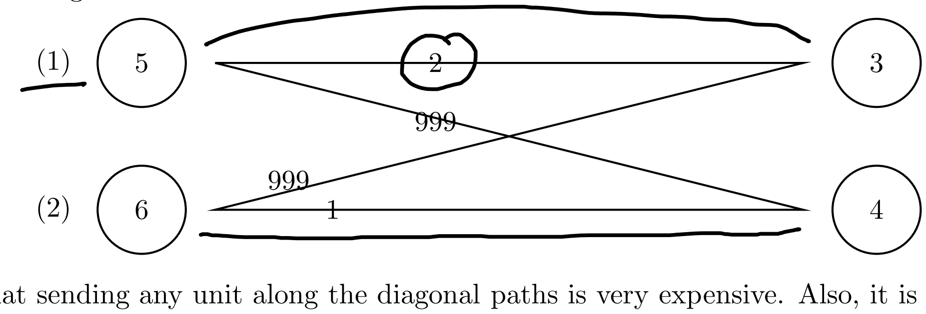
left with the source S_1. Now it is your wish whether to send it to some sink or to keep it at the source. This is as good as sending it to a new sink. So introduce a virtual sink T_{n+1} with demand $(\sum a_i - \sum b_j)$. Put $c_{i,n+1} = \min\{\theta_i, c_{i1}, \cdots, c_{in}\}$. Note that $c_{i,n+1} = \theta_i$ means that any surplus product at S_i goes to the storage and $c_{i,n+1} = c_{i1}$ means that the surplus product at S_i goes to T_1. This gives us a btp. Draw picture.

d) `(Retrieving the solution)` Suppose that X is a minimum solution of the btp with cost α.

We define a solution Y of the ubtp of the same cost in the following way. For $j = 1, \ldots, n$, define

$$y_{ij} = \begin{cases} x_{ij} + x_{i,n+1} & \text{if } c_{i,j} = c_{i,n+1} < \theta_i,\ j \text{ is smallest} \\ x_{ij} & \text{if } c_{i,n+1} = \theta_i \ \text{ or if } c_{i,j} = c_{i,n+1} < \theta_i,\ j \text{ is not smallest.} \end{cases}$$

[20.3] **Example** Consider the following ubtp. Here the availabilities and the demands are written inside the circles. The storage costs are written left to the sources with brackets. The cost of per unit transportation is written on the edges.

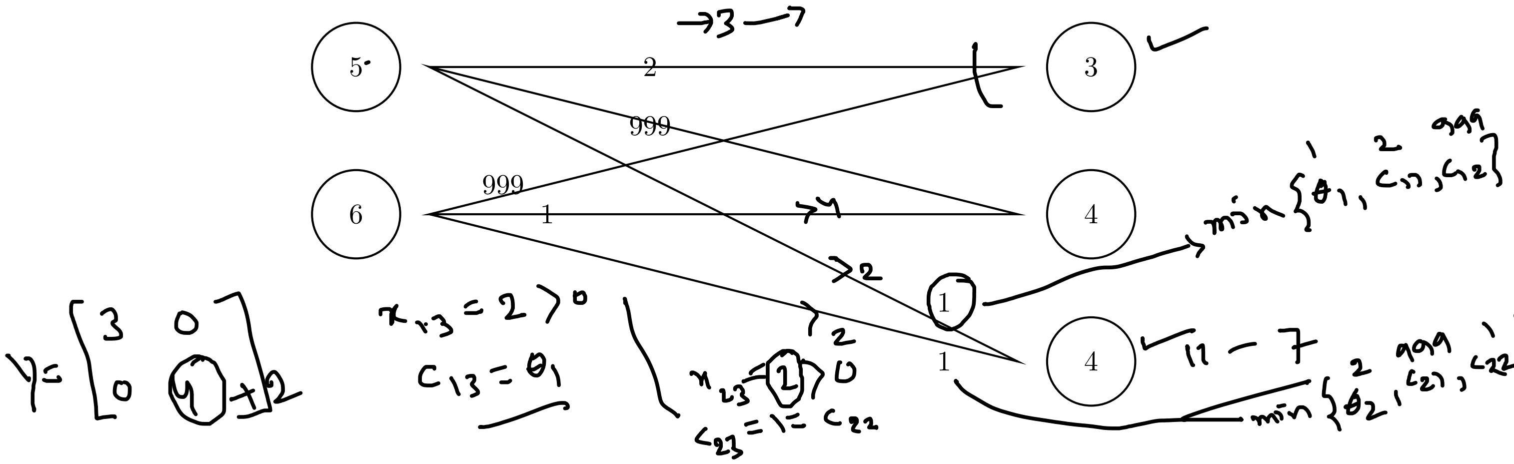


It is clear that sending any unit along the diagonal paths is very expensive. Also, it is better that we keep 2 units at source one. As the transportation cost from source $S2$ to sink $T2$ is cheaper than the storage cost at $S2$, we should send all 6 units to $T2$. So the minimum solution will be $Y = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$. The corresponding

