

Risk Neutral Pricing Black-Scholes Formula Lecture 19

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Risk Neutral Valuation: Two-Horse Race Example

- One horse has 20% chance to win another has 80% chance
- \$10000 is put on the first one and \$50000 on the second

If odds are set 4-1:

- Bookie may gain \$10000 (if first horse wins)
- Bookie may loose \$2500 (if second horse wins)
- Bookie expects to make $0.2 * (10000) + 0.8 * (-2500) = 0$

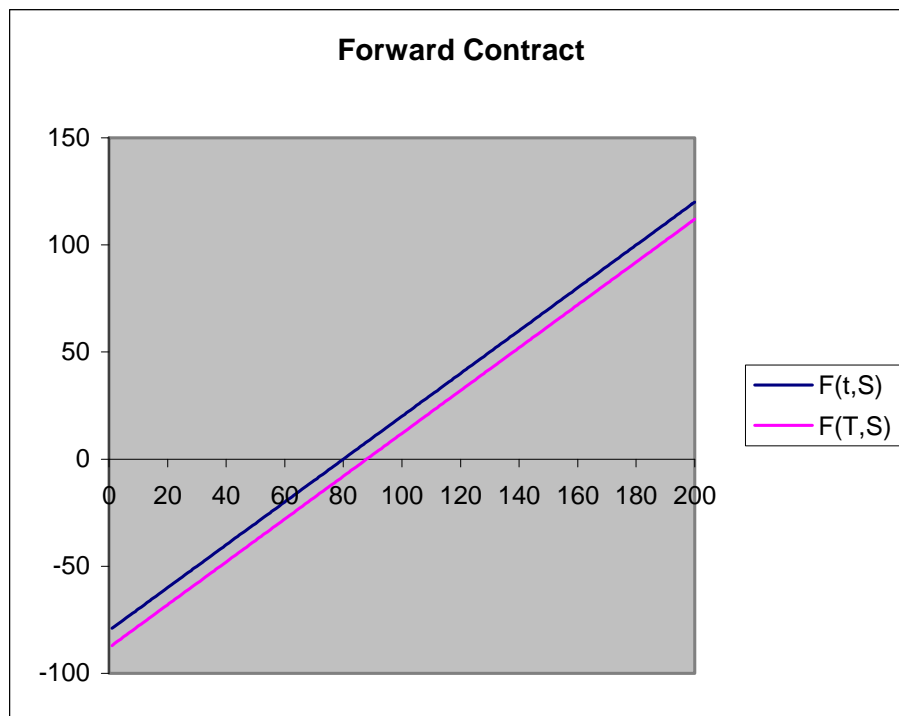
If odds are set 5-1:

- Bookie will not lose or gain money no matter which horse wins

Risk Neutral Valuation : Introduction

We are interested in finding prices of various *derivatives*.

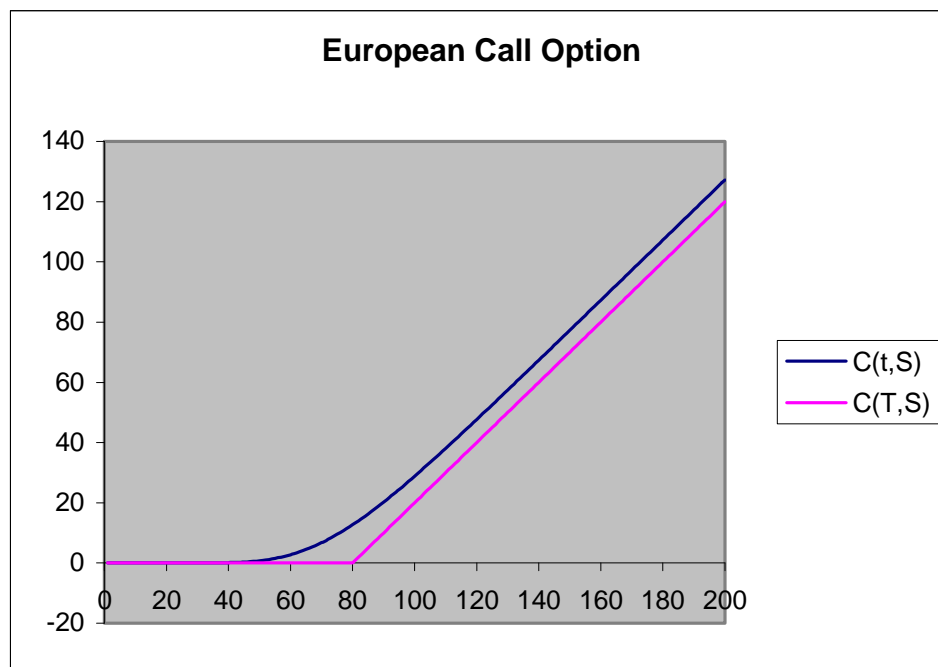
Forward contract pays $S-K$ at time T :



