$c^T x \leq \alpha$ $\quad a_1 x_1 + \dots + c_n x_n \leq 5$

$\{x \mid a^T x \leq 1, \forall a \in P\} = P^*$

$a^T x = \langle a, x \rangle$

6 Lecture 6

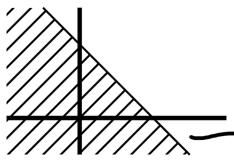
[6.1] **Theorem** A nonempty bounded polyhedron K is a polytope.

Proof. As K is closed, bounded and convex, by [5.3], $K = \mathsf{conv}(E)$, where E is the set of vertices of K. By [5.33] and [5.24], E is nonempty and finite. Hence K is a polytope. ∎

[6.2] **Definition** Let $\emptyset \neq P \subseteq \mathbb{R}^n$. We call $\boldsymbol{P^*} := \{x \mid a^t x \leq 1, \forall a \in P\}$ the POLAR DUAL or simply the DUAL of P. We simply write a^* to mean $\{a\}^*$.

[6.3] **Example** Take $a = [1 \quad 1]^t$. Then $a^* = \{x \mid a^t x \leq 1\}$ is the half-space $x + y \leq 1$. It is unbounded.

$x_1 + x_2 \leq 1$



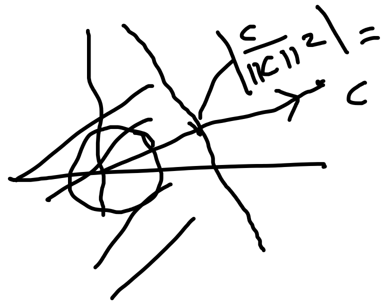
$\to x_1 + x_2 = 1$

[6.4] **Fact** If $0 \neq c \in \mathbb{R}^n$, then $c^t y = 1$ is a hyperplane that passes through $\frac{c}{\|c\|^2}$ (the vector along c of norm $1/\|c\|$) and c^* is the halfspace that contains 0.‼ This is why some people call the polar dual the RECIPROCAL DUAL.

$c^* =$ $\quad c^t \frac{c}{\|c\|^2} = 1$

$0 \in (c^*)^0$

$\left\| \frac{c}{\|c\|^2} \right\| = \frac{1}{\|c\|}$

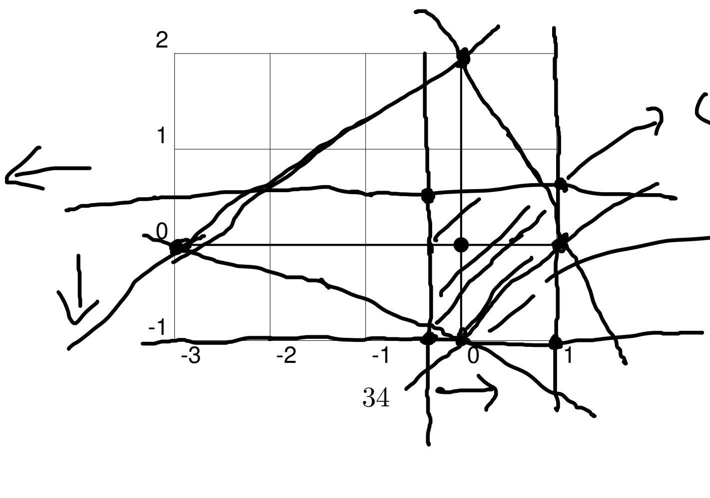


[6.5] **Fact** For any nonzero $c \in \mathbb{R}^n$, the dual c^* always has 0 in its interior. Is $B_\delta(0)$, $\delta = 1/\|c\|$, in the interior?‼ Hence intersection of finitely many of them will also contain 0 in its interior.

[6.6] **Fact** Let S be nonempty. Then $S^* = \bigcap_{a \in S} a^*$. That is, S^* is the intersection of the duals of each point in S. Follows from the definition.

[6.7] **Fact** If $A \subseteq B$, then $B^* \subseteq A^*$. Follows from the definition.

