EXERCISES ON STOCHASTIC CALCULUS IN FINANCE ◎

KARTHIK IYER

This collection of solutions is based on assignments for a graduate level course on stochastic calculus for quantitative finance that I attended. The course roughly covered Brownian motion, Stochastic Calculus for processes without jumps, Risk Neutral Pricing, Connections to PDEs (Kolmogorov forward and backward equations) and Stochastic Optimization (HJB equations).

All questions, cited theorems, definitions and equations are from the (excellent) book '*Stochastic Calculus for Finance II, Continuous-Time Models,* ' by Steven Shreve. A bold-faced equation or a theorem number refers to the corresponding equation/theorem in the book.

All errors are entirely mine. Please let me know if you spot any. Comments, suggestions and ideas are always welcome.

Exercises

(1.5) When dealing with double Lebesgue integrals, just as in double Riemann integrals, the order of integration can be reversed. The only assumption required is that the function being integrated be either non-negative or integrable. Here is an application of this fact.

Let X be a non negative random variable with cumulative distribution function $F(x) = \mathbb{P}(X \leq x)$. Show that

$$\mathbb{E}(X) = \int_0^\infty (1 - F(x))\, dx \tag{0.1}$$

by showing

$$\int_\Omega \int_0^\infty \mathbb{1}_{[0,X(\omega))}(x)\, dx\, d\mathbb{P}(\omega) \tag{0.2}$$

is equal to both $\mathbb{E}(X)$ and $\int_0^\infty (1 - F(x))\, dx$

Proof. We first note that $g(x,\omega) = \mathbb{1}_{[0,X(\omega))}(x)$ is a non-negative function on $(0,\infty)\times\Omega$. Fubini-Tonelli theorem gives us

$$\int_\Omega \int_0^\infty \mathbb{1}_{[0,X(\omega))}(x)\, dx\, d\mathbb{P}(\omega) = \int_0^\infty \int_\Omega \mathbb{1}_{[0,X(\omega))}(x)\, d\mathbb{P}(\omega)\, dx \tag{0.3}$$

Let us look at the LHS of (0.3). We have

$$\int_\Omega \int_0^\infty \mathbb{1}_{[0,X(\omega))}(x)\,dx\,d\mathbb{P}(\omega) = \int_\Omega [\int_0^\infty \mathbb{1}_{[0,X(\omega))}(x)\,dx\,]d\mathbb{P}(\omega)$$
$$= \int_\Omega X(\omega)d\mathbb{P}(\omega)$$
$$= \mathbb{E}(X) \text{ (by definition of } \mathbb{E}(X)) \tag{0.4}$$

Before we proceed with the RHS of (0.3), let us first note that, by the finite additivity of the given probability measure and the fact that X is a non-negative random variable, we have $1 - F(x) = \mathbb{P}(X > x) = \mathbb{P}(\omega : X(\omega) > x \geq 0)$. RHS of (0.3) gives us

$$\int_0^\infty \int_\Omega \mathbb{1}_{[0,X(\omega))}(x)\,d\mathbb{P}(\omega)\,dx = \int_0^\infty \mathbb{P}(\omega : 0 \leq x < X(\omega))\,dx$$
$$= \int_0^\infty (1 - F(x))\,dx \tag{0.5}$$

(0.4), (0.5) and (0.3) together give the desired result. ∎

(1.9) Suppose X is a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, A is a set in $\mathcal{F}$, and for every Borel set B of $\mathbb{R}$, we have

$$\int_A \mathbb{1}_B(X(\omega))d\mathbb{P}(\omega) = \mathbb{P}(A) \cdot \mathbb{P}\{X \in B\} \tag{0.6}$$

then we say that X is independent of the event A. Show that if X is independent of an event A, then

$$\int_A g(X(\omega))d\mathbb{P}(\omega) = \mathbb{P}(A) \cdot \mathbb{E}g(X) \tag{0.7}$$

for every non-negative, Borel measurable function g.

Proof. We use the 'standard machine' trick to prove the result. First we prove for $g = \mathbb{1}_C(x)$ where C is a Borel measurable subset of $\mathbb{R}$. By hypothesis (0.6), we have

$$\int_A \mathbb{1}_C(X(\omega))d\mathbb{P}(\omega) = \mathbb{P}(A) \cdot \mathbb{P}\{X \in C\} \tag{0.8}$$

Since the random variable $\mathbb{1}_C(X)$ takes only two values, 0 and 1, its expectation is $\mathbb{E}\mathbb{1}_C(X) = 1 \cdot \mathbb{P}\{X \in C\} + 0 \cdot \mathbb{P}\{X \in C^c\} = \mathbb{P}\{X \in C\}$. We thus have from (0.8)

$$\int_A \mathbb{1}_C(X(\omega))d\mathbb{P}(\omega) = \mathbb{P}(A) \cdot \mathbb{P}\{X \in C\} = \mathbb{P}(A) \cdot \mathbb{E}\mathbb{1}_C(X) \tag{0.9}$$

Next we choose $g = \sum_{i=1}^{k} c_i \mathbb{1}_{C_i}$ where $\{C_i\}_{i=1}^{k}$ are disjoint subsets of $\mathbb{R}$ and all $c_i \geq 0$.

$$\begin{aligned}
\int_A g(X(\omega))d\mathbb{P}(\omega) &= \int_A \sum_{i=1}^{k} c_i \mathbb{1}_{C_i}(X(\omega))d\mathbb{P}(\omega) \\
&= \sum_{i=1}^{k} c_i \int_A \mathbb{1}_{C_i}(X(\omega))d\mathbb{P}(\omega) \text{ (linearity of the Lebesgue integral)} \\
&= \sum_{i=1}^{k} c_i \mathbb{P}(A) \cdot \mathbb{E}\mathbb{1}_{C_i}(X) \text{ (by (0.9))} \\
&= \mathbb{P}(A) \cdot \mathbb{E}g(X) \text{ (by linearity of expectation)} \qquad (0.10)
\end{aligned}$$

Next, let g be a arbitrary non negative Borel measurable function and choose a sequence of simple increasing non negative functions $h_n \to g$. By monotone convergence theorem applied on each side and using (0.10), we have

$$\begin{aligned}
\int_A g(X(\omega))d\mathbb{P}(\omega) &= \lim_{n\to\infty} \int_A h_n(X(\omega))d\mathbb{P}(\omega) \\
&= \lim_{n\to\infty} \mathbb{P}(A) \cdot \mathbb{E}(h_n(X)) = \mathbb{P}(A) \cdot \mathbb{E}(g(X)) \qquad (0.11)
\end{aligned}$$

■

(1.10) Let $\mathbb{P}$ be the uniform (Lebesgue) measure on $\Omega = [0,1]$. Define

$$Z(\omega) = \begin{cases} 0 & \text{if } 0 \leq \omega < \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} \leq \omega \leq 1 \end{cases}$$

For $A \in \mathcal{B}[0,1]$, define

$$\tilde{\mathbb{P}}(A) = \int_\Omega Z(\omega)d\mathbb{P}(\omega)$$

(a) Show that $\tilde{\mathbb{P}}$ is a probability measure.

Proof. Instead of proving the result by invoking the definition of a probability measure, we will just use Theorem 1.6.1. To ensure the conclusion of Theorem 1.6.1, i.e that $\tilde{\mathbb{P}}$ is a probability measure, we need to show that $\mathbb{E}Z = 1$. The fact that $\mathbb{P}$ is the Lebesge measure on $[0,1]$ along with the definition of the random variable Z gives us $\mathbb{E}Z = \int_{\frac{1}{2}}^{1} 2\,dx = 1$ ■

(b) Show that if $\mathbb{P}(A) = 0$, then $\tilde{\mathbb{P}}(A) = 0$. We say that $\tilde{\mathbb{P}}$ is *absolutely continuous* with respect to $\mathbb{P}$.

Proof. If $\mathbb{P}(A) = 0$, then by definition of the integral, $\int_A Z(\omega)d\mathbb{P}(\omega) = 0$ for any random variable Z. (We can justify this assertion, for example, by using the 'standard machine' trick for the random variable Z.) Hence $\tilde{\mathbb{P}}(A) = 0$. ■

(c) Show that there is a set A for which $\tilde{\mathbb{P}}(A) = 0$ but $\tilde{\mathbb{P}}(A) > 0$. In other words, $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are not equivalent.

Proof. Choose any set A supported in $[0, \frac{1}{2}]$, for instance take $A = [0, \frac{1}{4}]$. $\mathbb{P}(A) = \frac{1}{4}$ as $\mathbb{P}$ is the usual Lebesgue measure. Note that on A, $Z = 0$ and hence $\int_A Z(\omega) d\mathbb{P}(\omega) = 0$ (By Theorem 1.3.4). Thus $\tilde{\mathbb{P}}(A) = 0$.

Note that this happens because Z is not $\mathbb{P}$ almost surely non-zero. ■

(1.11) In Example 1.6.6, we began with a standard normal random variable X under a measure $\mathbb{P}$. According to Exercise 1.6, this random variable has the moment-generating function

$$\mathbb{E}e^{uX} = e^{\frac{1}{2}u^2} \text{ for all } u \in \mathbb{R}$$

The moment-generating function of a random variable determines its distribution. In particular, any random variable that has moment generating function $e^{\frac{1}{2}u^2}$ must be standard normal.

In Example 1.6.6, we also defined $Y = X + \theta$, where θ is a constant, we set $Z = e^{-\theta X - \frac{1}{2}\theta^2}$, and we defined $\tilde{\mathbb{P}}$ by the formula

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F}$$

We showed by considering its cumulative distribution function that Y is a standard normal variable under $\tilde{\mathbb{P}}$. Give another proof that Y is standard normal under $\tilde{\mathbb{P}}$ by verifying the moment-generating function formula

$$\tilde{\mathbb{E}}e^{uY} = e^{\frac{1}{2}u^2} \text{ for all } u \in \mathbb{R}$$

Proof. Consider any $u \in \mathbb{R}$. We have

$$\begin{aligned}
\tilde{\mathbb{E}}e^{uY} &= \int_\Omega e^{uY(\omega)} Z(\omega) d\mathbb{P}(\omega) \text{ (By eqn 1.6.4)} \\
&= \int_\Omega e^{u(X+\theta)} e^{-\theta X - \frac{1}{2}\theta^2} d\mathbb{P}(\omega) \\
&= \int_\mathbb{R} e^{ux+u\theta} e^{-\theta x - \frac{1}{2}\theta^2} e^{-\frac{1}{2}x^2} dx \text{ (By Theorem 1.5.2)} \\
&= e^{\frac{1}{2}u^2} \int_\mathbb{R} e^{-\frac{1}{2}(x+\theta-u)^2} dx \\
&= e^{\frac{1}{2}u^2} \int_\mathbb{R} e^{-\frac{1}{2}(x)^2} dx \text{ (Change of variables)} \\
&= e^{\frac{1}{2}u^2}
\end{aligned}$$

■

(1.14) Let X be a non-negative random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the *exponential distribution*, which is

$$\mathbb{P}\{X \leq a\} = 1 - e^{-\lambda a},\ a > 0$$

where $\lambda > 0$ is a constant. Let $\tilde{\Lambda}$ be another positive constant. Define

$$Z = \frac{\tilde{\lambda}}{\lambda} e^{-(\lambda - \tilde{\lambda})X}$$

Define $\tilde{\mathbb{P}}$ by

$$\tilde{\mathbb{P}}(A) = \int_A Z\, d\mathbb{P} \text{ for all } A \in \mathcal{F}$$

Proof. Let us first note that for any $a \geq 0$,

$$\mathbb{P}\{X \leq a\} = 1 - e^{-\lambda a} = \lambda \int_0^a e^{-\lambda x}\, dx \tag{0.12}$$

Using properties of the measure $\mathbb{P}$ and properties of the Lebesgue integral, it is easy to see that

$$\mathbb{P}\{X \in B\} = \lambda \int_B e^{-\lambda x}\, dx \tag{0.13}$$

where B is any interval in $[0, \infty)$.

(To show that (0.13) is valid for *all* $B \in \mathcal{B}$, we define a collection $\mathcal{S}$ of sets in $[0, \infty)$ for which (0.13) holds. This collection contains all intervals and it can be shown that it is actually a σ-algebra (using properties of $\mathbb{P}$ and the Lebesgue integral). Thus (0.13) can be shown to be valid for all Borel subsets of $\mathbb{R}$.)

Now that we have a density function for X, computations become simpler.

(a) Show that $\tilde{\mathbb{P}}(\Omega) = 1$.

$$\begin{aligned}
\tilde{\mathbb{P}}(\Omega) &= \int_\Omega Z(\omega) d\mathbb{P}(\omega) \\
&= \int_\Omega \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda} - \lambda)X(\omega)} d\mathbb{P}(\omega) \\
&= \int_0^\infty \tilde{\lambda} e^{-\tilde{\lambda} x}\, dx \text{ (By (0.13) and Theorem 1.5.2)} \\
&= 1
\end{aligned} \tag{0.14}$$

(b) Compute the cumulative distribution function

$$\tilde{\mathbb{P}}\{X \leq a\} \text{ for } a \geq 0$$

for the random variable X under the probability measure $\tilde{\mathbb{P}}$.

We have

$$\begin{aligned}\tilde{\mathbb{P}}\{X \le a\} &= \int_\Omega \mathbb{1}_{X(\omega)\le a} Z(\omega) d\mathbb{P}(\omega)\\ &= \int_\Omega \mathbb{1}_{X(\omega)\le a} \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)X(\omega)} d\mathbb{P}(\omega)\\ &= \int_0^\infty \mathbb{1}_{x\le a} \tilde{\lambda} e^{-\tilde{\lambda}x} dx \text{ (By (0.13) and Theorem 1.5.2)}\\ &= 1 - e^{-\tilde{\lambda}a}. \end{aligned} \tag{0.15}$$

∎

(2.2) Independence of random variables can be affected by changes of measure. To illustrate thus point, consider the space of two coin tosses $\Omega_2 = \{HH, HT, TH, TT\}$, and let stock prices be given by $S_0 = 4$, $S_1(H) = 8$, $S_1 = 2$, $S_2(HH) = 16$, $S_2(HT) = S_2(TH) = 4$, $S_2(TT) = 1$.

Consider two probability measures given by

$$\tilde{\mathbb{P}}(HH) = \frac{1}{4}, \tilde{\mathbb{P}}(HT) = \frac{1}{4}, \tilde{\mathbb{P}}(TH) = \frac{1}{4}, \tilde{\mathbb{P}}(TT) = \frac{1}{4}$$
$$\mathbb{P}(HH) = \frac{4}{9}, \mathbb{P}(HT) = \frac{2}{9}, \mathbb{P}(TH) = \frac{2}{9}, \mathbb{P}(TT) = \frac{1}{9}$$

Define the random variable $X = 1$ if $S_2 = 4$, $X = 0$ otherwise.

(a) List all the sets in $\sigma(X)$.

Solution. $[\{\emptyset, \Omega_2, HH \cup TT, HT \cup TT\}]$ ∎

(b) List all the sets in $\sigma(S_1)$.

Solution. $[\{\emptyset, \Omega_2, HH \cup HT, TH \cup TT\}]$ ∎

(c) Show that $\sigma(X)$ and $\sigma(S_1)$ are independent under $\tilde{\mathbb{P}}$.

Solution. We only show one set of equalities. Other cases follow by a similar computation.

$$\begin{aligned}\frac{1}{4} &= \tilde{\mathbb{P}}(HH) = \tilde{\mathbb{P}}(\{HH \cup TT)\} \cap \{HH \cup HT\})\\ &= \tilde{\mathbb{P}}(\{HH \cup TT)\}) \cdot \tilde{\mathbb{P}}(\{HH \cup TT\}) = \frac{1}{2} \cdot \frac{1}{2}\end{aligned}$$

∎

(d) Show that $\sigma(X)$ and $\sigma(S_1)$ are not independent under $\mathbb{P}$.

Solution. $\mathbb{P}(\{HH \cup HT\} \cdot \mathbb{P}(\{HH \cup TT\}) \neq P(\{HH\})$. ∎

(e) Under $\mathbb{P}$, we have $\mathbb{P}\{S_1 = 8\} = \frac{2}{3}$ and $\mathbb{P}\{S_1 = 2\} = \frac{1}{3}$. Explain intuitively why, if you are told that $X = 1$, you would want to revise your estimate of the distribution of S_1.

