1 General Sampling Methods

With the introduction of random number generators behind us, we assume the availability of an ideal sequence of random numbers. More precisely, we assume the availability of a sequence $U_1, U_2, \ldots$ of independent random variables, each satisfying,

$$P(U_i \leq u) = \begin{cases} 0 \ , \ u < 0 \\ u \ , \ 0 \leq u \leq 1 \\ 1 \ , \ u > 1, \end{cases}$$

i.e., uniformly distributed between 0 and 1. A typical simulation uses methods for transforming samples from the uniform distribution to samples from other distributions. The two most widely used general techniques are:

1. Inverse Transform Method.
2. Acceptance Rejection Method.

1.1 Inverse Transform Method

The inverse transform method is based on the following theorem.

Theorem 1. *Let F be a CDF. Define the quasi-inverse of F by*

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\} \quad \textit{for } 0 < u < 1.$$

Let $U \sim U(0, 1)$ and $X = F^{-1}(U)$. Then, the CDF of X is F.

Before going in to the proof of the theorem, let us first discuss the inverse transform method. Suppose that we want a sample from a cumulative distribution function $F(x)$, *i.e.,* we want to generate a random variable X with the property that $P(X \leq x) = F(x)$ for all $x \in \mathbb{R}$. Using the Theorem 1, we have the following algorithm.

Algorithm 1 Inverse Transform Method

1: Generate U from $U(0, 1)$ distribution. ▷ Using some random number generator.
2: Set $X = F^{-1}(U)$.
3: Return X.

In principle, we can use this algorithm for generation of all non-uniform random variables. However, there are computational aspects. We generally use this algorithm if F^{-1} is in closed form and easy to compute.

Example 1 (Exponential Distribution). The exponential distribution with mean θ has distribution

$$F(x) = 1 - e^{-x/\theta}, \ x \geq 0.$$

Inverting yields

$$X = -\theta \log(1 - U).$$

This can be implemented also as $X = -\theta \log(U)$, since U and $(1-U)$ have the same distribution. $||$

Example 2 (Arc Sin Law). Consider the CDF

$$F(x) = \frac{2}{\pi} \text{ arc sin } \sqrt{x} \ , \ 0 \leq x \leq 1.$$

The inverse transform method for sampling from this distribution is:

$$X = \sin^2\left(\frac{U\pi}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos(U\pi), \ U \sim U(0,1),$$

using the identity, $2\sin^2(t) = 1 - \cos(2t)$ for $0 \leq t \leq \pi/2$. $||$

Example 3 (Rayleigh Distribution). We consider the Rayleigh Distribution:

$$F(x) = 1 - e^{-2x(x-b)} \ , \ x \geq b.$$

Solving the equation $F(x) = u \ , \ u \in (0,1)$, results in a quadratic with roots:

$$x = \frac{b}{2} \pm \frac{\sqrt{b^2 - 2\log(1-u)}}{2}.$$

The inverse is given by the larger of the two roots. Replacing U with $(1-U)$ we get,

$$X = \frac{b}{2} + \frac{\sqrt{b^2 - 2\log(U)}}{2}.$$ $||$

Remark 1. Note that even if the inverse of F is not known explicitly, the inverse transform method is still applicable through numerical evaluation of F^{-1}. Computing $F^{-1}(u)$ is equivalent to finding a root x of the equation $F(x) - u = 0$. For a CDF F with PDF f, Newton's method for finding roots produces a sequence of iterates:

$$x_{n+1} = x_n - \frac{F(x_n) - u}{f(x_n)},$$

given a starting point x_0. †

Now, we will discuss the prove of the Theorem 1. To prove the above theorem, we need the following lemmas.

Lemma 1. *F and F^{-1} both are non-decreasing.*

Proof: F is a non-decreasing function (a properties of CDF). Now, we will prove that F^{-1} is a non-decreasing function. Let $0 < u_1 < u_2 < 1$. Then

$$\begin{aligned}
&\{x \in \mathbb{R} : F(x) \geq u_2\} \subseteq \{x \in \mathbb{R} : F(x) \geq u_1\} \\
\Longrightarrow\ &\inf\{x \in \mathbb{R} : F(x) \geq u_2\} \geq \inf\{x \in \mathbb{R} : F(x) \geq u_1\} \\
\Longrightarrow\ &F^{-1}(u_1) \leq F^{-1}(u_2).
\end{aligned}$$

□

Lemma 2. *$F\,F^{-1}(u) \geq u$ for all $u \in (0, 1)$.*

Proof: As F is a right continuous function (a property of CDF), the infimum of the set

$$\{x \in \mathbb{R} : F(x) \geq u\}$$

belongs to the set. That means

$$F^{-1}(u) \in \{x \in \mathbb{R} : F(x) \geq u\}.$$

Therefore, we have the lemma. □

Lemma 3. *$F^{-1}\,F(x) \leq x$ for all $x \in \mathbb{R}$.*

Proof: Notice that

$$F^{-1}\,F(x) = \inf\{y \in \mathbb{R} : F(y) \geq F(x)\}.$$

Moreover, $x \in \{y \in \mathbb{R} : F(y) \geq F(x)\}$, which proves the lemma. □

Lemma 4. *For $x \in \mathbb{R}$ and $0 < u < 1$, $F(x) \geq u$ if and only if $F^{-1}(u) \leq x$.*

Proof: Suppose that $F^{-1}(u) \leq x$. Applying F on both sides, we get $F(x) \geq F\,F^{-1}(u) \geq u$. The first inequality is due to Lemma 1 and the last inequality is due to Lemma 2.

