# 1 Risk Neutral Pricing

## 1.1 Change of Measure

**Fact:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $Z$ be a random variable such that $\mathbb{P}(Z > 0) = 1$ and $\mathbb{E}(Z) = 1$. Define a set function $\tilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ by

$$\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}.$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Further $\mathbb{P}(A) = 0$ if and only if $\tilde{\mathbb{P}}(A) = 0$. Two such measures are called equivalent. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be probability space. Let $X$ be a random variable defined on it such that $X \sim N(0,1)$. Define another random variable $Y$ by $Y = X + \theta$, $\theta \in \mathbb{R}$. Then under $\mathbb{P}$, $Y \sim N(\theta, 1)$. Define a new probability measure $\tilde{\mathbb{P}}$ by $\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}$, where $Z = e^{-\theta X - \frac{1}{2}\theta^2}$. What is the distribution of $Y$ under $\tilde{P}$.

$$\begin{aligned}\tilde{\mathbb{E}}(e^{tY}) &= \int e^{tY} d\tilde{\mathbb{P}} = \int e^{tY} Z d\mathbb{P} = \mathbb{E}(e^{tY} Z) = \mathbb{E}(e^{t(X+\theta)} e^{-\theta X - \frac{1}{2}\theta^2}) = e^{t\theta - \frac{1}{2}\theta^2} \mathbb{E}(e^{(t-\theta)X}) \\ &= e^{t\theta - \frac{1}{2}\theta^2} e^{\frac{1}{2}(t-\theta)^2} = e^{\frac{t^2}{2}}.\end{aligned}$$

Thus $Y \sim N(0,1)$ under $\tilde{\mathbb{P}}$. Thus we can change the mean of a random variable by changing the measure appropriately.Now let us try to do such an exercise for a stochastic process. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathcal{F}_t, 0 \le t \le T$. Further suppose $Z$ be a random variable such that $\mathbb{P}(Z > 0) = 1$ and $\mathbb{E}(Z) = 1$. Define a new probability measure $\tilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ by

$$\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}.$$

Now define the Radon-Nikodym derivative process $Z(t) = \mathbb{E}[Z|\mathcal{F}_t]$. Now for $s < t$,

$$\mathbb{E}[Z(t)|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Z|\mathcal{F}_s] = Z(s).$$

Thus $Z(\cdot)$ is a martingale.

**Lemma 1.1.** *Let $t$ satisfying $0 \le t \le T$ be given and let $Y$ be an $\mathcal{F}_t$ measurable random variable. Then $\tilde{\mathbb{E}}(Y) = \mathbb{E}(YZ(t))$.*

**Proof:**

$$\tilde{\mathbb{E}}(Y) = \mathbb{E}(YZ) = \mathbb{E}(\mathbb{E}[YZ|\mathcal{F}_t]) = \mathbb{E}(Y\mathbb{E}[Z|\mathcal{F}_t]) = \mathbb{E}(YZ(t)).$$

□

**Lemma 1.2.** *Let $s$ and $t$ satisfying $0 \le s \le t \le T$ be given and let $Y$ be an $\mathcal{F}_t$ measurable random variable. Then*

$$\tilde{\mathbb{E}}[Y|\mathcal{F}_s] = \frac{1}{Z(s)} \mathbb{E}[YZ(t)|\mathcal{F}_s].$$

**Proof:** It is clear that $\frac{1}{Z(s)} \mathbb{E}[YZ(t)|\mathcal{F}_s]$ is $\mathcal{F}_s$. Thus in order to show the above we need to show that,

$$\int_A \frac{1}{Z(s)} \mathbb{E}[YZ(t)|\mathcal{F}_s] d\tilde{\mathbb{P}} = \int_A Y d\tilde{\mathbb{P}},$$

for all $A \in \mathcal{F}_s$. The left hand side is equal to

$$\begin{aligned}\tilde{\mathbb{E}}\left(1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\right) &= \mathbb{E}\left(1_A \mathbb{E}[YZ(t)|\mathcal{F}_s]\right) \\ &= \mathbb{E}\left(1_A YZ(t)\right) = \tilde{\mathbb{E}}(1_A Y) = \int_A Y d\tilde{\mathbb{P}}\,,\end{aligned}$$

where the first and third equalities are by Lemma above.

