4 Lecture 4

Supporting and separating hyperplanes

Why do we need supporting and separating hyperplanes? We need them to characterize vertices. Vertices are important as the set of optimal solutions to a given lpp, if nonempty, always contains a vertex. (We shall see this in the fundamental theorem of linear programming.) These two concepts are very useful.

[4.1] **Definition** Let $S \subseteq \mathbb{R}^n$ and $w \in \partial S$ (boundary). A hyperplane H containing w and containing S in one of its closed halfspaces is called a SUPPORTING HYPERPLANE of S at w. (With an appropriate point of view, it would look like as if the set S is lying just above the hyperplane H touching it at w.) For our convenience, for some time, we shall use the nonstandard term ***H*** POSITIVELY SUPPORTS ***S*** AT ***w*** to mean that 'H supports S at w and $S \subseteq H_+$'. The term ***H*** NEGATIVELY SUPPORTS ***S*** AT ***w*** has a similar meaning.

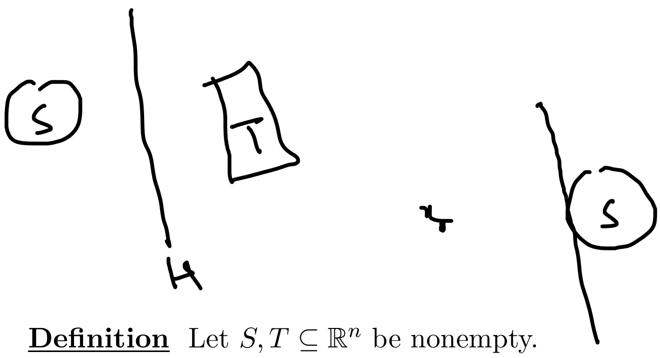
$S \subseteq \{x \mid c^t x \geq \alpha\}$

[4.2] **Recall** Recall that any point of S is either an interior point or a boundary point. If $w \in S^\circ$, then it is not possible to have a supporting hyperplane of S at w.‼ So, if H supports S at w, then w must be in ∂S.[6]

[4.3] **Example** The hyperplane $x + y = 1$ supports the region $S = \mathsf{conv}(e_1, e_2, e_3) \subseteq \mathbb{R}^3$ negatively at any point on the line segment $[e_1, e_2]$. Can you give a hyperplane that supports S positively at e_3 only?

[4.4] **Fact** a) It is easy to see that if $H_i : c_i^t x = \alpha_i$, $i = 1, \ldots, k$ support S at w positively, and $\lambda_1, \ldots, \lambda_k > 0$, then $H : (\sum \lambda_i c_i)^t x = (\sum \lambda_i \alpha_i)$ also supports S at w positively.

b) Thus, if there are two linearly independent hyperplanes supporting S at a, then there are infinitely many hyperplanes supporting S at a.



$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$

$S \subseteq H_-$

$\begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1$

[4.5] **Definition** Let $S, T \subseteq \mathbb{R}^n$ be nonempty.

a) A hyperplane H is said to SEPARATE S and T, if S is contained in one closed halfspace and T is contained in the other.

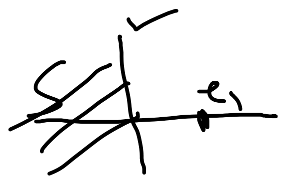
b) We say H STRICTLY SEPARATES S and T if S is contained in one open halfspace and T is contained in the other.

c) We say H STRONGLY SEPARATES S and x, if S is contained in one closed half space and x is contained in the other open halfspace.

[4.6] **Examples**

1. In $\mathbb{R}^2$, the hyperplane $x_1 = 0$ separates $S = \{x \mid x_1 \leq 0\}$ and $T = \{x \mid x_1 \geq 0\}$.

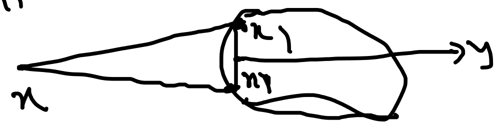
[6] Indeed if, $H : c^t x = \alpha$ supports S positively at a and $B_\epsilon(a) \subseteq S$, then $x := a - \frac{c\epsilon}{2\|c\|} \in B_\epsilon(a)$. But then $c^t x < \alpha$, a contradiction.



2. In $\mathbb{R}^2$, the hyperplane $x_1 = 0$ strictly separates the sets $S = \{x \mid x_1 < -1\}$ and $T = \{x \mid x_1 > 0\}$.
3. In $\mathbb{R}^2$, the hyperplane $x_1 = 0$ strongly separates the set $S = \{x \mid x_1 \le 0\}$ and the point e_1.

