1 Basics of Probability Theory

1.1 Infinite Probability Spaces

• Used to model a random experiment with infinitely many possible outcomes.

• Two examples to keep in mind:

i) Choosing a number from the interval $[0, 1]$

ii) Tossing a coin infinitely many times.

For i) the sample space is $[0, 1]$. For ii) the sample space is the set of all infinite sequences of heads and tails. One important issue with infinite sample spaces is that the classical definition fails. For example, what is the probability that you choose a number less than or equal to 1/2. In this case both the total number of outcomes as well as the number of favorable outcomes are infinite. This leads us to what is called the axiomatic definition of probability. Before that we need one more definition.

Definition 1.1. *Let Ω be a non-empty set and let $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a $\sigma-$algebra provided:*
i) the empty set ϕ belongs to $\mathcal{F}$
ii) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ (closed under complementation)
iii) whenever a sequence of sets $A_1, A_2, \ldots$ belongs to $\mathcal{F}$ then $\cup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$ (closed under countable union).

Two trivial examples $\mathcal{F}_1 = \mathcal{P}(\Omega)$, $\mathcal{F}_2 = \{\phi, \Omega\}$.

Result: Arbitrary intersection of $\sigma-$algebras is again a $\sigma-$algebra.

Proof: Exercise.

Exercise: Is arbitrary union of $\sigma-$algebras again a $\sigma-$algebra ?

Definition 1.2. *Let $\mathcal{A}$ be a collection of subsets of Ω. The $\sigma-$algebra generated by $\mathcal{A}$ is given by*

$$\sigma(\mathcal{A}) \doteq \bigcap \mathcal{F} \,,$$

where the intersection is over all $\sigma-$algebras containing $\mathcal{A}$. Thus this is the smallest $\sigma-$algebra containing $\mathcal{A}$.

Example 1.3. *Let A be a non-empty subset of Ω. Then $\sigma(\{A\}) = \{\phi, A, A^c, \Omega\}$.*

Example 1.4. *Let Θ be the collection of open subsets of $\mathbb{R}$ then $\sigma(\Theta) = \mathcal{B}(\mathbb{R})$ is called Borel $\sigma-$algebra. The sets in this $\sigma-$algebra are called Borel sets.*

Exercise: Let A be a non-empty subset of Ω. Let B be another non-empty subset of Ω such that $A \cap B \neq \phi$ and $A \cup B \neq \Omega$. Find $\sigma(\{A, B\})$.

Exercise: Suppose $\Omega = [0, 1]$. Find the $\sigma-$algebra generated by the collection of singletons.

Definition 1.5 (Axiomatic Definition)**.** *Let Ω be a non-empty set and $\mathcal{F}$ be a $\sigma-$algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a function that to every set $A \in \mathcal{F}$ assigns a number in $[0,1]$, called the probability of the event A and denoted by $\mathbb{P}(A)$. We require:*
i) $\mathbb{P}(\Omega) = 1$
ii) whenever $A_1, A_2, \dots$ is a sequence of disjoint sets in $\mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum \mathbb{P}(A_n)$.

• The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called the probability space.
• The first requirement is just a normalizing property.
• The second requirement is called the countable additivity property.
The following are easy consequences of the definition.

- $\mathbb{P}(\phi) = 0$.
- Finite additivity holds.
- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
- If $A \subset B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$. This is called the monotonicity property.
- If $\{A_i\}$ is a sequence of events not necessarily disjoint then $\mathbb{P}(\cup_{i=1}^{\infty})A_i \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ (countable subadditivity).

Result: Let $A_1 \subset A_2 \subset A_3 \dots$ be an increasing sequence of events. Then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$. This is called the continuity from below property.
Proof: Exercise.
Result: Let $A_1 \supset A_2 \supset A_3 \dots$ be a decreasing sequence of events. Then $\mathbb{P}(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$. This is called the continuity from above property.
Proof: Exercise.

Definition 1.6. *Let Ω be a non-empty set and let $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a algebra provided:*
i) the empty set ϕ belongs to $\mathcal{F}$
ii) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ (closed under complementation)
iii) If A_1, A_2 belongs to $\mathcal{F}$ then $A_1 \cup A_2$ also belongs to $\mathcal{F}$ (closed under finite union).

Example 1.7. *Let $\mathcal{L}$ be the collection of all finite disjoint unions of all intervals of the form $(-\infty, a], (a, b], (b, \infty], \phi, \Omega$. Then $\mathcal{L}$ is an algebra not a $\sigma-$algebra.*

Example 1.8. *Let Ω be an infinite set and $\mathcal{L}$ be the collection of all subsets of Ω which are finite or have finite complement. Then $\mathcal{L}$ is an algebra not a $\sigma-$algebra.*

Theorem 1.9. *(Caratheodary extension theorem)*
Let Ω be a non-empty set and $\mathcal{F}_0$ be an algebra on it. Let $\mathbb{P}_0$ be a probability measure on $\mathcal{F}_0$. Then there is a unique probability measure $\mathbb{P}$ on $\sigma(\mathcal{F}_0)$ such that

$$\mathbb{P}_0(A) = \mathbb{P}(A) \;\forall\; A \in \mathcal{F}_0.$$

A Probabilistic Model for Tossing a Coin Infinitely Many Times:

• $\Omega = \{\omega = \{\omega_n\} : \omega_n = H or T\}$.

