MA 101 (Mathematics I)

Hints/Solutions for Practice Problem Set - 1

Ex.1(a) State TRUE or FALSE giving proper justification: If both (x_n) and (y_n) are unbounded sequences in $\mathbb{R}$, then the sequence $(x_n y_n)$ cannot be convergent.
Solution: The given statement is FALSE, since both $(x_n) = (1, 0, 2, 0, 3, 0, ...)$ and $(y_n) = (0, 1, 0, 2, 0, 3, ...)$ are unbounded sequences in $\mathbb{R}$ but the sequence $(x_n y_n) = (0, 0, 0, ...)$ is convergent.

Ex.1(b) State TRUE or FALSE giving proper justification: If both (x_n) and (y_n) are increasing sequences in $\mathbb{R}$, then the sequence $(x_n y_n)$ must be increasing.
Solution: The given statement is FALSE, since both $(x_n) = (-\frac{1}{n})$ and $(y_n) = (n^2)$ are increasing sequences in $\mathbb{R}$ but the sequence $(x_n y_n) = (-n)$ is not increasing.

Ex.1(c) State TRUE or FALSE giving proper justification: If (x_n), (y_n) are sequences in $\mathbb{R}$ such that (x_n) is convergent and (y_n) is not convergent, then the sequence $(x_n + y_n)$ cannot be convergent.
Solution: The given statement is TRUE. If $(x_n + y_n)$ is convergent, then since (x_n) is also convergent, $(y_n) = (x_n + y_n) - (x_n)$ must be convergent, which is not true.

Ex.1(d) State TRUE or FALSE giving proper justification: A monotonic sequence (x_n) in $\mathbb{R}$ is convergent iff the sequence (x_n^2) is convergent.
Solution: The given statement is TRUE. If (x_n) is convergent, then by the product rule, $(x_n^2) = (x_n x_n)$ is also convergent. Conversely, let (x_n^2) be convergent. Then (x_n^2) is bounded, *i.e.* there exists $M > 0$ such that $|x_n^2| \leq M$ for all $n \in \mathbb{N}$. This gives $|x_n| \leq \sqrt{M}$ for all $n \in \mathbb{N}$. So (x_n) is bounded. Since it is given that (x_n) is monotonic, we can conclude that (x_n) is convergent.

Ex.1(e) State TRUE or FALSE giving proper justification: If (x_n) is an unbounded sequence of nonzero real numbers, then the sequence $(\frac{1}{x_n})$ must converge to 0.
Solution: The given statement is FALSE. The sequence $(x_n) = (1, 2, 1, 3, 1, 4, ...)$ is not bounded, but $\frac{1}{x_n} \not\to 0$, because $(\frac{1}{x_n})$ has a subsequence $(1, 1, ...)$ converging to 1.

Ex.1(f) State TRUE or FALSE giving proper justification: If $x_n = (1 - \frac{1}{n}) \sin \frac{n\pi}{2}$ for all $n \in \mathbb{N}$, then the sequence (x_n) is not convergent although it has a convergent subsequence.
Solution: The given statement is TRUE. We have $x_{2n} = (1 - \frac{1}{2n}) \sin n\pi = 0$ and $x_{4n+1} = (1 - \frac{1}{4n+1}) \sin(2n\pi + \frac{\pi}{2}) = 1 - \frac{1}{4n+1}$ for all $n \in \mathbb{N}$. Hence $x_{2n} \to 0$ and $x_{4n+1} \to 1$. Thus (x_n) has two convergent subsequences (x_{2n}) and (x_{4n+1}) with different limits and therefore (x_n) is not convergent.

Ex.1(g) State TRUE or FALSE giving proper justification: If both the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ of real numbers are convergent, then the series $\sum\limits_{n=1}^{\infty} x_n y_n$ must be convergent.
Solution: The given statement is FALSE. Taking $x_n = y_n = \frac{(-1)^n}{\sqrt{n}}$ for all $n \in \mathbb{N}$, we find that both the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ are convergent by Leibniz's test (since $(\frac{1}{\sqrt{n}})$ is a decreasing sequence of positive real numbers with $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$), but we know that the series $\sum\limits_{n=1}^{\infty} x_n y_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Ex.1(h) State TRUE or FALSE giving proper justification: If $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$f(x) > 0$ for all $x \in \mathbb{Q}$, then it is necessary that $f(x) > 0$ for all $x \in \mathbb{R}$.
Solution: The given statement is FALSE, because if $f(x) = |x - \sqrt{2}|$ for all $x \in \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in \mathbb{Q}$, but $f(\sqrt{2}) = 0$.

Ex.1(i) State TRUE or FALSE giving proper justification: There exists a continuous function from $(0,1)$ onto $(0,\infty)$.
Solution: The given statement is TRUE. The function $f : (0,1) \to (0,\infty)$, defined by $f(x) = \frac{x}{1-x}$ for all $x \in (0,1)$, is continuous. Also, f is onto, because if $y \in (0,\infty)$, then $x = \frac{y}{1+y} \in (0,1)$ such that $f(x) = y$.

Ex.1(j) State TRUE or FALSE giving proper justification: There exists a continuous function from $[0,1]$ onto $(0,1)$.
Solution: The given statement is FALSE. If possible, let there exist a continuous function $f : [0,1] \to (0,1)$ which is onto. Then there exists $x_0 \in [0,1]$ such that $f(x_0) \le f(x)$ for all $x \in [0,1]$. Since $0 < \frac{1}{2}f(x_0) < 1$ and since f is onto, there exists $c \in [0,1]$ such that $f(c) = \frac{1}{2}f(x_0)$. From above, we get $f(x_0) \le f(c)$, *i.e.* $f(x_0) \le \frac{1}{2}f(x_0)$, which is not possible, since $f(x_0) > 0$. Hence there does not exist any continuous function from $[0,1]$ onto $(0,1)$.

Ex.1(k) State TRUE or FALSE giving proper justification: There exists a continuous function from $(0,1)$ onto $[0,1]$.
Solution: The function $f : (0,1) \to [0,1]$, defined by $f(x) = |\sin(2\pi x)|$ for all $x \in (0,1)$, is continuous. Since $f(\frac{1}{2}) = 0$ and $f(\frac{1}{4}) = 1$, by the intermediate value theorem, for each $k \in (0,1)$, there exists $c \in (\frac{1}{4}, \frac{1}{2})$ such that $f(c) = k$. Hence f is onto.

Ex.1(l) State TRUE or FALSE giving proper justification: If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, then there must exist $c \in \mathbb{R}$ such that $f(c) = c$.
Solution: The given statement is TRUE. Since f is bounded, there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in \mathbb{R}$. Let $g(x) = f(x) - x$ for all $x \in \mathbb{R}$. Since f is continuous, $g : \mathbb{R} \to \mathbb{R}$ is continuous. If $f(-M) = -M$ or $f(M) = M$, then we get the result by taking $c = -M$ or $c = M$ respectively. Otherwise $g(-M) = f(-M) + M > 0$ and $g(M) = f(M) - M < 0$. Hence by the intermediate value theorem, there exists $c \in (-M, M)$ such that $g(c) = 0$, *i.e.* $f(c) = c$.

Ex.1(m) State TRUE or FALSE giving proper justification: If both $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous at 0, then the composite function $g \circ f : \mathbb{R} \to \mathbb{R}$ must be continuous at 0.
Solution: The given statement is FALSE. If $f(x) = x + 1$ for all $x \in \mathbb{R}$ and if
$g(x) = \begin{cases} 2 & \text{if } x \in \mathbb{R} \setminus \{1\}, \\ 3 & \text{if } x = 1, \end{cases}$
then both $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous at 0, but $g \circ f : \mathbb{R} \to \mathbb{R}$ is not continuous at 0, since $(g \circ f)(x) = \begin{cases} 2 & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 3 & \text{if } x = 0, \end{cases}$
so that $\lim\limits_{x \to 0} (g \circ f)(x) = 2 \ne 3 = (g \circ f)(0)$.

Ex.1(n) State TRUE or FALSE giving proper justification: If $f : \mathbb{R} \to \mathbb{R}$ is not differentiable at $x_0 \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is not differentiable at $f(x_0)$, then $g \circ f : \mathbb{R} \to \mathbb{R}$ cannot be differentiable at x_0.
Solution: The given statement is FALSE. If $f(x) = |x|$ for all $x \in \mathbb{R}$ and if $g(x) = \begin{cases} 1 & \text{if } x \ge 0, \\ -1 & \text{if } x < 0, \end{cases}$
then $f : \mathbb{R} \to \mathbb{R}$ is not differentiable at 0 and $g : \mathbb{R} \to \mathbb{R}$ is not differentiable at $f(0) = 0$, but $(g \circ f)(x) = 1$ for all $x \in \mathbb{R}$, so that $g \circ f$ is differentiable at 0.

Ex.1(o) State TRUE or FALSE giving proper justification: If $f : \mathbb{R} \to \mathbb{R}$ is such that $\lim\limits_{h \to 0} \frac{f(x+h) - f(x-h)}{h}$ exists (in $\mathbb{R}$) for every $x \in \mathbb{R}$, then f must be differentiable on $\mathbb{R}$.

Solution: The given statement is FALSE. Let $f(0) = 1$ and $f(x) = 0$ if $x(\neq 0) \in \mathbb{R}$. Then for every $x \in \mathbb{R}$, $\lim\limits_{h\to 0} \frac{f(x+h)-f(x-h)}{h} = \lim\limits_{h\to 0} \frac{0-0}{h} = 0$, but f (being not continuous at 0) is not differentiable at 0.

Ex.2(a) Using the definition of convergence of sequence, examine whether the sequence $\left(n + \frac{3}{2}\right)$ is convergent.
Solution: If possible, let $(n + \frac{3}{2})$ be convergent. Then there exist $\ell \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $|n + \frac{3}{2} - \ell| < 1$ for all $n \geq n_0 \Rightarrow n < \ell - \frac{1}{2}$ for all $n \geq n_0$, which is not true. Therefore the given sequence is not convergent.

Ex.2(b) Using the definition of convergence of sequence, examine whether the sequence $\left((-1)^n \frac{3}{n+2}\right)$ is convergent.
Solution: Let $\varepsilon > 0$. For all $n \in \mathbb{N}$, we have $|(-1)^n \frac{3}{n+2} - 0| = \frac{3}{n+2} < \frac{3}{n}$. There exists $n_0 \in \mathbb{N}$ such that $n_0 > \frac{3}{\varepsilon}$. Hence $|(-1)^n \frac{3}{n+2} - 0| < \frac{3}{n_0} < \varepsilon$ for all $n \geq n_0$ and so the given sequence is convergent (with limit 0).

Ex.2(c) Using the definition of convergence of sequence, examine whether the sequence $\left((-1)^n(1 - \frac{1}{n})\right)$ is convergent.
Solution: If possible, let the given sequence (x_n) (say) be convergent with limit ℓ. Then there exists $m \in \mathbb{N}$ such that $|x_n - \ell| < \frac{1}{4}$ for all $n \geq m \Rightarrow |x_{2m} - \ell| < \frac{1}{4}$ and $|x_{2m+1} - \ell| < \frac{1}{4} \Rightarrow |1 - \frac{1}{2m} - \ell| < \frac{1}{4}$ and $|1 + \ell - \frac{1}{2m+1}| < \frac{1}{4} \Rightarrow 2 - (\frac{1}{2m} + \frac{1}{2m+1}) < \frac{1}{2} \Rightarrow \frac{3}{2} < \frac{1}{2m} + \frac{1}{2m+1} \leq \frac{1}{2} + \frac{1}{2} = 1$, which is a contradiction. Therefore the given sequence is not convergent.

Ex.2(d) Using the definition of convergence of sequence, examine whether the sequence $\left(\frac{3n^2+\sin n-4}{2n^2+3}\right)$ is convergent.
Solution: Let $\varepsilon > 0$. For all $n \in \mathbb{N}$, we have $|\frac{3n^2+\sin n-4}{2n^2+3} - \frac{3}{2}| = \frac{|2\sin n-17|}{4n^2+6} < \frac{19}{4n^2}$. There exists $n_0 \in \mathbb{N}$ such that $n_0 > \frac{\sqrt{19}}{2\sqrt{\varepsilon}}$. Hence $|\frac{3n^2+\sin n-4}{2n^2+3} - \frac{3}{2}| < \frac{19}{4n_0^2} < \varepsilon$ for all $n \geq n_0$ and so the given sequence is convergent (with limit $\frac{3}{2}$).

Ex.2(e) Using the definition of convergence of sequence, examine whether the sequence $\left(\frac{2\sqrt{n}+3n}{2n+3}\right)$ is convergent.
Solution: Let $\varepsilon > 0$. For all $n \in \mathbb{N}$, we have $|\frac{2\sqrt{n}+3n}{2n+3} - \frac{3}{2}| = \frac{|4\sqrt{n}-9|}{4n+6} < \frac{4\sqrt{n}+9}{4n} < \frac{1}{\sqrt{n}} + \frac{9}{\sqrt{n}} = \frac{10}{\sqrt{n}}$. There exists $n_0 \in \mathbb{N}$ such that $n_0 > \frac{100}{\varepsilon^2}$. Hence $|\frac{2\sqrt{n}+3n}{2n+3} - \frac{3}{2}| < \frac{10}{\sqrt{n_0}} < \varepsilon$ for all $n \geq n_0$ and so the given sequence is convergent (with limit $\frac{3}{2}$).

Ex.3(a) Let a, b, c be distinct positive real numbers and let $x_n = (a^n + b^n + c^n)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: Let $\alpha = \max\{a, b, c\}$. Then $\alpha^n \leq a^n + b^n + c^n \leq 3\alpha^n$ for all $n \in \mathbb{N}$. So $\alpha \leq x_n \leq 3^{\frac{1}{n}}\alpha$ for all $n \in \mathbb{N}$. Since $3^{\frac{1}{n}} \to 1$, $3^{\frac{1}{n}}\alpha \to \alpha$. Hence by sandwich theorem, it follows that the sequence (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = \alpha$.

Alternative solution: Let $\alpha = \max\{a, b, c\}$. Then $\alpha^n \leq a^n + b^n + c^n = \alpha^n[(\frac{a}{\alpha})^n + (\frac{b}{\alpha})^n + (\frac{c}{\alpha})^n] \leq \alpha^n[(\frac{a}{\alpha})^n + (\frac{b}{\alpha})^n + (\frac{c}{\alpha})^n]^n$ for all $n \in \mathbb{N}$. So $\alpha \leq x_n \leq \alpha[(\frac{a}{\alpha})^n + (\frac{b}{\alpha})^n + (\frac{c}{\alpha})^n]$ for all $n \in \mathbb{N}$. Since $(\frac{a}{\alpha})^n + (\frac{b}{\alpha})^n + (\frac{c}{\alpha})^n \to 1$, by sandwich theorem, it follows that (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = \alpha$.

