1 Lecture 1

What are mathematical programming problems?

We shall discuss optimization (maximization or minimization) a given real valued function f over a given set $T \subseteq \mathbb{R}^n$. Most often, the set T is described by a finite set of equations and inequations.

[1.1] **Definitions** The problem of optimizing $f(x)$ over the set $T = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, h_j(x) = 0, i = 1, \dots m, j = 1, \dots, p\}$ is called a MATHEMATICAL PROGRAMMING PROBLEM(mpp). It is convenient to express this problem in the following form (here s.t. means subject to)

$$\text{(mpp)} \qquad \begin{array}{ll} \text{optimize} & f(x) \longrightarrow \text{objective fn} \\ \hline & g_i(x) \geq 0, \quad i = 1, \cdots, m \\ \text{s.t.} & h_j(x) = 0, \quad j = 1, \cdots, p \\ & x \in \mathbb{R}^n \end{array}$$

→ constraints

T → feasible set

The function f is called the OBJECTIVE FUNCTION. The expressions $g_i(x) \geq 0$, $h_j(x) = 0$ are called the CONSTRAINTS. The set $T = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, h_j(x) = 0\}$ is called the FEASIBLE SET. A point $x \in T$ is called a FEASIBLE SOLUTION.

[1.2] **Remark** A feasible solution actually a solution for the system of equations and inequations given as constraints. Usually f, g_i, h_j are in $\mathcal{C}(\mathbb{R}^n)$ or $\mathcal{C}^1(\mathbb{R}^n)$, where $\mathcal{C}(X)$ means the set of continuous functions on X and $\mathcal{C}^1(X)$ means the set of functions with first partial derivatives continuous on X.

[1.3] **Definitions** We call the mpp INCONSISTENT if the feasible set $T = \emptyset$, otherwise it is called CONSISTENT.

[1.4] **Definitions** If f, g_i, h_j are linear (functions of the form $c^t x + d$), then the mpp is called a LINEAR PROGRAMMING PROBLEM (lpp), otherwise it is called a NONLINEAR PROGRAMMING PROBLEM (nlpp).

[1.5] **Definitions** Let f be a real valued function defined on a set $T \subseteq \mathbb{R}^n$ and $x_0 \in T$. We call x_0 a GLOBAL/ABSOLUTE MINIMUM if $f(x_0) \leq f(x)$, $\forall x \in T$. We call x_0 a STRICT GLOBAL MINIMUM if $f(x_0) < f(x)$, $\forall x \in T, x \neq x_0$. We call x_0 a LOCAL MINIMUM if $\exists\, \delta > 0$, such that $f(x_0) \leq f(x)$, $\forall x \in T \cap B_\delta(x_0)$. Here $\boldsymbol{B_\delta(x)}$:= $\{x \in \mathbb{R}^n \mid \|x\| < \delta\}$. The terms STRICT LOCAL MINIMUM, GLOBAL MAXIMUM, LOCAL MAXIMUM, STRICT GLOBAL MAXIMUM and STRICT LOCAL MAXIMUM are defined similarly.

a pt of loc min of f over S if

$\exists\ B_\delta(a)$ s.t. $x \in B_\delta(a) \cap S$

$\Rightarrow f(x) \geq f(a)$

Applications

Optimization finds uses in many places. An important part is the integer programming problems which is a very important study in computer science. Below we give a few examples to show how optimization is used in many subjects.

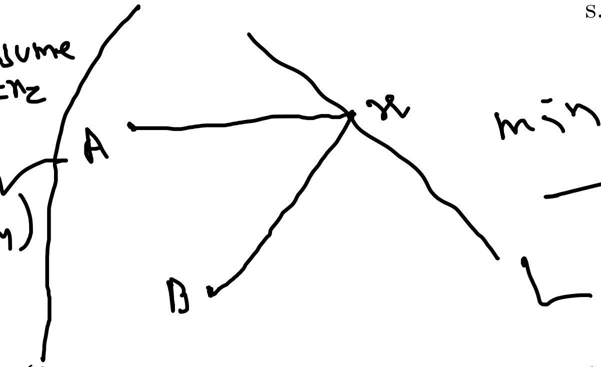
[1.6] **Example** (`Heron's problem : an optimization problem in geometry`) Consider the points $A(0,2), B(2,3)$ and the line $L: y=-x$. The problem asks to solve $\begin{array}{ll} \min & |AC|+|BC| \\ \text{s.t.} & C \text{ is on } L. \end{array}$

(x_1, x_2, x_3, x_4), assume $x_1 \neq x_2$

$x_1 x_2 x_3 x_4$

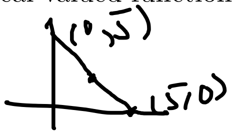
$\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, x_4\right)$

