x_r

$w^*+\alpha d$

$\bar{c}_r<0$

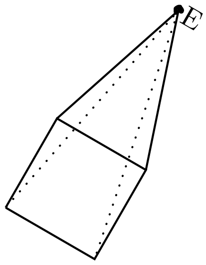
12 Lecture 12

[12.1] **Remark** It is clear that each nonbasic variable gives us a direction to move out of w^* and the directions given by the different nonbasic variables are linearly independent (solely due to the value 1 taken by that nonbasic variable). However, they may not result in an actual movement due to degeneracy. (We will see many examples later.)

Also they may not show you all possible available directions. For example, consider minimizing $x_1+x_2+x_3$ on the square pyramid with vertices $A(1,1,0)$, $B(2,1,0)$, $C(2,2,0)$, $D(1,2,0)$ and $E(1.5,1.5,1)$.

PT:

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	0	2	-1	-1	0	0	0	2
	2	0	1	0	1	0	0	4
	0	2	1	0	0	1	0	4
	2	0	-1	0	0	0	1	2
$*$	1	1	1	0	0	0	0	$*$



Point $E \to E^* = (1.5,1.5,1,0,0,0,0)$. We can take a basis (x_1,x_2,x_3,x_7). In any case, we will have only 3 nonbasic variables and they will show only three of the four available directions at E^*.

opt $\dfrac{c^Tx}{Ax=b,\ x\geq 0}$ $\quad w^* \to$ bfs, basis B $\quad$ ST

bv	$x_1 - x_r \cdots x_n$	$\bar b$
$\vdots$	$B^{-1}A=\bar A$	$B^{-1}b$
	$\bar c^T - c_B^T B^{-1}A$	$-f(w^*)$

$x_r \to$ non basic var. $d \to$ direction given by x_r :

$d_B \begin{bmatrix} -B^{-1}A_{:r} \end{bmatrix} \to \bar{A}_{:r}$

$d_c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow x_r$ pos

$f(w^*+\alpha d) = f(w^*) + \alpha \bar{c}_r$

[12.2] **Theorem** Let $A \in M_{m,n}$ have rank m. Consider $\begin{array}{ll}\text{opt} & \underline{f(x)=c^tx} \\ \text{s.t.} & Ax=b,\ x\geq 0.\end{array}$ Consider the simplex table at a bfs w^* of $Ax=b$ with the basis matrix B. Let x_r be a nonbasic variable.

a) Then the direction given by x_r is the vector

$$d \text{ with } d_B = -\overline{A}_{:r},\ d_r = 1, \text{ and other entries zero.}$$

b) Also $f(w^*+\alpha d) = f(w^*) + \alpha\bar{c}_r$. That is, for moving α units in the direction d given by x_r, the objective function $f(x)$ changes by $\alpha\bar{c}_r$.

[12.3] **Lemma** (Moving out of a vertex : unboundedness of objective function) Let rank $A \in M_{m,n}$ have rank m. Consider the simplex table at a bfs w^* of the slpp $Ax=b$. Suppose that $\bar{c}_r < 0$. If $\overline{A}_{:r} \leq 0$, then the objective function is not bounded below on the feasible set T^* of the slpp. (So you cannot get a minimum.)

$\bar{c}_r<0$, $-\bar{A}_{:r} \geq 0 \Rightarrow d \geq 0 \Rightarrow \boxed{w^*+\alpha d} \geq 0$