**Theorem 1.3.** *(Girsanov) Let $W(t), t \in [0,T]$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_t, t \in [0,T]$ be a filtration for this Brownian motion. Let $\theta(t), t \in [0,T]$ be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right\} \quad \textit{and}$$

$$\tilde{W}(t) = W(t) + \int_0^t \theta(u)du\,.$$

*Assume that $\int_0^T \mathbb{E}(Z^2(t)\theta^2(t))dt < \infty$. Set $Z = Z(T)$. Then $\mathbb{E}(Z) = 1$, and under the probability measure $\tilde{\mathbb{P}}$ given by $\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}$, $\tilde{W}(t)$ is a Brownian motion.*

**Proof:** By Levy's theorem we need to show that $\tilde{W}$ is a $\tilde{\mathbb{P}}$ martingale and $[\tilde{W}, \tilde{W}](t) = t$.

$$\begin{aligned}d\tilde{W}(t) &= dW(t) + \theta(t)dt \\ \Rightarrow d\tilde{W}(t)d\tilde{W}(t) &= dW(t)dW(t) = dt\,.\end{aligned}$$

Thus it remains to show that $\tilde{W}$ is a martingale under $\tilde{\mathbb{P}}$. Now by Ito's formula,

$$dZ(t) = -\theta(t)Z(t)dW(t)\,.$$

Thus $Z(t)$ is a martingale. Hence $\mathbb{E}(Z) = \mathbb{E}(Z(T)) = \mathbb{E}(Z(0)) = 1$. Since $Z(t)$ is a martingale,

$$Z(t) = \mathbb{E}[Z(T)|\mathcal{F}_t] = \mathbb{E}[Z|\mathcal{F}_t]\,.$$

Thus $Z(t)$ is a Radon Nikodym derivative process and the above two lemmas are applicable. Now by Ito's product rule,

$$\begin{aligned}d(\tilde{W}(t)Z(t)) &= \tilde{W}(t)dZ(t) + Z(t)d\tilde{W}(t) + dZ(t)d\tilde{W}(t) \\ &= -\theta(t)\tilde{W}(t)Z(t)dW(t) + Z(t)dW(t) + Z(t)\theta(t)dt - Z(t)\theta(t)dt \\ &= (-\tilde{W}(t)\theta(t) + 1)Z(t)dW(t)\,.\end{aligned}$$

Thus $\tilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$. Hence using the lemma above,

$$\tilde{\mathbb{E}}[\tilde{W}(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\tilde{\mathbb{E}}[\tilde{W}(t)Z(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\tilde{W}(s)Z(s) = \tilde{W}(s)\,.$$

Hence the proof. □

## 1.2 Black Scholes Market with Single Stock

Let $W(t), t \in [0,T]$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_t, t \in [0,T]$ be a filtration for this Brownian motion. Consider a stock price process satisfying,

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\,,$$

where $\alpha(t)$ and $\sigma(t)$ are appropriate adapted processes. Such a process is called a generalized geometric Brownian motion. Using Ito's formula we get,

$$\begin{aligned} d\log(S(t)) &= \frac{1}{S(t)}dS(t) - \frac{1}{2S^2(t)}\sigma^2(t)S^2(t)dt \\ &= (\alpha(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW(t) \,. \end{aligned}$$

Thus,

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - \frac{\sigma^2(s)}{2})ds\right\} .$$

In addition assume that we have an adapted interest rate process $R(t)$. We define the discount process by

$$D(t) = e^{-\int_0^t R(s)ds} \,.$$

Thus $dD(t) = -R(t)D(t)dt$. Thus by Ito's product rule the discounted stock price process is

$$D(t)S(t) = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - R(s) - \frac{\sigma^2(s)}{2})ds\right\} ,$$

and its differential is

$$\begin{aligned} d(D(t)S(t)) &= (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\ &= \sigma(t)D(t)S(t)(\theta(t)dt + dW(t)) \,, \end{aligned}$$

where $\theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}$. Now using this $\theta(t)$ we define the measure $\tilde{\mathbb{P}}$ via Girsanov's theorem. Under the new measure

$$d\tilde{W}(t) = dW(t) + \theta(t)dt$$

is a Brownian motion. Thus

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\tilde{W}(t) \,.$$

Hence under $\tilde{\mathbb{P}}$, $D(t)S(t)$ is a martingale. This measure $\tilde{\mathbb{P}}$ is called the risk neutral measure. Replacing $W(t)$ by $\tilde{W}(t)$ we see that $S(t)$ satisfies

