# 2 Lecture 2

## Conversion of lpp to slpp

We need a standard form, as we want to use the computer. The idea is to develop an algorithm so that the computer can solve the problem in the standard form. Problems in other forms can be converted to the standard form. Different texts adopt different definitions of a standard form of an lpp. For us, a STANDARD LPP is of the form

$$\text{(slpp)} \qquad \begin{array}{ll} \min & z = c^t x \\ \text{s.t.} & Ax = b,\ x \geq 0, b \geq 0. \end{array}$$

We can convert an lpp it to a slpp using the following techniques.

**Converting lpp to slpp.**

a) Convert a constraint of the form $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ to $a_{i1}x_1 + \cdots + a_{in}x_n + s_i = b_i$ with a nonnegative variable $s_i$. Such variables are called SLACK VARIABLES.

b) Convert a constraint of the form $a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$ to $a_{i1}x_1 + \cdots + a_{in}x_n - s_i = b_i$ with a nonnegative variable $s_i$. Such variables are called SURPLUS VARIABLES.

c) If $b_i \leq 0$, then multiply the corresponding constraint by $-1$.

d) If a variable $x_i$ is unrestricted then replace it by $x_{i1} - x_{i2}$, in the whole problem, where $x_{i1}$ and $x_{i2}$ are two new nonnegative variables.[a]

e) If a variable $x_i$ is nonpositive replace it with $-y_i$, where $y_i \geq 0$, in the whole problem.

f) If $x_i \geq k$ replace it with $y_i + k$, where $y_i \geq 0$, in the whole problem.

g) Maximize $f(x)$ can be converted to minimize $-f(x)$ as $\max\limits_{x \in T} f(x) = -\min\limits_{x \in T} -f(x)$.

[a] Recall that every real valued function on $S$ is the difference of two nonnegative functions.

$\max f \equiv -\min -f$

**[2.1]** **Example** Write the slpp for the lpp:

$$\begin{array}{ll} \max & x_1 + 2x_2 \\ \text{s.t.} & -x_1 + 2x_2 \geq -4, x_1 + x_2 = 2,\ x_1 \geq 2, x_2 \in \mathbb{R}. \end{array}$$

$\min \ -x_1 - 2x_{21} + 2x_{22}$

$x_2 = x_{21} - x_{22}$

$-x_1 + 2x_{21} - 2x_{22} \geqslant -4,\ x_1 + x_{21} - x_{22} = 2,\ x_1 \geqslant 2,\ x_{21}, x_{22} \geqslant 0$

$\equiv \min \ -((x_1 - 2) + 2) - 2x_{21} + 2x_{22}$

$-x_1' - 2x_{21} + 2x_{22} - 2$

*Answer.* First, convert to minimization problem and then replace $x_2$ by $x_{21} - x_{22}$.

$$\begin{array}{ll} \min & -x_1 - 2x_{21} + 2x_{22} \\ \text{s.t.} & -x_1 + 2x_{21} - 2x_{22} \geq -4, x_1 + x_{21} - x_{22} = 2,\ x_1 \geq 2, x_{21}, x_{22} \geq 0. \end{array}$$

Then replace the first inequation by an equation using a surplus variable.

$$\begin{array}{ll} \min & -x_1 - 2x_{21} + 2x_{22} \\ \hline \text{s.t.} & -x_1 + 2x_{21} - 2x_{22} - s_1 = -4, x_1 + x_{21} - x_{22} = 2,\ x_1 \ge 2, x_{21}, x_{22}, s_1 \ge 0. \end{array}$$

Next write $x_1 = x_1' + 2$ as $x_1 \ge 2$.

$$\begin{array}{ll} \min & -x_1' - 2x_{21} + 2x_{22} - 2 \\ \hline \text{s.t.} & -x_1' + 2x_{21} - 2x_{22} - s_1 = -2, x_1' + x_{21} - x_{22} = 0,\ x_1', x_{21}, x_{22}, s_1 \ge 0. \end{array}$$

