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7 Lecture 7

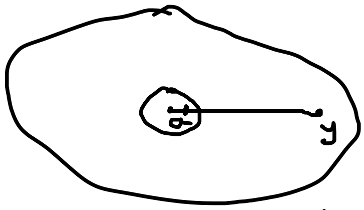
Fundamental theorems of linear programming

We now supply some results describing the behavior of the objective function on the feasible set. Here FTLP means the **fundamental theorems of linear programming**.

[7.1] **Fact** (Local minimum = global minimum) For a linear objective function on a convex set, a local minimum is a global minimum. That is, if v is a local minimum of $f(x) = c^t x$ on a convex set T. Then v is an absolute minimum.

✓ $x \in B_\varepsilon(a) \cap T$ | $y \in T$.

$f(x) \geq f(a)$



$z = a + \frac{\varepsilon}{2} \frac{y-a}{\|y-a\|} \in B_\varepsilon(a) \cap T$

$\cancel{f(a)} + \frac{\varepsilon}{2\|y-a\|} f(y-a) = f(z) \geq \cancel{f(a)} \Rightarrow f(y) - f(a) \geq 0.$

Proof. As v is a local minimum, $f(v)$ is minimum on some $B_\epsilon(v) \cap T$. Let $x \in T$ be any other point. Then $w := v + \frac{\epsilon}{2} \frac{x-v}{\|x-v\|} \in B_\epsilon(v)$. Hence, $c^t w \geq c^t v$, that is, $c^t x \geq c^t v$. ∎

[7.2] **Theorem** Consider minimizing $f(x) = c^t x$ over the set $T = \{x \mid Ax \leq b, x \geq 0\}$. Then the following statements are true.

✓a) The set T is closed, convex, and bounded below.‼

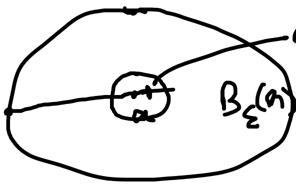
✓b) The set of minimum solutions is closed, convex, and bounded below.‼

✓c) If a (absolute) minimum solution exists, then the minimum is also attained at a vertex of T.

$f(v) \geq f(a)$; $=$ ✓

Assume $f(v) > f(a)$ $\Rightarrow \Leftarrow$

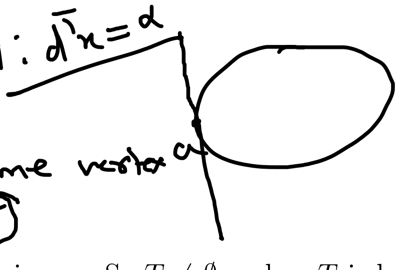
$a \in \partial T$



v $\quad$ $a + \frac{\varepsilon}{2} \frac{a-v}{\|a-v\|}$ $\quad$ $B_\varepsilon(a)$

$H \cap T = \{x \mid Ax \leq b, x \geq 0, d^T x \leq \alpha, -d^T x \leq -\alpha\}$

$H: d^T x = \alpha$



The min $f(x)$ on $H \cap T$ is attained at some vertex w of $H \cap T$ a

$f(w) \leq f(a)$ =

Proof. Use induction on n. Let $a \in T$ be a point of minimum. So $T \neq \emptyset$ and as T is bounded below it has a vertex, say v. If $f(v) = f(a)$, we done. So let $f(v) > f(a)$.

Case I. Let $a \in T^\circ$. Then let some $B_\epsilon(a) \subseteq T$. Consider $w := a + \frac{\epsilon}{2} \frac{a-v}{\|a-v\|} \in B_\epsilon(a)$. We see that $f(w) < f(a)$. This is a contradiction.

Case II. Let $a \in \partial T$. Take a supporting hyperplane $H : d^t x = r$ at a. Then $T \cap H = \{x \mid Ax \leq b, d^t x \leq r, -d^t x \leq -r\}$ is a polyhedron. It is bounded below. It contains a. It can be seen as subset of an $n-1$ dimensional space. By induction hypothesis, the minimum is obtained at a vertex v of $T \cap H$. But v is also a vertex of T. ■

d) (FTLP-I) If $T \neq \emptyset$ and bounded, then the minimum is attained at a vertex.

Proof. As $f(x)$ is continuous and T is compact, a minimal solution exists. By c), we are done.

e) (FTLP-II) If $T \neq \emptyset$ and f is bounded below, then the minimum is attained at a vertex.

$\exists\, d \in T$ we will stop

$\{a + \lambda(b-a) \mid \lambda \geq 0\}$

$f(d) < f(a)$

full ray or $[a,\ \underset{= d}{a + \lambda_0 (b-a)}]$

H $\quad H \cap T$

vertex of $H \cap T$

$\boxed{a \text{ is not pt of min } \Rightarrow \exists\, w \text{ vertex s.t } f(w) < f(a)}$

$\min_{x \in H \cap T} f(x) = f(w)$

Proof. In view of c), it is enough to prove the existence of a minimum solution.

