Monte Carlo Simulation (MA 323)

Quiz-I: Model Solution

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Q1

(a) $x_0 = 7,\ x_1 = 6,\ x_2 = 1,\ x_3 = 8,\ x_4 = 11,\ x_5 = 10,$

$x_6 = 5,\ x_7 = 12,\ x_8 = 15,\ x_9 = 14,\ x_{10} = 9,\ x_{11} = 0$

$x_{12} = 3,\ x_{13} = 2,\ x_{14} = 13,\ x_{15} = 4,\ x_{16} = 7.$

Period $= 16.$ ——— 1

(b) $x_0 = 5,\ x_1 = 12,\ x_2 = 15,\ x_3 = 14,\ x_4 = 9,\ x_5 = 0,\ x_6 = 3,$

$x_7 = 2,\ x_8 = 13,\ x_9 = 4,\ x_{10} = 7,\ x_{11} = 6,\ x_{12} = 1,\ x_{13} = 8$

$x_{14} = 11,\ x_{15} = 10,\ x_{16} = 5$

Period $= 16$ ——— 1

(c) $x_0 = 5,\ x_1 = 9,\ x_2 = 13,\ x_3 = 1,\ x_4 = 5$

Period $= 4.$ ——— 1

Q2 To implement acceptance-rejection algorithm, we need to find the supremum of the function

$$h(x) = \frac{e^{-x^2/2}}{e^{-\lambda x}} = e^{\lambda x - x^2/2} \quad \text{for } x > 0.$$

$$\Rightarrow \ln h(x) = \lambda x - x^2/2$$

$$\Rightarrow \frac{d}{dx} \ln h(x) = \lambda - x$$

Now, $\frac{d}{dx} \ln h(x) = 0 \Rightarrow \lambda - x = 0 \Rightarrow x = \lambda.$

Moreover, $\frac{d^2}{dx^2} \ln h(x) = -1 < 0 \quad \forall x > 0.$

Thus, $\sup_{x>0} h(x) = h(\lambda) = e^{\lambda^2/2}.$

Therefore,

$$e^{-x^2/2} \leq e^{\lambda^2/2} e^{-\lambda x}$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} e^{-x^2/2} \leq \frac{1}{\lambda} \sqrt{\frac{2}{\pi}} e^{\lambda^2/2} \times \lambda e^{-\lambda x}.$$

Take $c = \frac{1}{\lambda} \left(\frac{2}{\pi}\right)^{1/2} e^{\lambda^2/2}$, then we have

$f(x) \leq c\, g(x)$, where $g(x) = \lambda e^{-\lambda x} \quad x > 0.$ —— 2

Now, the following algorithm can be used to generate from $f(x)$.

Repeat
  generate u_1 from $U(0,1)$.
  set $x = -\frac{1}{\lambda} \ln u_1$.
  generate u_2 from $U(0,1)$.
until $u_2 \leq e^{\lambda x - x^2/2 - \lambda^2/2}$. —— 1
return x.

Acceptance probability of the algorithm is

$$P_\lambda = \frac{1}{c} = \lambda \left(\frac{\pi}{2}\right)^{1/2} e^{-\lambda^2/2}.$$

$\Rightarrow \ln P_\lambda = K + \ln \lambda - \lambda^2/2$, K indep. of λ.

$\Rightarrow \frac{d}{d\lambda} \ln P_\lambda = \frac{1}{\lambda} - \lambda.$

Now, $\frac{d}{d\lambda} \ln P_\lambda = 0 \Rightarrow \lambda = 1$ as $\lambda > 0.$

Moreover, $\frac{d^2}{d\lambda^2} \ln P_\lambda = -1 - \frac{1}{\lambda^2} < 0 \quad \forall \lambda > 0.$

Thus, P_λ is max at $\lambda = 1$ and $P_1 = \left(\frac{\pi}{2}\right)^{1/2} e^{-1/2}$.

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$\boxed{Q3}$ The CDF is

$$F(x) = cp\left[1 + (1-p) + \cdots + (1-p)^{x-1}\right]$$

$$= c\left[1 - (1-p)^{x}\right] \quad \text{for } x = 1, 2, \ldots, n,$$

where $c = \dfrac{1}{1-(1-p)^{n}}$.

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~~The following algorithm can be used to generate from $f(x)$.~~

~~Step 1: Generate u from $U(0,1)$~~

~~Step 2: Find $k \in \{2, \ldots, n\}$ such that~~

$$\cancel{\frac{1-(1-p)^{k}}{1-(1-p)^{n+1}} < u \leq \frac{1-(1-p)^{k+1}}{1-(1-p)^{n+1}}}.$$

~~Step3: Return~~

Let $q_0 = 0$, $q_k = F(k)$, $k = 1, 2, \ldots, n$.

Now, the following algorithm can be used to generate from $f(\cdot)$.

Step-1: Generate u from $U(0,1)$.

Step-2: Find $k \in \{1, 2, \ldots, n\}$ such that $q_{k-1} < u \leq q_k$

Step-3: Return k.

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