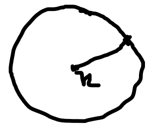
[4.7] **Theorem** `(Point of shortest distance)` Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set and $x \notin S$. Then there exists a unique $x_0 \in S$, such that $\|x - x_0\| = \min\limits_{z\in S} \|x - z\|$.

$\|u+v\| \le \|u\| + \|v\|$



$\|x-x_0\| \le \|x-y\| = \|\frac{x-x_0}{2} + \frac{x-x_1}{2}\|$

$\|x-x_0\| \ne \lambda(x-x_1)\|$ $\le \|\frac{x-x_0}{2}\| + \|\frac{x-x_1}{2}\| = \|x-x_0\|$



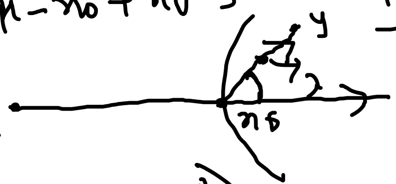
Proof. The existence of a point x_0 of minimum distance is a routine exercise in calculus. To show the uniqueness, let $x_1 \in S$, $x_1 \ne x_0$ be such that $\|x - x_1\| = \|x - x_0\|$. Take the midpoint $y = \frac{x_0+x_1}{2}$ of x_0 and x_1. Then, as y is a point in S,

$$\|x - x_0\| \le \|x - y\| = \|\frac{(x - x_0) + (x - x_1)}{2}\| \le \frac{\|x - x_0\| + \|x - x_1\|}{2} = \|x - x_0\|.$$

So we must have equality in the previous inequality. That is, $\|(x - x_0) + (x - x_1)\| = \|x - x_0\| + \|x - x_1\|$. Hence $x - x_0 = \lambda(x - x_1)$, for some $\lambda \ge 0$. But as $\|x - x_0\| = \|x - x_1\| \ne 0$, we get $\lambda = 1$. So $x_0 = x_1$. ■

[4.8] **Exercise+** `(Geometric proof to the uniqueness of` x_0 `in [4.7] using angles)` Let $S \subseteq \mathbb{R}^n$ be a closed convex set, $x \notin S$ and $x_0 \in S$ be a point of minimum distance. Let $y \in S$, $y \ne x_0$. Then show that $\langle x_0 - x, y - x_0\rangle \ge 0$. That is, the angle between $x_0 - x$ and $y - x_0$, is at most 90°. Now use it to prove to the uniqueness of x_0 in [4.7].

$\|x-x_0\|^2 \le \|x-y\|^2 = \|x-x_0+x_0-y\|^2 = \|x-x_0\|^2 + \|x_0-y\|^2 + 2\langle x-x_0, x_0-y\rangle$



$\langle x_0-x, y-x_0\rangle \ge 0$

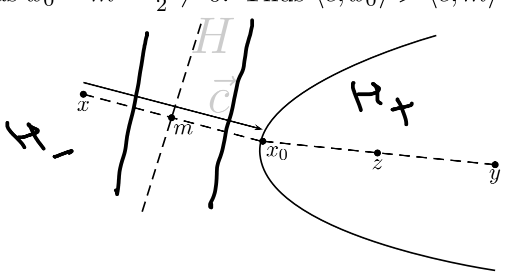
$-\|x_0-y\|^2 \le 2\langle x-x_0, x_0-y\rangle$

$y_\lambda = x_0 + \lambda(y-x_0)$

$-\lambda^2\|y-x_0\|^2 \le 2\langle x-x_0, -\lambda(y-x_0)\rangle = 2\langle x_0-x, y-x_0\rangle$

[4.9] **Theorem** `(Strict separation)` Let $\emptyset \ne S \subseteq \mathbb{R}^n$ be a closed convex set and $x \notin S$. Then there exists a hyperplane H which strictly separates x and S.

Proof. Let $x_0 \in S$ be closest to x. Put $c = x_0 - x$, $m = \frac{x+x_0}{2}$, and $\alpha = \langle c, m\rangle$. Define $H : c^t y = \alpha$. We have $\langle c, x_0 - m\rangle > 0$ as $x_0 - m = \frac{c}{2} \ne 0$. Thus $\langle c, x_0\rangle > \langle c, m\rangle = \alpha$. Similarly, $\langle c, m\rangle > \langle c, x\rangle$.



