Assignment 4 Q50 (june 2018)

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50) Let X and Y be i.i.d uniform (0,1) random variables. Let Z = max(X,Y) and W = min(X,Y). Then P((Z-W)> 0.5) =

Ans
Cumulative distribution function of Z -

$$\begin{aligned} F_Z(z) &= P(Z < z) \\ &= P(max(X, Y) < z) \\ &= P(X < z, Y < z) \\ &= P(X < z) \times P(Y < z) \\ &= F_X(z) \times F_Y(z) \end{aligned} \tag{1}$$

We know the cumulative distribution function (cdf) of Uniform distribution is-

$$F_X(x) = \begin{cases} 0 & \text{for x<0} \\ x & \text{for } 0 \le \text{x} \le 1 \\ 1 & \text{for x>1} \end{cases} \tag{2}$$

Substituting in eq(1) -

$$F_Z(z) = \begin{cases} 0 & \text{for z<0} \\ z^2 & \text{for } 0 \le \text{z} \le 1 \\ 1 & \text{for z>1} \end{cases} \tag{3}$$

Probability distribution function of Z -

$$f_Z(z) = \frac{d}{dz} F_Z(z) \tag{4}$$

$$\Rightarrow f_Z(z) = \begin{cases} 0 & \text{for z<0} \\ 2z & \text{for } 0 \le \text{z} \le 1 \\ 0 & \text{for z>1} \end{cases} \tag{5}$$

Cumulative distribution function of W -

$$
\begin{aligned}
F_W(w) &= 1 - P(W > w) \\
&= 1 - P(min(X, Y) > w) \\
&= 1 - P(X > w, Y > w) \\
&= 1 - P(X > w) \times P(Y > w) \\
&= 1 - ((1 - F_X(w)) \times (1 - F_Y(w))) \\
&= F_X(w) + F_Y(w) - F_X(w)F_Y(w)
\end{aligned} \tag{6}
$$

Substituting cdf (uniform distribution)-

$$
\Rightarrow F_W(w) = \begin{cases} 0 & \text{for w<0} \\ 2w - w^2 & \text{for } 0 \leq \text{w} \leq 1 \\ 1 & \text{for w>1} \end{cases} \tag{7}
$$

Solving probability -

$$
\begin{aligned}
P((Z - W) > 0.5) &= P(W < Z - 0.5) \\
&= E[F_W(Z - 0.5)] \\
&= \int F_W(Z - 0.5) f_Z(z) dz \\
&= \int_0^1 (z - 0.5)(2.5 - z)(2z) dz \\
&= \frac{1}{4}
\end{aligned} \tag{8}
$$

$$
\therefore P((Z - W) > 0.5) = \frac{1}{4} \tag{9}
$$

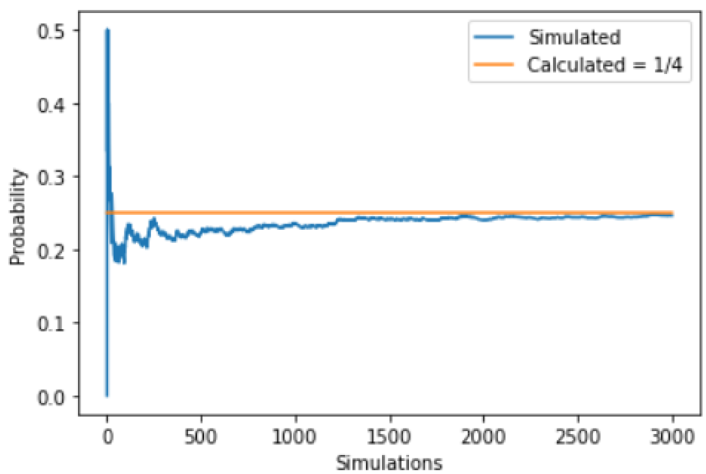

Figure 1: Final Probability vs number of simulations