Primal $\min c^T x \to f_p$, $\underline{Ax \geq b, x \geq 0}$ T_p Dual $\max y^T b \to f_d$, $\underline{y^T A \leq c^T, y \geq 0}$ T_d

8 Lecture 8

Duality via Farka's lemma

$A_{m \times n}$

[8.1] **Definition**

a) Given an lpp, there is another lpp called the DUAL of the first lpp. Here $A \in M_{m,n}$.

$$\text{Primal lpp: } \begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \geq b,\ x \geq 0 \end{array} \qquad \text{Dual lpp: } \begin{array}{ll} \max & y^t b \\ \text{s.t.} & y^t A \leq c^t,\ y \geq 0. \end{array} \tag{3}$$

The first one is sometimes called the PRIMAL lpp. It is in $\mathbb{R}^n$ and the second one is in $\mathbb{R}^m$.

b) We call the set $\boldsymbol{T_p}:= \{x \mid Ax \geq b, x \geq 0\}$ the primal feasible set and the set $\boldsymbol{T_d}:= \{y \mid y^t A \leq c^t, y \geq 0\}$ the dual feasible set. We call the function $\boldsymbol{f_p(x)}:= c^t x$, the primal objective function and $\boldsymbol{f_d(x)}:= y^t b$, the dual objective function.

c) The dual for an lpp given in another form is obtained by first converting it to the above form.

[8.2] **Example** Dual of $\begin{array}{ll} \min & 2x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 \geq 2, x_i \geq 0 \end{array}$ is $\begin{array}{ll} \max & 2y \\ \text{s.t.} & y \leq 2,\ y \leq 1,\ y_i \geq 0. \end{array}$

$\min \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 2$

$\max 2y$, $y \begin{bmatrix} 1 & 1 \end{bmatrix} \leq \begin{bmatrix} 2, 1 \end{bmatrix}$

[8.3] **Fact** The dual of the dual problem is the primal.

$\min 2x_1 + x_2$, $x_1 + x_2 \geq 2$

$\max -\begin{bmatrix} x_1, x_2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $-\begin{bmatrix} x_1, x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq -2$

Dual

$\min -2y$, $\cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix} y \leq \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$, $-\begin{bmatrix} 1 \\ 1 \end{bmatrix} y \geq \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

Proof. We show this for the form given in (3). The proof for the others is similar. The dual

$$\begin{array}{ll} \max & y^t b \\ \text{s.t.} & y^t A \leq c^t,\ y \geq 0 \end{array} \equiv \begin{array}{ll} \min & -b^t y \\ \text{s.t.} & (-A)^t y \geq (-c),\ y \geq 0 \end{array}.$$

It's dual is

$$\begin{array}{ll} \max & z^t(-c) \\ \text{s.t.} & z^t(-A^t) \leq (-b^t),\ z \geq 0 \end{array} \equiv \begin{array}{ll} \min & c^t z \\ \text{s.t.} & Az \geq b,\ z \geq 0 \end{array},$$

$y_0 \in T_d$, $x_0 \in T_p$

$f_d(y_0) = y_0^T b \leq y_0^T A x_0 \leq c^T x_0 = f_p(x_0)$

the primal. ∎

[8.4] <u>**Fact**</u> `(Dual functional value is always smaller than the primal functional value.)` Consider the primal and the dual lpp's given in (3). Let $x_0 \in T_p$, $y_0 \in T_d$. Then $f_d(y_0) \leq f_p(x_0)$.

Proof. We have $f_d(y_0) = y_0^t b \leq y_0^t A x_0$ (as $b \leq Ax_0$) $\leq c^t x_0$ (as $y_0^t A \leq c$) $= f_p(x_0)$. ∎

The following corollary is immediate.

[8.5] <u>**Corollary**</u> Let $x_0 \in T_p$, $y_0 \in T_d$ and $f_d(y_0) = f_p(x_0)$. Then x_0 is a minimum for the primal and y_0 is a maximum for the dual.

