The Fundamentals of Logic

Predicate Logic

Lecture-3

Topic covered:Lecture-2

1. Introduction to FoLT (Completed)
2. What are Logic- Propositional (Completed)
3. Types of operators for Logic (Completed)
4. Fuzzy Logic(Completed)
5. Propositional Equivalences(Completed)
6. Predicates and Quantifiers
7. Rules of Inference
8. Introduction to proofs
9. Normal forms

Practice Question

- P and Q are two propositions. Which of the following logical expressions are equivalent?

I. $P \vee \sim Q$

II. $\sim(\sim P \wedge Q)$

III. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

IV. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$

(A) Only I and II
(B) Only I, II and III
(C) Only I, II and IV
(D) All of I, II, III and IV

Predicate Logic

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.
- Propositional logic (recall) treats simple *propositions* (sentences) as atomic entities.
- In contrast, *predicate* logic distinguishes the *subject* of a sentence from its *predicate.*

Applications of Predicate Logic

It is *the* formal notation for writing perfectly clear, concise, and unambiguous mathematical *definitions*, *axioms*, and *theorems* for *any* branch of mathematics.

Predicate logic with function symbols, the "=" operator, and a few proof-building rules is sufficient for defining *any* conceivable mathematical system, and for proving anything that can be proved within that system!

Practical Applications of Predicate Logic

- It is the basis for clearly expressed formal specifications for any complex system.
- It is basis for *automatic theorem provers* and many other Artificial Intelligence systems.
  - *E.g.* automatic program verification systems.
- Predicate-logic like statements are supported by some of the more sophisticated *database query engines* and *container class libraries*
  - these are types of programming tools.

Subjects and Predicates

- In the sentence "The dog is sleeping":
  - The phrase "the dog" denotes the *subject* - the *object* or *entity* that the sentence is about.
  - The phrase "is sleeping" denotes the *predicate*- a property that is true **of** the subject.
- In predicate logic, a *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).

More About Predicates

- Convention: Lowercase variables x, y, z... denote objects/entities; uppercase variables P, Q, R… denote propositional functions (predicates).
- Keep in mind that the *result of applying* a predicate P to an object x is the *proposition* $P(x)$. But the predicate P **itself** (*e.g.* P=“is sleeping”) is **not** a proposition (not a complete sentence).
  - *E.g.* if $P(x)$ = “x is a prime number”,
    $P(3)$ is the *proposition* “3 is a prime number.”

Propositional Functions

- Predicate logic *generalizes* the grammatical notion of a predicate to also include propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take.
  - *E.g.* let $P(x,y,z)$ = "x gave y the grade z", then if x="Mike", y="Mary", z="A", then $P(x,y,z)$ = "Mike gave Mary the grade A."

E.g:1

Let *P (x)* denote the statement "$x > 3$." What are the truth values of *P* (4) and *P* (2)?

We obtain the statement *P* (4) by setting $x = 4$ in the statement "$x > 3$." Hence,
P (4), which is the statement "4 > 3," is true. However, *P* (2), which is the statement "2 > 3,"
is false.

E.g:2

Let $Q(x, y)$ denote the statement "$x = y + 3$."
What are the truth values of the propositions
$Q(1, 2)$ and $Q(3, 0)$?

- To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "1 = 2 + 3," which is false. The statement $Q(3, 0)$ is the proposition "3 = 0 + 3," which is true. ▲

E.g:3

Consider the statement
if $x > 0$ **then** $x := x + 1$.

Universes of Discourse (U.D.s)

- The power of distinguishing objects from predicates is that it lets you state things about *many* objects at once.
- E.g., let $P(x)$="$x+1>x$". We can then say, "For *any* number x, $P(x)$ is true" instead of $(\mathbf{0}+1>\mathbf{0}) \wedge (\mathbf{1}+1>\mathbf{1}) \wedge (\mathbf{2}+1>\mathbf{2}) \wedge \ldots$
- The collection of values that a variable x can take is called x's *universe of discourse*.