Next, make the constant in all the equations nonnegative.

$$\text{slpp:}\ \begin{array}{ll} \min & -x_1' - 2x_{21} + 2x_{22} - 2 \\ \hline \text{s.t.} & x_1' - 2x_{21} + 2x_{22} + s_1 = 2, x_1' + x_{21} - x_{22} = 0,\ x_1', x_{21}, x_{22}, s_1 \ge 0. \end{array}$$

**[2.2]** **Matrix form** When the variables and the constants are nonnegative, the conversion looks like

$\in \mathbb{R}^n$

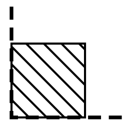
$$\begin{array}{ll} \min & c^t x \\ \hline \text{s.t.} & \begin{array}{l} A_{m\times n}x = b_1 \\ B_{k\times n}x \le b_2 \\ C_{r\times n}x \ge b_3 \\ x \ge 0, b_i \ge 0 \end{array} \end{array} \quad \mapsto \quad \begin{array}{ll} \min & \begin{bmatrix} c^t & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \hline \text{s.t.} & \begin{bmatrix} A & 0 & 0 \\ B & I_k & 0 \\ C & 0 & -I_r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ & x, y, z \ge 0, b_i \ge 0. \end{array} \tag{1}$$

$x_0 \to (x_0, y, z)$

We shall refer to the form on the rhs as the MATRIX FORM. Notice that one feasible set is in $\mathbb{R}^n$ whereas the other is in $\mathbb{R}^{n+k+r}$. Does a minimum solution of the rhs give us a minimum solution of the lhs? Yes. Are we creating a larger set by this conversion? No. Both the sets are in bijection, as shown below.

**[2.3]** **Theorem** In (1), let $T$ and $T'$ be the feasible sets for the lpp and slpp, respectively. Define $f: T \to T'$ as $f(x) = (x, b_2 - A_2x, A_3x - b_3)$. It is a bijection. Moreover, the values of the objective functions at $x$ and $f(x)$ are the same. In particular, the minimum values of the objective functions are the same.‼

**[2.4]** **Favorite example** Consider $\begin{array}{ll} \min & x_1 + x_2 \\ \hline \text{s.t.} & x_1 \le 1, x_2 \le 1, x_i \ge 0. \end{array}$



$x_1 + x_3 = 1$

$x_2 + x_4 = 1$ $\quad x_1, x_2, x_3, x_4 \ge 0$

The slpp is
$$\begin{array}{ll} \min & \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ \hline \text{s.t.} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_i \ge 0. \end{array}$$

## Some exercises

**[2.5]** **Exercise+** Write the slpp of $\begin{array}{ll} \max & x_1 - 2x_2 + 3x_3 \\ \hline \text{s.t.} & x_1 + x_2 \le 5, 2x_2 + 3x_3 \ge 2, x_1 \ge 0, x_2 \le 0 \end{array}$ in the matrix form.

**[2.6]** **Exercise+** Convert $\begin{array}{ll} \max & d^t x \\ \hline \text{s.t.} & Ax \ge a,\ Bx \le b,\ Cx = c,\ x \in \mathbb{R}^n,\ a, b, c \ge 0 \end{array}$ to slpp. Write in matrix form.

**[2.7]** **Exercise** Consider the lpp $\begin{array}{ll} \max & c^t x \\ \hline \text{s.t.} & Ax \ge b, x \ge 0 \end{array}$, where $b \le 0$. Write the corresponding slpp.

**[2.8]** **No-pen-exercise** `(Feasible regions under addition of a new constraint)` Suppose that $\alpha$ is the minimal value of $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \leq b, x \geq 0 \end{array}$. Assume that a new constraint is added. What happens to the feasible region? What can happen to the new optimal value, assuming it exists?

