* Linear Algebra - 2 *

Matrices used in fields →

1. linear systems
2. Linear Transformation
3. Eigenvalues & Eigenvectors
4. Graph Theory
5. Markov Chains
6. Computer graphics
7. Control theory
8. Optimization

Recap

Linear Algebra

- Vectors
- Matrix
  - Computation
  - Intuition

Matrices?

- A matrix is a rectangular ~~way~~ array of no.s, symbols or expressions arranged in rows and columns. The numbers, symbols or expressions are called the elements of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Order of matrix

\# rows X # columns.

$\boxed{2 \times 3}$

Here $\boxed{2 \times 3}$ is shape or order of matrix.

Types of Matrix

1) row Matrix → row vector

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}_{1 \times 4}$$

2) Col Matrix

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$$

3) Square Matrix -

- no. of rows = no. of columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$$

a_{11}, a_{22} ; a_{11}, a_{22}, a_{33}

- diagonal of square matrix in which value of $\boxed{i=j}$ means rows = columns.

i = row
j = column

4) Diagonal Matrix → square

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$$

Here, diagonal elements are non zero. & non-diagonal elements are zero.

$i=j \neq 0$

5) Scalar Matrix → diagonal matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

all diagonal elements are equal.

Scanned with OKEN Scanner

6) **Identity Matrix** -

In which all diagonal elements are 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} \rightarrow \text{order} \rightarrow 3$$

7) **Zero Matrix** -

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* **Matrix Equality** -

$A = B$

1) $\boxed{\text{order should be same.}}$
if A is 2×2 then B should 2×2.

2) $\boxed{A_{ij} = B_{ij}}$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}} = \underset{B}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}}$$

- If Both conditions are true then matrices are equal.

* Scalar Operations -

- Scalar is single no. 2, 3, -5
- Two ways to do scalar Operations

By add | By multiply

- $\boxed{K+A}$ = $K = 2$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\rightarrow K + A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

- $\boxed{K \cdot A}$ = $K = 2$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\rightarrow KA = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

- By using scalar matrix one can make negative

$A = -A$

$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ $K = -1$

Now, $\boxed{K.A} = -1 \times \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$

Now this matrix is $\boxed{-A}$
Also called as Negative of a Matrix.

Rule -

Suppose you have $\boxed{A, B}$ K

You can also use above scalar operation as $K(A+B) = KA + KB$

So, Both gives same result.

* Matrix Addition & Subtraction -

- add
- sub
- multi
- div

① for addition,
order should be same of both matrices
A B → order same

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}} + \underset{B}{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}_{2\times2}}$$

$$= \begin{bmatrix} 1+5 = 6 & 2+6 = 8 \\ 7+3 = 10 & 4+8 = 12 \end{bmatrix}_{2\times2}$$

Scanned with OKEN Scanner

Date

② for subtraction-

To use substraction in matrices, you have to use
Scalar multiplication than
Scalar matrix addition

- eg → $A - B$ (for perform this),
↓
① Scalar multiplication

$A + (-1) B$

So, B become negative B

② Then matrix addition

$A + (-1) B$

- Some rules to remember -

① $A + B = B + A$ → Commutative

② $(A+B)+C = A+(B+C)$ → Associative

- 1 Additive Identity -

$A + \boxed{X} = A$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \underset{(2\times 2)}{\boxed{X}} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

So, for above X is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2] <u>Additive Inverse</u>-

$$A - \boxed{X} = 0$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - \boxed{X} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Use negative of A to get result.

* <u>Matrix Multiplication</u>-

- Remember even order is not same then also multiplication can be done between two matrices.

- Matrix multiplication only possible if A's no. of columns is equal to B's no. of rows.

-
$$\begin{array}{cc} A & B \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times & \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} \end{array} = \begin{bmatrix} & \\ & \end{bmatrix}$$

2×2 $\quad$ 2×2 $\quad$ 2×2

This should same

- Result's shape will be A's no. of rows X B's no. of column

Date

eg: - ∘ $1 \times 3 \neq 1 \times 3$

∘ $2 \times \underline{3} = \underline{3} \times 3$ ✓

Shape will be 2×3

row vector ↓ col^n vector ↙

∘ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3 \times 3}$ (Doing dot product)

$= \begin{bmatrix} (1\times1+2\times4+3\times7) & 36 & 42 \\ (4\times1+5\times4+6\times7) & 71 & 96 \end{bmatrix}$

$= \begin{bmatrix} 30 & 36 & 42 \\ 66 & 71 & 96 \end{bmatrix}$

∴

∘ Now understand Rules -

→ $\boxed{A \cdot B \,!= B \cdot A}$

∘ Here, placing matters means

Suppose $\underset{2 \times \underline{3}}{A}$ & $\underset{\underline{3} \times 3}{B}$, here multiplication happens but

if it like this $\underset{3 \times \underline{3}}{B}$ & $\underset{\underline{2} \times 3}{A}$

then multiplication doesn't happen.