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\tilde{W}(t) \,,$$

or equivalently

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)d\tilde{W}(s) + \int_0^t (R(s) - \frac{\sigma^2(s)}{2})ds\right\} .$$

Consider an agent who begins with an initial capital $X(0)$ and at each time t, $0 \leq t \leq T$, holds $\Delta(t)$ shares of stock, investing or borrowing at the interest rate $R(t)$ as necessary to finance this. The differential of the porfolio is given by,

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\ &= R(t)X(t)dt + \Delta(t)(\alpha(t) - R(t))S(t)dt + \Delta(t)\sigma(t)S(t)dW(t) \\ &= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)(\theta(t)dt + dW(t)) \,. \end{aligned}$$

So by Ito's product rule,

$$d(D(t)X(t)) = \Delta(t)\sigma(t)D(t)S(t)(\theta(t)dt + dW(t)) = \Delta(t)\sigma(t)D(t)S(t)d\tilde{W}(t) \,.$$

Thus the discounted portfolio value process is a martingale under $\tilde{\mathbb{P}}$.

While deriving the Black-Scholes-Merton equation for the price of a European call option, we asked what initial capital $X(0)$ and portfolio process $\Delta(t)$ an agent would need in order to hedge a short position in the call, i.e., in order to have $X(T) = (S(T) - K)^+$. We generalise the question in this chapter. Let $V(T)$ be an $\mathcal{F}_T$ measurable random variable representing the pay-off at time $T$ of an European derivative security. We are interested to know what initial capital $X(0)$ and portfolio process $\Delta(t)$ an agent would need in order to hedge a short position in this derivative security, i.e., in order to have $X(T) = V(T)$. So the question is whether this is at all possible. Now if this can be done then the fact that the discounted portfolio process is a martingale under the risk neutral measure $\tilde{\mathbb{P}}$ implies,

$$D(t)X(t) = \tilde{\mathbb{E}}[D(T)X(T)|\mathcal{F}_t] = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t]\,.$$

Now since $X(t)$ is the value of the hedging portfolio at time $t$, by no-arbitrage argument this should be the price of the derivative security at time $t$. Thus if we denote the price of the derivative security at time t, by $V(t)$, the $V(t)$ must be given by

$$V(t) = \frac{1}{D(t)}\tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t]\,.$$

This is known as the risk neutral valuation formula.

Now let us use this formula to re-obtain the BSM price of an European call. Thus for this part we assume constant volatility $\sigma$ and constant interest rate $r$. Thus by the risk neutral valuation formula the call price should be

$$c(t, S(t)) = \tilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}_t]\,,$$

where $S(t)$ satisfies

$$S(t) = S(0)\exp\{\sigma\tilde{W}(t) + (r - \sigma^2/2)t\}\,.$$

Thus

$$\begin{aligned}S(T) &= S(t)\exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \sigma^2/2)\tau\}\\ &= S(t)\exp\{-\sigma\sqrt{\tau}Y + (r - \sigma^2/2)\tau\}\end{aligned}$$

where $Y$ is the standard normal random variable $Y = -\frac{\tilde{W}(T)-\tilde{W}(t)}{\sqrt{T-t}}$ and $\tau = T - t$. Thus $S(T)$ is the product of $\mathcal{F}_t$ measurable random variable $S(t)$ and the random variable $\exp\{-\sigma\sqrt{\tau}Y + (r - \sigma^2/2)\tau\}$ which is independent of $\mathcal{F}_t$. Thus by Independence lemma,

$$\begin{aligned}c(t, x) &= \tilde{\mathbb{E}}[e^{-r\tau}(x\exp\{-\sigma\sqrt{\tau}Y + (r - \sigma^2/2)\tau\} - K)^+]\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-r\tau}(x\exp\{-\sigma\sqrt{\tau}y + (r - \sigma^2/2)\tau\} - K)^+ e^{-y^2/2}dy\,.\end{aligned}$$