## Convex sets

Why do we need convex sets? The graphical method is not useful for problems involving more variables. As people wanted to develop an algorithm, a study of the properties of the feasible set $T = \{x \mid Ax = b, x \geq 0\}$ (which is convex) is required for that. Mostly this study is required to prove the correctness of many useful steps involved in the algorithm.

**[2.9]** **Recall** Let $x, y \in \mathbb{R}^n$. Recall that the LINE SEGMENT $[\boldsymbol{x}, \boldsymbol{y}]$ joining $x$ and $y$ is the set $\{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$.

**[2.10]** **Definition** A set $S \subseteq \mathbb{R}^n$ is called CONVEX, if the line segment joining each pairs of points in $S$ remains inside $S$.

**[2.11]** **Example** Empty set, singleton set, straight line segment, straight line, a circular disc including any part of the boundary, and any subspace of $\mathbb{R}^n$ are convex.‼

**[2.12]** **Fact** Let $\{S_\alpha\}$ be any collection of convex sets in $\mathbb{R}^n$. Then $\cap_\alpha S_\alpha$ is convex.‼

**[2.13]** **Fact** It follows from the definition that the set $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ for a fixed $A_{m \times n}$ and $b$ is convex.

**[2.14]** **Fact** Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then the open unit ball in this norm is convex. What would happen if we include a part of the boundary? $B_1(a)$

**[2.15]** **Definition** We call $S \subseteq \mathbb{R}^n$ a CONE if nonnegative multiples of each point in $S$ are in $S$. A CONVEX CONE is a cone which is convex.

**[2.16]** **Example** Empty set, $\{0\}$, $\mathbb{R}^n_+ = \{x \mid x_i \geq 0\}$ and any subspace are some convex cones in $\mathbb{R}^n$.

**[2.17]** **Fact** The sets $\{x \in \mathbb{R}^n \mid x_i > 0\} \cup \{0\}$ and $\{x \in \mathbb{R}^n \mid Ax \leq 0\}$ for a fixed $A_{m \times n}$ are convex cones.

$(x,y)$ $>0$ $>0$ $\cup \{0\}$

**[2.18]** **Recall** Recall that, a LINEAR COMBINATION of points $x_1, \cdots, x_n \in \mathbb{R}^m$ is a point $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$, where $\alpha_i \in \mathbb{R}$. The combination is called a NONTRIVIAL LINEAR COMBINATION if some $\alpha_i \neq 0$. We do not consider linear combinations of infinitely many vectors.

**[2.19]** **Recall** Recall that, if $\emptyset \neq S \subseteq \mathbb{R}^m$, then by $\mathsf{span}(S)$ we denote the collection of all linear combinations of elements of $S$. Conventionally, $\mathsf{span}(\emptyset) = \{0\}$.

**[2.20]** **Recall** Recall that $\mathsf{span}(S)$ is always a subspace of $\mathbb{R}^m$.

**[2.21]** **Definition** A linear combination $\sum \alpha_i x_i$ is called an AFFINE COMBINATION if $\sum \alpha_i = 1$. If $\emptyset \neq S \subseteq \mathbb{R}^m$, then by $\mathsf{\mathbf{aff}}(\boldsymbol{S})$, we denote the collection of all affine combinations of elements of $S$.

**[2.22]** **Fact** It is easy to see that $\mathsf{aff}(S)$ is an AFFINE SUBSPACE (that is, a translated subspace) of $\mathbb{R}^m$. In fact, if $s_0 \in S$, then

$$\mathsf{aff}(S) = s_0 + \mathsf{span}(S - s_0),$$

where $S - s_0 := \{x - s_0 \mid x \in S\}$. This is illustrated below.

**[2.23]** **Example** In $\mathbb{R}^3$, the plane $x + y + z = 1$ is $\mathsf{aff}(e_1, e_2, e_3)$. It is nothing but plane $x + y + z = 0$ translated by $e_1$, that is, $e_1 + \mathsf{span}(e_2 - e_1, e_3 - e_1)$.