$$S(t)=80, K=88.41, T=2 \text{ (years)}$$

Risk Neutral Valuation: Introduction

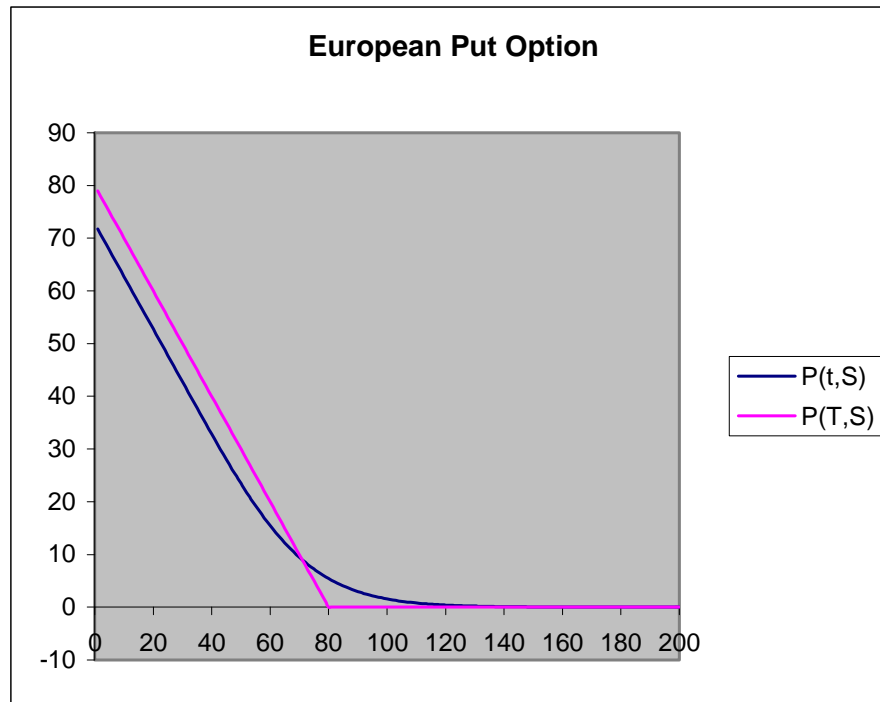
European Call option pays $\max(S-K, 0)$ at time T



$$S(t)=80, K=80, T=2 \text{ (years)}$$

Risk Neutral Valuation: Introduction

European Put option pays $\max(K-S, 0)$ at time T



$$S(t)=80, K=80, T=2 \text{ (years)}$$

Risk Neutral Valuation: Introduction

- Given current price of the stock and assumptions on the dynamics of stock price, there is no uncertainty about the price of a derivative
- The price is defined only by the price of the stock and not by the risk preferences of the market participants
- Mathematical apparatus allows to compute current price of a derivative and its risks, given certain assumptions about the market

Risk Neutral Valuation: Replicating Portfolio

Consider *Forward* contract which pays $S-K$ in time dt . One could think that its strike K should be defined by the “real world” transition probability p :

