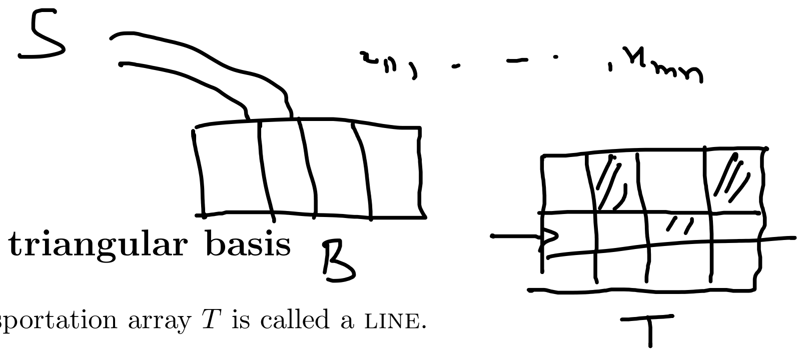


17 Lecture 17

Lower triangular basis

[17.1] **Definition** A row/column of a transportation array T is called a LINE.

[17.2] **Class workout** `(Understanding the array)` Consider a BTP. Let S be a set of some variables. Take the corresponding submatrix B of A and draw the transportation array T highlighting the variables from S.

1. For $i \leq m$, assume that ith row in B is zero. Can the ith row of T contain any highlighted variable?

   $\underline{x_{i*}} \Longrightarrow$ col below $\left[\frac{e_i}{*}\right] \longrightarrow$ is a col of B.

   Answer. No. Any variable in this will have the form $x_{i,*}$. Hence it's column in B will be of the form $\left[\frac{e_i}{*}\right]$. In that case, B should have a 1 in the ith row.

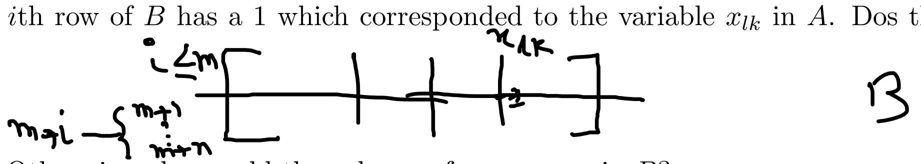
2. For $i \leq n$, let the $m+i$th row in B be zero. Can the ith column of T contain any highlighted variable?

   Answer. No.

3. T/F? The number of 1's in ith row of B equals the number of highlighted variables in the corresponding line of T.

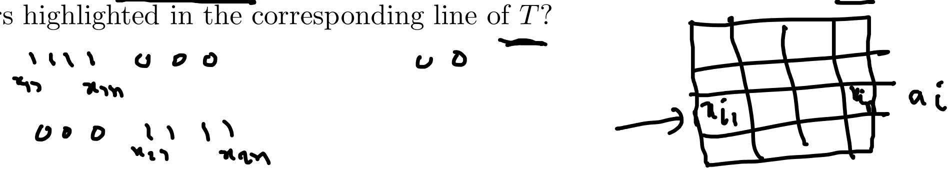
   Answer. True. The argument is similar.

4. Suppose that ith row of B has a 1 which corresponded to the variable x_{lk} in A. Dos this mean x_{lk} is highlighted?

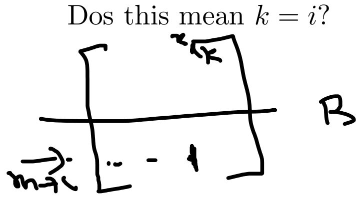
   Answer. Yes. Otherwise why would the column of x_{lk} appear in B?

5. Suppose that $i \leq m$ and that the ith row of B has a 1 which corresponded to the variable x_{ik} in A. Does this mean x_{ik} appears highlighted in the corresponding line of T?

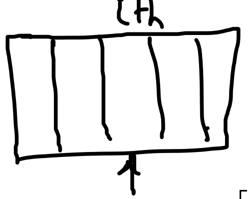
   

   Answer. Yes. The corresponding line is the ith row of T and x_{ik} is already highlighted.

6. Suppose that $i \leq n$ and that the $m+i$th row of B has a 1 which corresponded to the variable x_{lk} in A. Dos this mean $k = i$?

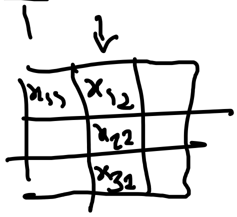
   $\left[\frac{*}{e_i}\right] = \left[\frac{x_l}{e_k}\right] \Rightarrow i = k$

   Answer. Yes. At $m+i$th row, if we find a 1, then it must be due to the column structure $\left[\frac{*}{e_i}\right]$. But below x_{lk} we have $\left[\frac{e_l}{e_k}\right]$. So $k = i$.

$m+i$ th row of $B \longleftrightarrow i$ th col of T

x_{1i}, x_{2i}, $\vdots$, x_{mi} $\quad \boxed{x_{li}}$



7. Continue. Does this mean x_{lk} appears highlighted in the corresponding line of T?

Answer. Yes. The corresponding line is the ith column of T and x_{li} is already highlighted.