Solution. We first note that under $\mathbb{P}$, X and S_1 are dependent (as shown in part d) above). In fact, a simple calculation shows that $\mathbb{P}(S_1 = 8|X = 1) = \frac{1}{2}$ and $\mathbb{P}(S_1 = 2|X = 1) = \frac{1}{2}$. Intuitively, we are finding the *conditional* distribution here. Since we are conditioning on the knowledge of a random variable i.e we already have some prior information, the sample space shrinks and we need to revise the distribution of S_1. ■

(2.6) Consider a probability space $\Omega = \{a, b, c, d\}$. Let $\mathcal{F}$ be all possible subsets of Ω. Define a probability measure $\mathbb{P}$ by specifying that $\mathbb{P}\{a\} = \frac{1}{6}, \mathbb{P}\{b\} = \frac{1}{3}, \mathbb{P}\{c\} = \frac{1}{4}, \mathbb{P}\{d\} = \frac{1}{4}$. Next we define two random variables, X and Y by the formulas

$$X(a) = X(b) = 1 \ X(c) = X(d) = -1$$
$$Y(a) = 1, Y(b) = -1, Y(c) = 1, Y(d) = -1$$

We then define $Z = X + Y$.

(a) List the sets in $\sigma(X)$.

Solution. It is easy to see that $\sigma(X) = \{\emptyset, \Omega, \{a, b\}, \{c, d\}]$. ■

(b) Determine $\mathbb{E}[Y|X]$. And verify the partial averaging property.

Solution. A quick calculation gives us
$\mathbb{E}[Y|X](a) = -\frac{1}{3}$, $\mathbb{E}[Y|X](b) = -\frac{1}{3}$, $\mathbb{E}[Y|X](c) = 0$, $\mathbb{E}[Y|X](d) = 0$
Let us now verify the partial averaging property. We have,

$$\int_{\{a,b\}} \mathbb{E}[Y|X] d\mathbb{P}(\omega) = -\frac{1}{3} \cdot \frac{1}{2} = -\frac{1}{6} = \int_{\{a,b\}} Y d\mathbb{P}(\omega)$$
$$\int_{\{c,d\}} \mathbb{E}[Y|X] d\mathbb{P}(\omega) = 0 = \int_{\{c,d\}} Y d\mathbb{P}(\omega)$$
$$\int_{\{a,b,c,d\}} \mathbb{E}[Y|X] d\mathbb{P}(\omega) = -\frac{1}{3} \cdot \frac{1}{2} = -\frac{1}{6} = \int_{\{a,b,c,d\}} Y d\mathbb{P}(\omega)$$
$$\int_{\emptyset} \mathbb{E}[Y|X] d\mathbb{P}(\omega) = 0 = \int_{\emptyset} Y d\mathbb{P}(\omega) \text{ (Vacuously true)}$$

■

(c) Determine $\mathbb{E}[Z|X]$. And verify the partial averaging property.

Solution. $\mathbb{E}[Z|X](a) = \frac{2}{3}$, $\mathbb{E}[Z|X](b) = \frac{2}{3}$, $\mathbb{E}[Z|X](c) = -1$, $\mathbb{E}[Z|X](d) = -1$, Let us now verify the partial averaging property.

$$\int_{\{a,b\}} \mathbb{E}[Z|X] d\mathbb{P}(\omega) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3} = \int_{\{a,b\}} Z d\mathbb{P}(\omega)$$

$$\int_{\{c,d\}} \mathbb{E}[Z|X] d\mathbb{P}(\omega) = -\frac{1}{2} \cdot 1 = -\frac{1}{2} = \int_{\{c,d\}} Z d\mathbb{P}(\omega)$$

$$\int_{\{a,b,c,d\}} \mathbb{E}[Z|X] d\mathbb{P}(\omega) = -\frac{1}{6} = \int_{\{a,b,c,d\}} Z d\mathbb{P}(\omega)$$

$$\int_{\emptyset} \mathbb{E}[Z|X] d\mathbb{P}(\omega) = 0 = \int_{\emptyset} Z d\mathbb{P}(\omega) \text{ (Vacuously true)}$$

■

(d) Compute $\mathbb{E}[Z|X] - \mathbb{E}[Y|X]$. Explain why you get X.

Solution. As we can see from the computations in part (b) and (c) above, the random variable $\mathbb{E}[Z|X] - \mathbb{E}[Y|X] = X$ on Ω. Alternatively, by linearity of expectation and definition of Z, $\mathbb{E}[Z|X] - \mathbb{E}[Y|X] = \mathbb{E}[X|X] = X$. The last equality follows by 'Taking out what is known property' and the fact that $\mathbb{E}[1|X] = \int_\Omega 1 = \mathbb{P}(\Omega) = 1$. ■

(2.7) Let Y be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-sigma algebra of $\mathcal{F}$. Based on the information in $\mathcal{G}$, we can form the estimate $\mathbb{E}[Y|\mathcal{G}]$ of Y and define the error of the estimation Err= $Y - \mathbb{E}[Y|\mathcal{G}]$. This is a random variable with expectation zero and some variance Var(Err). Let X be some other $\mathcal{G}$ measurable random variable, which we can regard as another estimate of Y. Show that

$$\text{Var(Err)} \le \text{Var(Y-X)}$$

In other words, the estimate $\mathbb{E}[Y|\mathcal{G}]$ minimizes the error among all estimates based on the information in $\mathcal{G}$.

Proof. Let us first note that $S := \mathbb{E}[Y|\mathcal{G}]$ is $\mathcal{G}$ measurable. We thus have

$$\begin{aligned} \mathbb{E}[S\mathbb{E}[Y|\mathcal{G}]] &= \mathbb{E}[\mathbb{E}[SY|\mathcal{G}]] \text{ (Taking out what is known)} \\ &= \mathbb{E}[SY] \\ &= \mathbb{E}[Y\mathbb{E}[Y|\mathcal{G}]] \end{aligned} \tag{0.16}$$

Moreover, using the fact that X is $\mathcal{G}$ measurable, we have

$$\begin{aligned} &\mathbb{E}[\mathbb{E}[Y|\mathcal{G}](X+\mu)] \\ &= \mathbb{E}[\mathbb{E}[(X+\mu)Y|\mathcal{G}]] \text{ (Taking out what is known)} \\ &= \mathbb{E}[(X+\mu)Y] \end{aligned} \tag{0.17}$$

Let $\mu = \mathbb{E}[Y - X]$. We have

$$\begin{aligned}
&\mathbb{E}[(Y - X - \mu)^2] \\
&= \mathbb{E}[((Y - \mathbb{E}[Y|\mathcal{G}]) + (\mathbb{E}[Y|\mathcal{G}] - X - \mu))]^2 \\
&\geq \mathbb{E}[((Y - \mathbb{E}[Y|\mathcal{G}]))^2] + 2\mathbb{E}[(Y - \mathbb{E}[Y|\mathcal{G}])(\mathbb{E}[Y|\mathcal{G}] - X - \mu))] \text{ (By linearity of } \mathbb{E}) \\
&= \mathbb{E}[((Y - \mathbb{E}[Y|\mathcal{G}]))^2] \text{ (Cross terms cancel; by (0.16) and (0.17))}
\end{aligned}$$

(We have to assume, for cancellation of cross terms, that all terms of the form $\mathbb{E}[ST] < \infty$ where S and T are different random variables that are relevant to our computation.) ■

(2.10) Let X and Y be random variables (on some unspecified probability space $(\Omega, \mathcal{F}, \mathbb{P})$), assume they have a joint density $f_{X,Y}(x,y)$, and assume $\mathbb{E}|Y| < \infty$. In particular, for every Borel subset C of $\mathbb{R}^2$, we have

$$\mathbb{P}\{(X,Y) \in C\} = \int_C f_{X,Y}(x,y)\,dx\,dy$$

Define

$$g(x) = \int_{-\infty}^{\infty} \frac{y\, f_{X,Y}(x,y)}{f_X(x)}\,dy$$

where $f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,\eta)\,d\eta$ is the *marginal density* of X, and we must assume it is strictly positive for every x. We want to show

$$\mathbb{E}[Y|X] = g(X)$$

Since $g(X)$ is obviously $\sigma(X)$ measurable, to verify that $\mathbb{E}[Y|X] = g(X)$, we need only check that the partial-averaging property is satisfied. For every Borel-measurable function h mapping $\mathbb{R}$ to $\mathbb{R}$ and satisfying $\mathbb{E}|h(X)| < \infty$, we have

$$\mathbb{E}h(X) = \int_{\mathbb{R}} h(x)\, f_X(x)\,dx \tag{0.18}$$

Similarly, if h is a function of both x and y, then

$$\mathbb{E}h(X,Y) = \int_{\mathbb{R}}\int_{\mathbb{R}} h(x,y)\, f_{X,Y}(x,y)\,dx\,dy \tag{0.19}$$

whenever (X,Y) has a joint density $f_{X,Y}(x,y)$. You may use both (0.18) and (0.19) in your solution to this problem.

Show the partial averaging property

$$\int_A g(X)d\mathbb{P} = \int_Y d\mathbb{P}$$

Proof. We would like to use (0.19) in our proof. For that, we need to verify that $\mathbb{E}|g(X)| < \infty$. Let us first get that out of the way.

$$\begin{aligned}\mathbb{E}|g(X)| &= \int_{\Omega} |g(X)| d\mathbb{P}(\omega) \\ &= \int_{\mathbb{R}} |g(x)|\, f_X(x)\, dx \text{ (By Theorem 1.5.2)} \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |y|\, f_{X,Y}(x,y)\, dx\, dy \text{ (By definition of g)} \\ &< \infty \text{ (As } \mathbb{E}|Y| < \infty \text{ by hypothesis)} \end{aligned} \tag{0.20}$$

For any $A \in \sigma(X)$, we let B be the Borel-measurable subset of $\mathbb{R}$ so that $A = \{X \in B\}$. The existence and uniqueness of such a set follows by the definition of $\sigma(X)$.

$$\begin{aligned}\int_A g(X)(\omega) d\mathbb{P}(\omega) &= \int_{\Omega} g(X) \mathbb{1}_A d\mathbb{P}(\omega) \\ &= \int_{-\infty}^{\infty} g(x) \mathbb{1}_B\, f_X(x)\, dx \text{ (By Theorem 1.3.4 iii and (0.20))}\end{aligned} \tag{0.21}$$

We also have,

$$\begin{aligned}\int_A Y\, d\mathbb{P} &= \int_{\Omega} \mathbb{1}_A(\omega) Y(\omega)\, d\mathbb{P}(\omega) \\ &= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} y\, \mathbb{1}_B(x)\, f_{X,Y}(x,y)\, dx\, dy \text{ (By (0.19))} \\ &= \int_{-\infty}^{\infty} \mathbb{1}_B [\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\, dy]\, dx \end{aligned} \tag{*}$$

The last equality is valid by Fubini's theorem (which, of course can be applied as $\mathbb{E}|Y| < \infty$.)

$$* = \int_{-\infty}^{\infty} g(x)\, \mathbb{1}_B(x)\, f_X(x)\, dx \text{ (By the definition of } g) \tag{0.22}$$

■

(3.1) According to Definition 3.3.3(iii), for $0 \leq t \leq u$, the Brownian motion increment $W(u) - W(t)$ is independent of the σ-algebra $\mathcal{F}(t)$. Use this property and property (i) of that definition to show that for $0 \leq t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

Proof. Let X be a random variable defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us first observe that if $\mathcal{G}$ is a sub-sigma algebra of $\mathcal{H}$ and X is independent of $\mathcal{H}$, then X is independent of $\mathcal{G}$. Indeed, if $A \in \sigma(X)$ and $B \in \mathcal{G}$, then $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

This equality follows because B is also in $\mathcal{H}$ and by assumption, X is independent of $\mathcal{H}$.

With this observation in mind, let us prove that $W(u_2)-W(u_1)$ is also independent of $\mathcal{F}(t)$. Note that, by definition of Brownian motion the increment $W(u_2)-W(u_1)$ is independent of $\mathcal{F}(u_1)$. Since the collection $\mathcal{F}(s)$, $s \geq 0$ is a filtration, $\mathcal{F}(t)$ is a sub sigma algebra of $\mathcal{F}(u_1)$ and the observation above proves that $W(u_2)-W(u_1)$ is also independent of $\mathcal{F}(t)$. ∎

(3.2) Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$ be a filtration for this Brownian motion. Show that $W^2(t)-t$ is also a martingale.

Proof. Let $0 \leq s \leq t$ be given. We first use linearity for expectation and get

$$\begin{aligned}&\mathbb{E}[W^2-t|\mathcal{F}(s)] = \mathbb{E}[(W(s)-W(t))^2+2W(s)W(t)-W^2(s)-t|\mathcal{F}(s)]\\ &= \mathbb{E}[(W(t)-W(s))^2|\mathcal{F}(s)]+2\mathbb{E}[W(s)W(t)|\mathcal{F}(s)]-\mathbb{E}[W^2(s)|\mathcal{F}(s)]-t \end{aligned} \tag{0.23}$$

Since, the increment $W(t)-W(s)$ is independent of $\mathcal{F}(s)$ and has mean 0 and variance $t-s$,

$$\mathbb{E}[(W(t)-W(s))^2|\mathcal{F}(s)] = \mathbb{E}[(W(t)-W(s))^2] = t-s \tag{0.24}$$

As $W(s)$ is $\mathcal{F}(s)$ measurable,

$$\mathbb{E}[W(s)^2|\mathcal{F}(s)] = W^2(s) \tag{0.25}$$

Moreover,

$$\begin{aligned}&2\mathbb{E}[W(s)W(t)|\mathcal{F}(s)] = 2\mathbb{E}[W(s)(W(t)-W(s))+W^2(s)|\mathcal{F}(s)]\\ &= 2W(s)\mathbb{E}[W(t)-W(s)|\mathcal{F}(s)]+2W^2(s) \text{ (Taking out what is known and (0.25))}\\ &= 2W^2(s) \text{ [Since } W(t)-W(s) \text{ is } \mathcal{F}(s) \text{ independent with mean 0]}\end{aligned} \tag{0.26}$$

From (0.23), (0.24), (0.25), (0.26), we get

$$\mathbb{E}[W^2-t|\mathcal{F}(s)] = W^2(s)-s \tag{0.27}$$

thereby proving $W^2(t)-t$ is also a martingale. (It is trivially true that $W^2(t)-t$ is $\mathcal{F}(t)$ adapted stochastic process.) ∎

(3.4) **Other variations of Brownian motion** Theorem 3.4.3 asserts that if T is a positive number and we choose a partition Π with points $0=t_0<t_1<t_2<\ldots<t_n=T$, then as the number n of partition points approaches ∞ and the length of the longest sub-interval approaches 0, the sample quadratic variation

$$\sum_{j=0}^{n-1}(W(t_{j+1}-W(t_j))^2$$

approaches T for almost every path of the Brownian motion W. In Remark 3.4.5, we further showed that $dW(t)dW(t)=dt$ and $dW(t)dt=0$, $dtdt=0$.

(a) Show that as the number of partition points approaches ∞ and the length of the longest sub interval approaches 0, the sample first variation

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

approaches ∞ for almost every path of the Brownian motion W.

Proof. Let $\Omega' \subset \Omega$ denote the full measure set for which the quadratic variation of the Brownian motion up to time T equals T. From now on, we assume all our computations are on this set Ω'.

Let $\delta(n) = max_{0 \leq k \leq n-1} |W(t_{j+1} - W(t_j)|$. Since $W(t)$ is a continuous on $[0,T]$, $\delta(n) \to 0$ as $n \to \infty$. Moreover, $\delta(n)$ cannot be 0 since in that case $W(t)$ would be a constant violating the assumption that $W(t)$ needs to have variance t. We can hence safely divide by $\delta(n)$ and get

$$\sum_{j=0}^{n-1} |W(t_{j+1} - W(t_j)| \geq \frac{\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2}{\delta(n)} \tag{0.28}$$

Taking limits on both sides as $n \to \infty$, the we see that the RHS of (0.28) approaches $\frac{T}{\delta(n)} \to \infty$. ■

(b) Show that as the number n of partition points approaches ∞ and the length of the longest sub-interval approaches 0, the sample cubic variation

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3$$

approaches 0 for almost every path of the Brownian motion W.

