1 Lebesgue Integral

Definition:- A random variable $s : \Omega \to [0, \infty)$ is defined by $s(\omega) = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}(\omega)$, $\omega \in \Omega$, where n is some positive integer, $a_1, a_2, \cdots, a_n$ are non-negative real-numbers, $A_i \in \mathcal{F}$ for every i; $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} A_i = \Omega$. Such a function s is called a non-negative simple random variable. We say that $\sum_{i=1}^{n} a_i \mathcal{X}_{A_i}(\omega)$ is the standard representation of s if $a_1, a_2, \cdots, a_n$ are all distinct. We denote by $\mathbb{L}_0^+$ the class of all non-negative simple random variables on $(\Omega, \mathcal{F})$.

Examples:

- If $s(\omega) \equiv c$ for some $c \in [0, \infty)$, then $s \in \mathbb{L}_0^+$.
- For $A \subset \Omega$, consider $\mathcal{X}_A : \Omega \to [0, \infty)$, the indicator function of the set A, i.e.,

$$\mathcal{X}(\omega) = \begin{cases} 0 \text{ if } \omega \notin A \\ 1 \text{ if } \omega \in A. \end{cases}$$

  Then $\mathcal{X}_A \in \mathbb{L}_0^+$ iff $A \in \mathcal{F}$.
- Let $A, B \in \mathcal{F}$. then $s = \mathcal{X}_A \mathcal{X}_B \in \mathbb{L}_0^+$ since $s = \mathcal{X}_{A \cap B}$.
- Let $A, B \in \mathcal{F}$. If $A \cap B = \emptyset$, then clearly, $\mathcal{X}_A + \mathcal{X}_B = \mathcal{X}_{A \cup B} \in \mathbb{L}_0^+$.

Definition:- For $s \in \mathbb{L}_0^+$ with a representation $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$, we define $\int_\Omega s(\omega) d\mathbb{P}(\omega)$, the integral of s with respect to $\mathbb{P}$, by $\int_\Omega s(\omega) d\mathbb{P}(\omega) := \sum_{i=1}^{n} a_i \mathbb{P}(A_i)$.

We should check that $\int_\Omega s(\omega) d\mathbb{P}(\omega)$ is well defined i.e., if $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i} = \sum_{j=1}^{m} b_i \mathcal{X}_{B_j}$ where $\{A_1, A_2, \cdots, A_n\}$ and $\{B_1, B_2, \cdots, B_m\}$ are partitions of Ω by elements of $\mathcal{F}$, then

$$\sum_{i=1}^{n} a_i \mathbb{P}(A_i) = \sum_{b=1}^{m} b_j \mathbb{P}(B_j).$$

For this, we note that we can write

$$s = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mathcal{X}_{A_i \cap B_j} = \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mathcal{X}_{A_i \cap B_i}.$$

Thus if $A_i \cap B_j \neq \emptyset$ then $a_i = b_j$. Hence using finite additivity of $\mathbb{P}$,

$$\begin{aligned} \sum_{i=1}^{n} a_i \mathbb{P}(A_i) &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mathbb{P}(A_i \cap B_j) \\ &= \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mathbb{P}(A_i \cap B_j) = \sum_{j=1}^{m} b_j \mathbb{P}(B_j). \end{aligned}$$

Thus $\int_\Omega s(\omega) d\mathbb{P}(\omega)$ is independent of the representation of $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$.

Proposition 1.1. *For $s, s_1, s_2 \in \mathbb{L}_0^+$ and $\alpha \in \mathbb{R}$ with $\alpha \geq 0$, the following hold:*

(i) $0 \leq \int_\Omega s \; d\mathbb{P} < +\infty$

(ii) $\alpha s \in \mathbb{L}_0^+$ *and* $\int_\Omega (\alpha s) \; d\mathbb{P} = \alpha \int_\Omega s \; d\mathbb{P}$

(iii) $s_1 + s_2 \in \mathbb{L}_0^+$ *and* $\int_\Omega (s_1 + s_2) \; d\mathbb{P} = \int_\Omega s_1 \; d\mathbb{P} + \int_\Omega s_2 \; d\mathbb{P}$.

Proof. Statements (i) and (ii) are obvious.

