CSE 558 : Assignment - 2 (Abhishek Sushil 2021441)

Q1.

a) if $R_1 = 1$ Valid $R_2 = 1, 2, 3, 4, 5, 6$ or $\frac{6}{6} = 1$

$R_1 = 2$ " " $= 2, 4, 6$ or $\frac{1}{2} = \frac{3}{6}$

$R_1 = 3$ $= 3, 6$ or $\frac{2}{6} = \frac{1}{3}$

$R_1 = 4$ $= 4$ or $\frac{1}{6}$

$R_1 = 5$ $= 5$ or $\frac{1}{6}$

$R_1 = 6$ $= 6$ or $\frac{1}{6}$

$$\text{Prob} = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot\frac{1}{2} \text{ - - - - - } \frac{1}{6}\cdot\frac{1}{6}$$

$$= \frac{1}{6}\left[\frac{6}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right]$$

$$= \frac{1}{6}\left[\frac{14}{6}\right]$$

$$= \frac{7}{18}$$

b) here if we use the principle that if a 'p' is rolled where $p < \frac{k}{2}$ then every p^{th} element is a multiple b/w 1 & k

& if $p > \frac{k}{2}$ there is exactly 1 multiple (p itself)

& if $p = \frac{k}{2}$ " 2 multiples ($\frac{k}{2}$ & k)

$\therefore$ for upto $\frac{k}{2}$ $(\leq \frac{k}{2})$ [Note $R_n \rightarrow$ 'n' is Rolled]

$$\frac{1}{k}\left(\overset{R_1}{\frac{k}{k}} + \overset{R_2}{\frac{(\frac{k}{2})}{k}} + \overset{R_3}{\frac{(\frac{k}{3})}{k}} \cdots\cdots \overset{R_{k/2}}{\frac{\frac{k}{k/2}}{k}}\right) \Big\} \ \frac{k}{2} \text{ rolls}$$

beyond $\frac{k}{2} \rightarrow$ $$\frac{1}{k}\left(\underset{R_{\frac{k}{2}+1}}{\frac{1}{k}} \cdots\cdots \underset{R_k = R_{\frac{k}{2}+\frac{k}{2}}}{\frac{1}{k}}\right) \Big\} \ \frac{k}{2} \text{ rolls}$$

Scanned with CamScanner

$$\text{Prob} = \frac{1}{k}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \cdots\cdots \frac{2}{k}\right) + \frac{1}{k}\left(\frac{1}{k} \times \frac{k}{2}\right)$$

$$= \frac{1}{k}\left(1 + \frac{1}{2} + \frac{1}{3} \cdots\cdots \frac{2}{k} + \frac{1}{2}\right)$$

$$= \frac{1}{k}\left(1 + \frac{1}{2} \cdots\cdots \frac{1}{(k/2)} + \frac{1}{2}\right)$$

$$= \frac{1}{k}\left(H_{k/2} + \frac{1}{2}\right)$$

H_n = Harmonic sum of n

$$= 1 + \frac{1}{2} + \frac{1}{3} \cdots\cdots \frac{1}{n}$$

Scanned with CamScanner

Q2.

a) To find the probability atleast 2 birthdays match we can first work out the opposite . i.e no 2 birth-days match.

1st child can have any birthday $= \frac{365}{365}$

2nd " " except the above $= \frac{364}{365}$

and so on for 23 children

$$p_1 = \frac{(365-0)(365-1)\cdots(365-22)}{(365)^{23}}$$

$$= p_1 \times \frac{(365-23)!}{(365-23)!}$$

$$= \frac{365!}{(365)^{23}(342)!}$$

$\therefore$ general formula for 'n' ppl

$$\frac{365!}{(365)^n(365-n)!}$$

Now we need to find $p_{ans} = 1 - p_1$

We know $e^x \approx 1 + x \quad \therefore e^{-a/365} \approx 1 - \frac{a}{365}$

$$p_1 = \left(1-\frac{0}{365}\right)\cdot\left(1-\frac{1}{365}\right)\cdots\left(1-\frac{22}{365}\right)$$

$$= e^0 \cdot e^{-1/365} \cdots e^{-22/365} \qquad n = 23$$

$$= e^{-\frac{(n-1)(n)}{2\times 365}}$$

$$p_1 \approx e^{-\frac{22\times 23}{730}}$$

$$p_1 \approx e^{-\frac{253}{365}}$$

$$\therefore\ 1 - p_1 = p_{ans} = 1 - e^{-\frac{253}{365}}$$

$$= 1 - 0.4999$$

$$\approx 0.50001$$

Hence proved

Scanned with CamScanner

b) find n people such that $p_{ans} > \frac{3}{4}$

or $p_1 = 1 - p_{ans} < \frac{1}{4}$

p_1 from earlier

↓

$$p_1 = \frac{365!}{(365)^n (365-n)!} < \frac{1}{4}$$

On solving $n =$ using approximation

$$p_{ans} \approx 1 - e^{-\frac{n(n-1)}{2 \cdot 365}} \approx 1 - e^{-\frac{n^2}{730}}$$

$$\therefore e^{-\frac{n^2}{730}} < \frac{1}{4}$$

$$\frac{-n^2}{730} < \ln \frac{1}{4}$$

$$\frac{-n^2}{730} < - \ln 4$$

$$\frac{n^2}{730} > \ln 4$$

$$n^2 > 730 \times 1.386$$

$$n^2 > 1012 \text{ approx.}$$

$$n > 31.8$$

$$\therefore n = 32$$

Scanned with CamScanner

2.c) Each of the 'n' people has 2 birthdays (that can't overlap w/ each other) i.e. $R_i \neq F_i$

So the first person can have ${}^{365}C_2$ possible R_i, F_i combinations. Note: We dont need to do ${}^{365}P_2$ as we dont care for the R_i & F_i as different entities as long as they are distinct.