$f_d(y) \leq f_p(x_0)$, $f_d(y_0) = f_p(x_0)$

[8.6] <u>**Primal-dual Theorem.**</u> Consider the primal and the dual lpp's given in (3). Then the primal lpp has a minimum iff the dual lpp has a maximum. In case the optimums exist, both their values are equal.

Suppose min sol for the primal is attained. $\{Ax \geq b, x \geq 0\}$

$\Rightarrow$ min soln is attained at a vertex w of T_p.

$= \{x \mid \begin{bmatrix} A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ 0 \end{bmatrix}\}$

$\min c^T x$, $\tilde{A} x \geq \tilde{b}$

$\Rightarrow c^T = \boxed{\lambda^T \tilde{A}_w}$ ($\exists\ \lambda \geq 0$ vector)

$\tilde{A} x \geq \tilde{b}$

$= \tilde{\lambda}^T \tilde{A}_w = \tilde{\lambda}_A^T A + \tilde{\lambda}_I^T I \geq \tilde{\lambda}_A^T A$

Take $y = \tilde{\lambda}_A^T$. Then $y^T A \leq c^T$, $y \geq 0$

$f_d(y) = y^T b = \tilde{\lambda}_A^T b = \tilde{\lambda}^T \tilde{b}$ $\quad \tilde{\lambda}^T = [\tilde{\lambda}_A^T \ \tilde{\lambda}_I^T] \begin{bmatrix} b \\ 0 \end{bmatrix}$

$= \lambda^T \tilde{b}_w = \lambda^T \tilde{A}_w w = c^T w = f_p(w)$

$\Rightarrow y = \tilde{\lambda}_A^T$ is a dual max.

Proof. Assume that the primal has a minimum solution. By FTLP, the minimum is attained at a vertex, say w. Rewrite the primal feasible set.

$$T_p = \{x \mid \begin{bmatrix} A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ 0 \end{bmatrix}\} = \{x \mid \tilde{A}x \geq \tilde{b}\} \quad \text{(say)}.$$

As w is a minimal vertex, by [7.5], there is a vector $\lambda \geq 0$ such that $c^t = \lambda^t \tilde{A}_w$. But then

$$c^t = \lambda^t \tilde{A}_w = \tilde{\lambda}^t \tilde{A} \text{ (where } \tilde{\lambda} \text{ is extended from } \lambda \text{ by putting some zeros)} \quad = \begin{bmatrix}\tilde{\lambda}_A^t & \tilde{\lambda}_I^t\end{bmatrix}\begin{bmatrix}A\\ I\end{bmatrix} \geq \tilde{\lambda}_A^t A.$$

That is, $y = \tilde{\lambda}_A \in T_d$. Also the value

$$y^t b = \tilde{\lambda}_A^t b = \begin{bmatrix}\tilde{\lambda}_A^t & \tilde{\lambda}_I^t\end{bmatrix}\begin{bmatrix}b\\ 0\end{bmatrix} = \tilde{\lambda}^t \tilde{b} = \lambda^t \tilde{b}_w \text{ (remaining entries of } \tilde{\lambda} \text{ are zeros)} \quad = \lambda^t \tilde{A}_w w = c^t w.$$

So y is a dual maximum.

Conversely, let $\gamma \geq 0$ be a dual maximum solution of the value p. Convert this problem into a minimization problem and proceed as in the first paragraph to complete the proof. ■

Some exercises

[8.7] **Exercise(M)** `(Illustration of [8.6].)` Take the convex unit cube T in $\mathbb{R}^3_+$ with vertices at $(1,1,1)$ and $(2,2,2)$. Write this set as $\{x \mid A_{6\times 3}x \geq b\}$. Consider minimizing $f(x) = x_1 + 2x_2 + 3x_3$ over the set $T = \{x \mid Ax \geq b, x \geq 0\}$. (Note that T is the same unit cube.) Follow the steps given and illustrate [8.6].