$(1, -1, 0)$
$(1, 0, -1)$

**[2.24]** **Definition** A linear combination $\sum \alpha_i x_i$ is called a NONNEGATIVE COMBINATION if each $\alpha_i \geq 0$. If $\emptyset \neq S \subseteq \mathbb{R}^m$, then by $\mathsf{cone}(S)$ we denote the collection of all nonnegative combinations of elements of $S$.

$s^x = \lambda_1 s_1 + \cdots + \lambda_k s_k$

**[2.25]** **Fact** It is easy to see that $\mathsf{cone}(S)$ is always a convex cone in $\mathbb{R}^m$. So it is called the CONE GENERATED by $S$.

**[2.26]** **Definition** An affine nonnegative combination is called a CONVEX COMBINATION. If $\emptyset \neq S \subseteq \mathbb{R}^m$, then by $\mathsf{conv}(S)$ we denote the collection of all convex combinations of elements of $S$. It is called the CONVEX HULL of $S$.

**[2.27]** **Fact** It is easy to see that $\mathsf{conv}(S)$ is always a convex subset of $\mathbb{R}^m$.

aff$(a, b)$

a

b

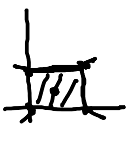
**[2.28]** **Example** Let $a, b \in \mathbb{R}^n$, $a \neq b$. Then $\mathsf{conv}(a, b)$ (that is, $\mathsf{conv}(\{a, b\})$) is the line segment $[a, b]$. All affine combinations of $a$ and $b$ constitute the LINE passing through $a$ and $b$. All linear combinations of $a$ and $b$ constitute a subspace of dimension 1 or 2. All nonnegative combinations of $a$ and $b$ can form a cone like structure or a ray or a line.

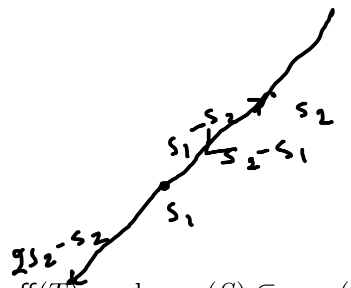
**[2.29]** **Fact** Let $x_1, \ldots, x_k$ be linear combinations of elements of $X_1, \ldots, X_k$, respectively. Then a linear combination $\alpha_1 x_1 + \cdots + \alpha_k x_k$ can be seen as a linear combination of elements of $\overset{k}{\underset{i=1}{\cup}} X_i$. That is, a linear combination of linear combinations is also a linear combination. (Similar statements are also valid for affine, nonnegative and convex combinations.)

**[2.30]** **Exercise-** Think about the combinations of three distinct non-collinear points.

**[2.31]** **Example** $\circ$ The set $\mathsf{conv}(e_1, e_2, e_3)$ is the closed triangular plate with corner at $e_1, e_2, e_3$.

$\circ$ Our favorite set is $\mathsf{conv}(0, e_1, e_2, e_1 + e_2)$. It is also $\mathsf{conv}(0, e_1, e_2, e_1 + e_2, \frac{e_1+e_2}{2})$.





**[2.32]** **Facts** ∘ The inclusion $S \subseteq \mathsf{conv}(S)$ always holds.

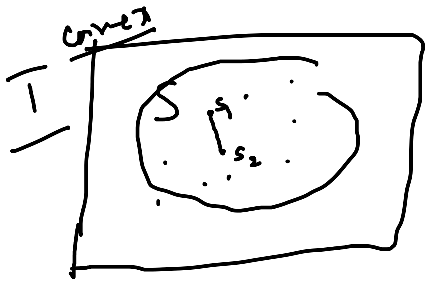
- ∘ If $S \subseteq T$, then $\mathsf{conv}(S) \subseteq \mathsf{conv}(T)$, $\mathsf{span}(S) \subseteq \mathsf{span}(T)$, $\mathsf{aff}(S) \subseteq \mathsf{aff}(T)$, and $\mathsf{cone}(S) \subseteq \mathsf{cone}(T)$.
- ∘ Let $S \subseteq \mathbb{R}^n$ be nonempty. Then $S = \mathsf{conv}(S)$ iff $S$ is convex. A proof follows by using induction.
- ∘ Let $S \subseteq \mathbb{R}^n$ be nonempty. Then $S = \mathsf{cone}(S)$ iff $S$ is a convex cone.
- ∘ Let $S \subseteq \mathbb{R}^n$ be nonempty. Then $S = \mathsf{span}(S)$ iff $S$ is a subspace.
- ∘ Let $S \subseteq \mathbb{R}^n$ be nonempty. Then $S = \mathsf{aff}(S)$ iff $S$ is an affine subspace.