Proof. Let Ω', $\delta(n)$ be as in the previous part. As before, we assume all our computations are on this set Ω'. We have,

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \leq \delta(n) \cdot [\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j)^2)] \tag{0.29}$$

Taking limits on both sides as $n \to \infty$, the we see that the RHS of (0.29) approaches $\delta(n) \cdot T \to 0$. ■

Remark: This exercise in particular tells us why only the quadratic variation of a Brownian motion is worth studying. In fact, we can show, very similar to what we showed in part a) that, for any $0 < \epsilon < 2$, the ϵth variation, $\sum_{j=0}^{n-1} |W(t_{j+1} - W(t_j)|^{\epsilon} \to \infty$ as $n \to \infty$. We can also show, very similar to to what we showed in part b) that for any $2 < \epsilon$, the ϵth variation, $\sum_{j=0}^{n-1} |W(t_{j+1} - W(t_j)|^{\epsilon} \to 0$ as $n \to \infty$. Both the claims follow by observing that if $\delta(n) \to 0$ as $n \to \infty$ then so does $\delta(n)^{\epsilon}$ for any $\epsilon > 0$.

(3.5) **Black-Scholes-Merton formula** Let the interest rate r and the volatility $\sigma > 0$ be constant. Let

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}$$

be a geometric Brownian motion with mean rate of return r, where the initial stock price $S(0)$ is positive. Let K be a positive constant. Show that, for $T > 0$,

$$\mathbb{E}\left[e^{-rT}(S(T)-K)^+\right] = S(0)N(d_+(T,S(0))) - Ke^{-rT}N(d_-(T,S(0))),$$

where

$$d_\pm = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S(0)}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)T\right],$$

and N is the standard normal cumulative distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2}\,dz = \int_{-y}^{\infty} e^{-\frac{1}{2}z^2}\,dz. \tag{0.30}$$

Proof. Fix a $T > 0$. We begin by noting that $\frac{dS}{dW} > 0$ implying that S is an increasing function of W. Hence, $S(T) > K$ iff $W(T) > \frac{1}{\sigma}\left[\log\left(\frac{K}{S(0)}\right) + \left(\frac{1}{2}\sigma^2 - r\right)T\right] = x_0$. With this observation and the fact that $W(T)$ is a normal random variable with mean 0 and variance T, we have

$$\mathbb{E}\left[e^{-rT}(S(T)-K)^+\right] = \frac{S(0)}{\sqrt{2\pi T}}\int_{x_0}^{\infty} e^{-\frac{1}{2}\sigma^2T+\sigma x-\frac{x^2}{2T}}\,dx - \frac{1}{\sqrt{2\pi T}}Ke^{-rT}\int_{x_0}^{\infty} e^{-\frac{x^2}{2T}}\,dx$$

$$= \frac{S(0)}{\sqrt{2\pi T}}\int_{x_0}^{\infty} e^{-\left(\frac{\sigma\sqrt{T}}{\sqrt{2}}+\frac{x}{\sqrt{2T}}\right)^2}\,dx - \frac{1}{\sqrt{2\pi T}}Ke^{-rT}\int_{x_0}^{\infty} e^{-\frac{x^2}{2T}}\,dx$$

$$= \frac{S(0)}{\sqrt{2\pi}}\int_{y_0}^{\infty} e^{-\frac{1}{2}y^2}\,dy - \frac{1}{\sqrt{2\pi}}Ke^{-rT}\int_{z_0}^{\infty} e^{-\frac{z^2}{2}}\,dz \tag{*}$$

where $y_0 = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{K}{S(0)}\right) - \left(\frac{1}{2}\sigma^2 + r\right)T\right]$ and $z_0 = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{K}{S(0)}\right) - \left(-\frac{1}{2}\sigma^2 + r\right)T\right]$. The last equality follows by change of variables $\frac{y}{\sqrt{2}} = \frac{\sigma\sqrt{T}}{\sqrt{2}} + \frac{x}{\sqrt{2T}}$ and $z = \frac{x}{\sqrt{T}}$.

Note that $-y_0 = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S(0)}{K}\right) + \left(\frac{1}{2}\sigma^2 + r\right)T\right] = d_+(T,S(0))$ and $z_0 = d_-(T,S(0))$. Using (0.30), (*) can be written as

$$(*) = \frac{S(0)}{\sqrt{2\pi}}\int_{-\infty}^{d_+(T,S(0))} e^{-\frac{1}{2}y^2}\,dy - Ke^{-rT}\frac{1}{\sqrt{2\pi}}\int_{d_-(T,S(0))} e^{-\frac{1}{2}z^2}\,dz$$

$$= S(0)N(d_+(T,S(0))) - Ke^{-rT}N(d_-(T,S(0)))$$

■

(3.6) Let $W(t)$ be a Brownian motion and let $\mathcal{F}(t), t \geq 0$ be an associated filtration.

(a) For $\mu \in \mathbb{R}$, consider the Brownian motion with drift μ,

$$X(t) = \mu t + W(t)$$

Show that for any Borel measurable function $f(y)$, and for any $0 \le s < t$, the function

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} f(y) e^{-\frac{-(y-x-\mu(t-s))^2}{2(t-s)}} dy$$

satisfies $\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$, and hence X has the Markov property. We may write $g(x)$ as $g(x) = \int_{\mathbb{R}} f(y) p(\tau, x, y)\, dy$, where $\tau = t - s$ and

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{-(y-x-\mu\tau)^2}{2\tau}}$$

is the *transition density* for the Brownian motion with drift μ.

Proof. Clearly $X(t)$ is an adapted stochastic process with respect to the filtration $\mathcal{F}(t)$. Note that,

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = \mathbb{E}[f(W(t) - W(s) + \mu(t-s) + s\mu + W(s))|\mathcal{F}(s)].$$

The random variable $W(t) - W(s) + \mu(t-s)$ is $\mathcal{F}(s)$ independent (sigma algebra generated by a random variable X is the same as the sigma algebra generated by $X + c$ for any constant c) and the random variable $s\mu + W(s)$ is $\mathcal{F}(s)$ measurable. This permits us to apply the Independence Lemma, Lemma 2.3.4.

In order to compute the expectation above, we replace $W(s) + \mu s = X(s)$ by a dummy variable x and take the unconditional expectation of the remaining random variable. Since, $W(t) - W(s) + \mu(t-s)$ is normally distributed with mean $\mu(t-s)$ and variance $t - s$, we define

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} f(w+x) e^{-\frac{-(w-\mu(t-s))^2}{2(t-s)}} dw \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} f(y) e^{-\frac{-(y-x-\mu(t-s))^2}{2(t-s)}} dy \text{ (Changing variables } w + x \to y) \qquad (0.31) \end{aligned}$$

The Independence Lemma states that if we now take the function $g(x)$ defined in (0.31) and replace the dummy variable x by the random variable $X(s)$ then $\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$ and hence X has Markov property. ■

(b) For $\nu \in \mathbb{R}$ and $\sigma > 0$, consider the *geometric Brownian motion*

$$S(t) = S(0) e^{\sigma W(t) + \nu t}$$

Set $\tau = t - s$ and

$$\rho(\tau, x, y) = \frac{1}{\sigma y \sqrt{2\pi\tau}} e^{\frac{-(\log(\frac{y}{x}) - \nu\tau)^2}{2\sigma^2\tau}}$$

Show that for any Borel measurable function $f(y)$ and for any $0 \le s < t$ the function $g(x) = \int_0^\infty f(y) \rho(\tau, x, y)\, dy$ satisfies $\mathbb{E}[f(S(t))|\mathcal{F}(s)] = g(S(s))$ and hence S has Markov property and $\rho(\tau, x, y)$ is its transition density.

Proof. Clearly, $S(t)$ is an adapted stochastic process with respect to filtration $\mathcal{F}(t)$. Note that,

$$\mathbb{E}[f(S(t))|\mathcal{F}(s)] =$$
$$\mathbb{E}[f\,(S(0)(exp(\sigma(W(t)-W(s))+\nu(t-s)+\nu s+\sigma W(s)))\,|\mathcal{F}(s)]$$

The random variable $\sigma(W(t)-W(s))+\nu(t-s)$ is $\mathcal{F}(s)$ independent and the random variable $\nu s+\sigma W(s)$ is $\mathcal{F}(s)$ measurable. This permits us to apply the Independence Lemma, Lemma 2.3.4.

In order to compute the expectation above, we replace $S(0)(e^{W(s)+\nu s}) = S(s)$ by a dummy variable x and take the unconditional expectation of the remaining random variable. Since, $\sigma(W(t)-W(s))+\nu(t-s)$ is normally distributed with mean $\nu(t-s)$ and variance $\sigma^2(t-s)$, we define

$$\begin{aligned} g(x) &= \frac{1}{\sigma\sqrt{2\pi(t-s)}}\int_{\mathbb{R}} f(xe^w)e^{\frac{-(w-\nu(t-s))^2}{2\sigma^2(t-s)}}\,dw \\ &= \frac{1}{\sigma y\sqrt{2\pi(t-s)}}\int_{\mathbb{R}^+} f(y)e^{\frac{-(\log(\frac{y}{x})-\nu(t-s))^2}{2\sigma^2(t-s)}}\,dy \\ &\text{(Changing variables } xe^w \to y) \end{aligned} \tag{0.32}$$

The Independence Lemma states that if we now take the function $g(x)$ defined in (0.32) and replace the dummy variable x by $S(s)$ then $\mathbb{E}[f(S(t))|\mathcal{F}(s)] = g(S(s))$ and hence S has Markov property and moreover

$$\rho(\tau,x,y) = \frac{1}{\sigma y\sqrt{2\pi\tau}}e^{\frac{-(\log(\frac{y}{x})-\nu\tau)^2}{2\sigma^2\tau}}$$

is its transition density. ■

(3.7) Theorem 3.6.2 provides the Laplace transform of the density of the first passage time for Brownian motion. This problem derives the analogous formula for Brownian motion with drift. Let W be a Brownian motion. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \le t < \infty$, we define

$$X(t) = \mu t + W(t),$$
$$\tau_m = min\{t \ge 0; X(t) = m\}$$

As usual, we set $\tau_m = \infty$ if $X(t)$ never reaches the level m. Let σ be a positive number and set $Z(t) = e^{\sigma X(t)-(\sigma\mu+\frac{1}{2}\sigma^2)t}$

(a) Show $Z(t)$, $t \ge 0$, is a martingale.

Proof. Clearly, $Z(t)$ is an adapted stochastic process with respect to the filtration $\mathcal{F}(t)$ where $\mathcal{F}(t)$ denotes the filtration of sub-sigma algebras of $\mathcal{F}$ with respect to which W is a martingale. We have, for $0 \le s \le t$,

$$\mathbb{E}[Z(t)|\mathcal{F}(s)] = \mathbb{E}[e^{\sigma X(t)-(\sigma\mu+\frac{1}{2}\sigma^2)t}|\mathcal{F}(s)] = \mathbb{E}[e^{\sigma W(t)-\frac{1}{2}\sigma^2 t}|\mathcal{F}(s)]$$
$$= \mathbb{E}[e^{\sigma(W(t)-W(s))+\sigma W(s)-\frac{1}{2}\sigma^2 t}|\mathcal{F}(s)]$$
$$= \mathbb{E}[e^{\sigma(W(t)-W(s))}|\mathcal{F}(s)]\cdot e^{\sigma W(s)-\frac{1}{2}\sigma^2 t} \text{ (Taking out what is known)}$$
$$= e^{\sigma W(s)-\frac{1}{2}\sigma^2 t}\cdot \mathbb{E}[e^{\sigma(W(t)-W(s))}] \text{ (Independence of } W(t)-W(s) \text{ w.r.t to } \mathcal{F}(s))$$
$$= e^{\sigma W(s)-\frac{1}{2}\sigma^2 t} e^{\frac{1}{2}\sigma^2(t-s)} \text{ (By (3.2.13) in text)}$$
$$= e^{\sigma W(s)-\frac{1}{2}\sigma^2 s} = e^{\sigma X(s)-(\sigma\mu+\frac{1}{2}\sigma^2)s}$$
$$= Z(s)$$

■

(b) Use (a) to conclude that

$$\mathbb{E}[e^{\{\sigma X(t\wedge\tau_m)-\left(\sigma\mu+\frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\}}] = 1,\ t \ge 0. \tag{0.33}$$

Proof. A martingale that is frozen at a stopping time is still a martingale and must have constant expectation. In particular, $1 = Z_0 = \mathbb{E}[Z_{t\wedge\tau_m}|\mathcal{F}_0] = \mathbb{E}[Z_{t\wedge\tau_m}]$. ■

(c) Now suppose that $\mu \ge 0$. Show that for $\sigma > 0$,

$$\mathbb{E}[e^{\{\sigma m-\left(\sigma\mu+\frac{1}{2}\sigma^2\right)\tau_m\}}\mathbb{1}_{\{\tau_m<\infty\}}] = 1$$

Use this fact to show that $\mathbb{P}\{\tau_m < \infty\} = 1$ and to obtain the Laplace transform

$$\mathbb{E}e^{-\alpha\tau_m} = e^{m\mu-m\sqrt{2\alpha+\mu^2}}\ \forall\alpha > 0 \tag{0.34}$$

Proof. The random variable $e^{\sigma(X(t\wedge\tau_m))} \in [0, e^{\sigma m}]$ for all $t \ge 0$. Hence,

$$0 \le \mathbb{E}[e^{\sigma X(t\wedge\tau_m)}] \le e^{\sigma m}\ \forall t \ge 0 \tag{0.35}$$

If $\tau_m < \infty$, then for large enough t, $t\wedge\tau_m = \tau_m$ and the term $e^{(-\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = e^{(-\sigma\mu-\frac{1}{2}\sigma^2)\tau_m}$. If $\tau_m = \infty$, then $e^{(-\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = e^{(-\sigma\mu-\frac{1}{2}\sigma^2)t}$ which goes to 0 as $t \to \infty$ as $\sigma\mu + \frac{1}{2}\sigma^2 > 0$. We combine these 2 cases by writing

$$\lim_{t\to\infty} e^{(-\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = \mathbb{1}_{\tau_m<\infty} e^{(-\sigma\mu-\frac{1}{2}\sigma^2)\tau_m}$$

The notation $\mathbb{1}_{\tau_m<\infty}$ denotes a random variable which is 1 if random variable $\tau_m < \infty$ and otherwise takes the value 0.

If $\tau_m < \infty$, then $e^{\sigma X(t\wedge\tau_m)} = e^{\sigma X(\tau_m)} = e^{\sigma m}$ for large enough t. If $\tau_m = \infty$, then we know from (0.35) that $0 \le e^{\sigma X(t\wedge\tau_m)} \le e^{\sigma m}$ as t becomes large. This ensures that $e^{\sigma X(t\wedge\tau_m)}e^{(-\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} \to 0$ as $t \to \infty$. We combine all these observations above in to

$$\lim_{t\to\infty} e^{\sigma X(t\wedge\tau_m)-(\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = \mathbb{1}_{\tau_m<\infty} e^{\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m} \tag{0.36}$$

Taking expectations on both sides in (0.36) and using (0.33) gives us

$$\mathbb{E}[\mathbb{1}_{\tau_m<\infty}e^{\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m}] = 1,\ t \geq 0. \tag{0.37}$$

The switching of limits and integrals in the last step can be justified by Lebesgue dominated convergence theorem.

We now take limits as $\sigma \to 0$ in (0.37) and use Dominated Convergence theorem to get

$$\mathbb{E}[\mathbb{1}_{\tau_m<\infty}] = 1$$

Thus, $\mathbb{P}\{\tau_m < \infty\} = 1$. Because τ_m is finite almost surely, we may drop the indicator of this event in (0.37) to obtain

$$\mathbb{E}[e^{-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m}] = e^{-\sigma m},\ t \geq 0. \tag{0.38}$$

We let α be a positive constant and set $(\sigma\mu + \frac{1}{2}\sigma^2) = \alpha$. Thus, from (0.38), we get

$$\mathbb{E}[e^{-\alpha\tau_m}] = e^{m\mu-m\sqrt{2\alpha+\mu^2}}\ \forall\alpha > 0 \tag{0.39}$$

(The other root is discarded as σ would be then negative.) ■

(d) Show that if $\mu > 0$, then $\mathbb{E}\tau_m < \infty$. Obtain a formula for $\mathbb{E}\tau_m$.