$2 \times (1) + 3 \times 2 + 6 \times 1 = 14$

cost will be $(1) \times 2 + 3 \times 2 + 6 \times 1 = 14$. Assume that, we do not know this.

We write the corresponding btp, by introducing a new sink $T3$ of demand $\sum a_i - \sum b_j = 4$. The cost c_{13} set to $\min\{\theta_1, c_{11}, c_{12}\} = 1$ and the cost c_{23} set to $\min\{\theta_2, c_{21}, c_{22}\} = 1$.



We can now use our algorithm to find a minimum solution. In this particular example, as it will not involve the edges of cost 999, the only way to meet the demand of $T1$ is from $S1$ and the rest should go to $T3$. From $S2$ we send 4 units to $T2$ and 2 to $T3$. So a minimum solution of the btp is $X = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 4 & 2 \end{bmatrix}$.

The moment we have got a minimum solution X for the btp, first let Y be the matrix obtained from X by deleting the last column. Now just look at the last column of X, that is, the entries $x_{i,n+1}$.

Here $x_{1,3} = 2 > 0$. Observe that $c_{1,3} = \min\{\theta_1, c_{11}, c_{12}\} = \theta_1$. So, do nothing to Y. (That amount will automatically stay at $S1$.)

Here $x_{2,3} = 2 > 0$. Observe that $c_{2,3} = \min\{\theta_2, c_{21}, c_{22}\} = c_{22}$. So, redefine $y_{22} = x_{22} + x_{23} = 6$.

So, our solution to the ubtp is $Y = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$. Of course, we expected this to happen.

e)`(Arguing that it is indeed the correct solution)` The idea here is to

i) find a solution of the btp from a minimum solution of the ubtp, of the same cost (hence proving that the minimum cost solution of the btp will be cheaper) and

ii) find a solution of the ubtp from a minimum solution of the btp, of the same cost (hence proving that the minimum cost solution of the ubtp will be cheaper).

In view of the previously discussed Item d), we already have ii). Now to see i), let Y be a minimum solution for the ubtp.

There are two types of sources you will notice. One which has some units left at it and one which does not. The first idea is to move those units in the storages to the sink T_{n+1}. Similarly, observe the sinks that have got extra units. Return back those extra units (along the edges in which it got them, proportionately) and then move those units to the sink T_{n+1} with the respective costs. This gives us a solution of the btp of the same cost.

STEP1 argument: Suppose $A_1 = a_1 - y_{11} - \cdots - y_{1n} > 0$. Then we must have $\theta_1 = \min\{\theta_1, c_{11}, \cdots, c_{1n}\}$, otherwise, we could send that extra amount at S_1 to another place decreasing the overall cost.

Thus, in this case that extra amount at S_1 may be viewed as being taken to T_{n+1} with a transportation cost $c_{1,n+1}$. We perform this operation at all sources S_i that have some extra product A_i left at it.

STEP2 argumet: Suppose that the sink T_1 has received $B_1 > b_1$. Assume that only $y_{11}, \ldots, y_{k1} > 0$. So $y_{11} + \ldots + y_{k1} > b_1$. Notice that we must have $c_{11} = \min\{\theta_1, c_{11}, \ldots, c_{1n}\}$. [If $c_{11} > \theta_1$, we could have kept a little amount at S_1, instead of sending it to T_1 and decrease the cost. If $c_{11} > c_{12}$, we could have sent a little more amount to T_2 instead of sending it to T_1.]

Similarly, we must have $c_{k1} = \min\{\theta_k, c_{k1}, \ldots, c_{kn}\}$.

The above two mean we can reduce the transportation from sources $S_1, \ldots, S_k$ to T_1 proportionately and divert the residual to T_{n+1}. In fact, we can do a similar scaling at all other sources too, as those transportation are zero.

