Assume that the statements hold up to stage $k-1$. At a stage k, if a row is struck off, then

$$x_{i_k j_k} = a_{i_k} - \sum \text{already found basic variables in that row.}$$

By induction hypothesis, the values of the already found variables in that row are in the required form. So the value of $x_{i_k j_k}$ has the required form. Argument is similar, if a column is struck off at stage k. ■

[18.14] **Corollary** (`Nonnegative integer bfs`) Suppose that in a BTP, all a_i's and b_j's are nonnegative integers. Then in any bfs, the values of the variables are nonnegative integers. In particular, if all a_i and b_j are 1, then the value any variable in that bfs is 0 or 1.

Proof. As the values of the variables are sums of the $\pm a_i$'s and $\pm b_j$'s, they must be integers. As it is a bfs, they must be nonnegative. For the next question, the value any variable in that bfs is 0 or 1, as the value of a variable (in a bfs) cannot be more than the a_i's and b_j's. ■

The minimality test

$A_{m \times n}$ drop last row $\tilde{A}$

$\tilde{B} \to$ basis matrix

[18.15] **Discussion** Consider a BTP with a bfs w. Let $\tilde{A}$ be the matrix obtained from A by dropping the last row and $\tilde{B}$ be a basis matrix for w (this is a submatrix of $\tilde{A}$). For minimality, we need to check the nonnegativity of the vector $\bar{c}^t = c^t - c^t_{\tilde{B}} \tilde{B}^{-1} \tilde{A}$.

a) Suppose that we have a vector $\alpha^t = \begin{bmatrix} p_1 & \cdots & p_m & q_1 & \cdots & q_{n-1} \end{bmatrix}$ such that $c^t - \alpha^t \tilde{A} \cancel{\geq 0}$ with basic entries 0. In view of our discussions on lpp, we know that $\alpha^t = c^t_{\tilde{B}} \tilde{B}^{-1}$ and $\bar{c}^t = c^t - \alpha^t \tilde{A}$.

old LPP $\min c^T x$, $Ax = b$ $B \to$ basis matrix, $x \to$ bfs $\bar{c}^t = c^t - c_B^T B^{-1} A$

$c^t - c^t_{\tilde{B}} \tilde{B}^{-1} \tilde{A} \geq 0$

b) Suppose that we have a vector $\beta^t = \begin{bmatrix} u_1 & \cdots & u_m & v_1 & \cdots & v_{n-1} & 0 \end{bmatrix}$ such that $c^t - \beta^t A \geq 0$ with basic entries 0. As the last entry of β is 0, we see that

$$\beta^t = \begin{bmatrix} \alpha^t & 0 \end{bmatrix} \quad \text{and} \quad \bar{c}^t = c^t - \alpha^t \tilde{A} = c^t - \beta^t A.$$

$c^t - \alpha^t \tilde{A}$ has basic entries 0 $\overset{?}{\Rightarrow}$ $\alpha^t = c^t_{\tilde{B}} \tilde{B}^{-1}$? yes ✓

Recall $c^t - \alpha^t A = [c_B^T \; c_C^T] - \alpha^T [B \; C]$ has basic entries 0 means $c_B^T = \alpha^T B$. so $\alpha^T = c_B^T B^{-1}$

$c^t - \alpha^t \tilde{A} = c^t - \underbrace{[\alpha^t \; 0]}_{\beta} A$

Idea: find a β s.t. $c^T - \beta^t A$ has basic entries 0. Then $c^T - \beta^t A = \bar{c}^T$.

$\bar{c}^T = c^T - c_B^T B^{-1} A$ | $c^T - \alpha^T A$ → has basic entries 0. $= \bar{c}^T$?

$A_{m+n} \to \tilde{A}$ $\quad c^T - \alpha^T \tilde{A}$ → has basic entries 0

$= c^T - [\alpha^T \ 0] A$ → is -

c) Assume that we have done b). Let $x_{i,j}$ be a basic variable. We must have $\bar{c}_{i,j} = 0$. That is, $c_{i,j} - u_i - v_j = 0$.

This means, $c_{i,j} = u_i + v_j$ on a basic square of the transportation array.