$$p(S_1-K)+(1-p)(S_2-K)=pS_1+(1-p)S_2-K$$

$$K_0 = pS_1 + (1-p)S_2$$

If $p=1/2$, $K_0=(S_1+S_2)/2$

Risk Neutral Valuation: Replicating Portfolio

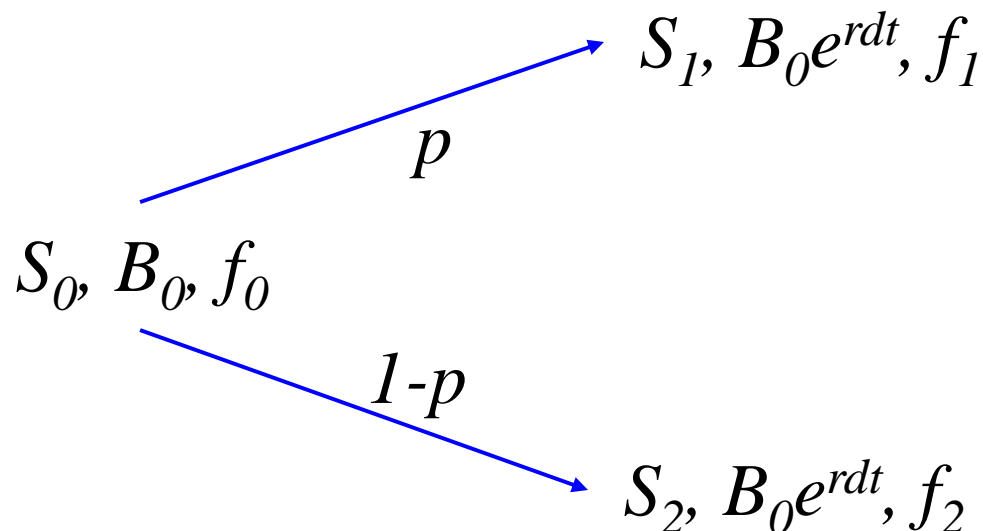
Consider the following strategy:

1. Borrow $\$S_0$ to buy the stock. Enter *Forward* contract with strike K_0
 2. In time dt deliver stock in exchange for K_0 and repay $\$S_0 e^{rdt}$
- If $K_0 > S_0 e^{rdt}$ we made riskless profit
 - If $K_0 < S_0 e^{rdt}$ we definitely lost money
- $\Rightarrow K_0 = S_0 e^{rdt}$

Current price of a derivative claim is determined by current price of a portfolio which exactly replicates the payoff of the derivative at the maturity

Risk Neutral Valuation: One step binomial tree

Suppose our economy includes stock S , riskless money market account B with interest rate r and derivative claim f . Assume that only two outcomes are possible in time dt :



Risk Neutral Valuation: One step binomial tree

For a general derivative claim f , find a and b such that

$$f_1 = aS_1 + bB_0e^{rdt}$$

$$f_2 = aS_2 + bB_0e^{rdt}$$

Then

$$f_0 = aS_0 + bB_0$$

Easy to see that

$$a = \frac{f_1 - f_2}{S_1 - S_2}, b = \frac{S_1f_2 - S_2f_1}{(S_1 - S_2)B_0e^{rdt}}$$

$$f_0 = e^{-rdt} \left(S_0e^{rdt} \frac{f_1 - f_2}{S_1 - S_2} + \frac{S_1f_2 - S_2f_1}{S_1 - S_2} \right)$$

Risk Neutral Valuation: One step binomial tree

One should notice that

$$f_0 = e^{-rdt} \left(f_1 \frac{S_0 e^{rdt} - S_2}{S_1 - S_2} + f_2 \frac{S_1 - S_0 e^{rdt}}{S_1 - S_2} \right)$$

$$f_0 = e^{-rdt} (f_1 q + f_2 (1 - q))$$

where

$$q = (S_0 e^{rdt} - S_2) / (S_1 - S_2), \quad 0 < q < 1$$

Moreover

$$S_1 q + S_2 (1 - q) = e^{rdt} S_0$$

Risk Neutral Valuation: Continuous case

$$f_t = e^{-r(T-t)} E_Q[f_T]$$

Q is the risk neutral (martingale) measure under which

$$S_0 = e^{-rt} E_Q[S_t]$$

Black-Scholes equation

Assume that the stock has log-normal dynamics:

$$dS = \mu S dt + \sigma S dW$$

Where dW is normally distributed with mean 0 and standard deviation \sqrt{dt} (i.e. W is a Brownian Motion)

We want to find a replicating portfolio such that

$$df = a dS + b dB$$

Black-Scholes equation

Use *Ito's formula*:

$$df(S, t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2$$

$$(dS)^2 = \sigma^2 S^2 dt$$

(analogous to first order Taylor expansion, up to dt term)