Now

$$\begin{aligned}&x\exp\{-\sigma\sqrt{\tau}y + (r - \sigma^2/2)\tau\} - K > 0\\ \Longleftrightarrow\ & y < \frac{1}{\sigma\sqrt{\tau}}[\log(x/K) + (r - \sigma^2/2)\tau] = d_-(\tau, x)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}
c(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}(x\exp\{-\sigma\sqrt{\tau}y + (r-\sigma^2/2)\tau\} - K)e^{-y^2/2}dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} xe^{-\frac{y^2}{2}-\sigma\sqrt{\tau}y-\frac{\sigma^2\tau}{2}}dy - \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau-\frac{y^2}{2}}dy \\
&= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}}dy - e^{-r\tau}KN(d_-(\tau,x)) \\
&= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)+\sigma\sqrt{\tau}} e^{-\frac{z^2}{2}}dz - e^{-r\tau}KN(d_(\tau,x)) \\
&= xN(d_+(\tau,x)) - e^{-r\tau}KN(d_-(\tau,x))\,,
\end{aligned}$$

where $d_+(\tau,x) = d_-(\tau,x) + \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}[\log(x/K) + (r+\sigma^2/2)\tau]$. Thus we have obtained the same formula.

**Exercise:** Using risk neutral valuation formula find the price of a forward contract on the stock price $S(\cdot)$ with strike price K and maturity $T$. (Recall the payoff at maturity is $S(T) - K$.)

Risk neutral evaluation formula was derived under the assumption that if an agent begins with the correct initial capital, then there exists a portfolio process $\delta(t)$ such that the agent's portfolio value at time $T$ will be $V(T)$. We will now verify the assumption. The existence of a hedging portfolio depends on the following theorem.

**Theorem 1.4.** *(Martingale Representation Theorem) Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_t, 0 \le t \le T$ be the filtration generated by this Brownian motion. Let $M(t), 0 \le t \le T$ be a martingale with respect to $\mathcal{F}_t$. Then there is an adapted process $\Gamma(t), 0 \le t \le T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u), \quad 0 \le t \le T\,.$$

**Exercise:** Let $\mathcal{F}_t$ be the filtration generated by the Brownian motion $W(\cdot)$. Find the martingale representation for the following martingales:

- $M_t = \mathbb{E}[W^2(T)|\mathcal{F}_t]$, $t \in [0,T]$.
- $M_t = \mathbb{E}[W^3(T)|\mathcal{F}_t]$, $t \in [0,T]$.
- $M_t = \mathbb{E}[e^{W(T)}|\mathcal{F}_t]$, $t \in [0,T]$.

Now we return to the hedging problem. Define $V(t)$ by the risk neutral evaluation formula,

$$D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t]\,.$$

Then $D(t)V(t)$ is a martingale with respect to $\mathcal{F}_t$. Now it is also known that for any portfolio value process $X(t)$ we have,

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)d\tilde{W}(u)\,.$$

By MRT, there exists $\Gamma(t), 0 \le t, \le T$ such that

$$D(t)V(t) = V(0) + \int_0^t \Gamma(u)d\tilde{W}(u)\,.$$

So if we want $X(t) = V(t)$ for all $t \in [0,T]$ , we choose $X(0) = V(0)$ and $\Delta(t)$ satisfying,

$$\Delta(t) = \frac{\Gamma(t)}{\sigma(t)D(t)S(t)}\,.$$

## 1.3 Black Scholes Market with Multiple Stocks

Now we will extend our market to the case of multiple stocks driven by multiple Brownian motions.