∴ $A \cdot B \,!= B \cdot A$

Scanned with OKEN Scanner

Date

→ $(AB)C = A(BC)$

Associative rule follows.

→ $A(B+C) = AB + BC$

Distributive law/rule

→ Multiplicative identity.

$$\boxed{A\, X = A}$$

I

eg → $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \boxed{X} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

A

What type of matrix it is by which answer will be A.

A

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$\boxed{AI = A}$$

- In deep learning, matrix multiplication and backpropagation are mostly used.

* Transpose of Matrix -

- Here, row's become col^n and col^n becomes row.

Page No.
Date

eg → • $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

• $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

• $C = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 4 \end{bmatrix}_{2\times3}$

$C^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 4 \end{bmatrix}_{3\times2}$

• **Symmetric matrix -**

$\boxed{A = A^T}$

In which, no changes shown in result after transpose.

eg → $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}_{2\times2}$

$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}_{2\times2}$

Page No.
Date

- **Skew symmetric matrix** -

$$\boxed{A^T = -A}$$

- **Rules**

→ $(A^T)^T = A$

→ $(A+B)^T = A^T + B^T$

→ $(AB)^T = B^T . A^T$

* Determinant -

- It is a scalar value computed from a square matrix that carries important information about the matrix.

- It has several uses in linear algebra, including determining the invertibility of a matrix, finding the solⁿ to systems of linear eqⁿ and calculating the volume scaling factor for linear transformation.

By the way, this is matrix eqⁿ

- $\boxed{A \quad X = B}$

Suppose, we know the values of matrix A.

Don't know the value

Know the value

Had to find X matrix / these values.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boxed{X}} = \underbrace{\begin{bmatrix} 4 \\ 11 \end{bmatrix}}_{B}$$

To find X matrix, had to

$$X = \frac{B}{A}$$

- But B & A is matrices not scalar. In matrices, division doesn't happen. But multiplication can done.
- Multiplication is possible

So, we can do,

$$X = B.A^{-1}$$

We had B but don't have A^{-1} value.

Note – $\frac{1}{A} = A^{-1}$

- now problem is how to calculate inverse A (A^{-1}).

- Suppose given A matrix is invertible then it should be squared matrix only and even if both cond^n satisfies we can't tell that it can be invertible.

$$\boxed{A^{-1} = \frac{1}{\text{determinant}(A)} \text{Adj}(A)}$$

- Note – To inverse non-square matrix it can be done by pseudoinverse sometimes.

- Now goal to learn
  1. Determinant
  2. minor
  3. Cofactor
  4. Adjoint.

  By these 4 things we can do Inverse.

1. Determinant -

$\boxed{1 \times 1}$

$A = \begin{bmatrix} 1 \end{bmatrix}_{1 \times 1}$

determinant(A) $= \boxed{1}$

$\boxed{2 \times 2}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$

$\det(A) = \Delta = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$= 1 \times 4 - 3 \times 2$

$= \boxed{-2}$

$\boxed{2 \times 2}$

$B = \begin{bmatrix} -2 & 5 \\ -6 & 7 \end{bmatrix}_{2 \times 2}$

$\det(B) = \Delta = \begin{vmatrix} -2 & 5 \\ -6 & 7 \end{vmatrix}$

$= -14 - (-30)$

$= 30 - 14$

$= \boxed{16}$

- Now how to decide if inverse of a matrix is possible.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$

Rule 1. Square $\boxed{\checkmark}$

Rule 2. $\det(A) \neq 0$ if $\det(A)$ is zero then you can't calculate inverse and called as singular matrix.

- only non-singular matrices inverses can be done.

Date

$\boxed{3 \times 3}$

$\downarrow$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$$

$$\det(A) = \Delta = \begin{vmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \\ 7 \quad 8 \quad 9 \end{vmatrix}$$

→ Here, we done only one single col^n. but that's enough. See next pages, you'll understand.

$$= 1 \times \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \times \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \times \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= -3 + 12 - 9$$

$$= 0$$

So, $\det(A) = 0$ → singular matrix means inverse can't done

② **Minor** –

- Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth col^n.

- It is denoted by M_{ij}.

Suppose,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$$

a_{11} has its own minor

$M_{11} = a_{22}$

$M_{12} = a_{21}$

$M_{21} = a_{12}$

$M_{22} = a_{11}$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

for a_{11}, delete its ith & jth row, coln. Means determinant of a_{22} is the minor of a_{11}.

This way, is apply to all.

Note → $|a_{22}|$ is scalar so ans is a_{22}.