For (iii), let $s_1 = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$ and $s_2 = \sum_{j=1}^{m} b_j \mathcal{X}_{B_j}$. Then we can write $s_1 = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \mathcal{X}_{A_i \cap B_j}$ and $s_2 = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j \mathcal{X}_{A_i \cap B_j}$. Thus $s_1 + s_2 = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i + b_j) \mathcal{X}_{A_i \cap B_j}$. Hence $s_1 + s_2 \in \mathbb{L}_0^+$ and using these representations, it is clear that $\int_\Omega (s_1 + s_2) d\mathbb{P} = \int_\Omega s_1 d\mathbb{P} + \int_\Omega s_2 d\mathbb{P}$. □

Excercise:- Let $s_1, s_2 \in \mathbb{L}_0^+$. Then prove the followings

1. Let $s_1 \geq s_2$. Set $\phi = s_1 - s_2$. Show that $\phi \in \mathbb{L}_0^+$.
2. If $s_1 \geq s_2$, then $\int s_1 d\mathbb{P} \geq \int s_2 d\mathbb{P}$.

Proposition 1.2. *Let $X : \Omega \to \mathbb{R}$ a non-negative bounded random variable, then there exists a sequence $\{s_n\}_{n \geq 1}$ of random variables in $\mathbb{L}_0^+$ such that $\mathbb{P}\{\omega \in \Omega : \lim_{n\to\infty} s_n(\omega) = X(\omega)\} = 1$.*

Proof. Let X be bounded by M. Then the sets $A_k^n = \{\omega : \dfrac{(k-1)M}{2^n} \leq X(\omega) < \dfrac{kM}{2^n}\}$, $1 \leq k \leq 2^n$. Then $\{A_k^n\}$ are disjoint, $A_k^n \in \mathcal{F}$ and have union $\cup_{k=1}^{2^n} A_k^n = \Omega$. We define function s_n on Ω by

$$s_n(\omega) = \sum_{k=1}^{2^n} \frac{M(k-1)}{2^n} \mathcal{X}_{A_k^n}(\omega).$$

Clearly, $s_n \in \mathbb{L}_0^+$ and it is easy to check that for every n,

$$s_n(\omega) \leq s_{n+1}(\omega), \; \forall \omega \in \Omega.$$

If $\omega \in A_k$ for some k, $1 \leq k \leq 2^n$. Then

$$s_n(x) = \frac{(k-1)M}{2^n}$$

and $X(\omega) \in \left[\frac{(k-1)M}{2^n}, \frac{kM}{2^n}\right)$. Thus we have $s_n(\omega) \leq X(\omega)$ and $X(\omega) - s_n(\omega) \leq \frac{M}{2^n}$. In other words, $\lim_{n\to\infty} s_n(\omega) = X(\omega)$. □

Consider the case n=1

Then $A_1^1 = [a, a_1] \cup (a_2, b]$

$A_2^1 = [a_1, a_2]$

$s_1 = 0.\mathcal{X}_{A_1^1} + \frac{M}{2}\mathcal{X}_{A_2^1}$.

Consider the case n=2

$A_1^2 = [a, a_2^{'}) \cup (a_2^{''}, b]$, $A_2^2 = [a_2^{'}, a_1) \cup (a_2, a_2^{''}]$, $A_3^2 = [a_1, c_2) \cup (c_2^{'}, a_2]$, $A_4^2 = [c_2, c_2^{'}]$.