**Theorem 1.5.** *(Girsanov, multiple dimensions) Let $W(t) = (W_1(t), \dots, W_d(t)), t \in [0,T]$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_t, t \in [0,T]$ be a filtration for this Brownian motion. Let $\Theta(t) = (\Theta_1(t), \dots, \Theta_d(t)), t \in [0,T]$ be an adapted d-dimensional process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^t ||\Theta||^2(u)du\right\} \quad \textit{and}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du\,.$$

*Assume that $\int_0^T \mathbb{E}(Z^2(t)||\Theta||^2(t))dt < \infty$. Set $Z = Z(T)$. Then $\mathbb{E}(Z) = 1$, and under the probability measure $\tilde{\mathbb{P}}$ given by $\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}$, $\tilde{W}(t)$ is a d-dimensional Brownian motion.*

**Theorem 1.6.** *(MRT, multiple dimensions) Let $\mathcal{F}_t, 0 \le t \le T$ be the filtration generated by the $d - dimensional$ Brownian motion. Let $M(t), 0 \le t \le T$ be a martingale with respect to $\mathcal{F}_t$. Then there is a d-dimensional adapted process $\Gamma(t) = (\Gamma_1(t), \dots, \Gamma_d(t)), 0 \le t \le T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u) \cdot dW(u), \quad 0 \le t \le T\,.$$

Consider a market with $m$ stocks, each satisfying the stochastic differential equation

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\,,$$

for $i = 1, 2, \dots, m$ and where $W = (W_1, W_2, \dots, W_d)$ is a $d-$dimensional Brownian motion. Set $\sigma_i(t) = \sqrt{\sum_{j=1}^d \sigma_{ij}^2(t)}$, which we assume is never zero. Define $B_i(t) = \sum_{j=1}^d \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)} dW_j(u)$, $i = 1, 2, \dots, m$. Each $B_i(t)$ is a continuous martingale and

$$dB_i(t)dB_i(t) = \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}dt = dt.$$

Thus each $B_i(t)$ is a Brownian motion. In terms of $B_i(t)$ we have,

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sigma_i(t)dB_i(t)\,.$$

**Exercise:** Use Ito's product rule to show that $Cov(B_i(t), B_k(t)) = \mathbb{E}\left(\int_0^t \frac{\sum_{j=1}^d \sigma_{ij}(u)\sigma_{kj}(u)}{\sigma_i(u)\sigma_k(u)} du\right)$.

Thus $S_i(t)$s are also correlated. We assume an adapted interest rate process and define the discount process by $D(t) = e^{-\int_0^t R(u)du}$ or in differential form, $dD(t) = -D(t)R(t)dt$. So by Ito's product rule,

$$\begin{aligned} d(D(t)S_i(t)) &= D(t)S_i(t)[(\alpha_i(t) - R(t))dt + \sigma_i(t)dB_i(t)] \\ &= D(t)S_i(t)[(\alpha_i(t) - R(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)]\,. \end{aligned}$$

**Definition 1.7.** *A probability measure $\tilde{\mathbb{P}}$ is said to be a risk neutral measure if*

*1. $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent.*

*2. Under $\tilde{\mathbb{P}}$, the discounted stock price process $D(t)S_i(t)$ is a martingale for all $i = 1, 2, \ldots, m$.*

If we can rewrite

$$d(D(t)S_i(t)) = D(t)S_i(t)[(\alpha_i(t) - R(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)]$$

as

$$d(D(t)S_i(t)) = D(t)S_i(t)\sum_{j=1}^{d} \sigma_{ij}(t)[\Theta_j(t)dt + dW_j(t)]$$

for some $\Theta_j$, then we can use multi-dimensional Girsanov theorem to construct an equivalent probability measure $\tilde{\mathbb{P}}$ under which $\tilde{W} = (\tilde{W}_1, \ldots, \tilde{W}_d)$ where $d\tilde{W}_j(t) = \Theta_j(t)dt + dW_j(t)$, is a Brownian motion. Thus under $\tilde{\mathbb{P}}$

$$d(D(t)S_i(t)) = D(t)S_i(t)\sum_{j=1}^{d} \sigma_{ij}(t)d\tilde{W}_j(t)\,,$$

and hence $D(t)S_i(t)$ is a martingale.Thus equating dt terms we see that finding a risk neutral measure boils down to finding processes $\Theta_j(t)$ that satisfy

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\Theta_j(t), i = 1, 2, \ldots, m.$$

These are called the market price of risk equations. ($m$ equations in $d$ unknown processes.) If it is not possible to solve the market price of risk equations, then there is an arbitrage opportunity lurking in the model. We will not see a proof of this result but we will see an example illustrating this. Before coming to the example let us define arbitrage mathematically.

