$S \subseteq \mathbb{R}^n$ S is open if for each $x \in S$ $\exists\, \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq S$. T is closed if T^c is open

$\overline{[0,1)} = [0,1]$

$(x_n) \to x$

$[0,1)^c = (-\infty, 0) \cup [1, \infty)$

S is bounded if $\exists\, M > 0$ s.t. $S \subseteq B_M(0)$

11 Lecture 11

Information obtained from the simplex table

[11.1] **Benefits of the simplex table** a) It shows the revised problem in simple tabular form.

b) It also gives us many geometric information which helps us to get a better bfs, in case we are not an optimal bfs.

c) The tabular structure is very convenient to generate the next simplex table by doing the elementary algebraic manipulations, without bothering about the actual vertices. So it is easily implementable on a computer.

[11.2] **Discussion** This discussion is based on an example. A general discussion is similar.

a) Consider the unit cube T in $\mathbb{R}^3$ with sides e_1, e_2 and e_3. Imagine minimizing $f(x) := x_1 + x_2 + x_3$. We know that it will be at $(0, 0, 0)$.

b) Write the lpp, slpp and the problem table.

$$\min \frac{x_1 + x_2 + x_3}{x_1 \le 1,\ x_2 \le 1,\ x_3 \le 1,\ x \ge 0} \qquad \mathbb{R}^3$$

slpp: $\mathbb{R}^6$

$$\min \frac{c^T x}{Ax = b,\ x \ge 0} \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

PT:

bv	x_1	x_2	x_3	x_4	x_5	x_6	b
	1	0	0	1	0	0	1
	0	1	0	0	1	0	1
	0	0	1	0	0	1	1
$-f$	1	1	1	0	0	0	*

Answer. Lpp:
$$\min \frac{x_1 + x_2 + x_3}{\text{s.t. } x_1 \le 1, x_2 \le 1, x_3 \le 1, x_i \ge 0.}$$

Slpp: $\min\ c^t x$ s.t. $Ax = b, x \geq 0$ where $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $c^t = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$.

PT:

bv	x_1	x_2	x_3	x_4	x_5	x_6	b
	1	0	0	1	0	0	1
	0	1	0	0	1	0	1
	0	0	1	0	0	1	1
$*$	1	1	1	0	0	0	$*$

c) Let $T \subseteq \mathbb{R}^3$ be the feasible set of the lpp and $T^* \subseteq \mathbb{R}^6$ be the feasible set of the slpp. Let $w = (1,1,0) \in T$

1) What is the corresponding point w^* in T^*?

2) Is w^* a bfs of $Ax = b$?

3) Write an ordered basis that corresponds to w^*?

$w^* = (1,1,0,0,0,1) \to$ bfs

An ordered basis is (x_1, x_2, x_6) or (x_2, x_1, x_6)

Answer. 1) Using the slpp, we see that $w^* = (1, 1, 0, 0, 0, 1)$.

2) Yes, as it is nonnegative and the columns of the matrix A corresponding to the nonzero entries in w^* are linearly independent. (Alternately, you can also say, in view of [9.4] that since w was a vertex of T, the point w^* is a vertex of T^*.)

3) We can take $B = (x_1, x_2, x_6)$ or $B = (x_2, x_1, x_6)$ or any of 3! ways of writing x_1, x_2, x_6.

$w^* = (1,1,0,0,0,1)$

d) Form the simplex table at the ordered basis $B = (x_2, x_1, x_6)$.

bv	x_1	x_2	x_3	x_4	x_5	x_6	b
$\to x_2$	1	0	0	1	0	0	1
x_1	0	1	0	0	1	0	1
x_6	0	0	1	0	0	1	1
$-f$	1	1	1	0	0	0	0

bv	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	0	1	0	0	1	0	1
x_1	1	0	0	1	0	0	1
x_6	0	0	1	0	0	1	1
$-f$	1	1	1	0	0	0	0

$\bar{c} - R_1 - R_2 \Rightarrow$

bv	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	0	1	0	0	1	0	1
$\to x_1$	1	0	0	1	0	0	1
x_6	0	0	1	0	0	1	1
$-f$	0	0	1	-1	-1	0	$-2 \to -f(w^*)$

$\to \bar{c}^T$

$w^* = (1,1,0,0,0,1)$
$f(w^*) = 2$

bv	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	0	1	0	0	1	0	1
x_1	1	0	0	1	0	0	1
x_6	0	0	1	0	0	1	1
$-f$	0	0	1	-1	-1	0	-2

(7)

e) From this table, we see that $-f(w^*) = -2$. Indeed, we had $f(w) = 2$.

f) Select a nonbasic variable, say x_4. We are at $w^* = (1, 1, 0, 0, 0, 1)$. The value of x_4 is now 0.