Black-Scholes equation

$$df = a dS + b dB$$

Substitute dS , df , $dB = rBdt$ and $(dS)^2$

$$\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dW = (a \mu S + b r B) dt + a \sigma S dW$$

Compare terms

$$a = \frac{\partial f}{\partial S}, \quad b r B = \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2$$

Black-Scholes equation

$bB = f - aS$ is deterministic and as $dB = rBdt$

$$d(f - aS) = r(f - aS)dt$$

Substituting once again $df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 dt$ and $a = \frac{\partial f}{\partial S}$

we obtain the **Black-Scholes equation**

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial f}{\partial S}rS - rf = 0$$

Fisher Black, Myron Scholes – paper 1973

Myron Scholes, Robert Merton – Nobel Prize 1997

Black-Scholes equation

- Any tradable derivative satisfies the equation
- There is no dependence on actual drift μ
- We have a hedging strategy (replicating portfolio)
- By a change of variables Black-Scholes equation transforms into heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

Black-Scholes equation

Boundary and *final* conditions are determined by the pay-off of a specific derivative

For European Call

$$C(S, T) = \max(S - K, 0)$$

$$C(0, t) = 0, C(\infty, t) \cong S$$

For European Put

$$P(S, T) = \max(K - S, 0)$$

$$P(0, t) = Ke^{-r(T-t)}, P(\infty, t) = 0$$

Black-Scholes equation

For European Call/Put the equation can be solved analytically

$$C_t = e^{-r(T-t)} \left(e^{r(T-t)} S N(d_1) - K N(d_2) \right)$$

$$P_t = e^{-r(T-t)} \left(K N(-d_2) - e^{r(T-t)} S N(-d_1) \right)$$

where

$$d_1 = \frac{\ln(S / K) + (r + \sigma^2 / 2)(T - t)}{\sigma \sqrt{T - t}}$$

$$d_2 = \frac{\ln(S / K) + (r - \sigma^2 / 2)(T - t)}{\sigma \sqrt{T - t}}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$$

Black-Scholes: Risk Neutral Valuation

$$f_t = e^{-r(T-t)} E_Q[f_T]$$

Q is the risk neutral measure under which

$$dS = rSdt + \sigma SdW$$

$$PDF(S_T) = \frac{1}{\sigma S \sqrt{2\pi T}} \exp\left(-\frac{(\ln(S_T / S_t) - (r - \sigma^2 / 2)(T - t))^2}{2\sigma^2(T - t)}\right)$$

Black-Scholes equation

For more complicated options or more general assumptions numerical methods have to be used:

- Finite difference methods
- Tree methods (equivalent to explicit scheme)
- Monte Carlo simulations

Black-Scholes equation: Conclusions

Modern financial services business makes use of

- PDE
- Numerical methods
- Stochastic Calculus
- Simulations
- Statistics
- Much, much more