a) First write the problem in the form $\begin{array}{ll}\min & c^t x\\ \hline \text{s.t.} & \tilde{A}x \geq \tilde{b}\end{array}$.

b) At which vertex w of T does f attain the minimum?

c) Write $\tilde{A}_w$, $\tilde{b}_w$ and a $\tilde{\lambda}$. Write $\hat{\lambda}$ and $\hat{\lambda}_A$.

d) Write the other problem.

e) Is $\hat{\lambda}_A^t$ a maximal solution of this problem?

f) Is it a vertex?

g) Now add one more halfspace $x_1 + x_2 + x_3 \geq 3$. It does not make any changes to the set T. But it changes our matrices.

h) Write $\tilde{A}_w$, $\tilde{b}_w$ and a $\tilde{\lambda}$. Write $\hat{\lambda}$ and $\hat{\lambda}_A$. Write the other problem. Is $\hat{\lambda}_A^t$ a maximal solution of this problem? Is it a vertex?

[8.8] **Exercise(E)** Show that the dual of $\begin{array}{ll}\max & b^t y\\ \hline \text{s.t.} & A^t y \leq c\end{array}$ is $\begin{array}{ll}\min & c^t x\\ \hline \text{s.t.} & Ax = b, x \geq 0\end{array}$ and vice-versa.

[8.9] **Exercise(E)** Write the dual of $\begin{array}{ll}\min & c^t x\\ \hline \text{s.t.} & A_1 x \leq b_1,\ A_2 x = b_2,\ A_3 x \geq b_3,\ x \geq 0.\end{array}$

A quick recap of some necessary concepts.

1. A set $S \subseteq \mathbb{R}^n$ is called CONVEX, if the line segment joining each pair of points in S lies completely in S.
2. A linear combination $\lambda_1 x_1 + \cdots + \lambda_k x_k$ is called a CONVEX COMBINATION of $x_1, \ldots, x_k$ if $\lambda_1 + \cdots + \lambda_k = 1$ and $\lambda_i \geq 0$ for each i.
3. Let $\emptyset \neq S \subseteq \mathbb{R}^n$. By $\mathsf{conv}\, S$ we denote the CONVEX HULL of S, which is the collection of all convex combinations of points of S.

QUIZ I Sunday 4-5

7.55
5.10
by email

4. Let S be convex. A point $v \in S$ is called a VERTEX if it cannot be written as convex combination of two other points of S. That is, if $v \notin \mathsf{conv}(S \setminus \{v\})$.
5. Let $c \in \mathbb{R}^n$ be nonzero and $\alpha \in \mathbb{R}$. The set $H := \{x \mid c^t x = \alpha\}$ is a translated $n-1$-dimensional subspace. It is called a HYPERPLANE. In short we write this hyperplane as $H : c^t x = \alpha$. If $\alpha = 0$, then it is called a LINEAR HYPERPLANE.
6. Let $H : c^t x = \alpha$ be a hyperplane. The set $\{x \mid c^t x \leq \alpha\}$ and $\{x \mid c^t x \geq \alpha\}$ are called the closed HALF-SPACES generated by H.
7. Hyperplanes and half-spaces are convex sets.
8. Intersection of convex sets is convex.
9. A set which is the intersection of some finitely many (at least one) closed half-spaces is called a POLYHEDRON. So a polyhedron is a convex set.
10. Consider minimizing $f(x) := c^t x$ over the feasible set $T := \{x \mid Ax = b, x \geq 0\}$. The following are known about T.
    (a) The set T is a polyhedron, which is sometimes bounded and sometimes not.
    (b) When T is nonempty, it has at least one vertex.
    (c) The function $f(x)$ on T is either unbounded below or it is bounded below. If it is unbounded below, then we do not have a minimum solution of the problem.
    (d) So suppose that T is nonempty and f is bounded below on T. Then we will find at least one vertex $w \in T$ where f is minimized. (There can be other points of minimum too.)
11. Let $A \in M_{m \times n}$ and $T = \{x \mid Ax \geq b\}$ be nonempty polyhedron. Take a point $w \in T$. Then w is a vertex iff $\mathsf{rank}\, A_w = n$, where A_w is the matrix formed by taking those rows of A corresponding to the equalities in $Ax \geq b$.
12. Consider minimizing $f(x) = c^t x$ over $T = \{x \mid Ax \geq b\}$. Let $w \in T$ be a vertex. Then f is minimized at w iff c^t is a nonnegative combination of the rows of A_w.

