DEPARTMENT OF MATHEMATICS
Indian Institute of Technology, Guwahati

EndSem | MA321
23-11-2021 | Instructor : Sukanta Pati
Time : 14:00–17.00 | Maximum Score : 40

Write appropriate and precise justifications, with readable handwriting. Use pencils for convenience.

Submit in the codetantra portal only. The portal will close by 17:15. Start submission at 17:01 to avoid problems.

If you are submitting to my email, do it before time. There may be deductions for late submissions.

1. Consider the first six prime numbers $p_1, \ldots, p_6$, the next eight prime numbers $q_1, \ldots, q_8$ and the next ten prime numbers $r_1, \ldots, r_{10}$. For any point $(x, y, z) \in \mathbb{R}^3$, define

$$f(x, y, z) = |x - p_1| + \cdots + |x - p_6| + |y - q_1| + \cdots + |y - q_8| + |z - r_1| + \cdots + |z - r_{10}|.$$

Optimize it, using the techniques you have learned in this course. $\boxed{10}$

Answer. The function is not differentiable at the points (x, y, z), if $x = p_i$ or $y = q_j$ or $z = r_k$, for some i, j, k. Call the set of these points S. At all other points it is twice differentiable (being linear).

Let us first find a critical point. Note that

$D_x f(x, y, z) =$ number of p_i left to $x -$ number of p_i right to x.

Equating it to zero we see that $x \in (p_3, p_4)$. Similarly, $y \in (q_4, q_5)$ and $z \in (r_5, r_6)$.

Now let (x, y, z) be a critical point. Select a $\delta > 0$ small such that Let $B_\delta(x, y, z)$ does not contain any of the points of S. It now follows that, for each point in $B_\delta(x, y, z)$, the Hessian H_f is the zero matrix. That is, H_f is a psd matrix in a neighborhood. Hence each of these points are local minimums. [5]

Now consider a point $a = (x_0, y_0, z_0)$ with $x_0 < p_3$. Take $0 < \epsilon < p_3 - x_0$ and $b = (p_i + \epsilon, y_0, z_0)$, then $f(b) - f(a) \leq -\epsilon < 0$. So a cannot be a local minimum.

Similarly, if $x_0 > p_4$, it cannot be a local minimum. So $x_0 \in [p_3, p_4]$. Similarly, $y_0 \in [q_4, q_5]$ and $z_0 \in [r_5, r_6]$.

That is, we are looking at the points of $T = [p_3, p_4] \times [q_4, q_5] \times [r_5, r_6]$. As the function is continuous and the function is a constant (either evaluate the function of use MVT on a coordinate as the set is convex) in T°, we see that all the points of T are local minimums of the same value. And there are no other local minimums. [5]

Since this continuous function is bounded below and goes to infinity as we move far away form the points in S, we must have an absolute minimum. But then it must be a local minimum too. Hence, points in $T = [p_3, p_4] \times [q_4, q_5] \times [r_5, r_6]$ are the absolute minimums and there are no other local minimums. The minimum value is

$$|p_3 - p_1| + \cdots + |p_3 - p_6| + |q_4 - q_1| + \cdots + |q_4 - q_8| + |r_5 - r_1| + \cdots + |r_5 - r_{10}| = 193. \quad \text{[Bonus 1]}$$

2. Let the Earth be represented by the solid sphere $x^2+y^2+z^2 \le 1$. Consider the seven sisters (these are seven stars far away from us) to be seven fixed points $A(a_1,a_2,a_3)$, $B(b_1,b_2,b_3)$, $C(c_1,c_2,c_3)$, $D(d_1,d_2,d_3)$, $E(e_1,e_2,e_3)$, $F(f_1,f_2,f_3)$, $G(g_1,g_2,g_3)$ in $\mathbb{R}^3$ with large positive coordinates. Given any point $P=(x,y,z)$ on the surface S of the Earth, the sum of the squares of the distances of P from the seven sisters is computed and it is called $f(P)$. How do we find a point P where $f(P)$ is the minimum and a point P where $f(P)$ is the maximum? $\boxed{10}$

Answer. ◦ Notice that $f(P)=(x-a_1)^2+(y-a_2)^2+(z-a_3)^2+\cdots+(x-g_1)^2+(y-g_2)^2+(z-g_3)^2 = 7+\|A\|^2+\cdots+\|G\|^2-2(a_1+\cdots+g_1)x-\cdots-2(a_3+\cdots+g_3)z$.