S = closed convex

$x \notin S$

$c = x_0 - x$, $\alpha = c^T m$

Let $y \in S$. By [4.8], $\langle x_0 - x, y - x_0\rangle \ge 0$. That is, $c^t(y - x_0) \ge 0$ or $c^t y \ge c^t x_0 > \alpha$. ■

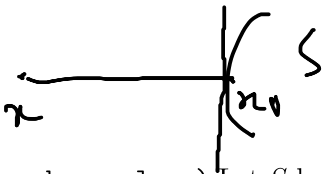
$H = \{z \mid c^T z = \alpha\}$

$c^T x_0 > c^T \frac{x+x_0}{2}$

$c^T \frac{x_0}{2} > \frac{c^T x}{2}$

$c^T c > 0$

$c^T x_0 > c^T x$



[4.10] **Theorem** `(Strong separation by a supporting hyperplane)` Let S be nonempty, closed, convex and $x \notin S$. Then there exists a supporting hyperplane of S which separates S and x strongly.‼

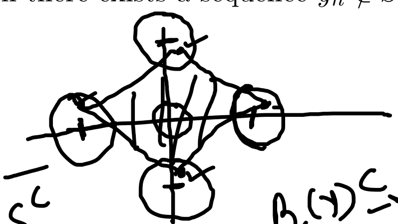
[4.11] **Lemma** `(Supporting hyperplane of a cone)` Every supporting hyperplane of a nonempty closed convex cone is linear.‼

[4.12] **Facts** `(Some properties of convex sets)`

1. Let $n \geq 2$ and take $0 < \epsilon < n^{-2}$. Select $2n$ points $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ such that $\|x_i - e_i\| \leq \epsilon$ and $\|y_i + e_i\| \leq \epsilon$. Geometrically, these are $2n$ points one of which is close to $\pm e_i$'s. Then 0 is an interior point of $P = \mathsf{conv}(x_1, \cdots, x_n, y_1, \cdots, y_n)$.[7]
2. ✓ If $S \subseteq \mathbb{R}^n$ is convex and $B_\epsilon(y) \subseteq \overline{S}$, then $B_\epsilon(y) \subseteq S$.‼ (Use previous item.) Hence, if $a \in \partial S$, then $a \in \partial\overline{S}$.
3. Let $S \subseteq \mathbb{R}^n$ and $y \in \partial S$. Then by definition each $B_\epsilon(y)$ will contain a point of S^c and hence there exists a sequence $y_n \in S^c$ such that $y_n \to y$.
   - But there may not exist a sequence y_n outside $\overline{S}$ such that $y_n \to y$. See for example, $\mathbb{Q}$.
   - However, if in addition S is given convex, then there exists a sequence $y_n \notin \overline{S}$ such that $y_n \to y$.‼

a is bd pt of S
Then a is a bd pt of $\overline{S}$.

$\partial S = \overline{S} \cap \overline{S^c}$



$B_\epsilon(y) \subseteq S$? ✓

$B_\epsilon(y) \subseteq \overline{S}$

$\mathbb{Q}^2$

$\overline{S} = \mathbb{R}^2$

$B_1(0)$

The following is a very important, useful and fundamental result.

[4.13] **Theorem** `(Existence of a supporting hyperplane at any boundary point)` Let S be convex and $y \in \partial S$. Then there is a supporting hyperplane of S at y.[8]

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[7] *Proof.* If $0 \notin P^\circ$, then either $0 \notin P$ or $0 \in \partial P$. In case $0 \notin P$, there is a hyperplane $H : c^t x = 0$ which keeps P in one half space, say in H_+. In case $0 \in \partial P$, we see by [3.54] that $0 \in \partial\,\mathsf{cone}(P)$ and hence $\mathsf{cone}(P) \neq \mathbb{R}^n$. So, there is a point outside. Hence we have a supporting hyperplane H of $\mathsf{cone}(P)$ such that $\mathsf{cone}(P) \subseteq H_+$. But then H must be a linear hyperplane. As $0 \in \partial P$, this hyperplane H is supporting P. So, in this case also, we have a hyperplane $H : c^t x = 0$ which keeps P in H_+.

Consider the coordinate of c of the largest magnitude. Say it is c_1. Clearly, $c_1 \neq 0$. Assume first that $c_1 < 0$. Then consider the point x_1. We are supposed to have $c^t x_1 \geq 0$. However, $|c_2x_2 + \ldots + c_nx_n| \leq |c_1|\frac{n-1}{n^2}$ and $|c_1x_1| > |c_1|\frac{n^2-1}{n^2}$. Hence $c^t x_1$ must be negative, a contradiction.