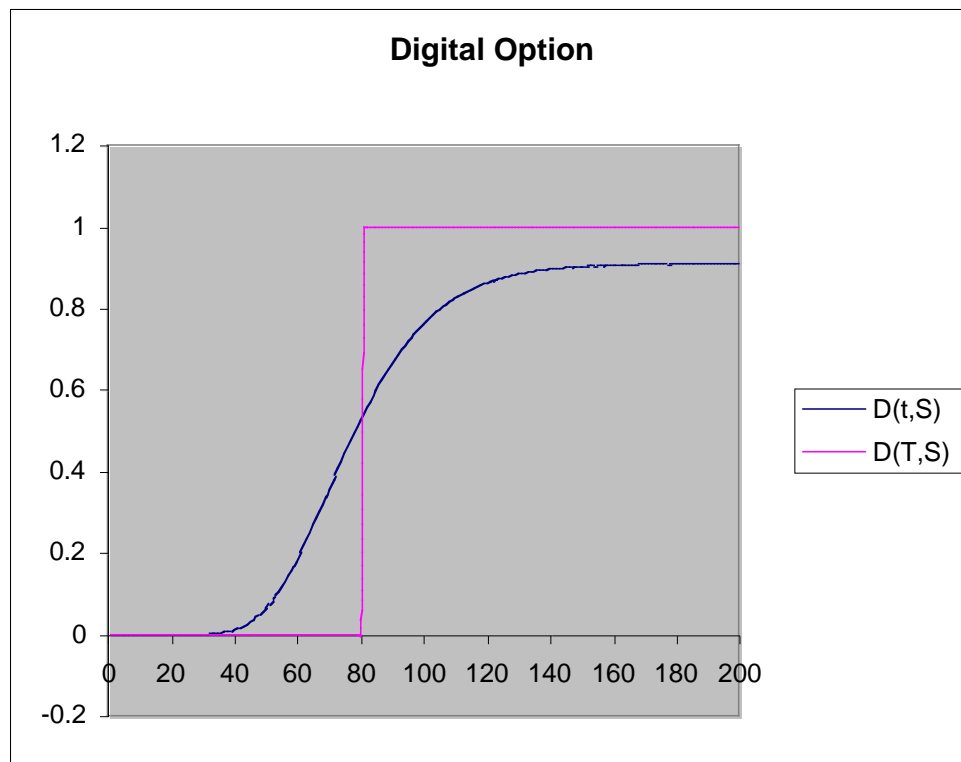
Risk Neutral Valuation: Example

IBM US \$ sC 81.14 +.85 B sho Equity OMON											
As of Mar8 DELAYED Vol 5,570,000 Op 80.27 T Hi 81.60 N Lo 80.25 P											
Template List		Edit		Contract Months		Security List		IBM US Equity Go			
Option Monitor: INTL BUSINESS MACHINES CORP											
Center 81.14 Number of Strikes 18 -or- % from Center Exchange C (Composite)											
CALLS						PUTS					
Ticker	Strike	Bid	Ask	Last	Volume	Ticker	Strike	Bid	Ask	Last	Volume
IBZ 22 APR 06 (Contract Size: 100)						IBZ 22 APR 06 (Contract Size: 100)					
1) IBZ+DH	40.00	41.30	41.50	41.10	y	18) IBZ+PH	40.00		.05		
2) IBZ+DI	45.00	36.40	36.50	35.50	y	19) IBZ+PI	45.00		.05		
3) IBZ+DJ	50.00	31.40	31.50	31.20	y	20) IBZ+PJ	50.00		.05	.05	y
IBM 22 APR 06 (Contract Size: 100)						IBM 22 APR 06 (Contract Size: 100)					
4) IBM+DK	55.00	26.40	26.60	25.40	y	21) IBM+PK	55.00		.05		
5) IBM+DL	60.00	21.40	21.60	21.20	y	22) IBM+PL	60.00		.05	.05	y
6) IBM+DM	65.00	16.50	16.60	16.30	y 3	23) IBM+PM	65.00		.05	.10	y
7) IBM+DN	70.00	11.60	11.70	11.50	y 21	24) IBM+PN	70.00	.05	.10	.10	y 10
8) IBM+DO	75.00	6.80	7.00	7.10	y 224	25) IBM+PO	75.00	.25	.35	.25	y 72
9) IBM+DP	80.00	2.90	3.00	2.90	y 1469	26) IBM+PP	80.00	1.25	1.35	1.30	y 1513
10) IBM+DQ	85.00	.70	.80	.80	y 1699	27) IBM+PQ	85.00	4.10	4.20	4.20	y 838
11) IBM+DR	90.00	.10	.15	.15	y 913	28) IBM+PR	90.00	8.70	8.90	9.40	y 5
12) IBM+DS	95.00		.05	.05	y 8	29) IBM+PS	95.00	13.70	13.90	14.80	y
13) IBM+DT	100.00		.05	.05	y	30) IBM+PT	100.00	18.70	18.90	19.90	y
14) IBM+DA	105.00		.05	.05	y	31) IBM+PA	105.00	23.70	23.90	23.70	y
15) IBM+DB	110.00		.05	.05	y	32) IBM+PB	110.00	28.70	28.90	29.80	y
16) IBM+DC	115.00		.05	.05	y	33) IBM+PC	115.00	33.70	33.90	34.80	y
17) IBM+DD	120.00		.05	.05	y	34) IBM+PD	120.00	38.70	38.90	39.80	y
Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410						Hong Kong 852 2977 6000 Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2006 Bloomberg L.P.					
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Risk Neutral Valuation: Example

Digital option pays 1 if $S > K$ at time T



$$S(t)=80, K=80, T=2 \text{ (years)}$$

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