26 Lecture 26

Recalling few results of functions of several variables

[26.1] **About continuity**

a) Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. TFAE. (The following are equivalent.)

i) The function f is continuous at a.

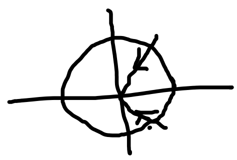
ii) For each unit vector $u \in \mathbb{R}$, we have $\lim_{t\to 0+} f(a + tu) = f(a)$.

$u=1$, $u=-1$

b) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. The following are not equivalent.

i) The function f is continuous at a.

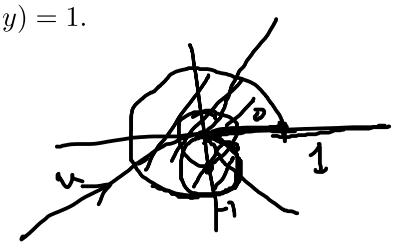
ii) For each unit vector $u \in \mathbb{R}^n$, we have $\lim_{t\to 0+} f(a + tu) = f(a)$.



To see this define

$$f(x, y) = \begin{cases} \frac{y^2}{x}, & \text{if } x \neq 0 \\ 0 & \text{else.} \end{cases}$$

Notice that for each unit vector $u \in \mathbb{R}^2$, we have $\lim_{t\to 0+} f(0 + tu) = f(0)$. However, f is not continuous at 0 because if we approach 0 along the curve (imagine a sequence of points on this curve converging $(0, 0)$) $y = \sqrt{x}$, the limit is 1, that is, $\lim_{\substack{x\to 0+ \\ y=\sqrt{x}}} f(x, y) = 1$.



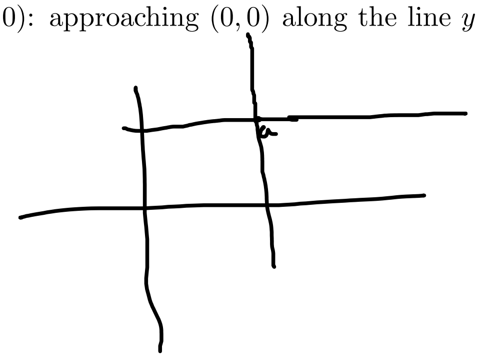
$\lim_{t\to 0} f(tu) = 0$ $\forall u$ in the unit circle

$3x_1^2 x_2^{x_3}$ 3 x_2 -5

c) Let $P(x)$ and $Q(x)$ be polynomials in $x = (x_1, \ldots, x_n)$. Then the function $\frac{P(x)}{Q(x)}$ is continuous wherever it is defined on $\mathbb{R}^n$.

d) Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that both $g(y) = f(a, y)$, $h(x) = f(x, b)$ are continuous for each $a, b \in \mathbb{R}$. Even then, $f(x, y)$ my not be continuous.

To see this, consider the function $f(x, y) = \frac{xy}{x^2+y^2}$, $f(0, 0) = 0$. If $a \neq 0$ then $f(a, y) = \frac{ay}{a^2+y^2}$ is continuous on $\mathbb{R}$. If $a = 0$ then $f(a, y) = 0$ is continuous on $\mathbb{R}$. Similarly $f(x, b)$ is continuous for any fixed b. The function f is not continuous at $(0, 0)$: approaching $(0, 0)$ along the line $y = mx$, the limit is $\frac{m}{1+m^2}$, which changes with m.



