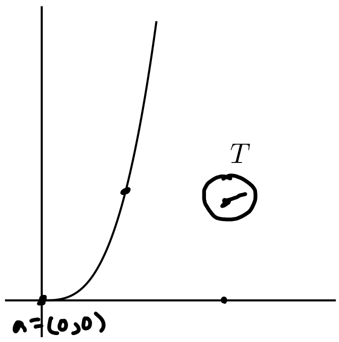
✓

34 Lecture 34

[34.1] **Example** Consider $T = \{x \in \mathbb{R}^2 \mid g_1(x) = x_1^3 - x_2 \geq 0,\, g_2(x) = x_1 \geq 0,\, g_3(x) = x_2 \geq 0\}$. Show that ktcq1 holds at all points in T.

CQ ✓



a , $\mathcal{D}(a)$

$\forall\, d \in \mathcal{D}(a)$

Answer. The region is shown here.

a) There are four types of points here: interior points, point only on x-axis, point only on the curve, and $(0,0)$ which is common.

b) Do we know that ktcq1 holds at each interior point? Yes.

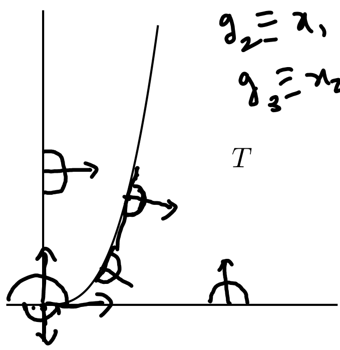
c) Let $a = (0,0)$. Find $\mathcal{D}(a)$ and check whether all those directions are tangents of some $\mathcal{C}^1$-curves.

Before that try to draw the normals and guess $\mathcal{D}(a)$. It is important.

$g_1 = x_1^3 - x_2 \geq 0$

$g_2 = x_1 \geq 0$

$g_3 = x_2 \geq 0$



Guessing $\mathcal{D}(a) = \left\{ \begin{bmatrix} \geq 0 \\ 0 \end{bmatrix} \right\}$

$$\mathcal{D}(a) = \left\{ d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \;\middle|\; \underbrace{\begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \geq 0}_{d_2 \leq 0},\; d_1 \geq 0,\; d_2 \geq 0 \right\}$$

$$= \left\{ d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \;\middle|\; d_1 \geq 0,\; d_2 = 0 \right\}$$

Let $d = \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$, $d_1 \geq 0$. $\quad \alpha(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$, $t \in [0,1]$

$$= \begin{bmatrix} td_1 \\ 0 \end{bmatrix}$$

$$\lim_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t} = \lim_{t \to 0+} \frac{t \begin{bmatrix} d_1 \\ 0 \end{bmatrix}}{t} = \begin{bmatrix} d_1 \\ 0 \end{bmatrix} = d.$$

So ktcq1 holds

Here $A(a) = \{1, 2, 3\}$. So

$$\mathcal{D}(a) = \{d \mid \nabla g_i^t(a)d \geq 0, i \in A(a)\} = \left\{ d \mid \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} d \geq 0 \right\} = \{d \mid d_1 \geq 0, d_2 = 0\}.$$

Let $d = \begin{bmatrix} d_1 \\ 0 \end{bmatrix} \in \mathcal{D}(a)$. We take $\alpha(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$, $t \in [0, 1]$. Then the curve is in T and

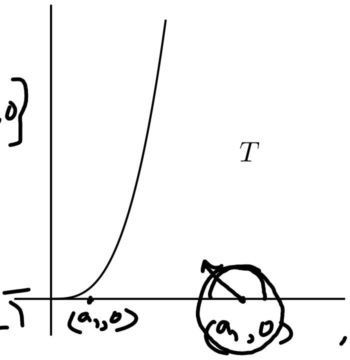
$$\lim_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t} = \lim_{t \to 0+} \frac{t \begin{bmatrix} d_1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}}{t} = \begin{bmatrix} d_1 \\ 0 \end{bmatrix} = d.$$

So we see that kctq1 holds at a.

d) Let $a = (a_1, 0)$, $a_1 > 0$. Find $\mathcal{D}(a)$ and check whether all those directions are tangents of some $\mathcal{C}^1$-curves.