**Definition 1.8.** *An arbitrage is a portfolio value process $X(t)$ satisfying $X(0) = 0$ and there exists some $T > 0$ such that,*

$$\mathbb{P}(X(T) \geq 0) = 1, \quad \mathbb{P}(X(T) > 0) > 0\,.$$

**Exercise:** (i) Suppose the market has an arbitrage. So there is a portfolio value process satisfying $X_1(0) = 0$ and $\mathbb{P}(X_1(T) \geq 0) = 1, \quad \mathbb{P}(X_1(T) > 0) > 0$ for some $T > 0$. Show that if $X_2(0)$ is positive, then there exists a portfolio value process $X_2(t)$ starting at $X_2(0)$ and satisfying

$$\mathbb{P}(X_2(T) \geq \frac{X_2(0)}{D(T)}) = 1, \quad \mathbb{P}(X_2(T) > \frac{X_2(0)}{D(T)}) > 0\,.$$

(ii) Suppose that the market has a portfolio process $X_2(t)$ such that $X_2(0)$ is positive and the above holds. Then show that the market has an arbitrage.

**Example:** Suppose there are two stocks ($m = 2$) and one Brownian motion ($d = 1$) and suppose further that all co-efficients are constants. Thus

$$dS_1(t) = \alpha_1 S_1(t)dt + \sigma_1 S_1(t)dW(t) \quad \text{and}$$

$$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)dW(t)\,.$$

Then the market price of risk equations are

$$\alpha_1 - R = \sigma_1 \Theta, \alpha_2 - R = \sigma_2 \Theta\,.$$

These have a solution if and only if,

$$\frac{\alpha_1 - R}{\sigma_1} = \frac{\alpha_2 - R}{\sigma_2}\,.$$

Suppose this is not the case. Suppose

$$\frac{\alpha_1 - R}{\sigma_1} > \frac{\alpha_2 - R}{\sigma_2}\,.$$

Define

$$\mu = \frac{\alpha_1 - R}{\sigma_1} - \frac{\alpha_2 - R}{\sigma_2} > 0\,.$$

Suppose that at each time $t$ an agent holds $\Delta_1(t) = \frac{1}{S_1(t)\sigma_1}$ shares of stock 1 and $\Delta_2(t) = -\frac{1}{S_2(t)\sigma_2}$ shares of stock 2, borrowing or investing as necessary at the interest rate $R$ to setup and maintain this portfolio. The initial capital required to take the stock positions is $\frac{1}{\sigma_1} - \frac{1}{\sigma_2}$. If this is positive then we borrow and if this is negative then we invest. So the initial capital required to setup this portfolio is 0, i.e., $X(0) = 0$. Now

$$\begin{aligned}
d(X(t)) &= \Delta_1(t) dS_1(t) + \Delta_2(t) dS_2(t) + R(X(t) - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t))dt \\
&= \frac{\alpha_1 - R}{\sigma_1} dt + dW(t) - \frac{\alpha_2 - R}{\sigma_2} dt - dW(t) + RX(t)dt \\
&= \mu dt + RX(t)dt\,.
\end{aligned}$$

The differential of the discounted portfolio is

$$d(e^{-Rt}X(t)) = \mu e^{-Rt} dt\,.$$

Hence

$$\begin{aligned}
e^{-Rt}X(t) &= \frac{\mu}{R}(1 - e^{-Rt}) \\
\text{implies} \quad X(t) &= \frac{\mu}{R}(e^{Rt} - 1)\,.
\end{aligned}$$

Thus this is an arbitrage opportunity.
Now consider an agent who begins with an initial capital of $X(0)$ and at each time $t$, holds $\Delta_i(t)$ shares of stock $S_i$, investing and borrowing from the market as necessary. Thus the differential of the portfolio is given by

$$\begin{aligned}
dX(t) &= \sum_{i=1}^{m} \Delta_i(t) dS_i(t) + R(t)(X(t) - \sum_{i=1}^{m} \Delta_i(t) S_i(t))dt \\
&= R(t)X(t)dt + \sum_{i=1}^{m} \Delta_i(t)(dS_i(t) - R(t)S_i(t)dt) \\
&= R(t)X(t)dt + \sum_{i=1}^{m} \frac{\Delta_i(t)}{D(t)}(D(t)dS_i(t) - D(t)R(t)S_i(t)dt) \\
&= R(t)X(t)dt + \sum_{i=1}^{m} \frac{\Delta_i(t)}{D(t)} d(D(t)S_i(t))\,.
\end{aligned}$$