Ex.3(b) Let $x_n = \frac{1-n+(-1)^n}{2n+1}$ for all $n \in \mathbb{N}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $x_n = \frac{\frac{1}{n}-1+\frac{(-1)^n}{n}}{2+\frac{1}{n}}$ for all $n \in \mathbb{N}$. Since $\frac{1}{n} \to 0$ and $\frac{(-1)^n}{n} \to 0$, by the limit rules for algebraic operations, (x_n) is convergent with $\lim\limits_{n\to\infty} x_n = \frac{0-1+0}{2+0} = -\frac{1}{2}$.

Ex.3(c) Let $|\alpha| > 1$, $k > 0$ and $x_n = \frac{n^k}{\alpha^n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} (1+\frac{1}{n})^k \frac{1}{|\alpha|} = \frac{1}{|\alpha|} < 1$. Hence (x_n) converges to 0.

Ex.3(d) Let $x_n = \frac{p(n)}{2^n}$ for all $n \in \mathbb{N}$, where $p(x)$ is a polynomial in the real variable x of degree 5. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: The highest power of n in each of $p(n)$ and $p(n+1)$ is 5 and the coefficient of n^5 in $p(n)$ and $p(n+1)$ is same. Hence dividing both numerator and denominator by n^5 and using the fact that $\frac{1}{n} \to 0$, it follows that $\lim\limits_{n\to\infty} \left|\frac{p(n+1)}{p(n)}\right| = 1$ and consequently $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} \left|\frac{p(n+1)}{2^{n+1}} \cdot \frac{2^n}{p(n)}\right| = \frac{1}{2} < 1$. This implies that (x_n) is convergent with limit 0.

Ex.3(e) Let $x_n = \frac{3.5.7.\cdots.(2n+1)}{2.5.8.\cdots.(3n-1)}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} \frac{2n+3}{3n+2} = \lim\limits_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{2}{n}} = \frac{2}{3} < 1$. Hence (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = 0$.

Ex.3(f) Let $x_n = \frac{1}{n}\sin^2 n$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: Since $0 \le \frac{1}{n}\sin^2 n \le \frac{1}{n}$ for all $n \in \mathbb{N}$ and since $\frac{1}{n} \to 0$, by sandwich theorem, (x_n) is convergent with limit 0.

Ex.3(g) Let $x_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $0 \le x_n \le \frac{n}{(n+1)^2}$ for all $n \in \mathbb{N}$ and $\frac{n}{(n+1)^2} = \frac{\frac{1}{n}}{(1+\frac{1}{n})^2} \to \frac{0}{(1+0)^2} = 0$. Hence by sandwich theorem, it follows that (x_n) is convergent with limit 0.

Ex.3(h) $x_n = \frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $(1+2+\cdots+n)\frac{n}{n^3+n} \le x_n \le (1+2+\cdots+n)\frac{n}{n^3+1}$ for all $n \in \mathbb{N}$. Also, $(1+2+\cdots+n)\frac{n}{n^3+n} = \frac{1+\frac{1}{n}}{2(1+\frac{1}{n^2})} \to \frac{1}{2}$ and $(1+2+\cdots+n)\frac{n}{n^3+1} = \frac{1+\frac{1}{n}}{2(1+\frac{1}{n^3})} \to \frac{1}{2}$. Hence by sandwich theorem, (x_n) is convergent with limit $\frac{1}{2}$.

Ex.3(i) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n+1}}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $\frac{n+1}{\sqrt{n^2+n+1}} \le x_n \le \frac{n+1}{\sqrt{n^2+1}}$ for all $n \in \mathbb{N}$. Also, $\frac{n+1}{\sqrt{n^2+n+1}} = \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} \to 1$ and $\frac{n+1}{\sqrt{n^2+1}} = \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}}} \to 1$. Hence by sandwich theorem, (x_n) is convergent with limit 1.

Ex.3(j) Let $x_n = \frac{1}{\sqrt{n}}(\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}})$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: Since $x_n = \frac{1}{2\sqrt{n}}(\sqrt{3}-1+\sqrt{5}-\sqrt{3}+\cdots+\sqrt{2n+1}-\sqrt{2n-1}) = \frac{1}{2\sqrt{n}}(\sqrt{2n+1}-1) = \frac{1}{2}(\sqrt{2+\frac{1}{n}} - \frac{1}{\sqrt{n}})$ for all $n \in \mathbb{N}$ and since $\frac{1}{n} \to 0$, by the limit rules for algebraic operations, (x_n) is convergent with $\lim\limits_{n\to\infty} x_n = \frac{1}{2}(\sqrt{2+0}-0) = \frac{1}{\sqrt{2}}$.

Ex.3(k) Let $x_n = (\frac{\sin n+\cos n}{3})^n$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $0 \le |x_n| \le (\frac{2}{3})^n$ for all $n \in \mathbb{N}$. Since $(\frac{2}{3})^n \to 0$, by sandwich theorem, it follows that $|x_n| \to 0$ and consequently (x_n) is convergent with limit 0.

Ex.3(l) Let $x_n = \sqrt{4n^2+n} - 2n$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: For all $n \in \mathbb{N}$, $\sqrt{4n^2+n} - 2n = \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4+\frac{1}{n}}+2}$. Since $\frac{1}{n} \to 0$, by the limit rules for algebraic operations, (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = \frac{1}{\sqrt{4+0}+2} = \frac{1}{4}$.

Ex.3(m) Let $x_n = \sqrt{n^2+n} - \sqrt{n^2+1}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: For all $n \in \mathbb{N}$, we have $x_n = \frac{n-1}{\sqrt{n^2+n}+\sqrt{n^2+1}} = \frac{1-\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{1}{n^2}}}$. Since $\frac{1}{n} \to 0$, by the limit rules for algebraic operations, (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = \frac{1-0}{\sqrt{1+0}+\sqrt{1+0}} = \frac{1}{2}$.

Ex.3(n) Let $x_1 = 1$ and $x_{n+1} = 1 + \sqrt{x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $x_2 = 2 > x_1$. Also, if $x_{k+1} > x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} = 1 + \sqrt{x_{k+1}} > 1 + \sqrt{x_k} = x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$. So (x_n) is increasing. Again, $x_1 < 3$ and if $x_k < 3$ for some $k \in \mathbb{N}$, then $x_{k+1} = 1+\sqrt{x_k} < 1+\sqrt{3} < 3$. Hence by the principle of mathematical induction, $x_n < 3$ for all $n \in \mathbb{N}$. So (x_n) is bounded above. Consequently (x_n) is convergent. If $\ell = \lim\limits_{n\to\infty} x_n$, then $x_{n+1} \to \ell$ and since $x_{n+1} = 1 + \sqrt{x_n}$ for all $n \in \mathbb{N}$, we get $\ell = 1+\sqrt{\ell} \Rightarrow \ell = \frac{3+\sqrt{5}}{2}$ or $\frac{3-\sqrt{5}}{2}$. Since $x_n \geq 1$ for all $n \in \mathbb{N}$, $\ell \geq 1$ and so $\ell = \frac{3+\sqrt{5}}{2}$.

Ex.3(o) Let $x_1 = 4$ and $x_{n+1} = 3 - \frac{2}{x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $x_1 > 2$ and if we assume that $x_k > 2$ for some $k \in \mathbb{N}$, then $x_{k+1} > 3 - 1 = 2$. Hence by the principle of mathematical induction, $x_n > 2$ for all $n \in \mathbb{N}$. Therefore (x_n) is bounded below. Again, $x_2 = \frac{5}{2} < x_1$ and if we assume that $x_{k+1} < x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} - x_{k+1} = 2(\frac{1}{x_k} - \frac{1}{x_{k+1}}) < 0 \Rightarrow x_{k+2} < x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Therefore (x_n) is decreasing. Consequently (x_n) is convergent. Let $\ell = \lim\limits_{n\to\infty} x_n$. Then $\lim\limits_{n\to\infty} x_{n+1} = 3 - \frac{2}{\lim\limits_{n\to\infty} x_n} \Rightarrow \ell = 3 - \frac{2}{\ell}$ (since $x_n > 2$ for all $n \in \mathbb{N}$, $\ell \neq 0$) $\Rightarrow (\ell-1)(\ell-2) = 0 \Rightarrow \ell = 1$ or $\ell = 2$. But $x_n > 2$ for all $n \in \mathbb{N}$, so $\ell \geq 2$. Therefore $\ell = 2$.

Alternative solution: For all $n \in \mathbb{N}$, we have $|x_{n+2} - x_{n+1}| = \frac{2}{|x_{n+1}||x_n|}|x_{n+1} - x_n|$. Also, as shown in the above solution, $x_n > 2$ for all $n \in \mathbb{N}$. Hence $|x_{n+2} - x_{n+1}| \leq \frac{1}{2}|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. It follows that (x_n) is a Cauchy sequence in $\mathbb{R}$ and hence (x_n) is convergent. To show that $\lim\limits_{n\to\infty} x_n = 2$, we proceed as in the above solution.

Ex.3(p) Let $x_1 = 0$ and $x_{n+1} = \sqrt{6 + x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $x_2 = \sqrt{6} > x_1$. Also, if $x_{k+1} > x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} = \sqrt{6 + x_{k+1}} > \sqrt{6 + x_k} = x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$. So (x_n) is increasing. Again, $x_1 < 3$ and if $x_k < 3$ for some $k \in \mathbb{N}$, then $x_{k+1} = \sqrt{6 + x_k} < \sqrt{6+3} = 3$. Hence by the principle of mathematical induction, $x_n < 3$ for all $n \in \mathbb{N}$. So (x_n) is bounded above. Consequently (x_n) is convergent. If $\ell = \lim\limits_{n\to\infty} x_n$, then $x_{n+1} \to \ell$ and since $x_{n+1} = \sqrt{6 + x_n}$ for all $n \in \mathbb{N}$, we get $\ell = \sqrt{6 + \ell} \Rightarrow \ell^2 - \ell - 6 = 0 \Rightarrow \ell = 3$ or -2. Since $x_n \geq 0$ for all $n \in \mathbb{N}$, $\ell \geq 0$ and so $\ell = 3$.

Ex.3(q) Let $x_1 > 1$ and $x_{n+1} = \sqrt{x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $x_1 > 1$ and if we assume that $x_k > 1$ for some $k \in \mathbb{N}$, then $x_{k+1} = \sqrt{x_k} > 1$. Hence by the principle of mathematical induction, $x_n > 1$ for all $n \in \mathbb{N}$. Again, $x_2 = \sqrt{x_1} \leq x_1$ and if we assume that $x_{k+1} \leq x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} = \sqrt{x_{k+1}} \leq \sqrt{x_k} = x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. Thus (x_n) is decreasing and

bounded below. Consequently (x_n) is convergent. If $\ell = \lim_{n\to\infty} x_n$, then $\lim_{n\to\infty} x_{n+1} = \ell$ and since $x_{n+1} = \sqrt{x_n}$ for all $n \in \mathbb{N}$, we get $\ell = \sqrt{\ell} \Rightarrow \ell^2 = \ell \Rightarrow \ell = 0$ or 1. Since $x_n > 1$ for all $n \in \mathbb{N}$, $\ell \geq 1$ and so we must have $\ell = 1$.

Ex.4 Let (x_n), (y_n) be sequences in $\mathbb{R}$ such that $x_n \to x \in \mathbb{R}$ and $y_n \to y \in \mathbb{R}$. Show that $\lim_{n\to\infty} \max\{x_n, y_n\} = \max\{x, y\}$.
Solution: We know that $\max\{x_n, y_n\} = \frac{1}{2}(x_n + y_n + |x_n - y_n|)$ for all $n \in \mathbb{N}$. Since $x_n \to x$ and $y_n \to y$, $x_n + y_n \to x + y$ and $|x_n - y_n| \to |x - y|$. Consequently $\lim_{n\to\infty} \max\{x_n, y_n\} = \frac{1}{2}(x + y + |x - y|) = \max\{x, y\}$.

Ex.5 If a sequence (x_n) of positive real numbers converges to $\ell \in \mathbb{R}$, then show that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\ell}$.
Solution: In view of Ex.2 of Tutorial Problem Set, we get $\ell \geq 0$. If $a \geq 0$ and $b \geq 0$, then $\frac{1}{2}(a + b - |a - b|) = \min\{a, b\} \leq \sqrt{ab}$ and hence it follows that $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|}$. Let $\varepsilon > 0$. Since $x_n \to \ell$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon^2$ for all $n \geq n_0$. Therefore using the inequality obtained above, we get $|\sqrt{x_n} - \sqrt{\ell}| \leq \sqrt{|x_n - \ell|} < \varepsilon$ for all $n \geq n_0$. Consequently $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\ell}$.

Ex.6 Let (x_n) be a convergent sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = \ell \neq 0$. Show that there exists $n_0 \in \mathbb{N}$ such that $x_n \neq 0$ for all $n \geq n_0$.
Solution: Since $x_n \to \ell$ and $|\ell| > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \frac{1}{2}|\ell|$ for all $n \geq n_0$. If for some $n \geq n_0$, $x_n = 0$, then we obtain $|\ell| < \frac{1}{2}|\ell|$, which is not possible. Hence $x_n \neq 0$ for all $n \geq n_0$.

Ex.7 If $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ for all $n \in \mathbb{N}$, then show that the sequence (x_n) convergent.
Solution: For all $n \in \mathbb{N}$, we have $x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \geq \frac{2}{2n+2} - \frac{1}{n+1} = 0 \Rightarrow x_{n+1} \geq x_n$ for all $n \in \mathbb{N} \Rightarrow (x_n)$ is increasing. Also, $x_n \leq \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1$ for all $n \in \mathbb{N} \Rightarrow (x_n)$ is bounded above. Therefore (x_n) is convergent.

