Identification of Linear Elastic Constants in a 2D Linear Elastic Plate

APL104: Solid Mechanics - Fall 2024

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1. Introduction

This project focuses on identifying the linear elastic constants of a 2D square plate made of homogeneous isotropic material under quasi-static displacement-controlled loading. The material exhibits linear elasticity, characterized by a linear relationship between stress and strain within its elastic limit, described by its Lame’s constants, lambda (λ) and mu (μ).

The boundary value problem (BVP) involves the square plate with an elliptical hole, subjected to Dirichlet boundary conditions on its edges. Displacement data and reaction forces are measured over five loading steps using full-field digital image correlation (DIC) and load cells, respectively.

This experiment intends to identify the Lame’s constant values based on the parameter identification method and optimization algorithm.

2. Derivation of Governing Equations

The governing equations are derived by combining the stress-strain relationship (Hooke's law) and the stress-equilibrium equations with the strain compatibility equations. The material is assumed to be isotropic and linearly elastic. These equations link the displacements to the stress components, forming a set of partial differential equations that describe the system's response under applied boundary conditions.

ASSESSMENT (PART I)

{ DERIVATION OF GOVERNING EQNS }

(A)
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \cancelto{0}{\gamma_x} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \cancelto{0}{\gamma_y} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \cancelto{0}{\gamma_z} = 0$$

(Stress - Equilibrium equations)

(B)
$$\sigma_{xx} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{xx}$$
$$\sigma_{yy} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{yy}$$
$$\sigma_{zz} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{zz}$$
$$\tau_{xy} = 2\mu\epsilon_{xy}$$
$$\tau_{yz} = 2\mu\epsilon_{yz}$$
$$\tau_{zx} = 2\mu\epsilon_{zx}$$

(Stress - Strain Relation)

(C)
$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

(Strain - compatibility eq^{n}s)

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial y}\right); \quad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \quad \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)$$

using set (C) eq^{n}s in set (B),

$$\sigma_{xx} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_x}{\partial x} \quad \text{(Similar for } y, z\text{)}$$

$$\tau_{xy} = \cancel{2}\mu \frac{1}{\cancel{2}}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \quad \text{(Similar for } y, z\text{)}$$

—— (D)

To obtain the set of three governing equations we substitute (D) in (A).

using set ⓓ in Ⓐ.

$$\Rightarrow \frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+2\mu\frac{\partial u_x}{\partial x}\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)\right]=0$$

$$\Rightarrow (\lambda+2\mu)\frac{\partial^2}{\partial x^2}u_x+\lambda\frac{\partial^2 u_y}{\partial x\partial y}+\lambda\frac{\partial^2 u_z}{\partial x\partial z}$$

$$+\mu\frac{\partial^2}{\partial y^2}u_x+\mu\frac{\partial^2 u_x}{\partial z^2}+\mu\frac{\partial^2 u_y}{\partial y\partial x}+\mu\frac{\partial^2 u_z}{\partial x\partial z}=0$$

$$\Rightarrow (\lambda+2\mu)\frac{\partial^2}{\partial x^2}(u_x)+\mu\frac{\partial^2}{\partial y^2}(u_x)+\mu\frac{\partial^2}{\partial z^2}(u_x)+$$

$$(\lambda+\mu)\frac{\partial^2}{\partial x\partial y}(u_y)+(\lambda+\mu)\frac{\partial^2}{\partial x\partial z}(u_z)=0.$$

$$\Rightarrow \boxed{(\lambda+2\mu)\frac{\partial^2}{\partial x^2}(u_x)+\mu\left[\frac{\partial^2}{\partial y^2}(u_x)+\frac{\partial^2}{\partial z^2}(u_x)\right]+(\lambda+\mu)\left[\frac{\partial^2}{\partial x\partial y}(u_y)+\frac{\partial^2}{\partial x\partial z}(u_z)\right]=0} \quad —①$$

Similarly,

$$\Rightarrow \boxed{(\lambda+2\mu)\frac{\partial^2}{\partial y^2}(u_y)+\mu\left[\frac{\partial^2}{\partial x^2}(u_y)+\frac{\partial^2}{\partial z^2}(u_y)\right]+(\lambda+\mu)\left[\frac{\partial^2}{\partial x\partial y}(u_x)+\frac{\partial^2}{\partial y\partial z}(u_z)\right]=0} \quad —②$$

$$\Rightarrow \boxed{(\lambda+2\mu)\frac{\partial^2}{\partial z^2}(u_z)+\mu\left[\frac{\partial^2}{\partial x^2}(u_z)+\frac{\partial^2}{\partial y^2}(u_z)\right]+(\lambda+\mu)\left[\frac{\partial^2}{\partial x\partial z}(u_x)+\frac{\partial^2}{\partial y\partial z}(u_y)\right]=0.} \quad —③$$

①, ②, ③ represent the set of three governing equations.

