14 Lecture 14

Dual simplex method

[14.1] **Why? What is it?**

a) This is another method to solve the slpp. It is fundamental in nature, that is, it required in theoretical discussions.

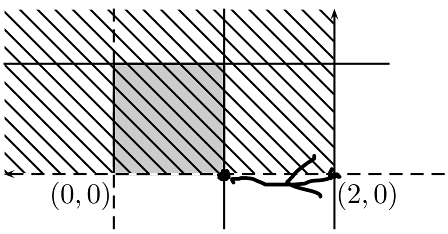
b) Geometrically, the simplex method starts at a vertex and moves to other vertices via edges, until it reaches a minimum.

c) In other words, at every stage of the simplex method, the simplex table has $\overline{b} \geq 0$, and the method generates table after table, until $\overline{c} \geq 0$.

d) Geometrically, the dual simplex method starts at a minimal corner point (at which the objective function is minimum over the affine cone) out side the feasible set and reaches a minimal vertex via a sequence of minimal corner points.

e) In other words, at every stage of the dual simplex method, the simplex table has $\overline{c} \geq 0$, and the method generates table after table, until $\overline{b} \geq 0$.

[14.2] **Discussion** Consider $\begin{array}{ll} \min & \underline{x_2} \\ \text{s.t.} & x_1 \leq 1, x_2 \leq 1, x_1 \leq 2, x_i \geq 0 \end{array}$. Notice that the constraint $x_1 \leq 2$ is not really changing the feasible set, but do not bother about it now.



a) Write the slpp.

bv	x_1	x_2	x_3	x_4	x_5	$\overline{b}$
x_3	1	0	1	0	0	1
x_4	0	1	0	1	0	1
x_5	1	0	0	0	1	2
	0	1	0	0	0	0

$(2,0) \to (2,0,-1,1,0)$

b) The function $f(x) = x_2$ is minimized at $w = (2, 0)$ over the affine cone (shaded region). However, this point is outside the feasible set. What is the corresponding solution w^* of the slpp? Is it a basic solution? For which basis?

c) Write the simplex table for this basis.

ST

bv	x_1	x_2	x_3	x_4	x_5	$\overline{b}$	
x_3	0	0	1	0	$\boxed{-1}$	-1	$\rightarrow$
x_4	0	1	0	1	0	1	
x_1	1	0	0	0	1	2	
	0	1	0	0	0	0	

$\cancel{c_1x_1 + c_2x_2 + \dots + c_5x_5 = 1}$

bv	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_5 ~~x_3~~	0	0	-1	0	$\boxed{1}$	1
x_4	0	1	0	1	0	1
x_1	1	0	0	0	1	2
	0	1	0	0	0	0

d) What does the simplex table tell us? we are at $(2, 0, -1, 1, 0)$

And at this pt $\bar{c} \geq 0$.

Plan If we can move out of this point towards the feasible set, via corner-points, at which the objective function remains minimized over their affine cones, then whichever feasible point we reach first, will be a point of minimum. In other words, we want to get another simplex table keeping $\bar{c} \geq 0$, while trying to make $\bar{b}$ nonnegative.

e) Look at the simplex table. Choose a basic variable for which the entry of $\bar{b}$ is negative.

f) Multiply that row by -1. What is the effect? Write the table to see it.

bv	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_5	0	0	-1	0	1	1
x_4	0	1	0	1	0	1
x_1	1	0	1	0	0	1
	0	1	0	0	0	0

$x = (1, 0, 0, 1, 1)$ 2

$\bar{c} \geq 0$

g) So we must get another basic variable whose column can be changed to e_1. Which variable shall I choose?

h) What if no such choice is there?

i) Prepare the next simplex table.

j) How did we move in the picture?