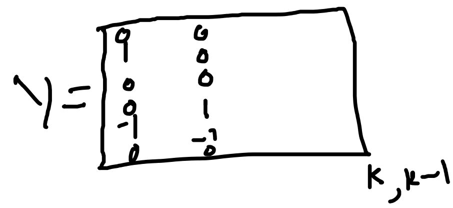
[17.3] <u>**Definition**</u> Let $A \in M_{m,n}$. Recall that A is called LOWER TRIANGULAR if each $a_{i,j}$ with $j > i$ is 0.

[17.4] <u>**Lemma**</u> Let $k \geq 2$ and $Y \in M_{k,k-1}$. Assume that

a) each column of Y has only two nonzero entries, one 1 and one -1,

~~b)~~ each row of Y is a linear combination of the remaining rows and

c) $\operatorname{rank} Y = k - 1$.



Then there exist permutation matrices P and Q such that PYQ is lower triangular.

Proof. $\cancel{k=2}$ $Y = \begin{bmatrix}1\\-1\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & \end{bmatrix}$, let $k > 2$.

a) $R_1 + R_2 + \dots + R_m = 0$

$R_i = -R_1 - \dots$

Y has a row with exactly one nonzero entry.

$$PYQ = \left[\begin{array}{c|c} \pm 1 & 0 \cdots 0 \\ \hline * & X \end{array}\right] \quad X: k-1 \times k-2$$

$$\begin{bmatrix}1 & 0\\ 0 & P'\end{bmatrix} P Y Q \begin{bmatrix}1 & 0\\ 0 & Q'\end{bmatrix} \to \text{Lower triangular.}$$

$P' X Q' \to$ lower triangular

$Y_r \to$ from Y by deleting r th row. $\to \operatorname{rank} Y_r = k-1$

$Y_r = \begin{bmatrix} | & | & | & | & | \end{bmatrix} \to \exists$ a col which has exactly one nonzero entry.

at most $2(k-1)-1$ nonzero entries.

The lemma is true for $k = 2$. Let $k > 2$.

Suppose that we could show that there is row with exactly one nonzero entry. Then, we can find permutation matrices P_1 and Q_1 such that $P_1 Y Q_1$ has the form $\left[\begin{array}{c|c} 1 & 0 \\ \hline * & X \end{array}\right]$. It is not difficult to see that X is a smaller sized matrix satisfying the hypothesis of the lemma.[10] Hence by induction, we can find permutation matrices P_x and Q_x such that $T_x := P_x X Q_x$ is lower triangular. Thus

$$\left(\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_x \end{array}\right] P_1\right) Y \left(Q_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q_x \end{array}\right]\right) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_x \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline * & X \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q_x \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline * & P_x X Q_x \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline * & T_x \end{array}\right]$$

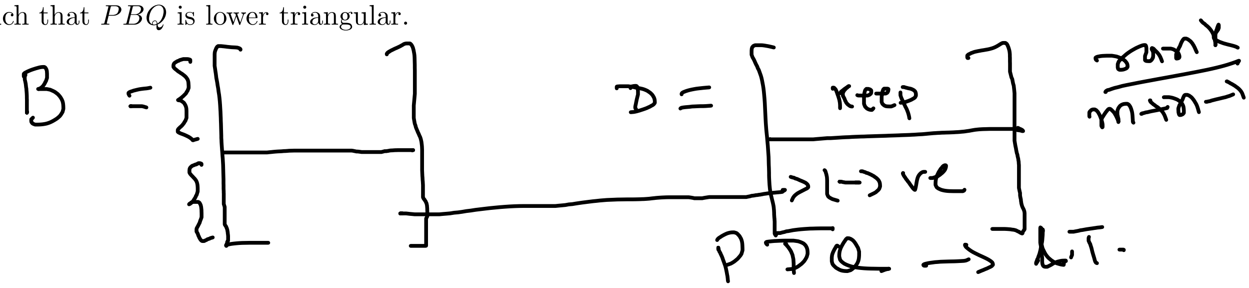
is lower triangular. As the product of permutation matrices is a permutation matrix[11], the proof will be complete.

So we only need to show that Y has a row with a nonzero entry. Towards that, since each row of Y is a combination of the remaining rows, the deletion of any row from Y keeps the rank unchanged. Let Y_r be the matrix obtained from Y by deleting the rth row. As $\mathsf{rank}\, Y = k-1$, we see that Y_r is nonsingular.

Is it possible that each column of Y_r still contains two nonzero entries? No, otherwise, it would be singular, as the column sum is zero. Hence Y_r can have at most $2(k-1)-1$ nonzero entries.

As Y_r is nonsingular, it cannot have a zero row and as it has $k-1$ many rows, it must have a row with exactly one nonzero entry. ■

[17.5] **Theorem** `(Any basis is a lower triangular basis.)` Let $(x_{i_1,j_1}, \cdots, x_{i_{m+n-1},j_{m+n-1}})$ be a basis for (12). Form B by taking the corresponding columns of A. Then there exist permutation matrices P and Q such that PBQ is lower triangular.