[6.8] **Example** Take $S = \{e_1, 2e_2, -3e_1, -e_2\}$. Draw S^* and $(S^*)^*$.



$\to (1, \frac{1}{2}) \to x_1 + \frac{x_2}{2} = 1$

$\to S^*$

$S^* = \mathrm{conv}(S)^*$ $\qquad$ $S \subseteq \mathrm{conv}(S)$

$\mathrm{conv}(S)^* \subseteq S^*$

[6.9] **Lemma** `(Dual of a nonempty set is the dual of its convex hull)` Let $\emptyset \neq S \subseteq \mathbb{R}^n$. Then $\mathsf{conv}(S)^* = S^*$. In particular, dual of a polytope is a polyhedron.

$x \in S^*$. To show $x \in \mathrm{conv}(S)^*$.

is $x^T y \le 1$ $\quad \forall y \in \mathrm{conv}(S)$

Let $y = \lambda_1 s_1 + \cdots + \lambda_k s_k$

$x^T(\lambda_1 s_1 + \cdots + \lambda_k s_k)$

$= \lambda_1 x^T s_1 + \cdots + \lambda_k x^T s_k$

$\le \lambda_1 + \cdots + \lambda_k = 1$ ✓

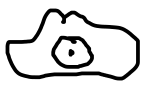
Proof. As $S \subseteq \mathsf{conv}(S)$, we have $\mathsf{conv}(S)^* \subseteq S^*$. Now let $z \in S^*$. We show that $z \in x^*$ for each $x \in \mathsf{conv}(S)$. Towards that, let $x = \sum \lambda_i s_i \in \mathsf{conv}(S)$. As $z \in S^*$, we have $z^t s_i \le 1$ for each i. Hence

$$z^t x = z^t(\sum \lambda_i s_i) = \sum \lambda_i (z^t s_i) \le 1$$

and so $z \in x^*$. Hence $S^* \subseteq \mathsf{conv}(S)^*$. The next statement is immediate. ∎

[6.10] **Lemma** `(Dual of set with zero as an interior point is bounded.)` Let $P \subseteq \mathbb{R}^n$ such that $B_\epsilon(0) \subseteq P$. Then P^* is bounded.

Proof. Take $0 \neq x \in P^*$. As $\frac{\epsilon x}{2\|x\|} \in B_\epsilon(0) \subseteq P$, we have $\frac{\epsilon}{2\|x\|} x^t x \le 1$. That is, $\|x\| \le \frac{2}{\epsilon}$. ∎

✓ $B_\varepsilon(0) \subseteq$  $S \Rightarrow S^*$ bounded. Take $x \in S^*$.

$\langle x, \frac{\varepsilon}{2}\frac{x}{\|x\|}\rangle \le 1 \Rightarrow \frac{\varepsilon}{2}\|x\| \le 1 \Rightarrow \|x\| \le \frac{2}{\varepsilon}$

[6.11] **Lemma** If a polytope P contains $B_r(0)$, then P^* is a polytope. Also P^{**} is a polytope and $P^{**} = P$.

$B_r(0) \subseteq P \Rightarrow P^*$ is a bounded polyhedron $\Rightarrow P^*$ is a polytope.

As $0 \in (P^*)^\circ$, $\Rightarrow (P^*)^*$ is a polytope. $\quad P = P^{**}$

Let $a \in P$. $\langle a, x\rangle \le 1 \ \forall x \in P^* \Rightarrow a \in P^{**} \to P \subseteq P^{**}$.

$b \in P^{**} \setminus P \Rightarrow \Leftarrow$

$\exists c \neq 0$ s.t.

$c^T b > \alpha$, $c^T x < \alpha \ \forall x \in P$

$0 < \alpha$

$\frac{c}{\alpha} \in P^*$ $\qquad$ $b \notin P^{**}$ $\qquad$ $(\frac{c}{\alpha})^T b > 1$, $(\frac{c}{\alpha})^T x < 1 \ \forall x \in P$

Proof. Let $P = \mathsf{conv}\{x_1, \cdots, x_t\} \subseteq \mathbb{R}^n$ with $B_r(0) \subseteq P$. Then P^* is a bounded polyhedron, hence a polytope. Again, as P^* also has 0 in its interior, we see that P^{**} is a polytope.

