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Example: Let $f(x) = x^4 - 4x^3 + 10$ for all $x \in [1, 4]$. Then for the partition $P = \{1, 2, 3, 4\}$ of $[1, 4]$,
$U(f, P) = 11$ and $L(f, P) = -40$.

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$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$, where
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Riemann integral: If Upper integral = Lower integral, then f is Riemann integrable on $[a, b]$ and the common value is the Riemann integral of f on $[a, b]$, denoted by $\int_a^b f$.

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Remark: Let $f : [a,b] \to \mathbb{R}$ be bounded. Let there exist a sequence (P_n) of partitions of $[a,b]$ such that $L(f, P_n) \to \alpha$ and $U(f, P_n) \to \alpha$. Then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = \alpha$.

Riemann's criterion for integrability: A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ iff for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

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Properties of Riemann integrable functions:

Example: $\frac{1}{3\sqrt{2}} \leq \int_0^1 \frac{x^2}{\sqrt{1+x}}\, dx \leq \frac{1}{3}$

First fundamental theorem of calculus: Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and let $F(x) = \int_a^x f(t)\, dt$ for all $x \in [a, b]$. Then $F : [a, b] \to \mathbb{R}$ is continuous.

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Second fundamental theorem of calculus: Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. If there exists a differentiable function $F : [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$, then $\int\limits_a^b f(x)\, dx = F(b) - F(a)$.

Riemann sum: $S(f,P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$,

where $f : [a,b] \to \mathbb{R}$ is bounded,

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Example: $\lim_{n \to \infty} [\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}] = \log 2$.

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Convergence of Type I improper integrals:

Let $f \in \mathcal{R}[a, x]$ for all $x > a$. If $\lim_{x \to \infty} \int_a^x f(t)\,dt$ exists in $\mathbb{R}$, then $\int_a^\infty f(t)\,dt$ converges and $\int_a^\infty f(t)\,dt = \lim_{x \to \infty} \int_a^x f(t)\,dt$.

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Otherwise, $\int\limits_a^\infty f(t)\,dt$ is divergent.

Similarly, we define convergence of $\int\limits_{-\infty}^{b} f(t)\,dt$ and $\int\limits_{-\infty}^{\infty} f(t)\,dt$.

Examples: (a) $\int_1^{\infty} \frac{1}{t^p}\, dt$ converges iff $p > 1$.

(b) $\int_{-\infty}^{\infty} e^t\, dt$ (c) $\int_0^{\infty} \frac{1}{1+t^2}\, dt$

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Comparison test: Let $0 \le f(t) \le g(t)$ for all $x \ge a$. If $\int\limits_{a}^{\infty} g(t)\, dt$ converges, then $\int\limits_{a}^{\infty} f(t)\, dt$ converges.

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(a) If $\ell \ne 0$, then $\int\limits_a^\infty f(t)\,dt$ converges iff $\int\limits_a^\infty g(t)\,dt$ converges.

(b) If $\ell = 0$, then $\int\limits_a^\infty f(t)\,dt$ converges if $\int\limits_a^\infty g(t)\,dt$ converges.

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Integral test for series: Let $f : [1,\infty) \to \mathbb{R}$ be a positive decreasing function. Then $\sum_{n=1}^\infty f(n)$ converges iff $\int_1^\infty f(t)\,dt$ converges.

Dirichlet's test: Let $f : [a, \infty) \to \mathbb{R}$ and $g : [a, \infty) \to \mathbb{R}$ such that

(a) f is decreasing and $\lim_{t \to \infty} f(t) = 0$, and

(b) g is continuous and there exists $M > 0$ such that $\left| \int_a^x g(t)\, dt \right| \leq M$ for all $x \geq a$.

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Convergence of Type II and mixed type improper integrals:

Example: $\int\limits_0^1 \frac{1}{t^p}\,dt$ converges iff $p < 1$.

Lengths of smooth curves:

(a) Let $y = f(x)$, where $f : [a, b] \to \mathbb{R}$ is such that f' is continuous.

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(b) Let $x = \varphi(t)$, $y = \psi(t)$, where $\varphi : [a, b] \to \mathbb{R}$ and $\psi : [a, b] \to \mathbb{R}$ are such that φ' and ψ' are continuous.

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(c) Let $r = f(\theta)$, where $f : [\alpha, \beta] \to \mathbb{R}$ is such that f' is continuous.

Then $L = \int\limits_\alpha^\beta \sqrt{r^2 + (f'(\theta))^2}\, d\theta$

Examples:

(a) The length of the curve $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ from $x = 0$ to $x = 3$ is 12.

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(c) The length of the curve $x = e^t \sin t$, $y = e^t \cos t$, $0 \le t \le \frac{\pi}{2}$, is $\sqrt{2}(e^{\frac{\pi}{2}} - 1)$.

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(c) The length of the curve $x = e^t \sin t$, $y = e^t \cos t$, $0 \le t \le \frac{\pi}{2}$, is $\sqrt{2}(e^{\frac{\pi}{2}} - 1)$.

(d) The length of the cardioid $r = 1 - \cos\theta$ is 8.

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Area between two curves: If $f, g : [a, b] \to \mathbb{R}$ are continuous and $f(x) \ge g(x)$ for all $x \in [a, b]$, then we define the area between $y = f(x)$ and $y = g(x)$ from a to b to be

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Example: The area above the x-axis which is included between the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 2ax$, where $a > 0$, is $(\frac{3\pi - 8}{12})a^2$.

Area in polar coordinates: Let $f; [\alpha, \beta] \to \mathbb{R}$ be continuous. We define the area bounded by $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ to be $\frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta$.

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Example: A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \leq x \leq 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Then the volume of the solid is 16.

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Example: The volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2 = 4ax$ about the x-axis, and bounded by the section $x = x_1$.