Answer. a) The equivalent slpp is $\min\ x_2$.

$$\text{s.t.} \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \; x_i \geq 0$$

b) The corresponding solution of the slpp is $w^* = (2, 0, -1, 1, 0)$. It is a basic solution with a basis (x_1, x_3, x_4).

c) The simplex table at the ordered basis (x_3, x_4, x_1) is

bv	x_1	x_2	x_3	x_4	x_5	$\overline{b}$
x_3	0	0	1	0	-1	-1
x_4	0	1	0	1	0	1
x_1	1	0	0	0	1	2
$-f$	0	1	0	0	0	0

d) The simplex table tells us that (look at the $\overline{c}$ row) we are at a corner point where the objective function is minimized over the affine cone. Since $\overline{b}$ has a negative entry, it means, we are outside the feasible set.

e) We choose x_3.

f) That negative entry of $\overline{b}$ changes to positive. But the basic column below x_3 which was e_1 has changed to $-e_1$.

g) We choose a variable that has positive values in that row. Here we choose x_5.

h) If no such choice is there, it means the system is inconsistent that is, the feasible set is empty. If all the entries in that row are nonpositive but the $\overline{b}$ entry is positive, it means you cannot have a nonnegative solution to satisfy the corresponding equation.

i) Next simple table is at the ordered basis (x_5, x_4, x_1):

bv	x_1	x_2	x_3	x_4	x_5	$\overline{b}$
x_5	0	0	-1	0	1	1
x_4	0	1	0	1	0	1
x_1	1	0	1	0	0	1
$-f$	0	1	0	0	0	0

j) We moved from $(2, 0)$ to $(1, 0)$.

Dual simplex method Inputs: A, b, c and a basic solution w at which $\overline{c} \geq 0$.

a) Form the initial table for w.

b) If $\overline{b} \geq 0$, then conclude that the current basic solution is a minimum.

c) If $\overline{b}_i < 0$ and $\overline{A}_{i:} \geq 0$, then conclude that the feasible set is empty.[a]

d) Otherwise find s such that $\delta = \frac{\overline{c}_s}{|\overline{a}_{is}|} = \min\limits_{a_{ir}<0} \frac{\overline{c}_r}{|\overline{a}_{ir}|}$. Then x_s is the entering variable and the variable corresponding to ith row is the outgoing one. Add δ multiple of ith row to $\overline{c}$, make $\overline{a}_{is} = 1$ and other elements in that column 0. Go to b).[b]

[a] That equation would mean a 'nonnegative amount of each variables added, gives us a negative number', not possible.
[b] As there are finitely many basic solutions, the algorithm must terminate if cycling is avoided.

[14.3] **Example** Consider $\min\ 2x_1 + x_2$ s.t. $x_1 + x_2 - x_3 = 2,\ x_2 + x_4 = 1\ x_i \geq 0.$ Write the simplex table for the basis (x_3, x_4). We employ dual simplex method to get a new simplex table and proceed.

bv	x_1	x_2	x_3	x_4	$\overline{b}$
x_3	-1	$\boxed{-1}$	1	0	-2
x_4	0	1	0	1	1
$-f$	2	1	0	0	0

$w = (0, 0, -2, 1)$

$\overline{c} \geq 0 \Rightarrow$ obj fn is min over aff cone

$\min \left\{ \frac{2}{|-1|}, \frac{1}{|-1|} \right\}$ ✓

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	1	1	-1	0	2
x_4	$\boxed{-1}$	0	1	1	-1
	1	0	1	0	-2

$w = (0, 2, 0, -1)$

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	0	1	0	1	1
x_1	1	0	-1	-1	1
$-f$	0	0	2	1	-3

$w = (1, 1, 0, 0)$

$\bar{b} \geq 0 \rightarrow$ bfs

Answer.

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_3	-1	$\boxed{-1}$	1	0	-2
x_4	0	1	0	1	1
$-f$	2	1	0	0	0

$\rightarrow$

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	1	1	-1	0	2
x_4	$\boxed{-1}$	0	1	1	-1
$-f$	1	0	1	0	-2

$\rightarrow$

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	0	1	0	1	1
x_1	1	0	-1	-1	1
$-f$	0	0	2	1	-3

Since $\bar{b} \geq 0$, the current basic solution is a bfs and it is a minimum.