3. Parameter identification methodology

3.1 Formulation of optimization problem

To identify Lame's constants, I first used Scipy's interpolation model to estimate the displacement functions ‘u_x’ and ‘u_y’ using a meshgrid over ‘x’ and ‘y’. To compute the required strains ‘ϵxx’ and ‘ϵyy’, I calculated the partial derivatives ‘∂ux/∂x’ and ‘∂uy/∂y’. To further calculate the integral I used the trapezoid function of scipy.integrate library.

The integral of ‘σyy dx’ should be equal to reaction force R2, and the integral of ‘σxx dy’ should be equal to R4. Since the net stresses were known along the top and right boundaries, the optimization problem simplifies to a system of two linear equations in two variables, making it optimum to solve for lambda (λ) and mu (μ).

The constraints for this optimization include:

1. Non-negative values for Lame's constants: λ≥0 and μ≥0.
2. The Neumann boundary conditions, requiring the integrals of the stress components over the boundary to be equal to the displacement control loads.

3.2 Parameter Identification Algorithm

Initially, I assumed a cubic fit for the displacement functions ‘u_x’ and ‘u_y’ with respect to ‘x’ and ‘y’. From these displacement functions, I calculated the strain tensor at each point using strain-displacement relations. The stress at any general point was then expressed in terms of Lame’s constants lambda (λ) and mu (μ).

To apply the Neumann boundary conditions, I calculated the integrals of the stresses over the boundaries x = 1 and y = 1, ensuring equilibrium conditions were met. Solving the two equations derived from the boundary conditions provided an estimate for Lame’s constants.

$$\int_{\text{top edge}} \sigma_{yy}\, dx = R_2 \qquad \int_{\text{right edge}} \sigma_{xx}\, dy = R_4$$

$$R_4 = \int_0^1 \left[\lambda \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + 2\mu \frac{\partial u_x}{\partial x} \right] dy \quad R_2 = \int_0^1 \left[\lambda \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + 2\mu \frac{\partial u_y}{\partial y} \right] dx$$

$$R_4 = \lambda \int_0^1 \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) dy + 2\mu \int_0^1 \frac{\partial u_x}{\partial x}\, dy \qquad R_2 = \lambda \int_0^1 \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) dx + 2\mu \int_0^1 \frac{\partial u_y}{\partial y}\, dx$$

$$R_4 = \lambda A_4 + 2\mu B_4 \qquad R_2 = \lambda A_2 + 2\mu B_2$$

However, when I used the fitted displacement functions for calculating the integrals, the resulting values of the constants were negative, thus violating the essential constraint condition. To resolve this, I switched to directly calculating the integrals using the given experimental data, and then solving the two equations, which gave me more accurate values for Lame’s constants.

4. Results

4.1 Identified Lame’s Constants

After implementing the identification algorithm, the values of Lame’s constants (λ and μ) are extracted.

The average of identified Lame’s Constants are:

- **Lame’s first constant (λ):** 123886424212.93498 Pa, (approx. **123.88 GPa**)
- **Lame’s second constant (μ):** 80671499291.83545 Pa, (approx. **80.67 GPa**)

Lame’s Constants for different load steps

1ST LOAD STEP:

- Lame’s first constant λ: 123883832096.86609
- Lame’s second constant μ: 80673343554.46678

2ND LOAD STEP:

- Lame’s first constant λ: 123883832099.85689
- Lame’s second constant μ: 80673343548.19536

3RD LOAD STEP:

- Lame’s first constant λ: 123888717291.63132
- Lame’s second constant μ: 80669085012.20647

4TH LOAD STEP:

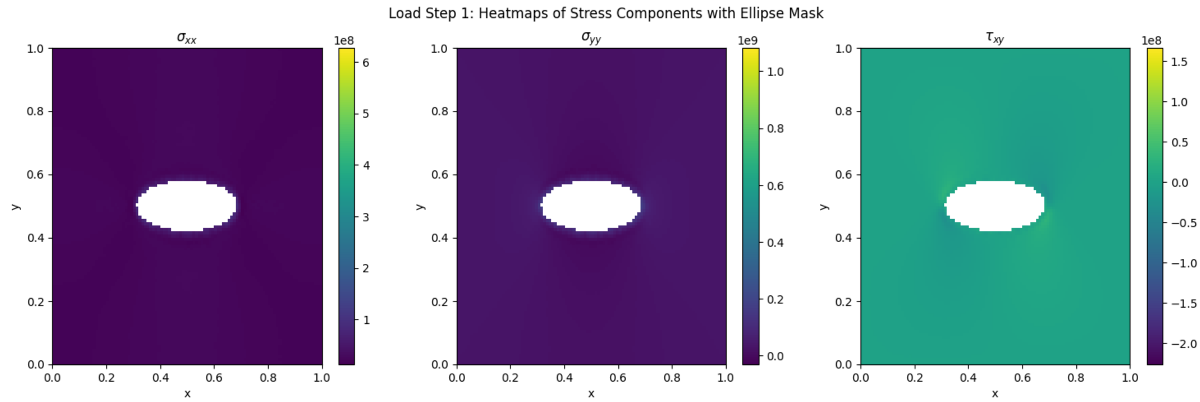
- Lame’s first constant λ: 123887869779.1054
- Lame’s second constant μ: 80670862170.79884

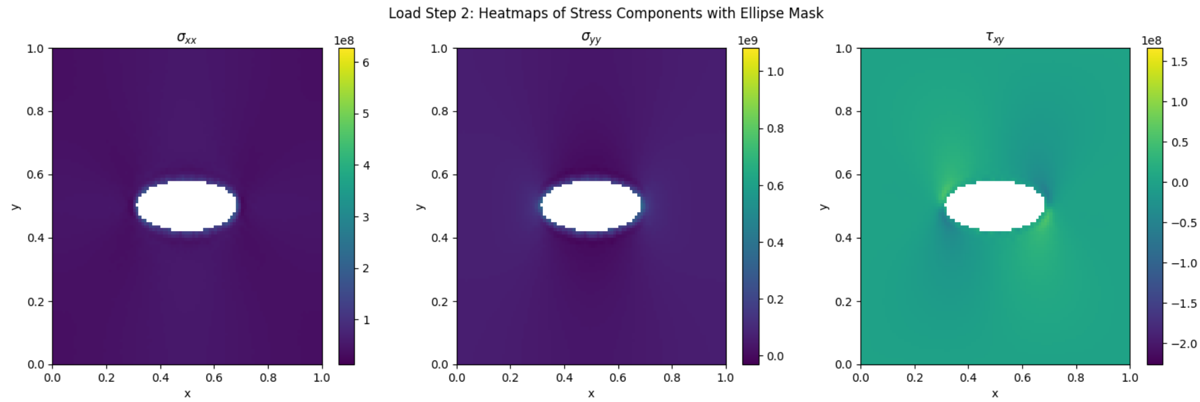
5TH LOAD STEP:

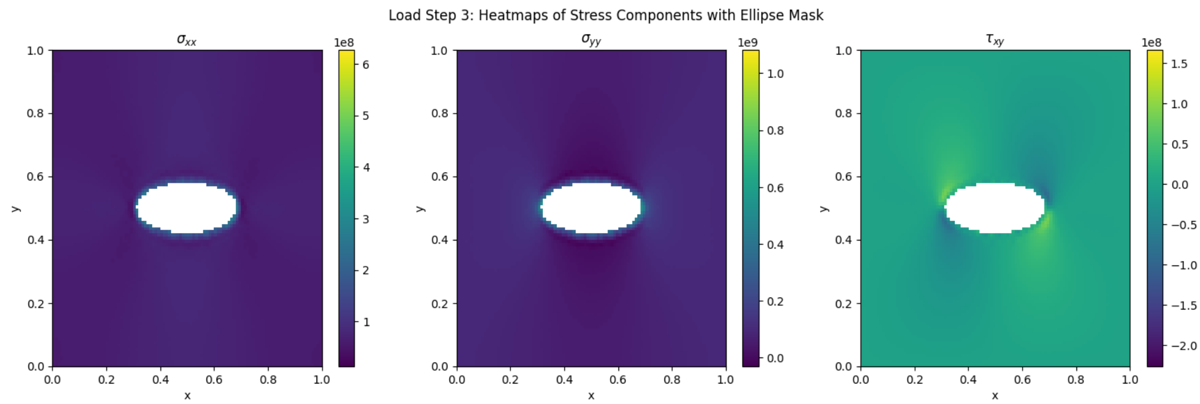
- Lame’s first constant λ: 123887869797.2153
- Lame’s second constant μ: 80670862173.50984