Now, see for 3×3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Scanned with OKEN Scanner

Date

- $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$

- $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

- $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$

⋮

- $M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

- This is how, we calculate minor.

(3) Cofactor -

- Cofactor of an element of a_{ij} of a determinant is defined by $A_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the minor of a_{ij}.
- Cofactor is denoted by capital A.
- Every item has cofactor.
- Suppose you have matrix B.

$$B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Now, you had to find a_{11} cofactor then,

for, $A_{11} = (-1)^2 M_{11}$

$i=1 \quad j=1$

for, $A_{21} = (-1)^{2+1} M_{21} = -M_{21}$

- Basically, signs reverse alternatively

- Suppose, for 3x3 matrix,

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{bmatrix}$$

$$A_{32} = (-1)^{3+2} M_{32}$$

$i \quad j$

you know, how to find minor.

Note →

Determinant = Sum of the product of element of any row (or col^n) with their corresponding cofactor.

$$\boxed{\det = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}}$$

Scanned with OKEN Scanner

(4) Adjoint -

Suppose you ~~have~~ have matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

↓

you find every items cofactor.

↓

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

↓

Then you transpose it

↓

$$adj(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

This is your adjoint of A.

* Inverse of Matrix -

$$\boxed{A^{-1} = \frac{1}{\det(A)} \, adj(A)}$$

- An inverse matrix is a matrix that, when multiplied by original matrix, results in the identity matrix.

$$\boxed{A \cdot A^{-1} = I}$$

- The inverse matrix is defined only for square matrices (matrices with the same number of rows and col^n's) and not all square matrices has an inverse.

- A matrix is invertible if and only if it is non-singular, meaning its determinant is non zero. If $|A| = 0$, A is called a singular matrix, and it does not have an inverse.

- Now suppose, $A X = B$.

Don't know value of X.

You know values of A & B.

So, inverse A both sides.

$$A^{-1} \cdot A X = A^{-1} B$$

$$\because A \cdot A^{-1} = A^{-1} \cdot A = I \text{ .. (Commutative law)}$$

$$I X = A^{-1} B$$

→ multiplicative identity.

$$\boxed{X = A^{-1} \cdot B}$$ ← That's how we solve / calculate eq^n.

- Inverse matrices play a crucial role in linear algebra and have many appl^n, such as solving systems of linear eq^ns, finding the sol^n to a matrix eq^n, and performing various matrix operations.

- Several methods for finding the inverse of a matrix, including Gaussian elimination, the adjugate method, and LU decomposition.

Just note, $\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$

* <u>Solving a system of linear eq^n</u> -

Q.1 $\begin{bmatrix} x+y=5 \\ 4x+3y=15 \end{bmatrix}$

$\downarrow$

$$\underset{2\times2}{\begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

$$\underset{2\times1}{\begin{bmatrix} x+y \\ 4x+3y \end{bmatrix}} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{matrix} x+y=5 \\ 4x+3y=15 \end{matrix}}$$

Means, you can write it into matrix form.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 5 \\ 15 \end{bmatrix}}_{B}$$

$\boxed{AX = B}$ matrix form

$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = \boxed{A^{-1}B}$$

- $A = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$

$\Rightarrow \boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$

- $\det(A) = \Delta = 3-4 = \boxed{-1}$

- Cofactor $= \begin{bmatrix} (-1)^{1+1}3 & (-1)^3 4 \\ (-1)^3 1 & (-1)^4 1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & -4 \\ -1 & 1 \end{bmatrix}^T$$

- Adjoint $= \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$

So, $A^{-1} = \frac{1}{-1} \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$

$$\boxed{A^{-1} = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}}$$

Scanned with OKEN Scanner

Now, $X = A^{-1} B$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}_{2\times 2} \cdot \begin{bmatrix} 5 \\ 15 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} -15+15 \\ 20 - 15 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{\therefore x = 0 \ \& \ y = 5}$$

Q.2 $x + 3y + 4z = 5$
$6x + 4y + z = 16$
$2x + 2y + 2z = 11$

→ ① Matrix form,

$$\underbrace{\begin{bmatrix} 1 & 3 & 4 \\ 16 & 4 & 1 \\ 2 & 2 & 2 \end{bmatrix}}_{\boxed{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\boxed{X}} = \underbrace{\begin{bmatrix} 5 \\ 16 \\ 11 \end{bmatrix}}_{\boxed{B}}$$

$$\boxed{AX = B}$$

$$\boxed{X = A^{-1} B}$$

let's solve by program,

```
import numpy as np
A = np.array([[1,3,4],[6,4,1],[2,2,2]])
B = np.array([5,6,11])
A_inv = np.linalg.inv(A)
np.dot(A_inv, B)
```

o/p → array([10.125, -13.375, 8.75]).

x y z