The Fundamentals of Logic

Introduction to proofs

Lecture-5/6

Topic covered

1. Introduction to FoLT (Completed)
2. What are Logic- Propositional (Completed)
3. Types of operators for Logic (Completed)
4. Fuzzy Logic(Completed)
5. Propositional Equivalences(Completed)
6. Predicates and Quantifiers (Completed)
7. Rules of Inference (Completed)
8. Introduction to proofs
9. Normal forms

Example -1

- Which one of the following propositional logic formulas is TRUE when exactly two of p, q, and r are TRUE?

(A) $\big((p \leftrightarrow q) \land r\big) \lor (p \land q \land \sim r)$

(B) $(\sim (p \leftrightarrow q) \land r) \lor (p \land q \land \sim r)$

(C) $\big((p \rightarrow q) \land r\big) \lor (p \land q \land \sim r)$

(D) $(\sim (p \leftrightarrow q) \land r) \land (p \land q \land \sim r)$

(A) A
(B) B
(C) C
(D) D

Rule of Inference revision Example -2

1. A student in this class has not read the book.
2. Everyone in this class, passed the first exam.

- Is the Conclusion True: “Someone who passed the first exam has not read the book”?

C(x) := x is a student in the class   B(x) := x has read the book
P(x) := x passed the first exam

$$\exists x\,(\ \mathrm{C}(x) \wedge \neg \mathrm{B}(x)\ ),$$
$$\forall x\,(\ \mathrm{C}(x) \rightarrow \mathrm{P}(x)\ )$$
$$\exists x\,(\ \mathrm{P}(x) \wedge \neg\ \mathrm{B}(x)\ )$$

Step	**Reason**
1. $\exists x\,(\,C(x) \land \neg B(x)\,)$	Premise
2. $C(a) \land \neg B(a)$	Existential Instantiation
3. $C(a)$	Simplification by (2)
4. $\forall x\,(\,C(x) \rightarrow P(x)\,)$	Premise
5. $C(a) \rightarrow P(a)$	Universal Instantiation
6. $P(a)$	Modus Ponens by (3) and (5)
7. $\neg B(a)$	Simplification by (2)
8. $P(a) \land \neg B(a)$	Conjunction by (6) and (7)
9. $\exists x\,(\,P(x) \land \neg B(x)\,)$	Existential Generalization

Which one of the following is **NOT** logically equivalent to $\neg\exists x(\forall y(\alpha)\wedge\forall z(\beta))$?