Claim: Let $a \in T$ be any point. Suppose that a is not a point of minimum. Then there is a vertex v with $f(v) < f(a)$.

Proof of the claim: As a is not minimal, let b be a point with $f(b) < f(a)$. Consider $x_\alpha = b + \alpha(b-a)$, $\alpha \geq 0$. This is moving from in the direction $b - a$. As $\alpha \to \infty$, we have $f(x_\alpha) \to -\infty$. This is not possible, as f is bounded below. So $\exists\, \alpha_0 \geq 0$ such that $x_{\alpha_0} \in \partial T$ and $\forall \alpha > \alpha_0$, $x_\alpha \notin T$. Let H be a supporting hyperplane to T at x_{α_0}. Using induction, the minimum of f over $T \cap H$ is attained at a vertex v of $T \cap H$. Notice that $f(v) \leq f(x_{\alpha_0}) \leq f(b) < f(a)$. So the claim is justified.

Now, as $T \neq \emptyset$, there is a point $a \in T$. If a is not a point of minimum, then apply the claim to get a vertex v_1 such that $f(v_1) < f(a)$. Again, if v_1 is not minimal, then apply the claim to get a vertex v_2 such that $f(v_2) < f(v_1)$. Notice that $v_2 \neq v_1$. Continue. If none the vertices obtained in this sequence is minimal, then there will be infinitely many vertices of T, which is not true. Hence, the process must stop a minimal vertex. ■

Some exercises

[7.3] **Exercise(E)** (Illustrate FTLP-II.) Take the convex cube T in $\mathbb{R}^3_+$ with a vertex at 0. Consider minimizing $f(x) = x_1 + 2x_2 + 5x_3$ and take $a = e_1 + e_2 + e_3$. Take $b = (.6, .2, 1)$. What boundary point x_α do you get? What is the supporting hyperplane? What is the minimal vertex for f over $T \cap H$? Is that the minimal vertex for f over T?

[7.4] **Exercise(E)** (Does two points of min and two points of max imply two points of each possible value?) Suppose that, in an lpp we have more than one points of minimum value m and more than one points of maximum value M. Can there be a value $\alpha \in (m, M)$ which is attained by only one point?

Which objective functions are minimized at a vertex?

[7.5] **Theorem** (Also give a graphical interpretation.) Let w be a vertex of $T = \{x \mid Ax \geq b\}$ and consider minimizing $f(x) = c^t x$ over T. The following statements are true.

equalities in $\underbrace{\left[\quad\right]}_{A} w \geq \underbrace{\left[\,\right]}_{b}$

a) If $c^t = \lambda^t A_w$, $\lambda \geq 0$, that is, c^t a nonnegative combination of the rows of A_w, then w is a minimum.

$x \in T$ $c^T x = \lambda^T A_w x \geq \lambda^T b_w = \lambda^T A_w w = c^T w$

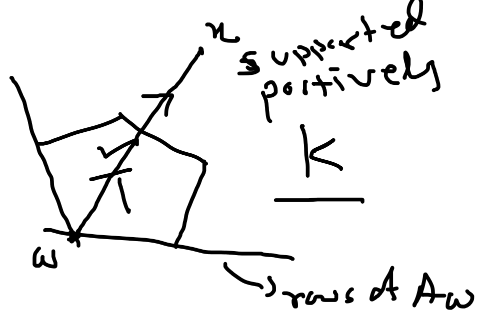
$Ax \geq b$

$A_w x \geq b_w$

Proof. Take any $x \in T$. Then $c^t x = \lambda^t A_w x \geq \lambda^t b_w = \lambda^t A_w w = c^t w$. So w is a point of minimum in T. ∎

b) If w is a minimum, then c^t is a nonnegative linear combination of the rows of A_w.

$\exists\ \lambda \geq 0$ s.t. $c^T = \lambda^T A_w$?



