36 Lecture 36

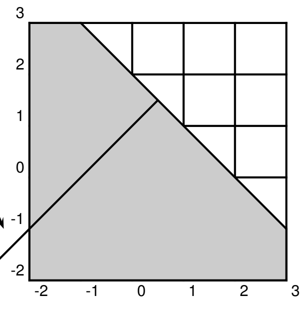
[36.1] **Example** Solve $\min \; f(x) = (x_1 - 1)^2 + x_2 - 2$
s.t. $g(x) \equiv 2 - x_1 - x_2 \geq 0, \; h(x) \equiv x_2 - x_1 - 1 = 0.$ $\longrightarrow T.$

- constraints are linear. so KTCQ1 holds at each feasible pt.
- By KTNC, if a is a pt of local minimum, then it must be a KT point.
- Find KT points: $\exists\, \lambda \geq 0, \; w \in \mathbb{R}$

$$\begin{bmatrix} 2x_1 - 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ -1 \end{bmatrix} + w \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \lambda(2 - x_1 - x_2) \geq 0, \quad a \in T$$

- check if $\lambda = 0$ is possible. If $\lambda = 0$, then $w = 1$. so $x_1 = .5, x_2 = 1.5$
- check if $\lambda > 0$. If $\lambda > 0$, then $x_1 + x_2 = 2$, $x_2 - x_1 = 1 \Rightarrow x_2 = 1.5, x_1 = .5$.
  As $\lambda > 0$, we get $w > 1$. $2x_1 - 2 = -\lambda - w < -1 \Rightarrow 2x_1 < 1 \Rightarrow \Leftarrow$
- so $(.5, 1.5)$ is the only KT point with $\lambda = 0, w = 1$.

This ray is our feasible region. We do not need it though.



- Apply KTSC $H_L(a, \lambda, w) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ not pd, psd in a nbhd
  so it is a lc min.

$D_+(a) = \left\{ d \;\middle|\; \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \geq 0, \; \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \right\} = \left\{ d \;\middle|\; \begin{matrix} d_1 + d_2 \leq 0 \\ d_1 = d_2 \end{matrix} \right\} = \left\{ \begin{bmatrix} -t \\ -t \end{bmatrix} \;\middle|\; t \geq 0 \right\}$

$\longrightarrow \begin{bmatrix} -t & -t \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -t \\ -t \end{bmatrix} = 2t^2 > 0$ for $t \neq 0$

so by KTSC a), a is a st lc min.

∘ Constraints are linear. So KTCQ1 holds at each feasible point. So by KTNC, each point of local minimum is a KT point. Denote the feasible region by T.

∘ We find KT points:

$$\begin{bmatrix} 2(x_1 - 1) \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ -1 \end{bmatrix} + w \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lambda(2 - x_1 - x_2) = 0, \qquad x \in T.$$

∘ Check if $\lambda = 0$.

If $\lambda = 0$, then $w = 1$. Equating the first row we get $x_1 = .5$. Applying $h(x)$, we get $x_2 = 1.5$. Thus $(.5, 1.5)$ being feasible, is a KT point with $\lambda = 0$, $w = 1$.

∘ Check if $\lambda > 0$ gives more KT points.

If $\lambda > 0$, then $x_1 + x_2 = 2$. Using $h(x)$, we get $x_2 = 1.5$ and $x_1 = .5$. But as $w > 1$, we must have $2x_1 < 1$, not possible. So we do not get any more KT points.

∘ So $a = (.5, 1.5)$ is the only KT point with $\lambda = 0$, $w = 1$. Use KTSC.

We have $H_L = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. It is psd in a neighborhood of a. So by KTSC c), a is a local minimum.

More information: We have g is active with $\lambda = 0$. So $A_+(a) = \emptyset$. So

$$\mathcal{D}_+(a) = \mathcal{D}(a) = \{d \mid d_1 + d_2 \leq 0, d_1 = d_2\}.$$

Take an arbitrary nonzero $d = \begin{bmatrix} -t \\ -t \end{bmatrix} \in \mathcal{D}_+(a)$. Then $t > 0$. So we have $d^t H(L)_a d = 2t^2 > 0$. By KTSC a), a is a strict local minimum.