$f(x) = x_1^2 + x_1x_2$

$f(a+h) - f(a) = (a_1+h_1)^2 + (a_1+h_1)(a_2+h_2) - a_1^2 - a_1a_2$

$= 2a_1h_1 + h_1^2 + a_1h_2 + a_2h_1 + h_1h_2$

$= 2a_1h_1 + a_2h_1 + a_1h_2 + h_1^2 + h_1h_2$

$= \begin{bmatrix} 2a_1+a_2 & a_1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ + higher degree terms

[26.2] About differentiability

a) Consider a polynomial $f(x) = x_1^2 + x_1x_2$. Then

$$f(a+h) - f(a) = 2a_1h_1 + a_2h_1 + a_1h_2 + h_1^2 + h_1h_2.$$

Notice that, the linear terms can also be written as $\begin{bmatrix} 2a_1 + a_2 & a_1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$. → coeff of lin terms in $f(a+h)-f(a)$

b) So

the coefficient matrix of linear terms in $f(a+h) - f(a)$ is $A = \begin{bmatrix} 2a_1 + a_2 & a_1 \end{bmatrix}$.

$-(a_1+h_1)(a_2+h_2)$

c) Thus, for $f(x) = x_1^2 - x_1x_2 + x_3^2$ defined on $\mathbb{R}^3$ and $a = (1, 2, 3)$, the coefficient matrix of the linear terms in h in the expression $f(a+h) - f(a)$ is $\begin{bmatrix} 2a_1 - a_2 & -a_1 & 2a_3 \end{bmatrix}$.

To get this quickly, try to imagine the terms that could give you $(\cdots)h_1$.

$\begin{bmatrix} 2a_1 - a_2 & -a_1 & 2a_3 \end{bmatrix}$

d) Let $f(x)$ be a real polynomial in $x = (x_1, \ldots, x_n)$ and $a \in \mathbb{R}^n$. Let A be the coefficient of linear terms in h of $f(a+h) - f(a)$. Then $\lim\limits_{\|h\| \to 0} \frac{|f(a+h)-f(a)-Ah|}{\|h\|} = 0$. [14]

$\lim\limits_{\|h\| \to 0} \left| \frac{h_1^2}{\|h\|} \right| \le \lim\limits_{\|h\| \to 0} |h_1| = 0$

e) Let $E \subseteq \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$, and $a \in E$. Suppose that there is a matrix $A_{1\times n}$ such that

$$\lim_{\|h\| \to 0} \frac{|f(a+h) - f(a) - Ah|}{\|h\|} = 0.$$

Then we say f is DIFFERENTIABLE at a and write $f'(a) = A$. We sometimes call A, the TOTAL DERIVATIVE of f at a. The GRADIENT $\bigtriangledown f(a)$ of f at a is the vector $f'(a)^t$.

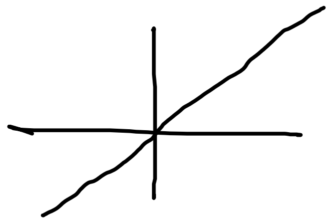
f) Thus for $f(x, y, z) = x^2 + y^2 + xyz$, we have $f'(x, y, z) = \begin{bmatrix} 2x + yz & 2y + xz & xy \end{bmatrix}$.

[14] *Proof.* Note that $f(a+h) - f(a) - Ah$ is a polynomial with terms of degree two or more in h. So $\lim\limits_{\|h\| \to 0} \frac{|f(a+h)-f(a)-Ah|}{\|h\|} = 0$. Note that $\lim\limits_{\|h\| \to 0} \frac{|h_1h_2|}{\|h\|} = \lim\limits_{\|h\| \to 0} \frac{|h_1|}{\|h\|} |h_2| = 0$. ■

[26.3] <u>About directional derivatives</u>

a) On your way to school, the height of the road increases and the road goes as if it is the line $f(x) = x$. What is the slope of the road you are facing?

b) What is the slope you will face on your way back?



c) So, the slope changes according to which direction we are facing.

d) Let $E \subseteq \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$, $a \in E$ and $0 \neq u \in \mathbb{R}^n$. The DIRECTIONAL DERIVATIVE $\boldsymbol{D_u f(a)}$ of f at a in the direction of u is defined as

$$D_u f(a) := \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t},$$

$D_u f(a) = \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$

provided the limit exists. It means the 'instantaneous rate of change of f where unit step means u'.