1. Suppose that we want to increase the value of x_4 by 1, keeping the other nonbasic variables unchanged.
2. That is, we are looking at a point $x := w^* + d$, where $d_4 = 1, d_3 = 0, d_5 = 0$.
3. We want the new point x to satisfy $\overline{A}x = \overline{b}$ and $x \geq 0$, so that we stay in the feasible region. Can you write d?

$$\underbrace{w^* + \alpha d}_{w^*_\alpha} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} d_1 \\ d_2 \\ 0 \\ 1 \\ 0 \\ d_6 \end{bmatrix} \to \quad \bar{A} w^*_\alpha = \bar{b} \equiv \underbrace{\bar{A} w^*}_{\bar{b}} + \bar{A}\alpha d = \bar{b}$$

If such a d exists, it must satisfy $\bar{A}d = 0$

$$\bar{A}_{:1} d_1 + \bar{A}_{:2} d_2 + \cancel{\bar{A}_{:3} d_3} + \bar{A}_{:4} d_4 + \cancel{\bar{A}_{:5} d_5} + \bar{A}_{:6} d_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} d_2 \\ d_1 \\ d_6 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad w^*_\alpha = \begin{bmatrix} 1-\alpha \\ 1 \\ 0 \\ \alpha \\ 0 \\ 1 \end{bmatrix}$$

Answer. We want $\overline{A}(w^* + \alpha d) = \overline{b}$. As $\overline{A}w^*$ is already $\overline{b}$, we must have $\overline{A}d = 0$. That is, we must have

$$d_1\overline{A}_{:1} + d_2\overline{A}_{:2} + d_3\overline{A}_{:3} + d_4\overline{A}_{:4} + d_5\overline{A}_{:5} + d_6\overline{A}_{:6} = 0.$$

As $A_{:2} = e_1$, $A\colon 1 = e_2$, $A_6 = e_3$ and $A_{:3} = A_{:5} = 0$, we have

$$d_1\overline{A}_{:1} + d_2\overline{A}_{:2} + d_6\overline{A}_{:6} + d_4\overline{A}_{:4} = 0 \quad \text{that is,} \quad \begin{bmatrix} d_2 \\ d_1 \\ d_6 \end{bmatrix} = -\overline{A}_{:4} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

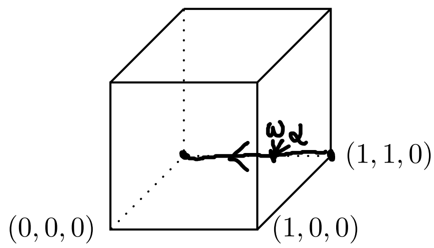
g) What is the form of a general point $w^*_\alpha = w^* + \alpha d$ in that direction?

Answer. A general point in the direction of d is

$$w^*_\alpha = w^* + \alpha d = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ 1 \\ 0 \\ \alpha \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

h) We want to travel inside T^*. So entries of w^*_α must be nonnegative. What is the maximum value α_{max} of α possible? Locate the points $w_\alpha \in T$ corresponding to w^*_α for $\alpha \in [0, \alpha_{max}]$. What do you see?

$\alpha_{max} = 1$



Answer. We must have $\alpha_{max} = 1$ and $w_\alpha = \begin{bmatrix} 1-\alpha \\ 1 \\ 0 \end{bmatrix}$. That is, we are heading from $w = (1,1,0)$ towards $u = (0,1,0)$ on the edge (imagine increasing α slowly).

$c^T x - c^T w^* = \bar{c}^T x$

i) Evaluate the cost difference between w^*_α and w^*.

$c^T w^*_\alpha - c^T w^* = 2 - \alpha - 2 = -\alpha = \bar{c}_4 \alpha$

$\bar{c}^T w^*_\alpha = \begin{bmatrix} & \bar{c}^T & \\ 0 & 0 & \bar{c}_3\ \bar{c}_4\ \bar{c}_5\ 0 \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \bar{c}_4 \alpha$

$w^* + \alpha \begin{bmatrix} d_1 \\ d_2 \\ 0 \\ 1 \\ 0 \\ d_6 \end{bmatrix}$

$\Rightarrow f(w^*_{\alpha_{max}}) - f(w^*) = -1$

Answer. The cost difference is

$$c^t w^*_\alpha - c^t w^* = \bar{c}^t w^*_\alpha = \sum \bar{c}_i (w^*_\alpha)_i = \bar{c}_4 \alpha,$$

as that except fourth entry of w^*_α which is α, all other nonzero entries are basic places for w^* (hence in $\bar{c}$, those places will be 0).

j) Express the meaning of the previous equation in words.