$\bar{c} \geq 0 \qquad c^T - \beta^T A = c^T - [u_1 \cdots u_m \ v_1 v_2 \cdots v_n] \ \ v_n = 0$

d) Suppose that we are searching for a vector $\beta^t = \begin{bmatrix} u_1 & \cdots & u_m & v_1 & \cdots & v_{n-1} & v_n = 0 \end{bmatrix}$ such that the basic entries of $c^t - \beta^t A$ are 0. Knowing that the basic squares form a tree and $v_n = 0$, we can compute u_i's and v_j's uniquely.

x_{ij} → non basic var

$\bar{c}_{ij} = c_{ij} - [u_1 \cdots u_m \ v_1 \cdots v_n] \begin{bmatrix} e_i \\ e_j \end{bmatrix}$

$= c_{ij} - u_i - v_j = 0$

e) Suppose that we have computed a vector $\beta^t = \begin{bmatrix} u_1 & \cdots & u_m & v_1 & \cdots & v_{n-1} & v_n = 0 \end{bmatrix}$ such that the basic entries of $c^t - \beta^t A$ are 0. In order to conclude that the current basis is a minimal basis, we must have $c^t - \beta^t A \geq 0$.

This means, for a nonbasic square $x_{i,j}$, we must have $c_{i,j} - u_i - v_j \geq 0$.

If we have that, then w is a minimum. These u_i's and v_j's are called the SIMPLEX MULTIPLIERS.

[ ] In view of this discussion, we give an algorithm to check the minimality of a basis.

Algorithm to check the minimality of a basis w.

a) Highlight the basic squares in the transportation table.

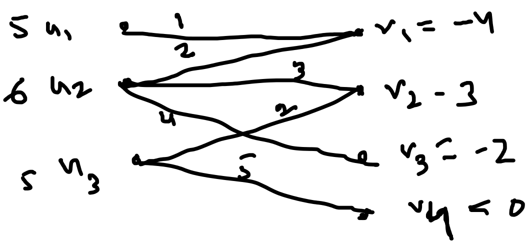
b) Write $u_1, \cdots, u_m$ at the right end of the rows in the table in top to bottom order.

c) Write $v_1, \cdots, v_n$ at the bottom end of the table in left to right order.

d) Make any one of u_i's or v_j's equal to 0. You may use $v_n = 0$. A normal convention is to choose the u_i or v_j whose line contains many basic variables.

e) Compute the remaining u_i's and v_j's, using $c_{ij} = u_i + v_j$ in basic squares.

f) If $c_{ij} \geq u_i + v_j$ in all nonbasic squares, then w is a minimum.



[18.16] **Example** Consider the btp and the highlighted bfs.

$c_{ij} = u_i + v_j$

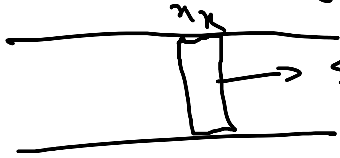
$c_{ij} - u_i - v_j = 0$

40 / 4	50 / 1	10 / 2	6	9	100	u_1 = 6
6	4	60 / 3	60 / 5	7	120	u_2 = 7
6	2	3	30 / 5	90 / 7	120	u_3 = 7
40 v_1	50 v_2	70 v_3	90 v_4	90 v_5		
-2	-5	-4	-2	0		

$B = [6\ 7\ 7\ -2\ -5\ -4\ -2\ 0]$

Is it minimal?

$$\bar{c} = \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$



x_k $\to \le 0$ will not come

-2

unbounded situation

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Answer. We take $u_1 = 0$. The other simplex multipliers are $v_1 = 4$, $v_2 = 1$, $v_3 = 2$, $u_2 = 1$, $v_4 = 4$, $u_3 = 1$, $v_5 = 6$. On each nonbasic square we have $\bar{c}_{i,j} = c_{i,j} - u_i + v_j \geq 0$. So this bfs is minimal.

To get an improvised bfs

[18.17] **Discussion** `(To get an improvised bfs)`

a) Suppose that at a bfs with basis S we found that $\bar{c}_{i,j} < 0$. What shall we do? In order to answer that, we need a few more observations.

b) The objective function in a BTP is bounded below.