Proof. We differentiate (0.34) with respect to α (differentiation under integral sign is permitted by use of Lebesgue dominated convergence theorem) to get

$$\mathbb{E}[\tau_m e^{-\alpha\tau_m}] = e^{m\mu-m\sqrt{2\alpha+\mu^2}}\frac{m}{\sqrt{2\alpha+\mu^2}} \tag{0.40}$$

Taking limit as $\alpha \to 0$ on both sides of (0.40) and using Monotone convergence theorem gives us

$$\mathbb{E}[\tau_m] = \frac{m}{\mu}$$

In particular, we have $\mathbb{E}\tau_m < \infty$ ■

(e) Now suppose that $\mu < 0$. Show that, for $\sigma > -2\mu$,

$$\mathbb{E}[e^{\{\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m\}}\mathbb{1}_{\{\tau_m<\infty\}}] = 1$$

Use this fact to show that if $\mathbb{P}\{\tau_m < \infty\} = e^{-2x|\mu|}$, which is strictly less than 1, and to obtain the Laplace transform

$$\mathbb{E}e^{-\alpha\tau_m} = e^{m\mu-m\sqrt{2\alpha+\mu^2}}\ \forall\alpha > 0$$

Proof. The random variable $e^{\sigma(X(t\wedge\tau_m))} \in [0, e^{\sigma m}]$ for all $t \geq 0$. Hence,

$$0 \leq \mathbb{E}[e^{\sigma X(t\wedge\tau_m)}] \leq e^{\sigma m}\ \forall t \geq 0 \tag{0.41}$$

If $\tau_m < \infty$, then for large enough t, $t \wedge \tau_m = \tau_m$ and the term $e^{(-\sigma\mu - \frac{1}{2}\sigma^2)(t\wedge\tau_m)} = e^{(-\sigma\mu-\frac{1}{2}\sigma^2)\tau_m}$. If $\tau_m = \infty$, then $e^{(-\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = e^{(-\sigma\mu-\frac{1}{2}\sigma^2)t}$ which goes to 0 as $t \to \infty$ as $\sigma\mu + \frac{1}{2}\sigma^2 > 0$. We combine these 2 cases by writing

$$\lim_{t\to\infty} e^{(-\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = \mathbb{1}_{\tau_m<\infty} e^{(-\sigma\mu-\frac{1}{2}\sigma^2)\tau_m}$$

The notation $\mathbb{1}_{\tau_m<\infty}$ denotes a random variable which is 1 if random variable $\tau_m < \infty$ and otherwise takes the value 0.

If $\tau_m < \infty$, then $e^{\sigma X(t\wedge\tau_m)} = e^{\sigma X(\tau_m)} = e^{\sigma m}$ for large enough t. If $\tau_m = \infty$, then we know from (0.41) that $0 \le e^{\sigma X(t\wedge\tau_m)} \le e^{\sigma m}$ as t becomes large. This ensures that $e^{\sigma X(t\wedge\tau_m)} e^{(-\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} \to 0$ as $t \to \infty$. We combine all these observations above in to

$$\lim_{t\to\infty} e^{\sigma X(t\wedge\tau_m)-(\sigma\mu-\frac{1}{2}\sigma^2)(t\wedge\tau_m)} = \mathbb{1}_{\tau_m<\infty} e^{\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m} \tag{0.42}$$

Taking expectations on both sides in (0.42) and using (0.33) gives us

$$\mathbb{E}[\mathbb{1}_{\tau_m<\infty} e^{\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m}] = 1,\, t \ge 0. \tag{0.43}$$

The switching of limits and integrals in the last step can be justified by Lebesgue dominated convergence theorem.

We now take limits as $\sigma \to -2\mu$ in (0.43) and use Lebesgue dominated Convergence theorem to get

$$\mathbb{E}[\mathbb{1}_{\tau_m<\infty}] = e^{2m\mu} = e^{-2m|\mu|}$$

Thus, $\mathbb{P}\{\tau_m < \infty\} = e^{-2m|\mu|}$. From (0.43) we have,

$$\mathbb{E}[\mathbb{1}_{\{\tau_m<\infty\}} e^{-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m}] = e^{-\sigma m},\, t \ge 0. \tag{0.44}$$

We let α be a positive constant and set $(\sigma\mu + \frac{1}{2}\sigma^2) = \alpha$. Thus, from (0.44), we get

$$\mathbb{E}[\mathbb{1}_{\{\tau_m<\infty\}} e^{-\alpha\tau_m}] = e^{m\mu - m\sqrt{2\alpha+\mu^2}} \;\forall \alpha > 0 \tag{0.45}$$

(The other root is discarded as σ would then be negative.)

On the set $\{\tau_m = \infty\}$, the random variable $e^{-\alpha\tau_m} = 0$. Thus, $\mathbb{E}[\mathbb{1}_{\{\tau_m=\infty\}} e^{-\alpha\tau_m}] = 0$. Hence, we have

$$\mathbb{E}[e^{-\alpha\tau_m}] = \mathbb{E}[\mathbb{1}_{\tau_m<\infty} e^{-\alpha\tau_m}] + \mathbb{E}[\mathbb{1}_{\tau_m=\infty} e^{-\alpha\tau_m}] = e^{m\mu - m\sqrt{2\alpha+\mu^2}} \;\forall \alpha > 0 \tag{0.46}$$

∎

(4.1) Suppose $M(t)$, $0 \le t \le T$ is a martingale with respect to some filtration $\mathcal{F}(t)$, $0 \le t \le T$. Let $\Delta(t)$, $0 \le t \le T$, be a simple process adapted to $\mathcal{F}(t)$. For $t \in [t_k, t_{k+1})$, define the Stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(t) - M(t_k)]$$

We think of $M(t)$ as price of an asset at time t and $\Delta(t_j)$ as the number of shares of the asset held by an investor between times t_j and t_{j+1}. Then $I(t)$ is the capital gains that accrue to the investor between times 0 and t. Show that $I(t)$, $0 \le t \le T$ is an a martingale.

Proof. Clearly, $I(s)$ adapted to $\mathcal{F}(s)$. Consider $0 \le s \le t$. When $s = 0$ or $s = t$, it is easy to see that $\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$. We thus consider $0 < s < t$.

Let $s \in (0,t)$ be such that $s \in [t_j, t_{j+1})$ for $0 \le j \le k$ with the convention that $t_{k+1} = t$. Then,

$$I(t) = \sum_{i=0}^{j-1} \Delta(t_i)[M(t_{i+1}) - M(t_i)] + \sum_{i=j}^{k} \Delta(t_i)[M(t_{i+1}) - M(t_i)]$$

Note that $\sum_{i=0}^{j-1} \Delta(t_i)[M(t_{i+1}) - M(t_i)]$ is $\mathcal{F}(s)$ measurable as $t_j \le s$, $M(t)$ is a martingale with respect to the filtration $\mathcal{F}(t)$ and $\Delta(t)$ is a simple process adapted to $\mathcal{F}(t)$.

$$\sum_{i=j}^{k} \Delta(t_i)[M(t_{i+1}) - M(t_i)] = \Delta(t_j)[M(t_{j+1}) - M(s) + M(s) - M(t_j)] +$$

$$\sum_{i=j+1}^{k} \Delta(t_i)[M(t_{i+1}) - M(t_i)]$$

$$= \Delta(t_j)[M(s) - M(t_j)] + \Delta(t_j)[M(t_{j+1}) - M(s)] + \sum_{i=j+1}^{k} \Delta(t_i)[M(t_{i+1}) - M(t_i)] \tag{0.47}$$

The first term on the right hand side of the last equality in (0.47) is $\mathcal{F}(s)$ measurable. Moreover, by 'taking out what is known' property, $\mathbb{E}[\Delta(t_j)M(t_{j+1})|\mathcal{F}(s)] = \mathbb{E}[\Delta(t_j)\mathbb{E}[M(t_{j+1})|\mathcal{F}(s)] = \Delta(t_j)\mathbb{E}[M(t_{j+1})|\mathcal{F}(s)] = \Delta(t_j)M(s)$. The last equality follows from the fact that $M(t)$ is a martingale.

We also have,

$$\begin{aligned}\mathbb{E}[\sum_{i=j+1}^{k} \Delta(t_i)(M(t_{i+1}) - M(t_i))|\mathcal{F}(s)] &= \sum_{i=j+1}^{k} \mathbb{E}[\Delta(t_i)(M(t_{i+1}) - M(t_i))|\mathcal{F}(s)] \\ &= \sum_{i=j+1}^{k} \Delta(t_i)(M(s) - M(s)) = 0 \end{aligned} \tag{0.48}$$

The last equality in (0.48) follows because, for $j+1 \le i \le k$, $\Delta(t_i)$ is a constant and $\mathbb{E}[M(t_i)|\mathcal{F}(s)] = M(s)$ as $M(t)$ is a martingale.

Combining the observations obtained from (0.47) and (0.48), we can say

$$\mathbb{E}[\sum_{i=j}^{k} \Delta(t_i)[M(t_{i+1}) - M(t_i)|\mathcal{F}(s)] = \Delta(t_j)(M(s) - M(t_j)) \tag{0.49}$$

(0.49) along with a previously observed fact that $\sum_{i=0}^{j-1} \Delta(t_i)[M(t_{i+1}) - M(t_i)]$ is $\mathcal{F}(s)$ measurable gives us

$$\mathbb{E}[I(t)|\mathcal{F}(s)] = \sum_{i=0}^{j-1} \Delta(t_i)[M(t_{i+1}) - M(t_i)] + \Delta(t_j)(M(s) - M(t_j)) = I(s)$$

showing $I(t), 0 \leq t \leq T$ is a martingale. ■

(4.2) Let $W(t)$, $0 \leq t \leq T$, be a Brownian motion, and let $\mathcal{F}(t)$, $0 \leq t \leq T$ be an associated filtration. Let $\Delta(t)$, $0 \leq t \leq T$ be a non random simple process. For $t \in [t_k, t_{k+1})$ define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

(a) Show that whenever $0 \leq s < t \leq T$, the increment $I(t) - I(s)$ is independent of $\mathcal{F}(s)$.

Proof. We only need to show $I(t_k) - I(t_l)$ is independent of $\mathcal{F}(t_l)$ whenever t_k and t_l are two partition points with $t_l < t_k$.
$I(t_k) - I(t_l) = \sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$.

Since $W(t)$ is a Brownian motion, each of the increments $W(t_{j+1}) - W(t_j)$ is independent of $\mathcal{F}(t_j)$ for $l \leq j \leq k-1$ and hence independent of $\mathcal{F}(t_l)$ as $F(t)$ is a filtration. Moreover since, $\Delta(t_j)$ is a non-random quantity, $\Delta(t_j)[W(t_{j+1}) - W(t_j)]$ is also independent of $\mathcal{F}(t_l)$ for $l \leq j \leq k-1$. (The sigma algebra generated by a random variable X is the same as the sigma algebra generated by cX if $c \neq 0$. If $c = 0$, that particular term does not enter our computations and hence our claim still holds.)

Since, $I(t_k) - I(t_l)$ is a sum of random variables each of which is independent with respect to $\mathcal{F}(t_l)$, $I(t_k) - I(t_l)$ is independent of $\mathcal{F}(t_l)$. (If X and Y are independent of a sigma algebra $\mathcal{F}$, then so is $f(X,Y)$ for any Borel-measurable function f.) ■

(b) Show that whenever $0 \leq s < t \leq T$, the increment $I(t) - I(s)$ is a normally distributed random variable with mean 0 and variance $\int_s^t \Delta^2 u \, du$

Proof. Note that it suffices to show that$I(t_k) - I(t_l)$ is normally distributed with zero mean and variance $\int_{t_l}^{t_k} \Delta^2 u \, du$whenever t_k and t_l are two partition points with $t_l < t_k$.
$I(t_k) - I(t_l) = \sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$.

Note that $\Delta(t_j)$ is non-random and constant on each sub-interval $[t_j, t_{j+1})$. Since $W(t)$ is a Brownian motion, the increments $W(t_{j+1}) - W(t_j)$ are independent normal random variables with zero mean and variance $t_{j+1} - t_j$ for $l \leq j \leq k-1$. Hence, $\sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$ being a sum of independent normal random variables is itself normal with zero mean and variance $\sum_{j=l}^{k-1}\Delta^2(t_j)(t_{j+1}-t_j) = \int_{t_l}^{t_k}\Delta^2(u)\,du$ ∎

(c) Use (a) and (b) to show $I(t)$, $0 \le t \le T$ is a martingale.

Proof. Clearly, $I(s)$ adapted to $\mathcal{F}(s)$. Consider $0 \le s \le t$. When $s = 0$ or $s = t$, it is easy to see that $\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$. We thus consider $0 < s < t$.

$$\begin{aligned}
\mathbb{E}[I(t)|\mathcal{F}(s)] &= \mathbb{E}[I(t)-I(s)|\mathcal{F}(s)] + \mathbb{E}[I(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[I(t)-I(s)] + I(s) \text{ (By part a) and since } I(s) \in \mathcal{F}(s)) \\
&= I(s) \text{ (By part b)}
\end{aligned} \tag{0.50}$$

∎

(d) Show that $I^2(t) - \int_0^t \Delta^2(u)\,du$, $0 \le t \le T$ is a martingale.

Proof. Let $0 \le t \le$.Clearly, $I^2(t) - \int_0^t \Delta^2(u)\,du$ is adapted to $\mathcal{F}(t)$. Fix $0 \le s \le t$. By part a) and part b)

$$\mathbb{E}[(I(t)-I(s))^2|\mathcal{F}(s)] = \mathbb{E}[(I(t)-I(s))^2] = \int_s^t \Delta^2(u)\,du \tag{0.51}$$

Moreover, by part a) and 'taking out what is known' property,

$$\begin{aligned}
\mathbb{E}[I(s)I(t)|\mathcal{F}(s)] &= \mathbb{E}[I(s)(I(t)-I(s))|\mathcal{F}(s)] + \mathbb{E}[I(s)^2|\mathcal{F}(s)] \\
&= I(s)\mathbb{E}[I(t)-I(s)|\mathcal{F}(s)] + \mathbb{E}[I(s)^2|\mathcal{F}(s)] \\
&= I^2(s)
\end{aligned} \tag{0.52}$$

We also have,

$$\begin{aligned}
\mathbb{E}[(I(t)-I(s))^2|\mathcal{F}(s)] &= \mathbb{E}[(I(t)^2|\mathcal{F}(s)] + \mathbb{E}[I(s))^2|\mathcal{F}(s)] - 2\mathbb{E}[(I(t)I(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[(I(t)^2|\mathcal{F}(s)] - I^2(s) \text{ (By (0.52))}
\end{aligned} \tag{0.53}$$

(0.51) and (0.53) give us,

$$\begin{aligned}
\int_s^t \Delta^2(u)\,du &= \mathbb{E}[(I(t)^2|\mathcal{F}(s)] - I^2(s) \\
\int_0^t \Delta^2(u)\,du - \int_0^s \Delta^2(u)\,du &= \mathbb{E}[(I(t)^2|\mathcal{F}(s)] - I^2(s) \\
I^2(s) - \int_0^s \Delta^2(u)\,du &= \mathbb{E}[(I(t)^2|\mathcal{F}(s)] - \int_0^t \Delta^2(u)\,du
\end{aligned} \tag{0.54}$$

In the RHS of the last equality in (0.54), we can take the Lebesgue integral inside the conditional expectation as the process $\Delta(u)$ is non random thus showing $I^2(t) - \int_0^t \Delta^2(u)\,du$, $0 \le t \le T$ is a martingale. ∎

(4.6) Let $S(t) = S(0)exp\{\sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)t\}$ be a geometric Brownian motion. Let p be a positive constant. Compute $d(S^p(t))$, the differential of $S(t)$ raised to the power p.

