MA 372 : Stochastic Calculus for Finance

July - November 2021

Department of Mathematics, Indian Institute of Technology Guwahati

Total Marks: 40 Mid-Semester Examination Duration: 75 minutes

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- Answer **all** questions.
- Justify all your answers. Answers without justification carry no marks.

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1. Set $\Omega = \{a, b, c, d\}, \mathcal{F} = 2^{\Omega}, \mathbb{P}(\{a\}) = 1/8, \mathbb{P}(\{b\}) = 1/8, \mathbb{P}(\{c\}) = 1/2, \mathbb{P}(\{d\}) = 1/4$. Then $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. We next define two random variables, X and Y , by the formulas $X(a) = X(d) = 2, X(b) = X(c) = 4$ and $Y(a) = Y(c) = 1, Y(b) = Y(d) = 2$.
   (i) List the sets in $\sigma(X)$.
   (ii) Compute $E[XY|X]$.
   (iii) Compute $E[X|Y]$. [2+4+4]

2. Let $W(t),\ t \geq 0$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let
   $$X(t) = W(t + 4) - W(4).$$
   Check whether the process $X(t),\ t \geq 0$ is a standard Brownian motion. [6]

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X(t) : t \geq 0$ be a supermartingale with respect to the filtration $\mathcal{F}_t : t \geq 0$ and $\mathbb{E}(X(t)) = 6 \quad \forall\ t$. Then check whether the process $X(t) : t \geq 0$ is a martingale with respect to the filtration $\mathcal{F}_t : t \geq 0$. [6]

4. Let X be a standard normal random variable and let Z be an independent random variable of X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying $\mathbb{P}\{Z = 0\} = \mathbb{P}\{Z = 1\} = 1/2$. Let $Y = ZX$.
   (i) Find the covariance of X^2 and Y^2 .
   (ii) Are X and Y independent?
   (iii) Find the correlation between X^2 and $Y_1 := Y^2 - E[Y^2|X^2]$. [4+3+6]

5. Let $W(t),\ t \geq 0$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $a > 0, b > 0$, define
   $$\tau = \min\{t \geq 0 | W(t) = -a \text{ or } W(t) = b\}.$$
   Assume that $\mathbb{E}(W(\tau)) = 0$. Find the distribution of $W(\tau)$.
   ($W(t)(w) := W(t, w)$ and $W(\tau)(w) := W(\tau(w), w)$)

   [5]