(A) $\forall x(\exists z(\neg\beta)\rightarrow\forall y(\alpha))$

(B) $\forall x(\forall z(\beta)\rightarrow\exists y(\neg\alpha))$

(C) $\forall x(\forall y(\alpha)\rightarrow\exists z(\neg\beta))$

(D) $\forall x(\exists y(\neg\alpha)\rightarrow\exists z(\neg\beta))$

(A) A

(B) B

(C) C

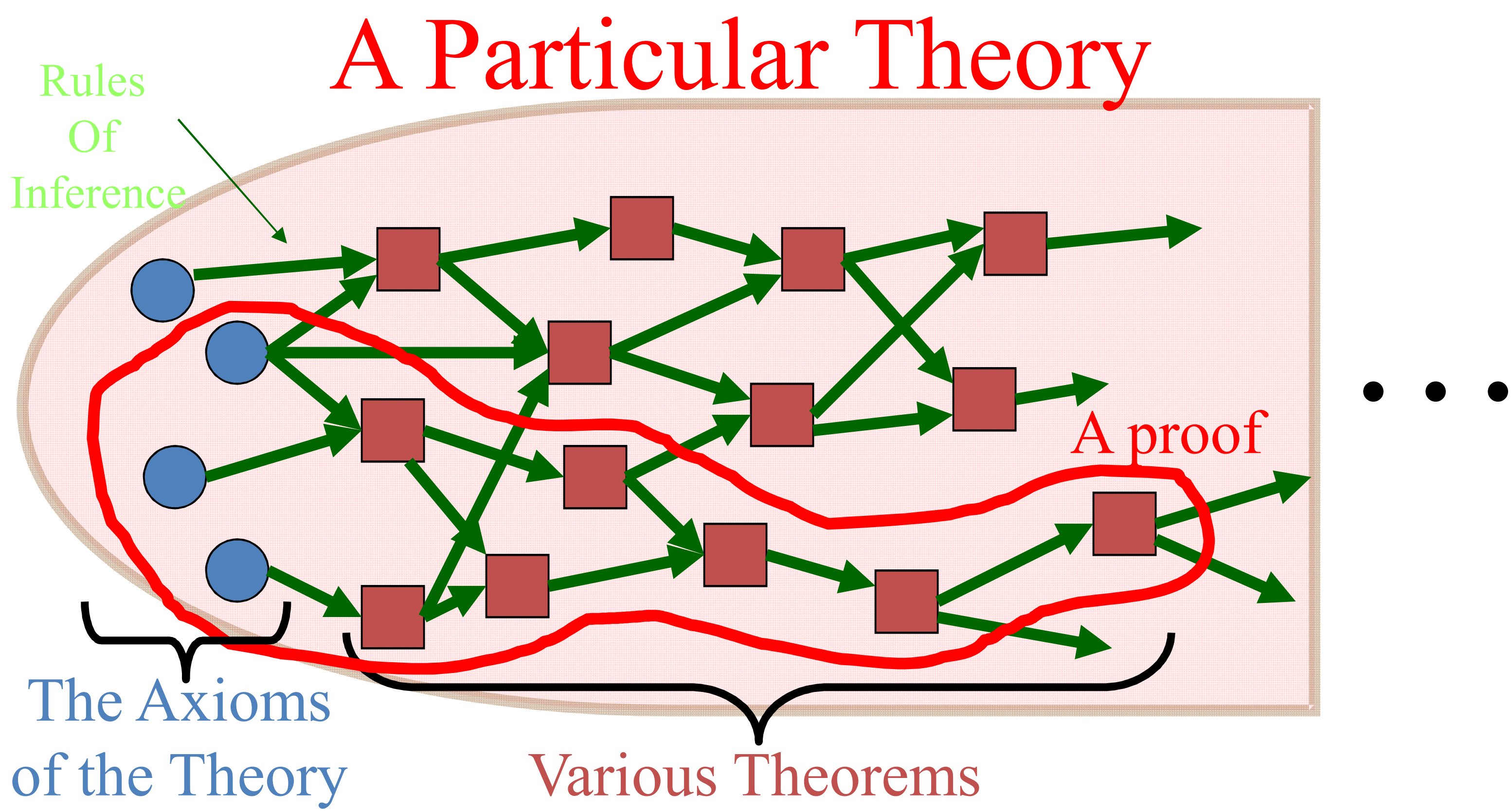
(D) D

Introduction to proofs

Nature & Importance of Proofs

- In mathematics, a *proof* is:
  - A sequence of statements that form an argument.
  - Must be *correct* (well-reasoned, logically valid) and *complete* (clear, detailed) that rigorously & undeniably establishes the truth of a mathematical statement.
- Why must the argument be correct & complete?
  - *Correctness* prevents us from fooling ourselves.
  - *Completeness* allows anyone to verify the result.

Visualization of Proofs



Formal Proofs

- A formal proof of a conclusion *C*, given premises $p_1, p_2, \ldots, p_n$ consists of a sequence of *steps*, each of which applies some inference rule to premises or to previously-proven statements (as hypotheses) to yield a new true statement (the conclusion).
- A proof demonstrates that *if* the premises are true, *then* the conclusion is true (i.e., valid argument).

Formal Proof - Example

- Suppose we have the following premises:
  "It is not sunny and it is cold."
  "if it is not sunny, we will not swim"
  "If we do not swim, then we will boat."
  "If we boat, then we will be home early."
- Given these premises, prove the theorem
  "We will be home early" using inference rules.

Proof Example *cont.*

- Let us adopt the following abbreviations:

  sunny = **"It is sunny"**; *cold* = **"It is cold"**;
  swim = **"We will swim"**; *boat* = **"We will boat"**;
  early = **"We will be home early"**.

- Then, the premises can be written as:
  (1) $\neg sunny \wedge cold$ (2) $\neg sunny \rightarrow \neg swim$
  (3) $\neg swim \rightarrow boat$ (4) $boat \rightarrow early$

Proof Example *cont*.

Step	Proved by
1. $\neg sunny \wedge cold$	Premise #1.
2. $\neg sunny$	Simplification of 1.
3. $\neg sunny \rightarrow \neg swim$	Premise #2.
4. $\neg swim$	Modus tollens on 2,3.
5. $\neg swim \rightarrow boat$	Premise #3.
6. $boat$	Modus ponens on 4,5.
7. $boat \rightarrow early$	Premise #4.
8. $early$	Modus ponens on 6,7.