$f(w^*+\alpha d) = f(w^*) + \alpha\bar{c}_r \to -\infty$ as $\alpha \to \infty$

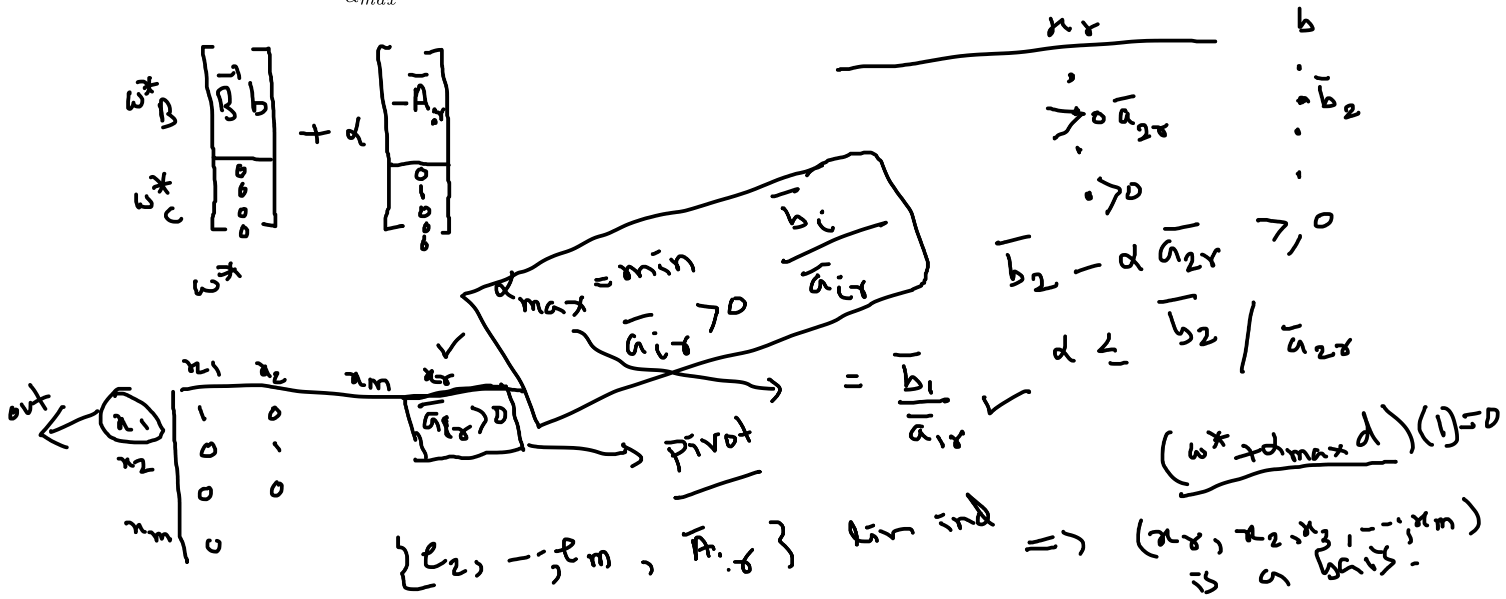
Proof. The direction d given by x_r is nonnegative. So $w^*_\alpha = w^* + \alpha d \in T^*$ for each $\alpha \geq 0$. Hence

$$f(w^*_\alpha) = c^t w^*_\alpha = f(w) + \alpha \overline{c}_r \to -\infty \qquad \text{as } \alpha \to \infty.$$

■

[12.4] <u>**Lemma**</u> `(Moving out of a vertex : better bfs)` Let $\mathsf{rank}\, A_{m\times n} = m$. Consider the slpp $Ax = b$ with a feasible set T^* and a simplex table at a bfs w^* of $Ax = b$. Suppose that $\overline{c}_r < 0$ and let d be the direction given by x_r. Assume that $\overline{A}_{:r} \not\leq 0$.

1. Then there is a maximum number $\alpha_{max} \geq 0$ such that for each $t \in [0, \alpha_{max}]$, we have $w^*_t \in T^*$.
2. The solution $w^*_{\alpha_{max}} = w^* + \alpha_{max} d$ is a better bfs.



Proof. As $\overline{A}_{:r}$ has a positive entry, we see that d has a negative entry and hence w^*_α will have negative entries for large values of α. Now α_{max} can be easily determined by

$$\alpha_{max} = \min_{\overline{a}_{i,r} > 0} \frac{\overline{b}_i}{\overline{a}_{i,r}}.$$

Hence $f(w^*_{\alpha_{max}}) = f(w^*) + \alpha_{max}\overline{c}_r \leq f(w^*)$.