Thus

$$
\begin{aligned}
d(D(t)X(t)) &= \sum_{i=1}^{m} \Delta_i(t) d(D(t)S_i(t)) \\
&= \sum_{i=1}^{m} \Delta_i(t) D(t) S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t) d\tilde{W}_j(t) \\
&= \sum_{j=1}^{d} \left( \sum_{i=1}^{m} \Delta_i(t) D(t) S_i(t) \sigma_{ij}(t) \right) d\tilde{W}_j(t) \,.
\end{aligned}
$$

Thus under the risk neutral measure $\tilde{\mathbb{P}}$, the discounted portfolio process is also a martingale.

## 1.4 Fundamental Theorems of Asset Pricing

**Theorem 1.9.** *(First Fundamental Theorem of Asset Pricing) If a market model has a risk neutral measure then it does not admit any arbitrage.*

**Proof:** If a market model has a risk neutral measure $\tilde{\mathbb{P}}$, then every discounted portfolio value process is a martingale under $\tilde{\mathbb{P}}$. In particular, every portfolio value process satisfies $\tilde{\mathbb{E}}(D(T)X(T)) = X(0)$ for all $T > 0$. Let $X(t)$ be a portfolio value process with $X(0) = 0$. Suppose there exists $T > 0$ such that $\mathbb{P}(X(T) \geq 0) = 1$, i.e., $\mathbb{P}(X(T) < 0) = 0$. Since $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent, $\tilde{\mathbb{P}}(X(T) < 0) = 0$. Thus $\tilde{\mathbb{P}}(X(T) \geq 0) = 1$.
Claim: $\tilde{\mathbb{P}}(X(T) > 0) = 0$. If not, then $\tilde{\mathbb{P}}(X(T) > 0) > 0$, implies $\tilde{\mathbb{E}}(D(T)X(T)) > 0$, which is a contradiction. Hence the claim. By equivalence of $\mathbb{P}$ and $\tilde{\mathbb{P}}$, $\mathbb{P}(X(T) > 0) = 0$. Since $X(t)$ was any portfolio, there cannot exist an arbitrage opportunity. $\square$

**Definition 1.10.** *A market model is said to be complete if every derivative security can be hedged.*

Suppose that the market model has a risk neutral measure. That means, we have been able to solve the market price of risk equations, used the resulting $\Theta_i$s to define the risk neutral measure $\tilde{\mathbb{P}}$ via Girsanov's theorem. Further suppose that the filtration is generated by the d-dimensional Brownian motion $W(t)$. Let $V(T)$ be an $\mathcal{F}_T$ measurable random variable representing the payoff of some time T maturity derivative security. We want to know whether it is possible to hedge a short position in this derivative security. Define the process $V(t), 0 \leq t \leq T$ by

$$
D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t] \,.
$$

Then $D(t)V(t)$ is a martingale under $\tilde{\mathbb{P}}$ and so by martingale representation theorem there exists a d-dimensional process $\Gamma(t) = (\Gamma_1(t), \ldots, \Gamma_d(t))$ such that for all $t \in [0, T]$,

$$
D(t)V(t) = V(0) + \sum_{j=1}^{d} \int_0^t \Gamma_j(u) d\tilde{W}_j(u) \,.
$$

Now for any portfolio value process $X(t)$ we have

$$
d(D(t)X(t)) = \sum_{j=1}^{d} \left( \sum_{i=1}^{m} \Delta_i(t) D(t) S_i(t) \sigma_{ij}(t) \right) d\tilde{W}_j(t) \,.
$$

Thus

$$
D(t)X(t) = X(0) + \sum_{j=1}^{d} \int_0^t \left( \sum_{i=1}^{m} \Delta_i(u) D(u) S_i(u) \sigma_{ij}(u) \right) d\tilde{W}_j(u) \,.
$$

