28 Lecture 28

Unconstrained optimization

[28.1] **Discussion**

a) Our general problem has the form

$$(P1) \qquad \begin{array}{ll} \min & \underline{f(x)} \\ \text{s.t.} & x \in T \subseteq \mathbb{R}^n. \end{array}$$

$T \to$ feasible set

b) Let $T \subseteq \mathbb{R}^n$ be nonempty and $a \in T$.

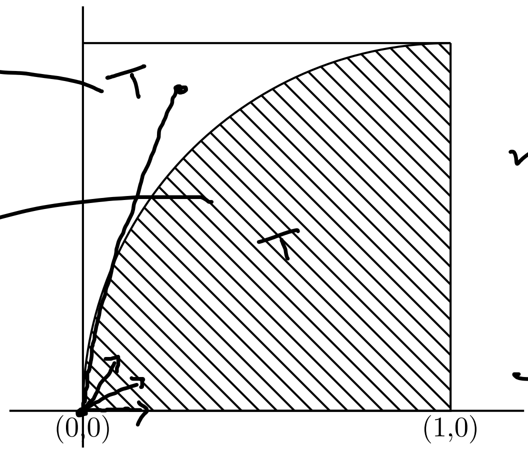
c) A FEASIBLE DIRECTION at a is a vector d (allow $d = 0$) such that we can travel from a in that direction some positive amount and while staying inside T. That is, $\exists \delta > 0$ such that $a + \theta d \in T$, $\forall \theta \in (0, \delta)$.

T $\vec{a}$ $\qquad \exists\ \delta > 0$ s.t $\forall\ t \in [0, \delta)$ $a + td \in T$

d) For example, consider the shaded region in the following figure.

only feasible directions available at $(0,0)$ are positive multiples $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

part of circle center at $(1,0)$ & radius 1.



$a = (0,0)$

$d = \begin{bmatrix} >0 \\ \geq 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$d = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin D((0,0))$

$a = (1,0)$

$d = \begin{bmatrix} \leq 0 \\ \geq 0 \end{bmatrix}$

Then any vector of the form $(> 0, \geq 0)$ is a (nontrivial) feasible direction at $(0, 0)$.

Any vector of the form $(\leq 0, \geq 0)$ is a feasible direction at $(1, 0)$.

If we consider the upper unshaded region, then the only nontrivial feasible direction available at $(0, 0)$ is $(0, 1)$ (or its positive multiples).

e) By $\boldsymbol{D(a)}$, we denote the set of all feasible directions available at a. (Remember that, 0 is allowed as a feasible direction.)

f) We shall use $\overline{D}(a)$ to denote the closure of $D(a)$.

$\odot \subseteq T$, $B_\varepsilon(a) \subseteq T$

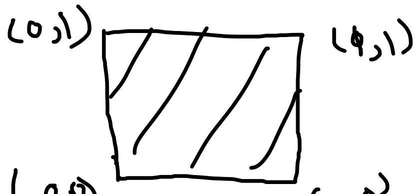
$a + td \in B_\varepsilon(a)$

$\forall\ t \in [0, \frac{\varepsilon}{\|d\|}]$

[28.2] **Workout** a)T/F? If $a \in T^\circ$, then $D(a) = \mathbb{R}^n$.

b) T/F? Consider the square plate with vertices at $0, e_1, e_2, e_1 + e_2$. Then $D(0) = \mathsf{cone}(e_1, e_2)$.

c) T/F? Consider the square plate with vertices at $0, e_1, e_2, e_1 + e_2$. Then $D(e_1) = \mathsf{cone}(-e_1, e_2)$.



$\begin{bmatrix} \geq 0 \\ \geq 0 \end{bmatrix} = \text{cone}(e_1, e_2)$

f is diff'ble $a \in T$

T, a, d → a feasible

$D_d f(a) < 0$.