**[2.33]** **Fact** Let $\emptyset \neq S \subseteq \mathbb{R}^n$. Then $\mathsf{conv}(S)$ is the intersection of all convex sets containing $S$. It follows as $\mathsf{conv}(S)$ is itself convex and it contains $S$. (Similar statements involving $\mathsf{span}(S)$, $\mathsf{cone}(S)$ and $\mathsf{aff}(S)$ are true.)

$\cap T \supseteq \mathrm{conv}(S)$

$\downarrow$

$\mathrm{conv}(S)$



$T \supseteq \mathrm{conv}(S)$

$\Downarrow$

$\mathrm{conv}(S)$

**[2.34]** **Exercise**

1. T/F? The cone generated by our favorite set is $\mathbb{R}^2_+$.
2. T/F? The set $\mathsf{cone}(e_1, e_1 + e_2)$ in $\mathbb{R}^2$ is the region in the first quadrant between the line $x = y$ and the $x$-axis.
3. T/F? The set $\mathsf{cone}(-e_1, e_1)$ is the $x$-axis.
4. T/F? The cone generated by $\{\pm e_1, \pm e_2\}$ is the $xy$-plane.

**[2.35]** **Exercise+** `(A simple fact)` a) Let $V$ and $W$ be two subspaces of a vector space $X$ and $x, y \in X$. Suppose that $x + V = y + W$. Then show that $V = W$. (Imagine it in $\mathbb{R}^3$ to get a geometrical feeling.)

b) Suppose that $a_1, a_2, \ldots, a_9 \in \mathbb{R}^{50}$ and that $\{a_2 - a_1, a_3 - a_1, \ldots, a_9 - a_1\}$ is linearly independent. Must $\{a_1 - a_9, a_2 - a_9, \ldots, a_8 - a_9\}$ be linearly independent?

$\mathrm{aff}(a_1, a_2, \ldots, a_9) = a_1 + \mathrm{span}(\;) = a_9 + \mathrm{span}(\;)$

$V \qquad W$

$\Rightarrow V = W$

$a + W = b + V$

$b \in a + W \Rightarrow b - a \in W$

$v \in V \Rightarrow b + v \in b + V$

$b + v \in a + W$

$\Rightarrow b - a + v \in W \Rightarrow v \in W$

**[2.36]** **Definitions** Fix $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{R}^n$. If $\{p_1 - p_0, \ldots, p_k - p_0\}$ is linearly independent, then we say that $S$ is AFFINE INDEPENDENT, otherwise it is called AFFINE DEPENDENT. If $S = \{p_0, p_1, \ldots, p_k\}$ is affine independent, then we call $\mathsf{conv}(S)$ a SIMPLEX of dimension $k$.

**[2.37]** **Example** The sets $S_1 = \{0, e_1, e_2\} \subseteq \mathbb{R}^2$ and $S_2 = \{e_1, e_2, e_3\} \subseteq \mathbb{R}^3$ are affine independent. Whereas related to our favorite set, the set $\{0, e_1, e_2, e_1 + e_2\}$ is affine dependent.

**[2.38]** **Example** A point is regarded as a zero dimensional simplex. A closed line segment is a 1-dimensional simplex. A closed triangular plate is a 2-dimensional simplex and a closed solid tetrahedron is a 3-dimensional simplex.

