Monte Carlo Simulation (MA323)
A Model Report
Quiz I

1. (a) Let X be a random variable that denotes the amount of rain on a given day. Let Y be another random variable defined as follows.

$$Y = \begin{cases} 1 & \text{if there is rain on the day} \\ 0 & \text{if there is no rain on the day.} \end{cases}$$

Then, for $x \geq 0$,

$$\begin{aligned} P\left(X \leq x\right) &= P\left(X \leq x | Y = 1\right) P\left(Y = 1\right) + P\left(X \leq x | Y = 0\right) P\left(Y = 0\right) \\ &= \frac{1}{5}\left(1 - e^{-2x}\right) + \frac{4}{5}. \end{aligned}$$

Thus, the cumulative distribution function of X is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \frac{1}{5}e^{-2x} & \text{if } x \geq 0. \end{cases}$$

Clearly, $F(\cdot)$ has a jump at $x = 0$ and size of the jump is $\frac{4}{5}$ and $F(\cdot)$ is continuous at all $x \neq 0$. Therefore, the following algorithm can be used to generate a sample from the cumulative distribution function $F(\cdot)$.

Generate U from $U(0,\, 1)$ distribution.
if $U \leq \frac{4}{5}$ **then**
    Set $X = 0$.
else
    Set $X = -\frac{1}{2}\left(\ln 5 + \ln(1 - u)\right)$.
end if
Return X.

The above algorithm is implemented in R software and 1000 random numbers are generated. The seed is 123. The histogram of the generated random numbers are given in the Figure 1.

(b) Based on the random numbers generated in the previous part, the average of amount of rain is 0.0956.

2. For $(x,\, y) \in \mathbb{R}^2$, the joint probability density function of X and Y is

$$\begin{aligned} f(x, y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-2x-1}{2}\right)^2} \\ &= \frac{1}{4\pi} e^{-g(x,\, y)}, \end{aligned}$$

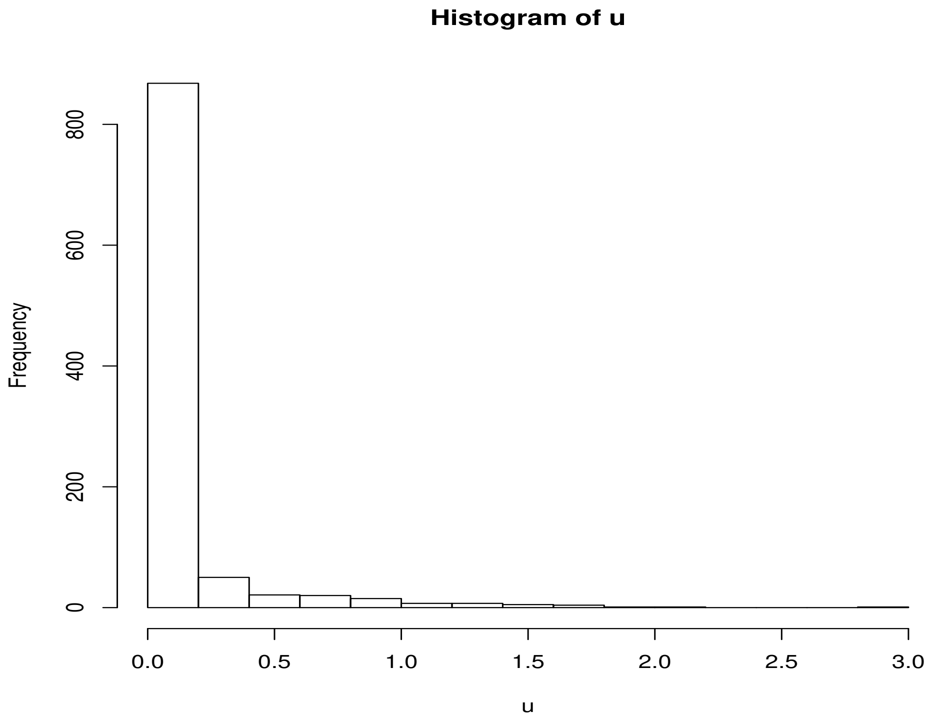

Figure 1: Histogram of Generated Random Numbers

where $g(x,\, y) = x^2 + \left(\frac{y-1}{2\sqrt{2}}\right)^2 - \frac{2\times\frac{1}{\sqrt{2}}}{2\sqrt{2}} x(y-1)$. Thus, $(X,\, Y) \sim N_2\,(\boldsymbol{\mu},\, \Sigma)$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \Sigma = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}.$$

Now, we can use the following algorithm to generate random numbers from the joint distribution of X and Y.

1: Generate Z_1 and Z_2 from $N(0,\, 1)$ distribution using Box-Muller method.
2: Set $X = Z_1$.
3: Set $Y = 1 + 2Z_1 + 2Z_2$ (using Cholesky Factorization).
4: Return $(X,\, Y)$.

The above algorithm is implemented in R software with seed 123. Based on the generated random numbers, the sample means of X and Y are 0.019 and 0.98, respectively. The sample variances of X and Y are 0.945 and 7.696, respectively. The sample correlation coefficient is 0.688.

3. (a) As $1 + \frac{x^2}{3.2} > 1$ for all $x \in \mathbb{R}$,

$$\begin{aligned} &\left(1 + \frac{x^2}{3.2}\right)^{-2.1} \le \left(1 + \frac{x^2}{3.2}\right)^{-1} \\ \implies &\frac{\Gamma(2.1)}{\sqrt{3.2\pi}\Gamma(1.6)}\left(1 + \frac{x^2}{3.2}\right)^{-2.1} \le \frac{\Gamma(2.1)\sqrt{3.2\pi}}{\sqrt{3.2\pi}\Gamma(1.6)}\frac{1}{\sqrt{3.2\pi}}\left(1 + \frac{x^2}{3.2}\right)^{-1} \\ \implies &f(x) \le cg(x) \end{aligned}$$

for all $x \in \mathbb{R}$, where $c = \frac{\sqrt{\pi}\Gamma(2.1)}{\Gamma(1.6)}$ and $g(x) = \frac{1}{\sqrt{3.2}\pi\left(1+\frac{x^2}{3.2}\right)}$. Now, to generate from $g(\cdot)$, note that the corresponding cumulative distribution function is

$$G(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{\sqrt{3.2}}\right) \quad \text{for all } x \in \mathbb{R}.$$

Therefore, $G^{-1}(\cdot)$ is given by

$$G^{-1}(u) = \sqrt{3.2}\tan\left[\pi\left(u - \frac{1}{2}\right)\right] \quad \text{for all } u \in (0,\, 1).$$

Hence, we can use the following algorithm to generate random numbers from $f(\cdot)$.

repeat
    Generate U_1 from $U(0,\, 1)$.
    Set $X = \sqrt{3.2}\tan\left[\pi\left(U_1 - \frac{1}{2}\right)\right]$.
    Generate U_2 from $U(0,\, 1)$.
until $U_2 \leq \left(1 + \frac{X^2}{3.2}\right)^{-1.1}$
Return X.

(b) The acceptance probability is $\frac{1}{c} = \frac{\Gamma(1.6)}{\sqrt{\pi}\Gamma(2.1)} \simeq 0.482$.

(c) The proportion of acceptance based on 1000 random numbers generated using previous algorithm (implemented in R software) with seed 123 is 0.478.