Q1

$\Omega = \{1,2,3,4\}$

Let $\mathcal{F}_1 = \{\phi, \Omega, \{1\}, \{2,3,4\}\}$.

$\mathcal{F}_2 = \{\phi, \Omega, \{2\}, \{1,3,4\}\}$.

Then $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \Omega, \{1\}, \{2\}, \{1,3,4\}, \{2,3,4\}\}$.

$\{1\} \in \mathcal{F}_1 \cup \mathcal{F}_2$ and $\{2\} \in \mathcal{F}_1 \cup \mathcal{F}_2$

But $\{1\} \cup \{2\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

Q2

$\mathbb{P}([1/2, 1]) = 1/2$.

$$\widetilde{\mathbb{P}}((1/2, 1]) = \int_{1/2}^{1} Z(\omega)\, d\mathbb{P}(\omega)$$

$$= 0.$$

$\Rightarrow$ $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are not equivalent probability measure.

Q3

$\mathbb{P}(\mathbb{N}) = 1$

$\mathbb{P}(\{m\}) = 0$. suppose $\mathbb{P}$ is a probability measure

then, $\mathbb{P}(\mathbb{N}) = \sum_{k=1}^{\infty} \mathbb{P}(\{k\})$

$\Rightarrow 1 = 0$ (contradiction)

Therefore $\mathbb{P}$ is not a Probability measure.

Q4 suppose $x_n \downarrow x$ in $[0,1]$. Then

$$A \cap [0,x] = \bigcap_{n \geq 1} (A \cap [0,x_n])$$

so $f(x) = \mathbb{P}(A \cap [0,x]) = \lim_{n\to\infty} \mathbb{P}(A \cap [0,x_n]) = \lim_{n\to\infty} f(x_n)$.

On the other hand, if $x_n \uparrow x$, then

$$A \cap [0,x) = \bigcup_{n \geq 1} (A \cap [0,x_n])$$

Because $\mathbb{P}(\{x\}) = 0$, we have

$$\mathbb{P}(A \cap [0,x]) = \mathbb{P}(A \cap [0,x)) = \lim_{n\to\infty} \mathbb{P}(A \cap [0,x_n])$$

$$\Rightarrow f(x) = \lim_{n\to\infty} f(x_n).$$

Therefore f is continuous at x

Note that $f(0) = 0$ and $f(1) = \frac{1}{2}$ and f is continuous on $[0,1]$. Therefore $\exists\ \hat{x} \in [0,1]$ such that $f(\hat{x}) = \frac{1}{4}$.

Let $B = A \cap [0,\hat{x}]$

Then $\mathbb{P}(B) = f(\hat{x}) = \frac{1}{4}$.

Q5 If $x < -2$, then $\mathbb{P}(\{X \leq x\}) = 0$.

If $-2 \leq x < 1$, then $\mathbb{P}(\{X \leq x\}) = \frac{1}{2} = \mathbb{P}(B)$

If $1 \leq x < 2$, then $\mathbb{P}(\{X \leq x\}) = \mathbb{P}(A) + \mathbb{P}(B) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

If $x \geq 2$, then $\mathbb{P}(\{X \leq x\}) = 1$.

$$F_X(x) = \begin{cases} 0 & \text{if } x < -2 \\ \frac{1}{2} & \text{if } -2 \leq x < 1 \\ \frac{5}{6} & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$