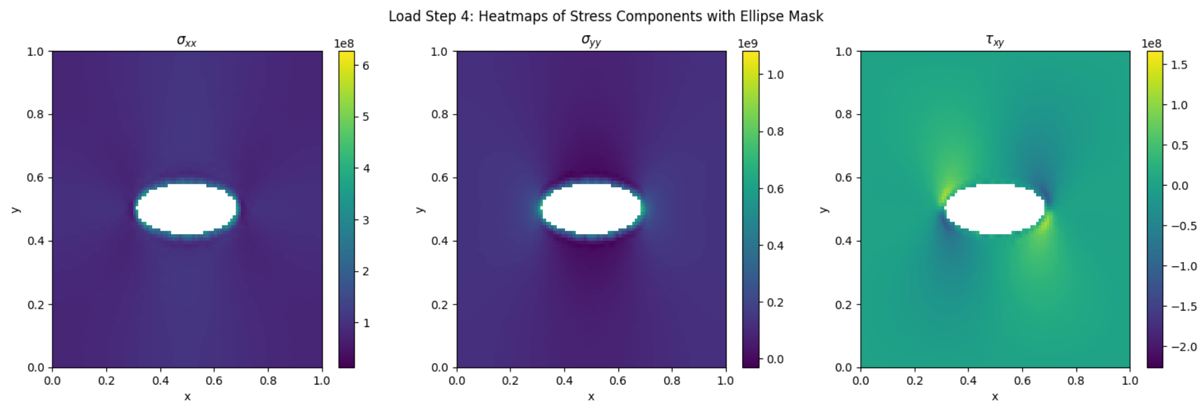
4.2 Stress Distribution Across Load Steps

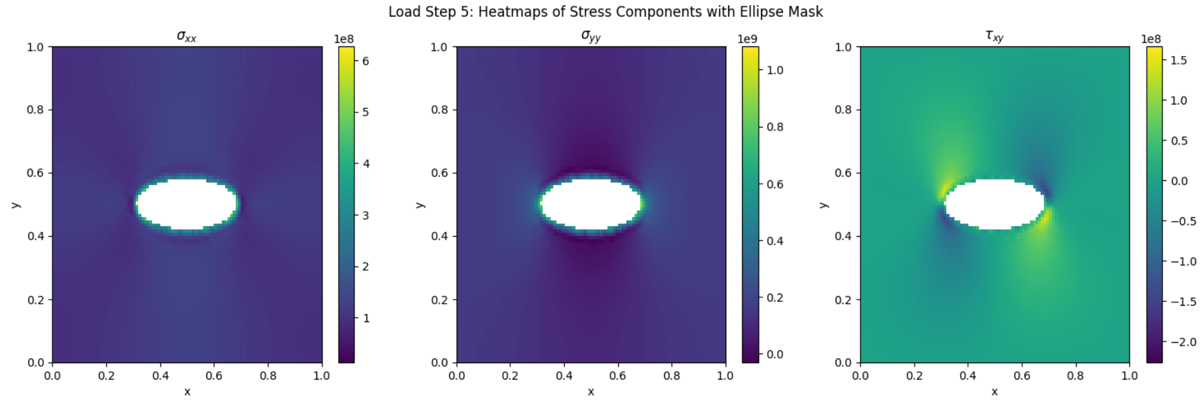
Below are the heat maps/contour maps showing stress distribution for their respective load steps.











5. Conclusion

In conclusion, the project effectively identified the linear elastic constants of the 2D plate, demonstrating the reliability of the parameter identification method. The results highlight the importance of accurate material characterization for predicting stress distributions and optimizing designs in solid mechanics.

The identified Lamé's constants are approximately
123.88 GPa for λ and **80.67 GPa** for μ.

References

- **Gemini AI. (2024).** "Assistance in Code Development."
- **OpenAI. (2024). ChatGPT Model**. "Assistance in Developing the Solid Mechanics Approach."