So it is enough to optimize $-ax-by-cz$ over $S \equiv x^2+y^2+z^2-1=0$, where $a=a_1+\cdots+g_1$, $b=a_2+\cdots+g_2$, $c=a_3+\cdots+g_3$. [2]

◦ Matrix $J=\begin{bmatrix}2x & 2y & 2y\end{bmatrix}$ has full rank throughout S. So by LNC, a local optimum must be a KT point. [1]

◦ Now, we find KT points. We have $L \equiv f-wh = -ax-by-cz-w(x^2+y^2+z^2-1)$. The KT conditions are $\bigtriangledown L=0$ and $(x,y,z)\in S$. From the first one we get

$$\begin{bmatrix}-a\\-b\\-c\end{bmatrix} = w\begin{bmatrix}2x\\2y\\2z\end{bmatrix}.$$

[1]

◦ Note that $w=0$ is not possible. Because, if $w=0$, then $a=b=c=0$, not possible, as they are positive numbers. So

$$x=\frac{-a}{2w},\ y=\frac{-b}{2w},\ z=\frac{-c}{2w}.$$

Since $x^2+y^2+z^2=1$, we have $w=\pm\frac{\sqrt{a^2+b^2+c^2}}{2}=\pm\frac{\alpha}{2}$ (say).[1]

◦ The bordered Hessian matrix is $M=\begin{bmatrix}0 & 2x & 2y & 2z\\ 2x & -2w & 0 & 0\\ 2y & 0 & -2w & 0\\ 2z & 0 & 0 & -2w\end{bmatrix}$.[1]

◦ As $n-p=3-1=2$, we find two leading minors starting from $\det M$. Try adding suitable multiples of row two to row one, etc.

$$\det\begin{bmatrix}0 & 2x & 2y\\ 2x & -2w & 0\\ 2y & 0 & -2w\end{bmatrix} = 8w(x^2+y^2),$$

$$\det\begin{bmatrix}0 & 2x & 2y & 2z\\ 2x & -2w & 0 & 0\\ 2y & 0 & -2w & 0\\ 2z & 0 & 0 & -2w\end{bmatrix} = -16w^2(x^2+y^2)-16w^2z^2=-16w^2(x^2+y^2+z^2).$$

[2]

∘ Take $w = \frac{\alpha}{2}$. So

$$x = \frac{-a}{\sqrt{a^2+b^2+c^2}},\ y = \frac{-b}{\sqrt{a^2+b^2+c^2}},\ z = \frac{-c}{\sqrt{a^2+b^2+c^2}}.$$

The determinants change sign starting from $(-1)^n = (-1)^3$. So this is a point of strict local maximum. [1]

∘ Take $w = \frac{-\alpha}{2}$. So

$$x = \frac{a}{\sqrt{a^2+b^2+c^2}},\ y = \frac{b}{\sqrt{a^2+b^2+c^2}},\ z = \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

The determinants have the same sign of $(-1)^p = (-1)^1$. So this is a point of strict local minimum. [1]

∘ But as there are only two KT points, we see that they are absolute maximum and absolute minimum. [Bonus 1]

3. Consider the region

$$S = \{x \in \mathbb{R}^4 \mid x_1^2+x_2^2+x_3^2+x_4^2 \le 25,\ x_1^2+x_2^2+x_3^2+x_4^2 \ge 16,\ x_1+x_2+x_3+x_4 = 8\}.$$

Show that KTCQ1 holds at each point in S. $\boxed{3}$

Use KT theory to maximize $x_1+x_2-x_3-x_4$ over S. $\boxed{7}$

Answer.

∘ Take $g_1 \equiv 25 - x_1^2 - \cdots - x_4^2 \ge 0$, $g_2 \equiv x_1^2 + \cdots + x_4^2 - 16 \ge 0$, $h \equiv x_1 + \cdots + x_4 - 8 = 0$.

∘ At any point in S other than $(2,2,2,2)$, we have at most one of g_1 and g_2 active. So $\bigtriangledown g_i$ (active) and $\bigtriangledown h$ are linearly independent.

In fact, if $\bigtriangledown g_i = \alpha \bigtriangledown h$, then we must get $2x = \alpha(1,1,1,1)$. Substituting in $h(x) = 0$, we get $x = (2,2,2,2)$. A contradiction.

So KTCQ1 holds at these points. [1]

∘ At $a = (2,2,2,2)$, g_2 is active. We have

$$\mathcal{D}(a) = \{d \mid \bigtriangledown g_2^t d \ge 0, \bigtriangledown h^t d = 0\} = \{d \mid d_1+d_2+d_3+d_4 = 0\}.$$

Let $d \in \mathcal{D}(a)$ be nonzero. Then for all small positive t, we have $g_1(a+td) > 0$ and $h(a+td) = 0$. Now,

$$g_2(a+td) = (2+td_1)^2 + \cdots + (2+td_4)^2 - 16 = t^2(d_1^2+\cdots+d_4^2) + 4t(d_1+\cdots+d_4) > 0.$$

So for $t \in (0, 3/\|d\|]$ (comes from g_1), we have $\alpha(t) = a + td$ in the feasible reason. And we have $\lim\limits_{t\to 0+} \frac{\alpha(t)-\alpha(0)}{t} = d$. Hence KTCQ1, holds at $(2,2,2,2)$. [2]

∘ Consider minimizing $f(x) \equiv -x_1 - x_2 + x_3 + x_4$ over S. As KTCQ1 holds at each point of S, by KTNC every point of local minimum must be a KT point.[1]

∘ To find KT points.