$\left(\frac{x_1+x_2}{2}\right)^2 x_3 x_4$



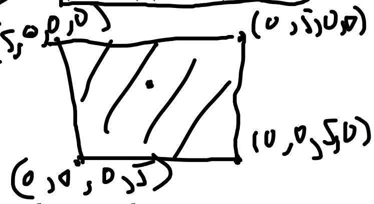
$\begin{array}{ll} \min & f(x) = d(A,x) + d(B,x) \\ & x \in L \end{array}$

[1.7] **Example+** (`An optimization problem in algebra: AM-GM inequality`) Recall that a continuous real valued function on a compact subset of $\mathbb{R}^n$ attains its bounds. Use that and $\boxed{(x+k)(x-k) \le x^2}$, to solve



$$\begin{array}{ll} \max & x_1x_2x_3x_4 \\ \text{s.t.} & x_1+x_2+x_3+x_4=5,\ x_i \ge 0. \end{array}$$

$x = \left(\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}\right)$



Derive the am-gm inequality.

[1.8] **Example+** (`An optimization problem in graphs`) Let G be a simple graph on vertices $1, 2, \ldots, 50$ and A be its adjacency matrix ($a_{ij}=1$ if i is adjacent to j, otherwise $a_{ij}=0$). Let $(A^k)_{i,j}$ mean the (i,j)th entry of A^k. What will a solution to $\begin{array}{ll} \min & k \\ \text{s.t.} & (A^k)_{3,13} > 0, k \in \mathbb{N} \end{array}$ give us?

[1.9] **Example** (`Kepler : another optimization problem in Geometry`) What is the largest area of a rectangle inscribed in a circle of unit radius? Can you try the equivalent problem in $\mathbb{R}^3$?

[1.10] **Example** (`An optimization problem from physics (Snell's law)`) Light travels faster in mediums that are less dense. Also light takes the path of least time. Consider light traveling from $A(0,a)$ to $B(b,-d)$ and that above x-axis we have one medium where light travels with a velocity v_1 and below x-axis we have another medium where light travels with a lower velocity v_2. At which point $X(x,0)$ shall it pass the x-axis?

[1.11] **Example+** (`An optimization problem from linear algebra (Cauchy-Schwarz)`) Does the minimum value of $f(x,y) = \|x\|\|y\| - |\langle x, y\rangle|$ exist over $x, y \in \mathbb{C}^n$? What is it?

Some exercises

[1.12] **Exercise** Imagine a large V shaped area and draw a point inside that area. Imagine that a person is standing at the point. The person wants to move and touch the the left boundary and then the right boundary and then come back to the original position. In what ways should the person move in order to minimize the total distance traveled?

[1.13] **Exercise** There is a nail fixed in the vertical wall (yz-plane) at a height 5 meters above the horizontal floor. A painting of height is 6 meters is hanged with the nail so that it extends above the nail by an amount as it extends below the nail. A living particle P is on the floor just below the nail. It moves parallel to x-axis and places itself at a distance t meters from the wall so as to maximize the viewing angle (BPT) it makes with the bottom (B) and top (T) of the painting. After that the painting is replaced by another one of height 4 meters. The particle has to move to a new position on the floor in order to maximize the viewing angle. How much distance shall it move?

Modeling

Given a practical problem, the very first task is to express it as an mpp. This is called modeling.

> To model a problem.
> a) Identify the variables.
> b) Write the objective function.
> c) Write the constraints.

[1.14] **Example** `(A linear model)` Doctors advice me to take a minimum of 10mg of vitamin A, 10mg of vitamin B and 18mg of vitamin C, daily. There are two type of pills available in the market.

pill	A content	B content	C content	cost
red	4	2	2	1
blue	2	4	18	2

What amount (real number) of pills do I take to meet the requirements while minimizing the cost?

$r \to$ no of red pills
$b \to$ no of blue pills

$$\min \; r + 2b$$
$$4r + 2b \geq 10$$
$$2r + 4b \geq 10$$
$$2r + 18b \geq 18$$
$$r, b \geq 0$$

[1.15] **Example** `(A nonlinear model)` A gambler has Rs. 2000 to play. There are 3 places for stake. There are 3 outcomes. Depending on the outcome the return per unit is shown in the table. In what way the gambler should play in order to maximize the minimum return?