We now show that $w^*_{\alpha_{max}}$ is also a bfs. For simplicity, assume that $(x_1, \dots, x_m)$ is the basis for w^* and $\alpha_{max} = \min_{\overline{a}_{i,r}>0} \frac{\overline{b}_i}{\overline{a}_{i,r}}$ is attained at $i = 1$. Consider the columns of $\overline{A}$ corresponding to $x_r, x_2, \cdots, x_m$. Since $\overline{a}_{1,r} > 0$, these columns are linearly independent. So $w^*_{\alpha_{max}}$ is a bfs. ■

Some exercises

[12.5] <u>**Exercise(E)**</u> Suppose that a minimum for $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, x \geq 0 \end{array}$ exists. Suppose also that we are at a nondegenerate bfs w and that $\overline{c}_r < 0$. Must we have $\delta > 0$? Can the new bfs be degenerate?

[12.6] <u>**Exercise(M)**</u> `(Converse of test of optimality.)` Let $\mathsf{rank}\, A_{m\times n} = m$. Consider the simplex table for the slpp $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, x \geq 0 \end{array}$ at a bfs w. The optimality test says that 'if $\overline{c} \geq 0$, then w is a point of minimum'.

a) Give an example to show that the converse is not true in general.

b) Argue that if w is a minimum nondegenerate bfs, then $\overline{c}$ must be nonnegative.

c) Conclude that, if $\overline{c}$ has a negative entry for a minimal bfs w, then w must be degenerate.

[12.7] **Exercise(E)** `(Forming the simplex table.)` Consider $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & A_{m\times n}x = b,\ x \geq 0 \end{array}$, rank $A = m$.
Let w be a bfs of $Ax = b$ for some basis. Students X and Y are forming the simplex table. Student X first converts A to $\overline{A}$ so that he has identity matrix under the basic variables. Then he takes c^t; finds entries of c^t corresponding to the basic variables; makes them zero by subtracting appropriate multiples of rows of $\overline{A}$. He claims that he has got the simplex table at the given basis. Student Y first takes c^t; finds entries of c^t corresponding to the basic variables; makes them zero by subtracting appropriate multiples of rows of A. Then he converts A to $\overline{A}$. He claims that he has got the simplex table at the given basis. Who is correct?

[12.8] **Application(M)** `(When does` $\{x \mid A_{m,n}x \leq 0\}$ `have a nontrivial point?)` We wish to find a nontrivial point in the set $P = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$. Convert this to an lpp.

[12.9] **Application(H)** `(Using lpp to separate two finite sets.)` Consider two sets $V = \{v_1, \ldots, v_k\}$ and $W = \{w_1, \ldots, w_m\}$ in $\mathbb{R}^n$. We wish to find a hyperplane $H : a^t x = b$ which separates them. Convert this to a lpp.

Simplex Algorithm

Developed by George Dantzig (dan-sig) in 1947, this algorithm assumes that we have an slpp with the matrix $A_{m\times n}$ of rank m and a bfs w to start with. At each bfs, it calculates $\overline{c}$ and checks for the minimality and unboundedness. If none of these are met, then the algorithm moves in a direction given by a nonbasic variable x_r for which $\overline{c}_r < 0$, and reaches another bfs (vertex).

