1 Generation from Mixture Distribution

Let K be a fixed positive integer and $\pi_1, \pi_2, \ldots, \pi_K$ be non-negative real numbers such that $\sum_{i=1}^{K} \pi_i = 1$. Also, let $f_1, f_2, \ldots, f_K$ be K PDFs. Then, it is easy to see that

$$f(x) = \sum_{i=1}^{K} \pi_i f_i(x), \quad x \in \mathbb{R} \tag{1}$$

is a PDF. The corresponding CDF is given by

$$F(x) = \sum_{i=1}^{K} \pi_i F_i(x), \quad x \in \mathbb{R},$$

where F_i is the CDF corresponding to PDF f_i. The PDF given in (1) is called mixture PDF and corresponding distribution is called mixture distribution. In this section, we will discuss the method of generation from PDF of the form (1). Let $q_0 = 0$ and $q_k = \pi_1 + \pi_2 + \ldots + \pi_k$ for $k = 1, 2, \ldots, K$. Now, the following algorithm can be used to generate random numbers from the PDF (1).

Algorithm 1 Generation from mixture distribution

1: generate U from $U(0, 1)$
2: **if** $q_{k-1} < U \leq q_k$ **then**
3: Generate X from f_k.
4: **end if**
5: **return** X

Let us discuss the justification of the algorithm. For $x \in \mathbb{R}$, we have

$$\begin{aligned} P(X \leq x) &= \int_0^1 P(X \leq x | U = u)\, du \\ &= \sum_{k=1}^{K} \int_{q_{k-1}}^{q_k} P(X \leq x | U = u)\, du \\ &= \sum_{k=1}^{K} \int_{q_{k-1}}^{q_k} F_k(x) du \\ &= \sum_{k=1}^{K} F_k(x)\,(q_k - q_{k-1}) \\ &= \sum_{k=1}^{K} \pi_k F_k(x). \end{aligned}$$

2 Generating Sample Paths of a Brownian Motion

By a standard one-dimensional Brownian Motion on $[0, T]$, we mean a stochastic process $\{W(t) : 0 \leq t \leq T\}$ with the following properties :

1. $W(0) = 0$.
2. The mapping $t \mapsto W(t)$ is, with probability 1, a continuous function on $[0, T]$.
3. The increments $\{W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_k) - W(t_{k-1})\}$ are independent for any k and any $0 \leq t_0 < t_1 < \cdots < t_k \leq T$.
4. $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ for any $0 \leq s < t \leq T$.

A consequence of (1) and (4) is that

$$W(t) \sim \mathcal{N}(0, t) \;\text{ for }\; 0 < t \leq T.$$

For constants μ and $\sigma > 0$, we call a process $X(t)$, a Brownian motion with drift μ and diffusion coefficient σ^2 (abbreviated $X \sim BM(\mu, \sigma^2)$) if $(X(t) - \mu t)/\sigma$ is a standard Brownian motion. Thus we may construct X from a standard Brownian Motion from W by setting,

$$X(t) = \mu t + \sigma W(t).$$

In discussing the simulation of $BM(0,\, 1)$, we mostly focus on simulating values

$$\{W(t_0),\, W(t_1),\, W(t_2),\, \ldots,\, W(t_n)\} \text{ or } \{X(t_0),\, X(t_1),\, X(t_2),\, \ldots,\, X(t_n)\}$$

at a fixed set of times $0 = t_0 < t_1 < t_2 < \cdots < t_n$. Because Brownian Motion has independent normally distributed increments, simulating the $W(t_i)$ or $X(t_i)$ from their increments is straightforward. Let $Z_1, Z_2, Z_3, \ldots, Z_n$ be independent standard normal variables generated using any of the standard methods. For a standard Brownian Motion we set : $t_0 = 0$ and $W(0) = 0$. Subsequent values can be generated using:

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} \cdot Z_{i+1} \;,\; i = 0, 1, \ldots, n-1.$$

For $X \sim BM(\mu, \sigma^2)$ with constants μ and σ and given $X(0)$ we set,

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} \cdot Z_{i+1} \;,\; i = 0, 1, \ldots, n-1.$$

The methods above are exact at the time points $t_1, t_2, \ldots, t_n$, but subject to discretization error, compared to the true Brownian motion.