Simplex algorithm.

[8.10] **The idea behind the simplex algorithm.**

a) If we can make a vertex of $\{x \mid Ax = b, x \geq 0\}$, a solution of the system $Ax = b$ having certain algebraic (verifiable by computer) properties, then that will help.

b) Then we will need a method (doable by the computer) to test the minimality of such solutions.

c) If it is not minimal, we will need a method so that the computer moves to a new vertex.

d) As there are finitely many vertices, this should terminate, as long as we do not create a loop while implementing.

[8.11] **Remark** `(Now onward we consider the set` $\{x \mid Ax = b, x \geq 0\}$`, where` A `has full row rank.)` Here are two reasons for that.

$A \in M_{m\times n}$ $\mathrm{rank}\, A = m$

1) Recall that $f(x) = c^t x$ is minimized at a vertex w of a certain set if $c^t = \lambda^t A_w$ has a nonnegative solution. This is as good as asking to find a nonnegative solution of $(A_w^t)x = c$. Of course A_w^t has full row rank.

2) Consider a system $A_{m\times n}x = b$. We can use GJE (Gauss Jordan elimination), to check whether it is consistent. We can use GJE on A^t to find a set of maximal linearly independent rows of A. Suppose that the system $Ax = b$ is consistent and $\mathsf{rank}\, A = k < m$. Consider the submatrix A' of A made using a set of k rows

which constitute a basis for the row space. Choose the corresponding subvector b' of b for those rows. Then the set

$$\{x \mid Ax = b, x \geq 0\} = \{x \mid A'x = b', x \geq 0\}.$$

Hence, our problem is to minimize $c^t x$ over $\{A'x = b', x \geq 0\}$. In view of this, we can assume without loss that A has full row rank.

Basic solutions and basic feasible solutions

To study the simplex algorithm, we first need to know what are basic solutions and basic feasible solutions.

[8.12] **Definition** (`Basis matrix`)

Let $A \in M_{m,n}$ have rank m. Consider the system $Ax = b$. An ordered tuple $(x_{i_1}, \ldots, x_{i_m})$ is called a BASIS if the matrix $B = [A_{:i_1} \quad \cdots \quad A_{:i_m}]$ is nonsingular. Then B is also called a BASIS MATRIX. In fact, the columns of B form a basis for the column space of A.

[8.13] **Example** Consider $Ax = b$ where $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\left.\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_4 &= 1 \end{aligned}\right\}$$

a) The ordered tuple (x_1, x_3) is not a basis.

b) The ordered tuple (x_1, x_2) is a basis and the corresponding basis matrix is $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(x_1, x_2) is a basis and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$ basis matrix

(x_2, x_1) " $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow$ "

[8.14] **Definition** (`Basic variables`) The elements of a basis are called the BASIC VARIABLES and the remaining are called the NONBASIC VARIABLES.

[8.15] **Example** In the previous example item b), x_1, x_2 are the basic variables and x_3, x_4 are the nonbasic variables.

$(x_4, x_1) \rightarrow$ basis $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ bv x_1, x_4

[8.16] **Notation** Let $A \in M_{m,n}$ with rank m and consider the system $Ax = b$. Let $B = [A_{:i_1} \quad \cdots \quad A_{:i_m}]$ be a basis matrix. We use

$\boldsymbol{C}$ to denote the submatrix of A (we do not use $\overline{A}$ for this) obtained by deleting the columns of B.