Proof. Note that $X(t) = \sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)t$ is an Itô process with $dX(t) = \sigma dW(t) + (\alpha - \frac{1}{2}\sigma^2)dt$ and $dX(t)dX(t) = \sigma^2 dt$. Using the Itô-Doeblin formula (**4.4.23**) with the function $f(x) = S(0)^p e^{xp}$ gives us

$$\begin{aligned} d(S^p(t)) &= pS^p(t)dX(t) + \frac{1}{2}p^2 S^p(t)dX(t)dX(t) \\ &= pS^p(t)[\sigma dW(t) + (\alpha - \frac{1}{2}\sigma^2)dt + \frac{p}{2}\sigma^2 dt] \\ &= pS^p(t)[\sigma dW(t) + (\alpha + \frac{p-1}{2}\sigma^2)dt] \end{aligned}$$

■

(4.7) (a) Compute $dW^4(t)$ and then write $W^4(T)$ as the sum of an ordinary Lebesgue integral and an Itô integral.

Proof. Using Itô-Doeblin formula (**4.4.1**) with the function $f(x) = x^4$ gives us

$$dW^4(t) = 4W^3(t)\,dW(t) + 6W^2(t)\,dt$$

Thus, by (**4.4.2**), since $W(0) = 0$,

$$W^4(T) = \int_0^T 4W^3(t)\,dW(t) + \int_0^T 6W^2(t)\,dt \tag{0.55}$$

■

(b) Take expectations on both sides of the formula you obtained in part (a) and show $\mathbb{E}[W^4(T)] = 3T^2$.

Proof. Taking expectations on both sides of (0.55) (and using Fubini's theorem), we have

$$\begin{aligned} \mathbb{E}[W^4(T)] = \mathbb{E}[\int_0^T 4W^3(t)\,dW(t)] \\ + \int_0^T 6\mathbb{E}[W^2(t)]\,dt \end{aligned} \tag{0.56}$$

The first term above is zero as $I(t) = \int_0^t 4W^3(u)\,dW(u)$ is a martingale. Thus $\mathbb{E}[I(t)] = \mathbb{E}[I(t)|\mathcal{F}(0)] = I(0) = 0$. Since $W(t)$ is a Brownian motion, $\mathbb{E}[W^2(t)] = t$. Thus, (0.56) simplifies to

$$\mathbb{E}[W^4(T)] = \int_0^T 6t\,dt = 3T^2$$

■

(c) Use the method of (a) and (b) to find a formula for $\mathbb{E}[W^6(T)]$.

Proof. Using Itô-Doeblin formula (**4.4.1**) with the function $f(x) = x^6$ gives us

$$dW^6(t) = 6W^5(t)\,dW(t) + 15W^4(t)\,dt \tag{0.57}$$

Thus, by (**4.4.2**), since $W(0) = 0$,

$$W^6(T) = \int_0^T 6W^5(t)\,dW(t) + \int_0^T 15W^4(t)\,dt \tag{0.58}$$

Taking expectations on both sides of (0.58) (and using Fubini's theorem), we have

$$\begin{aligned}\mathbb{E}[W^6(T)] = \mathbb{E}[&\int_0^T 6W^5(t)\,dW(t)] \\ &+ \int_0^T 15\mathbb{E}[W^4(t)]\,dt\end{aligned} \tag{0.59}$$

The first term above is zero as $I(t) = \int_0^t 6W^5(u)\,dW(u)$ is a martingale. Thus $\mathbb{E}[I(t)] = \mathbb{E}[I(t)|\mathcal{F}(0)] = I(0) = 0$. By part b), (0.59) simplifies to

$$\mathbb{E}[W^6(T)] = \int_0^T 45t^2\,dt = 15T^3$$

■

(4.8) (**Solving the Vasicek equation**) The Vasicek interest rate stochastic differential equation (**4.4.32**) is

$$dR(t) = (\alpha - \beta R(t))\,dt + \sigma\,dW(t)$$

where α, β and σ are positive constants. The solution to this equation is given in Example (**4.4.10**). This exercise shows how to derive this solution.

(a) Use (**4.4.32**) and the Itô-Doeblin formula to compute $d(e^{\beta t}R(t))$. Simplify it so that you have a formula for $d(e^{\beta t}R(t))$ that does not involve $R(t)$.

Proof. Let $f(t,x) = e^{\beta t}x$. Then $f_t(t,x) = \beta f(t,x)$, $f_x(t,x) = e^{\beta t}$, $f_{xx}(t,x) = 0$. Using Itô-Doeblin formula (**4.4.23**) with $f(t,x) = e^{\beta t}x$ gives

$$\begin{aligned}d(e^{\beta t}R(t)) &= \beta e^{\beta t}R(t) + e^{\beta t}dR(t) \\ &= e^{\beta t}(\alpha\,dt + \sigma\,dW(t))\end{aligned} \tag{0.60}$$

■

(b) Integrate the equation you obtained in part a) and solve for $R(t)$ to obtain (**4.4.33**).

Proof. We integrate both sides in (0.60) between 0 and t to get

$$e^{\beta t}R(t) - R(0) = \frac{\alpha}{\beta}(e^{\beta t-1}) + \sigma \int_0^t e^{\beta s} dW(S) \tag{0.61}$$

Adding $R(0)$ and multiplying by $e^{\beta t}$ on both sides in (0.61) gives us the desired equation (**4.4.33**). ∎

(4.9) In this exercise, we will verify that a given function solves the Black-Scholes-Merton PDE with the right terminal and boundary conditions.

(a) First verify the equation

$$Ke^{-r(T-t)}N'(d_-) = xN'(d_+)$$

Proof. For any $y \in \mathbb{R}$, $N'(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$. Let $\tau = T - t$. We have,

$$\begin{aligned} Ke^{-r(T-t)}N'(d_-) &= \frac{1}{\sqrt{2\pi}}Ke^{-r(T-t)}e^{-\frac{d_-^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}}Ke^{-r(T-t)}e^{-\frac{d_+^2+\sigma^2\tau-2d_+\sigma\sqrt{\tau}}{2}} \end{aligned} \tag{0.62}$$

Since

$$e^{-\frac{d_+^2+\sigma^2\tau-2d_+\sigma\sqrt{\tau}}{2}} = e^{-\frac{d_+^2}{2}-\frac{\sigma^2\tau}{2}+\log(\frac{x}{K})+(r+\frac{1}{2}\sigma^2)\tau}$$

(0.62) reduces to

$$Ke^{-r(T-t)}N'(d_-) = x\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}} = xN'(d_+).$$

∎

(b) Show that $c_x = N(d_+)$. This is the *delta* of the option.

Proof. Using chain rule, we get

$$c_x = xN(d_+) + xN'(d_+)d_+' - Ke^{-r(T-t)}N'(d_-)d_-'.$$

By part (a),

$$xN'(d_+)d_+' - Ke^{-r(T-t)}N'(d_-)d_-' = xN'(d_+)(d_+' - d_-') = 0.$$

Hence, $c_x = N(d_+)$. ∎

(c) Show that

$$c_t = -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t)}}N'(d_+)$$

This is the *theta* of the option.

Proof. An application of chain rule gives us

$$
\begin{aligned}
c_t &= xN'(d_+)\frac{d_+}{dt} - Ke^{-r(T-t)}N'(d_-)\frac{d_-}{dt} - rKe^{-r(T-t)}N(d_-) \\
&= -rKe^{-r(T-t)}N(d_-) + xN'(d_+)\frac{d}{dt}\sigma\sqrt{T-t} \text{ (By part a)} \\
&= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t)}}N'(d_+)
\end{aligned}
$$

■

(d) Use the formulas above to show that c satisfies (**4.10.3**).

Proof. By part (b) and part (c) and using the notation $\tau = T - t$,

$$
\begin{aligned}
&c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} \\
&= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+) + rxN(d_+) + \frac{1}{2}N'(d_+)\frac{\sigma x}{\sqrt{\tau}} \\
&= rc(t,x)
\end{aligned}
$$

■

(e) Show that for $x > K$, $\lim_{t\to T} d_\pm = \infty$, but for $) < x < K$, $\lim_{t\to T} d_\pm = -\infty$. Use these equalities to derive the terminal condition (**4.10.4**).

Proof. $t \to T$ is equivalent to $\tau \to 0$. As $\tau \to 0$, $\frac{1}{\sigma\sqrt{\tau}}(r + \frac{1}{2}\sigma^2)\tau \to 0$ and $\frac{1}{\sigma\sqrt{\tau}}\log(\frac{x}{K}) \to \pm\infty$ as $\tau \to 0$ depending on whether $x > K$ or $0 < x < K$. Thus for $x > K$, $\lim_{t\to T} d_+ = \infty$, but for $0 < x < K$, $\lim_{t\to T} d_+ = -\infty$.

Since $d_- = d_+ - \sigma\sqrt{\tau}$ and $\sigma\sqrt{\tau} \to 0$ as $\tau \to 0$, d_- also has the same limiting behavior as d_+ when $t \to T$ (or equivalently $\tau \to 0$). This finishes the proof of the first part.

It is easy to see that $N(y) \to 0$ as $y \to -\infty$ and $N(y) \to 1$ as $y \to \infty$. Thus, when $x > K$ and we take the limit $t \to T$ i.e $\lim_{t\to T} d_\pm = \infty$, we get

$$\lim_{t\to T} c(t,x) = x - K$$

Likewise, when $0 < x < K$ and we take the limit $t \to T$ i.e $\lim_{t\to T} d_\pm = -\infty$, we get

$$\lim_{t\to T} c(t,x) = 0$$

We thus get the terminal condition (**4.10.4**). ■

(f) Show that for $0 \le t < T$, $\lim_{x\to 0} d_\pm = -\infty$. Use this fact to verify that the first part of boundary condition (**4.10.5**) as $x \to 0$.

Proof. Fixing a $0 \le t < T$ is equivalent to fixing a $0 < \tau \le T$. Since $\lim_{x\to 0}\log(\frac{x}{K}) = -\infty$, we get $\lim_{x\to 0} d_\pm = -\infty$.

Since $N(y) \to 0$ as $y \to -\infty$, we see that $\lim_{x\to 0} c(t,x) = 0$. ■

(g) Verify that

$$\lim_{x\to\infty}[c(t,x)-(x-Ke^{-r(T-t)})]=0 \quad ,0\le t\le T$$

Proof. Since $\lim_{x\to\infty}\log(\frac{x}{K})=\infty$, $\lim_{x\to\infty}d_{\pm}=\infty$.

Let us first show that $\lim_{x\to\infty}\frac{N(d_+)-1}{x^{-1}}=0$. Since $\lim_{x\to\infty}N(d_+)=1$, this is an indeterminate form $\frac{0}{0}$, and L'Hópital's rule implies that the limit is

$$\lim_{x\to\infty}\frac{\frac{d}{dx}[N(d_+-1]}{\frac{d}{dx}x^{-1}}$$

The above expression evaluates to

$$\begin{aligned}&-\frac{x}{\sigma\sqrt{2\pi\tau}}e^{-\frac{d_+(x)^2}{2}}\\ &=-\frac{Ke^{-\tau(r+\frac{1}{2}\sigma^2)}}{\sigma\sqrt{2\pi\tau}}e^{\sigma\tau d_+-\frac{d_+^2}{2}}\end{aligned}$$

Since, $\lim_{x\to\infty}\sigma\tau d_+-\frac{d_+^2}{2}=-\infty$,

$$\lim_{x\to\infty}-\frac{Ke^{-\tau(r+\frac{1}{2}\sigma^2)}}{\sigma\sqrt{2\pi\tau}}e^{\sigma\tau d_+-\frac{d_+^2}{2}}=0$$

Moreover, $\lim_{x\to\infty}N(d_-(\tau,x))=1$. Putting the above observations together, we get

$$\lim_{x\to\infty}[c(t,x)-(x-Ke^{-r(T-t)})]=0$$

∎

(4.10) **Self-Financing Trading**

(a) In continuous time, let $M(t)=e^{rt}$ be the price of a share of the money market account at time t, let $\Delta(t)$ denote the number of shares of stock held at time t, and let $\Gamma(t)$ denote the number of shares of the money market held at time t, so that the total portfolio value at time t is

$$X(t)=\Delta(t)s(t)+\Gamma(t)M(t)$$

Use (**4.10.16**) and (**4.10.9**) and derive the continuous-time self-financing condition

$$S(t)d\Delta(t)+dS(t)d\Delta(t)+M(t)d\Gamma(t)+dM(t)d\Gamma(t)=0$$

Proof. We have, by Itô's product rule

$$\begin{aligned}dX(t)&=d\Delta(t)S(t)+S(t)d\Delta(t)+d\Delta(t)dS(t)+d\Gamma(t)M(t)\\ &\quad+\Gamma(t)dM(t)+d\Gamma(t)dM(t)\end{aligned}\tag{0.63}$$

Equating (0.63) and $dX(t)=\Delta(t)dS(t)+r(X(t)-\Delta(t)S(t))dt$ and using that $dM(t)=rM(t)$, we obtain

$$S(t)d\Delta(t)+dS(t)d\Delta(t)+M(t)d\Gamma(t)+dM(t)d\Gamma(t)=0$$

∎

(b) Replace (**4.10.17**) by its corrected version (**4.10.21**) and use the continuous-time self-financing condition you derived in part a) to derive (**4.10.18**)

Proof. Since $N(t) = \Gamma(t)M(t)$, by Itô's product formula,

$$dN(t) = d\Gamma(t)M(t) + dM(t)\Gamma(t) + d\Gamma(t)dM(t) \tag{0.64}$$

Equating (0.64) and (**4.10.21**) and using the continuous-time self-financing condition from part (a), we get

$$\begin{aligned} &c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))dS(t)dS(t) \\ &= \Delta(t)dS(t) + dM(t)\Gamma(t) \end{aligned} \tag{0.65}$$

Since $\Delta(t) = c_x(t,S(t))$, $dS(t)dS(t) = \sigma^2 S^2 dt$ and $dM(t)\Gamma(t) = r(t)M(t)\Gamma(t)dt = rN(t)dt$, (0.65) reduces to

$$rN(t)dt = \left[c_t(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt.$$

■

(4.11) Let

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

denote the price for a European call, expiring at time T with strike price K, where

$$d_\pm(T-t,x) = \frac{1}{\sigma_1\sqrt{T-t)}}\left[\log(\frac{x}{K}) + \left(r \pm \frac{\sigma_1^2}{2}\right)(T-t)\right].$$

This option price assumes the underlying stock is a geometric Brownian motion with volatility $\sigma > 0$. For this problem, we take this to be the market price of the option.

Suppose, however, that the underlying asset is really a geometric Brownian motion with volatility $\sigma_2 > \sigma_1$, i.e,

$$dS(t) = \alpha S(t)dt + \sigma_2 s(t)dW(t)$$

Consequently, the market price of the call is incorrect.

We set up a portfolio whose value at each time t we denote by $X(t)$. We begin with $X(0) = 0$. At each time t, the portfolio is long one European call, is short $c_x(t,S(t))$ shares of stock, and thus has a cash position

$$X(t) - c(t,S(t)) + S(t)c_x(t,S(t)),$$

which is invested at the constant interest rate r. We remove cash from this portfolio at a rate $\frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2(t)c_{xx}(t,S(t))$. Therefore the differential value of this portfolio is

$$\begin{aligned} dX(t) &= dc(t,S(t)) - c_x(t,S(t))dS(t) \\ &\quad + r[X(t) - c(t,S(t)) + S(t)c_x(t,S(t))]dt \\ &\quad - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2(t)c_{xx}(t,S(t))dt \quad 0 \le t \le T. \end{aligned}$$

Show that $X(t) = 0$ for all $t \in [0, T]$. In particular, because $c_{xx}(t, S(t)) > 0$ and $\sigma_2 > \sigma_1$, we have an arbitrage opportunity; we can start with zero initial capital, remove cash at a positive rate between times 0 and T, and at time T have zero liability.