Outcomes	Place 1	Place 2	Place 3
1	-5	1	1
2	-7	6	10
3	13	-2	-6

$x_1 \to$ place 1, $x_2 \to$ pl 2, x_3

$$\max \; f(x) = \min\{-5x_1 + x_2 + x_3, -7x_1 + 6x_2 + 10x_3, 13x_1 - 2x_2 - 6x_3\}$$

$$x_1 + x_2 + x_3 = 2000, \quad x_i \geq 0$$

max $\cancel{z}\ g(x_1, x_2, x_3, z) = z$

$z \le -5x_1 + x_2 + x_3 \to -5x_1 + x_2 + x_3 - z \ge 0$

$z \le -7x_1 + 6x_2 + 10x_3$

$z \le 13x_1 - 2x_2 - 6x_3$

$x_1 + x_2 + x_3 = 2000, \ x_i \ge 0$

Some exercises

[1.16] <u>**Exercise**</u> A shop supplies card boards of length 1 meter and of any ordered width between 1cm and 1m (in multiples of 1cm). It has a machine which produces square card boards of side 1 meter. The shopkeeper has to cut the card boards to meet the demand. On a particular day the shop receives an order of 30 boards of width 65cm, 50 boards of width 55cm and 70 boards of width 22cm. The shopkeeper is interested in using minimum possible square boards to reduce the production cost.

1. Formulate the problem.
2. A piece of card board which is cut but is not supplied for the order is a waste. Formulate the problem if the shopkeeper wants to minimize the waste.
3. Will these two give different solutions?

[1.17] <u>**Exercise**</u> Country A has p_i ships at ith port, $i = 1, \ldots, n$ and country B requires q_j ships at jth port, $j = 1, \ldots, m$. It is given that $\sum p_i = \sum q_j$. It takes d_{ij} number of days for a ship from ith port in A reach the jth port in B. The ships are to be moved so that the total number of days is minimum. Formulate the problem.

[1.18] <u>**Exercise**</u> `(Optimal location)` There are m shops in a city (a map in $\mathbb{R}^2$). Suppose that the roads in the city are such that any two roads are either parallel or perpendicular. You want to build a supply center in a place so that the maximum distance of a shop from the center is minimum. Formulate this problem. This is easy. It is challenging to simplify the problem.

[1.19] <u>**Exercise**</u>

1. There are 5 jobs and 5 persons. We have to pay a price c_{ij} to make the ith person do the jth job. We want to get all our jobs done with minimum price. Model it.
2. There are 5 jobs and 5 persons. We have to pay a price c_{ij} (positive number) to make the ith person do the jth job. We want to get all our jobs done with minimum price such that each person gets only one job. Model it.
3. There are 5 jobs and 5 persons. We have to pay a price c_{ij} to make the ith person do the jth job. Our boss will be happy even if we get 3 jobs done. So we want to get three of our jobs done with minimum price. Model it.
4. There are 5 jobs and 5 persons. We have to pay a price c_{ij} to make the ith person do the jth job. Our boss will be happy even if we get 3 jobs done. So we want to get three of our jobs done with minimum price such that no persons gets more than on job. Model it.

[1.20] <u>**Exercise**</u> Suppose that n machines are available and each machine can do any of the two types of jobs in unit time. If x machines are used to do the ith ($i = 1, 2$) job, goods worth $g_i(x)$ can be produced.

The machines are subject to attrition so that after doing the ith job only $a_i(x)$ machines become useless. The process gets repeated with the remaining machines and it is required to maximize the total worth of goods produced in N unit time. Formulate the problem.

[1.21] **Exercise** A machine produces paper in reels of length 100m and width 180 cm. A customer orders a number of reels each of length 100 m but of varying width.

Width	No of reels ordered
80	150
40	125
25	175

a) How should the reels be cut to the required widths, to minimize the trimming loss? Formulate the problem.

b) Suppose a student approaches to minimize $\sum \alpha_i$ with the same constraints. Will this have the same solution as the previous problem?

c) If we use $\geq$ in stead of $=$ in the constraints, can we get a different answer?

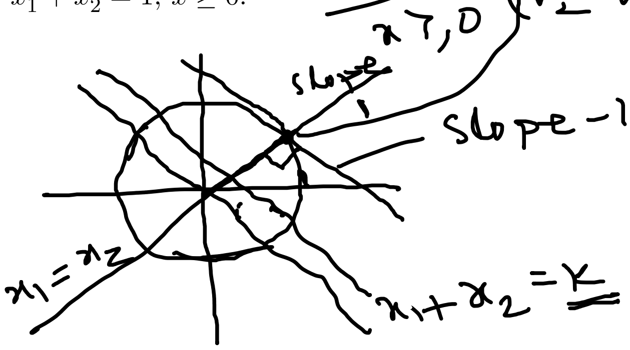
[1.22] **Exercise** An airline operates its fleet of m types of aircrafts on n different routes. Suppose $a_i (i = 1, 2, \cdots, m)$ is the number of aircrafts of type i in the fleet and $b_j (j = 1, \cdots, n)$ is number of passengers requiring passage on the jth route in a given period. Also suppose that one aircraft of type i can accommodate p_{ij} passengers at an operating cost c_{ij} if it is assigned to the jth route during the period. How should the airline assign its various aircrafts to different routes in order to satisfy the passengers demand at the least operating cost? Formulate this problem.