Now assume that $F(x) \geq u$. Then applying F^{-1} on the both sides, $F^{-1}(u) \leq F^{-1}\,F(x) \leq x$. The first inequality is due to Lemma 1 and the last inequality is due to Lemma 3. □

Proof (of Theorem 1): Using the Lemma 4, proof of the Theorem 1 is very simple. Let us try to find the CDF of X. For $x \in \mathbb{R}$, the CDF of X is

$$P(X \leq x) = P\left(F^{-1}(U) \leq x\right) = P(U \leq F(x)) = F(x).$$

The first equality is due to the definition of X, the second is due to Lemma 4, and the last is due to the fact that $U \sim U(0, 1)$. □

1.2 Discrete Distribution

Let us start with an example.

Example 4 (Bernoulli Distribution)**.** Suppose that we want to generate random number from Bernoulli distribution with probability of success p. Thus, $P(X = 0) = 1 - p$ and $P(X = 1) = p$. The corresponding CDF is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Therefore,

$$F^{-1}(u) = \begin{cases} 0 & \text{if } 0 < u < 1-p \\ 1 & \text{if } 1-p \le u < 1. \end{cases}$$

Thus, generate U from $U(0,\,1)$, and then return 0 if $U < 1-p$, return 1 otherwise. $||$

In the case of a discrete distribution with finite support, the evaluation of F^{-1} reduces to a table look up. Note that CDF of a discrete random variable is a step function. Consider, for example, a discrete random variable whose possible values are $c_1 < c_2 < c_3 < \cdots < c_N$. Let p_i be the probability attached to c_i , $i = 1,\,2,\,3\,\ldots,\,N$ and set $q_0 = 0$. Also,

$$q_i = \sum_{j=1}^{i} p_j \ , \ i = 1,\,2,\,3\,\ldots,\,N.$$

These are cumulative probabilities associated with the c_i, that is, $q_i = F(c_i)$, $i = 1,\,2,\,\ldots,\,N$. To sample from this distribution, we can use the following algorithm.

Algorithm 2 Inversion Method for Discrete Random Variable with Finite Support

1: Generate a uniform $U \sim U(0,\,1)$.
2: Find $K \in \{1, 2, \ldots, N\}$ such that $q_{K-1} < U \le q_K$.
3: Return c_K.

If the discrete random variable takes countably infinite values, then table look up does not make sense. When there are infinitely many values, their description can only be a mathematical one. However, sometimes other transformation may help.

Example 5 (Geometric Distribution)**.** Suppose that we want to generate random number from Geometric distribution with success probability p. The PMF is given by

$$P(X = i) = p(1-p)^i \quad \text{for } i = 0,\,1,\,2,\,\ldots.$$

Note that in this case the support is countably infinite. Let Y be an exponential random variable with mean $\frac{1}{\lambda}$ and $W = \lfloor Y \rfloor$. Then it is easy to see that

$$P(W = i) = e^{-i\lambda}\left(1 - e^{-\lambda}\right) \quad \text{for } i = 0,\,1,\,2,\,\ldots.$$

Thus, W has a Geometric distribution with success probability $1 - e^{-\lambda}$. We can use this result to generate random number from a Geometric distribution using the following steps. Generate U from $U(0,\,1)$ distribution. Then, set $X = \left\lfloor \frac{\ln U}{\ln(1-p)} \right\rfloor$. $||$

1.3 Conditional Distribution

Suppose X has distribution F and consider the problem of sampling X conditional on $a < X \leq b$ with $F(a) < F(b)$. Note that the CDF of X conditional on $a < X \leq b$ is given by

$$P(X \leq x | a < X \leq b) = \begin{cases} 0 & \text{if } x < a \\ \frac{P(a<X\leq x)}{P(a<X\leq b)} = \frac{F(x)-F(a)}{F(b)-F(a)} & \text{if } a < X \leq b \\ 1 & \text{if } x > b. \end{cases}$$

Now, using the inverse transform method, this is no more difficult than generating X unconditionally. We can follow the following algorithm for this purpose.

Algorithm 3 Generation from Conditional/Truncated Distribution

1: Generate U from $U(0, 1)$ distribution.
2: Set $X = F^{-1}(F(a) + (F(b) - F(a))U)$.
3: Return X.