Some exercises

[14.4] **Exercise(E)** Consider $\begin{array}{ll} \min & x_2 - 5x_1 \\ \text{s.t.} & x_1 \leq 2, x_2 \leq 2, x_1 + x_2 \leq 5, x_1 - x_2 \leq 3, x_i \geq 0. \end{array}$

a) Write the slpp.

b) Write the basic solution that corresponds to $x_1 = 4, x_2 = 1$ specifying the basis.

c) Apply dual simplex method starting from this solution to find a minimum.

d) Draw the feasible region and the nearby region showing the points of movement.

[14.5] **Exercise(E)** a) Suppose that we want to minimize $3x + 2y$ on $\mathsf{conv}(0, e_1, e_2, 2e_1 + e_2) \subseteq \mathbb{R}^2$. Write the corresponding lpp removing the redundant constraints. Write the corresponding slpp.

b) Draw the feasible region and highlight the point $(0, -1)$.

c) Write the simplex table at $(0, -1)$.

d) Use dual simplex method to proceed and highlight your movement in your figure.

Complexity of simplex algorithm

The simplex method has exponential complexity. However, it has been observed that in most real life problems, the simplex method is very efficient. The following is an example supporting the first statement.

[14.6] **Example** (V. Klee, and G. Minty, 1971) Consider the problem for any fixed $0 < \delta < \frac{1}{2}$.

$$\begin{array}{rl} \min & -x_n \\ \hline & x_1 \leq 1, \\ \text{s.t.} & \delta x_{i-1} \leq x_i \leq 1 - \delta x_{i-1}, \quad i = 2, \ldots, n, \\ & x_i \geq 0. \end{array}$$

Here we have a $(2n-1) \times (3n-1)$ matrix. If we start with the bfs 0 and apply simplex method, then we need $2^n - 1$ iterations, as it visits all $2^n - 1$ vertices.

[14.7] **Illustration of the previous example**

For $n = 3$, $\delta = \frac{1}{3}$, we have PT:

bv	x_1	x_2	x_3	y_1	y_2	y_3	y_4	y_5	$\overline{b}$
*	1	0	0	1	0	0	0	0	1
*	1/3	−1	0	0	1	0	0	0	0
*	1/3	1	0	0	0	1	0	0	1
*	0	1/3	−1	0	0	0	1	0	0
*	0	1/3	1	0	0	0	0	1	1
$-f$	0	0	−1	0	0	0	0	0	*

If the computer starts with the basis $(x_1, x_2, x_4, x_6, x_8)$, then the following is a possible path: $(x_1, x_2, x_4, x_6, x_8) \to (x_1, x_2, x_3, x_6, x_8) \to (x_1, x_2, x_3, x_5, x_8) \to (x_4, x_2, x_3, x_5, x_8) \to (x_4, x_2, x_3, x_5, x_7) \to (x_1, x_2, x_3, x_5, x_7) \to (x_1, x_2, x_3, x_6, x_7) \to (x_1, x_4, x_3, x_6, x_7)$. Seven iterations.

Sensitivity analysis

Consider minimizing $c^t x$ over $\{x \mid Ax = b, x \geq 0\}$. Suppose that w is a minimum bfs. Modify the problem by doing one the followings: a) add a new variable, b) add a new constraint, c) change the vector b, d) change the cost factor c. Shall we redo all the computations? Are there other ways of approach?

[14.8] **Standard example.** We shall consider this problem for the study of the above perturbations.

$$(A) \qquad \begin{array}{rl} \min & -2x_1 - 4x_2 - 3x_3 \\ \hline \text{s.t.} & x_1 + x_2 + 3x_3 \leq 4,\ 2x_1 + x_2 + x_3 \leq 3,\ x_i \geq 0 \end{array}$$

The problem table (I) and the optimal table (O) are shown below.