Ex.8(a) Let $x_1 = 1$ and $x_{n+1} = \frac{2+x_n}{1+x_n}$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) in $\mathbb{R}$ is Cauchy (and hence convergent). Also, find the limit.
Solution: Since $x_{n+1} = 1 + \frac{1}{1+x_n}$ for all $n \in \mathbb{N}$, we have $|x_{n+2} - x_{n+1}| = |\frac{1}{1+x_{n+1}} - \frac{1}{1+x_n}| = \frac{|x_{n+1}-x_n|}{|1+x_{n+1}||1+x_n|}$ for all $n \in \mathbb{N}$. Also, $x_1 = 1$ and if we assume that $x_k \geq 1$ for some $k \in \mathbb{N}$, then $x_{k+1} = 1 + \frac{1}{1+x_k} \geq 1$. Hence by the principle of mathematical induction, $x_n \geq 1$ for all $n \in \mathbb{N}$. Consequently $|x_{n+2} - x_{n+1}| \leq \frac{1}{4}|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. It follows that (x_n) is Cauchy and hence (x_n) converges. Let $\ell = \lim_{n\to\infty} x_n$. Then $\lim_{n\to\infty} x_{n+1} = \ell$ and so we get $\ell = 1 + \frac{1}{1+\ell} \Rightarrow \ell^2 = 2 \Rightarrow \ell = \sqrt{2}$ or $-\sqrt{2}$. Since $x_n \geq 1$ for all $n \in \mathbb{N}$, we must have $\ell \geq 1$ and so $\ell = \sqrt{2}$.

Ex.8(b) Let $x_1 > 0$ and $x_{n+1} = 2 + \frac{1}{x_n}$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) in $\mathbb{R}$ is Cauchy (and hence convergent). Also, find the limit.
Solution: We have $|x_{n+2} - x_{n+1}| = |\frac{1}{x_{n+1}} - \frac{1}{x_n}| = \frac{|x_{n+1}-x_n|}{|x_{n+1}||x_n|}$ for all $n \in \mathbb{N}$. Also, $x_2 = 2 + \frac{1}{x_1} > 2$ and if we assume that $x_k > 2$ for some $k \geq 2$, then $x_{k+1} = 2 + \frac{1}{x_k} > 2$. Hence by the principle of mathematical induction, $x_n > 2$ for all $n \geq 2$. Consequently $|x_{n+2} - x_{n+1}| \leq \frac{1}{4}|x_{n+1} - x_n|$ for all $n \geq 2$. It follows that (x_n) is Cauchy and hence (x_n) converges. Let $\ell = \lim_{n\to\infty} x_n$. Then $\lim_{n\to\infty} x_{n+1} = \ell$ and so we get $\ell = 2 + \frac{1}{\ell} \Rightarrow \ell^2 - 2\ell - 1 = 0 \Rightarrow \ell = 1 \pm \sqrt{2}$. Since $x_n > 2$ for all $n \geq 2$, we must have $\ell \geq 2$ and so $\ell = 1 + \sqrt{2}$.

Ex.9(a) If $x_n = (-1)^n n^2$ for all $n \in \mathbb{N}$, then examine whether the sequence (x_n) has a convergent subsequence?
Solution: If possible, let the given sequence have a convergent subsequence $((-1)^{n_k} n_k^2)$. Then $((-1)^{n_k} n_k^2)$ must be bounded. So there exists $M > 0$ such that $|(-1)^{n_k} n_k^2| \leq M$ for all $k \in \mathbb{N} \Rightarrow n_k^2 \leq M$ for all $k \in \mathbb{N}$, which is not possible, since (n_k) is a strictly increasing sequence of positive integers. Therefore the given sequence cannot have any convergent subsequence.

Ex.9(b) If $x_n = (-1)^n \frac{5n \sin^3 n}{3n-2}$ for all $n \in \mathbb{N}$, then examine whether the sequence (x_n) has a convergent subsequence.
Solution: Since $|x_n| = \frac{5}{3-\frac{2}{n}} |\sin n|^3 \leq 5$ for all $n \in \mathbb{N}$, the sequence (x_n) is bounded and hence by Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence.

Ex.10 If $a, b \in \mathbb{R}$, then show that the series $a + (a+b) + (a+2b) + \cdots$ is not convergent unless $a = b = 0$.
Solution: Let $s_n = a + (a+b) + \cdots + a + (n-1)b = n[a + \frac{1}{2}(n-1)b]$ for all $n \in \mathbb{N}$. If $b \neq 0$, then the sequence $(a + \frac{1}{2}(n-1)b)$ is not bounded and so the sequence (s_n) is not bounded, which implies that (s_n) is not convergent. If $b = 0$, then the sequence $(s_n) = (na)$ is not bounded and hence is not convergent if $a \neq 0$. Thus the given series is not convergent (*i.e.* (s_n) is not convergent) if $a \neq 0$ or $b \neq 0$.
If $a = b = 0$, then the series becomes $0 + 0 + \cdots$, which is clearly convergent.

Ex.11(a) Examine whether the series $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.
Solution: Taking $x_n = \frac{n!}{n^n}$ for all $n \in \mathbb{N}$, we find that $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} (\frac{n}{n+1})^n = \lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$. Hence by the ratio test, the given series is convergent.

Ex.11(b) Examine whether the series $\sum\limits_{n=1}^{\infty} \frac{(2n)!}{n^n}$ is convergent.
Solution: For all $n \in \mathbb{N}$, we have $\frac{(2n)!}{n^n} = \frac{2n}{n} \cdot \frac{2n-1}{n} \cdots \frac{n+1}{n} \cdot n! \geq 1$. Hence $\lim\limits_{n\to\infty} \frac{(2n)!}{n^n} \neq 0$ and consequently the given series is not convergent.

Ex.11(c) Examine whether the series $\sum\limits_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$ is convergent.

Solution: Since $0 \leq \frac{1}{n} \sin \frac{1}{n} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test, the given series is convergent.

Ex.11(d) Examine whether the series $\sum\limits_{n=1}^{\infty} \sqrt{\frac{2n^2+3}{5n^3+1}}$ is convergent.

Solution: Let $x_n = \sqrt{\frac{2n^2+3}{5n^3+1}}$ and $y_n = \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \sqrt{\frac{2+\frac{3}{n^2}}{5+\frac{1}{n^3}}} = \sqrt{\frac{2}{5}} \neq 0$ and since $\sum\limits_{n=1}^{\infty} y_n$ is not convergent, by limit comparison test, $\sum\limits_{n=1}^{\infty} x_n$ is not convergent.

Ex.11(e) Examine whether the series $\sum\limits_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$ is convergent.
Solution: Taking $x_n = \frac{n^n}{2^{n^2}}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{n}{2^n} = 0 < 1$ (since $\lim\limits_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$). Hence by the root test, the given series is convergent.

Ex.11(f) Examine whether the series $\sum\limits_{n=1}^{\infty} ((n^3+1)^{\frac{1}{3}} - n)$ is convergent.
Solution: Taking $x_n = (n^3+1)^{\frac{1}{3}} - n \geq 0$ and $y_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$, we have

$\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{n^2(n^3+1-n^3)}{(n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2} = \lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n^3})^{2/3}+(1+\frac{1}{n^3})^{1/3}+1} = \frac{1}{3}$. Since $\sum\limits_{n=1}^{\infty} y_n$ is convergent, $\sum\limits_{n=1}^{\infty} x_n$ is also convergent by limit comparison test.

Ex.11(g) Examine whether the series $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ is convergent.

Solution: Let $x_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$ and $y_n = \frac{1}{n^{3/2}}$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$ and since $\sum\limits_{n=1}^{\infty} y_n$ is convergent, by limit comparison test, $\sum\limits_{n=1}^{\infty} x_n$ is convergent.

Ex.11(h) Examine whether the series $\sum\limits_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ is convergent.

Proof: Taking $x_n = (\frac{n}{n+1})^{n^2}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$. Hence by the root test, the given series is convergent.

Ex.11(i) Examine whether the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$ is convergent.

Solution: For $n \in \mathbb{N}$, the inequality $\frac{\sqrt{n+1}+1}{n+2} < \frac{\sqrt{n}+1}{n+1}$ is equivalent to the inequality $(n+1)^{\frac{3}{2}} < (n+2)\sqrt{n}+1$. Since $n(n+2)^2-(n+1)^3 = n^2+n-1 > 0$ for all $n \in \mathbb{N}$, we get $(n+1)^{\frac{3}{2}} < (n+2)\sqrt{n}+1$ for all $n \in \mathbb{N}$ and hence $\frac{\sqrt{n+1}+1}{n+2} < \frac{\sqrt{n}+1}{n+1}$ for all $n \in \mathbb{N}$. Consequently the sequence $\left(\frac{\sqrt{n}+1}{n+1}\right)$ is decreasing. Also, $\frac{\sqrt{n}+1}{n+1} = \frac{\frac{1}{\sqrt{n}}+\frac{1}{n}}{1+\frac{1}{n}} \to 0$. Hence by Leibniz's test, the given series converges.

Alternative method for showing decreasing: Let $f(x) = \frac{\sqrt{x}+1}{x+1}$ for all $x \geq 1$. Then $f : [1,\infty) \to \mathbb{R}$ is differentiable and $f'(x) = \frac{1-x-2\sqrt{x}}{2\sqrt{x}(x+1)^2} \leq 0$ for all $x \geq 1$. Hence f is decreasing on $[1,\infty)$ and so $f(n+1) \leq f(n)$ for all $n \in \mathbb{N}$.

Ex.12 Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$ is convergent.

Solution: For $x = 0$, the given series becomes $0 + 0 + \cdots$, which clearly converges. We now assume that $x \neq 0$. Then $\lim\limits_{n\to\infty} |\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}| = \lim\limits_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$. So by the ratio test, $\sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$ is absolutely convergent and hence convergent. Therefore the given series is convergent for all $x \in \mathbb{R}$.

Ex.13 Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{(x+2)^n}{3^n\sqrt{2n+1}}$ is convergent.

Solution: If $x = -2$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x(\neq -2) \in \mathbb{R}$ and let $a_n = \frac{(x+2)^n}{3^n\sqrt{2n+1}}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{3}|x+2|$. Hence by ratio test, $\sum\limits_{n=1}^{\infty} a_n$ is convergent (absolutely) if $\frac{1}{3}|x+2| < 1$, *i.e.* if $x \in (-5,1)$ and is not convergent if $\frac{1}{3}|x+2| > 1$, *i.e.* if $x \in (-\infty,-5) \cup (1,\infty)$. If $x = -5$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$ is convergent by Leibniz test, since $(\frac{1}{\sqrt{2n+1}})$ is a decreasing sequence of positive real numbers and $\lim\limits_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$. Again, if $x = 1$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$ is not convergent by limit comparison test, since $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is not convergent and $\lim\limits_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1}} = \frac{1}{\sqrt{2}} \neq 0$. Therefore the set of all $x \in \mathbb{R}$ for which $\sum\limits_{n=1}^{\infty} a_n$ is convergent is $[-5,1)$.

Ex.14 Show that the series $\sum\limits_{n=1}^{\infty} \frac{a^n}{a^n+n}$ is convergent if $0 < a < 1$ and is not convergent if $a > 1$.

Solution: If $0 < a < 1$, then $0 < \frac{a^n}{a^n+n} < a^n$ for all $n \in \mathbb{N}$ and $\sum\limits_{n=1}^{\infty} a^n$ is convergent. Hence by comparison test, $\sum\limits_{n=1}^{\infty} \frac{a^n}{a^n+n}$ is convergent if $0 < a < 1$. Again, if $a > 1$, then $\frac{a^n}{a^n+n} = \frac{1}{1+\frac{n}{a^n}} \to 1 \neq 0$ and hence $\sum\limits_{n=1}^{\infty} \frac{a^n}{a^n+n}$ is not convergent if $a > 1$. (We have used that $\lim\limits_{n\to\infty} \frac{n}{a^n} = 0$, which follows from the fact that $\lim\limits_{n\to\infty} \frac{n+1}{a^{n+1}} \cdot \frac{a^n}{n} = \frac{1}{a} < 1$.)

Ex.15 If $0 < x_n < \frac{1}{2}$ for all $n \in \mathbb{N}$ and if the series $\sum\limits_{n=1}^{\infty} x_n$ converges, then show that the series $\sum\limits_{n=1}^{\infty} \frac{x_n}{1-x_n}$ converges.

Solution: Since $0 < x_n < \frac{1}{2}$ for all $n \in \mathbb{N}$, we have $0 < \frac{x_n}{1-x_n} < 2x_n$ for all $n \in \mathbb{N}$. Also, since $\sum\limits_{n=1}^{\infty} 2x_n$ converges, by comparison test, $\sum\limits_{n=1}^{\infty} \frac{x_n}{1-x_n}$ converges.

Ex.16 Let (x_n), (y_n) be sequences in $\mathbb{R}$ such that $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$. Find out (with justification) the true statement(s) from the following.

(a) If the series $\sum\limits_{n=1}^{\infty} y_n$ converges, then the series $\sum\limits_{n=1}^{\infty} x_n$ must converge.

(b) If the series $\sum\limits_{n=1}^{\infty} x_n$ converges, then the series $\sum\limits_{n=1}^{\infty} y_n$ must converge.

(c) If the series $\sum\limits_{n=1}^{\infty} y_n$ converges absolutely, then the series $\sum\limits_{n=1}^{\infty} x_n$ must converge absolutely.

(d) If the series $\sum\limits_{n=1}^{\infty} x_n$ converges absolutely, then the series $\sum\limits_{n=1}^{\infty} y_n$ must converge absolutely.

Solution: By comparison test, $\sum\limits_{n=1}^{\infty} |x_n|$ is convergent (*i.e.* $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent) if $\sum\limits_{n=1}^{\infty} |y_n|$ is convergent (*i.e.* $\sum\limits_{n=1}^{\infty} y_n$ is absolutely convergent) and so (c) is true. Again, we know that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not convergent. Also, since $(\frac{1}{n})$ is a decreasing sequence of positive real numbers with $\frac{1}{n} \to 0$, by Leibniz's test, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent. Hence to see that (a) is false, we can take $x_n = \frac{1}{n}$, $y_n = \frac{(-1)^n}{n}$ for all $n \in \mathbb{N}$ and to see that (b) and (d) are false, we can take $x_n = \frac{1}{n^2}$, $y_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.

Ex.17 If a series $\sum\limits_{n=1}^{\infty} x_n$ is convergent but the series $\sum\limits_{n=1}^{\infty} x_n^2$ is not convergent, then show that the series $\sum\limits_{n=1}^{\infty} x_n$ is conditionally convergent.