That is,

$$y'_{11} = \frac{b_1}{B_1} y_{11}, \; \cdots, \; y'_{k1} = \frac{b_1}{B_1} y_{k1}, \; \cdots, y'_{k+1,1} = \cdots = y'_{n,1} = 0$$

Thus the final solution of the btp is given by

$$y'_{i,j} = \begin{cases} \frac{b_j}{B_j} y_{i,j} & \text{if } B_j > b_j, \; j = 1, \ldots, n \\ y_{i,j} & \text{if } B_j = b_j, \; j = 1, \ldots, n \end{cases}$$

and

$$y'_{i,n+1} = A_i + \sum_{j=1}^{n} [y_{i,j} - y'_{i,j}].$$

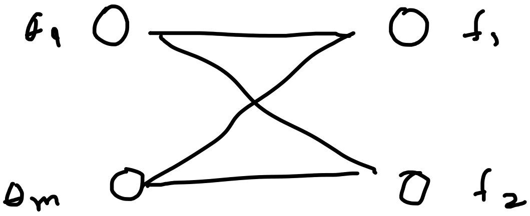
With this, we obtain a solution of the btp of the same cost.

21 Lecture 21

Below demand supply with storage cost and fine for unmet demands

a) Given a_i's, b_j's and the cost matrix C. Conditions: part of the product may remain at the source with a per unit storage cost θ_i at source S_i, sinks may get less while the fine amount for each unit of unmet demand of T_j is f_j. Problem: minimize the cost.

b) The lpp of the ubtp is



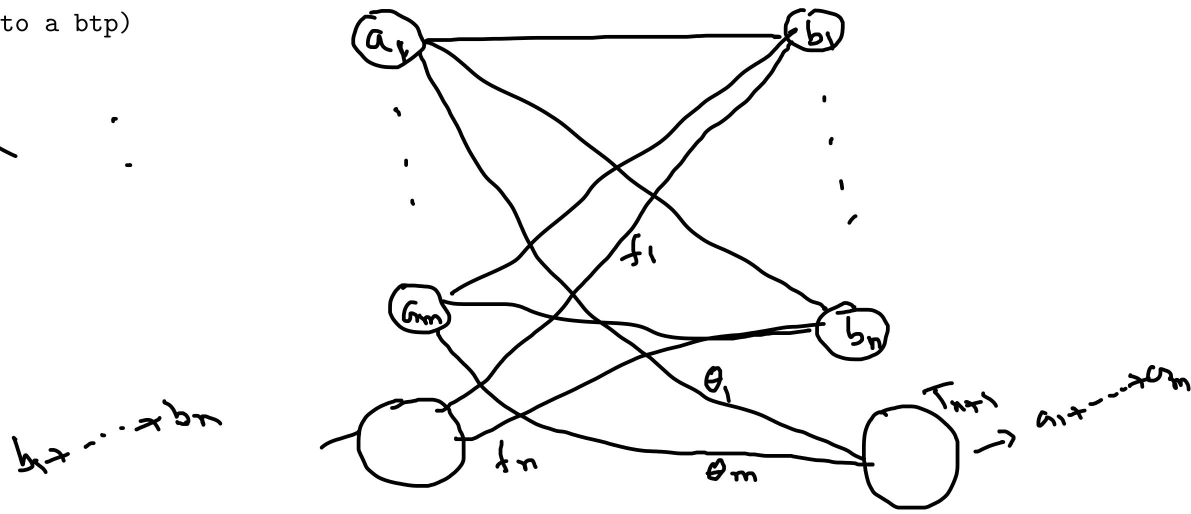
$$\sum_{i,j} c_{ij} x_{ij} + \sum_{i=1}^{m} \theta_i (a_i - x_{i1} - \cdots - x_{in}) + \sum_{j=1}^{n} f_j (b_j - x_{1j} - x_{2j} - \cdots - x_{mj})$$

$$\sum_j x_{ij} \le a_i, \quad \sum_i x_{ij} \le b_j, \quad x_{ij} \ge 0$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m} \theta_i\Big(a_i - \sum_{j=1}^{n} x_{ij}\Big) + \sum_{j=1}^{n} f_j\Big(b_j - \sum_{i=1}^{m} x_{ij}\Big) \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} \le a_i,\ \forall i, \qquad \sum_{i=1}^{m} x_{ij} \le b_j,\ \forall j, \qquad x_{ij} \ge 0. \end{aligned}$$

This problem is feasible as $x = 0$ is a feasible solution.

c)(Converting into a btp)