e) Thus, for $f(x, y, z) = x^2 + y^2 + xyz$, $a = (1, 2, 3)$, $u = (1, 0, 1)$ and $v = (0, 1, 1)$, we have

$$D_u f(a) = \lim_{t \to 0} \frac{(1 + t)^2 + 2^2 + (1 + t)2(3 + t) - 1^2 - 2^2 - 1.2.3}{t} = 10$$

and $D_v f(a) = 9$.

f) When $\boxed{u = e_i}$ the directional derivative $D_u f(a)$is called the PARTIAL DERIVATIVE $\underline{\boldsymbol{D_i f(a)}}$ of f with respect to the ith coordinate.

[26.4] <u>**Fact**</u> Suppose that $D_u f(a) = \beta$ and $\alpha \neq 0$. Then $D_{\alpha u} f(a)$ exists and $D_{\alpha u} f(a) = \alpha D_u f(a)$. [15]

[26.5] <u>**Fact**</u> Let $f : \mathbb{R}^n \to \mathbb{R}$ such that $D_u f(a)$ exists for each $u \neq 0$. Even then f need not be continuous at a. [16]

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[15]

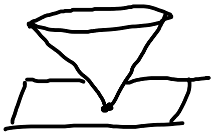
$$D_{\alpha u} f(a) = \lim_{t \to 0} \frac{f(a + t\alpha u) - f(a)}{t} = \alpha \lim_{t \to 0} \frac{f(a + t\alpha u) - f(a)}{\alpha t} = \alpha\beta = \alpha D_u f(a).$$

[16] Take $f(x, y) = \frac{y^2}{x}$ if $x \neq 0$ and 0, otherwise. Notice that $D_1 f(0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = 0$ and $D_2 f(0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = 0$. Take a unit vector $u = \begin{bmatrix} \frac{1}{\sqrt{m^2+1}} & \frac{m}{\sqrt{m^2+1}} \end{bmatrix}^t$ on the line $y = mx$, we have

$$D_u f(0) = \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0)}{t} = \frac{m^2}{\sqrt{m^2 + 1}}.$$

Thus the directional derivatives exist in all directions. We know that f is not continuous at 0.

[26.6] **Fact** Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous at a. Then $D_u f(a)$ may not exist even for a single $u \neq 0$. For example, take $f(x) = \|x\|$ on $\mathbb{R}^n$ and check at 0.



[26.7] **Fact** Let $E \subseteq \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$, $a \in E$, and $0 \neq u \in \mathbb{R}^n$. Suppose that $f'(a) = A = \begin{bmatrix} d_1 & \cdots & d_n \end{bmatrix}$. Then f is continuous at a and $D_u f(a)$ exists with $D_u f(a) = \langle \bigtriangledown f(a), u \rangle$. In particular, $D_i f(a) = d_i$. [17]

How to verify whether a function is differentiable at a point

1. Find all $D_i f(a)$ (if some $D_i f(a)$ does not exist, then conclude that f is not differentiable at a).
2. Form the matrix $A = \begin{bmatrix} D_1 f(a) & \cdots & D_n f(a) \end{bmatrix}$. Substitute A in the limit definition of the derivative and check if the limit is 0.
3. Conclude 'yes' if the limit is 0, otherwise conclude 'no'.
4. Take $f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$ We have $D_1 f(0) = 0$ and $D_2 f(0) = 0$. But it is not differentiable at 0 as $\lim\limits_{\|h\| \to 0} \left| \frac{f(h) - \begin{bmatrix} 0 & 0 \end{bmatrix} h}{\|h\|} \right|$ does not exist.

[26.8] **Fact** `(A sufficient condition for differentiability)` Let $E \subseteq \mathbb{R}^n$ be open and $f : E \to \mathbb{R}$. If $D_i f$ are continuous on E, then f is differentiable on E. This condition is not necessary for differentiability.[18]

[26.9] **Practice**

a) Let $f(x, y) = \min\{x, y\}$. Check the directional derivatives and differentiability at 0.

b) Let $f(x, y) = \min\{x^2, y^2\}$. Check the directional derivatives and differentiability at 0, $(1, 1)$ and $(1, 2)$.

