3b) $\sum_{1\le i\le n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j) \le P\left(\bigcup_{1\le i\le n} A_i\right) \le \sum_{1\le i\le n} P(A_i)$ ...... (i)

1. PART-A

Prove: $P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j) + \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k)$

1. It is trivial for $n = 1, 2 \& 3$ for $n=3$:

$P\left(\bigcup_{i=1}^{3} A_i\right) \le P(A_1) + P(A_2) + P(A_3) - \left[P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3)\right] + P(A_1 \cap A_2 \cap A_3)$

2. Say it is true for some $n = m$

$P\left(\bigcup_{i=1}^{m} A_i\right) \le \sum_{i=1}^{m} P(A_i) - \sum_{1\le i<j\le m} P(A_i \cap A_j) + \sum_{1\le i<j<k\le m} P(A_i \cap A_j \cap A_k)$ ...... (ii)

3. Now prove for $n = m+1$ (from slide 149)

$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\left\{\bigcup_{i=1}^{m} A_i\right\} \cup A_{m+1}\right) = P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left(\left\{\bigcup_{i=1}^{m} A_i\right\} \cap A_{m+1}\right)$ ...... (iii)

Replace ~~from~~ $P\left(\bigcup_{i=1}^{m} A_i\right)$ from (ii)

$\le \sum_{i=1}^{m} P(A_i) - \sum_{1\le i<j<m} P(A_i \cap A_j) + \sum_{1\le i<j<k<m} P(A_i \cap A_j \cap A_k) + P(A_{m+1}) - P\left(\bigcup_{i=1}^{m} A_i \cap A_{m+1}\right)$

Add these 2   Expand

Now Expand the last term from slide 150 but using (i)

$\le \sum_{i=1}^{m+1} P(A_i) - \sum_{1\le i<j<m} P(A_i \cap A_j) + \sum_{1\le i<j<k<m} P(A_i \cap A_j \cap A_k) - \left[\sum_{i=1}^{m} P(A_i \cap A_{m+1}) - \sum_{1\le i<j<m} P((A_i \cap A_j) \cap A_{m+1})\right]$

Now Expand the [ ] terms & combine

$= \sum_{i=1}^{m+1} P(A_i) - \sum_{1\le i<j<m+1} P(A_i \cap A_j) + \sum_{1\le i<j<k<m+1} P(A_i \cap A_j \cap A_k)$ ...... (iv)

from iii & iv

i.e.

$P\left(\bigcup_{i=1}^{m+1} A_i\right) \le \sum_{i=1}^{m+1} P(A_i) - \sum_{1\le i<j<m+1} P(A_i \cap A_j) + \sum_{1\le i<j<k<m+1} P(A_i \cap A_j \cap A_k)$

Hence, Proved

PART B:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{1\le i\le n} P(A_i) - \sum_{1\le i<j\le n} P(A_i\cap A_j) + \sum_{1\le i<j<k\le n} P(A_i\cap A_j\cap A_k) - \sum_{1\le i<j<k<l\le n} P(A_i\cap A_j\cap A_k\cap A_l)$$

1. This is trivial for $n=1,2,3 \& 4$; $n=4$ shown here:

$$P\left(\bigcup_{i=1}^{4} A_i\right) \geq \sum_{i=1}^{4} P(A_i) - \sum_{1\le i<j\le 4} P(A_i\cap A_j) + \sum_{1\le i<j<k\le 4} P(A_i\cap A_j\cap A_k) - P(A_1\cap A_2\cap A_3\cap A_4) \quad \text{--- (iv)}$$

2. Say it is true for some $n=m$

$$P\left(\bigcup_{i=1}^{m} A_i\right) \geq \sum_{i=1}^{m} P(A_i) - \sum_{1\le i<j\le m} P(A_i\cap A_j) + \sum_{1\le i<j<k\le m} P(A_i\cap A_j\cap A_k) - \sum_{1\le i<j<k<l\le m} P(A_i\cap A_j\cap A_k\cap A_l) \quad \text{--- (v)}$$

for ~~the~~ $n=m+1$ we can write it is (from slide 149)

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\left\{\bigcup_{i=1}^{m} A_i\right\}\cup A_{m+1}\right) = P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left(\left\{\bigcup_{i=1}^{m} A_i\right\}\cap A_{m+1}\right) \quad \text{---- (vi)}$$

Now put (v) in (vi)

$$\geq \sum_{i=1}^{m} P(A_i) - \sum_{1\le i<j\le m} P(A_i\cap A_j) + \sum_{1\le i<j<k\le m} P(A_i\cap A_j\cap A_k) - \sum_{1\le i<j<k<l\le m} P(A_i\cap A_j\cap A_k\cap A_l) + P(A_{m+1})$$
$$- P\left(\bigcup_{i=1}^{m} A_i\cap A_{m+1}\right) \quad \text{------- (vii)}$$

$\Rightarrow$ Expand this from (iv) prev part

from (iv)

$$P\left(\bigcup_{i=1}^{m} A_i\cap A_{m+1}\right) \geq \sum_{i=1}^{m} P(A_i\cap A_{m+1}) - \sum_{1\le i<j\le m} P((A_i\cap A_j)\cap A_{m+1}) \cancel{\sum_{1\le i<j<k} P(A_i\cap A_j\cap A_k)}$$
$$+ \sum_{1\le i<j<k\le m} P((A_i\cap A_j\cap A_k)\cap A_{m+1})$$

$\hookrightarrow$ substitute this in (vii) & add up terms

$$\geq \sum_{i=1}^{m+1} P(A_i) - \sum_{1\le i<j\le m+1} P(A_i\cap A_j) + \sum_{1\le i<j<k\le m+1} P(A_i\cap A_j\cap A_k) - \sum_{1\le i<j<k<l\le m+1} P(A_i\cap A_j\cap A_k\cap A_l) \quad \text{--- (viii)}$$

$\therefore$ Comparing vi & viii

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) \geq \sum_{i=1}^{m+1} P(A_i) - \sum_{1\le i<j\le m+1} P(A_i\cap A_j) + \sum_{1\le i<j<k\le m+1} P(A_i\cap A_j\cap A_k) - \sum_{1\le i<j<k<l\le m+1} P(A_i\cap A_j\cap A_k\cap A_l)$$

$\therefore$ Hence Proved