Proof. Let $v = \sum a_i$ (total amount of goods to be transported) and let m be the minimum of the per unit costs of transportation. Then the cost of any solution is at least mv. So the objective function is bounded below. ∎

c) Recall the simplex algorithm. When we have all entries of the column (for which $\bar{c}$ is negative) in the simplex table nonpositive, we concluded that the problem is unbounded below.

d) But such a situation will never happen for a BTP as we already know that the function is bounded below.

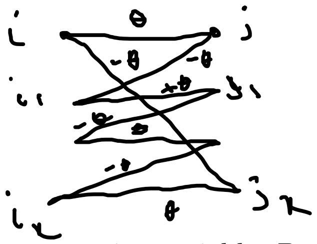
$\bar{c}_{ij} < 0$

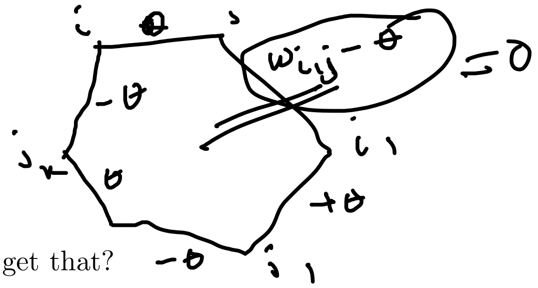
$x_{ij} \to$ non basic

$S = \{x_{i_1 j_1}, \ldots, x_{..}\} \to$ basis

Γ_S is a tree.

$S' = S \cup \{x_{ij}\}$ $\to$ unicyclic graph.





Hence we are going to get an outgoing variable. But how do we get that?

e) Let S be the current basis and suppose that $\overline{c}_{i,j} < 0$. This means $x_{i,j} \notin S$ and it is our entering variable.

f) So, we want to insert $x_{i,j}$ into the basis S. Let $S' = S \cup \{x_{i,j}\}$. As Γ_S is already a tree, we see that $\Gamma_{S'}$ will contain exactly one cycle. Since $\Gamma_{S'}$ is bipartite this cycle will have an even length.

g) Let $x_{i,j}, x_{i_1,j}, x_{i_1,j_1}, \ldots, x_{i_k,j_k}, x_{i,j_k}$ be the edges for that cycle, in that order.

h) Note that, currently $x_{i,j}$ is nonbasic and so it's current value is 0. If we want to give it a value θ, then it will force some changes on the other edges as follows.

Since $x_{i,j}$ and $x_{i_1,j}$ lie on the same column of the table (transportation array) T, we must decrease the value of $x_{i_1,j}$ by θ in order to satisfy the equation at the sink j. Similarly, the value of x_{i_1,j_1} must increase by θ. In general, the value of every alternate edge, starting from $x_{i,j}$ increases by θ and the remaining edges decrease by θ.

i) What is the maximum value of θ that could be given to $x_{i,j}$?

Recall that we always give the maximum possible value to the entering variable without making the other variables negative. So, we find the minimum of the edges where θ is getting subtracted. This is θ.

j) Hence, at least one of the old variables will become 0. Select any one of them as the outgoing variable, say it is $x_{l,k}$.

k) Let $S'' = S' \setminus \{x_{l,k}\}$. Notice that $x_{l,k}$ and $x_{i,j}$ were on the same cycle of $\Gamma_{S'}$, which is a connected graph with only one cycle. Hence $\Gamma_{S''}$ must be a tree. So S'' is a basis. Thus, we have got a new bfs.

l) What is the cost difference between the old and the new basis? From our discussions on simplex algorithm, we know that the cost of the new basis is $\overline{c}_{i,j}\theta$ plus the cost of the old basis.

m) In view of this discussion, we have the following algorithm to find an improvised basis.

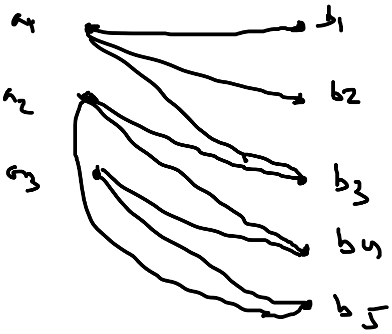
Algorithm for improvised bfs.

a) Suppose $\overline{c}_{ij} < 0$. Then x_{ij} is the entering variable.

b) Put $x_{ij} = \theta$ and make necessary adjustments in the values of other basic variables (by finding a cycle) so that the row sum and the column sum conditions are satisfied.