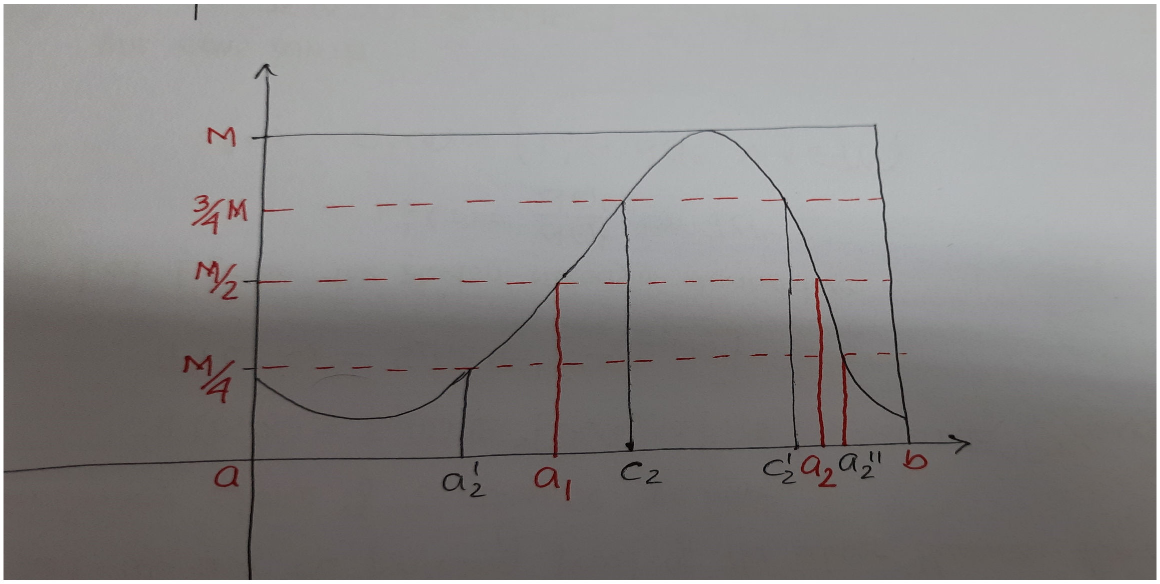

Figure 1:

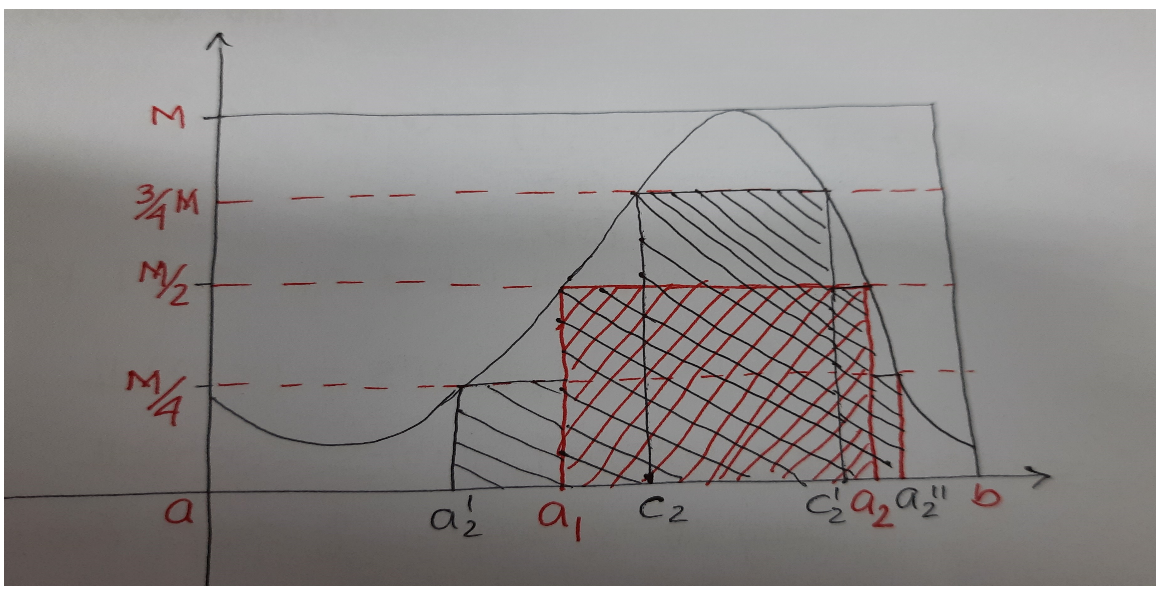

Figure 2:

$s_2 = 0.\mathcal{X}_{A_1^2} + \frac{M}{4}\mathcal{X}_{A_2^2} + \frac{M}{2}\mathcal{X}_{A_3^2} + \frac{3M}{4}\mathcal{X}_{A_4^2}$.

$$\begin{aligned}\int_\Omega s_1 \, d\mathbb{P}(\omega) &= 0.\mathbb{P}(A_1^{'}) + \frac{M}{2}\mathbb{P}(A_2^{'}) \\ &= \frac{M}{2}[a_2 - a_1].\end{aligned}$$

$$\int_\Omega s_2 \, d\mathbb{P}(\omega) = \frac{M}{4}[(a_1 - a_2^{'}) + (a_2^{''} - a_2)] + \frac{M}{2}[(c_2 - a_1) + (a_2 - c_2^{'})] + \frac{3M}{4}(c_2^{'} - c_2).$$