Graphical method

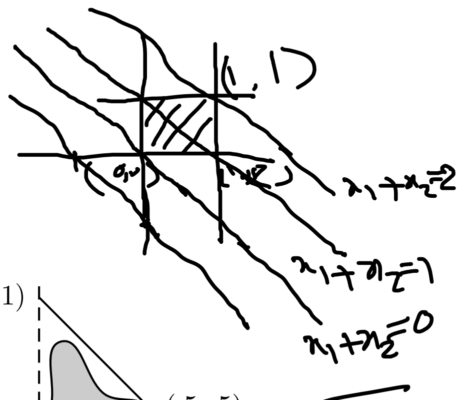
We can solve some MPPs that involve only two (or three) variables in this method. The method is relatively unimportant as we want to develop an algorithm so that the computer can deal with it. However, the method is illustrative in nature and it adds to understanding of the subject. It is described below.

> Graphical method to minimize $f(x)$ over a feasible region T.
>
> a) Plot the feasible region T.
>
> b) Draw the graph of $f(x) = k$.
>
> c) Observe the minimum value of k for which both graphs overlap.
>
> d) Justify.

[1.23] **Example** Solve using graphical method $\begin{array}{ll} \max & x_1 + x_2 \\ \hline \text{s.t.} & x_1^2 + x_2^2 = 1,\ x \geq 0. \end{array}$ $= \sqrt{2}$ $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

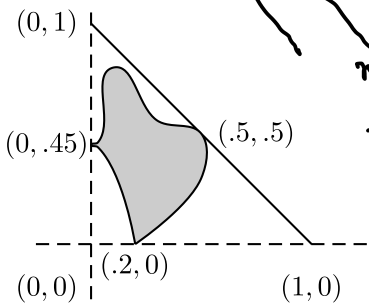


Some exercises



[1.24] **Exercise** Solve $\begin{array}{ll} \min & x_1 + x_2 \\ \text{s.t.} & x_1 \le 1, x_2 \le 1, x_i \ge 0 \end{array}$ by graphical method.

[1.25] **Exercise** Solve using graphical method: $\begin{array}{ll} \max & x_1 x_2 \\ \text{s.t.} & x_1 + x_2 = 5,\ x \ge 0. \end{array}$



[1.26] **Exercise** Find the maximum value of xy on the shaded region in the figure.

[1.27] **Exercise** Solve by graphical method: $\begin{array}{ll} \max & x_1 + 2x_2 \\ \text{s.t.} & x_1^2 + 2x_2^2 = 2, x_i \ge 0. \end{array}$

[1.28] **Exercise** Solve by graphical method: $\begin{array}{ll} \text{opt} & x^2 + y \\ \text{s.t.} & x^2 + y^2 = 1. \end{array}$

[1.29] **Exercise** Solve by graphical method: $\begin{array}{ll} \max & f(x) = (x_1 - 1)^2 + (x_2 - \frac{1}{2})^2 \\ \text{s.t.} & x_1 + x_2 \le 2,\ x_1, x_2 \ge 0. \end{array}$

[1.30] **Exercise** Solve by graphical method: $\begin{array}{ll} \max & x + 3y \\ \text{s.t.} & x^2 + 4y^2 = 4, x, y \in \mathbb{R}. \end{array}$

[1.31] **Exercise** Use graphical method to maximize $x_1 + x_2 - 2x_3 - x_4$ over the region $S = \{(x \in \mathbb{R}^4 \mid x_1^2 + 2x_2^2 + 3x_3^2 + x_4^2 \le 16\}$.

[1.32] **Exercise** Consider $\begin{array}{ll} \max & v = x_1^2 + x_2^2 \\ \text{s.t.} & x_1 + 4x_2 \le 5,\ 0 \le x_1 \le 1,\ x_2 \ge 0. \end{array}$ Use graphical method to solve it. Show that the point $x_1 = 0, x_2 = 5/4$ is a local, but not a global maximum. Show that the point $x_1 = 1, x_2 = 1$ is the global maximum.

[1.33] **Exercise** Suppose that product A costs 4 per unit and product B costs 3 per unit. Both are needed to produce the product C. When x units of A and y units of B are used, the total number of units of C produced by the production process is $4x^2 - 4xy + y^2 - 1$. How many units of A and B should be used to produce 8 units of C while minimizing the overall cost? Formulate this problem and solve graphically.