[28.3] **Convention** When we say $f : T \to \mathbb{R}$ is differentiable at $a \in T$, we mean that f is defined in some neighborhood $B_\epsilon(a)$, even if we are considering f on T only.

The following is the first observation related to directional derivative.

a, d $\quad D_d f(a) < 0$.

[28.4] **Lemma** Let $T \subseteq \mathbb{R}^n$, $a \in T$ and $d \in D(a)$. Suppose that $D_d f(a) < 0$. Then $\exists \delta > 0$ such that $f(a + \theta d) < f(a)$, for all $0 < \theta \leq \delta$.

Then a cannot be a loc min.

Proof.

a $\quad a + \varepsilon d$

Take $t < \varepsilon/2$

$D_d f(a) < 0 \Rightarrow \lim_{t \to 0} \frac{f(a+td) - f(a)}{t} < 0$

$\lim_{t \to 0+} \frac{f(a+td) - f(a)}{t} < 0 \Rightarrow$

As $D_d f(a) = \lim_{\theta \to 0} \frac{f(a+\theta d) - f(a)}{\theta} < 0$, there exists $\alpha > 0$ such that for each $t \in (0, \alpha)$ we have $\frac{f(a+td)-f(a)}{t} < 0$. That is, $f(a + td) - f(a) < 0$ for each $t \in (0, \alpha)$.

$f(a+td) - f(a) < 0$ when t is small.

As $d \in D(a)$, there exists $\beta > 0$ such that $a + td \in T$ for each $t \in (0, \beta)$. Taking $\delta = \min\{\alpha, \beta\}$, we are done. ∎

[28.5] **Corollary** Hence, if a is a point of local minimum, then $D(a) \cap \{d \mid D_d f(a) < 0\} = \emptyset$. In words, 'if a is a local minimum, then the set of feasible directions along which the directional derivative is negative, must be empty'.

a loc min $\Rightarrow$ no feasible direction d s.t $D_d f(a) < 0$.

[28.6] **First order necessary condition** (FONC) Let $T \subseteq \mathbb{R}^n$ and a be a local minimum of f. Assume that $f : T \to \mathbb{R}$ is differentiable at $a \in T$.

a) Then $D_d f(a) \geq 0$, for all $d \in \overline{D}(a)$.

b) Moreover, if $a \in T^\circ$, then $f'(a) = 0$.

$d \in D(a) \Rightarrow D_d f(a) \geq 0$

$d_1, d_2, \dots \to d$, $\quad D_{d_n} f(a) \geq 0$

$\langle \nabla f(a), d_n \rangle$ → taking lim

$D(a) = \mathbb{R}^n$

$d \neq 0$, $D_d f(a) \geq 0$

$\langle \nabla f(a), d \rangle \geq 0$ →

$(\nabla f(a))_1 \geq 0$

taking $d = e_1$

Proof. a) We already know that for each $d \in D(a)$, we have $\langle \nabla f(a), d \rangle \geq 0$. If d is a limit point of $D(a)$ then we have a sequence $d^k \in D(a)$ converging to d. So $\langle \nabla f(a), d \rangle = \lim_k \langle \nabla f(a), d^k \rangle \geq 0$.

b) If $a \in T^\circ$, then we know that $D(a) = \mathbb{R}^n$. Hence $\langle \nabla f(a), d \rangle \geq 0$, for all $d \in \mathbb{R}^n$. In particular, considering $d = \pm e_1, \dots, \pm e_n$, we get $\nabla f(a) = 0$. ∎

[28.7] **Definition** Let $E \subseteq \mathbb{R}^n$ be open and $f : E \to \mathbb{R}$ be in $\mathcal{C}^1(E)$.

$\nabla f(a)$ — $[0, 0, 0, \ldots, 0]$

a) A point $a \in E$ is called a STATIONARY POINT or a CRITICAL POINT[19] of f if $f'(a) = 0$.

b) A critical point a is called a SADDLE POINT if every open ball $B_\epsilon(a) \subseteq E$ contains two points x and y such that $f(x) < f(a) < f(y)$.

c) Thus, a critical point is either a point of local minimum or a point of local maximum or a saddle point.