Proof. Note that $m+n \geq 2$ and $\mathsf{rank}\, B = m+n-1$. Since each row of A is a combination of the remaining rows, each row of B will also be a combination of the remaining rows.

Let D be obtained from B by multiplying the last n rows with -1. Then $\mathsf{rank}\, D = \mathsf{rank}\, B$.[12] Notice that each column of D contains exactly one 1, exactly one -1 and the rest are 0. Also each row of D is a linear combination of the remaining rows. So, the theorem follows by the previous lemma. ■

[17.6] **Why do we need this lower triangular results?** It gives us a very useful STRIKE-OFF PROCEDURE for the transportation algorithm. First let us understand the previous results a little more.

a) From the previous result, we see that the matrix B contains a row that has exactly one nonzero entry.

b) If we delete that row and the column corresponding to the nonzero entry (in B), then the new matrix contains a row that has exactly one nonzero entry.

c) Again, if we delete that row and the column corresponding to the nonzero entry, then the new matrix contains a row that has exactly one nonzero entry and so on.

[10] Since the row above X is a zero row and in Y each row was a linear combination of the remaining rows, it follows that each row of X is also a linear combination of the remaining rows. It is easy to see that the other two conditions also hold.

[11] Try multiplying the all ones vector from left and right and use that they are invertible.

[12] As D is obtained from B by multiplying a nonsingular matrix.

d) It continues till the resulting matrix is empty.

This gives us a ‘test’ for a basis.

[17.7] **Strike-off Theorem** (Test for a basis) Let $S = \{x_{i_1,j_1}, \cdots, x_{i_{m+n-1},j_{m+n-1}}\}$ be a set of variables for (12). Highlight these variables in the transportation ~~table~~ array T. Let B be the submatrix of A corresponding to S. Then the following are equivalent.

a) The set S is a basis.

b) The table T can be completely struck off by repeated application of the rule:
‘find a line in T containing only one unstruck highlighted variable and strike that line off’.

a) ⟹ b)

S , B → basismatrix

Let ith row of B contain exactly one 1 ~~at~~ below x_{lk}

B' obtained by ~~deleting~~ striking off that row and column from B.

Then in T $\exists$ a line that contains ~~exactly~~ the only highlighted var x_{lk}.
Strike off that line.

Proof. a)⇒b) We employ induction. Base case is trivial. Suppose that S is a basis. By [17.5], B has a row with exactly one nonzero entry 1 which corresponds to a variable, say, x_{lk}. Correspondingly, the table T has a line containing only the highlighted variable x_{lk}. Let B' be obtained by striking off the row and column of B corresponding to this entry. Correspondingly, strike off the line of T containing only x_{lk} to obtain T'.

Observe that B' is the basis matrix for the basis $S \setminus \{x_{lk}\}$ of the array T'. We are done by induction hypothesis.

b)⇒a) We employ induction. Base case is trivial. Suppose that $x_{i,j}$ is the first highlighted variable we struck off on T. This means the corresponding row of B has only one nonzero entry 1 (below $x_{i,j}$).

** It also means that the column below $x_{i,j}$ does not belong to the span of the columns for the other variables in S.

We strike off that row and the column of B corresponding to $x_{i,j}$ to obtain B' and let T' be obtained from T by striking off the corresponding line.

By hypothesis T' can be completely struck off. By induction hypothesis, B' is a basis matrix for T'. By **, B is also a basis matrix. ∎

> Test for a set of $m+n-1$ variables to be a basis.
>
> First way. Form the matrix B by taking the columns $C_{i,j} = \begin{bmatrix} e_i \\ e_j \end{bmatrix}$ for a variable $x_{i,j}$. Find a row with only one nonzero entry. Delete that row and the column corresponding to the nonzero entry. Repeat. If you can finish the whole matrix in this way, then the given set is a basis. Otherwise not.
>
> Second way. Form the transportation table and highlight those variables. Find a line with only one unstruck variable and strike it. Repeat. If you can finish the whole table in this way, then the given set is a basis. Otherwise not.

[17.8] **Example** Consider a BTP with $m = 3, n = 3$. Is $(x_{11}, x_{12}, x_{21}, x_{23}, x_{32})$ a basis?

a) First write the matrix B and make the corresponding transportation array. Highlight these variables.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

b) If this is a basis, then there should be a row in B with one nonzero entry. (The respective variable is never going to be used in the matrix X mentioned in the earlier proof.) This means, we can strike off that row and column from B to get X. Correspondingly, strike off that row/column from the transportation table.

1 0 0 0 0

x_{23}

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

c) Repeat this process with the new matrices and tables.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

→ 5 th row

Yes it is a basis.

[17.9] **Example** Consider a BTP with $m = 3, n = 3$. Is $(x_{11}, x_{12}, x_{21}, x_{22}, x_{32})$ a basis?

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not possible

x_{11}	x_{12}	
x_{21}	x_{22}	
	x_{32}	

not possible.