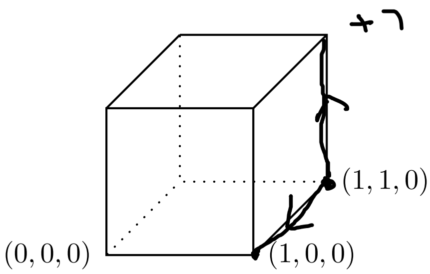
Answer. This means, for moving one step in the direction provided by the nonbasic variable x_4, the cost will change by $\bar{c}_4$.

[11.3] **Draw and explain below** Consider the simplex table as in the previous example.

bv	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	0	1	0	0	1	0	1
x_1	1	0	0	1	0	0	1
x_6	0	0	1	0	0	1	1
$-f$	0	0	1	-1	-1	0	-2

Write the points in the direction given by x_5. Compute α_{max}. Show the movement in the picture of T. Verify that the cost has changed by $\bar{c}_5 \alpha_{max}$ when you reach the next bfs.

$\begin{bmatrix} d_1 \\ d_2 \\ 0 \\ 0 \\ 1 \\ d_6 \end{bmatrix} \quad \begin{matrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{matrix} \qquad \begin{bmatrix} d_2 \\ d_1 \\ d_6 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$



$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-\alpha \\ 0 \\ 0 \\ \alpha \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1-\alpha \\ 0 \end{bmatrix} \xrightarrow{\alpha_{max}=1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Answer. We have

$$w^*_\alpha = w^* + \alpha d = \begin{bmatrix}1\\1\\0\\0\\0\\1\end{bmatrix} + \alpha \begin{bmatrix}0\\-1\\0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1-\alpha\\0\\0\\\alpha\\1\end{bmatrix}.$$

Here $\alpha_{max} = 1$.

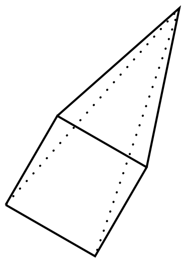
So in T, we are moving from $w = (1,1,0)$ to $v = (1,0,0)$. The cost difference is $c^t w - c^t v = 1$. We also have $c^t v = c^t w + \alpha_{max}\overline{c}_5$.

[11.4] **Remark** It is clear that each nonbasic variable gives us a direction to move out of w^* and the directions given by the different nonbasic variables are linearly independent (solely due to the value 1 taken by that nonbasic variable). However, they may not result in an actual movement due to degeneracy. (We will see many examples later.)

Also they may not show you all possible available directions. For example, consider minimizing $x_1+x_2+x_3$ on the square pyramid with vertices $A(1,1,0)$, $B(2,1,0)$, $C(2,2,0)$, $D(1,2,0)$ and $E(1.5,1.5,1)$.

PT:

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	0	2	−1	−1	0	0	0	2
	2	0	1	0	1	0	0	4
	0	2	1	0	0	1	0	4
	2	0	−1	0	0	0	1	2
*	1	1	1	0	0	0	0	*



Point $E \to E^* = (1.5, 1.5, 1, 0, 0, 0, 0)$. We can take a basis (x_1, x_2, x_3, x_7). In any case, we will have only 3 nonbasic variables and they will show only three of the four available directions at E^*.

[11.5] **Theorem** Let $A \in M_{m,n}$ have rank m. Consider $\begin{array}{rl} \text{opt} & f(x) = c^t x \\ \hline \text{s.t.} & Ax = b,\ x \geq 0. \end{array}$ Consider the simplex table at a bfs w^* of $Ax = b$ with the basis matrix B. Let x_r be a nonbasic variable.

a) Then the direction given by x_r is the vector

$$d \text{ with } d_B = -\overline{A}_{:r},\ d_r = 1, \text{ and other entries zero.}$$

b) Also $f(w^* + \alpha d) = f(w^*) + \alpha\overline{c}_r$. That is, for moving α units in the direction d given by x_r, the objective function $f(x)$ changes by $\alpha\overline{c}_r$.