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[17] To prove continuity note that by definition $\lim\limits_{\|h\| \to 0} \frac{|f(a+h) - f(a) - Ah|}{\|h\|} = 0$.
So $\lim\limits_{\|h\| \to 0} |f(a+h) - f(a) - Ah| = 0$ and $\lim\limits_{\|h\| \to 0} f(a+h) = f(a)$. So f is continuous.
To prove the next assertion note that as $\lim\limits_{\|h\| \to 0} \frac{|f(a+h) - f(a) - Ah|}{\|h\|} = 0$, in particular, taking $h = tu$, we have $\lim\limits_{t \to 0} \left| \frac{f(a+tu) - f(a) - tAu}{t} \right| = 0$. That is, $\lim\limits_{t \to 0} \left| \frac{f(a+tu) - f(a)}{t} - Au \right| = 0$. That is, $\lim\limits_{t \to 0} \frac{f(a+tu) - f(a)}{t} = Au$. That is, $D_u f(a) = Au = \langle \bigtriangledown f(a), u \rangle$. The proof is complete. ∎

[18] For example, take $f : (-1, 1) \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $f'(0) = 0$ and $f'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$. But f' is not continuous at 0.

[26.10] Higher order derivatives

a) Let $E \subseteq \mathbb{R}^n$ be open and $f : E \to \mathbb{R}$. Then f is said to be CONTINUOUSLY DIFFERENTIABLE on E, denoted $f \in \boldsymbol{\mathcal{C}^1(E)}$, if $D_i f$ are continuous on E.

b) Let $E \subseteq \mathbb{R}^n$ be open and $f : E \to \mathbb{R}$. Then the SECOND ORDER PARTIAL DERIVATIVE $\boldsymbol{D_{ij}f}$ is defined as $D_{ij}f = D_i(D_j f)$, if it exists.

$D_{ij} f := D_i (D_j f)$

c) It can happen that $D_{ij}f$ and $D_{ji}f$ both exist and unequal at a point. [19]

d) But, it is known from calculus that, if $D_{ij}f, D_{ji}f$ are continuous on E (open), then $D_{ij}f = D_{ji}f$ on E.

e) Let $E \subseteq \mathbb{R}^n$ be open and $f : E \to \mathbb{R}^m$. Then f is said to be TWICE CONTINUOUSLY DIFFERENTIABLE on E, denoted $f \in \boldsymbol{\mathcal{C}^2(E)}$, if all $D_{ij}f$ are continuous on E.

f) All polynomials in $x_1, \ldots, x_n$ infinitely differentiable functions.

g) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ function. The HESSIAN $\boldsymbol{H_f(a)}$ OR $\boldsymbol{H(a)}$ of f at a is the matrix (it is a real symmetric matrix as the function is $\mathcal{C}^2$)

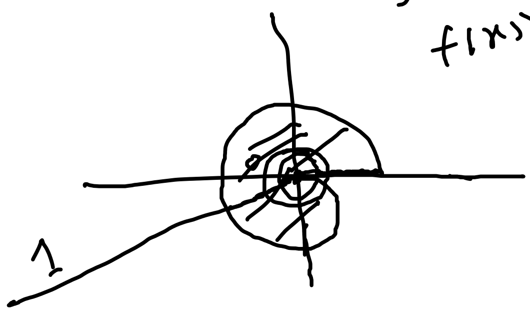
$$H(a) = \begin{bmatrix} D_{11}f & D_{12}f & \cdots & D_{1n}f \\ D_{21}f & D_{22}f & \cdots & D_{2n}f \\ \vdots & \vdots & \ddots & \vdots \\ D_{n1}f & D_{n2}f & \cdots & D_{nn}f \end{bmatrix}(a).$$

h) Thus the Hessian of $e^{x+y} - xyz$ at $(1, 1, 0)$ is $\begin{bmatrix} e^2 & e^2 & -1 \\ e^2 & e^2 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