Simplex Algorithm. Input: A, b, c^t and an initial bfs w.

a) Form a simplex table at w. Your table will be a permutation of rows of your friends table, depending on the order you have written the basic variables.

b) If $\overline{c}$ is nonnegative, then conclude that w is minimal. ✓

c) If $\overline{c}_r < 0$ and $\overline{A}_{:r} \leq 0$, then conclude that the problem is unbounded. ✓

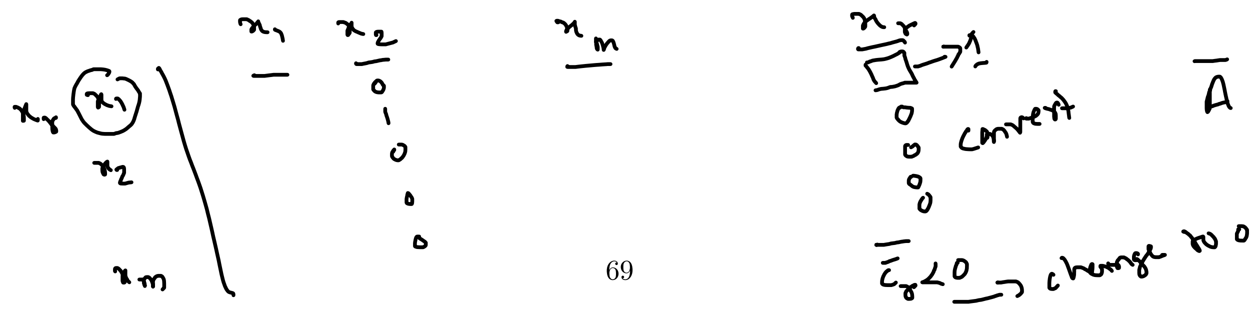
d) Otherwise, let x_r be a nonbasic variable to work with.[a] This x_r is called the INCOMING VARIABLE.

e) Evaluate $\delta = \min\limits_{\overline{a}_{i,r}>0} \frac{\overline{b}_i}{\overline{a}_{i,r}}$. (This is called the MINIMUM RATIO RULE.) If the minimum occurs at $\overline{a}_{i,r}$, then x_i is called the OUTGOING VARIABLE. The entry $\overline{a}_{i,r}$ is called a PIVOTAL ENTRY.

f) The new ordered basis is obtained by replacing x_i with x_r in the old ordered basis.

g) Form the new simplex table for the new basis. To do this, make the pivotal entry 1 and other entries in that column 0, using row operations on $\overline{A}$. Add a scalar multiple of the pivotal row to $\overline{c}$ so that the entry of $\overline{c}$ in the pivotal column is 0. This computes the relative cost $\overline{c}$. Go to b).

[a] A common practice here is to choose the x_r with $\overline{c}_r$ most negative. This has no theoretical justification. See [12.12].



[12.10] **Example** Consider $\min \; f(x_1, x_2) = -x_1 - x_2$ s.t. $x_1 + x_2 + x_3 = 2, x_1 - x_2 + x_4 = 1, x_2 + x_5 = 1, x_i \geq 0.$ Apply simplex method starting with the basis (x_3, x_4, x_5).

PT

bv	x_1	x_2	x_3	x_4	x_5	b
*	1	1	1	0	0	2
	1	-1	0	1	0	1
	0	1	0	0	1	1
	-1	-1	0	0	0	

ST

bv	x_1 ↓	x_2	x_3	x_4	x_5	$\bar{b}$
x_3	1	1	1	0	0	2
← x_4	$\boxed{1}$	-1	0	1	0	1
x_5	0	1	0	0	1	1
$-f$	-1	-1	0	0	0	0 → $-f(w^*)$

$w^* = (0, 0, 2, 1, 1) \to$ bfs, $f(w^*) = 0$

bv	x_1	x_2 ↓	x_3	x_4	x_5	$\bar{b}$
← x_3	0	$\boxed{2}$	1	-1	0	1
x_1	1	-1	0	1	0	1
x_5	0	1	0	0	1	1
$-f$	0	-2	0	1	0	1

$w^* = (1, 0, 1, 0, 1)$, $f(w^*) = -1$

bv	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_2	0	1	1/2	-1/2	0	1/2
x_1	1	0	1/2	1/2	0	3/2
x_5	0	0	-1/2	1/2	0	1/2
	0	0	1	0	0	2

$w^* = (\frac{3}{2}, \frac{1}{2}, 0, 0, \frac{1}{2})$ $f(w^*) = -2$

we are at a minimal bfs.