Common Fallacies

- A *fallacy* is an inference rule or other proof method that is not logically valid.
  - May yield a false conclusion!
- Fallacy of *affirming the conclusion*:
  - "$p \rightarrow q$ is true, and q is true, so p must be true." (No, because $\mathbf{F} \rightarrow \mathbf{T}$ is true.)
- Fallacy of *denying the hypothesis*:
  - "$p \rightarrow q$ is true, and p is false, so q must be false." (No, again because $\mathbf{F} \rightarrow \mathbf{T}$ is true.)

Common Fallacies - Examples

"If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics."

p: "You did every problem in this book"

q: "You learned discrete mathematics"

- Fallacy of *affirming the conclusion*:

  $p \rightarrow q$ *and* q does not imply p

- Fallacy of *denying the hypothesis*:

  $p \rightarrow q$ and $\neg p$ does not imply $\neg q$

Proof Methods

- Proving $p \rightarrow q$
  - *Direct* proof: Assume p is true, and prove q.
  - *Indirect* proof: Assume $\neg q$, and prove $\neg p$.
  - *Trivial* proof: Prove q true.
  - *Vacuous* proof: Prove $\neg p$ is true.
- Proving p
  - Proof by *contradiction:* Prove $\neg p \rightarrow (\mathrm{r} \wedge \neg \mathrm{r})$

  ($\mathrm{r} \wedge \neg \mathrm{r}$ is a contradiction); therefore $\neg p$ must be false.
- Prove $(a \vee b) \rightarrow p$
  - Proof by cases: prove $(a \rightarrow p)$ and $(b \rightarrow p)$.

Direct Proof Example

- **Definition:** An integer n is called *odd* iff n=2k+1 for some integer k; n is *even* iff n=2k for some k.
- **Axiom:** Every integer is either odd or even.
- **Theorem:** (For all numbers n) If n is an odd integer, then n^2 is an odd integer.
- **Proof:** If n is odd, then $n = 2k+1$ for some integer k. Thus, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Therefore n^2 is of the form $2j + 1$ (with j the integer $2k^2 + 2k$), thus n^2 is odd. □

Indirect Proof

- Proving $p \rightarrow q$
  - *Indirect* proof: Assume $\neg q$, and prove $\neg p$.

Indirect Proof Example

- **Theorem:** (For all integers n)
  If $3n+2$ is odd, then n is odd.
- **Proof:** Suppose that the conclusion is false, *i.e.*, that n is even. Then $n=2k$ for some integer k. Then $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$. Thus $3n+2$ is even, because it equals $2j$ for integer $j = 3k+1$. So $3n+2$ is not odd. We have shown that $\neg(n \text{ is odd}) \rightarrow \neg(3n+2 \text{ is odd})$, thus its contra-positive $(3n+2 \text{ is odd}) \rightarrow (n \text{ is odd})$ is also true. □

Another Example

- **Theorem:** Prove that if n is an integer and n^2 is odd, then n is odd.

Trivial Proof

- Proving $p \rightarrow q$
  - *Trivial* proof: Prove q true.

Trivial Proof Example

- **Theorem:** (For integers *n*) If *n* is the sum of two prime numbers, then either *n* is odd or *n* is even.
- **Proof:** *Any* integer *n* is either odd or even. So the conclusion of the implication is true regardless of the truth of the hypothesis. Thus the implication is true trivially. □

Vacuous Proof

- Proving $p \rightarrow q$
  - *Vacuous* proof: Prove $\neg p$ is true.

Vacuous Proof Example

- **Theorem:** (For all n) If n is both odd and even, then $n^2 = n + n$.
- **Proof:** The statement "n is both odd and even" is necessarily false, since no number can be both odd and even. So, the theorem is vacuously true. □

Proof by Contradiction

- Proving p
  - Assume $\neg p$, and prove that $\neg p \rightarrow (r \land \neg r)$
  - $(r \land \neg r)$ is a trivial contradiction, equal to **F**
  - Thus $\neg p \rightarrow$**F** is true only if $\neg p$=**F**

Contradiction Proof Example

- **Theorem:** Prove that $\sqrt{2}$ is irrational.

Proof by Cases

To prove $(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$

we need to prove

$$(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)$$

Prove that if *n* is an integer, then *n*^2 ≥ *n*.

