MA 372 : Stochastic Calculus for Finance

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Department of Mathematics, Indian Institute of Technology Guwahati

Exercises 1

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1. Let $\Omega = \{1, 2, 3, 4\}$. Is any of the following families of sets a sigma algebra?

$$
\begin{aligned}
\mathcal{F}_1 &= \{\phi, \{1, 2\}, \{3, 4\}\}, \\
\mathcal{F}_2 &= \{\phi, \Omega, \{1\}, \{2, 3, 4\}\{1, 2\}, \{3, 4\}\}, \\
\mathcal{F}_3 &= \{\phi, \Omega, \{1\}, \{2\}, \{1, 3, 4\}\{2, 3, 4\}\{1, 2\}, \{3, 4\}\}
\end{aligned}
$$

2. Let $\Omega = (0, 1)$. Is any of the following families of sets a sigma algebra?

$$
\begin{aligned}
\mathcal{F}_1 &= \{\phi, \Omega, (0, 1/2), (1/2, 1)\}, \\
\mathcal{F}_2 &= \{\phi, \Omega, (0, 1/2), [1/2, 1), (0, 2/3), [2/3, 1)\}, \\
\mathcal{F}_3 &= \{\phi, \Omega, (0, 2/3), [2/3, 1)\}
\end{aligned}
$$

3. Let $\Omega = \{1, 2, 3, 4\}$. Complete $\{\{2\}, \{3\}\}$ to obtain a sigma algebra (i.e.,$\sigma(\{2\}, \{3\}) =$?).

4. Show that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Give an example such that
$$\mathbb{P}(A \cup B) < \mathbb{P}(A) + \mathbb{P}(B).$$

5. Let Ω and $\tilde{\Omega}$ be arbitrary sets and let $X : \tilde{\Omega} \to \Omega$ be any function. Show that if $\tilde{\mathcal{F}}$ is a sigma algebra on $\tilde{\Omega}$, then $\mathcal{F} = \{A \subseteq \Omega : X^{-1}(A) \in \tilde{\mathcal{F}}\}$ is a sigma algebra on Ω.

6. Find an example of a function $X : \tilde{\Omega} \to \Omega$ and a sigma algebra $\tilde{\mathcal{F}}$ on $\tilde{\Omega}$ such that $\mathcal{F} = \{X(A) : A \in \tilde{\mathcal{F}}\}$ is not a sigma algebra. (Hint: Observe that $X(A) \setminus X(B)$ is not always equal to $X(A \setminus B)$, take $\tilde{\Omega} = \{1, 2, 3\}$.)

7. If $\mathcal{C} \subset \mathcal{D}$, then $\sigma(\mathcal{C}) \subset \sigma(\mathcal{D})$.

8. We call $f : \mathbb{R} \to \mathbb{R}$ a Borel measurable function if $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for any $B \in \mathcal{B}(\mathbb{R})$. Show that if f is Borel measurable and X is a real valued random variable, then the composition $|f(X)|$ is a real valued random variable.

9. Show that the distribution function $F_X(x) := \mathbb{P}(X \leq x)$ is non decreasing, right-continuous, and
$$\lim_{x \to +\infty} F(x) = 1, \lim_{x \to -\infty} F(x) = 0$$

10. Suppose that X is a random variable with density f_X. Show that
$$\frac{d}{dx} F_X(x) = f_X(x)$$
if f_X is continuous at x.

11. Let X be a continuous non-negative random variable with finite mean. Show that
$$EX = \int_0^\infty [1 - F(x)]dx$$
where F is the distribution function of X.

12. Are the following functions f densities? (Choose the constant c if necessary)
$$\begin{aligned} f(x) &= cx^n \text{ for x} \in (0,1) \text{ and zero otherwise.} \\ f(x) &= ce^{-\lambda x} \text{ for x} > 0 \text{ and zero otherwise.} \end{aligned}$$

13. Let $A_1, A_2, \cdots$ be a sequence of events such that $\mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots < \infty$ and let $B_n = A_n \cup A_{n+1} \cup \cdots$. Then $\mathbb{P}(B_1 \cap B_2 \cap \cdots) = 0$.

14. Let $\mathbb{P}$ be the Lebesgue measure on $\Omega = [0,1]$. Define
$$Z(w) = \begin{cases} 0 & \text{if } 0 \leq w < 1/2 \\ 2 & \text{if } 1/2 \leq w \leq 1 \end{cases}$$
For $A \in \mathcal{B}([0,1])$, define $\tilde{\mathbb{P}}(A) = \int_A Z(w)d\mathbb{P}(w)$.
(1) Show that $\tilde{\mathbb{P}}$ is a probability measure.
(2) Show that if $\mathbb{P}(A) = 0$, then $\tilde{\mathbb{P}}(A) = 0$.
(3) Show that there is a set A for which $\tilde{\mathbb{P}}(A) = 0$ but $\mathbb{P}(A) > 0$. In the other words, $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are not equivalent.

15. Let X be a non-negative random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with exponential distribution, which is
$$\mathbb{P}\{X \leq a\} = 1 - e^{-\lambda a}, a \geq 0,$$
where λ is a positive constant. Let $\tilde{\lambda}$ be another positive constant, and define
$$Z = \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda} - \lambda)X}$$
For $A \in \mathcal{F}$, define $\tilde{\mathbb{P}}(A) = \int_A Z(w)d\mathbb{P}(w)$.
(1) Show that $\tilde{\mathbb{P}}$ is a probability measure.
(2) Compute the distribution function for the random variable X under the probability measure $\tilde{\mathbb{P}}$.

16. Let $s_1, s_2 \in \mathbb{L}_0^+$. Prove the following:
(i) If $s_1 \geq s_2$, then $\int_\Omega s_1 d\mathbb{P} \geq \int_\Omega s_2 d\mathbb{P}$.
(ii) If $s_1 \geq s_2$, then $s_1 - s_2 \in \mathbb{L}_0^+$.

17. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a peobability space and $w_1, w_2 \in \Omega$. Show that $\mathbf{P}(A) = 1/2\delta_{\{w_1\}}(A) + 1/2\delta_{\{w_2\}}(A)$ is a probability measure. What is the largest set of measure 0 ? What is the smallest set of measure 1? ($\delta_{\{w\}}(A) = 1$ if $w \in A$ otherwise $\delta_{\{w\}}(A) = 0$, Dirac measure concentrated at w)

18. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures, then $\mathbb{P}(A) = \alpha_1\mathbb{P}_1(A) + \alpha_2\mathbb{P}_2(A)$ is also probability measure provided $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.