DEPARTMENT OF MATHEMATICS
Indian Institute of Technology, Guwahati

EndSem	MA321
23-11-2021	Instructor : Sukanta Pati
Time : 14:00–17.00	Maximum Score : 40

Write appropriate and precise justifications, with readable handwriting. Use pencils for convenience.

Submit in the codetantra portal only. The portal will close by 17:15. Start submission at 17:01 to avoid problems.

If you are submitting to my email, do it before time. There may be deductions for late submissions.

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1. Consider the first six prime numbers $p_1, \ldots, p_6$, the next eight prime numbers $q_1, \ldots, q_8$ and the next ten prime numbers $r_1, \ldots, r_{10}$. For any point $(x, y, z) \in \mathbb{R}^3$, define

$$f(x, y, z) = |x - p_1| + \cdots + |x - p_6| + |y - q_1| + \cdots + |y - q_8| + |z - r_1| + \cdots + |z - r_{10}|.$$

Optimize it, using the techniques you have learned in this course. $\boxed{10}$

2. Let the Earth be represented by the solid sphere $x^2 + y^2 + z^2 \le 1$. Consider the seven sisters (these are seven stars far away from us) to be seven fixed points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$, $D(d_1, d_2, d_3)$, $E(e_1, e_2, e_3)$, $F(f_1, f_2, f_3)$, $G(g_1, g_2, g_3)$ in $\mathbb{R}^3$ with large positive coordinates. Given any point $P = (x, y, z)$ on the surface S of the Earth, the sum of the squares of the distances of P from the seven sisters is computed and it is called $f(P)$. How do we find a point P where $f(P)$ is the minimum and a point P where $f(P)$ is the maximum? $\boxed{10}$

3. Consider the region

$$S = \{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 25,\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \ge 16,\ x_1 + x_2 + x_3 + x_4 = 8\}.$$

Show that KTCQ1 holds at each point in S. $\boxed{3}$

Use KT theory to maximize $x_1 + x_2 - x_3 - x_4$ over S. $\boxed{7}$

4. Let f, g_i, h_j be twice continuously differentiable and consider the problem

$$\begin{array}{ll} \min & f(x) \\ \hline \text{s.t.} & g_i(x) \ge 0,\ i = 1, \ldots, m,\ h_j(x) = 0,\ j = 1, \ldots, p,\ x_k \ge 0,\ k = 1, \ldots, n. \end{array}$$

Let a be a feasible point and define

$$L(x, \lambda, w) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) - \sum_{j=1}^{p} w_j h_j(x).$$

Prove that the following are equivalent. $\boxed{5+5}$

a) $Z(a) = \emptyset$.

b) $\exists\ \lambda_i \geq 0,\ i = 1, \ldots, m,\ w_j,\ j = 1, \ldots, p$ such that the following KT conditions are satisfied

$$\bigtriangledown L(a, \lambda, w) \geq 0, \qquad \lambda_i g_i(a) = 0, \forall i = 1, \ldots, m, \qquad a^t \bigtriangledown L(a, \lambda, w) = 0.$$

5. (a) Give an example of a function $f : \mathbb{R}^3 \to \mathbb{R}$ which is continuous at 0 such that all directional derivatives exist at 0 except one. No justifications required. $\boxed{\text{(2,-2) means correct 2, wrong -2}}$

   (b) Consider the Taylor series of $f(x, y, z) = \sin(xyz)$ about the origin. We add the coefficients of all the first degree, second degree and third degree terms. What will we get? No justifications required. $\boxed{(2,-2)}$

   (c) Give an example of a positive definite matrix $A \in M_5(\mathbb{C})$ which does not have a zero entry. No justifications required. $\boxed{(2,-2)}$

   (d) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function that satisfies the following condition at 0.

   'For each nonzero d, there exists a positive α_d such that for each $t \in (0, \alpha_d)$ we have $f(0) \leq f(td)$.'

   Must 0 be a point of local minimum? No justifications required. $\boxed{(2,-2)}$

   (e) Give an example of a twice continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ for which $(1, 2, 3)$ is a saddle point because its Hessian matrix is $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. No justifications required. $\boxed{(2,-2)}$

   (f) From $\mathbb{R}^2$ cut out the circular region $(x-1)^2 + (y-1)^2 \leq 1$. Also cut out the regions, $(x-1)^2 + (y+1)^2 \leq 1$, $(x+1)^2 + (y-1)^2 \leq 1$, and $(x+1)^2 + (y+1)^2 \leq 1$. Consider the region containing the origin. Is it true that KTCQ1 is satisfied at all points? $\boxed{(2,-2)}$