[36.2] **Example** `(One dimensional example.)` Consider $\begin{array}{ll} \min & x^2 \\ \text{s.t.} & x \geq 5, x \leq 6 \end{array}$ and $\begin{array}{ll} \max & x^2 \\ \text{s.t.} & x \geq 5, x \leq 6 \end{array}$.

Find the KT points for both the problems apply KTSC.

Answer. ~~min~~ · constraints Lin ⇒ KTCQ1 holds ⇒ pts of lc min must be KT pts. Find KT points. write $g_1 = x-5 \geq 0$, $g_2 = 6-x \geq 0$

KT cond: $\exists\ \lambda_i \geq 0$ s.t.
$2x = \lambda_1 - \lambda_2$, $\lambda_1(x-5) = 0$, $\lambda_2(6-x) = 0$, $x \in T$

· If $\lambda_1 = 0$ possible. If $\lambda_1 = 0$, then $2x = -\lambda_2$, ⇒ not possible.

· so $\lambda_1 > 0$. so $x = 5$. so $\lambda_2 = 0$. so $\lambda_1 = 10$. So $x = 5$ is a KT pt with $\lambda_1 = 10$, $\lambda_2 = 0$.

Apply KTSC $H_L(5) = [2]$ pd. ⇒ By KTSC b), 5 is a st lc min.

—x—

For max x^2, it is min $-x^2$. you get $x = 6$ is a KT pt with $\lambda_2 = 12$, $\lambda_1 = 0$. $H_L(6) = [-2]$ not pd, not psd. $A_+(6) = \{2\}$
$\mathcal{D}_+(6) = \{d \mid -d = 0\} = \{0\}$. So for every nonzero $d \in \mathcal{D}_+(6)$ we have $d^T H_L d > 0$. It is a st lc. min.

a) First consider the minimization problem. We have $L = x^2 - \lambda_1(x-5) - \lambda_2(6-x)$.

b) Constraints are linear. So ktcq1 holds at all feasible points. Any local minimum must be a kt point.

c) Find kt points:

$$2x = \lambda_1 - \lambda_2.$$

If $\lambda_2 > 0$, then $x = 6$. As $\lambda_1 g_1(a) = 0$, we must have $\lambda_1 = 0$. But then $12 = 0 - \lambda_2$ is not possible.

So $\lambda_2 = 0$. If $\lambda_1 > 0$, then $x = 5$. Notice that $x = 5$ is a kt point with $\lambda_1 = 10$ and $\lambda_2 = 0$. Notice that $H_L = [2]$ is pd. Hence the point $x = 5$ is a strict local minimum by ktsc.

A) Consider the maximization problem, that is, minimizing $-x^2$. Approaching similarly, we get $a = 6$ is the only KT point with $\lambda_1 = 0$ and $\lambda_2 = 12$.

B) Use KTSC: Here $H_L = -2$. So we cannot use parts b) and c) of KTSC.

C) However, we can use part a). Note that $A_+(6) = \{2\}$ and $\mathcal{D}_+(6) = \{0\}$. Hence, for each nonzero $d \in \mathcal{D}_+(6)$ we have $d^t H(L) d > 0$. So $x = 6$ satisfies KTSC part a), to be a point of strict local minimum for $-x^2$.

D) So $a = 6$ is a strict local maximum for $f(x) = x^2$.

E) You can conclude it to be absolute maximum, as the function being continuous on a compact set must attain its absolute maximum. That point has to be a KT point. But there is only one KT point. So $a = 6$ is also an absolute maximum.