Introduce a virtual source S_0 and a virtual sink T_0. The surplus amount at the S_i can be transported to T_0 with cost per unit transportation $c_{i0} = \theta_i$. Similarly the unmet demand of T_j can be obtained from the S_0 with cost per unit transportation being $c_{0j} = f_j$. We set $c_{00} = 0$, $a_0 = \sum\limits_{j=1}^{n} b_j$, and $b_0 = \sum\limits_{i=1}^{m} a_i$. The btp is

$$\begin{array}{ll} \min & \sum\limits_{i=0}^{n}\sum\limits_{j=0}^{n} c_{ij}x_{ij} \\ \hline \text{s.t.} & \sum\limits_{j=0}^{n} x_{ij} = a_i,\ \forall i = 1,\cdots,m; \qquad \sum\limits_{i=0}^{m} x_{ij} = b_j,\ \forall j = 1,\cdots,n; \qquad x_{ij} \geq 0 \\ & x_{00} + x_{01} + \cdots + x_{0n} = b_1 + \cdots + b_n,\ x_{00} + x_{10} + \cdots + x_{m0} = a_1 + \cdots + a_m. \end{array}$$

It is an exercise for the reader to argue that a minimum solution of the btp corresponds to a minimum solution of the ubtp.

[21.1] **Example** A company has plants in three cities Guwahati, Shillong, Jorhat with capacities of production (in tons) of 200, 250, and 300, respectively. The company has signed contracts to supply goods of weight (in tons) 150, 200, 250, and 150 to mills (1), (2), (3), and (4), respectively, every month. Due to strikes in a certain month, the production (in tons) in the plants at G, S, and J are 150, 210, and 250, respectively. According to the contracts, the company has to pay a penalty of rupees (in thousand) 2, 3, 1.5, and 2, to mills (1),(2),(3), and (4), respectively, for every ton of product not supplied. The cost (in thousand) of transportation of the product per ton is given below.

Cost matrix:

Plant	Mill (1)	(2)	(3)	(4)
G	1.5	2.5	2	1.1
S	2.0	3.0	1.5	1.2
J	1.8	3.2	1.8	1.0

How should the company supply all the product minimizing the total cost?

G	1.5	2.5	2	1.1	150
S	2	3	1.5	1.2	210
J	1.8	3.2	1.8	1	250
Virtual Source	2	3	1.5	2	140
	150	200	250	150	

150	0			-1.8
	200	10		-1.3
		240	10	-1
			140	0
3.3	4.3	2.8	2	

$$\bar{C} = \begin{bmatrix} 0 & 0 & 1 & .9 \\ 0 & 0 & 0 & .5 \\ -.5 & -1 & 0 & 0 \\ -1.3 & -1.3 & -1.3 & 0 \end{bmatrix}$$

Answer. Introduce a new source S_4 with production capacity 140 tons. Set cost of transportation per ton of goods from S_4 to mills (1), (2), (3), and (4), to be 2, 3, 1.5, and 2, respectively. We have a btp. We find an initial bfs, calculate the simplex multipliers and compute the relative cost.

150 1.5	0 2.5	 2	 1.1	150 .7
  2	200 3	 1.5	10 1.2	210 1.2
  1.8	 3.2	250 1.8	0 1	250 1
  2	 3	 1.5	140 2	140 2
150 .8	200 1.8	250 .8	150 0	

$$\overline{c} = \begin{bmatrix} 0 & 0 & .5 & .4 \\ 0 & 0 & -.5 & 0 \\ 0 & .4 & 0 & 0 \\ -.8 & -.8 & -1.3 & 0 \end{bmatrix}.$$

Entering variable is x_{43}. The cycle is given by the edges $x_{4,3}, x_{4,4}, x_{3,4}, x_{3,3}$, in that order. So $\theta = 140$. We calculate the simplex multipliers and the relative costs for the new bfs.

150 / 1.5	0 / 2.5	2	1.1	150 / .7
2	200 / 3	1.5	10 / 1.2	210 / 1.2
1.8	3.2	110 / 1.8	140 / 1	250 / 1
2	3	140 / 1.5	2	140 / .7
150 / .8	200 / 1.8	250 / .8	150 / 0	

$$\bar{c} = \begin{bmatrix} 0 & 0 & .5 & .4 \\ 0 & 0 & -.5 & 0 \\ 0 & .4 & 0 & 0 \\ .5 & .5 & 0 & 1.3 \end{bmatrix}.$$

Entering variable is x_{23}. The cycle is given by the edges $x_{2,3}, x_{2,4}, x_{3,4}, x_{3,3}$, in that order. The value of $\theta = 10$. Calculate the simplex multipliers and the relative costs for the new bfs.