Quantifier Expressions

- *Quantifiers* provide a notation that allows us to *quantify* (count) *how many* objects in the univ. of disc. satisfy a given predicate.
- "$\forall$" is the FOR$\forall$LL or *universal* quantifier. $\forall x\ P(x)$ means *for all* x in the u.d., *P* holds.
- "$\exists$" is the $\exists$XISTS or *existential* quantifier. $\exists x\ P(x)$ means there *exists* an *x* in the u.d. (that is, 1 or more) such that $P(x)$ is true.

The Universal Quantifier $\forall$

- Example:
  Let the u.d. of x be parking spaces at MNNIT.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *universal quantification of* $P(x)$, $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at MNNIT are full."
  - *i.e.*, "Every parking space at MNNIT is full."
  - *i.e.*, "For each parking space at MNNIT, that space is full."

e.g:4

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

- Because $P(x)$ is true for all real numbers x, the quantification
  $\forall x P(x)$
  is true.

Other words

- Besides “for all” and “for every,” universal quantification can be expressed in many other ways, including “all of,” “for each,” “given any,” “for arbitrary,” “for each,” and “for any.
- Avoid using “for any *x*” because it is often ambiguous as to whether “any” means “every” or “some.” In some cases, “any” is unambiguous, such as when it is used in negatives, for example, “there is not any reason to avoid studying.

e.g:5

- What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

- The statement $\forall x P(x)$ is the same as the conjunction
  $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$,
  because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement “$4^2 < 10$,” is false, it follows that $\forall x P(x)$ is false

The Existential Quantifier ∃

- Example:
  Let the u.d. of x be parking spaces at MNNIT.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *existential quantification of* $P(x)$, $\exists x\, P(x)$, is the *proposition*:
  - "Some parking space at MNNIT is full."
  - "There is a parking space at MNNIT that is full."

E.g:6

What do the statements $\forall x < 0\,(x^2 > 0)$, $\forall y \neq 0\,(y^3 \neq 0)$, and $\exists z > 0\,(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

E.g:7

- What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "x^2 < 10" and the domain consists of the positive integers not exceeding 4?

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
- $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.
- In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

E.g:8

Which one of the following options is CORRECT given three positive integers x, y and z, and a predicate?

$P(x) = \neg(x=1) \wedge \forall y(\exists z(x=y*z) \rightarrow (y=x) \vee (y=1))$

(A) P(x) being true means that x is a prime number
(B) P(x) being true means that x is a number other than 1
(C) P(x) is always true irrespective of the value of x
(D) P(x) being true means that x has exactly two factors other than 1 and x

Free and Bound Variables

- An expression like $P(x)$ is said to have a *free variable* x (meaning, x is undefined).
- A quantifier (either $\forall$ or $\exists$) *operates* on an expression having one or more free variables, and *binds* one or more of those variables, to produce an expression having one or more *bound variables*.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\, P(x,y)$ has 1 free variable, and one bound variable. [Which is which?]
- "$P(x)$, where x=3" is another way to bind x.
- An expression with <u>one or more</u> free variables is still only a predicate: *e.g.* let $Q(y) = \forall x\, P(x,y)$

Nesting of Quantifiers

Example: Let the u.d. of x & y be people.

Let $L(x,y)$="x likes y" (a predicate w. 2 f.v.'s)

Then $\exists y\, L(x,y)$ = "There is someone whom x likes." (A predicate w. 1 free variable, x)

Then $\forall x\, (\exists y\, L(x,y))$ =

"Everyone has someone whom they like."

(A ___________ with ___ free variables.)

Proposition 0

Quantifier Exercise

If $R(x,y)$="x relies upon y," express the following in unambiguous English:

$\forall x(\exists y\, R(x,y))=$

$\exists y(\forall x\, R(x,y))=$

$\exists x(\forall y\, R(x,y))=$

$\forall y(\exists x\, R(x,y))=$

$\forall x(\forall y\, R(x,y))=$

Everyone has *someone* to rely on.

There's a poor overburdened soul whom *everyone* relies upon (including himself)!

There's some needy person who relies upon *everybody* (including himself).

Everyone has *someone* who relies upon them.

Everyone relies upon *everybody*, (including themselves)!