Proof. Let us first note that since

$$\begin{aligned} dc(t,S(t)) - c_x(t,S(t)) &= c_t(t,S(t))dt + \frac{1}{2}c_{xx}dS(t)dS(t) \\ &= c_t(t,S(t))dt + \frac{1}{2}c_{xx}\sigma_2^2 S(t)dt \end{aligned}$$

we get

$$dX(t) = c_t(t,S(t))dt + r[X(t) - c(t,S(t)) + S(t)c_x(t,S(t))]dt + \frac{1}{2}\sigma_1^2 S^2(t)c_{xx}dt \tag{0.66}$$

The last equality implies $dX(t)dX(t) = 0$. Applying the Itô Doeblin formula **4.4.23** with $f(t,x) = e^{-rt}x$ gives us

$$d(e^{-rt}X(t)) = -re^{-rt}X(t)dt + e^{-rt}dX(t) \tag{0.67}$$

From (0.66) and (0.67), we get

$$\begin{aligned} d(e^{-rt}X(t)) &= e^{-rt}\left[c_t(t,S(t)) - rc(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma_1^2 S^2 c_{xx}(t,S(t))\right]dt \\ &= 0 \text{ (as } c \text{ solves Black-Scholes-Merton PDE) } \forall t \in [0,T] \end{aligned}$$

Since $X(0) = 0$, we thus get $e^{-rt}X(t) = 0$ for all $t \in [0,T]$ implying that $X = 0$ for all $t \in [0,T]$. ■

(4.18) Let a stock price be a geometric Brownian motion

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t),$$

and let r denote the interest rate. We define the *market price of risk* to be

$$\theta = \frac{\alpha - r}{\sigma}$$

and the *state price density process* to be

$$\zeta(t) = exp{-}\theta W(t) - (r + \frac{1}{2}\theta^2)t$$

(a) Show that

$$d\zeta(t) = -\theta\zeta(t)dW(t) - r\zeta(t)dt.$$

Proof. An application of Itô-Doeblin formula (**4.4.13**) with $f(x) = e^{-\theta x - (r + \frac{1}{2}\theta^2)t}$ gives us

$$\begin{aligned} d\zeta(t) &= -(r + \frac{1}{2}\theta^2)\zeta(t)dt - \theta\zeta(t)dW(t) + \frac{1}{2}\theta^2 dt \\ &= -\theta\zeta(t)dW(t) - r\zeta(t)dt \end{aligned}$$

■

(b) Let X denote the value of an investor's portfolio when he uses a portfolio process $\Delta(t)$. From **(4.5.2)**, we have

$$dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t).$$

Show that $\zeta(t)X(t)$ is a martingale.

Proof. By Itô's product rule,

$$d(\zeta(t)X(t)) = \zeta(t)dX(t) + d\zeta(t)X(t) + d\zeta(t)dX(t)$$

Using the fact that, $dtdt = dtdW(t) = 0$ and $dW(t)dW(t) = dt$, we get

$$d\zeta(t)dX(t) = -\sigma\theta\zeta(t)\Delta(t)S(t) \tag{0.68}$$

We also have,

$$dX(t)\zeta(t) = [rX(t)\zeta(t) + \Delta(t)(\alpha - r)\zeta(t)S(t)]dt + \sigma\Delta(t)S(t)\zeta(t)dW(t) \tag{0.69}$$

$$X(t)d\zeta(t) = -\theta\zeta(t)X(t)dW(t) - r\zeta(t)X(t)dt \tag{0.70}$$

Adding (0.68), (0.69), (0.70) and using that $\theta\sigma = \alpha - r$, we get

$$d(\zeta(t)X(t)) = [\sigma\Delta(t)S(t)\zeta(t) - \theta\zeta(t)X(t)]dW(t)$$

This last equality tells us that $\zeta(t)X(t)$ is an Itô integral and hence by Theorem (**4.3.1, part 4**) is a martingale. ■

(c) Let $T > 0$ be a fixed terminal time. Show that id an investor wants to begin with some initial capital $X(0)$ and invest in order to have portfolio value $V(T)$ at time T, where $V(T)$ is a given $\mathcal{F}(T)$ measurable random variable, then he must begin with initial capital

$$X(0) = \mathbb{E}[\zeta(T)V(T)].$$

In other words, the *present value* at time zero of the random payment $V(T)$ at time T is $\mathbb{E}[\zeta(T)V(T)]$. This justifies calling $\zeta(t)$ the state price density process.

Proof. Since by part (c), $\zeta(t)X(t)$ is a martingale, $\mathbb{E}[\zeta(T)X(T)] = \mathbb{E}[\zeta(T)X(T)|\mathcal{F}(0)] = \zeta(0)X(0)$.

Since $\zeta(0) = 1$ and we impose the condition $X(T) = V(T)$, we get

$$X(0) = \zeta(0)X(0) = \mathbb{E}[\zeta(T)X(T)] = \mathbb{E}[\zeta(T)V(T)].$$

■

(5.2) **State price density process**. Show that the risk-neutral pricing formula **(5.2.30)** may be re-written as

$$D(t)Z(t)V(t) = \mathbb{E}[D(T)Z(T)V(T)|\mathcal{F}(t)]. \tag{0.71}$$

Here $Z(t)$ is the Radon-Nikodym derivative process **(5.2.11)** when the market price of risk-process $\Theta(t)$ is given by **(5.2.21)** and the conditional expectation on the right-hand side of (0.16) is taken under the actual probability measure $\mathbb{P}$, not the risk-neutral measure $\tilde{\mathbb{P}}$. In particular, if for some $A \in \mathcal{F}(T)$, a derivative security pays off $\mathbb{1}_A$ then the value of this derivative security at time zero is $\mathbb{E}[D(T)Z(T)\mathbb{1}_A]$. The process $D(t)Z(t)$ appearing in (0.16) is called the *state price density process*.

Proof. Note that $Z(t)$ is a positive random variable. We have,

$$\begin{aligned} D(t)V(t) &= \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)], \ \ 0 \le t \le T \text{ (By } (\mathbf{5.2.30})\text{)} \\ &= \frac{1}{Z(t)}\mathbb{E}[D(T)V(T)Z(T)|\mathcal{F}(t)] \text{ (By Lemma } \mathbf{5.2.2}\text{)} \end{aligned}$$

This proves (0.71). ∎

(5.3) According to the Black-Scholes-Merton formula, the value at the zero of a European call on a stock whose initial price is $S(0) = x$ is given by

$$c(0,x) = xN(d_+(T,x)) - Ke^{-T}N(d_-(T,x)),$$

where

$$\begin{aligned} d_+(T,x) &= \frac{1}{\sigma\sqrt{T}}\left[\log(\frac{x}{k}) + \left(r + \frac{1}{2}\sigma^2\right)T\right], \\ d_-(T,x) &= d_+(T,x) - \sigma\sqrt{T}. \end{aligned}$$

The stock is modeled as a geometric Brownian motion with constant volatility $\sigma > 0$, the interest rate is constant r, the call strike is K, and the call expiration time is T. This formula is obtained by computing the discounted expected payoff of the call under the risk-neutral measure,

$$\begin{aligned} c(0,x) &= \tilde{\mathbb{E}}[e^{-rT}(S(T() - K)^+] \\ &= \tilde{\mathbb{E}}\left[e^{-rT}\left(xexp\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\} - K\right)^+\right], \end{aligned} \tag{0.72}$$

where $\tilde{W}$ is a Brownian motion under the risk-neutral measure $\tilde{\mathbb{P}}$. In Exercise 4.9(part b), the *delta* of this option is computed to be $c_x(0,x) = N(d_+(T,x))$. This problem provides an alternate way to compute $c_x(0,x)$.

(a) We begin with the observation that if $h(s) = (s-K)^+$, then

$$h'(s) = \begin{cases} 0 & s < K \\ 1 & s > K \end{cases}$$

If $s = K$, then $h'(s)$ is is undefined, but that will not matter in what follows because $S(T)$ has zero probability of taking the value K. Using the formula for $h'(s)$, differentiate inside the expected value in (0.72) to obtain a formula for $c_x(0,x)$.

Proof. We first note that $S(T) = S(0)exp\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\}$. Differentiating under integral sign (whose validity is justified since $\mathbb{P}\{S(T) = K\} = \tilde{\mathbb{P}}\{S(T) = K\} = 0$) gives us

$$\begin{aligned} c_x(0,x) &= \tilde{\mathbb{E}}[e^{-rT}exp\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\}\mathbb{1}_{S(T)>K}] \\ &= \tilde{\mathbb{E}}[e^{\sigma\tilde{W}(T) - \frac{1}{2}\sigma^2T}\mathbb{1}_{S(T)>K}] \end{aligned} \tag{0.73}$$

∎

(b) Show that the formula you obtained in part a) can be re-written as

$$c_x(0,x) = \hat{\mathbb{P}}(S(T) > K)$$

where $\hat{\mathbb{P}}$ is a probability measure equivalent to $\tilde{\mathbb{P}}$. Show that

$$\hat{W}(t) = \tilde{W}(t) - \sigma t$$

is a Brownian motion under $\hat{\mathbb{P}}$.

Proof. We first observe that $Z = e^{\sigma \tilde{W}(T) - \frac{1}{2}\sigma^2 T}$ is a random variable such that $\tilde{\mathbb{E}}Z = 1$. To prove this, we note that $\tilde{W}(T)$ is a normal random variable with mean zero and variance T. We then use a standard change of variables to see that $\tilde{\mathbb{E}}Z = 1$. We define a new probability measure $\hat{\mathbb{P}}$ by the formula

$$\hat{\mathbb{P}}(A) = \int_A Z(\omega) d\tilde{\mathbb{P}}(\omega) \ \ \forall A \in \mathcal{F}.$$

$\tilde{\mathbb{P}}$ and $\hat{\mathbb{P}}$ are in fact equivalent measures as Z is a strictly positive random variable. For any random variable X, we have $\hat{\mathbb{E}}X = \tilde{\mathbb{E}}[XZ]$. Hence, (0.72) reduces to

$$\hat{\mathbb{E}}[\mathbb{1}_{S(T)>K}] = \hat{\mathbb{P}}(S(T) > K).$$

To prove that $\hat{W}(t) = \tilde{W}(t) - \sigma t$ is a Brownian motion under $\hat{\mathbb{P}}$, we simply apply Girsanov's theorem (**Theorem 5.2.3**) with $\Theta(u) = -\sigma$. In particular, this implies that $\hat{W}(t)$ is a normal random variable under $\hat{\mathbb{P}}$ with mean zero and variance T. ■

(c) Rewrite $S(T)$ in terms of $\hat{W}(T)$, and then show that

$$\hat{\mathbb{P}}\{S(T) > K\} = \hat{\mathbb{P}}\{-\frac{\hat{W}(T)}{\sqrt{T}} < d_+(T,x)\} = N(d_+(T,x)).$$

Proof. By part (b),

$$\begin{aligned} S(T) &= S(0) exp\{\sigma \tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\} \\ &= S(0) exp\{\sigma \hat{W}(T) + \left(r + \frac{1}{2}\sigma^2\right)T\} \end{aligned}$$

Note that $S(T) > K$ iff $-\frac{\hat{W}(T)}{\sqrt{T}} < d_+(T,x)$. Hence,

$$\begin{aligned} \hat{\mathbb{P}}\{S(T) > K\} &= \hat{\mathbb{P}}\{-\frac{\hat{W}(T)}{\sqrt{T}} < d_+(T,x)\} \\ &= N(d_+) \end{aligned}$$

The last equality follows because $\frac{\hat{W}(T)}{\sqrt{T}}$ is a normal random variable under $\hat{\mathbb{P}}$ with mean zero and variance 1. ■

(5.5) Prove Corollary *5.3.2* by the following steps,

(a) Compute the differential of $\frac{1}{Z(t)}$, where $Z(t)$ is given in Corollary **5.3.2**.

Proof. $\frac{1}{Z(t)} = exp\{\int_0^t \theta(u)dW(u) + \frac{1}{2}\int_0^t \theta^2(u)\,du\}$. Using the Itô-Doeblin formula **4.4.23** with $f(x) = e^x$, we get

$$d\left(\frac{1}{Z(t}\right) = \frac{1}{Z(t)}\theta(t)dW(t) + \theta^2(t)\frac{1}{Z(t)}dt.$$

■

(b) Let $\tilde{M}(t)$, $0 \le t \le T$, be a martingale under $\tilde{\mathbb{P}}$. Show that $M(t) = Z(t)\tilde{M}(t)$ is a martingale under $\mathbb{P}$.

Proof. We first note that $Z(t)$ is a positive random variable for any $t \in [0, T]$. We have,

$$\begin{aligned}
&\tilde{\mathbb{E}}[\tilde{M}(t)|\mathcal{F}(s)] = \tilde{M}(s) \text{ (By hypothesis)}\\
&\tilde{\mathbb{E}}[\frac{M(t)}{Z(t)}|\mathcal{F}(s)] = \frac{M(s)}{Z(s)}\\
&\frac{1}{Z(s)}\mathbb{E}[M(t)|\mathcal{F}(s)] = \frac{M(s)}{Z(s)} \text{ (By Lemma 5.2.2)}\\
&\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s).
\end{aligned}$$

■

(c) According to Theorem *5.3.1*, there is an adapted process $\Gamma(u)$, $0 \le u \le T$, such that

$$M(t) = M(0) + \int_0^T \Gamma(u)dW(u) \;\; 0 \le t \le T$$

Write $\tilde{M}(t) = M(t) \cdot \frac{1}{Z(t)}$ and take its differential using Itô's product rule.

Proof. Itô's product rules gives us

$$d(\tilde{M}(t)) = \left(\frac{M(t)\theta(t) + \Gamma(t)}{Z(t)}\right)dW(t) + \left[\frac{M(t)\theta^2(t) + \theta(t)\Gamma(t)}{Z(t)}\right]dt \tag{0.74}$$

Since, $d\tilde{W}(t) = dW(t) - \theta(t)dt$, (0.74) reduces to

$$\begin{aligned}
d(\tilde{M}(t)) &= \left(\frac{M(t)\theta(t) + \Gamma(t)}{Z(t)}\right)d\tilde{W}(t)\\
&= \tilde{\Gamma}(t)d\tilde{W}(t) \qquad (0.75)
\end{aligned}$$

where $\tilde{\Gamma}(t)$ is a stochastic process adapted to the filtration generated by the Brownian motion $W(t)$. ■

(d) Show that the differential $\tilde{M}(t)$ is the sum of an adapted process, which we call $\tilde{\Gamma}(t)$, times $d\tilde{W}(t)$, and zero times dt. Integrate to obtain **(5.3.2)**.

Proof. We integrate (0.75) to obtain the conclusion of Corollary **5.3.2**. ■

(5.7) (a) Suppose a multi-dimensional market model as described in Section 5.4.2 has an arbitrage. In other words, suppose there is a portfolio value process satisfying $X_1(0) = 0$ and

$$\mathbb{P}\{X_1(T) \geq 0\} = 1, \ \mathbb{P}\{X_1(T) > 0\} > 0 \tag{0.76}$$

for some positive T. Show that if $X_2(0)$ is positive then there exists a portfolio value process $X_2(t)$ starting at $X_2(0)$ and satisfying

$$\mathbb{P}\{X_2(T) \geq \frac{X_2(0)}{D(T)}\}, \ \mathbb{P}\{X_2(T) > \frac{X_1(T)}{D(T)} > 0\} \tag{0.77}$$

Proof. Define $X_2(t) := X_1(t) + X_2(0)M(t)$, where $M(t) = \frac{1}{D(t)}$ denotes the money market account. Note that $X_2(T) \geq X_2(0)M(T)$ iff $X_1(T) \geq 0$ and $X_2(T) > X_2(0)M(T)$ iff $X_1(T) > 0$. We thus get (0.77) assuming (0.76). ∎

(b) Show that if a multidimensional market model has a portfolio vale process $X_2(t)$ such that $X_2(0)$ is positive and (0.17) holds, then the model has a portfolio value process $X_1(0)$ such that $X_1(0) = 0$ and (0.76) holds.

Proof. Define $X_1(t) := X_2(t) - X_2(0)M(t)$. It is easy to see that $X_2(T) \geq X_2(0)M(T)$ iff $X_1(0) \geq 0$ and $X_2(T) > X_1(T)M(0)$ iff $X_1(T) > 0$. Moreover, $X_1(0) = 0$. Thus (0.76) holds assuming (0.77). ∎

(5.8) **(Every strictly positive asset is a generalized Brownian motion)** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t), 0 \leq t \leq T$. Let $\mathcal{F}(t), 0 \leq t \leq T$ be the filtration generated by this Brownian motion. Assume that there is a unique risk-neutral measure $\tilde{\mathbb{P}}$, and let $\tilde{W}(t), 0 \leq t \leq T$, be the Brownian motion under $\tilde{\mathbb{P}}$ obtained by an application of Girsanov's theorem, Theorem **5.2.3**.

