0.1 Black-Scholes Market

In a Black-Scholes market an agent can invest in a money market(risk free) account which pays a constant rate of interest r and in a stock(risky) modeled by a geometric Brownian motion

$$S(t) = S(0)e^{\sigma W(t)+(\alpha-\frac{1}{2}\sigma^2)t}\,.$$

Thus $S(t)$ has log normal distribution. An amount x invested in the money market account gives a guaranteed return xe^{rt} after time t. The return on the stock is random. But $\mathbb{E}(S(t)) = S(0)e^{\alpha t}$. Thus α is the mean rate of return. Now,

$$S(t+1) = S(t)e^{\sigma(W(t+1)-W(t))+(\alpha-\frac{1}{2}\sigma^2)}\,.$$

Thus

$$\log(S(t+1)/S(t)) = \sigma(W(t+1)-W(t)) + (\alpha - \frac{1}{2}\sigma^2)\,.$$

Thus $\sqrt{Var(\log(S(t+1)/S(t)))} = \sigma$. The parameter σ is called the volatility of the stock.

Theorem 0.1. *Fix $t \geq 0$. As $n \to \infty$, the distribution of the scaled random walk $W^n(t)$ at the point t converges to the normal distribution with mean 0 and variance t.*

Consider a multi-period Binomial model on $[0,t]$, such that the stock price takes n steps per unit time. Assume that n and t are so chosen so that nt is an integer. So this is basically a nt period Binomial model. Suppose the up factor is $u_n = 1+\sigma/\sqrt{n}$ and the down factor is $d_n = 1-\sigma/\sqrt{n}$, where $\sigma > 0$. And suppose that the probabilities of going up and down are $1/2$ each. Then $S_n(t) = S(0)u_n^{H_{nt}}d_n^{T_{nt}}$, where H_{nt} is the number of upward movements and T_{nt} is the number of downward movements. Then $H_{nt}+T_{nt} = nt$ and $M_{nt} = H_{nt} - T_{nt}$ is the position of a simple random walk after nt steps. Thus $H_{nt} = \frac{1}{2}(nt+M_{nt})$ and $T_{nt} = \frac{1}{2}(nt - M_{nt})$. Thus

$$S_n(t) = S(0)(1+\sigma/\sqrt{n})^{\frac{1}{2}(nt+M_{nt})}(1-\sigma/\sqrt{n})^{\frac{1}{2}(nt-M_{nt})}\,.$$

Claim: As $n \to \infty$, the distribution of $S_n(t)$ converges to the distribution of $S(t) = S(0)e^{\sigma W(t)-\frac{1}{2}\sigma^2 t}$, where $W(t)$ is normal with mean 0 and variance t.
Proof of claim: Enough to show that $\log S_n(t)$ converges in distribution to $\log S(t)$. Now $\log(1+x) = x - x^2/2 + O(x^3)$. Then taking $x = \sigma/\sqrt{n}$ and $-\sigma/\sqrt{n}$ we get,

$$\begin{aligned}
\log S_n(t) &= \log S(0) + \frac{1}{2}(nt+M_{nt})(\sigma/\sqrt{n} - \sigma^2/(2n) + O(n^{-3/2})) \\
&\quad + \frac{1}{2}(nt-M_{nt})(-\sigma/\sqrt{n} - \sigma^2/(2n) + O(n^{-3/2})) \\
&= \log S(0) + nt(-\sigma^2/(2n) + O(n^{-3/2})) + M_{nt}(\sigma/\sqrt{n} + O(n^{-3/2})) \\
&= \log S(0) - \frac{\sigma^2 t}{2} + O(n^{-1/2}) + \sigma W^n(t) + O(n^{-1})W^n(t) \\
&\Rightarrow \log S(0) - \frac{\sigma^2 t}{2} + \sigma W(t) = \log S(t)\,.
\end{aligned}$$

By Ito's formula we have,

$$dS(t) = df(t, W(t)) = \alpha S(t)dt + \sigma S(t)dW(t)\,,$$

where $f(t,x) = S(0)e^{\sigma x + (\alpha - \frac{1}{2}\sigma^2)t}$. Suppose an investor has a portfolio with value $X(t)$ at time t, of which $\Delta(t)S(t)$ is invested in the stock and the remaining $X(t) - \Delta(t)S(t)$ in the money market account. Then the evolution of the portfolio is given by,

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ &= rX(t)dt + (\alpha - r)\Delta(t)S(t)dt + \sigma\Delta(t)S(t)dW(t) \end{aligned}$$

The three terms appearing above can be interpreted as follows:

- 1st term is an average rate of return r on the portfolio.
- 2nd term is the risk premium for investing in the stock.
- 3rd term is the volatility term proportional to the stock investment.