I instead treat B_1 & B_2 (identical distinct entities that cant overlap)

$P_1 \rightarrow \dfrac{{}^{365}C_2}{{}^{365}C_2}$ any day

$P_2 \rightarrow \dfrac{{}^{363}C_2}{{}^{365}C_2}$ any day apart from R_1 & F_1 ~~B_1 & B_2~~

$\vdots$

$P_r \rightarrow \dfrac{{}^{365-2(r-1)}C_2}{{}^{365}C_2}$

$\therefore \bar{p}(n)$ = Probability no birthdays overlap

$$= \prod_{i=0}^{n-1} \frac{{}^{365-2(i)}C_2}{{}^{365}C_2}$$

$$= \frac{{}^{365}C_2}{{}^{365}C_2} \cdot \frac{{}^{363}C_2}{{}^{365}C_2} \cdots\cdots \frac{{}^{365-2(n-1)}C_2}{{}^{365}C_2}$$

$$= \frac{365 \times 364}{365 \times 364} \times \frac{363 \times 362}{365 \times 364} \cdots\cdots \frac{\{365-2(n-1)\} \cdot \{364-2(n-1)\}}{365 \cdot 364}$$

$$= \left(\frac{365 \cdot 363 \cdot 361 \cdots \{365-2(n-1)\}}{365 \cdot 365 \cdot 365 \cdots}\right) \times \left(\frac{364 \cdot 362 \cdot 360 \cdots \{364-2(n-1)\}}{364 \cdot 364 \cdots\cdots 364}\right)$$

$$\left\{\left(1-\frac{0}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{2(n-1)}{365}\right)\right\}\cdot\left\{\left(1-\frac{0}{364}\right)\left(1-\frac{2}{364}\right)\cdots\left(1-\frac{2(n-1)}{364}\right)\right\}$$

$$= \left[e^{-0/365}\cdot e^{-2/365}\cdots e^{-2(n-1)/365}\right]\cdot\left[e^{-0/364}\cdot e^{-2/364}\cdots e^{-2(n-1)/364}\right]$$

$$= e^{-2\frac{[0+1\cdots(n-1)]}{365}}\cdot e^{-2\frac{[0+1\cdots(n-1)]}{364}}$$

$$= e^{-\frac{n(n-1)}{365}}\cdot e^{-\frac{n(n-1)}{364}}$$

$$= e^{-n(n-1)\left[\frac{1}{365}+\frac{1}{364}\right]}$$

$$= e^{\frac{-729\,n(n-1)}{132860}}$$

$\therefore$ Find $p_{ans}(n) \geq p \therefore 1-\bar{p}(n) \geq p$

$$\bar{p}(n) \leq 1-p$$

$$\ln(\bar{p}(n)) \leq \ln(1-p)$$

$$-729(n)(n-1) \leq 132860\ln(1-p)$$

$$-729n^2+729n \leq 132860\ln(1-p)$$

$$729n^2-729n+132860\ln(1-p) \geq 0$$

Solving

$$n \leq \underbrace{\frac{1+\sqrt{1-\frac{4\times132860\times\ln(1-p)}{729}}}{2}}$$

Q3.

(a) At most → One sided ∴ ≤ 1.64 (To not reject H_0)

$p_0 = 50\% = \frac{1}{2}$

$p = \frac{55}{n}$

$$z = \frac{p - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \leq 1.64$$

$$= \left(\frac{55}{n} - \frac{1}{2}\right) \times \frac{\sqrt{n}}{\sqrt{\frac{1}{2}\cdot\frac{1}{2}}} \leq 1.64$$

$$= \left(\frac{55}{n} - 0.5\right) \times 2\sqrt{n} \leq 1.64$$

$$= \frac{110}{\sqrt{n}} - \sqrt{n} \leq 1.64$$

$$110 - n \leq 1.64\sqrt{n}$$

$$n + 1.64\sqrt{n} - 110 \geq 0$$

replacing $\sqrt{n} = m$

$$m^2 + 1.64\,m - 110 \geq 0$$

$$\therefore \underset{\sim}{n} \geq 94.0915 \text{ (approx)}$$

$\therefore$ $n = 95$ mails (whole no. of emails)

At least 95 emails to not reject null hypothesis

b) More than $\therefore$ One sided thus 1.64

$H_0 = p_0 \leq 1/3$ (as per ques -> believe her ≡ reject H_0)

$H_1 = p_0 > 1/3$ (believe her)

$p = \frac{28}{n}$, now as $n \uparrow$ $p \downarrow$ $\therefore$ find the max n or the min p so that we can ~~be~~ believe her.

$$\left(\frac{28}{n} - \frac{1}{3}\right) \div \sqrt{\frac{1/3\,(2/3)}{n}} \geq 1.64$$

$$\left(\frac{28}{n} - \frac{1}{3}\right) \times \frac{3\sqrt{n}}{\sqrt{2}} \geq 1.64$$

$$\frac{84}{\sqrt{n}} - \sqrt{n} \geq 1.64\sqrt{2} = 2.32$$

$$84 - n \geq 2.32\sqrt{n}$$

$$0 \geq n + 2.32\sqrt{n} - 84$$

$$n \leq 65.258 \text{ (approx)}$$

$\therefore$ $n = 65$

Scanned with CamScanner