2. $\sigma_q^2 = \int \frac{f^2(x)\, p^2(x)}{q(x)}\, dx - \mu^2$

$p(x) = \frac{e^{-x/\theta}}{\theta}$ $\quad$ $f(x) = x$

$q(x) = \frac{1}{v} e^{-x/v}$

$I = \int_0^\infty \frac{x^2 e^{-2x/\theta}}{\theta^2}\, v e^{x/v}\, dx$

min I

$= \frac{v}{\theta^2} \int_0^\infty x^2 e^{-2x/\theta + x/v}\, dx$

~~after m~~

$= \frac{v}{\theta} \times \frac{2}{\left(\frac{2}{\theta} - \frac{1}{v}\right)^3}$

$\frac{dI}{dv} = 0 \Rightarrow \left(\frac{2}{\theta^2}\right)\left(\left(\frac{2}{\theta} - \frac{1}{v}\right)^{-3} + -3v\left(\frac{2}{\theta} - \frac{1}{v}\right)^{-4}\left(\frac{1}{v^2}\right)\right) = 0$

$1 - 3v\left(\frac{2}{\theta} - \frac{1}{v}\right)^{-1}\left(\frac{1}{v^2}\right) = 0$

$1 = \frac{3}{v\left(\frac{2}{\theta} - \frac{1}{v}\right)}$

$\frac{2v}{\theta} - 1 = 3$

$v = \frac{4\theta}{2} = \underline{2\theta}$

~~A-AB~~

(3) ~~Because~~ $X \sim U(0,1) \Rightarrow c = 0.5$

~~var~~ $\hat{\mu}_{anti} = \frac{1}{n} \sum_{i=1}^{n/2} (f(X_i) + f(\tilde{X}_i))$

$var(\hat{\mu}_{anti}) = \frac{n/2}{n} var(f(X) + f(\tilde{X})) = \frac{\sigma^2}{n}(1+\rho)$

$\rho = corr(f(X), f(\tilde{X}))$

~~For Antith~~ $f(x) = 10\, I_{[a, a+e]}$ $e = 0.01$

For antithetic to work $\rho < 0$ or $cov(f(x), f(\tilde{x})) < 0$

~~because~~

$cov(f(x), f(\tilde{x})) = E[f(x)f(1-x)] - E[f(x)]E[f(1-x)]$

$E[f(x)] = E[f(1-x)] = 10 \times 0.01 = 0.1$

$E[f(x)f(1-x)] - (0.1)(0.1) < 0$

$E(f(x)f(1-x)) < 0.01$

$\int_0^1 f(x)f(1-x)\, dx < 0.01$

$100 N = \int_0^1 100\, I_{(a,a+e]}\, I_{[1-a-e,\, 1-a)}\, dx < 0.01$

~~If $a+e \leq 1-a-e \Rightarrow N=0$~~

$N < \underline{10^{-4}}$

~~Four Cases~~

$$N = \int_0^1 I_{(a,a+e)}\, I_{[1-a-e,1-a)}\, dx.$$

Four cases.

(i) $a+e \le 1-a-e$

then $N=0$

Profitable

$2a \le 1-2e$

$a \le \frac{1-2e}{2}$

$a \le 0.5-e$.

$a \le 0.49$

(ii) $1-a \le a$ then $N=0$

$1 \le 2a$

Profitable

$0.5 \le a$

(iii) $a \le 1-a-e \le a+e \le 1-a$

$2a \le 1-e$

$a \le \frac{1-e}{2}$

$1-2e \le 2a$

$0.5-e \le a$

$2a \le 1-e$

$a \le \frac{1-e}{2}$

$0.5-e \le a \le 0.5-\frac{e}{2}$

$N = 1\times(a+e-(1-a-e))$

$= 2a+2e-1$

$2a+2e-1 < 10^{-4}$

$a < \frac{10^{-4}+1-2e}{2}$

$a < 0.5-e+\frac{10^{-4}}{2}$

∴ Profitable if

$$0.5 - e \le a < 0.5 - e + \frac{10^{-4}}{2}$$

Unprofitable if

$$0.5 - e + \frac{10^{-4}}{2} < a < 0.5 - \frac{e}{2}$$

(iv) $1-a-e \le a \le 1-a \le a+e$

$1-e \le 2a$

$0.5 - \frac{e}{2} \le a$

$a \le 0.5$

$1-e \le 2a$

$0.5 - \frac{e}{2} \le a$

$$\boxed{0.5 - \frac{e}{2} \le a \le 0.5}$$

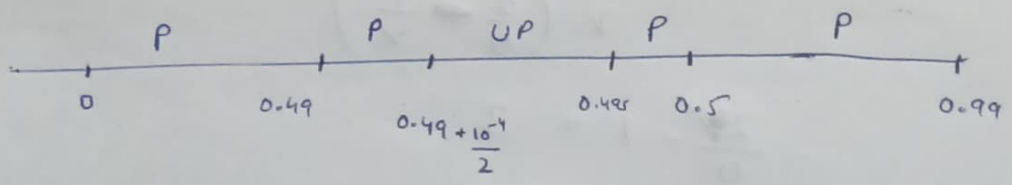
$$N = (\cancel{a}\, 1-a-e-(a))$$

$$= 1-2a-e$$

$$N < 10^{-4}$$

$1-2a-e < 10^{-4}$ ∴ Profitable

$0.5 - \frac{e}{2} - \frac{10^{-4}}{2} < a$



UP only if $a \in \left[0.49 + \frac{10^{-4}}{2},\ 0.495\right]$