Corollary **5.3.2** of the Martingale Representation Theorem asserts that every martingale $\tilde{M}(t), 0 \leq t \leq T$, under $\tilde{\mathbb{P}}$ can be written as a stochastic integral with respect to $\tilde{W}(t), 0 \leq t \leq T$.

Now, let $V(T)$ be an almost surely positive $\mathcal{F}(T)$ measurable random variable. According to the risk-neutral pricing formula **(5.2.31)**, the price at time t of a security paying $V(T)$ at time T is

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du} V(T) | \mathcal{F}(t)\right], 0 \leq t \leq T$$

(a) Show that there exists an adapted process $\tilde{\Gamma}(t), 0 \leq t \leq T$, such that

$$dV(t) = R(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)} d\tilde{W}(t), 0 \leq t \leq T. \tag{0.78}$$

Proof. Note that

$$D(t)V(t) = \tilde{\mathbb{E}}\left[e^{-\int_0^T R(u)du} V(T) | \mathcal{F}(t)\right], 0 \leq t \leq T.$$

It is easy to see that $\tilde{\mathbb{E}}\left[e^{-\int_0^T R(u)du} V(T)|\mathcal{F}(t)\right]$ is a martingale under $\tilde{\mathbb{P}}$. Thus, Corollary **(5.3.2)** of Martingale representation theorem gives us an existence of an adapted process $\tilde{\Gamma}(t)$ so that

$$d(D(t)V(t)) = \tilde{\Gamma}(t)d\tilde{W}(t), \tag{0.79}$$

(0.79) and Itô's product rule for $d(D(t)V(t))$ along with the fact that $dD(T) = -R(t)D(t)dt$ gives us (0.78). ■

(b) Show that, for each $t \in [0,T]$, the price of the derivative security $V(t)$ at time t is almost surely positive.

There might be a way to solve this using maximum principle for a parabolic equation by connecting a Markov process to a PDE as in Chapter 6. However, since $V(t)$ does not satisfy a stochastic PDE, it is not clear how we would bring a Markov process in to the picture.

Also, it is not clear how exactly we can bring a transition probability to use its properties. It will be great if some pointers can be given to complete this problem!

(c) Conclude from the previous two parts that there exists an adapted process $\sigma(t), 0 \le t \le T$, such that

$$dV(t) = R(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t), 0 \le t \le T. \tag{0.80}$$

In other words, prior to time T, the price of every asset with almost surely positive price at time T follows a generalized geometric Brownian motion.

Proof. Since $V(t)$ is almost surely positive for each $0 \le t \le t$, we can define an $\mathcal{F}(t)$ adapted process

$$\sigma(t) = \frac{\tilde{\Gamma}(t)}{D(t)V(t)}.$$

(0.80) thus follows from (0.78). ■

(5.11) **(Hedging a cash flow)**
Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \le t \le T$, be the filtration generated by this Brownian motion. Let the mean rate of return $\alpha(t)$, the interest rate $R(t)$, and the volatility $\sigma(t)$ be adapted processes, and assume that $\sigma(t)$ is never zero. Consider a stock price process whose differential is given by

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \le t \le T. \tag{0.81}$$

Suppose an agent must pay a cash flow at rate $C(t)$ at each time t, where $C(t), 0 \le t \le T$, is an adapted process. If the agent holds $\Delta(t)$ shares of stock at each time t, then the differential of her portfolio will be

$$dX(t) = \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt - C(t)dt. \tag{0.82}$$

Show that there is a non-random value of $X(0)$ and a portfolio process $\Delta(t), 0 \le t \le T$, such that $X(T) = 0$ almost surely.

Proof. From (0.82) it is easy to see that

$$d[D(t)X(t)+\int_0^t D(u)C(u)du] = \Delta(t)D(t)S(t)[(\alpha(t)-R(t))dt+\sigma(t)dW(t)]$$
$$= \Delta(t)D(t)S(t)\sigma(t)d\tilde{W}(t) \tag{0.83}$$

Here $\tilde{W}(t)$ is a Brownian motion under $\tilde{\mathbb{P}}$ where we apply the usual change of measure to risk-neutral probability $\tilde{\mathbb{P}}$.

Hence, we can say that $D(t)X(t)+\int_0^t D(u)C(u)du$ is a $\tilde{\mathbb{P}}$ martingale i.e

$$D(t)X(t)+\int_0^t D(u)C(u)du = \tilde{\mathbb{E}}[D(T)X(T)+\int_0^T D(u)C(u)du|\mathcal{F}(t)]$$
$$= \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du|\mathcal{F}(t)\right] (\text{ as } X(T)=0 \text{ a.s is what we impose}) \tag{0.84}$$

In particular, (0.84) tells us that $X(0) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du\right]$. Moreover, a quick application of iterated conditioning proves that $\tilde{M}(t) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du|\mathcal{F}(t)\right]$ is a $\tilde{\mathbb{P}}$ martingale. Applying Corollary (**5.3.2**), we get the existence of an $\mathcal{F}(t)$ adapted process $\tilde{\Gamma}(t)$ such that

$$\tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du|\mathcal{F}(t)\right] = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du\right]+\int_0^t \tilde{\Gamma}(u)d\tilde{W}(u). \tag{0.85}$$

(0.84) and (0.85) together give

$$D(t)X(t)+\int_0^t D(u)C(u)du = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du\right]+\int_0^t \tilde{\Gamma}(u)d\tilde{W}(u). \tag{0.86}$$

(0.83) and (0.86) together imply (as $\sigma(t)D(t)S(t)$ is never 0) that

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{D(t)S(t)\sigma(t)}. \tag{0.87}$$

■

(6.1) Consider the stochastic differential equation

$$dX(u) = (a(u)+b(u)X(u))du+(\gamma(u)+\sigma(u)X(u))dW(u), \tag{0.88}$$

where $W(u)$ is a Brownian motion relative to a filtration $\mathcal{F}(u), u \geq 0$, and we allow $a(u), b(u), \gamma(u)$ and $\sigma(u)$ to be processes adapted to this filtration. Fix an initial time $t \geq 0$ and an initial position $x \in \mathbb{R}$. Define

$$Z(u) = exp\{\int_u^t \sigma(v)dW(v)+\int_u^t \left(b(v)-\frac{1}{2}\sigma^2(v)\right)dv\},$$

$$Y(u) = x + \int_t^u \frac{\alpha(v) - \sigma(v)\gamma(v)}{Z(v)} dv + \int_t^u \frac{\gamma(v)}{Z(v)} dW(v).$$

(a) Show that $Z(t) = 1$ and

$$dZ(u) = b(u)Z(u)du + \sigma(u)Z(u)dW(u), u \geq t. \tag{0.89}$$

Proof. Clearly, $Z(t) = 1$. Applying Itô's formula with $f(x) = e^x$ and for $X(t) = \int_u^t \sigma(v)dW(v) + \int_u^t \left(b(v) - \frac{1}{2}\sigma^2(v)\right)dv$ gives us (0.89). ■

(b) By its very definition, $Y(u)$ satisfies $Y(t) = x$ and

$$dY(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)} du + \frac{\gamma(u)}{Z(u)} dW(u), u \geq t \tag{0.90}$$

Show that $X(u) = Y(u)Z(u)$ solves the stochastic differential equation (0.88) and satisfies the initial condition $X(t) = x$.

Proof. By the definition for $Y(u)$, it is clear that $Y(t) = x$. Since, $Z(t) = 1$ as shown in part (a), it follows that $X(t) = x$. We have for $u \geq t$

$$\begin{aligned} dY(u)Z(u) &= [a(u) - \sigma(u)\gamma(u)]du + \gamma(u)dW(u) \\ dZ(u)Y(u) &= b(u)Y(u)Z(u)du + \sigma(u)Z(u)Y(u)dW(u) \\ dZ(u)dY(u) &= \sigma(u)\gamma(u)du \text{ (From (0.32) and (0.33))} \end{aligned} \tag{0.91}$$

Itô's product rule and (0.91) show that $X(u) = Y(u)Z(u)$ solves the stochastic differential equation (0.88) and satisfies the initial condition $X(t) = x$. ■

(6.5) **(Two-dimensional Feynman-Kac)**

(a) With $g(t, x_1, x_2)$ and $f(t, x_1, x_2)$ defined by **(6.6.1)** and **6.6.2)**, show that $g(t, X_1(t), X_2(t))$ and $e^{-rt} f(t, X_1(t), X_2(t))$ are martingales.

Proof. By **(6.6.1)**, $g(t, x_1, x_2) = \mathbb{E}^{t, x_1, x_2} h(X_1(t), X_2(T))$. Since $(X_1(t), X_2(t)$ satisfies a stochastic differential equation system, $(X_1(t), X_2(t)$ is a Markov process and thus for $0 \leq u \leq T$, we obtain $\mathbb{E}[h(X_1(T), h(X_2)(T)|\mathcal{F}(u)] = g(t, X_1(u), X_2(u))$. Thus, for any $0 \leq s \leq t \leq T$, using iterated conditioning we get $\mathbb{E}[g(t, X_1(t), X_2(t)|\mathcal{F}(s)] = g(s, X_1(s), X_2(s)$.

Similarly, we observe that $e^{-rt} f(t, X_1(t), X_2(t) = \mathbb{E}[e^{-rT} h(X_1(T), X_2(T))|\mathcal{F}(t)]$. Again, using iterated conditioning, it is easy to see that $e^{-rt} f(t, X_1(t), X_2(t))$ is a martingale. ■

(b) Assuming that W_1 and W_2 are independent Brownian motions, use the Itô-Doeblin formula to compute the differentials of $g(t, X_1(t), X_2(t))$ and $e^{-rt} f(t, X_1(t), X_2(t))$, set the dt term to 0, and thereby obtain the partial differential equations **(6.6.3)** and **(6.6.4)**.

Proof. This is a routine (and tedious!) application of Itô Doeblin formula. We only mention the key points for brevity.

We use $dX_1(t)dX_1(t) = [\gamma_{11}^2+\gamma_{12}^2]dt$, $dX_2(t)dX_2(t) = [\gamma_{21}^2+\gamma_{22}^2]dt$ and $dX_1(t)dX_2(t) = [\gamma_{11}\gamma_{21}+\gamma_{12}\gamma_{22}]dt$ along with Itô-Doeblin formula and setting the dt term to zero to get **(6.6.3)**. In addition, we use Itô product formula and set the dt term to zero to get **(6.6.4)** ■

(c) Now consider the case that $dW_1(t)dW_2(t) = \rho dt$ where ρ is a constant. Compute the differentials of $g(t, X_1(t), X_2(t))$ and $e^{-rt}f(t, X_1(t), X_2(t))$, set the dt term to 0, and obtain the partial differential equations

$$\begin{aligned}
&g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + (\frac{1}{2}{\gamma^2}_{11} + \rho\gamma_{11}\gamma_{12} + \frac{1}{2}{\gamma^2}_{12})g_{x_1x_1}\\
&+ (\gamma_{11}\gamma_{21} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21} + \gamma_{12}\gamma_{22})g_{x_1x_2}\\
&+ (\frac{1}{2}\gamma_{21}^2 + \rho\gamma_{21}\gamma_{22} + \frac{1}{2}\gamma_{22}^2)g_{x_2x_2} = 0 \qquad (*)\\
&f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + (\frac{1}{2}{\gamma^2}_{11} + \rho\gamma_{11}\gamma_{12} + \frac{1}{2}{\gamma^2}_{12})f_{x_1x_1}\\
&+ (\gamma_{11}\gamma_{21} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21} + \gamma_{12}\gamma_{22})f_{x_1x_2}\\
&+ (\frac{1}{2}\gamma_{21}^2 + \rho\gamma_{21}\gamma_{22} + \frac{1}{2}\gamma_{22}^2)f_{x_2x_2} = rf. \qquad (**)
\end{aligned}$$

Proof. Again, we only mention the key points for brevity.

We use $dX_1(t)dX_1(t) = [\gamma_{11}^2 + \gamma_{12}^2 + 2\rho\gamma_{11}\gamma_{12}]dt$, $dX_2(t)dX_2(t) = [\gamma_{21}^2 + \gamma_{22}^2 + 2\rho\gamma_{21}\gamma_{22}]dt$ and $dX_1(t)dX_2(t) = [\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22} + \rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21}]dt$ along with Itô-Doeblin formula and setting the dt term to zero to get (*). In addition, we use Itô product formula and set the dt term to zero to get (**). ■

(6.2) **Solution of Hull-White model**
This exercise solves the ordinary differential equations **(6.5.8)** and **(6.5.9)** to produce the solutions $C(t,T)$ and $A(t,T)$ given in (**6.5.10)** and **(6.5.11)**.

(a) Use equation **(6.5.8)** with s replacing t to show that

$$\frac{d}{ds}[e^{-\int_0^s b(v)dv}C(s,T)] = -e^{-\int_0^s b(v)dv}.$$

Proof. By equation **(6.5.8)**, $\frac{d}{ds}[e^{-\int_0^s b(v)dv}C(s,T)] = e^{-\int_0^s b(v)dv}[c'(s,T) - b(s)c(s,T)] = -e^{-\int_0^s b(v)dv}$. ■

(b) Integrate the equation in part a) above from $s = t$ to $s = T$, and use the terminal condition $C(T,T)$ to obtain **(6.5.10)**.

Proof. Since $C(T,T) = 0$, integration by parts give us

$$e^{\int_0^t b(v)dv} c(t,T) = \int_t^T e^{-\int_0^s b(v)dv} ds$$

$$c(t,T) = \int_t^T e^{-\int_t^s b(v)dv} ds$$

■

(c) Replace t by s in **(6.5.9)**, integrate the resulting equation from $s = t$ to $s = T$, use the terminal condition $A(T,T) = 0$, and obtain **(6.5.11)**.

Proof. As indicated, we replace t by s in **(6.5.9)**, integrate the resulting equation from $s = t$ to $s = T$, use the terminal condition $A(T,T) = 0$ and obtain

$$A(t,T) = \int_t^T (a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C(s,T))ds \tag{0.92}$$

■

(6.4) **(Solution of Cox-Ingersoll-Ross model)**
This exercise solves the ordinary differential equations **(6.5.14)** and **(6.5.15)** to produce the solutions $C(t,T)$ and $A(t,T)$ given in **(6.5.16)** and **(6.5.17)**.

(a) Define the function

$$\phi(t) = exp\{\frac{1}{2}\sigma^2 \int_t^T C(u,T)du\}.$$

Show that

$$C(t,T) = -\frac{2\phi'(t)}{\sigma^2\phi(t)},$$
$$C'(t,T) = -\frac{2\phi''(t)}{\sigma^2\phi(t)} + \frac{1}{2}\sigma^2 C^2(t,T).$$

Proof. A quick application of chain rule gives $C(t,T) = -\frac{2\phi'(t)}{\sigma^2\phi(t)}$. We now use the usual product rule to get $C'(t,T) = -\frac{2\phi''(t)}{\sigma^2\phi(t)} + \frac{1}{2}\sigma^2 C^2(t,T)$. ■

(b) Use the equation **(6.5.14)** to show that

$$\phi''(t) - b\phi'(t) - \frac{1}{2}\sigma^2\phi(t) = 0.$$

Proof. We use equation **(6.5.14)** to get

$$-\frac{2\phi'(t)}{\sigma^2\phi(t)} + \frac{1}{2}\sigma^2 C^2(t,T) = -\frac{2b\phi'(t)}{\sigma^2\phi(t)} + \frac{1}{2}\sigma^2 C^2(t,T) - 1$$

Rearranging gives us the desired ODE. ■

(c) Show that $\phi(t)$ must be of the form

$$\phi(t) = \frac{c_1}{\frac{1}{2}b+\gamma}e^{-(\frac{1}{2}b+\gamma)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma}e^{-(\frac{1}{2}b-\gamma)(T-t)}$$

for some constants c_1 and c_2, where $\gamma = \frac{1}{2}\sqrt{b^2+2\gamma^2}$.