Solution: Since $\sum\limits_{n=1}^{\infty} x_n$ is convergent, $x_n \to 0$, and so there exists $n_0 \in \mathbb{N}$ such that $|x_n| < 1$ for all $n \geq n_0$. Hence $x_n^2 \leq |x_n|$ for all $n \geq n_0$. Since $\sum\limits_{n=1}^{\infty} x_n^2$ is not convergent, by comparison test, $\sum\limits_{n=1}^{\infty} |x_n|$ is not convergent. Consequently $\sum\limits_{n=1}^{\infty} x_n$ is conditionally convergent.

Ex.18(a) Examine whether the series $\sum\limits_{n=1}^{\infty} (-1)^n(\sqrt{n^2+1} - n)$ is conditionally convergent.

Solution: Let $x_n = \sqrt{n^2+1} - n$ for all $n \in \mathbb{N}$. Then $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n = \frac{1}{\sqrt{n^2+1}+n} = \frac{\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}}+1} \to 0$. Also, $x_{n+1} = \frac{1}{\sqrt{(n+1)^2+1}+(n+1)} < \frac{1}{\sqrt{n^2+1}+n} = x_n$ for all $n \in \mathbb{N}$, *i.e.* the sequence (x_n)

is decreasing. Therefore by Leibniz's test, $\sum\limits_{n=1}^{\infty}(-1)^{n+1}x_n$ is convergent and hence the given series is convergent.
Again, if $y_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2} \neq 0$. Since $\sum\limits_{n=1}^{\infty} y_n$ is not convergent, by limit comparison test, $\sum\limits_{n=1}^{\infty} x_n$ is not convergent, *i.e.* $\sum\limits_{n=1}^{\infty} |(-1)^n(\sqrt{n^2+1}-n)|$ is not convergent. Thus the given series is conditionally convergent.

Ex.18(b) Examine whether the series $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n^2+(-1)^n}$ is conditionally convergent.
Solution: By comparison test, the series $\sum\limits_{n=2}^{\infty} \left|\frac{(-1)^n}{n^2+(-1)^n}\right| = \sum\limits_{n=2}^{\infty} \frac{1}{n^2+(-1)^n}$ is convergent, since $0 < \frac{1}{n^2+(-1)^n} < \frac{2}{n^2}$ for all $n \geq 2$ and $\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ is convergent. Thus the given series is not conditionally convergent.

Ex.18(c) Examine whether the series $\sum\limits_{n=1}^{\infty}(-1)^n \frac{a^2+n}{n^2}$ (where $a \in \mathbb{R}$) is conditionally convergent.
Solution: Let $a \in \mathbb{R}$ and let $x_n = \frac{a^2+n}{n^2}$ for all $n \in \mathbb{N}$. Then $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n = \frac{a^2}{n^2} + \frac{1}{n} \to 0$. Also, $x_{n+1} = \frac{a^2}{(n+1)^2} + \frac{1}{n+1} < \frac{a^2}{n^2} + \frac{1}{n} = x_n$ for all $n \in \mathbb{N}$, *i.e.* the sequence (x_n) is decreasing. Therefore by Leibniz's test, it follows that the given series is convergent.
Again, if $y_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} (\frac{a^2}{n}+1) = 1 \neq 0$. Since $\sum\limits_{n=1}^{\infty} y_n$ is not convergent, by limit comparison test, $\sum\limits_{n=1}^{\infty} x_n$ is not convergent, *i.e.* $\sum\limits_{n=1}^{\infty} |(-1)^n \frac{a^2+n}{n^2}|$ is not convergent. Thus the given series is conditionally convergent.

Ex.19 Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{\log(n+1)}{\sqrt{n+1}}(x-5)^n$ is convergent.
Hint: If $x = 5$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x(\neq 5) \in \mathbb{R}$ and let $a_n = \frac{\log(n+1)}{\sqrt{n+1}}(x-5)^n$ for all $n \in \mathbb{N}$. Since $\lim\limits_{x\to\infty} \frac{\log(x+2)}{\log(x+1)} = 1$ (using L'Hôpital's rule), by sequential criterion of limits, we get $\lim\limits_{n\to\infty} \frac{\log(n+2)}{\log(n+1)} = 1$ and so $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-5|$. Hence by the ratio test, $\sum\limits_{n=1}^{\infty} a_n$ converges (absolutely) if $|x-5| < 1$, *i.e.* if $x \in (4,6)$ and diverges if $|x-5| > 1$, *i.e.* if $x \in (-\infty, 4) \cup (6, \infty)$. If $f(x) = \frac{\log x}{\sqrt{x}}$ for all $x > 0$, then $f : (0,\infty) \to \mathbb{R}$ is differentiable and $f'(x) < 0$ for all $x > e^2$. Hence f is decreasing on (e^2, ∞). Consequently the sequence $\left(\frac{\log n}{\sqrt{n}}\right)_{n=16}^{\infty}$ is decreasing. If $x = 6$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{\log(n+1)}{\sqrt{n+1}}$ diverges, by Cauchy's condensation test. Again, if $x = 4$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}(-1)^n \frac{\log(n+1)}{\sqrt{n+1}}$ converges, by Leibniz's test. Therefore the set of all $x \in \mathbb{R}$ for which the given series converges is $[4, 6)$.

Ex.20 Find all $x \in \mathbb{R}$ for which the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n(x+3)^n}{n5^n}$ is conditionally convergent.
Solution: If $x = -3$, then the given series becomes $0+0+\cdots$, which is clearly absolutely convergent. Let $x(\neq -3) \in \mathbb{R}$ and let $a_n = \frac{(-1)^n(x+3)^n}{n5^n}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5}|x+3|$. Hence by the ratio test, $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely if $\frac{1}{5}|x+3| < 1$, *i.e.* if $x \in (-8, 2)$ and diverges if $\frac{1}{5}|x+3| > 1$, *i.e.* if $x \in (-\infty, -8) \cup (2, \infty)$. If $x = -8$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges. If $x = 2$,

then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by Leibniz's test, but $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, *i.e.* $\sum\limits_{n=1}^{\infty} a_n$ is conditionally convergent. Therefore the set of all $x \in \mathbb{R}$ for which $\sum\limits_{n=1}^{\infty} a_n$ converges conditionally is $\{2\}$.

Ex.21 Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $|f(x)| \le |g(x)|$ for all $x \in \mathbb{R}$. If g is continuous at 0 and $g(0) = 0$, then show that f is continuous at 0.
Solution: Let $\varepsilon > 0$. Since g is continuous at 0, there exists $\delta > 0$ such that $|g(x)| = |g(x) - g(0)| < \varepsilon$ for all $x \in \mathbb{R}$ with $|x - 0| < \delta$. So $|f(x) - f(0)| \le |f(x)| + |f(0)| \le |g(x)| + |g(0)| = |g(x)| < \varepsilon$ for all $x \in \mathbb{R}$ with $|x - 0| < \delta$. Therefore f is continuous at 0.

Ex.22 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
Examine whether f is continuous at 0.
Solution: Let $x_n = \frac{2}{(4n+1)\pi}$ for all $n \in \mathbb{N}$. The sequence (x_n) in $\mathbb{R}$ converges to 0, but the sequence $(f(x_n)) = (2n\pi + \frac{\pi}{2})$ cannot converge because it is not bounded. Therefore f is not continuous at 0.

Ex.23 Give an example (with justification) of a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every point of $\mathbb{R}$ but $|f| : \mathbb{R} \to \mathbb{R}$ is continuous.
Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
If $x_0 \in \mathbb{Q}$, then there exists a sequence (t_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $t_n \to x_0$. Since $f(t_n) = -1$ for all $n \in \mathbb{N}$, $f(t_n) \to -1 \neq 1 = f(x_0)$. Hence f is not continuous at x_0. Again, if $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to x_0$. Since $f(r_n) = 1$ for all $n \in \mathbb{N}$, $f(r_n) \to 1 \neq -1 = f(x_0)$. Hence f is not continuous at x_0. Thus f is discontinuous at every point of $\mathbb{R}$.
However, $|f|(x) = |f(x)| = 1$ for all $x \in \mathbb{R}$ and so $|f| : \mathbb{R} \to \mathbb{R}$ is continuous.

Ex.24 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x) = x^2 + 5$ for all $x \in \mathbb{Q}$. Find $f(\sqrt{2})$.
Solution: There exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to \sqrt{2}$. Since f is continuous at $\sqrt{2}$, we have $f(\sqrt{2}) = \lim\limits_{n\to\infty} f(r_n) = \lim\limits_{n\to\infty} (r_n^2 + 5) = (\sqrt{2})^2 + 5 = 7$.

Ex.25 Evaluate $\lim\limits_{n\to\infty} \sin((2n\pi + \frac{1}{2n\pi}) \sin(2n\pi + \frac{1}{2n\pi}))$.
Solution: We have $(2n\pi + \frac{1}{2n\pi}) \sin(2n\pi + \frac{1}{2n\pi}) = 2n\pi \sin \frac{1}{2n\pi} + \frac{1}{2n\pi} \sin \frac{1}{2n\pi} \to 1$, since $|\frac{1}{2n\pi} \sin \frac{1}{2n\pi}| \le \frac{1}{2n\pi} \to 0 \Rightarrow \frac{1}{2n\pi} \sin \frac{1}{2n\pi} \to 0$ and $2n\pi \sin \frac{1}{2n\pi} = \frac{\sin \frac{1}{2n\pi}}{\frac{1}{2n\pi}} \to 1$, using $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. Since the sine function is continuous at 1, it follows that $\lim\limits_{n\to\infty} \sin((2n\pi + \frac{1}{2n\pi}) \sin(2n\pi + \frac{1}{2n\pi})) = \sin 1$.

Ex.26 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(0) > f(1) < f(2)$. Show that f is not one-one.
Solution: We choose $k \in \mathbb{R}$ such that $f(1) < k < \min\{f(0), f(2)\}$. Then by the intermediate value theorem, there exist $c_1 \in (0, 1)$ and $c_2 \in (1, 2)$ such that $f(c_1) = k$ and $f(c_2) = k$. Since $c_1 \neq c_2$, we conclude that f is not one-one.

Ex.27 Let $f : [0, 1] \to [0, 1]$ be continuous. Show that there exists $c \in [0, 1]$ such that $f(c) + 2c^5 = 3c^7$.
Solution: Let $g(x) = f(x) + 2x^5 - 3x^7$ for all $x \in [0, 1]$. Since f is continuous, $g : [0, 1] \to \mathbb{R}$ is continuous. If $f(0) = 0$ or $f(1) = 1$, then we get the result by taking $c = 0$ or $c = 1$ respectively. Otherwise $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$ (since it is given that $0 \le f(x) \le 1$ for all $x \in [0, 1]$). Hence by the intermediate value theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$, *i.e.* $f(c) = c$.

Ex.28 Show that there exists $c \in \mathbb{R}$ such that $c^{179} + \frac{163}{1+c^2+\sin^2 c} = 119$.
Solution: Let $f(x) = x^{179} + \frac{163}{1+x^2+\sin^2 x} - 119$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$f(-2) < 0$, $f(0) > 0$. Hence by the intermediate value theorem, there exists $c \in (-2, 0)$ such that $f(c) = 0$, *i.e.* $c^{179} + \frac{163}{1+c^2+\sin^2 c} = 119$.

Ex.29 Let $f, g : [-1, 1] \to \mathbb{R}$ be continuous such that $|f(x)| \leq 1$ for all $x \in [-1, 1]$ and $g(-1) = -1$, $g(1) = 1$. Show that there exists $c \in [-1, 1]$ such that $f(c) = g(c)$.
Solution: Let $\varphi(x) = f(x) - g(x)$ for all $x \in [-1, 1]$. Since f and g are continuous, $\varphi : [-1, 1] \to \mathbb{R}$ is continuous. If $f(-1) = -1$ or $f(1) = 1$, then we get the result by taking $c = -1$ or $c = 1$ respectively. Otherwise $\varphi(-1) = f(-1) + 1 > 0$ and $\varphi(1) = f(1) - 1 < 0$ (since it is given that $|f(x)| \leq 1$ for all $x \in [-1, 1]$). Hence by the intermediate value theorem, there exists $c \in (-1, 1)$ such that $\varphi(c) = 0$, *i.e.* $f(c) = g(c)$.

Ex.30 Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that
(a) if n is odd, then there exists unique $y \in \mathbb{R}$ such that $y^n = x$.
(b) if n is even and $x > 0$, then there exists unique $y > 0$ such that $y^n = x$.
Solution: Let $f(t) = t^n - x$ for all $t \in \mathbb{R}$, so that $f : \mathbb{R} \to \mathbb{R}$ is continuous.
(a) We first assume that n is odd. Then $\lim_{t\to\infty} f(t) = \infty$ and $\lim_{t\to-\infty} f(t) = -\infty$. So there exist $x_1 > 0$ and $x_2 < 0$ such that $f(x_1) > 0$ and $f(x_2) < 0$. By the intermediate value property of continuous functions, there exists $y \in (x_2, x_1)$ such that $f(y) = 0$, *i.e.* $y^n = x$. If possible, let there exist $u \in \mathbb{R}$ such that $u \neq y$ and $u^n = x$. Clearly either both u and y must be non-negative or both u and y must be negative. We consider the case $0 \leq y < u$. (Other cases can be handled similarly.) Then $x = y^n < u^n = x$, which is a contradiction. Thus the uniqueness of y is proved.
(b) We now assume that n is even and $x > 0$. Then $f(0) < 0$ and $\lim_{t\to\infty} f(t) = \infty$. So there exists $x_1 > 0$ such that $f(x_1) > 0$. By the intermediate value property of continuous functions, there exists $y \in (0, x_1)$ such that $f(y) = 0$ *i.e.* $y^n = x$. If possible, let there exist $u > 0$ such that $u \neq y$ and $u^n = x$. Without loss of generality, let $u > y$. Then $x = u^n > y^n = x$, which is a contradiction. This proves the uniqueness of y.

Ex.31 If $f : [0, 1] \to \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in [0, 1]$, then show that there exists $\alpha > 0$ such that $f(x) > \alpha$ for all $x \in [0, 1]$.
Solution: Since $f : [0, 1] \to \mathbb{R}$ is continuous, there exists $x_0 \in [0, 1]$ such that $f(x) \geq f(x_0)$ for all $x \in [0, 1]$. Choosing $\alpha = \frac{1}{2} f(x_0)$, we find that $\alpha > 0$ and $f(x) > \alpha$ for all $x \in [0, 1]$.