We shall often be interested in the discounted stock price $e^{-rt}S(t)$ and discounted portfolio value of an agent $e^{-rt}X(t)$. By Ito's formula,

$$\begin{aligned} d(e^{-rt}S(t)) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ &= (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)\,. \end{aligned}$$

$$\begin{aligned} d(e^{-rt}X(t)) &= -re^{-rt}X(t)dt + e^{-rt}dX(t) \\ &= \Delta(t)[(\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)] \\ &= \Delta(t)d(e^{-rt}S(t))\,. \end{aligned}$$

Discounting reduces the mean rate of return on the stock from α to $\alpha - r$. And discounting the portfolio value removes the average underlying rate of return. Basically, this makes the interest rate 0. So the change in discounted portfolio value is solely due to change in discounted stock price.

Definition 0.2. *An European call option on the stock $S(\cdot)$ is an agreement which gives its holder the right(but no obligation) to buy one unit of stock at time T(time of maturity) at a price K(strike price) from the seller(or writer) of the option.*

The payoff of an European call option is $(S(T) - K)^+$. So what should be the price of such an option? Black-Scholes and Merton argued that the option price should depend on the stock price, time to maturity, the model parameters and K. Out of these only two quantities are variable, stock price and time. For this reason we let $c(t,x)$ denote the value of the call option at time t, if the stock price at time t is $S(t) = x$. Our goal is to determine the function $c(t,x)$. By Ito's formula,

$$\begin{aligned} dc(t,S(t)) &= c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))dS(t)dS(t) \\ &= c_t(t,S(t))dt + c_x(t,S(t))\alpha S(t)dt + c_x(t,S(t))\sigma S(t)dW(t) + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)dt \\ &= [c_t(t,S(t)) + c_x(t,S(t))\alpha S(t) + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)]dt + c_x(t,S(t))\sigma S(t)dW(t) \end{aligned}$$

We next compute $d(e^{-rt}c(t,S(t))$. Again by Ito's formula,

$$\begin{aligned} & d(e^{-rt}c(t,S(t)) = -re^{-rt}c(t,S(t))dt + e^{-rt}dc(t,S(t)) \\ & = e^{-rt}[-rc(t,S(t)) + c_t(t,S(t)) + c_x(t,S(t))\alpha S(t) + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)]dt \\ & + e^{-rt}c_x(t,S(t))\sigma S(t)dW(t)\,. \end{aligned}$$

Now the option price should be such that starting with that amount, the seller of the option should be able to maintain a portfolio which will be equal to the option price at all times $t \in [0,T]$. Such a portfolio is called a hedging portfolio or a replicating portfolio. So if $X(t)$ denotes the value of the hedging portfolio at time t, then we want $X(t) = c(t, S(t))$ for all $t \in [0,T]$. This happens if and only if $e^{-rt}X(t) = e^{-rt}c(t,S(t))$. One way to ensure this equality is to make sure that $d(e^{-rt}X(t)) = d(e^{-rt}c(t,S(t)))$ for all $t \in [0,T)$ and $X(0) = c(0,S(0))$. This is because, integrating we get,

$$e^{-rt}X(t) - X(0) = e^{-rt}c(t,S(t)) - c(0,S(0)) \quad \forall\, t \in [0,T)\,.$$

Since $X(0) = c(0,S(0))$, we get the desired equality. The equality at $t = T$ is taken care of by continuity. So comparing the evolution of discounted portfolio and discounted option price we get

$$\begin{aligned}&\Delta(t)[(\alpha - r)S(t)dt + \sigma S(t)dW(t)]\\ &= [-rc(t,S(t)) + c_t(t,S(t)) + c_x(t,S(t))\alpha S(t) + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)]dt + c_x(t,S(t))\sigma S(t)dW(t)\,.\end{aligned}$$