Answer. PT:

bv	x_1	x_2	x_3	x_4	x_5	b
$*$	1	1	1	0	0	2
$*$	1	-1	0	1	0	1
$*$	0	1	0	0	1	1
$*$	-1	-1	0	0	0	$*$

Simplex tables at the given basis and the continuations using simplex method are shown below.

bv	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_3	1	1	1	0	0	2
x_4	1	-1	0	1	0	1
x_5	0	$\boxed{1}$	0	0	1	1
$-f$	-1	-1	0	0	0	-0

$\to$

bv	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_3	$\boxed{1}$	0	1	0	-1	1
x_4	1	0	0	1	1	2
x_2	0	1	0	0	1	1
$-f$	-1	0	0	0	1	1

$\to$

bv	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	0	1	0	-1	1
x_4	0	0	-1	1	2	1
x_2	0	1	0	0	1	1
$-f$	0	0	1	0	0	2

As $\bar{c} \geq 0$, we have reached the optimal solution. The minimal solution is $\begin{bmatrix}1 & 1 & 0 & 1 & 0\end{bmatrix}^t$, and the value of the function at this point is -2.

Some exercises

[12.11] **Exercise(M)** (`$\bar{c}^t$ may have a negative entry at a minimal basis`) Give an example of a simplex table at a basis, where the value of the objective function is minimum but the relative cost vector has a negative entry.

[12.12] **Exercise(M)** (`Most negative entry in $\bar{c}$ is not always good.`)

a) Consider $\min\ x_2 - 5x_1$ Draw the feasible region T.
s.t. $x_1 \le 2, x_2 \le 2, x_1 + x_2 \le 3, x_1 - x_2 \le 3, x_i \ge 0.$

b) Write the problem table and the basic solution that corresponds to $x_1 = 0, x_2 = 2$ specifying a basis. Choose the natural ordering while writing the basis.

c) Apply simplex algorithm starting from the basic solution obtained in b), choosing the most negative element in the relative cost.

d) Apply simplex algorithm starting from the basic solution obtained in b), NOT choosing the most negative element in the relative cost.

e) Is it true that choosing the most negative entry in $\bar{c}$ brings you to the minimum quickly?

f) In both the cases show the paths taken on the feasible region T.

g) Observe that $(2, -1)$ is a point lying outside the feasible region. What is the corresponding point of the slpp? Write a basis for this point and write the simplex table at this basis.

Observe: This point, though basic, is not a feasible solution. If we can move out of this point in a direction supplied by a nonbasic variable, then the value of the function cannot decrease. That is, if this point was a feasible solution, then the function would have been minimized at this point.

[12.13] **Exercise(E)** (`No way to guess the best outgoing variable, if there is a tie.`) Suppose that we are using simplex method and we are at the following table. We face a tie while choosing the outgoing variable. Continue from here in both the ways. See that one way is shorter and the other is longer, but there is no way to guess these beforehand.

bv	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_4	0	0	1	1	0	0	1
x_5	0	-2	1	0	1	0	0
x_6	0	2	-1	0	0	1	4
x_1	1	1	-1	0	0	0	2
$-f$	0	-1	0	0	0	0	-0

[12.14] **Exercise(E)** Fill in the blanks to compute the simplex table from the problem table for the ordered basis (x_2, x_5, x_6, x_1).

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{b}$
$*$	1	1	0	1	0	-1	0	3
$*$	1	-1	0	0	1	0	0	2
$*$	-1	1	0	0	0	1	-1	2
$*$	1	1	-1	0	0	0	1	2
$*$	-1	-1	1	0	0	1	0	$*$

$\rightarrow$

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{b}$
x_2				$\frac{1}{2}$				$\frac{5}{2}$
x_5				1				5
x_6				-1				-1
x_1				$-\frac{1}{2}$				$-\frac{1}{2}$
$*$				1				3

[12.15] **Exercise(E)** `(Understanding the simplex table.)` Suppose that we have only one $-$ve entry in $\overline{c}$ in a simplex table. Can we get more $-$ve entries in the next simplex table? To answer that, take the unit cube with two corners at $(0,0,0)$ and $(1,1,1)$. Cut it with the plane that passes through $(.5,0,0),(1,.5,0),(1,0,.5)$ and throw the smaller piece away. Let S be the the larger piece. We want to minimize x_3 over S.

a) Fill the table to write the problem table.