$g_3 = x_2 \geq 0$ $\quad A(a) = \{3\}$

$\mathcal{D}(a) = \{ d \mid [0 \ 1] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \geq 0 \} = \{ \begin{bmatrix} * \\ d_2 \end{bmatrix} \mid d_2 \geq 0 \}$



Let $d = \begin{bmatrix} d_1 \\ d_2 \geq 0 \end{bmatrix} \in \mathcal{D}(a)$

$\alpha(t) = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in B_\delta(a) \subseteq T$, $a_1 > 0$

$t \in [0, \varepsilon]$, where $\varepsilon = \frac{\delta}{2\|d\|}$

Let $O_1 = \{ x \mid g_1(x) > 0 \} \rightarrow$ open at

$O_2 = \{ x \mid g_2(x) > 0 \} \rightarrow$ open at

so $O_1 \cap O_2$ is open and $a \in O_1 \cap O_2$. so $B_\delta(a) \subseteq O_1 \cap O_2$

$\lim_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = d$. so kt(q1 holds here too

Here $A(a) = \{3\}$ and $\mathcal{D}(a) = \{d \mid d_2 \geq 0\}$. Draw picture to understand. Let $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathcal{D}(a)$.

`(A new style of argument. Every time we do not have to give` ϵ. `We can use existence.)`
Note that the set $O_1 := \{x \mid g_1(x) > 0\}$ is an open set (as g_1 is continuous) and a is in this set. So $\exists \epsilon_1 > 0$ such that $B_{\epsilon_1}(a) \subseteq O_1$. Similarly, considering $g_2(x)$, we get that $\exists \epsilon_2 > 0$ such that $B_{\epsilon_2}(a) \subseteq O_2$. Taking minimum, we get an $\epsilon > 0$ such that $B_\epsilon(a) \subseteq O_1 \cap O_2$. Now take

$$\alpha(t) = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \qquad t \in [0, \frac{\epsilon}{2\|d\|}].$$

It is routine to see (write them once on your notes) that kctq1 holds.

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

1) f is cts

2) $f^{-1}(O)$ is open for each open O

TFAE

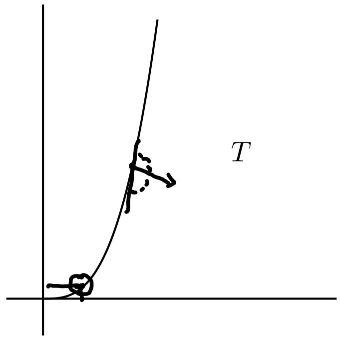
$g_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^3 - x_2$

$\mathbb{R}^2 \to \mathbb{R}$ $(0, \infty)$

3> $f^{-1}(K)$ is closed for each closed K

e) Let $a = (x > 0, x^3)$. Find $\mathcal{D}(a)$ and check whether all those directions are tangents of some $\mathcal{C}^1$-curves.

$a = (x, x^3)$ $\quad A(a) = \{1\}$ $\quad g = x_1^3 - x_2$



$\mathcal{D}(a) = \left\{ \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \;\middle|\; \begin{bmatrix} 3x_1^2 & -1 \end{bmatrix}\Big|_a d \geq 0 \right\}$

$= \underline{\left\{ \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \;\middle|\; 3x^2 d_1 - d_2 \geq 0 \right\}}$

$\alpha(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ $\quad t \in [0, \;]$ $\quad t < \frac{3x}{\|d\|}, x, x^3$

$\alpha(t) = \begin{bmatrix} \alpha(t)_1 \\ \alpha(t)_2 \end{bmatrix}$ $\quad \alpha(t)_1^3 - \alpha(t) \geq 0$ or

$(x + t d_1)^3 - x^3 - t d_2 \geq 0$

$\cancel{x^3} + t^3 d_1^3 + 3x t^2 d_1^2 + 3x^2 t d_1 - \cancel{x^3} - t d_2 \geq 0$