Exercises

[36.3] **<u>NoPen</u>** Let a be any feasible point for $\begin{array}{ll} \max & \underline{f(x)} \\ \text{s.t.} & g_i(x) \geq 0,\ i = 1, \ldots, m, \\ & h_j(x) = 0,\ j = 1, \ldots, p. \end{array}$

Suppose that $\mathcal{D}(a) \cap \{d \mid \bigtriangledown f(a)^t d > 0\} = \emptyset$. Is it necessary that a is a KT point?

[36.4] **<u>Exercise(M)</u>** Let a be a local minimum for

$$\begin{array}{ll} \min & \underline{f(x)} \\ \text{s.t.} & g_i(x) \geq 0,\ i = 1, \ldots, m,\ h_j(x) = 0,\ j = 1, \ldots, p,\ x_k \geq 0, k = 1, \ldots, n. \end{array}$$

a) Assume that $Z(a) = \emptyset$. Let

$$L(x, \lambda, w) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) - \sum_{j=1}^{p} w_j h_j(x).$$

Prove that $\exists\ \lambda_i \geq 0,\ i = 1, \ldots, m,\ w_j,\ j = 1, \ldots, p$ such that the following KT conditions are satisfied

$$\bigtriangledown L(a, \lambda, w) \geq 0, \qquad \lambda_i g_i(a) = 0, \forall i = 1, \ldots, m, \qquad a^t \bigtriangledown L(a, \lambda, w) = 0.$$

b) Is the converse of a) true?

c) Is $a = (0, \sqrt{2}, \sqrt{2})$ a KT point of

$$\begin{array}{ll} \min & \underline{f = x_1^3 - 6x_1^2 + 11x_1 + x_3} \\ \text{s.t.} & g_1 \equiv -x_1^2 - x_2^2 + x_3^2 \geq 0,\ g_2 \equiv x_1^2 + x_2^2 + x_3^2 - 4 \geq 0,\ g_3 \equiv 5 - x_3 \geq 0,\ x_i \geq 0 \end{array} \quad ?$$

[36.5] **<u>Practice</u>** Consider the problem $\begin{array}{ll} \text{opt} & \underline{x_1 + x_2^2 + x_3^3} \\ \text{s.t.} & x_1 + x_2 + x_3 = 1,\ x_i \geq 0. \end{array}$

a) Consider the minimization problem first and let $a = (0, 1, 0)$. Compute $D(a)$, $\mathcal{D}(a)$, $Z(a)$. Does KTCQ1 hold at a? Is a a KT point? Is it a local minimum?

b) Can $(0, 1, 0)$ be a point of local maximum?

c) Does KTCQ1 hold for every feasible point?

[36.6] **Practice** Consider the problem $\begin{array}{ll} \max & x^2+y \\ \text{s.t.} & x \geq 0,\ x^2+y^2=1. \end{array}$

a) Do you think, at each $a \in T$, KTCQ1 holds?

b) Find the KT points.

c) Conclude, whether these points are local maxima.

[36.7] **Practice** Find all local minimums for $\begin{array}{ll} \min & x^2+y^2+z^2-x-y-z \\ \text{s.t.} & 0 \leq x, y, z \leq 1. \end{array}$

[36.8] **Exercise(E)** Consider the problems $\begin{array}{ll} \min & x^4 \\ \text{s.t.} & |x| \leq 1 \end{array}$ and $\begin{array}{ll} \min & x^3 \\ \text{s.t.} & |x| \leq 1 \end{array}$.

a) Argue that $a=0$ is a KT point for both.

b) Now show that ktsc part a) is not a necessary condition. Hence, conclude that *if $d^tH(L)(a)d = 0$ for some nonzero $d \in \mathcal{D}_+(a)$, then a may or may not be a point of local minimum.*

c) However, show that we can still apply the ktsc part c) to conclude that 0 is a point of local minimum for the first One.

[36.9] **Exercise(M)** Consider the problem $\begin{array}{ll} \max & x_1x_2x_3x_4 \\ \text{s.t.} & x_1+x_2+x_3+x_4=5,\ x_i \geq 0. \end{array}$ We already know one way to find the absolute maximum. Find all KT points and give another solution using KT theory. Also search for all local maximums.

