Introduction to Combinatorial Game Theory

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Outline

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For any game position G we denote left options of game by $\eth^l$ and right options of game by $\eth^r$.
Thus any game position can be written as

$$G = \{\eth^l | \eth^r\}$$

Tic-Tac-Toe

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Result can be predicted after two initial moves only(one each player), given both play optimally thereafter.

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There is also a misere version in which one forces the opponent to place three cuts. And there are many variations of the game.

Hex

Hex is a strategy board game played on a hexagonal *nxn* grid.
One player tries to make a path from top to bottom and other from left to right.

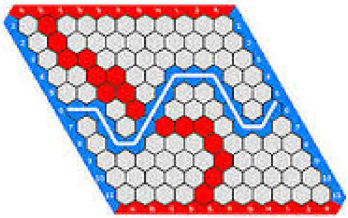

Figure: hex board.

Theorem

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CLAIM : HEX is a determined game.

Proof.

If the second player has a winning strategy, the first player could "steal" it by making an irrelevant move, and then follow the second player's strategy. If the strategy ever called for moving on the square already chosen, the first player can then make another arbitrary move. This ensures a first player win. Clearly such a strategy cannot exist. □

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- Control the center of the board.
- Since first player has advantage, few versions allow second player to swap with first player after first move.
- So choosing first move becomes tricky !

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Nim Sum : $a \oplus b =$ first write a and b in binary then add without carrying.

If nim sum of no. of coins in all the heaps is zero then G is called zero position.

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- We know that finally we'll have 0 coins left which is a zero nim sum position.
- Hence if we start from a zero nim sum position second player will loose and vice-versa.

Fundamental theorem of combinatorial Games

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- We denote by $o(\mathbb{G})$ the outcome class of $\mathbb{G}$.

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If $\mathbb{G}$ is an impartial game then $\mathbb{G}$ is in either $\mathcal{N}$ or $\mathcal{P}$.

Theorem

Suppose the positions of a finite impartial game can be parti-tioned into mutually exclusive sets A and B with the properties:

- *every option of a position in A is in B*
- *every position in B has at least one option in A.*

Then A is the set of $\mathcal{P}$ positions and B is the set of $\mathcal{N}$ positions.

Definition

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1. $\mathbb{G} + 0 = \mathbb{G}$
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$-\mathbb{G} ::= \{-\eth^R | -\eth^L\}$
The definition of negative corresponds exactly to reversing the roles of the two players.

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$=$ *is an equivalence relation.*

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Theorem

Let $\mathbb{G}$ *be any game and let* $\mathbb{Z} \in \mathcal{P}$ *be any game that is a second player win. Then outcome classes of* $\mathbb{G}$ *and* $\mathbb{G} + \mathbb{Z}$ *are the same.*

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These results give us an insight on how to actually compare games $\mathbb{G}$ and $\mathbb{H}$

- $\mathbb{G} > \mathbb{H}$ when L wins $\mathbb{G} - \mathbb{H}$
- $\mathbb{G} = \mathbb{H}$ when P wins $\mathbb{G} - \mathbb{H}$

- $\mathbb{G} < \mathbb{H}$ when R wins $\mathbb{G} - \mathbb{H}$
- $\mathbb{G} || \mathbb{H}$ when N wins $\mathbb{G} - \mathbb{H}$

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Theorem

The relation $\geq$ is a partial order on games.

- *Transitive :* $\mathbb{G} \geq \mathbb{H}$ *and* $\mathbb{H} \geq \mathbb{J}$ *then* $\mathbb{G} \geq \mathbb{J}$
- *Reflexive :* $\mathbb{G} \geq \mathbb{G}$
- *AntiSymmetry :* $\mathbb{G} \geq \mathbb{H}$ *and* $\mathbb{H} \geq \mathbb{G}$ *then* $\mathbb{G} = \mathbb{H}$

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The group containing all games form a partially ordered abelian group under $+$

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If

$$\mathbb{G} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, ...|\mathbb{H}, \mathbb{I}, \mathbb{J}, ...\}$$

and $\mathbb{B} \geq \mathbb{A}$ *then* $\mathbb{G} = \mathbb{G}'$ *where*

$$\mathbb{G}' = \{\mathbb{B}, \mathbb{C}, ...|\mathbb{H}, \mathbb{I}, \mathbb{J}, ...\}$$

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Definition

A Left option $\mathbb{A}$ of $\mathbb{G}$ can be considered to be reversible if $\mathbb{A}$ has a right option $\mathbb{A}^R$ such that $\mathbb{A}^R \leq \mathbb{G}$

Theorem

Fix a game

$$\mathbb{G} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, ...|\mathbb{H}, \mathbb{I}, \mathbb{J}, ...\}$$

and suppose for some Right option of $\mathbb{A}$*, call it* $\mathbb{A}^R$*,* $\mathbb{G} \geq \mathbb{A}^R$*. If we denote Left options of* $\mathbb{A}^R$ *by* $\{\mathbb{W}, \mathbb{X}, \mathbb{Y}, ...\}$*:*

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and define the new game

$$\mathbb{G}' = \{W, X, Y, ..., B, C, ...|H, I, J, ...\}$$

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☺☺☺☺Finally we have our result. ☺☺☺☺

Theorem

If $\mathbb{G}$ *and* $\mathbb{H}$ *are in canonical form and* $\mathbb{G} = \mathbb{H}$ *, then* $\mathbb{G} \cong \mathbb{H}$*(Isomorphic Games).*