1 Quasi-Monte Carlo

Quasi-Monte Carlo approximates the integral $\int_{[0,\,1)^m} f(\boldsymbol{x})d\boldsymbol{x}$ using

$$\int_{[0,\,1)^m} f(\boldsymbol{x})d\boldsymbol{x} \approx \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{x}_i)$$

for carefully and deterministically chosen points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

1.1 Sequences of Numbers with Low Discrepancy

One difficulty with random numbers is that they may fail to distribute uniformly. Here "uniform" is not meant in the stochastic sense of the distribution $U(0, 1)$, but has the meaning of equidistributiveness. The aim is to generate numbers for which the deviation from uniformity is minimal. This deviation is called "discrepancy". It would be desirable to find a compromise, picking samples points in some schemes such that the fineness advances but clustering is avoided. The sample points should fill the integration domain as uniformly as possible. Let $Q \subseteq [0, 1]^m$ be an arbitrary axially parallel m-dimensional rectangle in the unit cube $[0, 1]^m$ of $\mathbb{R}^m$. This Q is thus a product of m-intervals. Suppose a set of points $x_1, x_2, \ldots, x_M \in [0, 1]^m$. Let # denote the number of points, then the goal is,

$$\frac{\#\text{ of } x_i \in Q}{\#\text{ of all points}} \cong \frac{\mathrm{vol}(Q)}{\mathrm{vol}([0,1]^m)}.$$

for as many rectangles as possible.

Given a collection $\mathcal{A}$ of subsets of $[0,\,1)^m$, the discrepancy of the point set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$ relative to $\mathcal{A}$ is defined by

$$D(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N; \mathcal{A}) := \sup_{Q \in \mathcal{A}} \left| \frac{\#\text{ of } \boldsymbol{x}_i \in Q}{N} - \mathrm{vol}(Q) \right|.$$

Taking $\mathcal{A}$ be the collection of rectangles in $[0,\,1)^m$ of the form

$$\prod_{j=1}^{m} [u_j,\, v_j), \qquad 0 \le u_j < v_j < 1,$$

yields ordinary discrepancy $D(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$. Restricting $\mathcal{A}$ to rectangles of the form

$$\prod_{j=1}^{m} [0,\, v_j), \qquad 0 < u_j < 1$$

defines the star discrepancy $D^*(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$. It can be shown that

$$D^*(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \le D(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \le 2^m D^*(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N).$$

It is customary to reserve the informal term "low discrepancy" for methods that achieve a star discrepancy of $O((\log n)^m/n)$. The following introduce low discrepancy sequences, *viz.*, Van Der Corput sequence and Halton sequence.

Van Der Corput Sequence

For $i = 1, 2, \ldots$ let

$$i = \sum_{k=0}^{\infty} d_k b^k,$$

be the expression in base b (integer ≥ 2) with digits $d_k \in \{0, 1, \ldots, b-1\}$. Then the radical inverse function is defined by

$$\phi_b(i) := \sum_{k=0}^{\infty} d_k b^{-k-1}.$$

The base-b Van Der Corput sequence is given by

$$x_i := \phi_b(i).$$

Halton Sequence

Let $p_1, p_2, \ldots, p_m$ be a pairwise prime integers. The Halton sequence in m dimension is defined as follows:

$$\boldsymbol{x}_i := (\phi_{p_1}(i), \phi_{p_2}(i), \ldots, \phi_{p_m}(i)), i = 1, 2, \ldots$$

Usually one takes $p_1, p_2, \ldots, p_m$ to the first m prime numbers.