Next, if c_1 is positive, then work with $c^t y_1$ to get a contradiction. ∎

[8] *Proof.* Note that $y \in \partial\overline{S}$. So we will find a supporting hyperplane to $\overline{S}$ at y and that will serve as a supporting hyperplane to S at y. In view of this, we assume S is closed and $y \in \partial S$. As y is a boundary point, let (y_n) be a sequence of points in S^c converging to y. Using [4.7], let y_n' be the unique point in S closest to y_n. We know that $y_n \neq y_n'$. Put $c_n = \frac{y_n' - y_n}{\|y_n' - y_n\|}$ and $\alpha_n = c_n^t y_n'$. Then the hyperplane $H_n : c_n^t z = \alpha_n$ supports S at y_n' positively. That is,

$$c_n^t x \geq \alpha_n, \quad \forall x \in S, \text{where} \quad c_n = \frac{y_n' - y_n}{\|y_n' - y_n\|} \quad \text{and} \quad \alpha_n = c_n^t y_n'. \tag{2}$$

As the closed unit ball in $\mathbb{R}^n$ is compact, we will have a convergent subsequence of (c_n), say, $c_{n_k} \to c$. From the beginning, we could have considered the corresponding subsequence of (y_{n_k}). In view of that, we ca safely assume that $c_n \to c$. As $\|\cdot\|$ is continuous function, it follows that $\|c\| = 1$. Note that as $\|y_n - y_n'\| \leq \|y_n - y\| \to 0$. This means, $y_n' \to y$. Hence $c_n^t y_n' \to c^t y = \alpha$ (say). As $c_n^t x \geq \alpha_n$ for each $x \in S$, we see that $c^t x \geq \alpha$ for each $x \in S$. That is, $H = \{z \mid c^t z = \alpha\}$ passes through y and contains S in H_+. ∎

$c_n = \frac{y_n' - y}{\|y_n' - y\|} \in \mathbb{R}^n$

$y_n \dots y$ S

$\{c_n, c_2, \dots\}$

$y_n \to y_n'$ pt of min dist

$\|y_n - y_n'\| \le \|y_n - y\| \to 0$

$c_n \to c \Rightarrow \|c\| = 1$

$H_n: c_n^t z = \alpha_n = c_n^t\left(\frac{y_n + y_n'}{2}\right)$

$S \subseteq H_n^+$ | $H: c^t z = \alpha$

The previous results leads us to a very useful technique.

[4.14] **Lemma** (`Useful technique`) Let $S \subseteq \mathbb{R}^n$ be convex and $H = \{z \mid c^t z = \alpha\}$ be a supporting hyperplane of S. Put $T = S \cap H$. Then each vertex of T is also a vertex of S.

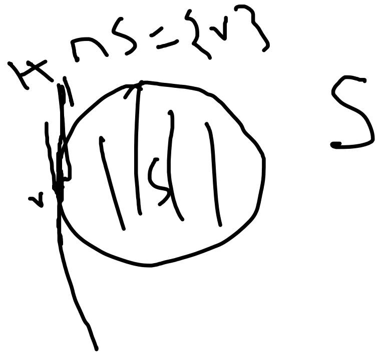
Proof. Let v be a vertex of T. Suppose that it is not a vertex of S. So $\exists x, y \in S$, $x \neq y$, and $\lambda \in (0,1)$ such that $v = \lambda x + (1-\lambda)y$. Assume, without loss, that $S \subseteq H_+$. We have

$$\alpha = c^t v = \lambda c^t x + (1-\lambda)c^t y \geq \lambda\alpha + (1-\lambda)\alpha = \alpha.$$

So we must have $c^t x = \alpha = c^t y$. Thus $x, y \in H$ and so v cannot be a vertex of T. ∎

[4.15] **Example**

1. This tells us that each point on the boundary of a closed circular disc C is a vertex. (Just take a tangent hyperplane H at a boundary point v. Then $H \cap C = \{v\}$. So v is a vertex of $H \cap C$. So v is a vertex of C.) Of course, we already knew this.
2. It also tells us that $0, e_1, e_2, e_1 + e_2$ are the vertices of our favorite set.



$x \in \partial S$

$H \to$ supp at x

$H \cap S \to$ pick a vertex w

[4.16] **Fact** Let $S \subseteq \mathbb{R}^n$ be a nonempty compact set. Then $\partial S \neq \emptyset$. (This follows from definition. You can give a more geometrical proof if S is given convex.)‼

[4.17] **Theorem** A nonempty compact convex set in $\mathbb{R}^n$ has a vertex.

Proof. Use induction on n. For $n = 1$, the statement is trivial. Assume the statement $\forall n < m$. Let S be a nonempty compact convex subset of $\mathbb{R}^m$. Then $\partial S \neq \emptyset$. Let $p \in \partial S$ and H a supporting hyperplane of S at p. Then $H \cap S$ may be seen as a nonempty compact convex set in $\mathbb{R}^{m-1}$. By induction hypothesis, $H \cap S$ has a vertex w. By [4.14], w is a vertex of S. ∎