To show $P \subseteq P^{**}$, take $a \in P$. So $a^t w \le 1$ for each $w \in P^*$. That is, $a \in P^{**}$.

To show the other inclusion, let $a \in P^{**} \setminus P$. As $a \notin P$ and P is a closed convex set, by the strict separation, $\exists b \neq 0$ and α such that $b^t a > \alpha$ and $b^t z < \alpha$, $\forall z \in P$. As $0 \in P$, we have $\alpha > 0$. Put $d = \frac{b}{\alpha}$ to get $d^t a > 1$ and $d^t z \le 1$, $\forall z \in P$. This means, $d \in P^*$ and $a^t d > 1$. So a cannot be in P^{**}, a contradiction. ∎

[6.12] Corollary

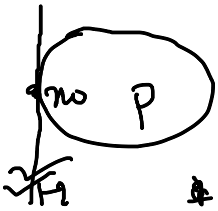
a) A polytope with an interior point 0 is a bounded polyhedron. (As P^{**} is a polyhedron.)

b) A polytope with an interior point is a bounded polyhedron. (Follows from translation.)

[6.13] Theorem Every polytope (whether it has an interior point or not) in $\mathbb{R}^n$ is a bounded polyhedron.

P does not have an int pt $\Rightarrow$ P is contained in a hyp plane.

$P = \text{conv}(x_1, x_2, \ldots, x_k)$. consider $x_0 = \frac{x_1 + x_2 + \cdots + x_k}{k} \in P$



$c^T x_0 = \alpha$ and $c^T x \geq \alpha \ \forall x \in P$

$\frac{c^T x_1}{k} + \cdots + \frac{c^T x_k}{k} = \alpha \Rightarrow c^T x_1 = \alpha, \ c^T x_k = \alpha$

Proof. Let $P = \mathsf{conv}(x_1, \ldots, x_t)$. If P is contained in a hyperplane of $\mathbb{R}^n$, we are done by induction. So, assume also that P is not contained in any hyperplane.

In this case the idea is to find an interior point. We claim that $x_0 := \frac{x_1 + \cdots + x_t}{t} \in P^\circ$.

Note that x_0 is already in P. So, if the claim is not true, then $x_0 \in \partial P$. So there is a hyperplane $H : c^t x = \beta$ supporting P positively at x_0. As

$$\beta = c^t x_0 = c^t \left[\frac{x_1}{t} + \frac{x_2}{t} + \cdots + \frac{x_t}{t} \right] \geq \beta,$$

we see that $c^t x_1 = \cdots = c^t x_t = \beta$. That is, $x_1, \ldots, x_t$ lie on H. Hence P lies on H. A contradiction. ∎

Some exercises

[6.14] Exercise(E) Let $P = \mathsf{conv}(0, e_1, e_2, e_3, e_1 + e_2 + e_3) \subseteq \mathbb{R}^3$. If $P^* = \{x \mid Ax \leq b\}$, then write A and b.

[6.15] Exercise(E) Let $P = \mathsf{conv}(0, e_1, e_2, e_3, e_1 + e_2 + e_3) \subseteq \mathbb{R}^3$. We know that each polytope is a bounded polyhedron, that is, we can write $P = \{x \mid Ax \leq b\}$ for some A and b. Write A and b.

[6.16] Exercise(E) `(Some interesting pictures of the dual)` a) Take $P = \{\alpha e_1 \mid \alpha \in \mathbb{R}\}$. Show that $P^* = P^\perp$.

b) Let $P = \overline{B_1(0)} \subseteq \mathbb{R}^2$ in $\|\cdot\|_1$. Draw P^* and notice that $P^* = \overline{B_1(0)}$ in $\|\cdot\|_\infty$. Draw P^{**} and notice that $P^{**} = P$.

c) Take $P = \overline{B_1(0)} \subseteq \mathbb{R}^2$ in $\|\cdot\|_2$. Draw P^*.