Ex.32 Give an example of each of the following.
(a) A function $f : [0, 1] \to \mathbb{R}$ which is not bounded.
(b) A continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ which does not attain $\sup\{f(x) : x \in \mathbb{R}\}$ as well as $\inf\{f(x) : x \in \mathbb{R}\}$.
(c) A continuous and bounded function $f : (0, 1) \to \mathbb{R}$ which attains both $\sup\{f(x) : x \in (0, 1)\}$ and $\inf\{f(x) : x \in (0, 1)\}$.
Hint: (a) If $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0, \end{cases}$
then $f : [0, 1] \to \mathbb{R}$ is not bounded.
(b) The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \frac{x}{1+|x|}$ for all $x \in \mathbb{R}$, is continuous and bounded. However, neither $\sup\{f(x) : x \in \mathbb{R}\} = 1$ nor $\inf\{f(x) : x \in \mathbb{R}\} = -1$ is attained by f at any point of $\mathbb{R}$.
(c) The function $f : (0, 1) \to \mathbb{R}$, defined by $f(x) = \sin(2\pi x)$ for all $x \in (0, 1)$, is continuous and bounded. Also, $\sup\{f(x) : x \in (0, 1)\} = 1 = f(\frac{1}{4})$ and $\inf\{f(x) : x \in (0, 1)\} = -1 = f(\frac{3}{4})$.

Ex.33 If $f(x) = x \sin x$ for all $x \in \mathbb{R}$, then show that $f : \mathbb{R} \to \mathbb{R}$ is neither bounded above nor bounded below.
Solution: If possible, let f be bounded above. Then there exists $M > 0$ such that $f(x) \leq M$ for all $x \in \mathbb{R}$ and hence $2n\pi + \frac{\pi}{2} = f(2n\pi + \frac{\pi}{2}) \leq M$ for all $n \in \mathbb{N}$. This gives $n \leq \frac{1}{2\pi}(M - \frac{\pi}{2})$ for all $n \in \mathbb{N}$, which is not possible. Hence f is not bounded above. Again, if possible, let f

be bounded below. Then there exists $K > 0$ such that $f(x) \geq K$ for all $x \in \mathbb{R}$ and hence $-2n\pi - \frac{3\pi}{2} = f(2n\pi + \frac{3\pi}{2}) \geq K$ for all $n \in \mathbb{N}$. This gives $n \leq -\frac{1}{2\pi}(K + \frac{3\pi}{2})$ for all $n \in \mathbb{N}$, which is not possible. Hence f is not bounded below.

Ex.34 Let p be an nth degree polynomial with real coefficients in one real variable such that $n(\neq 0)$ is even and $p(0) \cdot p^{(n)}(0) < 0$. Show that p has at least two real zeroes.
Solution: Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ for all $x \in \mathbb{R}$, where $a_i \in \mathbb{R}$ for $i = 0, 1, ..., n$, $n \in \mathbb{N}$ is even and $a_0 \neq 0$. Then p is infinitely differentiable (and so also continuous) and $p^{(n)}(0) = n!a_0$. Since $p(0) \cdot p^{(n)}(0) < 0$, we have $a_0a_n < 0$, *i.e.* a_0 and a_n are of different signs. Let us assume that $a_0 > 0$, so that $a_n < 0$. (The case $a_0 < 0$ and so $a_n > 0$ is almost similar.) Since $p(x) = a_0x^n(1 + \frac{a_1}{a_0} \cdot \frac{1}{x} + \cdots + \frac{a_{n-1}}{a_0} \cdot \frac{1}{x^{n-1}} + \frac{a_n}{a_0} \cdot \frac{1}{x^n})$ for all $x(\neq 0) \in \mathbb{R}$, we get $\lim\limits_{x\to\infty} p(x) = \infty$ and $\lim\limits_{x\to-\infty} p(x) = \infty$. So there exist $x_1 > 0$ and $x_2 < 0$ such that $p(x_1) > 0$ and $p(x_2) > 0$. Since $p(0) = a_n < 0$, by the intermediate value theorem, there exist $c_1 \in (x_2, 0)$ and $c_2 \in (0, x_1)$ such that $p(c_1) = 0$ and $p(c_2) = 0$.

Ex.35 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at 0 and let $g(x) = xf(x)$ for all $x \in \mathbb{R}$. Show that $g : \mathbb{R} \to \mathbb{R}$ is differentiable at 0.
Solution: Since $\lim\limits_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim\limits_{x\to 0} f(x) = f(0)$ (because f is continuous at 0), g is differentiable at 0.

Ex.36 Let $\alpha > 1$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x)| \leq |x|^\alpha$ for all $x \in \mathbb{R}$. Show that f is differentiable at 0.
Solution: We have $|f(0)| \leq |0|^\alpha = 0 \Rightarrow f(0) = 0$ and so $|\frac{f(x)-f(0)}{x-0}| \leq |x|^{\alpha-1}$ for all $x(\neq 0) \in \mathbb{R}$. Since $\lim\limits_{x\to 0} |x|^{\alpha-1} = 0$, by sandwich theorem for limit of functions, we get $\lim\limits_{x\to 0} |\frac{f(x)-f(0)}{x-0}| = 0$. It follows that $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ and consequently f is differentiable at 0.

Ex.37 Let $f(x) = x^2|x|$ for all $x \in \mathbb{R}$. Examine the existence of $f'(x)$, $f''(x)$ and $f'''(x)$, where $x \in \mathbb{R}$.
Solution: Here $f(x) = \begin{cases} x^3 & \text{if } x \geq 0, \\ -x^3 & \text{if } x < 0. \end{cases}$

Clearly $f : \mathbb{R} \to \mathbb{R}$ is differentiable at all $x(\neq 0) \in \mathbb{R}$ and $f'(x) = \begin{cases} 3x^2 & \text{if } x > 0, \\ -3x^2 & \text{if } x < 0. \end{cases}$

Also, $\lim\limits_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0+} x^2 = 0$ and $\lim\limits_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0-} (-x^2) = 0$.
Hence $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$.

Again, it is clear that $f' : \mathbb{R} \to \mathbb{R}$ is differentiable at all $x(\neq 0) \in \mathbb{R}$ and $f''(x) = \begin{cases} 6x & \text{if } x > 0, \\ -6x & \text{if } x < 0. \end{cases}$

Also, $\lim\limits_{x\to 0+} \frac{f'(x)-f'(0)}{x-0} = \lim\limits_{x\to 0+} 3x = 0$ and $\lim\limits_{x\to 0-} \frac{f'(x)-f'(0)}{x-0} = \lim\limits_{x\to 0-} (-3x) = 0$.
Hence $f''(0) = \lim\limits_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = 0$.

Finally, it is clear that $f'' : \mathbb{R} \to \mathbb{R}$ is differentiable at all $x(\neq 0) \in \mathbb{R}$ and $f'''(x) = \begin{cases} 6 & \text{if } x > 0, \\ -6 & \text{if } x < 0. \end{cases}$

Also, $\lim\limits_{x\to 0+} \frac{f''(x)-f''(0)}{x-0} = \lim\limits_{x\to 0+} 6 = 6$ and $\lim\limits_{x\to 0-} \frac{f''(x)-f''(0)}{x-0} = \lim\limits_{x\to 0-} (-6) = -6$.
Hence $\lim\limits_{x\to 0} \frac{f''(x)-f''(0)}{x-0}$ does not exist, *i.e.* $f'''(0)$ does not exist.

Ex.38 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2|\cos\frac{\pi}{x}| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
Examine whether f is differentiable (i) at 0 (ii) on $(0, 1)$.
Solution: (i) For each $\varepsilon > 0$, choosing $\delta = \varepsilon > 0$, we find that $\left|\frac{f(x)-f(0)}{x-0}\right| = |x||\cos\frac{\pi}{x}| \leq |x|$ for

all $x \in \mathbb{R}$ satisfying $0 < |x| < \delta$. Hence $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ and consequently f is differentiable at 0 (with $f'(0) = 0$).
(ii) Since $\lim\limits_{x\to \frac{2}{3}+} \frac{f(x)-f(\frac{2}{3})}{x-\frac{2}{3}} = \lim\limits_{x\to \frac{2}{3}+} \frac{-x^2\cos\frac{\pi}{x}-0}{x-\frac{2}{3}} = \frac{d}{dx}(-x^2\cos\frac{\pi}{x})|_{x=\frac{2}{3}}$ (applying L'Hôpital's rule) $=$ π and $\lim\limits_{x\to \frac{2}{3}-} \frac{f(x)-f(\frac{2}{3})}{x-\frac{2}{3}} = \lim\limits_{x\to \frac{2}{3}-} \frac{x^2\cos\frac{\pi}{x}-0}{x-\frac{2}{3}} = \frac{d}{dx}(x^2\cos\frac{\pi}{x})|_{x=\frac{2}{3}}$ (applying L'Hôpital's rule) $= -\pi$, $\lim\limits_{x\to \frac{2}{3}} \frac{f(x)-f(\frac{2}{3})}{x-\frac{2}{3}}$ does not exist and hence f is not differentiable at $\frac{2}{3} \in (0,1)$. Consequently f is not differentiable on $(0,1)$.

Ex.39(a) Examine whether $f : \mathbb{R} \to \mathbb{R}$, defined as below, is differentiable at 0.
$f(x) = \begin{cases} \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$
Solution: Since $\frac{f(\frac{1}{2^n})-f(0)}{1/2^n} = \frac{1}{2}$ and $\frac{f(\frac{1}{3^n})-f(0)}{1/3^n} = 0$ for all $n \in \mathbb{N}$, $\frac{f(\frac{1}{2^n})-f(0)}{1/2^n} \to \frac{1}{2}$ and $\frac{f(\frac{1}{3^n})-f(0)}{1/3^n} \to 0$. As $\frac{1}{2^n} \to 0$ and $\frac{1}{3^n} \to 0$, by the sequential criterion of limit, it follows that $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist. Consequently f is not differentiable at 0.

Ex.39(b) Examine whether $f : \mathbb{R} \to \mathbb{R}$, defined as below, is differentiable at 0.
$f(x) = \begin{cases} \frac{1}{4^n} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$
Solution: For all $x(\neq 0) \in \mathbb{R}$, we have $\left|\frac{f(x)-f(0)}{x-0}\right| \leq |x|$. Hence for each $\varepsilon > 0$, taking $\delta = \varepsilon > 0$, we find that $\left|\frac{f(x)-f(0)}{x-0}\right| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $0 < |x-0| < \delta$. Therefore $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ and consequently f is differentiable at 0 (with $f'(0) = 0$).

Ex.40 Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at 0 and $f(0) = f'(0) = 0$. Show that $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = \begin{cases} f(x)\sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$ is differentiable at 0.
Solution: Since $0 \leq \left|\frac{g(x)-g(0)}{x-0}\right| = \left|\frac{f(x)}{x}\right| |\sin\frac{1}{x}| \leq \left|\frac{f(x)}{x}\right|$ for all $x(\neq 0) \in \mathbb{R}$ and since $\lim\limits_{x\to 0}\left|\frac{f(x)}{x}\right| = \left|\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0}\right| = |f'(0)| = 0$, by the sandwich theorem for limits of functions, we get $\lim\limits_{x\to 0}\left|\frac{g(x)-g(0)}{x-0}\right| = 0$. It follows that $\lim\limits_{x\to 0} \frac{g(x)-g(0)}{x-0} = 0$ and consequently g is differentiable at 0 (with $g'(0) = 0$).

Ex.41 Let $f(x) = x^3 + x$ and $g(x) = x^3 - x$ for all $x \in \mathbb{R}$. If f^{-1} denotes the inverse function of f and if $(g \circ f^{-1})(x) = g(f^{-1}(x))$ for all $x \in \mathbb{R}$, then find $(g \circ f^{-1})'(2)$.
Solution: Since $f'(x) = 3x^2 + 1 \neq 0$ for all $x \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is one-one. Also, since f is an odd degree polynomial in $\mathbb{R}$, by the intermediate value property of continuous functions, $f : \mathbb{R} \to \mathbb{R}$ is onto. Hence $f^{-1} : \mathbb{R} \to \mathbb{R}$ exists and is differentiable. By chain rule and the rule for derivative of inverse, we get $(g \circ f^{-1})^{-1}(2) = g'(f^{-1}(2))(f^{-1})'(2) = g'(1)\frac{1}{f'(1)}$ (since $f(1) = 2$) $= \frac{1}{2}$.

Ex.42 If $a, b, c \in \mathbb{R}$, then show that the equation $4ax^3 + 3bx^2 + 2cx = a + b + c$ has at least one root in $(0,1)$.
Solution: Let $f(x) = ax^4 + bx^3 + cx^2 - (a+b+c)x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f(0) = 0 = f(1)$. Hence by Rolle's theorem, the equation $f'(x) = 0$, *i.e.* $4ax^3 + 3bx^2 + 2cx = a + b + c$ has at least one root in $(0,1)$.

Ex.43 If $a_0, a_1, ..., a_n \in \mathbb{R}$ satisfy $\frac{a_0}{1.2} + \frac{a_1}{2.3} + \cdots + \frac{a_n}{(n+1)(n+2)} = 0$, then show that the equation $a_0 + a_1x + \cdots + a_nx^n = 0$ has at least one root in $[0,1]$.
Solution: Let $f(x) = \frac{a_0}{1.2}x^2 + \frac{a_1}{2.3}x^3 + \cdots + \frac{a_n}{(n+1)(n+2)}x^{n+2}$ for all $x \in [0,1]$. Then $f : [0,1] \to \mathbb{R}$ is twice differentiable and $f'(x) = a_0x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$, $f''(x) = a_0 + a_1x + \cdots + a_nx^n$ for all

$x \in [0,1]$. Since $f(0) = 0 = f(1)$, by Rolle's theorem, there exists $c \in (0,1)$ such that $f'(c) = 0$. Again, since $f'(0) = 0$, by Rolle's theorem, there exists $\alpha \in (0,c)$ such that $f''(\alpha) = 0$. Thus the equation $a_0 + a_1x + \cdots + a_nx^n = 0$ has a root $\alpha \in [0,1]$.

Ex.44 Show that the equation $|x^{10} - 60x^9 - 290| = e^x$ has at least one real root.
Solution: Let $f(x) = |x^{10} - 60x^9 - 290| - e^x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(0) = 289 > 0$. Again, $\lim\limits_{x\to\infty} \frac{x^{10}-60x^9-290}{e^x} = \lim\limits_{x\to\infty} \frac{10!}{e^x}$ (using L'Hôpital's rule ten times) $= 0$. Hence there exists $M > 0$ such that $|\frac{x^{10}-60x^9-290}{e^x}| < 1$ for all $x > M$ and consequently $f(2M) < 0$. Therefore by the intermediate value property of continuous functions, the equation $f(x) = 0$ has at least one root in $(0, 2M)$. Hence the given equation has at least one real root.