(I)

bv	x_1	x_2	x_3	x_4	x_5	$\overline{b}$
*	1	1	3	1	0	4
*	2	1	1	0	1	3
$-f$	−2	−4	−3	0	0	*

(O)

bv	x_1	x_2	x_3	x_4	x_5	$\overline{b}$
x_2	2	1	1	0	1	3
x_4	−1	0	2	1	−1	1
$-f$	6	0	1	0	4	12

[14.9] **Addition of a new variable y**

∘ Revised problem :
$$\begin{array}{rl} \min & c^t x + c_{n+1} y \\ \hline \text{s.t.} & \begin{bmatrix} A & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = b,\ x_i \geq 0, y \geq 0. \end{array}$$

∘ In this case the minimal bfs w is still a bfs. Continue from the optimal simplex table by adding a new column $\overline{a} = B^{-1}a$ and put the corresponding relative cost $\overline{c}_{n+1} = c_{n+1} - c_B^t B^{-1} a$. If $\overline{c}_{n+1} < 0$, then continue with simplex method, otherwise conclude that the current bfs is optimal.

bv	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_2	2	1	1	0	1	3
x_4	−1	0	2	1	−1	1
—	6	0	1	0	4	12

[14.10] **Example** Let us introduce a new variable y to the problem table in [14.8].

bv	x_1	x_2	x_3	x_4	x_5	y	$\bar{b}$
$*$	1	1	3	1	0	1	4
$*$	2	1	1	0	1	−1	3
$-f$	−2	−4	−3	0	0	1	$*$

for y at the old opt basis

bv	x_1	x_2	x_3	x_4	x_5	y	$\bar{b}$
x_2	2	1	1	0	1	−1	3
x_4	−1	0	2	1	−1	2	1
—	6	0	1	0	4	~~1~~ −3	12

$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$\to B^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$1 - \begin{bmatrix} -4 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
$= 1 - 4 = -3$

The basis matrix for the final table is $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. So $B^{-1}a = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\bar{c}_y = c_y - c_B^t B^{-1}a = 1 - \begin{bmatrix} -4 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3$.

So our revised simplex table is table is

bv	x_1	x_2	x_3	x_4	x_5	y	$\bar{b}$
x_2	2	1	1	0	1	−1	3
x_4	−1	0	2	1	−1	2	1
$-f$	6	0	1	0	4	−3	12

.

Now, continue with simplex method.

[14.11] **Changing b to b'** Start with the final bfs. The vector $\bar{c}$ is unchanged. Compute $\overline{b'} = B^{-1}b'$ and the value of the function. If $\overline{b'} \geq 0$, then we have a minimum. Otherwise, use dual simplex method.

[14.12] **Changing c to c'** Start with the final simplex table. Compute $\overline{c'}^t = c'^t - c'_B B^{-1}A$ and the value of the function. If $\overline{c'} \geq 0$, then w is a minimum. If $\overline{c'}$ has some negative entries, then use simplex method.

[14.13] Addition of a new constraint of the form $a_{m+1}^t x \le \beta$ If old minimal bfs w satisfies the new constraint, then it is the minimum bfs for the new problem, as the feasible set of the new problem is a subset of that of the old problem. Otherwise, write the new constraint as $a_{m+1}^t x + x_{n+1} = \beta$ and put it in the old final simplex table and extend the basis by adding x_{n+1} to it. Make the entries of a_{m+1}^t in the old basic columns 0. In the process β changes to $\overline{\beta}$, where $\overline{\beta} < 0$. (Why? If $\overline{\beta} \ge 0$, then, as $(w, \overline{\beta})$ satisfy all the constraints, we see that w satisfies the new inequality, which is not possible in this case.) The $\bar{c}$ is extended by just a 0. Use dual simplex step.

[14.14] Addition of an equality constraint This can be handled appropriately. Either view it as addition of two inequality constraints or put one artificial variable.

[14.15] Example Let us introduce a constraint in the problem in [14.8].

$$\begin{array}{ll} \min & -2x_1 - 4x_2 - 3x_3 \\ \hline \text{s.t.} & x_1 + x_2 + 3x_3 \le 4,\ 2x_1 + x_2 + x_3 \le 3,\ x_1 + x_2 + x_3 \le 2,\ x_i \ge 0. \end{array}$$

We see that the old optimal solution w does not satisfy the third constraint. So we start with the following table and make the entries in the third row below basic variables 0.

bv	x_1	x_2	x_3	x_4	x_5	y	$\overline{b}$
x_2	2	1	1	0	1	0	3
x_4	−1	0	2	1	−1	0	1
y	1	1	1	0	0	1	2
$-f$	6	0	1	0	4	0	12

$\rightarrow$

bv	x_1	x_2	x_3	x_4	x_5	y	$\overline{b}$	
x_2	2	1	1	0	1	0	3	
x_4	−1	0	2	1	−1	0	1 .	
y	−1	0	0	0	−1	1	−1	$\rightarrow < 0$
$-f$	6	0	1	0	4	0	12	

We use dual simplex method here.