[6.17] Exercise(E) `(Vertices of` P^*`.)` Let $P = \mathrm{C}(e_1 - e_2, 2e_1, e_1 + e_2, -e_1) \subseteq \mathbb{R}^2$. Draw the dual P^* and write the vertices of P^*.

[6.18] Exercise(M) `(Converse of a result)` a) I have a set $P \subseteq \mathbb{R}^2$ for which P^* is bounded. Must P contain 0 as interior point?

b) I have a closed convex set $P \subseteq \mathbb{R}^2$ for which P^* is bounded. Must P contain 0 as an interior point?

[6.19] **Exercise(E)** Let $P = \mathsf{conv}\{x_1, \cdots, x_t\}$ be a polytope in $\mathbb{R}^n$ which is not contained in any hyperplane of $\mathbb{R}^n$. Then show that $\{x_2 - x_1, \cdots, x_t - x_1\}$ span $\mathbb{R}^n$.

[6.20] **Exercise(E)** `(`$P = P^{**}$ `can hold for other sets too.)` In $\mathbb{R}^2$ consider the set $S = \{x \mid x_1 \geq -1\}$. Let $y \in S^*$. Argue that $y(2)$ must be 0, as $\pm\alpha e_2 \in S$ for each $\alpha > 0$. Argue that $y(1)$ cannot be a positive number, as $\alpha e_1 \in S$ for each $\alpha > 0$. Argue that $y(1) \geq -1$. Conclude that $S^* = \{y \mid y(2) = 0, -1 \leq y(1) \leq 0\}$. Draw S^{**}.

[6.21] **NoPen** a) T/F? If S is a set of 50 points in $\mathbb{R}^2$ with positive y-coordinates, then S^* must be unbounded.

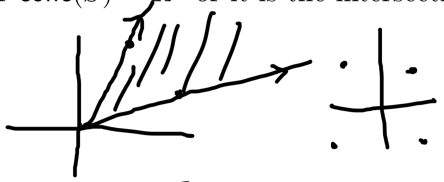
b) Is $(\mathbb{R}^3_+)^* = \mathbb{R}^3_-$?

Farka's Lemma

[6.22] **Why Farka's Lemma?** When does a nonnegative solution of $Ax = b$ exist? Farka's lemma gives an answer.

The following is a crucial result which is similar to that of [3.52]. We could not use that, as we do not know whether the convex hull contains the origin or not.

[6.23] **Lemma** `(Cone generated by a finite set is closed.)` Let $S = \{x_1, \ldots, x_k\} \subseteq \mathbb{R}^n$. Then either $\mathsf{cone}(S) = \mathbb{R}^n$ or it is the intersection of finitely many closed linear half spaces.



$x \in P = \mathrm{conv}(0, x_1, x_2, \ldots, x_k)$

$= \{x \mid Ax \leq b\}, \quad b \geq 0$

$K = \{x \mid Bx \leq 0\} = \mathrm{cone}(S) \ni x = \alpha\hat{x}$

$\alpha\hat{x} = \dfrac{1}{\lambda_1 + \cdots + \lambda_t}(\lambda_1 s_1 + \cdots + \lambda_t s_t)$

$A = \begin{bmatrix} B \\ C \end{bmatrix} x \leq \begin{bmatrix} 0 \\ >0 \end{bmatrix}$

$B\hat{x} \leq 0$

$B(\alpha\hat{x}) \leq 0$

$Bx \leq 0$

Proof. Take $P = \mathsf{conv}(0, x_1, \ldots, x_k)$. Being a polytope, it is a polyhedron, say, $P = \{x \mid Ax \leq b\}$. As $0 \in P$, we have $b \geq 0$. If $b > 0$ entrywise, then some $B_\epsilon(0) \subseteq P$. In that case, we get $\mathsf{cone}(S) = \mathbb{R}^n$.