[28.8] **Example** Let $f(x_1, x_2) = e^{-(x_1^2+x_2^2)}$. Find its critical points.

Answer.

$$\begin{bmatrix} -2x_1 e^{-(x_1^2+x_2^2)} \\ -2x_2 e^{-(x_1^2+x_2^2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We have $f'(x) = \begin{bmatrix} -2x_1 f & -2x_2 f \end{bmatrix}$. The only critical point is 0. ⌟

[28.9] **Practice** Find the critical points of

(i) $x^2 + 4xy - y^2 - 8x - 6y$ (ii) $x \sin y$ (iii) $(x - y)^4$.

[28.10] **Application of FONC** Optimize $f(x, y) = x^2 e^{-x^4-y^2}$.

Answer.

a) Find critical points:

$$\begin{bmatrix} 2x e^{-x^4-y^2} - 4x^5 e^{-x^4-y^2} \\ -2y x^2 e^{-x^4-y^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x - 4x^5 \\ -2yx^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = 0$ or $x = \pm\left(\frac{1}{2}\right)^{1/4}$ | $x = 0$ or $y = 0$

critical pts $(0, t)$, $\left(\pm\left(\frac{1}{2}\right)^{1/4}, 0\right)$

From $D_x f = e^{-x^4-y^2}[2x - 4x^5] = 0$, we get $x = 0, \pm\left(\frac{1}{2}\right)^{1/4}$.

From $D_y f = -2x^2 y e^{-x^4-y^2} = 0$, we get $x = 0$ or $y = 0$.

So, the critical points are $(0, y)$, $\left(\pm\left(\frac{1}{2}\right)^{1/4}, 0\right)$.

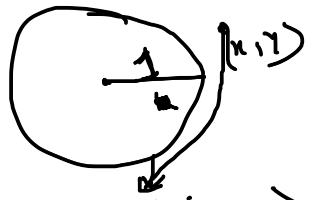
$f(x,y) = x^2 e^{-x^4-y^2} \geq 0$

$f(0,t) = 0 \to (0,t)$ are abs min.

[19] See Apostol, p377.

$f(x,y) = x^2 e^{-x^4-y^2}$

b) Conclude that $(0, y)$ are absolute minimums.[20]



$|(x,y)| = k > 1$

The value of f at $(0, y)$ is zero. Since $f \geq 0$, these are absolute minimums.

c) Conclude that f is very small outside large balls.

$$f(x,y) = \begin{cases} \dfrac{x^2}{e^{x^4+y^2}} \leq \dfrac{x^2}{e^{x^2+y^2}} \leq \dfrac{k^2}{e^{k^2}} \longrightarrow 0 & \text{if } |x| \geq 1 \\ \dfrac{x^2}{e^{x^4+y^2}} \leq \dfrac{1}{e^{y^2}} \leq \dfrac{1}{e^{k^2-1}} \longrightarrow 0 & |x| < 1 \end{cases}$$

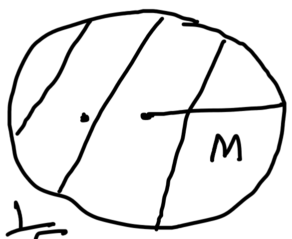
$\left(\pm\left(\frac{1}{2}\right)^{1/4}, 0\right)$

At the other critical points the value of f is $1/\sqrt{2e}$. Notice that, for $|(x, y)| = k > 1$ (that is, a point (x, y) on the k-circle), we have

$$f(x,y) = \frac{x^2}{e^{x^4+y^2}} \leq \frac{x^2+y^2}{e^{x^2+y^2}} = \frac{k^2}{e^{k^2}}, \text{ if } |x| \geq 1$$

and

$$f(x,y) = \frac{x^2}{e^{x^4+y^2}} \leq \frac{1}{e^{y^2}} \leq \frac{1}{e^{k^2-1}}, \text{ if } |x| < 1.$$



$\geq \frac{1}{\sqrt{2e}}$

Thus $f(x, y) \to 0$ as $|(x, y)| \to \infty$.

d) Conclude that the absolute maximum must exist.