Ex.45(a) Find the number of (distinct) real roots of the equation $x^2 = \cos x$.
Solution: Let $f(x) = x^2 - \cos x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable with $f'(x) = 2x + \sin x$ and $f''(x) = 2 + \cos x$ for all $x \in \mathbb{R}$. Since $f''(x) \neq 0$ for all $x \in \mathbb{R}$, as a consequence of Rolle's theorem, it follows that the equation $f'(x) = 0$ has at most one real root and hence the equation $f(x) = 0$ has at most two real roots. Again, since $f(-\frac{\pi}{2}) = \frac{\pi^2}{4} > 0$, $f(0) = -1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi^2}{4} > 0$, by the intermediate value property of continuous functions, the equation $f(x) = 0$ has at least one root in $(-\frac{\pi}{2}, 0)$ and at least one root in $(0, \frac{\pi}{2})$. Therefore the given equation has exactly two (distinct) real roots.

Ex.45(b) Find the number of (distinct) real roots of the equation $e^{2x} + \cos x + x = 0$.
Solution: Let $f(x) = e^{2x} + \cos x + x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) = 2e^{2x} + (1 - \sin x) > 0$ for all $x \in \mathbb{R}$. As a consequence of Rolle's theorem, the equation $f(x) = 0$ has at most one real root. Again, since $f(-\frac{\pi}{2}) = e^{-\pi} - \frac{\pi}{2} < 0$ and $f(0) = 2 > 0$, by the intermediate value property of continuous functions, the equation $f(x) = 0$ has least one root in $(-\frac{\pi}{2}, 0)$. Therefore the given equation has exactly one (distinct) real root.

Ex.46 Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable such that $f(0) = 0$, $f'(0) > 0$ and $f''(x) > 0$ for all $x \in \mathbb{R}$. Show that the equation $f(x) = 0$ has no positive real root.
Solution: Since $f''(x) > 0$ for all $x \in \mathbb{R}$, f' is strictly increasing on $\mathbb{R}$ and so $f'(x) > f'(0) > 0$ for all $x > 0$. This implies that f is strictly increasing on $[0, \infty)$ and so $f(x) > f(0) = 0$ for all $x > 0$. Thus the equation $f(x) = 0$ has no positive real root.

Ex.47 Show that between any two (distinct) real roots of the equation $e^x \sin x = 1$, there exists at least one real root of the equation $e^x \cos x + 1 = 0$.
Solution: Let $f(x) = \sin x - e^{-x}$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable (also continuous). Let $a, b \in \mathbb{R}$ with $a < b$ be such that $e^a \sin a = 1 = e^b \sin b$. Then $f(a) = 0 = f(b)$. By Rolle's theorem, there exists $c \in (a,b)$ such that $f'(c) = 0$, *i.e.* $\cos c + e^{-c} = 0 \Rightarrow e^c(\cos c + e^{-c}) = 0 \Rightarrow e^c \cos c + 1 = 0$. Thus $c \in (a,b)$ is a root of the equation $e^x \cos x + 1 = 0$.

Ex.48 Let $f(x) = 3x^5 - 2x^3 + 12x - 8$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is one-one and onto.
Solution: Here $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) = 15x^4 - 6x^2 + 12 = 15[(x^2 - \frac{1}{5})^2 + \frac{19}{25}] \neq 0$ for all $x \in \mathbb{R}$. As a consequence of the mean value theorem, $f : \mathbb{R} \to \mathbb{R}$ is one-one. Again, since f is an odd degree polynomial with real coefficients in one real variable, by Ex.12(c) of Tutorial Problem Set, $f : \mathbb{R} \to \mathbb{R}$ is onto.

Ex.49(a) Show that $\frac{x-1}{x} < \log x < x - 1$ for all $x (\neq 1) > 0$.
Solution: Let $f(x) = \log x$ for all $x > 0$. Then $f : (0, \infty) \to \mathbb{R}$ is differentiable and hence for each $x (\neq 1) \in (0, \infty)$, there exists c between 1 and x such that $f(x) - f(1) = (x-1)f'(c)$, *i.e* $\log x = \frac{x-1}{c}$. Since $\frac{1}{x} < \frac{1}{c} < 1$ if $x > 1$ and $1 < \frac{1}{c} < \frac{1}{x}$ if $0 < x < 1$, we get $\frac{x-1}{x} < \frac{x-1}{c} < x - 1$ for all $x (\neq 1) > 0$. Hence $\frac{x-1}{x} < \log x < x - 1$ for all $x (\neq 1) > 0$.

Ex.49(b) Show that $1+x<e^x<1+xe^x$ for all $x(\neq 0)\in\mathbb{R}$.
Solution: Let $f(x)=e^x$ for all $x\in\mathbb{R}$. Then $f:\mathbb{R}\to\mathbb{R}$ is differentiable and hence for each $x(\neq 0)\in\mathbb{R}$, by the mean value theorem, there exists c between 0 and x such that $f(x)-f(0)=xf'(c)$, *i.e.* $e^x-1=xe^c$. Since $1<e^c<e^x$ if $x>0$ and $e^x<e^c<1$ if $x<0$, we get $x<xe^c<xe^x$ for all $x(\neq 0)\in\mathbb{R}$. Hence $1+x<e^x<1+xe^x$ for all $x(\neq 0)\in\mathbb{R}$.

Ex.49(c) Show that $2\sin x+\tan x>3x$ for all $x\in(0,\frac{\pi}{2})$.
Solution: Let $f(x)=2\sin x+\tan x-3x$ for all $x\in[0,\frac{\pi}{2})$. Then $f:[0,\frac{\pi}{2})\to\mathbb{R}$ is twice differentiable and $f'(x)=2\cos x+\sec^2 x-3$ for all $x\in[0,\frac{\pi}{2})$, $f''(x)=2\sin x(\sec^3 x-1)>0$ for all $x\in(0,\frac{\pi}{2})$. Hence f' is strictly increasing on $[0,\frac{\pi}{2})$ and so $f'(x)>f'(0)=0$ for all $x\in(0,\frac{\pi}{2})$. Thus f is strictly increasing on $[0,\frac{\pi}{2})$ and so $f(x)>f(0)=0$ for all $x\in(0,\frac{\pi}{2})$. Consequently $2\sin x+\tan x>3x$ for all $x\in(0,\frac{\pi}{2})$.

Ex.49(d) Show that $(1+x)^\alpha\geq 1+\alpha x$ for all $x\geq -1$ and for all $\alpha>1$.
Solution: Let $\alpha>1$ and let $f(x)=(1+x)^\alpha-(1+\alpha x)$ for all $x\geq -1$. Then $f:[-1,\infty)\to\mathbb{R}$ is differentiable and $f'(x)=\alpha[(1+x)^{\alpha-1}-1]$ for all $x\geq -1$. Clearly $f'(x)\leq 0$ for all $x\in[-1,0]$ and $f'(x)\geq 0$ for all $x\in[0,\infty)$. Hence f is decreasing on $[-1,0]$ and increasing on $[0,\infty)$. So $f(x)\geq f(0)=0$ for $-1\leq x\leq 0$ and also $f(x)\geq f(0)=0$ for $x\geq 0$. Therefore $f(x)\geq 0$ for all $x\geq -1$, which proves the required inequality.

Ex.50(a) Determine all the differentiable functions $f:[0,1]\to\mathbb{R}$ satisfying the conditions $f(0)=0$, $f(1)=1$ and $|f'(x)|\leq\frac{1}{2}$ for all $x\in[0,1]$.
Solution: If possible, let $f:[0,1]\to\mathbb{R}$ be a differentiable function satisfying the given conditions. Then by the mean value theorem, there exists $c\in(0,1)$ such that $f'(c)=\frac{f(1)-f(0)}{1-0}=1$, which contradicts the given condition that $|f'(x)|\leq\frac{1}{2}$ for all $x\in[0,1]$. Therefore no such differentiable function can exist.

Ex.50(b) Determine all the differentiable functions $f:[0,1]\to\mathbb{R}$ satisfying the conditions $f(0)=0$, $f(1)=1$ and $|f'(x)|\leq 1$ for all $x\in[0,1]$.
Solution: Let f be such a function and let $g(x)=x-f(x)$ for all $x\in[0,1]$. Then $g:[0,1]\to\mathbb{R}$ is differentiable and $g'(x)=1-f'(x)\geq 0$ for all $x\in[0,1]$. Hence g is increasing on $[0,1]$ and since $g(0)=0=g(1)$, it follows that g is the constant function given by $g(x)=0$ for all $x\in[0,1]$, *i.e.* $f(x)=x$ for all $x\in[0,1]$. Also, if $f(x)=x$ for all $x\in[0,1]$, then f satisfies all the given conditions. Therefore there is exactly one function f satisfying the given conditions and it is given by $f(x)=x$ for all $x\in[0,1]$.

Ex.51 Let $f:[0,2]\to\mathbb{R}$ be differentiable and $f(0)=f(1)=0$, $f(2)=3$. Show that there exist $a,b,c\in(0,2)$ such that $f'(a)=0$, $f'(b)=3$ and $f'(c)=1$.
Solution: By Rolle's theorem, there exists $a\in(0,1)$ such that $f'(a)=0$. Again, by the mean value theorem, there exists $b\in(1,2)$ such that $f'(b)=\frac{f(2)-f(1)}{2-1}=3$. Hence by the intermediate value property of derivatives, there exists $c\in(a,b)$ such that $f'(c)=1$.

Ex.52(a) Evaluate the limit: $\lim\limits_{x\to 0}(\frac{1}{\sin x}-\frac{1}{x})$
Solution: We have $\lim\limits_{x\to 0}(\frac{1}{\sin x}-\frac{1}{x})=\lim\limits_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim\limits_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}$ (using L'Hôpital's rule)
$=\lim\limits_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}$ (using L'Hôpital's rule again) $=0$.

Ex.52(b) Evaluate the limit: $\lim\limits_{x\to 0}\frac{e^{-1/x^2}}{x}=0$
Solution: We have $\lim\limits_{x\to 0}\frac{e^{-1/x^2}}{x}=\lim\limits_{x\to 0}\frac{1/x}{e^{1/x^2}}=\lim\limits_{x\to 0}\frac{-1/x^2}{-\frac{2}{x^3}e^{1/x^2}}$ (applying L'Hôpital's rule) $=\lim\limits_{x\to 0}\frac{1}{2}xe^{-\frac{1}{x^2}}=0$.

Ex.52(c) Evaluate the limit: $\lim\limits_{x\to\infty} x(\log(1+\frac{x}{2}) - \log\frac{x}{2})$

Solution: $\lim\limits_{x\to\infty} x(\log(1+\frac{x}{2}) - \log(\frac{x}{2})) = \lim\limits_{x\to\infty} x\log\left(\frac{1+\frac{x}{2}}{\frac{x}{2}}\right) = \lim\limits_{x\to\infty} \frac{\log(1+\frac{2}{x})}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{-\frac{2}{x^2}}{-\frac{1}{x^2}(1+\frac{2}{x})}$ (using L'Hôpital's rule) $= \lim\limits_{x\to\infty} \frac{2}{1+\frac{2}{x}} = 2$.

Ex.52(d) Evaluate the limit: $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

Solution: If $f(x) = (1+x)^{\frac{1}{x}}$ for all $x \in (-1,1)\setminus\{0\}$, then $f : (-1,1)\setminus\{0\} \to \mathbb{R}$ is differentiable and $f'(x) = (1+x)^{\frac{1}{x}}\left[\frac{x-(1+x)\log(1+x)}{x^2(1+x)}\right]$ for all $x \in (-1,1)\setminus\{0\}$. Hence $\lim\limits_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} = \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} \cdot \lim\limits_{x\to 0}\frac{x-(1+x)\log(1+x)}{x^2(1+x)}$ (applying L'Hôpital's rule) $= e\lim\limits_{x\to 0}\frac{-\log(1+x)}{x(3x+2)}$ (using $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$ and applying L'Hôpital's rule in the second limit) $= -\frac{e}{2}$ (using $\lim\limits_{x\to 0}\frac{1}{x}\log(1+x) = 1$).

Ex.52(e) Evaluate the limit: $\lim\limits_{x\to\infty}\frac{2x+\sin 2x+1}{(2x+\sin 2x)(\sin x+3)^2}$

Solution: Let $x_n = n\pi$ and $y_n = (4n+1)\frac{\pi}{2}$ for all $n \in \mathbb{N}$. Then $x_n \to \infty$, $y_n \to \infty$ and $\lim\limits_{n\to\infty}\frac{2x_n+\sin 2x_n+1}{(2x_n+\sin 2x_n)(\sin x_n+3)^2} = \lim\limits_{n\to\infty}(\frac{1}{9}+\frac{1}{18n\pi}) = \frac{1}{9}$, $\lim\limits_{n\to\infty}\frac{2y_n+\sin 2y_n+1}{(2y_n+\sin 2y_n)(\sin y_n+3)^2} = \lim\limits_{n\to\infty}(\frac{1}{16}+\frac{1}{(4n+1)16\pi}) = \frac{1}{16}$. By the sequential criterion for existence of limits, it follows that $\lim\limits_{x\to\infty}\frac{2x+\sin 2x+1}{(2x+\sin 2x)(\sin x+3)^2}$ does not exist.

Ex.53 If $f : (0,\infty) \to (0,\infty)$ is differentiable at $a \in (0,\infty)$, then evaluate $\lim\limits_{x\to a}\left(\frac{f(x)}{f(a)}\right)^{\frac{1}{\log x - \log a}}$.

Solution: Let $g(x) = (\frac{f(x)}{f(a)})^{\frac{1}{\log x-\log a}}$ for all $x(\neq a) \in (0,\infty)$. Then $g(x) > 0$ for all $x(\neq a) \in (0,\infty)$ and we have $\lim\limits_{x\to a}\log g(x) = \lim\limits_{x\to a}\frac{\log f(x)-\log f(a)}{\log x-\log a} = \frac{\frac{d}{dx}(\log f(x)-\log f(a))|_{x=a}}{\frac{d}{dx}(\log x-\log a)|_{x=a}}$ (applying L'Hôpital's rule) $= a\frac{f'(a)}{f(a)}$. By the continuity of the exponential function, it follows that $\lim\limits_{x\to a} g(x) = e^{af'(a)/f(a)}$.