150 / 1.5	0 / 2.5	2	1.1	150 / .2
2	200 / 3	10 / 1.5	1.2	210 / .7
1.8	3.2	100 / 1.8	150 / 1	250 / 1
2	3	140 / 1.5	2	140 / .7
150 / 1.3	200 / 2.3	250 / .8	150 / 0	

$$\bar{c} = \begin{bmatrix} 0 & 0 & 1 & .9 \\ 0 & 0 & 0 & .5 \\ -.5 & -.1 & 0 & 0 \\ .0 & .0 & 0 & 1.3 \end{bmatrix}.$$

Entering variable is x_{31}. The cycle is given by the edges $x_{3,1}, x_{3,3}, x_{2,3}, x_{2,2}, x_{1,2}, x_{1,1}$, in that order. The value of $\theta = 100$. Calculate the simplex multipliers and the relative costs for the new bfs.

50 100 0 0
0 100 110 0
100 0 0 150
0 0 140 0

50 / 1.5	100 / 2.5	2	1.1	150 / .7
2	100 / 3	110 / 1.5	1.2	210 / 1.2
100 / 1.8	3.2	1.8	150 / 1	250 / 1
2	3	140 / 1.5	2	140 / 1.2
150 / .8	200 / 1.8	250 / .3	150 / 0	

sol'n for what

50 100 0 0
0 100 110 0
100 0 0 150

$$\bar{c} = \begin{bmatrix} 0 & 0 & 1 & .4 \\ 0 & 0 & 0 & 0 \\ 0 & .4 & .5 & 0 \\ 0 & 0 & 0 & .8 \end{bmatrix}.$$

Since the relative costs are all nonnegative we have found an optimum bfs to the btp:

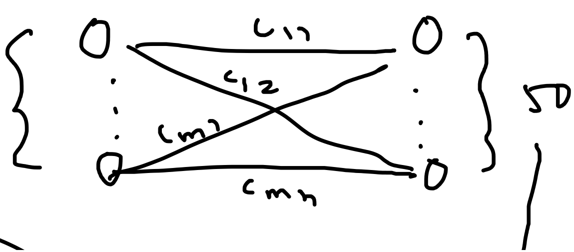
$$x_{11} = 50, x_{12} = 100, x_{22} = 100, x_{23} = 110, x_{31} = 100, x_{34} = 150, x_{43} = 140.$$

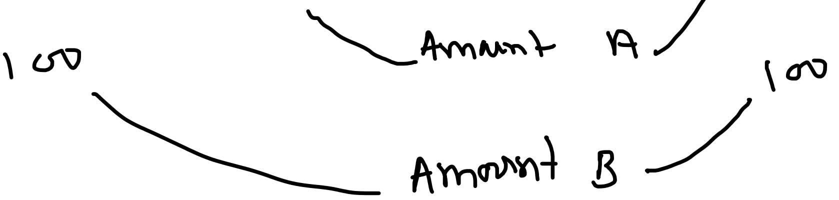
Accordingly, an optimum solution to the unbalanced problem is:

$$x_{11} = 50, x_{12} = 100, x_{22} = 100, x_{23} = 110, x_{31} = 100, x_{34} = 150$$

and the cost of transportation for this solution is 1330.

50





Transportation paradox

Take a btp $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & \cdots,\ \sum a_i = \sum b_j,\ c > 0,\ x_{ij} \geq 0 \end{array}$. Let α be the minimum cost. Consider a new btp by taking $a' \geq a$, $b' \geq b$, s.t. $\sum a'_i = \sum b'_j$ and let β be the minimum cost. Since the demand and supply have increased, one may think that $\beta \geq \alpha$. But this need NOT be true. For example, consider the btps below.

10	1	2
1	1	2
3	1	

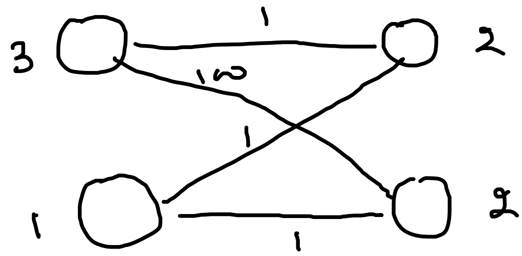
and

10	1	2
1	1	3
3	2	

.

The optimum cost for the first one is 13. However, the optimum cost for the one with higher demand and supply is 5.