Equating $dW(t)$ terms we get,

$$\Delta(t) = c_x(t,S(t)) \quad \forall\, t \in [0,T)\,.$$

Equating dt terms we get,

$$\alpha\Delta(t)S(t) - r\Delta(t)S(t) = -rc(t,S(t)) + c_t(t,S(t)) + c_x(t,S(t))\alpha S(t) + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)\,.$$

Implies,

$$rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t) \quad \forall\, t \in [0,T)\,.$$

Thus we need a continuous function $c(t,x)$ that is a solution of the Black-Scholes-Merton PDE

$$c_t(t,x) + rxc_x(t,S(t)) + \frac{1}{2}c_{xx}(t,x)\sigma^2 x^2 = rc(t,x) \quad \forall\, t \in [0,T)\,, x \geq 0,$$

with terminal condition $C(T,x) = (x-K)^+$.

Suppose we find a solution to the above pde, then if an investor starts with the initial capital $X(0) = c(0,S(0))$ and invests $\Delta(t)S(t)$ in stocks, with $\Delta(t)$ given by $c_x(t,S(t))$ and the remaining in the money market account, then his portfolio satisfies $X(t) = c(t,S(t))$ for all $t \in [0,t)$. Taking limit $t \uparrow T$ and using the fact that $X(t)$ and $c(t,S(t))$ are continuous we get,

$$X(T) = c(T,S(T)) = (S(T)-K)^+\,.$$

Thus the seller is perfectly hedged.

The BSM PDE is a partial differential equation of the type called backward parabolic. For such an equation, in addition to the terminal condition, one needs boundary conditions at $x = 0$ and $x = \infty$. For the boundary condition at $x = 0$, we take limit $x \downarrow 0$ in the BSM PDE to get,

$$rc(t,0) = c_t(t,0)\,.$$

This is an ODE in t, whose solution is

$$c(t,0) = e^{rt}c(0,0)\,.$$

Now $c(T,0) = (0-K)^+ = 0$. Thus $c(0,0) = 0$, implies $c(t,0) = 0$, for all $t \in [0,T]$. As $x \to \infty$, the function $c(t,x)$ grows without bound. In such a case we specify the boundary condition at $x = \infty$ by specifying the rate of growth. The boundary condition at $x \to \infty$ is given by

$$\lim_{x\to\infty}[c(t,x) - (x - e^{-r(T-t)}K)] = 0 \quad \forall\, t \in [0,T]\,.$$

For large x, the call is deep in money and very likely to end in money. In this case, the call option is like a forward contract. Now what is the price of a forward contract? Start with an initial capital of $x - Ke^{-rT}$, where $S(0) = x$. Borrow Ke^{-rT} and use the entire money to buy one unit stock. At time T, the value of this portfolio is $S(T) - K$ which is also the payoff of a forward contract. Thus by no-arbitrage principle the value of a forward contract at time t should be $S(t) - Ke^{-r(T-t)}$.

Theorem 0.3. *The solution to the BSM PDE with the specified terminal and boundary conditions is given by*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \quad 0 \le t < T,\ x > 0\,,$$

where $d_\pm(T-t,x) = \frac{1}{\sigma\sqrt{T-t}}[\log(x/K) + (r \pm \frac{\sigma^2}{2})(T-t)]$ *and N is the CDF of* $N(0,1)$. *Note that* $c(t,x)$ *is not defined for* $t = T$ *and* $x = 0$. *But* $c(t,x)$ *is defined in such a way that* $\lim_{t\to T} c(t,x) = (x-K)^+$ *and* $\lim_{x\downarrow 0} c(t,x) = 0$.

Exercise:

1. Verify that $Ke^{-r(T-t)}N'(d_-) = xN'(d_+)$.
2. Prove that $c_x(t,x) = N(d_+(T-t,x))$.
3. Prove that $c_t(t,x) = -rKe^{-r(T-t)}N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x))$.
4. Prove that $c_{xx}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x))$.
5. Use the above formulas to show that $c(\cdot,\cdot)$ satisfies the BSM PDE.

c_x is called the delta of the option, c_t is called the theta of the option and c_{xx} is called the gamma of the option. Since both N and N' are always positive, the option price is increasing in stock price, decreasing in time and convex in x.