Set $\mathbb{L}^+ = \{f : \Omega \to [0,\infty) : \exists$ an increasing sequence of random variables $\{s_n\}_{n\geq 1}$ in $\mathbb{L}_0^+$ such that $s_n(\omega)$ converges to $s(\omega)$ almost surely$\}$. For $f \in \mathbb{L}^+$, we define the integral of f w.r.t. $\mathbb{P}$ by

$$\int_\Omega f(\omega) d\mathbb{P}(\omega) := \lim_{n\to\infty} \int_\Omega s_n(\omega) d\mathbb{P}(\omega).$$

Proposition 1.3. *If $f \in \mathbb{L}^+$ and $s \in \mathbb{L}_0^+$ are such that $0 \leq s \leq f$, then $\int_\Omega s(\omega)d\mathbb{P}(\omega) \leq \int_\Omega f(\omega)d\mathbb{P}(\omega)$ and $\int_\Omega f(\omega)d\mathbb{P}(\omega) = \sup\{\int_\Omega s d\mathbb{P} | 0 \leq s \leq f, s \in \mathbb{L}_0^+\}$.*

Now for any random variable X, define

$$X^+(\omega) = \max\{X(\omega), 0\},\ X^- = \max\{-X(\omega), 0\}.$$

Then

$$X = X^+ - X^-.$$

We can define $\int_\Omega X^+ d\mathbb{P}(\omega)$ and $\int_\Omega X^- d\mathbb{P}(\omega)$ provided both of them are not infinite. Then we define

$$\int_\Omega X(\omega)d\mathbb{P}(\omega) = \int_\Omega X^+(\omega)d\mathbb{P}(\omega) - \int_\Omega X^-(\omega)d\mathbb{P}(\omega).$$

We say that X is integrable if both $\int_\Omega X^+(\omega)d\mathbb{P}(\omega)$ and $\int_\Omega X^-(\omega)d\mathbb{P}(\omega)$ are finite. If both are infinite, then $\int_\Omega X(\omega)d\mathbb{P}(\omega)$ is not defined. If $\int_\Omega X^+(\omega)d\mathbb{P}(\omega)$ is finite and $\int_\Omega X^-(\omega)d\mathbb{P}(\omega) = \infty$ then $\int_\Omega X(\omega)d\mathbb{P}(\omega) = -\infty$. If $\int_\Omega X^-(\omega)d\mathbb{P}(\omega)$ is finite and $\int_\Omega X^+(\omega)d\mathbb{P}(\omega) = \infty$ then $\int_\Omega X(\omega)d\mathbb{P}(\omega) = \infty$.

2 Comparison of Riemann and Lebesgue integrals:-

Let f be a bounded function defined on $\mathbb{R}$, and let $a < b$ be numbers.

1. The Riemann integral $\int_a^b f(x)dx$ is defined iff the set of points $x \in [a,b]$ where $f(x)$ is not continuous has Lebesgue measure zero.
2. If the Riemann integral $\int_a^b f(x)dx$ is defined, then f is Borel measurable and so the Lebesgue integral $\int_a^b f(x)dx$ is also defined and the Riemann and Lebesgue integrals agree.

Definition:- Let X be an integrable random variable. Then the expectation of X is defined by $\mathbb{E}[X] := \int_\Omega X(\omega)d\mathbb{P}(\omega)$.

If $X \geq 0$, then $\mathbb{E}[X]$ is always defined [can be $+\infty$ as well].

Examples:-

1. Consider the infinite independent coin-toss space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$ with $p = \frac{1}{2}$. Let

$$Y_n(\omega) = \begin{cases} 1 \text{ if } \omega_n = H \\ 0 \text{ if } \omega_n = T. \end{cases}$$

$$\begin{aligned} \mathbb{E}[Y_n] &= 1 \cdot \mathbb{P}(Y_n = 1) + 0 \cdot \mathbb{P}(Y_n = 0) \\ &= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

2. Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}[0,1]$ and let $\mathbb{P}$ be the Lebesgue measure on $[0,1]$. Consider the random variable

$$X(\omega) = \begin{cases} 1 \text{ if } \omega \text{ is irrational} \\ 0 \text{ if } \omega \text{ is rational.} \end{cases}$$

$$\begin{aligned} \mathbb{E}[X] =& 1 \cdot \mathbb{P}(X = 1) + 0 \cdot \mathbb{P}(X = 0) \\ &= 1 \cdot \mathbb{P}(\omega \in [0,1] \backslash \mathbb{Q}) + 0 \cdot \mathbb{P}(\omega \in [0,1] \cap \mathbb{Q}) \\ &= 1 \cdot 1 + 0 \cdot 0 = 1. \end{aligned}$$

Properties:-

1. If X takes only finitely many values $x_0, x_1, \cdots, x_n$, then

$$\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega) = \sum_{k=0}^{n} x_k \mathbb{P}(X = x_k).$$

2. The random variable X is integrable iff $\int_\Omega |X(\omega)| d\mathbb{P}(\omega) < \infty$.

3. If $X \leq Y$ and X and Y are integrable then

$$\int_\Omega X(\omega) d\mathbb{P}(\omega) \leq \int_\Omega Y(\omega) d\mathbb{P}(\omega).$$

   Note: $|X| = X^+ + X^-$, $X^+ \leq |X|$, $X^- \leq |X|$.

4. If α and β are real constant and X and Y are integrable or if α, β are non-negative constant and X and Y are non-negative. Then

$$\mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}[X] + \beta \mathbb{E}[Y].$$

Two important convergence theorems:-

Definition:- Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of random variables, all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be another random variable defined on the same space. We say that $X_1, X_2, \cdots$ converges to X almost surely and write $\lim_{n\to\infty} X_n = X$ a.s. if $\mathbb{P}\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = 1$.

Monotone convergence theorem:- Let $X_1, X_2, \cdots$ be a sequence of random variables converging almost surely to another random variable X. If $0 \leq X_1 \leq X_2 \leq X_3 \cdots$ almost surely, then

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Corollary 2.1. *Suppose the non-negative random variable X takes countable many values $x_0, x_1 \cdots$, then*

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} x_k \mathbb{P}(X = x_k).$$

Proof. Let $A_k = \{X = x_k\}$. Then X can be written as

$$X = \sum_{k=0}^{\infty} x_k \mathcal{X}_{A_k}.$$

Define

$$X_n = \sum_{k=0}^{n} x_k \mathcal{X}_{A_k}.$$

Then $0 \leq X_1 \leq X_2 \leq \cdots$ and $\lim_{n\to\infty} X_n = X$. Note that

$$\mathbb{E}[X_n] = \sum_{k=0}^{n} x_k \mathbb{P}(X = x_k).$$

Using Monotone convergence theorem, we obtain

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n] = \lim_{n\to\infty} \sum_{k=0}^{n} x_k \mathbb{P}(X = x_k) = \sum_{k=0}^{\infty} x_k \mathbb{P}(X = x_k).$$

$\square$

Dominated Convergence Theorem:- *let $X_1, X_2, \cdots$ be a sequence of random variables converging almost surely to a random variable X. If there is another random variable Y such that $\mathbb{E}[Y] < \infty$ and $|X_n| \leq Y$ almost surely for every n, then*

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

1. Consider the space $(\Omega, \mathcal{B}[0,1], \mathcal{L})$, where $\mathcal{L}$ is the Lebesgue measure. Define

$$X_n(\omega) = \begin{cases} n \text{ if } \omega \in [0, \frac{1}{n}) \\ 0 \text{ otherwise.} \end{cases}$$

   Then $\lim_{n\to\infty} X_n(\omega) = 0$ a.s.

$$\lim_{n\to\infty} \int_0^1 X_n(\omega) d\mathbb{P}(\omega) = \lim_{n\to\infty} n \cdot \frac{1}{n} = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} X_n(\omega) d\mathbb{P}(\omega)$$

2. Consider a sequence of normal densities, each with mean zero and and the n^{th} having variance $\frac{1}{n}$.

$$f_n(x) = \sqrt{\frac{n}{2\pi}} e^{\frac{-nx^2}{2}}.$$

   If $x \neq 0$, then $\lim_{n\to\infty} f_n(x) = 0$ but

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \sqrt{\frac{n}{2\pi}} = \infty.$$

   Therefore

$$f_n(x) \to 0 \text{ a.s.}$$

   and

$$\int_{-\infty}^{\infty} f_n(x) dx = 1 \neq 0 = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x) dx.$$