$f(x,y,z) =$

$f(x,y) = 0$

$f(x,y) = 1$



[19] For example, consider

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

[26.11] **Fact** `(Rolle's theorem : single variable)` Let f be continuous on $[a,b]$ and differentiable on (a,b) with $f(a)=f(b)$. Then $\exists c \in (a,b)$ where $f'(c)=0$.

[26.12] **Fact** `(Generalized mean value theorem)` Let f,g be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists c \in (a,b)$ where $f'(c)[g(b)-g(a)] = g'(c)[f(b)-f(a)]$.[20]

[26.13] **Fact** `(Taylor's theorem in one variable)` Let $E \subseteq \mathbb{R}$ be open, $f : E \to \mathbb{R}$ be in $\mathcal{C}^n(E)$ and $[a, a+x] \subseteq E$. Then $\exists t \in (a, a+x)$ such that

$$f(a+x) = f(a) + f'(a)x + \frac{1}{2!}f''(a)x^2 + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(a)x^{n-1} + \frac{1}{n!}f^{(n)}(t)x^n.$$

To recall a proof see this.[21]

$$f(a+x) = f(a) + f'(a)x + \frac{x^2}{2!}f''(a) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{x^n}{n!}f^{(n)}(t)$$

[20] Consider $H(x) = f(x)[g(b)-g(a)] - g(x)[f(b)-f(a)]$. Apply Rolle's theorem.

[21] For $r \in [0,x]$, consider

$$F(r) = f(a+r) + f'(a+r)(x-r) + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(a+r)(x-r)^{n-1}, \quad G(r) = (x-r)^n.$$

Then F, G are continuous on $[0,x]$ and differentiable on $(0,x)$. Apply generalized mean value theorem: $\exists c \in (0,x)$ such that $F'(c)[G(x)-G(0)] = G'(c)[F(x)-F(0)]$. So

$$\Big(f'(a+c) + f''(a+c)(x-c) - f'(a+c) + f^{(3)}(a+c)\frac{(x-c)^2}{2!} - f''(a+c)(x-c) + \cdots$$

$$+f^{(n)}(a+c)\frac{(x-c)^{n-1}}{(n-1)!} - f^{(n-1)}(a+c)\frac{(x-c)^{n-2}}{(n-2)!}\Big)[-x^n] = -n(x-c)^{n-1}\Big[f(a+x) - f(a) - f'(a)x - \cdots - f^{(n-1)}(a)\frac{x^{n-1}}{(n-1)!}\Big].$$

So

$$f^{(n)}(a+c)\frac{(x-c)^{n-1}}{(n-1)!}x^n = n(x-c)^{n-1}\Big[f(a+x) - f(a) - f'(a)x - \cdots - f^{(n-1)}(a)\frac{x^{n-1}}{(n-1)!}\Big].$$

That is, $f^{(n)}(a+c)\frac{x^n}{n!} = [f(a+x) - f(a) - f'(a)x - \cdots - f^{(n-1)}(a)\frac{x^{n-1}}{(n-1)!}]$. So, $\exists t := a+c \in (a, a+x)$ such that

$$f(a+x) = f(a) + f'(a)x + \frac{1}{2!}f''(a)x^2 + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(a)x^{n-1} + \frac{1}{n!}f^{(n)}(t)x^n.$$