[36.10] **Practice** Consider minimizing x^2+y^2+xy over the region $T = \{x^2+y^2 \geq 1,\ x^2+4y^2 \leq 4\}$. Show that a point of local minimum must be a KT point.

[36.11] **Practice** Consider minimizing x^2+y^2+xy over the region $T = \{x^2+y^2 \geq 1,\ x^2+y^2 \leq 4\}$. Assume that the points of local minimums are KT points. Find all KT points. Apply KTSC at these points.

$\exists\, w_j \in \mathbb{R}$ s.t.
$\nabla f = \sum w_i \nabla h_j$, $a \in T$
✓

Lagrange Method

$$\begin{bmatrix} \nabla h_1 & \nabla h_2 & \cdots & \nabla h_p \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_p \end{bmatrix} = \begin{bmatrix} \nabla f \end{bmatrix}$$
$$\downarrow$$
$$J^t w = \nabla f$$

Consider the problem with only equality constraints.

$$\text{(P3)} \qquad \begin{array}{ll} \text{opt} & f(x) \\ \text{s.t.} & h_j(x) = 0, j = 1, \ldots, p \end{array}.$$

The Lagrangian function here becomes $L(x,w) = f(x) - \sum w_j h_j(x)$. The matrix $J := \begin{bmatrix} h_1'(a) \\ \vdots \\ h_p'(a) \end{bmatrix}$ is called the JACOBIAN.

[36.12] **Lemma** `(Lagrangian necessary condition (LNC))` Let $f, h_j \in \mathcal{C}^1$ and a be a local optimum for

LICQ a
$\{\nabla g_i(a)$, active
$\nabla h_j(a)\} \to$ lin ind
$\Rightarrow$ KT(Q) holds at a.

If rank $J = p < n$.

KTNC ⇒ if KTCQ holds at a, and a is a lc min, then a must be KT pt.

(P3). If the Jacobian J has full rank, then $\exists\, w$ such that $\bigtriangledown L(a,w) = 0$, that is, a is a KT point.

Proof. Suppose that J has full rank. If $p \le n$, then rows of J are linearly independent. So by licq, ktcq1 holds at a. Thus a being a local minimum must be a kt point.

Now suppose that $p > n$. Assume without loss that the first n rows of J are linearly independent. Then $f'(a)$ being a vector in $\mathbb{R}^n$, is a combination $\sum_{j=1}^{n} w_j h_j'(a)$. Put $w_j = 0$, for $j = n+1, \ldots, p$. ■

[36.13] Theorem (Lagrangian sufficient condition (LSC1)) Let $f, h_j \in \mathcal{C}^2$ and suppose that $a \in T$ satisfies $\bigtriangledown L(a,w) = 0$. Then

a) a is a strict local minimum if for each $d \neq 0$ in $\{d \mid Jd = 0\}$ we have $d^t H_L(a) d > 0$.

b) a is a strict local maximum if for each $d \neq 0$ in $\{d \mid Jd = 0\}$ we have $d^t H_L(a) d < 0$.

Proof. Follows from ~~ktsc~~ KTSC a), as $\mathcal{D}_+(a) = \mathcal{D}(a) = \{d \mid Jd = 0\}$. ■

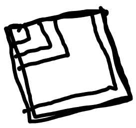
[36.14] Discussion a) We need to check whether $d^t H_L d > 0$, for any $d \neq 0$ satisfying $Jd = 0$.

b) See, if $\mathsf{rank}\, J = n$, then the above will not help, as $d = 0$ is the only vector which will satisfy $Jd = 0$.

c) So, assume $\mathsf{rank}\, J < n$. Let $\hat{J}$ be the submatrix obtained from J by keeping a maximal linearly independent set of rows. Then $\{d \mid \hat{J}d = 0\} = \{d \mid Jd = 0\}$.

d) Under the assumption that $\mathsf{rank}(J) = p < n$, using the symmetric nature of the matrix $H_L(a)$, a test was supplied by H. B. Mann, in 'Quadratic Forms with Linear Constraints', American Mathematical. Monthly (1943), pages 430–433. Accordingly, we have a sufficient condition for an optimal point.