For a vector $y \in \mathbb{R}^n$, we use

$\mathbf{y_B}$ to denote the vector $[y_{i_1} \quad \ldots \quad y_{i_m}]^t$, and

$\boldsymbol{y_C}$ to denote the subvector of y corresponding to C.

[8.17] **Example** Consider $Ax = b$ where $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Take $y = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^t$.

For the basis (x_1, x_2), we have $y_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $y_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$y_B = \begin{bmatrix} y(x_1) \\ y(x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $y_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(x_4, x_1) $y_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $y_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For the basis (x_2, x_1) we have $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $y_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

> How to solve $A_{m\times n}x = b$ effectively when rank $A = m$? Let B be a basis matrix (for the system $Ax = b$) and take any $y \in \mathbb{R}^n$. Then
>
> $$Ay = b \quad \Leftrightarrow \quad By_B + Cy_C = b \quad \Leftrightarrow \quad y_B = B^{-1}b - B^{-1}Cy_C. \tag{4}$$
>
> Thus, each selection of y_C will give us a y_B and hence a solution y. In fact, each y_C gives us a unique solution y, as y_C is a part of y.

$A_{m\times n} x = b$ $\quad Ax = b$, $B \to$ basis matrix

$$Bx_B + Cx_{\underset{n-m}{C}} = b \iff x_B = B^{-1}(b - Cx_C)$$

$\begin{bmatrix} Aw \\ \hline \bar{A}w \end{bmatrix}$

$[B \mid C]$

$x_1 + x_3 = 1$

[8.18] **Example** Consider $Ax = b$ where $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Take the basis (x_2, x_1). So $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Taking $y_C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we have

$x_2 + x_4 = 1$

$$y_B = \begin{bmatrix} y_2 \\ y_1 \end{bmatrix} = B^{-1}b - B^{-1}Cy_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

So $y = [0 \quad -1 \quad 1 \quad 2]^t$ is a solution of the system $Ax = b$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

[8.19] **Definitions** (Basic solutions)

a) Let B be a basis matrix. The unique solution of $Ax = b$ obtained by putting $x_C = 0$ (thus $x_B = B^{-1}b$) in (4) is called a BASIC SOLUTION of $Ax = b$ corresponding to the basis x_B (or to the basis matrix B).

b) A basic solution y (for some basis) is called a BASIC FEASIBLE SOLUTION (bfs), if $y \geq 0$.

c) A basic solution y is called NONDEGENERATE if all basic variables are nonzero, otherwise it is called DEGENERATE.

[8.20] **Example** Take $A = \begin{bmatrix} 2 & -2 & 1 & -1 & -1 & 0 & 2 \\ -1 & 1 & 1 & 3 & 0 & 1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The basic solution for the basis (x_5, x_6) is given by $y_B = B^{-1}b = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. So $y = [0 \quad 0 \quad 0 \quad 0 \quad -2 \quad 3 \quad 0]^t$ and it is not a bfs.

$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad x_B = B^{-1}(b - Cx_C) = B^{-1}b = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$x = (0, 0, 0, 0, -2, 3, 0)$ bs, not feasible

(x_1, x_7) $B = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$ $x_B = B^{-1} b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x = (0,0,0,0,0,0,1)$

degenerate

The basic solution for the basis (x_1, x_6) is $x = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 4 \quad 0]^t$. Here x is a nondegenerate bfs.

The basic solution for the basis (x_1, x_7) is $x = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^t$. Here x is a degenerate bfs.

The basic solution for the basis (x_2, x_7) is also the same $x = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^t$.

[8.21] **Fact** `(The same basic solution` y `of` $Ax = b$ `can correspond to two different bases)` .

