(1) $E[M(t)] = E[E[M(t) | \mathcal{F}_0]] = E[M(0)]$ $\forall$ t

Take $M(t) = (W(t))^3$

For ito formula take $f(t,x) = x^3$

$$df = 3x^2\,dx + 6x\,dt$$

$$f(t,x) - f(0,x) = \int 3x^2\,dx + \int 3x\,dt$$

Putting $x = W(t)$

$$(W(t))^3 - (W(0))^3 = \int_0^t 3(W(s))^2\,dW(s) + \int_0^t 3\,W(s)\,ds$$

Taking expectation both sides.

$$E[(W(t))^3] - E[(W(0))^3] = 0 + \int_0^t 3\,E[W(s)]\,ds$$

$$= 0 + 0$$

$$E[M(t)] - E[M(0)]$$

$$\therefore E[M(t)] = E[M(0)]$$

$$E[M(t) | \mathcal{F}_s] \qquad M(t) = (W(t))^3$$

$$= E[(W(t))^3 | \mathcal{F}_s]$$

$$= E[[(W(t) - W(s)) + W(s)]^3 | \mathcal{F}_s]$$

$$= E[(W(t) - W(s))^3 + (W(s))^3 + 3W(s)(W(t) - W(s))[W(t)] | \mathcal{F}_s]$$

$$= E[(W(t) - W(s))^3] + (W(s))^3 + 3W(s)\,E[(W(t) - W(s))\,W(t) | \mathcal{F}_s]$$

$$= 0 + (W(s))^3 + 3W(s)\,E[(W(t) - W(s))(W(t) - W(s) + W(s)) | \mathcal{F}_s]$$

$$= (W(s))^3 + 3W(s)\,E[(W(t) - W(s))^2] + W(s)\,E[W(t) - W(s)]$$

$$= (W(s))^3 + 3W(s)(t-s) + 0$$

$$= (W(s))^3 + 3W(s)(t-s) \neq M_s$$

$\therefore (W(t))^3$ is not a martingale.

2. (i) $c + \int_0^T f(s)\,dW(s) = D + \int_0^T g(s)\,dW(s)$

Taking Expectation both sides —

$c + 0 = D + 0$

$\therefore \underline{c = D}$

$\because$ ~~Expectation of ito integ~~

$\because$ ito integral is a martingale

~~and~~ $E[I(t)] = E[I(0)] = \underline{\underline{0}}$

(ii) $c - D = \int_0^T (g(s) - f(s))\,dW(s)$

$(g(s) - f(s))$ is a deterministic function.

$\therefore$ By Theorem, Ito integral of a deterministic integrand.

$\int_0^T (g(s) - f(s))\,dW(s)$ is normally distributed with expected value 0 and variance $\int_0^T (g(s) - f(s))^2\,ds$.

But left hand side is a constant. Therefore variance have to be zero.

$\therefore \int_0^T (g(s) - f(s))^2\,ds = 0$

~~By a proposition taught in class that: suppose $X \geq 0$ & $\int$~~

$\because (g(s) - f(s))^2 \geq 0$

$\therefore g(s) = f(s) \quad \forall\ s \in [0, T]$

~~$\Rightarrow \int_0^T (g(s) - f(s))\,dW(s) = 0 = C - D$~~

~~$\therefore \underline{\underline{C = D}}$~~

Q3

Take $f(x) = x^3$

By Ito,
$$W_t^3 = 3\int_0^t W^2 dW + 3\int_0^t W dt$$

$$\int_0^t W dt = \frac{W^3}{3} - \int_0^t W^2 dW$$

$$E\left[\int_0^t W dt\right] = 0 + 0 = 0$$

$$var\left[\int_0^t W dt\right) = E\left[\left(\int_0^t W dt\right)^2\right]$$

$$= \cancel{\mathbb{E}} \int_0^t \left(E[W^2]\right)^2 dt$$

$$= \int_0^t t^2 dt$$

$$= \frac{t^3}{3}$$

$$\therefore var(X(2)) = \frac{8}{3}$$

$$X(t) = \frac{W^3}{3} - \int_0^t W^2 dW$$

$\therefore$ ~~X would have normal distribution~~

$\therefore$ X would be $N\left(0, \frac{t^3}{3}\right)$

by matching MGF of X(t) with MGF of $N\left(0, \frac{t^3}{3}\right)$

(4) we know that, $dS(t) = \left(\sigma(t)\, dW(t) + \alpha(t)\, dt\right) S(t)$

this stochastic diff eqn gives -

$$S(t) = S(0) \exp\left\{ \int_0^t \sigma(s)\, dW(s) + \int_0^t \left(\alpha(s) - \frac{\sigma^2}{2}(s)\right) ds \right\}$$