[36.15] Theorem (Lagrangian sufficient condition 2 (LSC2)) Let $f, h_j \in \mathcal{C}^2$, $p < n$, and $a \in T$ satisfy $\bigtriangledown L(a,w) = 0$. Consider the BORDERED HESSIAN matrix $B_{n+p \times n+p} = \begin{bmatrix} 0 & J \\ J^t & H_L \end{bmatrix}$. Check the last $n - p$ leading principal minors starting with the determinant of B.



a) If they have alternate signs starting with $(-1)^n$, then a is a strict local maximum.

b) If all of them have the sign $(-1)^p$ then a is a strict local minimum.

$B = \begin{bmatrix} 0 & J \\ J^t & H_L \end{bmatrix}$

[36.16] Example Solve $\begin{array}{ll} \min & f = x + y \\ \text{s.t.} & x^2 + y^2 = 1. \end{array}$

• $J = [2x \;\; 2y] \to$ rank 1 over T.

LNC satisfied. so a pt of ~~opt~~ min must be a KT pt.

$-\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ with $w = -\frac{1}{\sqrt{2}}$ $\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = w \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow x = \frac{1}{2w} = y$, $\quad x^2+y^2=1$, $\quad \frac{1}{4w^2} + \frac{1}{4w^2} = 1$, $\quad 2w^2 = 1$, $\quad w = \pm \frac{1}{\sqrt{2}}$

✓ $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ with $w = \frac{1}{\sqrt{2}}$ $\quad w = \frac{1}{\sqrt{2}}$, $x = \frac{1}{2w} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Answer. Note that $f, g \in \mathcal{C}^2$.

∘ Matrix $J = \begin{bmatrix} 2x & 2y \end{bmatrix}$ has full rank throughout the feasible set. So a local minimum must be a KT point.

∘ Find KT points: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = w \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow x = y = \frac{1}{2w}$. Since $x^2 + y^2 = 1$, we have $w = \pm\frac{1}{\sqrt{2}}$.

$\dfrac{x+y}{x^2+y^2=1}$

∘ Two possibilities: $w = \frac{-1}{\sqrt{2}}, x = y = \frac{-1}{\sqrt{2}}$, and $w = \frac{1}{\sqrt{2}}, x = y = \frac{1}{\sqrt{2}}$.

$L = f - wh$

∘ The bordered Hessian matrix is $B = \begin{bmatrix} 0 & 2x & 2y \\ 2x & -2w & 0 \\ 2y & 0 & -2w \end{bmatrix}$.

$\begin{bmatrix} -\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$

$\begin{bmatrix} 0 & 2x & 2y \\ 2x & -2w & 0 \\ 2y & 0 & -2w \end{bmatrix}$

∘ At $w = \frac{1}{\sqrt{2}}, x = y = \frac{1}{\sqrt{2}}$, the matrix is $\begin{bmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \end{bmatrix}$. As $n - p = 1$, we only check one leading minor starting from $\det B$. Now, $\det B = 4\sqrt{2}$ has the sign $(-1)^n$. Hence $x = y = \frac{1}{\sqrt{2}}$ is a point of maximum.

∘ Similarly, the other point can be shown to be a point of minimum.

∘ Note that if H_L is pd, we can directly conclude that a KT point is a point of strict local minimum. You do not need to use the bordered Hessian.

[36.17] **Example** `(Self)` Solve $\begin{array}{ll} \min & f = x + y + z \\ \hline \text{s.t.} & x^2 + y^2 + z^2 = 1. \end{array}$

Answer. Note that $f, g \in \mathcal{C}^2$.