We have $L = -x_1 - x_2 + x_3 + x_4 - \lambda_1(25 - x_1^2 - x_2^2 - x_3^2 - x_4^2) - \lambda_2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - w(x_1 + x_2 + x_3 + x_4 - 8)$.

KT conditions: $\bigtriangledown L(x) = 0,\ \lambda_i g_i(x) = 0,\ \lambda_i \geq 0, w \in \mathbb{R},\ x \in S.$

Now

$$\bigtriangledown L = 0 \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \\ -2x_4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \\ 2x_4 \end{bmatrix} + w \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

[1]

∘ If $\lambda_1 > 0$, then g_1 is active and hence $\lambda_2 = 0$. Hence

$$(x_1, x_2, x_3, x_4) = (\frac{-1-w}{-2\lambda_1}, \frac{-1-w}{-2\lambda_1}, \frac{1-w}{-2\lambda_1}, \frac{1-w}{-2\lambda_1}) = (\frac{w+1}{2\lambda_1}, \frac{w+1}{2\lambda_1}, \frac{w-1}{2\lambda_1}, \frac{w-1}{2\lambda_1}).$$

Substituting in $h(x) = 0$, we get $w = 4\lambda_1$. Substituting in $g_1(x) = 0$, we get $\lambda_1 = 1/3$. Hence $w = 4\lambda_1 = 4/3$ and $x = (7/2, 7/2, 1/2, 1/2)$.

Here, $H(L, x, \lambda, w) = \frac{2}{3}I$ is pd. Hence this point is a point of strict minimum.[3]

∘ Let $\lambda_1 = 0$. If $\lambda_2 = 0$, then the first vector will be a scalar multiple of the last, which is not possible.

So $\lambda_2 > 0$ and g_2 is active. Hence

$$(x_1, x_2, x_3, x_4) = (\frac{-1-w}{2\lambda_2}, \frac{-1-w}{2\lambda_2}, \frac{1-w}{2\lambda_2}, \frac{1-w}{2\lambda_2}).$$

Substituting in $h(x) = 0$, we get $w = -4\lambda_2$. Substituting in $g_2(x) = 0$, we get $2 = 0$, an impossibility. Hence, this case is not possible.[2]

∘ Since the function is continuous and the set is compact, we must have an absolute minimum and it must be a KT point. So the point $x = (7/2, 7/2, 1/2, 1/2)$ is the absolute minimum. So, the maximum value of the original question is 6 attained at $(7/2, 7/2, 1/2, 1/2)$.[Bonus 1]

4. Let f, g_i, h_j be twice continuously differentiable and consider the problem

$$\begin{array}{ll} \min & f(x) \\ \hline \text{s.t.} & g_i(x) \geq 0,\ i = 1, \ldots, m,\ \ h_j(x) = 0,\ j = 1, \ldots, p,\ \ x_k \geq 0,\ k = 1, \ldots, n. \end{array}$$

Let a be a feasible point and define

$$L(x, \lambda, w) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) - \sum_{j=1}^{p} w_j h_j(x).$$

Prove that the following are equivalent. $\boxed{5+5}$

a) $Z(a) = \emptyset$.

b) $\exists\ \lambda_i \geq 0,\ i = 1, \ldots, m,\ w_j,\ j = 1, \ldots, p$ such that the following KT conditions are satisfied

$$\bigtriangledown L(a, \lambda, w) \geq 0, \quad \lambda_i g_i(a) = 0, \forall i = 1, \ldots, m, \quad a^t \bigtriangledown L(a, \lambda, w) = 0.$$

Answer.

a)⇒b). Put $g_{m+i}(x) = x_i$, for $i = 1, \ldots, n$. As $Z(a) = 0$, $\exists \lambda_i \geq 0$, $i = 1, \ldots, m+n$, w_j, $j = 1, \ldots, p$ such that

$$\bigtriangledown f(a) - \sum_{i=1}^{m+n} \lambda_i \bigtriangledown g_i(a) - \sum_j w_j \bigtriangledown h_j(a) = 0, \qquad \lambda_i g_i(a) = 0,\ i = 1, \ldots, m+n.$$

It follows that

$$\bigtriangledown f(a) - \sum_{i=1}^{m} \lambda_i \bigtriangledown g_i(a) - \sum_j w_j \bigtriangledown h_j(a) = \sum_{i=m+1}^{m+n} \lambda_i e_i \geq 0.$$