Definition 0.4. *An European put option on the stock* $S(\cdot)$ *is an agreement which gives its holder the right(but no obligation) to sell one unit of stock at time T(time of maturity) at a price K(strike price) to the seller(or writer) of the option.*

Thus the payoff of an European put option is $(K - S(T))^+$. Now again the question is what should be the price of a put option. The call price and the put price are related by the following relation.
Put-Call Parity: Let $p(t,x)$ denote the put option price at time t when the stock price is $S(t) = x$. Then

$$c(t,x) - p(t,x) = x - Ke^{-r(T-t)}\,.$$

Exercise: Using Put-Call parity show that the put price is given by

$$p(t,x) = Ke^{-r(T-t)}N(-d_-(T-t,x)) - xN(-d_+(T-t,x))\,.$$

0.2 Multivariable Stochastic Calculus

Definition 0.5. *A* $d-$*dimensional Brownian motion is a process* $W(t) = (W_1(t), W_2(t), \ldots, W_d(t))$ *with the following properties:*
a) Each $W_i(t)$ *is a Brownian motion.*
b) They are independent as stochastic processes.

Associated with a $d-$dimensional Brownian motion, the filtration $\{\mathcal{F}_t\}$ has the following properties:
i) For each t, the random vector $W(t)$ is $\mathcal{F}_t$ measurable.
ii) For $0 \leq t < u$, the vector of increments $W(u) - W(t)$ is independent of $\mathcal{F}_t$.
Since $W_i(t)$ is a Brownian motion for each i, $dW_i(t)dW_i(t) = dt$. What is $dW_i(t)dW_j(t)$ for $i \neq j$. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$. For $i \neq j$, define

$$C_\Pi = \sum_{k=0}^{n-1} (W_i(t_{k+1}) - W_i(t_k))(W_j(t_{k+1}) - W_j(t_k)) \, .$$

Now

$$\begin{aligned}
\mathbb{E}(C_\Pi) &= \mathbb{E}\left(\sum_{k=0}^{n-1} (W_i(t_{k+1}) - W_i(t_k))(W_j(t_{k+1}) - W_j(t_k)) \right) \\
&= \sum_{k=0}^{n-1} \mathbb{E}(W_i(t_{k+1}) - W_i(t_k))\mathbb{E}(W_j(t_{k+1}) - W_j(t_k)) = 0 \, .
\end{aligned}$$

Now

$$\begin{aligned}
C_\Pi^2 &= \sum_{k=0}^{n-1} (W_i(t_{k+1}) - W_i(t_k))^2 (W_j(t_{k+1}) - W_j(t_k))^2 \\
&\quad + 2 \sum_{l<k}^{n-1} (W_i(t_{l+1}) - W_i(t_l))(W_j(t_{l+1}) - W_j(t_l))(W_i(t_{k+1}) - W_i(t_k))(W_j(t_{k+1}) - W_j(t_k))
\end{aligned}$$

All the increments appearing in the cross term are independent of one another and have mean 0. Thus

$$\begin{aligned}
Var(C_\Pi) &= \mathbb{E}(C_\Pi^2) = \mathbb{E}\left[\sum_{k=0}^{n-1} (W_i(t_{k+1}) - W_i(t_k))^2 (W_j(t_{k+1}) - W_j(t_k))^2 \right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}(W_i(t_{k+1}) - W_i(t_k))^2 \mathbb{E}(W_j(t_{k+1}) - W_j(t_k))^2 = \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \leq ||\Pi|| T \to 0 \, ,
\end{aligned}$$

as $||\Pi|| \to 0$. Thus $[W_i, W_j](t) = 0$, in differential form, $dW_i(t)dW_j(t) = 0$.
Let $X(t)$ and $Y(t)$ be Ito processes, which means they are processes of the form,

$$\begin{aligned}
X(t) &= X(0) + \int_0^t \Theta_1(u)du + \int_0^t \sigma_{11}(u)dW_1(u) + \int_0^t \sigma_{12}(u)dW_2(u) \\
Y(t) &= Y(0) + \int_0^t \Theta_1(u)du + \int_0^t \sigma_{21}(u)dW_1(u) + \int_0^t \sigma_{22}(u)dW_2(u) \, ,
\end{aligned}$$

where Θ_i and σ_{ij} are adapted stochastic processes satisfying the required properties. The following theorem generalizes one dimensional Ito-Doeblin formula.