**[2.39]** **Fact** `(Similar to linear dependence)` The set $S = \{p_0, p_1, \ldots, p_k\}$ is affine dependent iff some $p_i$ is an affine combination of the remaining.

**[2.40]** **Fact** `(Similar to linear independence)` Let $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{R}^n$. Then $S$ is affine independent iff each $x \in \mathsf{aff}(S)$ can be written as an affine combination of $p_i$'s in a unique way. In particular, if $S$ is affine independent, then each $x \in \mathsf{conv}(S)$ can be written as a convex combination of $p_i$'s in a unique way.

## Some exercises

**[2.41]** **Exercise** Related to our favorite set in $\mathbb{R}^2$, the set $\{0, e_1, e_2, e_1 + e_2\}$ is affine dependent. So one element should be an affine combination of the rest. Verify this.

**[2.42]** **Exercise** `(Not every element will be an affine combination of the rest)` Take $S = \{0, e_1, e_2, 2e_1\}$. It is affine dependent. But we cannot express $e_2$ as an affine combination of the rest, as $e_2$ does not lie on the affine span of the other three. Does this contradict our previous results?

**[2.43]** **Exercise** Write our favorite set as a union of two simplices of dimension 2, in two different ways.

**[2.44]** **Exercise** Recall that our favorite set is $\mathsf{aff}(S)$, where $S = \{0, e_1, e_2, e_1 + e_2\}$ is affine dependent. So, there should be a point $x \in \mathsf{aff}(S)$ which can be written as an affine combination of points of $S$ in two different ways. Find such a point.

**[2.45]** **NoPen** a) T/F? The affine combinations of 5 points always form a subspace.

b) Take the points $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$. What do I get if I collect all the convex combinations? Affine combinations? Nonnegative combinations? Linear Combinations?

c) Is the convex hull of 5 different points in $\mathbb{R}^2$ always a pentagon?

d) Let $S \subseteq \mathbb{R}^4$ be a nonempty finite set. Is $\mathsf{conv}(S)$ necessarily a bounded set?

e) Let $\mathcal{A} = \{S \mid \mathsf{conv}(S) \neq S, S \subseteq \mathbb{R}^4 \text{ is convex}\}$. Can $\mathcal{A}$ have infinitely many elements?

f) Is there a bounded infinite subset of $\mathbb{R}^3$ such that $\mathsf{conv}(S)$ is open?

g) I have a set of 10 points in $\mathbb{R}^8$. Can it be affine independent?

h) Is a linearly independent set necessarily affine independent?

i) Can a set containing the zero vector be affine independent?

j)`(Imagine cones)` Determine the cones $\mathsf{cone}(S)$ for the following sets $S$.

$S = \{(1,1), (1,0), (0,0)\}$, $\qquad S = \{(1,0), (-1,0)\}$, $\qquad S = \{e_1, e_2, e_3\} \subseteq \mathbb{R}^3$

**[2.46]** **Exercise** `(Pictures of unit balls)` Draw $B_1(0)$ in $\mathbb{R}^2$ with respect to the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$. Can you imagine the pictures of the unit ball in $\mathbb{R}^3$ with respect to the above norms?

**[2.47]** **Exercise** `(Norms and convex sets)` Fix $0 < p < 1$. On $\mathbb{R}^n$, $n \geq 2$, define $\|x\|_p = \left[\sum_{i=1}^n |x_i|^p\right]^{1/p}$. Show that the set $S = \{x \mid \|x\|_p \leq 1\}$ is not convex. Conclude that $\|x\|_p$ is not a norm on $\mathbb{R}^n$. Why was $n \geq 2$ required?

**[2.48]** **Exercise** `(How to find out?)` Consider

$$p_0 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, p_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, p_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, p_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, p_4 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Which point is an affine combination of the rest? Give a general procedure that will work for many vectors in $\mathbb{R}^n$.