✓ $K = \{x \mid A_w x \geq b_w\}$

$z_\alpha = w + \alpha(x - w)$ $\alpha \geq 0$

$A_w z_\alpha \geq b_w \quad \Big| \quad \overline{A}_w w > \overline{b}_w$

continuity $\Rightarrow \exists\ \alpha > 0$ small s.t. $\overline{A}_w z_\alpha > \overline{b}_w$.

so $A z_\alpha \geq b \Rightarrow z_\alpha \in T \Rightarrow f(w) \leq f(w + \alpha(x - w)) \Rightarrow f(x) \geq f(w)$

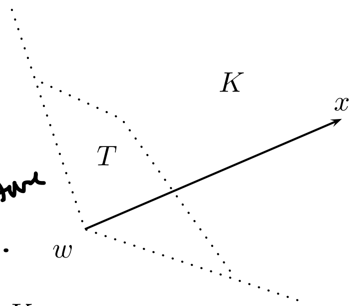
Proof. Suppose that w is a minimum. Note that the affine cone $K = \{x \mid A_w x \geq b_w\}$ contains T. (Observe "$A_w w = b_w$ and $Ax \geq b$" carefully. You that are looking at a set of supporting hyperplanes to T at w, containing T in their positive half spaces.)

$x \in K \Rightarrow$ ~~$f(w)$~~ $c^T w \leq c^T x$

$A_w x \geq A_w w \Rightarrow$ $c^T(x - w) \geq 0$

$A_w(x - w) \geq 0 \Rightarrow (x - w)^T c \geq 0$

$(x - w)^T A_w^T \geq 0$ $\equiv$ $A_w^T y = c$ has a nonnegative soln.



Claim: The objective function $f(x) = c^t x$ is minimum at w over K.

Proof of the claim: Take any $x \in K$. Then $A_w(w + \alpha(x - w)) \geq b_w$. As $\overline{A}_w w > \overline{b}_w$, by continuity, for a sufficiently small $\alpha > 0$, we have $\overline{A}_w(w + \alpha(x - w)) > \overline{b}_w$. That is, $w + \alpha(x - w) \in T$. Hence $c^t w \leq c^t(w + \alpha(x - w))$, implying that $c^t x \geq c^t w$. So the claim is justified.

By the claim, $x \in K \Rightarrow c^t x \geq c^t w$. In other words, $A_w(x - w) \geq 0 \Rightarrow c^t(x - w) \geq 0$. That is,

$$(x - w)^t A_w^t \geq 0 \Rightarrow (x - w)^t c \geq 0.$$

Hence, by Farka's lemma, $A_w^t z = c$ has a nonnegative solution. That is, c^t is a nonnegative combination of rows of A_w. ∎

c) If c^t is a positive combination of rows of A_w, then w is a strict minimum.

$c^T(x-w) = \underline{\lambda^T} A_w(x-w) > 0$ $\quad A_w(x-w) > 0$

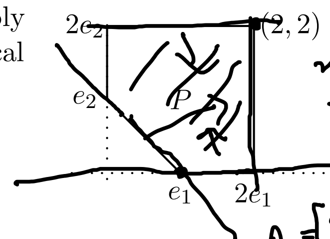
$x \in T,\ x \neq w$

$x \in K = \{x \mid A_w x \geq b_w\}$

$A_w(x-w) \geq 0$

Proof. Let $c^t = \lambda^t A_w$, $\lambda > 0$ and $x \in T$, $x \neq w$. Then $x \in K$. So $A_w(x-w) \gneq 0$, as $x - w \neq 0$ and $\mathsf{rank}\, A_w = n$. Hence $c^t(x-w) = \lambda^t A_w(x-w) > 0$. So w is a strict minimum over T (also over K). ∎

[7.6] **Example** In $\mathbb{R}^2$, consider $\mathsf{conv}((1,0),(2,0),(2,2),(0,2),(0,1))$. Supply answers to the following questions. You can use graphical method to visualize.



$x_1 \leq 2$

$x_2 \leq 2$

$A_w = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

1. Give α, β such that $f(x_1,x_2) = \alpha x_1 + \beta x_2$ is minimized at $w = (1,0)$.

$K = \left\{ x \mid \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

A_w $\quad [2\ \ 3]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \underline{f(x) = 2x_1 + 3x_2}$

Answer. Writing T in $Ax \geq b$ form, we get $A_w = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Hence $c^t = \lambda^t A_w$ for some $\lambda \geq 0$. For example, one such c^t is obtained by $c^t = [1 \quad 1]A_w = [1 \quad 2]$. That is, $f(x_1,x_2) = x_1 + 2x_2$ is such a function.

2. Give α, β such that $f(x) = \alpha x_1 + \beta x_2$ is strictly minimized at $(2,2)$. *Answer.* $f(x) = -3x_1 - 2x_2$.

[7.7] **Situation** Consider minimizing $f(x) = c^t x$ on $T = \{x \mid Ax \geq b\}$. Let w be a vertex of T. Then asking 'whether the function $f(x) = c^t x$ is minimized at w' is equivalent to asking 'whether $c^t = z^t A_w$ has a nonnegative solution z'. This is precisely what Farka's lemma talked about. But given c and w, how will we verify that?