Theorem 0.6. *Let $f(t, x, y)$ be a function with continuous partial derivatives $f_t, f_x, f_y, f_{xx}, f_{yy}, f_{xy}$. Let $X(t)$ and $Y(t)$ be Ito processes as above. Then the differential form of the two dimensional Ito-Doeblin formula is given by,*

$$\begin{aligned}
&df(t, X(t), Y(t)) = f_t(t, X(t), Y(t))dt + f_x(t, X(t), Y(t))dX(t) + f_y(t, X(t), Y(t))dY(t) \\
&+ \frac{1}{2} f_{xx}(t, X(t), Y(t))dX(t)dX(t) + \frac{1}{2} f_{yy}(t, X(t), Y(t))dY(t)dY(t) + f_{xy}(t, X(t), Y(t))dX(t)dY(t) \, .
\end{aligned}$$

Exercise: Write down the formula in integral form by expanding $dX(t)$, $dY(t)$, $dX(t)dX(t)$, $dY(t)dY(t)$ and $dX(t)dY(t)$.

Corollary 0.7. *(Ito's Product Rule) Let $X(t)$ and $Y(t)$ be Ito processes. Then*

$$d(X(t)Y(t)) = Y(t)dX(t) + X(t)dY(t) + dX(t)dY(t)\,.$$

A Brownian motion is a martingale with continuous sample paths and whose quadratic variation is $[W, W](t) = t$. It turns out that these properties characterize Brownian motion.

Theorem 0.8. *(Levy) Let $M(t), t \geq 0$ be a martingale relative to a filtration $\mathcal{F}_t$. Assume that $M(0) = 0$, $M(t)$ has continuous sample paths and $[M, M](t) = t$ for all $t \geq 0$. The $M(t)$ is a Brownian motion.*

Idea of the proof: By Ito's formula, we have

$$\begin{aligned} df(t, M(t)) &= f_t(t, M(t))dt + f_x(t, M(t))dM(t) + \frac{1}{2} f_{xx}(t, M(t))dM(t)dM(t) \\ &= f_t(t, M(t))dt + f_x(t, M(t))dM(t) + \frac{1}{2} f_{xx}(t, M(t))dt. \end{aligned}$$

Therefore, we have

$$f(T, M(T)) = f(0, M(0)) + \int_0^T \Big[f_t(t, M(t)) + \frac{1}{2} f_{xx}(t, M(t)) \Big] dt + \int_0^T f_x(t, M(t))dM(t).$$

Because $\{M(t)\}$ is a martingale, thus the stochastic integral $\int_0^T f_x(t, M(t))dM(t)$ is also a martingale relative to a the same filtration $\mathcal{F}_t$. Thus

$$\mathbb{E}[f(T, M(T))] = f(0, M(0)) + \mathbb{E} \int_0^T \Big[f_t(t, M(t)) + \frac{1}{2} f_{xx}(t, M(t)) \Big] dt.$$

Fix a number $u \in \mathbb{R}$ and define a function

$$g(t, x) = \exp\{ux - \frac{1}{2}u^2 t\}.$$

Then

$$g_t(t, x) = -\frac{1}{2}u^2 g(t, x),\ g_x(t, x) = ug(t, x),\ g_{xx}(t, x) = u^2 g(t, x).$$