$t^2 d_1^2 (t d_1 + 3x) + t \underbrace{(3x^2 d_1 - d_2)}_{\geq 0} \geq 0$

for all sufficiently small t, we have

Here $A(a) = \{1\}$ and

$$\mathcal{D}(a) = \left\{ d \mid \begin{bmatrix} 3a_1^2 \\ -1 \end{bmatrix} d \geq 0 \right\} = \{ d \mid 3a_1^2 d_1 \geq d_2 \}.$$

Taking $\alpha(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

Is $\alpha(t)$ in T? (Picture wise it appears that this is on the right side of tangent line at a. So $\alpha(t) \in T$ for small t. But this has to proved.)

As both coordinates of a are positive, for t small, both the coordinates of $\alpha(t)$ will also be positive. To see whether $\alpha(t)$ satisfies g_1, observe that

$$\begin{aligned} g_1(\alpha(t)) &= (a_1 + t d_1)^3 - (a_2 + t d_2) \\ &= a_1^3 - a_2 + t(3a_1^2 d_1 - d_2) + 3a_1 t^2 d_1^2 + t^3 d_1^3 \\ &= 0 \qquad + t(\geq 0) \qquad + (3a_1 + t d_1) t^2 d_1^2 \geq 0, \end{aligned}$$

if $3a_1 + t d_1 \geq 0$, which holds for all small $t \geq 0$, as $a_1 > 0$. So $\alpha(t) \in T$ for all small t.

As $\alpha(t) = a + td$, we have $\lim\limits_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t} = d$, as required. ⌟

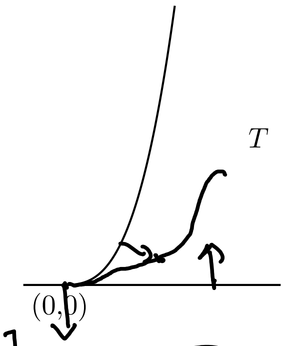
[34.2] **Important example** Consider the previous example removing g_2 from the constraints. We have

$$T = \{x \in \mathbb{R}^2 \mid g_1(x) = x_1^3 - x_2 \geq 0,\ g_3(x) = x_2 \geq 0\}.$$

$g_2 = x_1 \geq 0$

Notice that the feasible set is actually the same, only a superfluous constraint is removed.

$\alpha(t)_1^3 \geq \alpha(t)_2$



a) But now, at the feasible point $a = (0, 0)$, KTCQ1 does not hold. How?

$\mathcal{D}(a) = \left\{ \begin{bmatrix} d_1 \\ 0 \end{bmatrix} \right\}$

$d = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$ $\qquad \alpha(t) = \dfrac{\begin{bmatrix} \alpha(t)_1 \\ \alpha(t)_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}}{t} = \lim \begin{bmatrix} \dfrac{\alpha(t)_1}{t} \\ \dfrac{\alpha(t)_2}{t} \end{bmatrix}$

Notice that $\mathcal{D}(a) = \{d \mid d_2 = 0\}$ is a proper superset of the previous $\mathcal{D}(a)$. Take the direction $-e_1 \in \mathcal{D}(a)$. (Who stopped it earlier?) Do we have a continuously differentiable $\alpha(t) \in T$ such that $\alpha'(t) = -e_1$? No. How?

For, any $\alpha(t) = \begin{bmatrix} \alpha(t)_1 \\ \alpha(t)_2 \end{bmatrix} \in T$, we always have $\alpha(t)_1 \geq 0$. Hence, the first coordinate of

$$\alpha'(t) = \lim_{t \to 0+} \frac{\alpha(t)_1 - 0}{t}$$

must be ≥ 0.

b) This has happened, as we have removed an active constraint. This constraint was stopping many directions from entering $\mathcal{D}(a)$. But the removal of this constraint has resulted in inclusion of those directions in the linearizing cone. ⌟

To check whether ktcq1 holds at a point a, do we have to do it by the definition or is there a sufficient condition?