Solution: We can prove that $n^2 \geq n$ for every integer by considering three cases, when $n = 0$, when $n \geq 1$, and when $n \leq -1$. We split the proof into three cases because it is straightforward to prove the result by considering zero, positive integers, and negative integers separately.

Case (i): When $n = 0$, because $0^2 = 0$, we see that $0^2 \geq 0$. It follows that $n^2 \geq n$ is true in this case.

Case (ii): When $n \geq 1$, when we multiply both sides of the inequality $n \geq 1$ by the positive integer n, we obtain $n \cdot n \geq n \cdot 1$. This implies that $n^2 \geq n$ for $n \geq 1$.

Case (iii): In this case $n \leq -1$. However, $n^2 \geq 0$. It follows that $n^2 \geq n$.

Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$. ◀

Equivalence of a group of propositions

To prove a theorem that is a biconditional statement, that is, a statement of the form

$$(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p).$$

$$[p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n]$$

$$= [(p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_3) \wedge \ldots (p_n \rightarrow p_1)]$$

Example

Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

Example (5 Min)

Show that these statements about the integer n are equivalent:

$p1$: n is even.

$p2$: n - 1 is odd.

$p3$: n^2 is even.

exhaustive proofs

Some theorems can be proved by examining are relatively small number of examples

An exhaustive proof is a special type of proof by cases where each case involves checking a single example

Prove that *(n + 1)*^3 ≥ 3*n* if *n* is a positive integer with *n* ≤ 4

Solution: We use a proof by exhaustion. We only need verify the inequality $(n+1)^3 \geq 3^n$ when $n = 1, 2, 3$, and 4. For $n = 1$, we have $(n+1)^3 = 2^3 = 8$ and $3^n = 3^1 = 3$; for $n = 2$, we have $(n+1)^3 = 3^3 = 27$ and $3^n = 3^2 = 9$; for $n = 3$, we have $(n+1)^3 = 4^3 = 64$ and $3^n = 3^3 = 27$; and for $n = 4$, we have $(n+1)^3 = 5^3 = 125$ and $3^n = 3^4 = 81$. In each of these four cases, we see that $(n+1)^3 \geq 3^n$. We have used the method of exhaustion to prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$. ◀

Counterexamples

- When we are presented with a statement of the form $\forall x P(x)$ and we believe that it is false, then we look for a counterexample.
- We look for the x for which this can be false.
- Example
  - Is it true that "every positive integer is the sum of the squares of three integers?"

Mistakes in Proofs

Each step of a mathematical proof needs to be correct and the conclusion needs to follow logically from the steps that precede it.

What is wrong with this famous supposed "proof" that $1 = 2$?

"Proof:" We use these steps, where a and b are two equal positive integers.

Step	**Reason**
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

- Step 5 where we divided both sides by $a - b$.

Proving Existentials

- A proof of a statement of the form $\exists x\, P(x)$ is called an *existence proof*.
- If the proof demonstrates how to actually find or construct a specific element a such that $P(a)$ is true, then it is called a *constructive* proof.
- Otherwise, it is called a *non-constructive* proof.

Constructive Existence Proof

- **Theorem:** There exists a positive integer n that is the sum of two perfect cubes in two different ways:
  - equal to $j^3 + k^3$ and $l^3 + m^3$ where j, k, l, m are positive integers, and $\{j,k\} \neq \{l,m\}$
- **Proof:** Consider $n = 1729$, $j = 9$, $k = 10$, $l = 1$, $m = 12$. Now just check that the equalities hold.

Non-constructive Existence Proof

- **Theorem:**
  "There are infinitely many prime numbers."
- Any finite set of numbers must contain a maximal element, so we can prove the theorem if we can just show that <u>there is *no* largest prime number</u>.
- *i.e.*, Show that there exist irrational numbers x and y such that x^y is rational.

we know that $\sqrt{2}$ is irrational. Consider the number

$\sqrt{2}^{\sqrt{2}}$. If it is rational, we have two irrational numbers x and y with x^y rational, namely, $x = \sqrt{2}$ and $y = \sqrt{2}$. On the other hand if $\sqrt{2}^{\sqrt{2}}$ is irrational, then we can let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$

so that $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^2 = 2.$

This proof is an example of a nonconstructive existence proof because we have not found irrational numbers x and y such that x^y is rational. Rather, we have shown that either the pair $x = \sqrt{2}$, $y = \sqrt{2}$ or the pair $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$ have the desired property, but we do not know which of these two pairs works! ◀