In the question $\sigma(s) = 0.05$

$\alpha(s) = 0.1$ $\quad S(0) = 35$

$$\therefore S(t) = S(0) \exp\left\{ \int_0^t 0.05\, dW(s) + \int_0^t \left(0.1 - \frac{(0.05)^2}{2}\right) ds \right\}$$

$$= 35 \exp\left\{ 0.05\, W(t) + 0.09875\, t \right\}$$

To calculate $= P(S(5) < 48)$

$$P(S(5) < 48) = P\left(35 \exp\{0.05\, W(5) + 0.49375\} < 48\right)$$

$$\Rightarrow P\left(\frac{W(5)}{20} < \ln\left(\frac{48}{35}\right) - 0.49375\right)$$

we know $W(5)$ is $N(0,5) \Rightarrow \frac{W(5)}{\sqrt{5}}$ is $N(0,1)$

$$\therefore P\left(\frac{W(5)}{\sqrt{5}} < \frac{20}{\sqrt{5}}\left(\ln\left(\frac{48}{35}\right) - 0.49375\right)\right)$$

taking $Z = \frac{W(5)}{\sqrt{5}}$ $\quad = P(Z < -1.5911) = \underline{0.0558}$

(5) $X(t) = \int_0^t \sin(W_3(s)) dW_1(s) + \int_0^t \cos(W_3(s)) dW_2(s)$

$\therefore dX(t) = \sin(W_3(t)) dW_1(t) + \cos(W_3(t)) dW_2(t)$

$dX(t)\,dX(t) = \sin^2(W_3(t))\,dt + \cos^2(W_3(t))\,dt$

$= dt$

[using $dW_1 dW_2 = 0$
$dW_1 dW_1 = dt$
$dW_2 dW_2 = dt$]

$\therefore$ Quadratic variation
$[X(t), X(t)] = t$

$\because$ Ito integral ~~have~~ is continuous ~~paths~~ ~~almost~~
$\therefore X(t)$ [sum of ito integral] would also be continuous.

$\because$ Ito integral are martingale
$\therefore X(t)$ would be a martingale.

$X(0) = 0$

$\therefore$ By levy theorem $X(t)$ is a brownian motion.

Similarly $Y(t)$ would have quad. variation $[Y(t), Y(t)] = t$, it would be martingale and be cont. & $Y(0) = 0$.
$\therefore Y(t)$ is a BM.

$dY(t) = \cos(W_3(t)) dW_1(t) + \sin(W_3(t)) dW_2(t)$

$dX\,dY = 2\sin(W_3)\cos(W_3)\,dt$

$\neq dt$

[using $dW_1 dW_1 = dt$
$dW_2 dW_2 = dt$
$dW_1 dW_2 = 0$]

$\therefore (X(t), Y(t))$ is not a 2D brownian motion.

(6) For multidimensional BM,

$$dS_i(t) = \alpha_i(t) S_i(t) dt + S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t)$$

In our Question,

$R(t) = 5$, $\alpha_1 = 7$ $\alpha_2 = \mu$ $\sigma_{11} = 1$ $\sigma_{12} = 1$

$\sigma_{21} = 1$ $\sigma_{22} = \sigma$

By market price of risk equation

$$\alpha_1(t) - R(t) = \sigma_{11}\theta_1(t) + \sigma_{12}\theta_2(t)$$

$$\alpha_2(t) - R(t) = \sigma_{21}\theta_1(t) + \sigma_{22}\theta_2(t)$$

$$2 = \theta_1(t) + \theta_2(t)$$

$$\mu - 5 = \theta_1(t) + \sigma\theta_2(t)$$

$$\therefore \cancel{2} \; 7 - \mu = (1-\sigma)\theta_2(t).$$

$$\therefore \theta_2(t) = \frac{7-\mu}{1-\sigma}$$

$$\text{and } \theta_1 = 2 - \theta_2 = \frac{2 - 2\sigma - 7 + \mu}{1-\sigma}$$

$$= \frac{\mu - 2\sigma - 5}{1-\sigma} \qquad \text{if } \sigma \neq 1$$

i) Because value of θ_1 and θ_2 have been found if, we can define a risk measure. By 1st fundamental theorem of asset pricing, Market is arbitrage free. if $\sigma \neq 1$

(ii) If $\sigma \neq 1$, $\theta_1(t)$ and $\theta_2(t)$ are unique or Risk neutral measure unique $\Rightarrow$ market is Complete.

iii) $\mu = 8,\ \sigma = 2$

$\Rightarrow \theta_1(t) = 1$ and $\theta_2(t) = 1$

$$Z(t) = \exp\left\{ -\int_0^t dW_1(u) + dW_2(u) - \frac{1}{2}\int_0^t 2\,du \right\}$$

$$= \exp\left\{ -\int_0^t dW_1(u) - \int_0^t dW_2(u) - t \right\}$$

By Gersanov multidemension we have.

$$Q = \int_A Z\,dP$$

where

$$Z = Z(T) = \exp\left\{ -W_1(T) - W_2(T) - T \right\}$$

(iv) $\widetilde{W}_1(t) = W_1(t) + \int_0^t \theta_1(u)\,du$

$= W_1(t) + t.$

$\widetilde{W}_2 = W_2 + \int_0^t \theta_2(u)\,du$

$= W_2(t) + t$

$d\widetilde{W}_1 = dW_1 + dt \qquad d\widetilde{W}_2 = dW_2 + dt$

$dS_1(t) = S_1(7dt + d\widetilde{W}_1 - dt + d\widetilde{W}_2 - dt)$

$dS_1 = S_1(5dt + d\widetilde{W}_1 + d\widetilde{W}_2]$