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\overline{b}$
$*$	1							1
$*$		1						1
$*$			1					1
$*$							1	
$-f$								0

b) Write the simplex table for the vertex $(1,0,1)$. How many negative entries are there in $\overline{c}$?

c) Do you have 3 nonbasic variables? Write the directions vectors given by these three variables.

d) Which nonbasic variable corresponds to the movement from $(1,0,1)$ to $(0,0,1)$? What is the relative cost for this variable? What does that value mean?

e) Which nonbasic variable corresponds to the movement from $(1,0,1)$ to $(1,0,.5)$? What is the relative cost for this variable? What does that value mean?

f) I think the simplex table at $(1,0,.5)$ should have two $-$ve entries in $\overline{c}$? Why do I think so? Explain without using simplex tables.

[12.16] **NoPen** Suppose that we have a simplex table of $\begin{array}{ll}\min & \underline{c^t x} \\ \text{s.t.} & Ax=b, x\geq 0.\end{array}$ Which of these are true?

a) $c=0 \Rightarrow \overline{c}=0$.

b) $\overline{c}=0 \Rightarrow c=0$.

Cycling in simplex method

Does simplex algorithm terminate in finitely many steps? In general, NO.

If we assume that at each stage we get $\delta>0$ (this can happen if we have a nondegenerate bfs or for some other reasons too), then we get a strictly cheaper bfs taking us to a different vertex each time. As there are finitely many vertices (at most $\binom{m}{n}$ as we know), the process will stop in finitely many steps.

But what if we get $\delta=0$ for sequence of bfs's (must be degenerate)? It may so happen that the algorithm keeps on visiting them again and again, without ever reaching an optimal bfs. This occurrence is called CYCLING. Geometrically, it means that, the algorithm stays at the same (physical) vertex and only keeps on selecting a set of bases $B_1, B_2, \ldots, B_k, B_1$ in the following way.

At B_1 it sees a negative entry in $\overline{c}$, finds a new direction to move out, but as $\delta=0$, it only moves 0 amount in that direction. That is, it stays at the same vertex, only it changes its basis to B_2. At B_2 it does the same work, goes to B_3. It continues to get B_k and from B_k, it finds B_1. In order to avoid cycling, the following rule was suggested.

Bland's rule (smallest subscript rule) to avoid cycling In the case of a degenerate bfs do the following.
a) Select the entering variable with smallest possible subscript (an x_j with j smallest).
b) Select the outgoing variable with smallest possible subscript.

[12.17] **Theorem** (R. G. Bland, 1977.) Following the smallest subscript rule, the simplex method terminates in finitely many steps. Thus, if a minimum exists then there exists a bfs w with nonnegative relative cost factors.

Proof. (Self) Suppose that we have applied the rule and yet we get a cycle given by the bases $B_1, B_2, \ldots, B_k, B_1$ at a vertex w. Call a variable 'temporary', if it is present in some of these bases but not in all. Let x_t be the temporary variable with the largest index.

Select a stage where x_t goes out, say, B_1. Assume x_s enters here. (This means both x_s and x_t are temporary and so $s < t$.) After B_1, there will be a first stage where x_t enters, say at B_p.