[34.3] **Lemma** `(A sufficient condition for ktcq1)` Let a be a feasible point for (P2) with $D(a) = \mathcal{D}(a)$. Then ktcq1 holds at a. In particular, if all the constraints in (P2) are linear, then ktcq1 holds at all feasible points.

$D(a) = \mathcal{D}(a)$ ✓ $\qquad d \in \mathcal{D}(a) = D(a)$

$a + td \qquad t \in [0, \delta]$

Proof. Let $d \in \mathcal{D}(a)$, $d \neq 0$. So $\exists\ \delta$ such that $a + td \in T$, for $0 \le t \le \delta$. Define $\alpha(t) = a + td$. So ktcq1 holds at a. The next assertion follows from [31.5]. ■

LICQ

[34.4] **Lemma** (`licq : another sufficient condition for ktcq1 to hold`) Let a be a feasible point for (P2) with $f, g_i, h_j \in \mathcal{C}^2(T)$. If the set

$$F = \left\{ \bigtriangledown g_i(a),\ i \in A(a), \quad \bigtriangledown h_j(a),\ j = 1, \ldots, p \right\}$$

$\left\{ \nabla g_i(a)\ \forall \text{ active},\ \nabla h_j(a)\ \forall j \right\}$

is linearly independent, then ktcq1 holds at a.

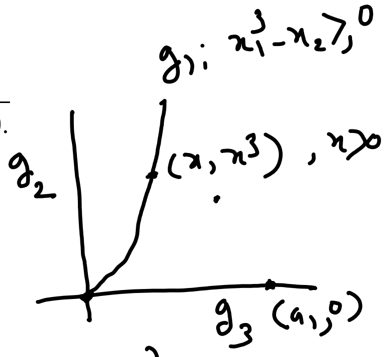
The proof is omitted. Interested reader may refer to 'Algorithmic principle of mathematical programming' by Faigle, Kern and Still, or to the book 'Mathematical programming techniques' by N S Kambo.

at $(0,0)$ $\quad F = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \rightarrow$ not lin ind

[34.5] **Example** Consider

$D(a) = \mathcal{D}(a)$

$$\begin{array}{ll} \min & f(x) = x_1 + x_2 \\ \text{s.t.} & g_1(x) = x_1^3 - x_2 \ge 0,\ g_2(x) = x_1 \ge 0,\ g_3(x) = x_2 \ge 0. \end{array}$$



At which points ktcq1 holds?

Answer.

$F = \{\ \}$ yes lin ind

so ktcq1 ✓

$\{\nabla g_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \rightarrow$ lin ind

ktcq1 ✓

$g_1 = x_1^3 - x_2$

$\left\{ \nabla g_1 = \begin{bmatrix} 3x_1^2 \\ -1 \end{bmatrix} \right\} \rightarrow$ yes lin ind.

so ktcq1 ✓

a) At $a = (0,0)$ we have $A(a) = \{1,2,3\}$ and $F = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, not linearly independent. We cannot conclude anything using licq. However $D(a) = \mathcal{D}(a)$. So ktcq1 holds here.

b) At $a = (a_1 > 0, 0)$ we have $A(a) = \{3\}$ and $F = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent. By licq, ktcq1 holds.

c) At $a = (t > 0, t^3)$ we have $A(a) = \{1\}$ and $F = \left\{ \begin{bmatrix} 3t^2 \\ -1 \end{bmatrix} \right\}$ is linearly independent. By licq, ktcq1 holds.

d) At points of interior, we have $A(a) = \emptyset$ and $F = \emptyset$ is linearly independent. By licq, ktcq1 holds.

> Have you noticed? The constraint qualification is actually a qualification (property) of the constraints and it has nothing to do with the objective function. However, the KT points depend on the objective function, as $\bigtriangledown f$ is used.

[34.6] **Exercise(E)** (`ktcq1 holds at a point of relative interior`) Let a be a feasible point for (P2) with $g_i(a) > 0$ for all i and with linear equality constraints. Then ktcq1 holds at a.