Consider $r > 100$. Then outside $\overline{B}_r(0)$, we have $|f(x, y)| < 0.1$. Inside $\overline{B}_r(0)$, we have some points, where the f-value is more than 0.1. The function being continuous, an absolute maximum of f must exist in $\overline{B}_r(0)$. As that f-value is more than 0.1, it follows that those points will be the over all absolute maximums of f.

e) Conclude that the remaining critical points are absolute maximums.

By FONC, these absolute maximums must be critical points. Hence $\left(\pm\left(\frac{1}{2}\right)^{1/4}, 0\right)$ are absolute maximums.

[20]The correct English is 'minima', but I will use 'minimums'. It is less complicated.

[28.11] **Application of FONC** Optimize $f(x, y) = x^2y$ in $T = \overline{\mathsf{conv}(0, e_1, e_2, e_1 + e_2)}$.

Answer. a) Find critical points.

From $D_xf = 2xy = 0$, $D_yf = 2x^2 = 0$, we get $x = 0$. No critical points in T°.

b) So, we should check f on the boundary.

c) Conclude that $(0, t)$ and $(t, 0)$, $0 \le t \le 1$, are absolute minimums.

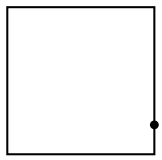
Notice that $f \ge 0$ on T. At points $(0, t)$ and $(t, 0)$, $0 \le t \le 1$, the f-value is 0. So, these are absolute minimums.

d) Conclude that point $(1, 1)$ is an absolute maximum.

See, the absolute maximum has to occur. It does not occur inside. It has to occur at the boundary. But among the boundary points, $(1, 1)$ has the largest f-value.

e) Can there be some other local optimums? For this, we need FONC. We show that below. Draw picture.

f) Let $a = (1, t)$, $0 < t < 1$. Compute $f'(a)$, $D(a)$.



Can we find a d such that $D_df(a) < 0$? What do we conclude?

Can we find a d such that $D_df(a) > 0$? What do we conclude?

We have $\bigtriangledown f(a) = \begin{bmatrix} 2t \\ 1 \end{bmatrix}$ and $D(a) = \{\begin{bmatrix} \le 0 \\ * \end{bmatrix}\}$. As $\bigtriangledown f(a)^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} > 0$, a is not a local maximum by FONC. As $\bigtriangledown f(a)^t \begin{bmatrix} 0 \\ -1 \end{bmatrix} < 0$, it is not a local minimum by FONC.

g) Let $a = (t, 1)$, $0 < t < 1$. Proceed similarly.

We have $\triangledown f(a) = \begin{bmatrix} 2t \\ t^2 \end{bmatrix}$ and $D(a) = \{ \begin{bmatrix} * \\ \leq 0 \end{bmatrix} \}$.

As $\triangledown f(a)^t e_1 > 0$ and $\triangledown f(a)^t(-e_1) < 0$, it does not satisfy FONC for being a local optimum.

h) Write the final conclusion.

The points $(0, t)$ and $(t, 0)$, $0 \leq t \leq 1$, are the absolute minimums, $(1, 1)$ is the absolute maximum. We do not have any other local optimums. ⌟

[28.12] **Practice** Consider $f(x, y) = x^2 y$ in $T = \overline{\mathsf{conv}(0, e_1, e_2, e_1 + e_2)}$. Apply FONC at points $a = (0, t), b = (t, 0)$, $0 < t < 1$ and at $(1, 1)$.

[28.13] **NoPen** In general, can we conclude that a is a point of local minimum using FONC at a?