Let (T1) and (Tp) denote the simplex tables at bases B_1 and B_p, respectively. Let $\overline{c}$ and $\tilde{c}$ be the relative cost vectors below the tables (T1) and (Tp), respectively.

$$(T1)\quad \begin{array}{c|ccccc|c} \text{bv} & x_1 & \cdots & x_s & \cdots & x_n & \overline{b} \\ \hline \vdots & & & \vdots & & & \vdots \\ x_r & & & \overline{a}_{is} & & & \overline{b}_i \\ \vdots & & & \vdots & & & \vdots \\ \hline -f & & & \overline{c}_s & & & -f(w) \end{array} \qquad (Tp)\quad \begin{array}{c|ccccc|c} \text{bv} & x_1 & \cdots & x_s & \cdots & x_n & \overline{b} \\ \hline \vdots & & & \vdots & & & \vdots \\ B_p & & & B_p^{-1}A_{:s} & & & B_p^{-1}b \\ \vdots & & & \vdots & & & \vdots \\ \hline -f & & & \tilde{c}_s & & & -f(w) \end{array}$$

Treating (Tp) as a problem table and taking the basis B_1, as x_s is entering, we see that in (T1), we have

$$\overline{c}_s = \tilde{c}_s - \tilde{c}_{B_1}^t B_1^{-1} A_{:s} < 0, \quad \text{that is,} \quad \tilde{c}_{B_1}^t B_1^{-1} A_{:s} > \tilde{c}_s \geq 0, \tag{9}$$

as in (Tp), x_t is entering and $s < t$. (If $\tilde{c}_s < 0$, then it should enter not x_t.)

Hence, by (9), we get $\tilde{c}_{B_1}^t B_1^{-1} A_{:s} > 0$. Hence, there is an index i such that $(\tilde{c}_{B_1}^t)_i (B_1^{-1} A_{:s})_i > 0$. Assume that, $(\tilde{c}_{B_1}^t)_i$ corresponds to the variable x_r. (This means x_r is a basic variable appearing at ith row in (T1).) Thus

$$\tilde{c}_r \overline{a}_{is} > 0. \tag{10}$$

As $\tilde{c}_r \neq 0$, we see that x_r is nonbasic at (Tp). So, x_r is a temporary variable and $r \leq t$.

Assume first that, $r = t$. In (Tp) x_t enters. So $\tilde{c}_t < 0$. So $\overline{a}_{is} < 0$. Hence in (T1), $x_r = x_t$ (appearing in ith row) cannot go out. A contradiction.

Now let $r < t$. At stage (Tp), we chose x_t to enter (not x_r). So by (10) $\tilde{c}_r > 0$ and so $\overline{a}_{is} > 0$. But then it means that x_r was available to be the outgoing variable in (T1). (Recall that as x_r and x_t are temporary variables, their values remain 0 throughout all the tables.) But we chose x_t. This is a contradiction. ■

[12.18] **Example** (Illustrate) Consider the problem given by the problem table below.

Problem table

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
*	1	2	3	4	1	0	0	0	0
*	4	1	2	3	0	1	0	0	0
*	3	4	1	2	0	0	1	0	0
*	2	3	4	1	0	0	0	1	0
$-f$	1	1	1	1	0	0	0	0	*

Verify that there is a possibility that the computer might keep on visiting the following sequence of bases; (x_6, x_2, x_1, x_8), (x_6, x_2, x_7, x_8), (x_4, x_2, x_7, x_8), (x_4, x_5, x_7, x_8), (x_4, x_5, x_1, x_8), (x_6, x_5, x_1, x_8), and back to the first basis. Check how Bland's rule would have helped you. You need to do the computations at home.

Answer.

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{b}$
x_6	0	0	18	22	$\frac{13}{2}$	1	$\frac{-7}{2}$	0	0
x_2	0	1	4	5	$\frac{3}{2}$	0	$\frac{-1}{2}$	0	0
x_1	1	0	-5	-6	-2	0	$\boxed{1}$	0	0
x_8	0	0	2	-2	$\frac{-1}{2}$	0	$\frac{-1}{2}$	1	0
$-f$	0	0	2	2	$\frac{1}{2}$	0	$\frac{-1}{2}$	0	0