As $\lambda_i g_i(a) = 0$, we see that when $\lambda_{m+i} > 0$ then $a_i = 0$, so that

$$a^t \Big[\bigtriangledown f(a) - \sum_{i=1}^{m} \lambda_i \bigtriangledown g_i(a) + \sum_j w_j \bigtriangledown h_j(a) \Big] = a^t \left[\sum_{i=m+1}^{m+n} \lambda_i e_i \right] = 0.$$

b)⇒a). Suppose that $\exists\ \lambda_i \geq 0$, $i = 1, \ldots, m$, w_j, $j = 1, \ldots, p$ such that these three conditions are satisfied. Define

$$\begin{bmatrix} \lambda_{m+1} \\ \vdots \\ \lambda_{m+n} \end{bmatrix} := \bigtriangledown L(a, \lambda, w).$$

Hence,

$$\bigtriangledown f(a) - \sum_{i=1}^{m+n} \lambda_i \bigtriangledown g_i(a) - \sum_j w_j \bigtriangledown h_j(a) = 0, \qquad \lambda_i g_i(a) = 0,\ i = 1, \ldots, m.$$

For $i = m+1, \ldots, m+n$, $\lambda_i g_i(a) = 0$ translates to $\lambda_i a_i = 0$, which holds due to the third part of the hypothesis.

Thus we have shown the existence of $\lambda_i \geq 0$, $i = 1, \ldots, m+n$ and w_j, $j = 1, \ldots, p$ such that

$$\bigtriangledown f(a) - \sum_{i=1}^{m+n} \lambda_i \bigtriangledown g_i(a) - \sum_j w_j \bigtriangledown h_j(a) = 0, \qquad \lambda_i g_i(a) = 0,\ i = 1, \ldots, m+n.$$

So $Z(a) = \emptyset$.

5. (a) Give an example of a function $f : \mathbb{R}^3 \to \mathbb{R}$ which is continuous at 0 such that all directional derivatives exist at 0 except one. No justifications required. $\boxed{\text{(2,-2) means correct 2, wrong -2}}$

   Answer. No such function can exist. If $D_d f(0)$ does not exist, then $D_{-d} f(0)$ will also not exist. Also $D_{5d} f(0)$ will also not exist.

   Suppose we take directions from the unit sphere and also we regard d and $-d$ as the same as they give us the same line. Then we can have an example. Take $f(x) = \begin{cases} x_1 & \text{if } x = (x_1, 0, 0), x_1 \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

   (b) Consider the Taylor series of $f(x, y, z) = \sin(xyz)$ about the origin. We add the coefficients of all the first degree, second degree and third degree terms. What will we get? No justifications required. $\boxed{\text{(2,-2)}}$

   Answer. 1

(c) Give an example of a positive definite matrix $A \in M_5(\mathbb{C})$ which does not have a zero entry. No justifications required. $\boxed{(2,\text{-}2)}$

Answer. $I + J$. Here $J = \mathbf{1}\mathbf{1}^t$ means all ones matrix and $\mathbf{1}$ is the all ones vector. In general for any nonzero vector v, $M = vv^t$ is psd and so $I + M$ is pd.

(d) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function that satisfies the following condition at 0.

'For each nonzero d, there exists a positive α_d such that for each $t \in (0, \alpha_d)$ we have $f(0) \le f(td)$.'

Must 0 be a point of local minimum? No justifications required. $\boxed{(2,\text{-}2)}$

Answer. No. See $f(x,y,z) = \begin{cases} -1 & \text{if } y < x^2, z = 0 \\ x^2 + y^2 + z^2 & \text{otherwise.} \end{cases}$

(e) Give an example of a twice continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ for which $(1,2,3)$ is a saddle point because its Hessian matrix is $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. No justifications required. $\boxed{(2,\text{-}2)}$

Answer. $(x-1)^2 + (x-1)(y-2) + (y-2)^2 - \frac{(z-3)^2}{2} = \frac{1}{2}\begin{bmatrix} x-1 & y-2 & z-3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix}$.

(f) From $\mathbb{R}^2$ cut out the circular region $(x-1)^2 + (y-1)^2 \le 1$. Also cut out the regions, $(x-1)^2 + (y+1)^2 \le 1$, $(x+1)^2 + (y-1)^2 \le 1$, and $(x+1)^2 + (y+1)^2 \le 1$. Consider the region containing the origin. Is it true that KTCQ1 is satisfied at all points? $\boxed{(2,\text{-}2)}$

Answer. No. Not satisfied at the four corner points. Consider the corner point $a = e_1$. Then $\mathcal{D}(a) = \{d \mid d_2 = 0\}$. So $p = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \in \mathcal{D}(a)$. However, we cannot get a $\mathcal{C}^1$ curve $\alpha(t)$ in the feasible region with $\lim\limits_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t} = p$, as the first coordinate of the numerator is always ≤ 0.