So if we start with an initial capital of $X(0) = V(0)$ and is able to choose portfolio processes $\Delta_1(t), \ldots, \Delta_m(t)$ such that the hedging equations

$$\frac{\Gamma_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t) S_i(t) \sigma_{ij}(t)$$

are satisfied for $j = 1, \ldots, d$ then we get,

$$D(t)X(t) = V(0) + \sum_{j=1}^{d} \int_0^t \Gamma_j(u) d\tilde{W}_j(u) = D(t)V(t)$$

for all $t \in [0, T]$. Thus $X(t) = V(t)$ for all $t \in [0, T]$, or in other words, $X(t)$ is a hedging portfolio.

**Theorem 1.11.** *(Second Fundamental Theorem of Asset Pricing) Consider a market model that has a risk neutral measure. The model is complete if and only if the risk neutral measure is unique.*

**Proof:** We first assume that the model is complete. Suppose the model has two risk neutral probability measures $\tilde{\mathbb{P}}_1$ and $\tilde{\mathbb{P}}_2$. Let $A$ be a set in $\mathcal{F}_T = \mathcal{F}$. Consider the derivative security with payoff $V(T) = 1_A \frac{1}{D(T)}$. Because the model is complete, a short position in this derivative security can be hedged, i.e., there exists a portfolio value process $X(t)$ with some initial condition $X(0)$ and satisfies $X(T) = V(T)$. Since both $\tilde{\mathbb{P}}_1$ and $\tilde{\mathbb{P}}_2$ are risk neutral measures, the discounted portfolio value process $D(t)X(t)$ is a martingale under both $\tilde{\mathbb{P}}_1$ and $\tilde{\mathbb{P}}_2$. Thus

$$\begin{aligned}
\tilde{\mathbb{P}}_1(A) &= \tilde{\mathbb{E}}_1(D(T)V(T)) = \tilde{\mathbb{E}}_1(D(T)X(T)) \\
&= X(0) = \tilde{\mathbb{E}}_2(D(T)X(T)) = \tilde{\mathbb{E}}_2(D(T)V(T)) = \tilde{\mathbb{P}}_2(A)\,.
\end{aligned}$$

Since $A$ was an arbitrary set in $\mathcal{F}$, we have that the measures are equal.
For the converse, suppose there is only one risk neutral measure. Thus the market price of risk equations has a unique solution. These equations are of the form $Ax = b$ where $A$ is the $m \times d$ dimensional matrix

$$A = \begin{bmatrix} \sigma_{11}(t) \ldots \sigma_{1d}(t) \\ \vdots \\ \sigma_{m1}(t) \ldots \sigma_{md}(t) \end{bmatrix},$$

$x$ is the d-dimensional column vector

$$x = \begin{bmatrix} \Theta_1(t) \\ \vdots \\ \Theta_d(t) \end{bmatrix},$$

and $b$ is the m-dimensional column vector

$$b = \begin{bmatrix} \alpha_1(t) - R(t) \\ \vdots \\ \alpha_m(t) - R(t) \end{bmatrix}.$$

Since by assumption this system of equation has a unique solution, so $KerA = \{x \in \mathbb{R}^d : Ax = \mathbf{0}\}$ must be trivial, i.e., $KerA = \{\mathbf{0}\}$. Thus the columns of $A$ are linearly independent. Thus $rankA = d = rankA^t$. Now in order to show that the market is complete we need to show that the hedging equations always has a solution. The hedging equations can be written in the form $A^t y = c$ where $y$ is the m-dimensional column vector

$$y = \begin{bmatrix} \Delta_1(t)S_1(t) \\ \vdots \\ \Delta_m(t)S_m(t) \end{bmatrix},$$

and $c$ is the d-dimensional column vector

$$c = \begin{bmatrix} \frac{\Gamma_1(t)}{D(t)} \\ \vdots \\ \frac{\Gamma_d(t)}{D(t)} \end{bmatrix}.$$

Now since $rank A^t = d$, we have $range A^t = \mathbb{R}^d$ and hence the completeness follows. $\square$