[8.22] **Lemma** `(How to recognize whether a given solution is a bfs?)` Let $\mathsf{rank}\, A_{m\times n} = m$ and w be a solution of $Ax = b$. Then the following are equivalent.

a) The solution w is a bfs of $Ax = b$.

b) The vector w is nonnegative and columns of A corresponding to the positive entries of w are linearly independent.

Proof. (Self) Let w be a bfs with respect to some basis matrix B. Then by definition, only the basic variables can be nonzero in w. So, the columns of A corresponding to the positive entries of w are columns of B only. Hence they are linearly independent.

Conversely, let the columns of A corresponding to the nonzero entries of w be linearly independent. As $\mathsf{rank}(A) = m$, we may add few more columns of A to obtain an invertible submatrix B of A. (How exactly the computer does this?) As we already have $w_C = 0$, we see that $w_B = B^{-1}b$, so w is indeed a basic solution. As $w \geq 0$, it is a bfs. ∎

[8.23] **Theorem** `(Bfs means a vertex)` A point w is a bfs of $Ax = b$ iff w is a vertex of the polyhedron $T = \{x \mid Ax = b, x \geq 0\}$.

$$\begin{bmatrix} A \\ -A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \equiv \{x \mid \tilde{A}x \geq \tilde{b}\}$$

$$\tilde{A}_w = \begin{bmatrix} A \\ -A \\ \text{part of } I \end{bmatrix} \text{ includes } \begin{bmatrix} B & C \\ -B & -C \\ 0 & \underbrace{I}_{n-m} \end{bmatrix} \qquad \begin{bmatrix} B & C \\ 0 & I \end{bmatrix} \rightarrow \text{rank } n$$

$\Rightarrow \text{rank } \tilde{A}_w = n$

$\Rightarrow w$ is a vertex.

Proof. Let w be a bfs of $Ax = b$ with a basis matrix, say, B. For simplicity, assume $A = [B|C]$. First write $T = \{x \mid A'x \le b'\}$, where $A' = \left[\begin{array}{c} A \\ \hline -A \\ \hline -I \end{array}\right] = \left[\begin{array}{cc} B & C \\ \hline -B & -C \\ \hline -I & 0 \\ \hline 0 & -I \end{array}\right]$ and $b' = \left[\begin{array}{c} b \\ \hline -b \\ \hline 0 \end{array}\right]$. The submatrix $\left[\begin{array}{cc} B & C \\ \hline 0 & -I \end{array}\right]$ has rank n and rows in this matrix correspond equalities in $A'w \le b'$. By [5.22], w is a vertex of T.

Conversely, let w be a vertex of T. This means $A'w \le b'$ and A'_w has rank n. Notice that $A'w \le b'$ itself implies that $Aw = b$. Hence A'_w must contain A, $-A$ and a few rows from $-I$. For simplicity, assume that A'_w contains the last $n - k$ rows of $-I$. (This means $w_{k+1} = \cdots = w_n = 0$.) As the rows in A'_w which are from $-A$ are negatives of those which are from A, we see that the matrix obtained from A'_w by deleting the rows of $-A$, also has rank n. But the matrix must look like

$$\left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ \vdots & & & & & \\ a_{m1} & \cdots & a_{mk} & a_{m,k+1} & \cdots & a_{mn} \\ \hline 0 & \cdots & 0 & -1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & 0 & 0 & \cdots & -1 \end{array}\right].$$

Further, as the rank of this matrix is n, we see the top left k columns are linearly independent. (These are columns of A corresponding to the nonzero entries in w.) So by [8.22], w is a bfs. ∎

[8.24] **Corollary** `(Existence of a bfs)` If $\{x \mid Ax = b, x \ge 0\} \ne \emptyset$, then $Ax = b$ has a bfs.

Proof. The set $T = \{x \mid Ax = b, x \ge 0\}$ being a nonempty polyhedron which is bounded below, has a vertex. By [8.23], a vertex of T is a bfs of $Ax = b$. ∎