Proof. As indicated, any solution $\phi(t)$ must be of the form $\phi(t) = a_1 e^{(\frac{b}{2}+\gamma)t} + a_2 e^{(\frac{b}{2}-\gamma)t}$ for some constants a_1 and a_2. We can re-write this in the form $\phi(t) = \frac{c_1}{\frac{1}{2}b+\gamma}e^{-(\frac{1}{2}b+\gamma)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma}e^{-(\frac{1}{2}b-\gamma)(T-t)}$ ■

(d) Show that

$$\phi'(t) = c_1 e^{-(\frac{1}{2}b+\gamma)(T-t)} - c_2 e^{-(\frac{1}{2}b-\gamma)(T-t)}.$$

Use the fact that $C(T,T) = 0$ to show that $c_1 = c_2$.

Proof. Clearly, $\phi'(t) = c_1 e^{-(\frac{1}{2}b+\gamma)(T-t)} - c_2 e^{-(\frac{1}{2}b-\gamma)(T-t)}$. Moreover, $C(T,T) = 0 \implies \phi'(T) = c_1 - c_2$ i.e $c_1 = c_2$. ■

(e) Show that

$$\phi(t) = \frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}[b\sinh(\gamma(T-t)) + 2\gamma\cosh(\gamma(T-t))],$$
$$\phi'(t) = -2c_1 e^{-\frac{1}{2}b(T-t)}\sinh(\gamma(T-t))$$

Conclude that $C(t,T)$ is given by **(6.5.16)**.

Proof. Part (d) above and a bit of algebraic manipulation give us

$$\phi(t) = c_1 e^{-\frac{1}{2}b(T-t)}\left[\frac{\frac{1}{2}b-\gamma}{\frac{1}{4}b^2-\gamma^2}e^{-\gamma(T-t)} - \frac{\frac{1}{2}b+\gamma}{\frac{1}{4}b^2-\gamma^2}e^{-\gamma(T-t)}\right]$$
$$\phi(t) = \frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}[b\sinh(\gamma(T-t)) + 2\gamma\cosh(\gamma(T-t))],$$
$$\phi'(t) = -2c_1 e^{-\frac{1}{2}b(T-t)}\sinh(\gamma(T-t))$$

Thus, we see that $c(t,T)$ is given by **(6.5.16)**. ■

(f) From **(6.5.15)** and **(6.9.8)**, we have

$$A'(t,T) = \frac{2a\phi'(t)}{\sigma^2\phi(t)}.$$

Replace t by s in this equation, integrate from $s = t$ to $s = T$, and show that $A(t,T)$ is given by **(6.5.17)**.

Proof. From **(6.5.15)** and **(6.9.8)**, we have $A'(t,T) = \frac{2a\phi'(t)}{\sigma^2\phi(t)}$. We replace t by s and integrate from $s = t$ to $s = T$ and use the fact that $A(T,T) = 0$ to get

$$A(t,T) = a\int_t^T C(u,T)du \tag{0.93}$$

From the expression obtained fro $C(t,T)$ obtained in part e), we get **(6.5.16)**. ∎

(6.7) **(Heston stochastic volatility model)**
Suppose that under a risk-neutral measure $\tilde{\mathbb{P}}$ a stock price is governed by

$$dS(t) = rS(t)dt + \sqrt{V(t))}S(t)d\tilde{W}_1(t),$$

where the interest rate r is constant and the volatility $\sqrt{V(t)}$ is itself a stochastic process governed by the equation

$$dV(t) = (a - bV(t))dt + \sigma\sqrt{V(t)}d\tilde{W}_2(t).$$

The parameters a, b and σ are positive constants, and $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ are correlated Brownian motions under $\tilde{\mathbb{P}}$ with

$$d\tilde{W}_1(t)d\tilde{W}_2(t) = \rho dt$$

for some $\rho \in (-1,1)$. Because the two-dimensional process $(S(t), V(t)$ is governed by the pair of stochastic differential equations **(6.9.23)** and **(6.9.24)**, it is a two-dimensional Markov process.

At time t, the risk neutral price of a call expiring at time $T \geq t$ in this stochastic volatility model is $c(t, S(t), V(t)) = \tilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - k)^+|\mathcal{F}(t)], 0 \leq t \leq T$.

This problem shows that $c(t,s,v)$ satisfies the partial differential equation

$$c_t + rsc_s + (a - bv)c_v + \frac{1}{2}s^2vc_ss + p\sigma svc_{sv} + \frac{1}{2}\sigma^2vc_{vv} = rc \tag{0.94}$$

in the region $0 \leq t < T$, $s \geq 0$, $v \geq 0$ with the terminal condition $C(T,s,v) = (s-K)^+$ for all $s \geq 0, v \geq 0$.

(a) Show that $e^{-rt}c(t, S(t), V(t))$ us a martingale under $\tilde{\mathbb{P}}$, and use this fact to show to obtain (0.94).

Proof. $e^{-rt}c(t,S(t),V(t)) = \tilde{\mathbb{E}}[e^{-rT}(S(T)-K)^+|\mathcal{F}(t)]$. Using iterated conditioning, for any $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}\tilde{\mathbb{E}}[e^{-rt}c(t,S(t),V(t))|\mathcal{F}(s)] &= \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[e^{-rT}(S(T)-K)^+|\mathcal{F}(t)]\mathcal{F}(s)]\\ &= \tilde{\mathbb{E}}[e^{-rT}(S(T)-K)^+|\mathcal{F}(s)] = e^{-rs}c(s,S(s),V(s)).\end{aligned}$$

This shows that $e^{-rt}c(t,S(t),V(t))$ is a martingale under $\tilde{\mathbb{P}}$.Observe that $dS(t)ds(t) = V(t)S^2(t)dt$, $dV(t)dV(t) = \sigma^2V(t)dt$ and $dS(t)dV(t) = \rho\sigma V(t)S(t)dt$. Next, we use the two-dimensional Itô-Doeblin formula **(4.6.8)**, the above observations and set the dt term to zero to get (0.94). ∎

(b) Suppose there are functions $f(t,x,v)$ and $g(t,x,v)$ satisfying

$$f_t + (r + \frac{v}{2})f_x + (a - bv - \rho\sigma v)f_v + \frac{1}{2}v f_{xx} + \rho\sigma v f_{xv} + \frac{1}{2}\sigma^2 v f_{vv} = 0, \tag{0.95}$$

$$g_t + (r - \frac{v}{2})g_x + (a - bv)g_v + \frac{1}{2}v g_{xx} + \rho\sigma v g_{xv} + \frac{1}{2}\sigma^2 v g_{vv} = 0, \tag{0.96}$$

in the region $0 \le t < T$, $-\infty < x < \infty$, and $v \ge 0$. Show that if we define

$$c(t,s,v) = sf(t,\log s, v) - e^{-r(T-t)}Kg(t,\log s, v),$$

then $c(t,s,v)$ satisfies the partial differential equation (0.94).

Proof. For brevity we suppress the dependence of c, f, s on their variables. Let $s = e^x$. We have

$$
\begin{aligned}
&c_t = sf_t - Kre^{-r(T-t)}g - Ke^{-r(T-t)}g_t\\
&rsc_s = rsf + rsf_x - rKe^{-r(T-t)}g_x\\
&(a-bv)c_v = (a-bv)sf_v - (a-bv)Ke^{-r(T-t)}g_v\\
&\frac{1}{2}s^2vc_{ss} = \frac{1}{2}svf_x + \frac{1}{2}svf_{xx} - \frac{1}{2}vKe^{-r(T-t)}g_{xx} + \frac{1}{2}vKe^{-r(T-t)}g_x\\
&\rho\sigma svc_{sv} = \rho\sigma svf_v + \rho\sigma svf_{xv} - K\rho\sigma ve^{-r(T-t)}g_{xv}\\
&\frac{1}{2}\sigma^2vc_{vv} = \frac{1}{2}\sigma^2vsf_{vv} - \frac{1}{2}\sigma^2vKe^{-r(T-t)}g_{vv}
\end{aligned} \tag{0.97}
$$

Adding all the equations in (0.97), using (0.95), (0.96) and the fact that $c = sf - e^{-r(T-t)}Kg$, we see that c satisfies (0.94). ■

(c) Suppose a pair of processes $(X(t), V(t))$ is governed by the the stochastic differential equations

$$
\begin{aligned}
dX(t) &= (r + \frac{V(t}{2})dt + \sqrt{V(t)}dW_1(t),\\
dV(t) &= (a - bV(t) + \rho\sigma V(t))dt + \sigma\sqrt{V(t)}dW_2(t),
\end{aligned}
$$

where $W_1(t)$ and $W_2(t)$ are Brownian motions under some probability measure $\mathbb{P}$ with $dW_1(t)dw_2(t) = \rho dt$. Define

$$f(t,x,v) = \mathbb{E}^{t,x,v}\mathbb{1}_{\{X(T)\ge\log K\}}.$$

Show that $f(t,x,v)$ satisfies the partial differential equation (0.95) and the boundary condition

$$f(T,x,v) = \mathbb{1}_{\{x\ge\log K\}}\ \forall x \in \mathbb{R}, v \ge 0.$$

Proof. Define $h(X(T),V(T)) = \mathbb{1}_{\{X(T)\ge\log K\}}$ where $h : \mathbb{R}\times[0,\infty) \to \mathbb{R}$. Clearly, h is Borel measurable. Because $(X(t), V(t)$ together solve a SDE system, $(X(t), V(t)$ is a Markov process and thus by a 2 dimensional analogue of Theorem **(6.3.1)**, $f(t, X(t), V(t)) = \mathbb{E}^{t,x,v}[h(X(T), V(T)|\mathcal{F}(t)]$ is a martingale.

We can now use the 2 dimensional Itô-Doeblin formula **(4.6.8)** and set the dt term to zero and obtain (0.95). Since $f(t, X(t), V(t)$ is a martingale, the terminal condition at $t = T$ is $f(T,x,v) = \mathbb{1}_{\{x \geq \log K\}}$. for all $(x,v) \in \mathbb{R} \times [0,\infty)$ i.e for all $(x.v)$ that can be reached by $(X(T), V(T)$. ■

(d) Suppose a pair of processes $(X(t), V(t))$ is governed by the the stochastic differential equations

$$dX(t) = (r - \frac{V(t}{2})dt + \sqrt{V(t)}dW_1(t),$$
$$dV(t) = (a - bV(t) + \rho\sigma V(t))dt + \sigma\sqrt{V(t)}dW_2(t),$$

where $W_1(t)$ and $W_2(t)$ are Brownian motions under some probability measure $\mathbb{P}$ with $dW_1(t)dw_2(t) = \rho dt$. Define

$$g(t,x,v) = \mathbb{E}^{t,x,v}\mathbb{1}_{\{X(T) \geq \log K\}}.$$

Show that $g(t,x,v)$ satisfies the partial differential equation (0.96) and the boundary condition

$$g(T,x,v) = \mathbb{1}_{\{x \geq \log K\}}\ \forall x \in \mathbb{R}, v \geq 0.$$

Proof. The proof for this part follows the same outline as in part (d). We first show that $g(t,X(t),V(t)) = \mathbb{E}[\mathbb{1}_{\{X(T) \geq \log K\}}|\mathcal{F}(t)]$ is a martingale. Then we apply the 2 dimensional Itô-Doeblin formula and set the dt term to zero to get (0.96). Since $g(t,X(t),V(t)$ is a martingale, the terminal condition at $t = T$ is $g(T,x,v) = \mathbb{1}_{\{x \geq \log K\}}$ for all $(x,v) \in \mathbb{R} \times [0,\infty)$ i.e for all $(x.v)$ that can be reached by $(X(T), V(T)$. ■

(e) Show that with $f(t,x,v)$ and $g(t,x,v)$ as in part c) and part d) above, the function $c(t,x,v)$ of part b) satisfies the terminal condition $C(T,s,v) = (s-K)^+$ for all $s \geq 0, v \geq 0$.

Proof. Let $s = e^x$. We have, $c(T,s,v) = sf(T,\log s,v) - Kg(T,\log s,v)$
$= (s-K)\mathbb{1}_{\{s \geq K\}}, \forall s \geq 0, v \geq 0.$
$= (s-K)^+, \forall s \geq 0, v \geq 0.$ ■

(6.10) **(Implying the volatility surface)**
Assume that a stock price evolves according to the stochastic differential equation

$$dS(u) = rS(u)dt + \sigma(u,S(u))S(u)d\tilde{W}(u),$$

where the interest rate r is constant, the volatility $\sigma(u,x)$ is a function of time and the underlying stock price, and $\tilde{W}$ is a Brownian motion under the risk-neutral probability measure $\tilde{\mathbb{P}}$. This is a special case of the stochastic differential equation **(6.9.46)** with $\beta(u,x) = rx$ and $\gamma(u,x) = \sigma(u,x)x$. Let $\tilde{p}(t,T,x,y)$ denote the transition density.

According to Exercise 6.9, the transition density $\tilde{p}(t,T,x,y)$ satisfies the Kolmogorov forward equation

$$\frac{\partial}{\partial T}\tilde{p}(t,T,x,y) = -\frac{\partial}{\partial y}(ry\tilde{p}(t,T,x,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(T,y)y^2\tilde{p}(t,T,x,y)). \quad (0.98)$$

Let

$$c(0,T,x,K) = e^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy \tag{0.99}$$

denote the time-zero price of a call expiring at time T, struck at K, when the initial stock price is $S(0) = x$. Note that

$$C_T(0,T,x,K) = -rc(0,T,x,K) + e^{-rT}\int_K^\infty (y-K)\tilde{p}_T(0,T,x,y)dy. \tag{0.100}$$

(a) Integrate once by parts to show that

$$-\int_K^\infty (y-K)\frac{\partial}{\partial y}(ry\tilde{p}(0,T,x,y))dy = \int_K^\infty ry\tilde{p}(0,T,x,y)dy. \tag{0.101}$$

You may assume that

$$\lim_{y\to\infty}(y-K)ry\tilde{p}(0,T,x,y) = 0. \tag{0.102}$$

Proof. As indicated, we integrate by parts once and use the fact that the boundary integral vanishes, (using (0.102)) to obtain (0.101). ∎

(b) Integrate by parts and then integrate again to show that

$$\frac{1}{2}\int_K^\infty (y-K)\frac{\partial^2}{\partial y^2}(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y))dy = \frac{1}{2}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K). \tag{0.103}$$

You may assume that

$$\begin{aligned}
&\lim_{y\to\infty}(y-K)\frac{\partial}{\partial y}(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)) = 0,\\
&\lim_{y\to\infty}\sigma^2(T,y)y^2\tilde{p}(0,T,x,y) = 0.
\end{aligned} \tag{0.104}$$

Proof. We integrate by parts twice to get

$$\begin{aligned}
&\frac{1}{2}\int_K^\infty (y-K)\frac{\partial^2}{\partial y^2}(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y))dy\\
&= -\frac{1}{2}\int_K^\infty \frac{\partial}{\partial y}(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y))dy \text{ (Using (0.104))}\\
&= \frac{1}{2}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K) \text{ (Using (0.104))}
\end{aligned} \tag{0.105}$$

∎

(c) Now use (0.100), (0.99), (0.98), (0.97), (0.103), and exercise 5.9 of Chapter 5 in that order to obtain the equation

$$\begin{aligned} &c_t(0,T,x,K) \\ &= e^{-rT}rK\int_K^\infty \tilde{p}(0,T,x,y)dy + \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K) \\ &= -rKc_K(0,T,x,K) + \frac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K). \end{aligned} \tag{0.106}$$

This is the end of the problem. Note that under the assumption that $c_{KK}(0,T,x,k) \neq 0$, (0.106) can be solved for the volatility term $\sigma^2(T,K)$ in terms of the quantities $c_T(0,T,x,K)$, $c_K(0,T,x,K)$, and $c_{KK}(0,T,x,K)$, which can be inferred from market prices.

Proof. As indicated, we first use (0.100), (0.99), (0.98), (0.101), (0.103) to get

$$\begin{aligned} c_t(0,T,x,K) = e^{-rT}rK&\int_K^\infty \tilde{p}(0,T,x,y)dy \\ &+ \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K) \end{aligned}$$

We now use the result of **Exercise 5.9** i.e $c_K(0,T,x,K) = -e^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy$ and $c_{KK}(),T,x,K) = e^{-rT}\tilde{p}(),T,x,K)$ to get (0.106). ∎

E-mail address: karthik2@uw.edu