3



100 + 3

3 ✓✓

2 ✓✓

2

3 + 2 = 5

Some exercises

[21.2] **Exercise(M)** Consider the btp

3	2	3	100
2	4	5	120
4	1	1	80
1	3	2	100
70	150	180	

We wish to get a solution so that the demand at sink 3 is met only from sources 2 or 4.

i) Put $c_{13} = c_{33} = m$, where m represents a large positive number and proceed as usual to get a solution. Show that this method is correct.

ii) Alternately, start with a bfs which does not involve x_{13} and x_{33}. Apply the algorithm, never select any of these two as incoming variables. Stop at a point where $\bar{c}$ is nonnegative at all other squares (these two squares may or may not be nonnegative). Show that this method is also correct.

[21.3] **Exercise(E)** The following is part of the optimal table for a btp.

30	70	
40		80
	80	
		100

The simplex multipliers for the sources are 0, -1, -1, and -4, respectively and for the sinks are 3, 2, and 6, respectively. Find the optimal cost.

[21.4] **NoPen** a) Let A be an upper triangular square matrix. Can we always view the system $Ax = b$ as a system $By = d$, where B is a lower triangular square matrix?

b) Suppose that we have a btp with integer supplies and demands. Is it necessary for a feasible solution of this btp to have all entries integers?

c) Suppose that we have btp with integer supplies and demands. Is it necessary for a bfs of this btp to have all entries integers?

d) Consider a btp and a bfs w. Suppose that one student computes the multipliers u_i, v_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$ by using $u_1 = 0$ and another student does that using $v_n = 0$. Do they necessarily get the same values for u_is and v_js?

e) Consider a btp and a bfs w. Suppose that one student computes the multipliers u_i, v_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$ by using $u_1 = 0$ and another student does that using $v_n = 0$. Do they get the same $\overline{c}$?

f) Consider a btp and a bfs w. Suppose that one student computes the multipliers u_i, v_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$ by using $u_1 = 0$ and gets $v_n = 5$. Another student does that using $v_n = 0$. What will be the value of u_1 for the second student?

g) Can a btp have an unbounded solution?

h) Can we construct a btp with integer demand and supply such that the set of feasible solutions T has a vertex with a non-integer entry?

i) Your friend views the process obtaining a better bfs for a btp as the same as obtaining a better bfs in simplex algorithm for a slpp.

'A variable x with $\overline{c} < 0$ is chosen in both the cases. A maximum possible value θ to this variable is given while satisfying the constraints. A basic variable which becomes 0 is termed as nonbasic.'

Therefore, he thinks that, in case of degeneracy, Bland's rule can be followed to avoid cycling while solving a btp. Do you agree?

j) Is it necessary that in a btp, each basis will give a feasible solution?

[21.5] **Exercise(E)** Consider the btp

1	2	3	4	5	100
6	7	8	9	10	100
5	4	3	2	1	60
6	7	8	9	10	100
40	60	80	80	100	

.

a) Argue that $\{x_{1,5}, x_{2,2}, x_{2,3}, x_{2,4}, x_{3,2}, x_{4,1}, x_{4,3}, x_{4,5}\}$ is a basis. Write down the corresponding solution. Is it a feasible solution?

b) Compute the relative cost at this vertex, taking $u_3 = 0$.

c) Obtain the next bfs taking the most negative entry. Is the cost of the new solution less than that of the old one? Would we have got a better solution, at this stage, had we selected the least negative entry in the relative cost matrix?

[21.6] **Exercise(E)** The following table show the production at three factories of a company, the demands at five of its outlets and the cost of transportation from factories to outlets. The data shows that the production is more than the demand at outlets.

4	1	2	6	9	100
6	4	3	5	7	120
5	2	6	4	8	120
40	60	60	80	80	

The company decides to transport all the production to the outlets while minimizing the overall cost of transportation. Write the corresponding btp in array format.

[21.7] **Exercise** A company has four factories and four outlets. It has to supply goods to the outlets. The cost of transportation per unit of goods, the availability and the capacity of the outlets is given below. This year due to lock down, the total production is less than the total capacity of the outlets. However, due to a lot of demand, the outlet 1 must be filled completely. Find the minimum cost for transporting all the goods to the outlets.

	Outlet 1	Outlet 2	Outlet 3	Outlet 4	Production
Factory 1	2	3	1	4	200
Factory 2	1	2	3	1	200
Factory 3	3	1	2	3	400
Factory 4	2	2	3	2	200
Capacity	400	300	300	200	