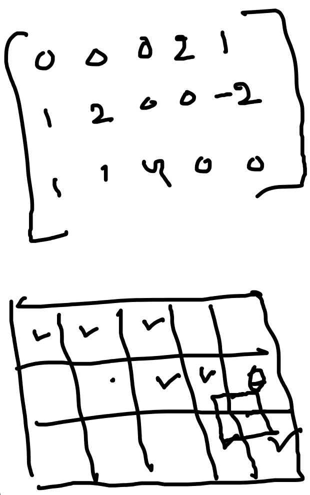
c) Determine the maximum value of θ so that at least one of the current basic variables gets a value 0. This is an outgoing variable.

[18.18] **Example** Consider the following btp and bfs. Find a minimum solution.



40 / 4	50 / 1	10 / 2	6	9	100	u_1 8
6	4	60 / 3	60 $-\theta$ / 5	θ / 7	120	u_2 9
5	2	6	30 $+\theta$ / 4	90 $-\theta$ / 8	120	u_3 8
40 v_1 -4	50 v_2 -7	70 v_3 -6	90 v_4 -4	90 / 0		

$$\overline{c} = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 & -2 \\ 1 & 1 & 4 & 0 & 0 \end{bmatrix}$$



a) Compute the simple multipliers and the relative cost. Repeat for the new bfs.

40 $-\theta$ / 4	50 / 1	10 $+\theta$ / 2	6	9	100	6
6	4	60 $-\theta$ / 3	5	60 $+\theta$ / 7	120	7
θ / 5	2	6	90 / 4	30 $-\theta$ / 8	120	8
40 -2	50 -5	70 -4	90 -4	90 / 0		

$$\overline{c} = \begin{bmatrix} 0 & 0 & 0 & 4 & 3 \\ 1 & 2 & 0 & 2 & 0 \\ -1 & -1 & 2 & 0 & 0 \end{bmatrix} \qquad \theta = 30$$

10 / 4	50 / 1	40 / 2	6	9	100
6	4	30 / 3	5	90 / 7	120
30 / 5	2	6	90 / 4	8	120
40	50	70	90	90 / 0	

$\overline{c} =$

4	1	2	6	9	100
6	4	3	5	7	120
5	2	6	4	8	120
40	50	70	90	90 0	

$\overline{c} =$

Answer. To start, calculate the simplex multipliers assuming $v_5 = 0$ and the $\overline{c}$ matrix..

40 4	50 1	10 2	6	9	100 8
6	4	60 3	60 5	7	120 9
5	2	6	30 4	90 8	120 8
40 -4	50 -7	70 -6	90 -4	90 0	

, and $\overline{c} = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 & -2 \\ 1 & 1 & 4 & 0 & 0 \end{bmatrix}.$

Entering variable is x_{25}. The cycle is formed by $x_{2,5}, x_{3,5}, x_{3,4}, x_{2,4}$, in that order. So $\theta = 60$. We compute the simplex multipliers ($v_5 = 0$) and the relative cost for the new bfs.

40 4	50 1	10 2	6	9	100 6
6	4	60 3	5	60 7	120 7
5	2	6	90 4	30 8	120 8
40 -2	50 -5	70 -4	90 -4	90 0	

$$\overline{c} = \begin{bmatrix} 0 & 0 & 0 & 4 & 3 \\ 1 & 2 & 0 & 2 & 0 \\ -1 & -1 & 2 & 0 & 0 \end{bmatrix}.$$

Let x_{32} enter. The cycle is formed by $x_{3,2}, x_{3,5}, x_{2,5}, x_{2,3}, x_{1,3}, x_{1,2}$, in that order. So $\theta = 30$. We compute the simplex multipliers and the relative cost for the new bfs.

40 4	20 1	40 2	6	9	100 6
6	4	30 3	5	90 7	120 7
5	30 2	6	90 4	8	120 7
40 -2	50 -5	70 -4	90 -3	90 0	

$$\overline{c} = \begin{bmatrix} 0 & 0 & 0 & 3 & 3 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \end{bmatrix}.$$

This bfs is optimal and the optimum cost is 1400.