[0.1] **Application(H)** `(Using lpp to separate two finite sets.)` Consider two sets $V = \{v_1, \ldots, v_k\}$ and $W = \{w_1, \ldots, w_m\}$ in $\mathbb{R}^n$. We wish to find a hyperplane $H : a^t x = b$ which separates them. Convert this to a lpp.

Answer. Write $V = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}$ and $W = \begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}$. We are looking for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^t V \leq b\mathbf{1}^t$ and $a^t W \geq b\mathbf{1}^t$. That is, we are looking for a $c^t = \begin{bmatrix} a^t & -b \end{bmatrix}$ satisfying $c^t \begin{bmatrix} V \\ \mathbf{1}^t \end{bmatrix} \leq 0$ and $c^t \begin{bmatrix} W \\ \mathbf{1}^t \end{bmatrix} \geq 0$. Thus, we are looking for a nontrivial c satisfying

$$\begin{bmatrix} V^t & \mathbf{1} \\ -W^t & -\mathbf{1} \end{bmatrix} c \leq 0,$$

which exists iff there is a nontrivial c satisfying

$$\begin{bmatrix} V^t & \mathbf{1} \\ -W^t & \mathbf{1} \end{bmatrix} c \leq 0, \quad -1 \leq c_i \leq 1.$$

If $\mathsf{rank}(A = \begin{bmatrix} V^t & \mathbf{1} \\ -W^t & -\mathbf{1} \end{bmatrix}) \leq n$, then (as A has $n+1$ columns) we apply GJE to find a nontrivial c such that $Ac = 0$. If $\mathsf{rank}\, A = n + 1$, use the previous question.

[0.2] **Exercise(M)** Is it true that cycling will occur only at a vertex of minimum cost?

Answer. No. Suppose that in $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, x, b \geq 0 \end{array}$ cycling occurs at a minimum vertex. Let $B_1, \ldots, B_k, B_1$ be a cycle of bases. Now add a new constraint $x_{n+1} \leq 2$, so the matrix changes to $A' = \begin{bmatrix} A & 0 & 8 \\ 0 & 1 & 1 \end{bmatrix}$. Change the cost function to $c^t x + x_{n+1}$. Let $B_{i,1}$ mean the basis B_i with x_{n+1}. Clearly, none of these $B_{i,1}$ is a minimum basis, as it has taken $x_{n+1} = 2$, instead of 0. But notice that we have a cycle $B_{1,1}, \ldots, B_{k,1}, B_{1,1}$ of bases.