Ex.54 Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x(\neq 0) \in \mathbb{R}, \\ 1 & \text{if } x = 0. \end{cases}$

Examine whether $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

Solution: Clearly f is differentiable at each $x(\neq 0) \in \mathbb{R}$ and $f'(x) = \frac{1}{x}\cos x - \frac{1}{x^2}\sin x$ for all $x(\neq 0) \in \mathbb{R}$. Also, $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\frac{\sin x - x}{x^2} = \lim\limits_{x\to 0}\frac{\cos x-1}{2x} = \lim\limits_{x\to 0}\frac{-\sin x}{2} = 0$ (using L'Hôpital's rule). So f is differentiable at 0 and $f'(0) = 0$. Again, it is clear that $f' : \mathbb{R} \to \mathbb{R}$ is continuous at each $x(\neq 0) \in \mathbb{R}$. Further, since $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0}\frac{x\cos x - \sin x}{x^2} = \lim\limits_{x\to 0}\frac{-x\sin x}{2x} = \lim\limits_{x\to 0}(-\frac{1}{2}\sin x) = 0 = f'(0)$ (using L'Hôpital's rule), f' is continuous at 0. Hence f is continuously differentiable.

Ex.55(a) Using Taylor's theorem, show that $|\sqrt{1+x} - (1+\frac{x}{2}-\frac{x^2}{8})| \le \frac{1}{2}|x|^3$ for all $x \in (-\frac{1}{2},\frac{1}{2})$.

Solution: Let $f(x) = \sqrt{1+x}$ for all $x \in (-\frac{1}{2},\frac{1}{2})$. Then f has derivatives of all orders in $(-\frac{1}{2},\frac{1}{2})$ and we have $f'(x) = \frac{1}{2\sqrt{1+x}}$, $f''(x) = -\frac{1}{4(1+x)^{3/2}}$ and $f'''(x) = \frac{3}{8(1+x)^{5/2}}$ for all $x \in (-\frac{1}{2},\frac{1}{2})$. By Taylor's theorem, for each $x \in (-\frac{1}{2},\frac{1}{2})$, there exists c between 0 and x such that $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(c) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\cdot\frac{1}{(1+c)^{5/2}}$. This gives $|\sqrt{1+x} - (1+\frac{x}{2}-\frac{x^2}{8})| = \frac{|x|^3}{16}\cdot\frac{1}{(1+c)^{5/2}} \le \frac{2^{5/2}}{16}|x|^3 = \frac{\sqrt{2}}{4}|x|^3 \le \frac{1}{2}|x|^3$.

Ex.55(b) Using Taylor's theorem, show that $1 - \frac{x^2}{2!} + \frac{x^4}{4!} > \cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ for all $x \in (0,\pi)$.

Solution: Let $f(x) = \cos x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, $f^{(5)}(x) = -\sin x$, $f^{(6)}(x) = -\cos x$ for all $x \in \mathbb{R}$. If $x \in (0,\pi)$, then by Taylor's theorem, there exist $c_1, c_2 \in (0,x)$ such that $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(c_1) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cos c_1$ and $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^5}{5!}f^{(5)}(0) + \frac{x^6}{6!}f^{(6)}(c_2) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\cos c_2$. Since $\cos c_1 < 1$ and $\cos c_2 < 1$, it follows that $1 - \frac{x^2}{2!} + \frac{x^4}{4!} > \cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$.

Ex.55(c) Using Taylor's theorem, show that $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for all $x \in (0, \pi)$.
Solution: Let $f(x) = \sin x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, $f^{(5)}(x) = \cos x$ for all $x \in \mathbb{R}$. If $x \in (0, \pi)$, then by Taylor's theorem, there exist $c_1, c_2 \in (0, x)$ such that $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(c_1) = x - \frac{x^3}{3!}\cos c_1$ and $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^5}{5!}f^{(5)}(c_2) = x - \frac{x^3}{3!} + \frac{x^5}{5!}\cos c_2$. Since $\cos c_1 < 1$ and $\cos c_2 < 1$, it follows that $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

Ex.56 Find the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} n!x^n$.
Solution: If $x = 0$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x(\neq 0) \in \mathbb{R}$ and let $a_n = n!x^n$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty$ and so there exists $n_0 \in \mathbb{N}$ such that $|\frac{a_{n+1}}{a_n}| > 2$ for all $n \geq n_0$. This gives $|a_n| > 2^{n-n_0}|a_{n_0}|$ for all $n \geq n_0$ and hence $\lim\limits_{n\to\infty} a_n \neq 0$. Consequently $\sum\limits_{n=1}^{\infty} a_n$ is not convergent. Therefore the radius of convergence of the given power series is 0.

Ex.57 Find the interval of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$.
Solution: If $x = 0$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x(\neq 0) \in \mathbb{R}$ and let $a_n = \frac{x^n}{n}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |x|$. Hence by ratio test, $\sum\limits_{n=1}^{\infty} a_n$ is convergent (absolutely) if $|x| < 1$, *i.e.* if $x \in (-1, 1)$ and is not convergent if $|x| > 1$, *i.e.* if $x \in (-\infty, -1) \cup (1, \infty)$. If $x = 1$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not convergent. Again, if $x = -1$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by Leibniz test, since $(\frac{1}{n})$ is a decreasing sequence of positive real numbers and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Therefore the interval of convergence of the given power series is $[-1, 1)$.

Ex.58 Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If there is a partition P of $[a, b]$ such that $L(f, P) = U(f, P)$, then show that f is a constant function.
Solution: Let $P = \{x_0, x_1, ..., x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$. Since $L(f, P) = U(f, P)$, we get $\sum\limits_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = 0$, where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ for $i = 1, 2, ..., n$. Since $M_i \geq m_i$ and $x_i - x_{i-1} > 0$ for $i = 1, 2, ..., n$, it follows that $M_i - m_i = 0$, *i.e.* $M_i = m_i$ for $i = 1, 2, ..., n$. This implies that f is constant on $[x_{i-1}, x_i]$ for each $i \in \{1, 2, ..., n\}$. Hence $f(x) = f(x_{i-1}) = f(x_i)$ for all $x \in [x_{i-1}, x_i]$ $(i = 1, 2, ..., n)$. Consequently $f(x) = f(a)$ for all $x \in [a, b]$. Therefore f is a constant function.

Ex.59(a) Evaluate the limit: $\lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} \sqrt{n^2 - k^2}$
Solution: Let $f(x) = \sqrt{1 - x^2}$ for all $x \in [0, 1]$. Considering the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$ of $[0, 1]$ for each $n \in \mathbb{N}$ (and taking $c_k = \frac{k}{n}$ for $k = 1, ..., n$), we find that
$S(f, P_n) = \sum\limits_{k=1}^{n} f(\frac{k}{n})(\frac{k}{n} - \frac{k-1}{n}) = \frac{1}{n^2}\sum\limits_{k=1}^{n} \sqrt{n^2 - k^2}$. Since $f : [0, 1] \to \mathbb{R}$ is continuous, f is Riemann integrable on $[0, 1]$ and hence
$\lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} \sqrt{n^2 - k^2} = \lim\limits_{n\to\infty} S(f, P_n) = \int\limits_0^1 f = \frac{1}{2}(x\sqrt{1 - x^2} + \sin^{-1} x)|_0^1 = \frac{\pi}{4}$.

Ex.59(b) Evaluate the limit: $\lim\limits_{n\to\infty} \frac{1}{n}[(n + 1)(n + 2)\cdots(n + n)]^{\frac{1}{n}}$
Solution: For each $n \in \mathbb{N}$, let $a_n = \frac{1}{n}[(n+1)(n+2)\cdots(n+n)]^{\frac{1}{n}} = [(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{n}{n})]^{\frac{1}{n}}$ and let $f(x) = \log(1 + x)$ for all $x \in [0, 1]$. Considering the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$ of $[0, 1]$

for each $n \in \mathbb{N}$ (and taking $c_k = \frac{k}{n}$ for $k = 1, ..., n$), we find that $S(f, P_n) = \sum_{k=1}^{n} f(\frac{k}{n})(\frac{k}{n} - \frac{k-1}{n}) = \frac{1}{n}\sum_{k=1}^{n} \log(1 + \frac{k}{n})$. Since $f : [0, 1] \to \mathbb{R}$ is continuous, f is Riemann integrable on $[0, 1]$ and hence $\lim_{n\to\infty}(\log a_n) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log(1+\frac{k}{n}) = \lim_{n\to\infty} S(f, P_n) = \int_0^1 \log(1+x)\, dx = \log \frac{4}{e}$ (integrating by parts). By the continuity of the exponential function, it follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\log a_n} = e^{\log \frac{4}{e}} = \frac{4}{e}$, which is the required limit.

Ex.59(c) Evaluate the limit: $\lim_{x\to 0} \frac{x}{1-e^{x^2}} \int_0^x e^{t^2}\, dt$.

Solution: Since the function $f : [-1, 1] \to \mathbb{R}$, defined by $f(x) = e^{x^2}$ for all $x \in [-1, 1]$, is continuous, by the first fundamental theorem of calculus, $\frac{d}{dx}\int_0^x e^{t^2}\, dt = e^{x^2}$ for all $x \in [-1, 1]$. Hence $\lim_{x\to 0} \frac{x}{1-e^{x^2}} \int_0^x e^{t^2}\, dt = \lim_{x\to 0} \frac{xe^{x^2} + \int_0^x e^{t^2}\, dt}{-2xe^{x^2}}$ (applying L'Hôpital's rule) $= \lim_{x\to 0} \frac{e^{x^2}+e^{x^2}+2x^2e^{x^2}}{-2e^{x^2}-4x^2e^{x^2}}$ (applying L'Hôpital's rule again) $= -1$.

Ex.59(d) Evaluate the limit: $\lim_{n\to\infty}\left(\frac{1^8+3^8+\cdots+(2n-1)^8}{n^9}\right)$.

Solution: Let $f(x) = 2^8x^8$ for all $x \in [0, 1]$. Considering the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$ of $[0, 1]$ for each $n \in \mathbb{N}$ and observing that $c_i = \frac{2i-1}{2n} = \frac{1}{2}(\frac{i-1}{n} + \frac{i}{n}) \in [\frac{i-1}{n}, \frac{i}{n}]$ for $i = 1, ..., n$, we find that $S(f, P_n) = \sum_{i=1}^{n} f(\frac{2i-1}{2n})(\frac{i}{n} - \frac{i-1}{n}) = \frac{1}{n}\sum_{i=1}^{n}(\frac{2i-1}{n})^8$. Since $f : [0, 1] \to \mathbb{R}$ is continuous, f is Riemann integrable on $[0, 1]$ and hence $\lim_{n\to\infty}\left(\frac{1^8+3^8+\cdots+(2n-1)^8}{n^9}\right) = \lim_{n\to\infty} S(f, P_n) = \int_0^1 f(x)\, dx = \frac{2^8x^9}{9}\Big|_{x=0}^1 = \frac{256}{9}$.

Ex.60 If $f : [-1, 1] \to \mathbb{R}$ is continuously differentiable, then evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f'(\frac{k}{3n})$.

Solution: Since f' is continuous on $[0, \frac{1}{3}]$, f' is Riemann integrable on $[0, \frac{1}{3}]$ and $\int_0^{\frac{1}{3}} f'(t)\, dt = \lim_{\|P_n\|\to 0} S(f', P_n)$, where for each $n \in \mathbb{N}$, $P_n = \{0, \frac{1}{3n}, \frac{2}{3n}, ..., \frac{n}{3n} = \frac{1}{3}\}$ is a partition of $[0, \frac{1}{3}]$ and $S(f', P_n) = \sum_{k=1}^{n}(\frac{k}{3n} - \frac{k-1}{3n})f'(\frac{k}{3n}) = \frac{1}{3n}\sum_{k=1}^{n} f'(\frac{k}{3n})$ (taking $c_k = \frac{k}{3n} \in [\frac{k-1}{3n}, \frac{k}{3n}]$ for $k = 1, ..., n$). So $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f'(\frac{k}{3n}) = 3\int_0^{\frac{1}{3}} f'(t)\, dt = 3[f(\frac{1}{3}) - f(0)]$.

Ex.61(a) Show that $\frac{\pi^2}{9} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx \le \frac{2\pi^2}{9}$.

Solution: Let $f(x) = \frac{x}{\sin x}$ for all $x \in (0, \frac{\pi}{2}]$. Then $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$ for all $x \in (0, \frac{\pi}{2}]$. If $g(x) = \sin x - x\cos x$ for all $x \in [0, \frac{\pi}{2}]$, then $g'(x) = x\sin x \ge 0$ for all $x \in [0, \frac{\pi}{2}]$ and so g is increasing on $[0, \frac{\pi}{2}]$. Hence for all $x \in [0, \frac{\pi}{2}]$, $g(x) \ge g(0) = 0$ and consequently $f'(x) \ge 0$ for all $x \in (0, \frac{\pi}{2}]$. Therefore f is increasing on $(0, \frac{\pi}{2}]$ and so $\frac{\pi}{3} = f(\frac{\pi}{6}) \le f(x) \le f(\frac{\pi}{2}) = \frac{\pi}{2}$. Since f is continuous on $[\frac{\pi}{6}, \frac{\pi}{2}]$, f is Riemann integrable on $[\frac{\pi}{6}, \frac{\pi}{2}]$ and therefore $\frac{\pi^2}{9} = \frac{\pi}{3}(\frac{\pi}{2} - \frac{\pi}{6}) \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx \le \frac{\pi}{2}(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{\pi^2}{6} \le \frac{2\pi^2}{9}$.

Ex.61(b) Show that $\frac{\sqrt{3}}{8} \le \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x}\, dx \le \frac{\sqrt{2}}{6}$.

Solution: Let $f(x) = \frac{\sin x}{x}$ for all $x \in (0, \frac{\pi}{2}]$. Then $f'(x) = \frac{x\cos x - \sin x}{x^2}$ for all $x \in (0, \frac{\pi}{2}]$. If

$g(x) = x\cos x - \sin x$ for all $x \in [0, \frac{\pi}{2}]$, then $g'(x) = -x\sin x \leq 0$ for all $x \in [0, \frac{\pi}{2}]$ and so g is decreasing on $[0, \frac{\pi}{2}]$. Hence for all $x \in [0, \frac{\pi}{2}]$, $g(x) \leq g(0) = 0$ and consequently $f'(x) \leq 0$ for all $x \in (0, \frac{\pi}{2}]$. Therefore f is decreasing on $(0, \frac{\pi}{2}]$ and so $\frac{3\sqrt{3}}{2\pi} = f(\frac{\pi}{3}) \leq f(x) \leq f(\frac{\pi}{4}) = \frac{2\sqrt{2}}{\pi}$. Since f is continuous on $[\frac{\pi}{4}, \frac{\pi}{3}]$, f is Riemann integrable on $[\frac{\pi}{4}, \frac{\pi}{3}]$ and therefore $\frac{\sqrt{3}}{8} = \frac{3\sqrt{3}}{2\pi}(\frac{\pi}{3} - \frac{\pi}{4}) \leq \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x}\, dx \leq \frac{2\sqrt{2}}{\pi}(\frac{\pi}{3} - \frac{\pi}{4}) = \frac{\sqrt{2}}{6}$.