[26.14] **Fact** `(Taylor's theorem in many variables)` Let $E \subseteq \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$ be in $\mathcal{C}^m(E)$ and $[a, a+x] \subseteq E$. Then $\exists t \in (0,1)$ such that

$$\begin{aligned} f(a+x) &= f(a) + \sum_i D_i f(a) x_i + \tfrac{1}{2!} \sum_{i,j} D_{ij} f(a) x_i x_j + \cdots + \tfrac{1}{(m-1)!} \sum_{i_1,\ldots,i_{m-1}} D_{i_1 \cdots i_{m-1}} f(a) x_{i_1} \cdots x_{i_{m-1}} \\ &\quad + \tfrac{1}{m!} \sum_{i_1,\ldots,i_m} D_{i_1 \cdots i_m} f(a+tx) x_{i_1} \cdots x_{i_m}. \end{aligned}$$

For a proof see this.[22]

$\sum_i D_i f(a) x_i \quad + \frac{1}{2!} \sum_{i,j} D_{ij} f(a) x_i x_j + \cdots$

[26.15] **Example** The first three terms of Taylor's expansion of the function $\sin(xy)$ about $(0,0)$ are

$$f(0) + [D_x f(0) x + D_y f(0) y] + \frac{1}{2!}[D_{xx} f(0) xx + D_{xy} f(0) xy + D_{yx} f(0) yx + D_{yy} f(0) yy]$$
$$= 0 + 0 + xy.$$

[26.16] **Fact** `(Another form of Taylor's theorem)` Let $E \subseteq \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$ be in $\mathcal{C}^m(E)$ and $[a, a+x] \subseteq E$. Then

$$\begin{aligned} f(a+x) &= f(a) + \sum_i D_i f(a) x_i + \tfrac{1}{2!} \sum_{i,j} D_{ij} f(a) x_i x_j + \cdots + \tfrac{1}{(m-1)!} \sum_{i_1,\ldots,i_{m-1}} D_{i_1 \cdots i_{m-1}} f(a) x_{i_1} \cdots x_{i_{m-1}} \\ &\quad + \tfrac{1}{m!} \sum_{i_1,\ldots,i_m} D_{i_1 \cdots i_m} f(a) x_{i_1} \cdots x_{i_m} + r(x), \end{aligned}$$

where $\lim_{\|x\| \to 0} \frac{r(x)}{\|x\|^m} = 0$.

For a proof see this.[23]

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[22] (Rudin-p243) Define $p(t) = a + tx$, for $0 \le t \le 1$. Define $h(t) = f(p(t))$. For any $t \in (0,1)$, by chain rule,

$$h'(t) = f'(p(t)) p'(t) = \sum_{i=1}^n D_i f(p(t)) x_i, \qquad h''(t) = \sum_{i,j} x_i x_j D_{ij} f(p(t)),$$

and so on. By Taylor's theorem in one variable, $\exists t \in (0,1)$ such that $\quad h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{(m)!}.$

[23] We have, using the previous version,

$$r(x) = \frac{1}{m!} \left(\sum_{i_1,\ldots,i_m} D_{i_1 \cdots i_m} f(a+tx) x_{i_1} \cdots x_{i_m} - \sum_{i_1,\ldots,i_m} D_{i_1 \cdots i_m} f(a) x_{i_1} \cdots x_{i_m} \right).$$

Let $\epsilon > 0$. As each $D_{i_1 \cdots i_m} f(x)$ is continuous at a, $\exists \delta > 0$ such that

$$\left| \sum_{i_1,\ldots,i_m} D_{i_1 \cdots i_m} f(a+tx) - \sum_{i_1,\ldots,i_m} D_{i_1 \cdots i_m} f(a) \right| < \epsilon$$

for all $t \in (0, \delta)$. Note that if $\|x\|^m = 1$, then the maximum possible value of $|x_{i_1} \cdots x_{i_m}|$ is 1. So for any $x \neq 0$, we have $|x_{i_1} \cdots x_{i_m}| \le \|x\|^m$. Hence

$$\left| \frac{r(x)}{\|x\|^m} \right| \le \frac{\epsilon}{m!} \sum_{i_1,\ldots,i_m} \left| \frac{x_{i_1} \cdots x_{i_m}}{\|x\|^m} \right| \le \frac{\epsilon n^m}{m!} \le \epsilon \times \text{bounded quantity}.$$

This completes the proof.