$\rightarrow$

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{b}$
x_6	$\frac{7}{2}$	0	$\frac{1}{2}$	$\boxed{1}$	$\frac{-1}{2}$	1	0	0	0
x_2	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{1}{2}$	0	0	0	0
x_7	1	0	-5	-6	-2	0	1	0	0
x_8	$\frac{1}{2}$	0	$\frac{-1}{2}$	-5	$\frac{-3}{2}$	0	0	1	0
$-f$	$\frac{1}{2}$	0	$\frac{-1}{2}$	-1	$\frac{-1}{2}$	0	0	0	0

$\rightarrow$

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{b}$
x_4	$\frac{7}{2}$	0	$\frac{1}{2}$	1	$\frac{-1}{2}$	1	0	0	0
x_2	$\frac{-13}{2}$	1	$\frac{1}{2}$	0	$\boxed{\frac{3}{2}}$	-2	0	0	0
x_7	22	0	-2	0	-5	6	1	0	0
x_8	18	0	2	0	-4	5	0	1	0
$-f$	4	0	0	0	-1	1	0	0	0

$\rightarrow$

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{b}$
x_4	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	0
x_5	$\frac{-13}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	1	$\frac{-4}{3}$	0	0	0
x_7	$\boxed{\frac{1}{3}}$	$\frac{10}{3}$	$\frac{-1}{3}$	0	0	$\frac{-2}{3}$	1	0	0
x_8	$\frac{2}{3}$	$\frac{8}{3}$	$\frac{10}{3}$	0	0	$\frac{-1}{3}$	0	1	0
$-f$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	$\frac{-1}{3}$	0	0	0

$\rightarrow$

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{b}$
x_4	0	-13	2	1	0	$\boxed{3}$	-4	0	0
x_5	0	44	-4	0	1	-10	13	0	0
x_1	1	10	-1	0	0	-2	3	0	0
x_8	0	-4	4	0	0	1	-2	1	0
$-f$	0	4	0	0	0	-1	1	0	0

$\rightarrow$

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{b}$	
x_6	0	$\frac{-13}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	1	$\frac{-4}{3}$	0	0	x_6
x_5	0	$\boxed{\frac{2}{3}}$	$\frac{8}{3}$	$\frac{10}{3}$	1	0	$\frac{-1}{3}$	0	0	x_2
x_1	1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0	0	x_1
x_8	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{-1}{3}$	0	0	$\frac{-2}{3}$	1	0	x_8
$-f$	0	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	$\frac{-1}{3}$	0	0	

We are back to the basis we started with. According to Bland's rule x_3 enters in the second table and x_2 goes out. The corresponding bases are $(x_6, x_2, x_1, x_8) \rightarrow (x_6, x_2, x_7, x_8) \rightarrow (x_6, x_3, x_7, x_8) \rightarrow (x_6, x_4, x_7, x_8) \rightarrow (x_6, x_5, x_7, x_8)$, at which $\overline{c} \geq 0$.

Some exercises

[12.19] <u>**Exercise(M)**</u> Is it true that cycling will occur only at a vertex of minimum cost?

[12.20] <u>**Exercise(E)**</u> `(If` w `is a minimal vertex, must there exist a basis for which` $\overline{c}$ `is nonnegative?)` Suppose that w is a vertex of $P = \{x \mid Ax = b,\ x \geq 0\}$ at which $f(x) = c^t x$ is minimum over P. Show that there exists a choice of basis such that the simplex method terminates at w.

[12.21] <u>**NoPen**</u> If `rank` $A_{m\times n} = m$, then with the Bland's rule, the simplex algorithm must stop within _____ iterations.

[12.22] <u>**Exercise(M)**</u> `(Matlab/Octave code for simplex table)` Let $A_{m\times n}$ have rank m. Consider a slpp min $c^t x$ s.t. $Ax = b, x \geq 0.$ Suppose that we already have the information that $x_i, i \in S$ form a basis. We are interested to see the simplex table for this basis, by giving the inputs A, b, c, S. Write a Matlab program to generate this table.