∘ Matrix $J = \begin{bmatrix} 2x & 2y & 2y \end{bmatrix}$ has full rank throughout the feasible set. So a local minimum is a KT point.

∘ To find KT points: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = w \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \Rightarrow x = y = z = \frac{1}{2w}$. Since $x^2 + y^2 + z^2 = 1$, we have $w = \pm\frac{\sqrt{3}}{2}$.

∘ Two possibilities: $w = \frac{-\sqrt{3}}{2}, x = y = z = \frac{-1}{\sqrt{3}}$, and $w = \frac{\sqrt{3}}{2}, x = y = z = \frac{1}{\sqrt{3}}$.

∘ The bordered Hessian matrix is $B = \begin{bmatrix} 0 & 2x & 2y & 2z \\ 2x & -2w & 0 & 0 \\ 2y & 0 & -2w & 0 \\ 2z & 0 & 0 & -2w \end{bmatrix}$.

∘ At $w = \frac{\sqrt{3}}{2}$, $x = y = z = \frac{1}{\sqrt{3}}$ the matrix is $B = \left[\begin{array}{ccc|c} 0 & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & -\sqrt{3} & 0 & 0 \\ \frac{2}{\sqrt{3}} & 0 & -\sqrt{3} & 0 \\ \hline \frac{2}{\sqrt{3}} & 0 & 0 & -\sqrt{3} \end{array}\right]$.

As $n - p = 2$, we find two leading minors starting from $\det B$, that is $\det B$ and the determinant of the upper-left 3×3 matrix C.

Now, $\det B = -\sqrt{3} \det C - \frac{2}{\sqrt{3}} 2\sqrt{3} = -12$, $\det C = \frac{8}{\sqrt{3}}$. The signs alter starting from the sign $(-1)^n$. Hence this is a point of local maximum.

∘ Similarly, the other point can be shown to be a point of local minimum.

Alternate. Note that, at $w = \frac{-\sqrt{3}}{2}$, H_L is pd. So by KTSC, we can directly conclude that this point is a point of strict minimum. Then use continuity over a compact set to conclude that the other point must be the absolute maximum.

Exercises

[36.18] **<u>Practice</u>** Use Lagrange multipliers to solve

$$\begin{array}{ll} \text{opt} & f(x) \equiv 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 \\ \text{s.t.} & h \equiv x_1 + x_2 + x_3 - 20 = 0. \end{array}$$

[36.19] **<u>Practice</u>** Use Lagrange multipliers to solve $\begin{array}{ll} \text{opt} & f \equiv xy \\ \text{s.t.} & h \equiv x^2 + 4y^2 - 1 = 0 \end{array}$.

[36.20] **<u>Practice</u>** Maximize $f(x, y, z) = x^2 + y^2 + z^2$, subject to $h_1 \equiv \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1 = 0$, and $h_2 \equiv x + y - z = 0$.

[36.21] **<u>Practice</u>** Consider the problem $\begin{array}{ll} \text{opt} & x^2 + y \\ \text{s.t.} & x^2 + y^2 = 1. \end{array}$

a) Solve it using graphical method.

b) Do you think each point of local optimum is a KT point? Why?

c) Find all KT points.

d) Conclude whether these points are local minimums or maximums.

e) Which of these are global maximums or minimums?

[36.22] **<u>Exercise</u>**

A) Consider an $n \times n$, $n \geq 4$ matrix of the form $\left[\begin{array}{c|ccc} 0 & 2 & 2 & 2 \\ \hline 2 & -1 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ 2 & 1 & 1 & -1 \end{array}\right]$. Show that its determinant is $(-1)^{n-1}2^n(n-1)$. You can subtract a multiple of remaining rows from first row. Another way is to use a known result. When D is invertible, we have

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}) = \det \begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix}$$

$$\Rightarrow \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \ \det(A - BD^{-1}C)$$

B) Solve $\begin{array}{ll} \text{opt} & x_1x_2x_3x_4 \\ \text{s.t.} & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4. \end{array}$