[26.17] **Corollary : Taylor-I** Let $E \subseteq \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$ be in $\mathcal{C}^2(E)$ and $[a, a+x] \subseteq E$. Then

$$f(a+x) = f(a) + \langle \bigtriangledown f(a), x \rangle + \frac{1}{2} x^t H(a) x + r(x),$$

$$\left(D_1 f(a) x_1 + D_2 f(a) x_2\right)$$

where $\lim\limits_{\|x\| \to 0} \frac{r(x)}{\|x\|^2} = 0$.

[26.18] **Corollary: Taylor-II** Let $E \subseteq \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$ be in $\mathcal{C}^2(E)$ and $[a, a+x] \subseteq E$. Then $\exists t \in (0,1)$ such that

$$f(a+x) = f(a) + \langle \bigtriangledown f(a), x \rangle + \frac{1}{2} x^t H(a+tx) x.$$

$$\frac{1}{2!}\Big[D_{11}f(a)\, x_1 x_1 + D_{12}f(a)\, x_1 x_2 + D_{21}f(a)\, x_2 x_1 + D_{22}f(a)\, x_2 x_2\Big]$$

$$[x_1\ x_2]\begin{bmatrix} D_{11}f & D_{12}f \\ D_{21}f & D_{22}f \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[26.19] **Taylor series** Let $f : \mathbb{R}^n \to \mathbb{R}$ be infinitely differentiable. Then the TAYLOR SERIES $\boldsymbol{T_f(x; a)}$ of f about the point a is defined as

$$f(a) + \sum_i D_i f(a)(x-a)_i + \frac{1}{2!} \sum_{i,j} D_{ij} f(a)(x-a)_i (x-a)_j + + \frac{1}{3!} \sum_{i,j,k} D_{ijk} f(a)(x-a)_i (x-a)_j (x-a)_k + \cdots$$

[26.20] **Example** Take $f(y,z) = y^2 z^4 + yz^3 - 5yz + 6$ and $a = (1,2)$. Find the coefficient of $(y-1)^2(z-2)^2$ in $T_f((y,z); a)$ in two different ways.

Answer. To apply Taylor's theorem, put $w = (y,z) - (1,2)$. Terms with degree 4 can only occur in

$$\frac{1}{4!} \sum_{i,j,k,l=1}^{2} D_{ijkl} f(a)\ w_i w_j w_k w_l.$$

We want $w_i w_j w_k w_l = (y-1)^2(z-2)^2$, which can be done in $\frac{4!}{2!2!}$ ways. Hence, the coefficient is

$$\frac{1}{4!}\binom{4}{2} D_{1,1,2,2} f(a) = \frac{1}{2!2!}(2.4.3.2^2) = 24.$$

$$(y-1)^2\ {}^4C_2\ (z-2)^2\ 2^2$$

∘ Alternately, note that

$$f(y,z) = (y-1+1)^2(z-2+2)^4 - (y-1+1)(z-2+2)^3 - 5(y-1+1)(z-2+2) + 6.$$

The coefficient for $(y-1)^2(z-2)^2$ can only come from the first term. When expanded using binomial expansion, it will look like $(y-1)^2\binom{4}{2}(z-2)^2 2^2$. So the required coefficient is 24.

easy to remember

To remember the coefficients of Taylor series of $f : \mathbb{R}^n \to \mathbb{R}$ at a.

a) Take $\alpha = (\alpha_1, \cdots, \alpha_n)$ where α_i are nonnegative integers and take $a = (a_1, \cdots, a_n)^t$.

b) Use the notations $\boldsymbol{D^{\alpha}} := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, $\quad \boldsymbol{x^{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $\quad \alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!$.

c) Then the coefficient of x^α in $T_f(x; a)$ is $\frac{1}{\alpha!} D^\alpha f(a)$.