Otherwise, let $\overline{A}$ be the submatrix of A obtained by taking the rows of A corresponding to the zeros in b. We claim that $\mathsf{cone}(S) = \{x \mid \overline{A}x \leq 0\}$. To see that, let $z \in \mathsf{cone}(S)$, $z \neq 0$. Then $\alpha z \in P$, for some $\alpha > 0$. So, $\overline{A}\alpha z \leq 0$ and hence $\overline{A}z \leq 0$. Conversely, let z satisfy $\overline{A}z \leq 0$. Choose $r > 0$, so that $A(z/r) \leq b$. (Why is this possible?) Thus $z/r \in P$ and so $z \in \mathsf{cone}(P) = \mathsf{cone}(S)$. ∎

Before we go to Farka's Lemma we introduce one more notation.

[6.24] **Definition** For a matrix A we use $\mathbf{conv}(\boldsymbol{A})$ and $\mathbf{cone}(\boldsymbol{A})$ to denote the convex hull and the convex cone generated by the columns of A, respectively.

[6.25] **Farka's Lemma** Let $A \in M_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. Then the following are equivalent.

$Ax = b$ has a nonnegative soln iff

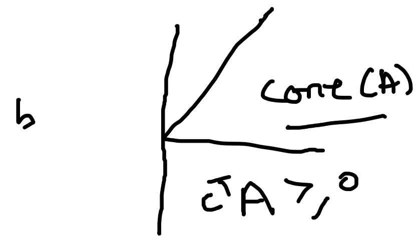
a) The system $Ax = b$ has a nonnegative solution.

b) For each $z \in \mathbb{R}^m$ satisfying $z^t A \geq 0$ we have $z^t b \geq 0$.

$z^t A \geq 0 \Rightarrow z^t b \geq 0$

$A x_0 = b, \quad x_0 \geq 0. \quad z^t A \geq 0. \quad z^t b = (z^t A) x_0 \geq 0$

b) holds. a) does not. $\equiv$ $b \notin \text{cone}(A)$



$\exists\, c \neq 0$ s.t. $c^T x \geq 0 \quad \forall x \in \text{cone}(A)$ and $c^T b < 0$. $\Rightarrow\Leftarrow$

Proof. a)$\Rightarrow$b). Assume that $Ay = b$, $y \geq 0$ holds. Let $z \in \mathbb{R}^m$ such that $z^t A \geq 0$. Then $z^t b = z^t(Ay) = (z^t A)y \geq 0$.

b)$\Rightarrow$a). Suppose b) holds for each $z \in \mathbb{R}^m$ and suppose, by the way of contradiction that $Ax = b$ has no nonnegative solution. This means, $b \notin \mathsf{cone}(A)$. Since $\mathsf{cone}(A)$ is closed (see [6.23]) and convex, by [4.10] and [4.11], there exists c such that $c^t b < 0$ and $c^t z \geq 0$, $\forall z \in \mathsf{cone}(A)$. In particular, we have $c^t A \geq 0$ and $c^t b < 0$, a contradiction. ∎

Some exercises

[6.26] <u>**Exercise(E)**</u> Consider the cone generated by $\{-e_1, e_1 + e_2, e_2 + e_3, e_3\}$ in $\mathbb{R}^4$. Write this cone as intersection of finitely many closed half-spaces.

[6.27] <u>**Exercise(H)**</u> `(Dichotomies)`

a) Let A be an $m \times n$ matrix. Use separation theorems to show that exactly one of the following systems has a solution: (i) $Ax > 0$, (ii) $A^t y = 0$, $y \geq 0$, $y \neq 0$.

b) Let A be an $m \times n$ matrix. Use Farka's lemma to show that exactly one of the following systems has a solution: (i) $Ax > 0$, (ii) $A^t y = 0$, $y \geq 0$, $y \neq 0$.

c) Let A be an $m \times n$ matrix and $c \in \mathbb{R}^m$. Prove that exactly one of the following systems has a solution: (i) $Ax \leq c$, (ii) $A^t y = 0$, $c^t y = -1$, $y \geq 0$.