In particular, we have

$$g_t(t, x) + \frac{1}{2} g_{xx}(t, x) = 0.$$

For this above function $g(t, x)$ we have

$$dg(t, M(t)) = u\, g(t, M(t))dM(t).$$

Therefore, we have

$$g(T, M(T)) = g(0, M(0)) + \int_0^T u\, g(t, M(t))dM(t).$$

Hence

$$g(t, M(t)) = \exp\{uM(t) - \frac{1}{2}u^2 t\}$$

is a martingale with respect to the filtration $\mathcal{F}_t$. Thus

$$\mathbb{E}[g(t, M(t))] = g(0, M(0)) = g(0, 0) = 1$$

Which implies that

$$\mathbb{E}\Big[\exp\{uM(t)-\frac{1}{2}u^2t\}\Big]=1.$$

Thus we have

$$\mathbb{E}\Big[\exp\{uM(t)\}\Big]=\exp\{\frac{1}{2}u^2t\}. \tag{1}$$

Hence $M(t)$ is normally distributed with mean zero and variance t.

Now for $t_1<t_2$ we have

$$\mathbb{E}\Big[\exp\{uM(t_2)-\frac{1}{2}u^2t_2\}|\mathcal{F}(t_1)\Big]=\exp\{uM(t_1)-\frac{1}{2}u^2t_1\}.$$

Which implies that

$$\mathbb{E}\Big[\exp\{u(M(t_2)-M(t_1))\}|\mathcal{F}(t_1)\Big]=\exp\{\frac{1}{2}u^2(t_2-t_1)\}. \tag{2}$$

Therefore, we have

$$\mathbb{E}\Big[\mathbb{E}\Big[\exp\{u(M(t_2)-M(t_1))\}|\mathcal{F}(t_1)\Big]\Big]=\mathbb{E}\Big[\exp\{u(M(t_2)-M(t_1))\}\Big]=\exp\{\frac{1}{2}u^2(t_2-t_1)\}.$$

Hence $M(t_2)-M(t_1)$ is normally distributed with mean zero and variance t_2-t_1.

Now for $u_1,u_2\in\mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{E}\Big[\exp\{u_1M(t_1)+u_2(M(t_2)-M(t_1))\}\Big]\\
&=\mathbb{E}\Big[\mathbb{E}\Big[\exp\{u_1M(t_1)+u_2(M(t_2)-M(t_1))\}|\mathcal{F}(t_1)\Big]\Big]\\
&=\mathbb{E}\Big[\exp\{u_1M(t_1)\}\mathbb{E}\Big[\exp\{u_2(M(t_2)-M(t_1))\}|\mathcal{F}(t_1)\Big]\Big]\\
&=\mathbb{E}\Big[\exp\{u_1M(t_1)\}\exp\{\frac{1}{2}u_2^2(t_2-t_1)\}\Big]\ \ \text{(by (2))}\\
&=\exp\{\frac{1}{2}u_2^2(t_2-t_1)\}\mathbb{E}\Big[\exp\{u_1M(t_1)\}\Big]\\
&=\exp\{\frac{1}{2}u_2^2(t_2-t_1)\}\exp\{\frac{1}{2}u_1^2t_1\}\ \ \text{(by (1))}.
\end{aligned}$$

Hence $M(t_1)$ and $M(t_2)-M(t_1)$ are independent.

Theorem 0.9. *(Levy) Let $M_1(t)$, $M_2(t)$ be martingales relative to a filtration $\mathcal{F}_t$. Assume that, for $i=1,2$, we have $M_i(0)=0$, $M_i(t)$ has continuous sample paths, $[M_i,M_i](t)=t$ for all $t\geq 0$. If in addition, $[M_1,M_2](t)=0$ for all $t\geq 0$ then (M_1,M_2) is a two dimensional Brownian motion.*

Exercise: Let $W_1(t)$ and $W_2(t)$ be two independent Brownian motions and $-1<\rho<1$. Show that $W_3(t)$ defined by

$$W_3(t)=\rho W_1(t)+\sqrt{1-\rho^2}W_2(t)$$

is a Brownian motion. Use Ito's product rule to find $\mathbb{E}(W_1(t)W_3(t))$.

Exercise: Let $W(t)$ be a Brownian motion and define

$$B(t)=\int_0^t sgn(W(s))dW(s)\,,$$

where

$$sgn(x)=\begin{cases}1 & \text{if } x\geq 0\\ -1 & \text{if } x<0.\end{cases}$$

Show that $B(t)$ is a Brownian motion.