• Let $\mathcal{F}_\infty$ be the $\sigma-$algebra generated by sets which can be described in terms of finitely many coin tosses. Let $A = \{\{\omega_n\} : \omega_1 = H\}$, $B = \{\{\omega_n\} : \omega_1 = H, \omega_2 = H\}$. Clearly $A, B \in \mathcal{F}_\infty$. Let $C = \{\{\omega_n\} : \lim_{n\to\infty} \frac{H_n(\omega)}{n} = 1/2\}$, where $H_n(\omega)$ is the number of heads in first n coin tosses. Clearly C is not determined by finitely many coin tosses. For fixed n and m, define

$$C_{n,m} = \{\omega : \left|\frac{H_n(\omega)}{n} - 1/2\right| \leq 1/m\}\,.$$

Clearly $C_{n,m} \in \mathcal{F}_\infty$. From the definition of limit

$$C = \cap_{m=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} C_{n,m}\,.$$

Thus $C \in \mathcal{F}_\infty$.

1.2 Random Variable

Let $\mathcal{B}(\mathbb{R})$ denote the Borel $\sigma-$algebra of $\mathbb{R}$, the $\sigma-$algebra generated by closed intervals.

Definition 1.10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is said to be a real-valued random variable if $X^{-1}(A) = \{\omega : X(\omega) \in A\} \in \mathcal{F}$ for all $A \in \mathcal{B}(\mathbb{R})$.*

Proposition 1.11. *Let $\mathcal{G}$ be a $\sigma-$algebra generated by a collection $\mathcal{S}$. If $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{S}$, then $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{G}$.*

Proof: Let

$$\mathcal{A} = \{A : X^{-1}(A) \in \mathcal{F}\}\,.$$

Then $\mathcal{S} \subset \mathcal{A}$. We will now show that $\mathcal{A}$ is a $\sigma-$algebra. $X^{-1}(A^c) = (X^{-1}(A))^c$ Thus if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$. Again, $X^{-1}(\cup A_i) = \cup X^{-1}(A_i)$. Thus if $A_i \in \mathcal{A}$ then $\cup A_i \in \mathcal{A}$. Hence we have shown that $\mathcal{A}$ is a $\sigma-$algebra, as a result $\sigma(\mathcal{S}) = \mathcal{G} \subset \mathcal{A}$. □

Fact: $\sigma-$algebra generated by open rays is $\mathcal{B}(\mathbb{R})$. Thus if $\mathcal{S} = \{(r, \infty) : r \in \mathbb{R}\}$, then $\sigma(\mathcal{S}) = \mathcal{B}(\mathbb{R})$.
Thus using the above proposition X is a random variable iff $X^{-1}(r, \infty) \in \mathcal{F}$ for all $r \in \mathbb{R}$.

Example 1.12. *Recall the independent, infinite coin-toss probability space $(\Omega, \mathcal{F}_\infty, \mathbb{P})$. Let $S : \Omega \to \mathbb{R}$ be defined by*

$$S(w) = \begin{cases} 8 & \text{if } \mathrm{w}_1 = \mathrm{H} \\ 2 & \text{if } \mathrm{w}_1 = \mathrm{T}. \end{cases}$$

Then

$$S^{-1}(A) = \begin{cases} \phi & \text{if } 8, 2 \notin \mathrm{A} \\ \Omega & \text{if } 8, 2 \in \mathrm{A} \\ A_H & \text{if } 8 \in \mathrm{A} \,\&\, 2 \notin \mathrm{A} \\ A_T & \text{if } 8 \notin \mathrm{A} \,\&\, 2 \in \mathrm{A}. \end{cases}$$

Also note that

$$S^{-1}(a, \infty) = \begin{cases} \phi & \text{if a} > 8 \\ A_H & \text{if } 2 < \text{a} \leq 8 \\ \Omega & \text{if a} \leq 2. \end{cases}$$

Therefore S is a random variable.

Lemma 1.13. *If X and Y are random variables then $X + Y$ is also a random variable.*

Proof:

$$X + Y > r$$
$$\Leftrightarrow X > r - Y$$
$$\Leftrightarrow \text{there exists a rational no. q such that } X > q > r - Y\,.$$

Thus $(X + Y)^{-1}(r, \infty) = \cup_{q \in \mathbb{Q}} X^{-1}(q, \infty) \cap Y^{-1}(r - q, \infty)$. □

Exercise: Let A be a subset of Ω. Then show that 1_A is a random variable if and only if $A \in \mathcal{F}$.

Exercise: Prove that if X is a random variable then X^2 is also a random variable. Prove that if X and Y are random variables then XY is also a random variable.

Exercise Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for all $B \in \mathcal{B}(\mathbb{R})$ (such a function is called Borel measurable function). Show that if X is a random variable then $f(X)$ is also a random variable.

Definition 1.14. *Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution measure of X is the probability measure μ_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that assigns to each Borel set B of $\mathbb{R}$ the measure $\mu_X(B) = \mathbb{P}(X \in B)$.*

Two examples:

Example 1.15. *Let $\Omega = [0, 1]$, $\mathcal{F} = \sigma-$algebra generated by closed intervals. Let $\mathbb{P}$ be the probability measure which assigns to each interval its length. Define $X(\omega) = \omega$ and $Y(\omega) = 1 - \omega$. Then*

$$\mu_X[a, b] = \mathbb{P}(X \in [a, b]) = \mathbb{P}[a, b] = b - a\,.$$

Now

$$\mu_Y[a, b] = \mathbb{P}(Y \in [a, b]) = \mathbb{P}(a \leq 1 - \omega \leq b) = \mathbb{P}[1 - b, 1 - a] = b - a\,.$$

Thus $\mu_X = \mu_Y$ (X and Y are said to be uniformly distributed.)