, they are not learned.
- For a set of training data $(\mathbf{x}_n, t_n)$, where $n = 1, \dots, N$, the posterior is given by

$$p(\mathbf{w} \,|\, \mathsf{t}) \propto p(\mathbf{w}) p(\mathsf{t} \,|\, \mathbf{w})$$

  where $\mathsf{t} = (t_1, \dots, t_N)^T$.

Bayesian Logistic Regression

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- Using our previous result for the cross-entropy function

$$E(\mathbf{w}) = -\ln p(\mathbf{t} \mid \mathbf{w}) = -\sum_{n=1}^{N} \{t_n \ln y_n + (1 - t_n) \ln(1 - y_n)\}$$

we can now calculate the log of the posterior

$$p(\mathbf{w} \mid \mathbf{t}) \propto p(\mathbf{w}) p(\mathbf{t} \mid \mathbf{w})$$

using the notation $y_n = \sigma(\mathbf{w}^T \phi_n)$ as

$$\begin{aligned} \ln p(\mathbf{w} \mid \mathbf{t}) = & -\frac{1}{2}(\mathbf{w} - \mathbf{m}_0)^T \mathbf{S}_0^{-1} (\mathbf{w} - \mathbf{m}_0) \\ & + \sum_{n=1}^{N} \{t_n \ln y_n + (1 - t_n) \ln(1 - y_n)\} \end{aligned}$$

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- To obtain a Gaussian approximation to

$$\ln p(\mathbf{w} \mid \mathbf{t}) = -\frac{1}{2}(\mathbf{w} - \mathbf{m}_0)^T \mathbf{S}_0^{-1} (\mathbf{w} - \mathbf{m}_0) + \sum_{n=1}^{N} \{ t_n \ln y_n + (1 - t_n) \ln(1 - y_n) \}$$

1. Find $\mathbf{w}_{MAP}$ which maximises $\ln p(\mathbf{w} \mid \mathbf{t})$. This defines the mean of the Gaussian approximation. (Note: This is a nonlinear function in $\mathbf{w}$ because $y_n = \sigma(\mathbf{w}^T \phi_n)$.)
2. Calculate the second derivative of the negative log likelihood to get the inverse covariance of the Laplace approximation

$$\mathbf{S}_N = -\nabla\nabla \ln p(\mathbf{w} \mid \mathbf{t}) = \mathbf{S}_0^{-1} + \sum_{n=1}^{N} y_n (1 - y_n) \phi_n \phi_n^T.$$

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- The approximated Gaussian (via Laplace approximation) of the posterior distribution is now

$$q(\mathbf{w} \mid \phi) = \mathcal{N}(\mathbf{w} \mid \mathbf{w}_{MAP}, \mathbf{S}_N)$$

where

$$\mathbf{S}_N = -\nabla\nabla \ln p(\mathbf{w} \mid \mathbf{t}) = \mathbf{S}_0^{-1} + \sum_{n=1}^{N} y_n(1-y_n)\phi_n\phi_n^T.$$