Some exercises

[14.16] NoPen

a) Consider $\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = b, x \geq 0 \end{array}$ with the minimum value α. Suppose that a new variable y is entered and the revised problem has an optimal value β. Is it true that $\beta \leq \alpha$?

b) Consider $\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = b, x \geq 0 \end{array}$ with the minimum value α. Suppose that a new constraint is entered and the revised problem has an optimal value β. Is it possible that $\beta < \alpha$?

c) Consider a problem $\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = b, x \geq 0. \end{array}$ Suppose that b is changed to b'. Is it necessary that the old optimal bfs is a bfs of the revised problem?

d) Consider a problem $\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = b, x \geq 0. \end{array}$ Suppose that c is changed to c'. Is it necessary that the old optimal bfs is a bfs of the revised problem?

e) Let w be an optimal nondegenerate bfs of $\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = b,\ x \geq 0. \end{array}$ Is it necessary that $\exists\, \epsilon > 0$ such that for all $b' \in B_\epsilon(b)$, the point w is an optimal bfs of $\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = b',\ x \geq 0 \end{array}$?

f) Let w be an optimal nondegenerate bfs of $\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = b,\ x \geq 0. \end{array}$ Is w necessarily an optimal bfs of $\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = \frac{b}{2},\ x \geq 0 \end{array}$?

[14.17] Exercise(E) Consider $\begin{array}{ll} \min & \underline{x_1 + x_2 - 2x_3 - x_4} \\ \text{s.t.} & x_1 + 3x_2 \leq 3,\ 3x_3 + x_4 \leq 7,\ x_i \geq 0. \end{array}$ Let T_p be the feasible set.

a) Is $(0, .5, 0, 7)$ a vertex of Ω_p? What about $(0, 0, 0, 7)$?

b) Write the slpp for the given lpp.

c) Introduce artificial variables to describe how the computer finds an initial bfs using simplex method.

d) Apply simplex method starting with the optimal bfs in (C).

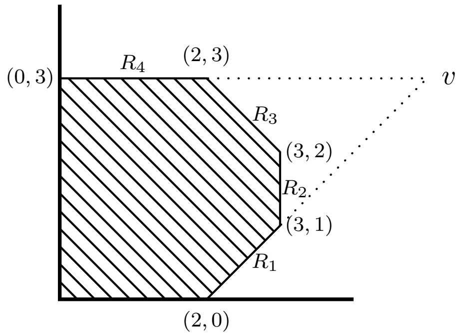
e) Is the solution unique?

f) Write the dual of the given lpp.

g) Solve the dual lpp by graphical method.

h) Replace the first equation in the given lpp by $x_1 + 3x_2 + x_4 \leq 3$. Let x_5 be the slack variable for this in-equation. Solve the new problem starting with the basis (x_5, x_4).

[14.18] Exercise(E) Consider the region given below. We want to minimize $-2x - y$ over this region.



a) Write the problem table using the line R_i for row i.

b) Write the simplex table at the basic solution corresponding to v.

c) Apply dual simplex method to solve it.

[14.19] **Exercise(E)** A person wants to donate money to four persons A,B,C,D. He has four astrologer advisors.

Astrologer 1: Donate whatever but twice the mount of A subtracted from the amount of the rest, should not exceed 100 (millions), otherwise something bad may happen to you.

Astrologer 2: Donate whatever but thrice the mount of B subtracted from the amount of the rest, should not exceed 100 (millions), otherwise something bad may happen to you.

Astrologer 3: The mount of B and C taken together, subtracted from the amount of the rest should not exceed 100 (millions), otherwise something bad may happen to you.

Astrologer 4: Twice the mount of D subtracted from the amount of the rest should not exceed 100 (millions), otherwise something bad may happen to you.

The person has just finished donating as much as possible. Do you know the amount he donated?