Ex.62 If $f : [a, b] \to \mathbb{R}$ is continuous, then show that there exists $c \in [a, b]$ such that $\int\limits_a^b f(x)\, dx = (b - a)f(c)$.
(This result is called the mean value theorem of Riemann integrals.)
Solution: Since f is continuous on $[a, b]$, f is Riemann integrable on $[a, b]$ and so $m(b - a) \leq \int\limits_a^b f(x)\, dx \leq M(b - a)$, where $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$. Since f is continuous on $[a, b]$, there exist $\alpha, \beta \in [a, b]$ such that $f(\alpha) = m$ and $f(\beta) = M$. Hence $f(\alpha) \leq \frac{\int\limits_a^b f(x)\, dx}{b-a} \leq f(\beta)$. By the intermediate value property of continuous functions, there exists c between α and β (both inclusive) such that $f(c) = \frac{\int\limits_a^b f(x)\, dx}{b-a}$, *i.e.* $\int\limits_a^b f(x)\, dx = (b - a)f(c)$.

Ex.63 Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous and let $g(x) \geq 0$ for all $x \in [a, b]$. Show that there exists $c \in [a, b]$ such that $\int\limits_a^b f(x)g(x)\, dx = f(c)\int\limits_a^b g(x)\, dx$.
(This result is called the generalized mean value theorem of Riemann integrals.)
Solution: Since f is continuous on $[a, b]$, f is bounded on $[a, b]$ and there exist $\alpha, \beta \in [a, b]$ such that $f(\alpha) = \inf\{f(x) : x \in [a, b]\}$ and $f(\beta) = \sup\{f(x) : x \in [a, b]\}$. We have $f(\alpha) \leq f(x) \leq f(\beta)$ for all $x \in [a, b] \Rightarrow f(\alpha)g(x) \leq f(x)g(x) \leq f(\beta)g(x)$ for all $x \in [a, b]$ (since $g(x) \geq 0$ for all $x \in [a, b]$). Since f, g are continuous on $[a, b]$, g, fg are Riemann integrable on $[a, b]$ and hence we obtain $f(\alpha)\int\limits_a^b g(x)\, dx \leq \int\limits_a^b f(x)g(x)\, dx \leq f(\beta)\int\limits_a^b g(x)\, dx$. If $\int\limits_a^b g(x)\, dx = 0$, then $\int\limits_a^b f(x)g(x)\, dx = 0$ and so we can choose any $c \in [a, b]$. If $\int\limits_a^b g(x)\, dx \neq 0$, then $\int\limits_a^b g(x)\, dx > 0$ and hence we get $f(\alpha) \leq \frac{\int\limits_a^b f(x)g(x)\, dx}{\int\limits_a^b g(x)\, dx} \leq f(\beta)$. By the intermediate value property of the continuous function f, there exists c between α and β (both inclusive) such that $f(c) = \frac{\int\limits_a^b f(x)g(x)\, dx}{\int\limits_a^b g(x)\, dx}$, *i.e.* $\int\limits_a^b f(x)g(x)\, dx = f(c)\int\limits_a^b g(x)\, dx$.

Ex.64 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $g(x) = \int\limits_0^x (x - t)f(t)\, dt$ for all $x \in \mathbb{R}$. Show that $g''(x) = f(x)$ for all $x \in \mathbb{R}$.
Solution: We have $g(x) = x\int\limits_0^x f(t)\, dt - \int\limits_0^x tf(t)\, dt$ for all $x \in \mathbb{R}$. Since f is continuous, by the first fundamental theorem of calculus, $g : \mathbb{R} \to \mathbb{R}$ is differentiable and $g'(x) = \int\limits_0^x f(t)\, dt + xf(x) - xf(x) = \int\limits_0^x f(t)\, dt$ for all $x \in \mathbb{R}$. Again, since f is continuous, by the first fundamental theorem of calculus, $g' : \mathbb{R} \to \mathbb{R}$ is differentiable and $g''(x) = f(x)$ for all $x \in \mathbb{R}$.

Ex.65 Let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 < x \leq 2, \end{cases}$ and let $F(x) = \int\limits_0^x f(t)\, dt$ for all $x \in [0, 2]$.

Is $F : [0,2] \to \mathbb{R}$ differentiable? Justify.

Solution: We have $F(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } 1 < x \le 2. \end{cases}$

Since $\lim\limits_{x\to 1-} \frac{F(x)-F(1)}{x-1} = 1 \neq 0 = \lim\limits_{x\to 1+} \frac{F(x)-F(1)}{x-1}$, F is not differentiable at 1 and hence $F : [0,2] \to \mathbb{R}$ is not differentiable.

Ex.66 If $f : [0,1] \to [0,1]$ is continuous, then show that the equation $2x - \int\limits_0^x f(t)\,dt = 1$ has exactly one root in $[0,1]$.

Solution: Let $g(x) = 2x - \int\limits_0^x f(t)\,dt - 1$ for all $x \in [0,1]$. Since f is continuous, by the first fundamental theorem of calculus, $g : [0,1] \to \mathbb{R}$ is differentiable and $g'(x) = 2 - f(x) > 0$ for all $x \in [0,1]$ (since $f(x) \le 1$ for all $x \in [0,1]$). As a consequence of Rolle's theorem, the equation $g(x) = 0$ has at most one root in $[0,1]$. Again, $g(0) = -1 < 0$ and $g(1) = 1 - \int\limits_0^1 f(t)\,dt \ge 0$ (since $f(t) \le 1$ for all $t \in [0,1] \Rightarrow \int\limits_0^1 f(t)\,dt \le 1$). If $g(1) = 0$, then 1 is the only root of the given equation in $[0,1]$. Otherwise $g(1) > 0$ and hence by the intermediate value property of the continuous function g, the equation $g(x) = 0$ has at least one root in (0,1). Thus the given equation has exactly one root in $[0,1]$.

Ex.67(a) Examine whether the improper integral $\int\limits_0^\infty e^{-t^2}\,dt$ is convergent.

Solution: Since $\int\limits_0^1 e^{-t^2}\,dt$ exists (in $\mathbb{R}$) as a Riemann integral, $\int\limits_0^\infty e^{-t^2}\,dt$ converges iff $\int\limits_1^\infty e^{-t^2}\,dt$ converges. Now $0 < e^{-t^2} \le e^{-t}$ for all $t \ge 1$. Also, since $\lim\limits_{x\to\infty} \int\limits_1^x e^{-t}\,dt = \lim\limits_{x\to\infty} (e^{-1} - e^{-x}) = e^{-1}$, $\int\limits_1^\infty e^{-t}\,dt$ converges. Hence by the comparison test, $\int\limits_1^\infty e^{-t^2}\,dt$ converges. By our remark at the beginning, $\int\limits_0^\infty e^{-t^2}\,dt$ is convergent.

Ex.67(b) Examine whether the improper integral $\int\limits_{-\infty}^\infty te^{-t^2}\,dt$ is convergent.

Solution: Since $\lim\limits_{x\to\infty} \int\limits_0^x te^{-t^2}\,dt = -\frac{1}{2} \lim\limits_{x\to\infty} e^{-t^2}|_0^x = \frac{1}{2} \lim\limits_{x\to\infty} (1 - e^{-x^2}) = \frac{1}{2}$, $\int\limits_0^\infty te^{-t^2}\,dt$ is convergent. Again, since $\lim\limits_{x\to-\infty} \int\limits_x^0 te^{-t^2}\,dt = -\frac{1}{2} \lim\limits_{x\to-\infty} e^{-t^2}|_x^0 = \frac{1}{2} \lim\limits_{x\to-\infty} (e^{-x^2} - 1) = -\frac{1}{2}$, $\int\limits_{-\infty}^0 te^{-t^2}\,dt$ is convergent. Therefore the given integral is convergent.

Ex.67(c) Examine whether the improper integral $\int\limits_0^1 \frac{dt}{\sqrt{t-t^2}}$ is convergent.

Solution: The given integral is convergent iff both $\int\limits_0^{\frac{1}{2}} \frac{dt}{\sqrt{t-t^2}}$ and $\int\limits_{\frac{1}{2}}^1 \frac{dt}{\sqrt{t-t^2}}$ are convergent. Let $f(t) = \frac{1}{\sqrt{t(1-t)}}$, $g(t) = \frac{1}{\sqrt{t}}$ and $h(t) = \frac{1}{\sqrt{1-t}}$ for all $t \in (0,1)$. Then $\lim\limits_{t\to 0+} \frac{f(t)}{g(t)} = \lim\limits_{t\to 0+} \frac{1}{\sqrt{1-t}} = 1$ and $\lim\limits_{t\to 1-} \frac{f(t)}{h(t)} = \lim\limits_{t\to 1-} \frac{1}{\sqrt{t}} = 1$. Since $\int\limits_0^{\frac{1}{2}} g(t)\,dt$ and $\int\limits_{\frac{1}{2}}^1 h(t)\,dt$ are convergent, by the limit comparison test, $\int\limits_0^{\frac{1}{2}} f(t)\,dt$ and $\int\limits_{\frac{1}{2}}^1 f(t)\,dt$ are convergent. Therefore the given integral is convergent.

Ex.68 Determine all real values of p for which the integral $\int_1^{\infty} t^p e^{-t}\,dt$ converges.

Solution: Let $p \in \mathbb{R}$ and let $f(t) = t^p e^{-t}$, $g(t) = \frac{1}{t^{[p]+2-p}}$ for all $t \geq 1$. Then $\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{t^{[p]+2}}{e^t} = 0$ (using L'Hôpital's rule $[p]+2$ times). Since $[p]+2-p > 1$, $\int_1^{\infty} g(t)\,dt$ converges and hence by the limit comparison test, $\int_1^{\infty} f(t)\,dt$ converges. Thus the given integral converges for all $p \in \mathbb{R}$.

Alternative solution: Let $p \in \mathbb{R}$ and let $f(t) = t^p e^{-t}$, $g(t) = \frac{1}{t^2}$ for all $t \geq 1$. Then $\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{t^{P-2}}{e^t} = 0$ (for $p > 2$, we use L'Hôpital's rule n times, where n is the least positive integer $\geq p-2$). Since $\int_1^{\infty} g(t)\,dt$ converges, by the limit comparison test, $\int_1^{\infty} f(t)\,dt$ converges. Thus the given integral converges for all $p \in \mathbb{R}$.

Ex.69 Find the area of the region enclosed by the curve $y = \sqrt{|x+1|}$ and the line $5y = x+7$.
Solution: Solving the equation $\frac{1}{5}(x+7) = \sqrt{x+1}$ for $x \geq -1$ and the equation $\frac{1}{5}(x+7) = \sqrt{-(x+1)}$ for $x < -1$, the x-coordinates of the points of intersection of the curve $y = \sqrt{|x+1|}$ and the line $5y = x+7$ are found to be -2, 3 and 8. Hence the required area is
$\int_{-2}^{-1} (\frac{x+7}{5} - \sqrt{-(x+1)})\,dx + \int_{-1}^{3} (\frac{x+7}{5} - \sqrt{x+1})\,dx + \int_{3}^{8} (\sqrt{x+1} - \frac{x+7}{5})\,dx = \frac{5}{3}$.

Ex.70 The region bounded by the parabola $y = x^2+1$ and the line $y = x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.
Solution: Solving $y = x^2+1$ and $y = x+3$, we obtain the x-coordinates of the points of intersection of the given parabola and the line as -1 and 2. Hence the required volume is
$\int_{-1}^{2} \pi((x+3)^2 - (x^2+1)^2)\,dx = \frac{117}{5}\pi$.

Ex.71 The region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ (where $a > 0$) is revolved about the x-axis to generate a solid. Find the volume of the solid.
Solution: Solving $y^2 = 4ax$ and $x^2 = 4ay$, we obtain the x-coordinates of the points of intersections of the two parabolas as 0 and $4a$. Hence the required volume is $\int_0^{4a} \pi(4ax - \frac{x^4}{16a^2})\,dx = \frac{96}{5}\pi a^3$.

Ex.72 Find the area of the region that is inside the circle $r = 2\cos\theta$ and outside the cardioid $r = 2(1-\cos\theta)$.
Solution: The given circle and the cardioid meet at three points corresponding to $\theta = 0$, $\theta = \frac{\pi}{3}$ and $\theta = -\frac{\pi}{3}$. By symmetry, the required area is $2\left(\frac{1}{2}\int_0^{\pi/3} 4\cos^2\theta\,d\theta - \frac{1}{2}\int_0^{\pi/3} 4(1-\cos\theta)^2\,d\theta\right) = 4(\sqrt{3} - \frac{\pi}{3})$.

Ex.73 Find the area of the region which is inside both the cardioids $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$, where $a > 0$.
Solution: The cardioids meet at three points corresponding to $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$. By symmetry, the required area is $4\int_0^{\pi/2} \frac{1}{2}a^2(1-\cos\theta)^2\,d\theta = \frac{1}{2}a^2(3\pi - 8)$.

Ex.74 Consider the funnel formed by revolving the curve $y = \frac{1}{x}$ about the x-axis, between $x = 1$ and $x = a$, where $a > 1$. If V_a and S_a denote respectively the volume and the surface area of the funnel, then show that $\lim_{a\to\infty} V_a = \pi$ and $\lim_{a\to\infty} S_a = \infty$.

Solution: For each $a > 1$, we have $V_a = \int\limits_1^a \frac{\pi}{x^2}\, dx = \pi(1 - \frac{1}{a})$ and $S_a = \int\limits_1^a \frac{2\pi}{x}\sqrt{1 + \frac{1}{x^2}}\, dx \geq \int\limits_1^a \frac{2\pi}{x}\, dx = 2\pi \log a$. Hence $\lim\limits_{a\to\infty} V_a = \pi$ and since $\lim\limits_{a\to\infty} \log a = \infty$, we get $\lim\limits_{a\to\infty} S_a = \infty$.