Still More Conventions

- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x>0\ P(x)$ is shorthand for
    "For all x that are greater than zero, $P(x)$."
    $=\forall x\ (x>0 \rightarrow P(x))$
  - $\exists x>0\ P(x)$ is shorthand for
    "There is an x greater than zero such that $P(x)$."
    $=\exists x\ (x>0 \wedge P(x))$

E.g:9

- Which one of the following is the most appropriate logical formula to represent the statement? "Gold and silver ornaments are precious". The following notations are used: G(x): x is a gold ornament S(x): x is a silver ornament P(x): x is precious
- The following notations are used:
  G(x): x is a gold ornament
  S(x): x is a silver ornament
  P(x): x is precious
  (A) $\forall x(P(x) \rightarrow (G(x) \wedge S(x)))$
  (B) $\forall x((G(x) \wedge S(x)) \rightarrow P(x))$
  (C) $\exists x((G(x) \wedge S(x)) \rightarrow P(x)$
  (D) $\forall x((G(x) \vee S(x)) \rightarrow P(x))$

More to Know About Binding

- $\forall x\ \exists x\ P(x)$ - x is not a free variable in $\exists x\ P(x)$, therefore the $\forall x$ binding isn't used.
- $(\forall x\ P(x)) \wedge Q(x)$ - The variable x is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a complete proposition!
- $(\forall x\ P(x)) \wedge (\exists x\ Q(x))$ – This is legal, because there are 2 different x's!

Quantifier Equivalence Laws

- Definitions of quantifiers: If u.d.=a,b,c,…
  $\forall x\, P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\, P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$
- From those, we can prove the laws:
  $\forall x\, P(x) \Leftrightarrow \neg\exists x\, \neg P(x)$
  $\exists x\, P(x) \Leftrightarrow \neg\forall x\, \neg P(x)$
- Which *propositional* equivalence laws can be used to prove this? (H.W)

E.g:10

- Which of the two are equivalent?

I. $\neg\forall x\,(P(x))$ II. $\neg\exists x\,(P(x))$

III. $\neg\exists x\,(\neg P(x))$ IV. $\exists x\,(\neg P(x))$

(A) I and III
(B) I and IV
(C) II and III
(D) II and IV

More Equivalence Laws

- $\forall x\ \forall y\ P(x,y) \Leftrightarrow \forall y\ \forall x\ P(x,y)$
  $\exists x\ \exists y\ P(x,y) \Leftrightarrow \exists y\ \exists x\ P(x,y)$
- $\forall x\ \mathbf{(}P(x) \wedge Q(x)\mathbf{)} \Leftrightarrow \mathbf{(}\forall x\ P(x)\mathbf{)} \wedge \mathbf{(}\forall x\ Q(x)\mathbf{)}$
  $\exists x\ \mathbf{(}P(x) \vee Q(x)\mathbf{)} \Leftrightarrow \mathbf{(}\exists x\ P(x)\mathbf{)} \vee \mathbf{(}\exists x\ Q(x)\mathbf{)}$

More Notational Conventions

- Quantifiers bind as loosely as needed: parenthesize $\forall x \Big(P(x) \wedge Q(x) \ \Big)$
- Consecutive quantifiers of the same type can be combined: $\forall x\ \forall y\ \forall z\ P(x,y,z) \Leftrightarrow \forall x,y,z\ P(x,y,z)$ or even $\forall xyz\ P(x,y,z)$
- All quantified expressions can be reduced to the canonical *alternating* form $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \ldots P(x_1, x_2, x_3, x_{4,} \ldots)$

Practice Example

Suppose the predicate $F(x, y, t)$ is used to represent the statement that person x can fool person y at time t. which one of the statements below expresses best the meaning of the formula $\forall x \exists y \exists t(\neg F(x, y, t))$?

(A) Everyone can fool some person at some time
(B) No one can fool everyone all the time
(C) Everyone cannot fool some person all the time
(D) No one can fool some person at some time

Review: Predicate Logic

- Objects x, y, z, …
- Predicates P, Q, R, … are functions mapping objects x to propositions $P(x)$.
- Multi-argument predicates $P(x, y)$.
- Quantifiers: $[\forall x\ P(x)] :\equiv$ "For all x's, $P(x)$."
  $[\exists x\ P(x)] :\equiv$ "There is an x such that $P(x)$."
- Universes of discourse, bound & free vars.