Some exercises

[7.8] **Exercise(E)** Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ -2 & -1 & 0 \\ 0 & -1 & -2 \\ -2 & 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 3 \\ 3 \\ 2 \end{bmatrix}$, and $P = \{x \mid Ax \leq b\}$. Notice that $v = \begin{bmatrix} .5 \\ 1.5 \\ 2.5 \end{bmatrix}$ is a vertex of P. Give a vector c such that $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \leq b \end{array}$ has a unique minimum at v.

[7.9] **Exercise(E)** Take the unit cube T in $\mathbb{R}^3$ with vertices at $(0,0,0)$ and $(1,1,1)$. For each vertex v of T, supply one function $\alpha x + \beta y + \gamma z$ which is minimized uniquely at v.

[7.10] **Exercise(E)** Let $P = \mathsf{conv}(0, e_1, e_2, e_3, e_1+e_2+e_3) \subseteq \mathbb{R}^3$. Give two examples of vectors c such that $f(x) = c^t x$ is strictly minimized at $e_1+e_2+e_3$.

[7.11] **Exercise(M)** Let $S = \{(0,0),(2,0),(2,1),(1,2),(0,2)\}$ and $T = \mathsf{conv}(S)$. Draw the unit circle (center at origin) and consider points $c = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ on this unit circle. Highlight those points c for which $f(x) = c^t x$ is maximized at $(1,2)$ on T.

[7.12] **Exercise(H)** (Converse of [7.5] b)) Let $T = \{x \mid Ax \geq b\}$ and suppose that w is a strict minimum for $f(x) = c^t x$ in T. Argue that w must be a vertex of T. Show that c^t can be written as a positive combination of rows of A_w. (Hint: can we assume, without loss, that A_w consists of the first k rows of A? Show that $\exists \lambda \geq 0$ with first entry positive such that $c^t = \lambda^t A_w$. Continue.)

[7.13] **Exercise(M)** (Doubly stochastic matrices) A matrix $A \in M_3(\mathbb{R})$ is called DOUBLY STOCHASTIC if its entries are nonnegative, the sum of the entries in each row is 1, and the sum of entries in each column is 1. Let P be the set of all doubly stochastic matrix of order 3.

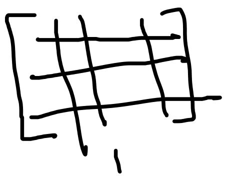
a) Show that P is a nonempty bounded polyhedron.

b) By FTLP, a linear function attains its minimum and maximum at vertices of P. Find the vertices of P.

c) Let $a, b \in \mathbb{R}^3$ be fixed vectors and $X \in P$. Is $f(X) = a^t X b$ a linear function?

d) Take a with $a_1 < a_2 < a_3$. Solve the problem $\begin{array}{ll} \text{opt} & a^t X \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ \text{s.t.} & X \in P \end{array}$ in two different ways.

[7.14] **Exercise(M)** Consider the set $S = \{e_1 + e_2, e_1 + e_3, e_2 + e_3\}$ in $\mathbb{R}^3$. We know that it forms a basis. So, each vector $[\alpha \quad \beta \quad \gamma]^t$ can be written as a linear combination of vectors in S. Do you think each vector $[\alpha \quad \beta \quad \gamma]^t$ with positive entries, can be written as a nonnegative linear combination of vectors in S? Argue that if α, β, γ are the sides of a triangle then the vector $[\alpha \quad \beta \quad \gamma]^t$ can be written as a nonnegative linear combination of vectors in S.



$a_{ij} \geq 0$ $\quad M_3(\mathbb{R})$ $\quad$ ~~$M_{m,n}(\mathbb{R})$~~

polyhedron $\quad \{X \in M_3 \mid X \text{ is doub stoch}\}$

$x_{11} + x_{12} + x_{13} = 1 \implies$ int 2 half spaces

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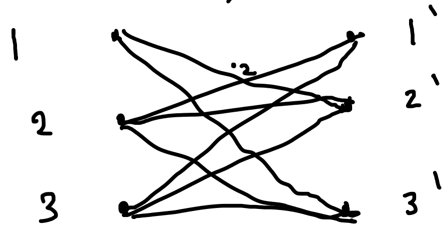
$$X \quad \begin{array}{c|ccc} & 1' & 2' & 3' \\ \hline 1 & 0 & .2 & .8 \\ 2 & .5 & .4 & .1 \\ 3 & .5 & .4 & .1 \end{array}$$

$$X_\varepsilon \quad \begin{bmatrix} & & \\ & .4+\varepsilon & .1-\varepsilon \\ & .1-\varepsilon & .1+\varepsilon \end{bmatrix}$$

$X_{-\varepsilon}$

Γ_X



$x = \frac{1}{2}(X_\varepsilon + X_{-\varepsilon})$