Example 0.1. *Let $\Omega = [0,1]$, $\mathcal{F} = \sigma-$algebra generated by closed intervals. Now suppose we define another probability measure $\tilde{\mathbb{P}}$ by*

$$\tilde{\mathbb{P}}[a,b] = \int_a^b 2\omega d\omega = b^2 - a^2.$$

Then $\tilde{\mu}_X[a,b] = b^2 - a^2$, whereas $\tilde{\mu}_Y[a,b] = (1-a)^2 - (1-b)^2$. Thus under $\tilde{\mathbb{P}}$, X and Y does not have the same distribution.

Definition 0.2. *The distribution function of a random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the function $F_X : \mathbb{R} \to [0,1]$ given by*

$$F_X(x) = \mathbb{P}(X \leq x)\,.$$

Proposition 0.3. *The distribution function of a random variable has the following properties:*

(1) $F_X(\cdot)$ *is non-decreasing and hence has only jump discontinuities.*

(2) $\lim\limits_{x\uparrow\infty} F_X(x) = 1,\ \lim\limits_{x\downarrow -\infty} F_X(x) = 0.$

(3) $\lim\limits_{h\downarrow 0} F_X(x+h) = F_X(x), \forall x \in \mathbb{R}$*, thus CDF is right continuous.*

(4) $\lim\limits_{h\downarrow 0} F_X(x-h) = F_X(x) - P(X = x), \forall x \in \mathbb{R}.$

Theorem 0.4. *Let F be a function from $\mathbb{R}$ to $[0,1]$ satisfying the properties of the above proposition, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X defined on it whose distribution function is F.*

Two Special Cases

• There exists a non-negative function f on $\mathbb{R}$ such that

$$\mu_X[a,b] = \mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx\,.$$

Thus

$$1 = \mathbb{P}(X \in \mathbb{R}) = \lim_{n\to\infty} \mathbb{P}(-n \leq X \leq n) = \lim_{n\to\infty} \int_{-n}^{n} f(x)dx = \int_{-\infty}^{\infty} f(x)dx\,.$$

• X takes only countably many values x_i. Define $p_i = \mathbb{P}(X = x_i)$. Then

$$\mu_X(B) = \sum_{\{i : x_i \in B\}} p_i\,.$$

In the first case X is said to have an absolutely continuous distribution with probability density function f and in the second case X is said to have a discrete distribution with probability mass function $\{p_i\}$.

Example: Consider the functions:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,, N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy\,.$$

Let X be uniformly distributed on $[0,1]$. Notice that N is a strictly increasing function. So it has an inverse N^{-1}. Define the random variable $Z = N^{-1}(X)$. Then

$$\begin{aligned}
\mu_Z[a,b] &= \mathbb{P}(\omega \in \Omega : a \leq Z(\omega) \leq b) \\
&= \mathbb{P}(\omega \in \Omega : a \leq N^{-1}(X)(\omega) \leq b) \\
&= \mathbb{P}(\omega \in \Omega : N(a) \leq NN^{-1}(X)(\omega) \leq N(b)) \\
&= \mathbb{P}(\omega \in \Omega : N(a) \leq X(\omega) \leq N(b)) \\
&= N(b) - N(a) = \int_a^b \varphi(x)dx\,.
\end{aligned}$$

The measure μ_X on $\mathbb{R}$ given by this formula is called the standard normal distribution. Any random variable that has this distribution, regardless of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which it is defined, is called a standard normal random variable.

0.1 Expectation

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We want to compute an "average value" of X, where we take the probabilities into account while computing the average.
If Ω is countable then we can simply define

$$\text{average value" of X} := \mathbb{E}(X) := \sum_{k=0}^{\infty} X(w_k)\mathbb{P}(X = w_k)\,,$$

where $\Omega = \{w_1, w_2, \ldots\}$
But if Ω is uncountable then we must think in terms of integrals.

1 Riemann integration

Partition: Let $[a,b]$ be a closed and bounded interval. A partition of $[a,b]$ is a finite sequence $P = (x_0, x_1, \cdots, x_n)$ of points of $[a,b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$. The family of all partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$ and the partition $P = (x_0, x_1, \cdots, x_n)$ is a member of $\mathcal{P}[a,b]$.
For example, $P = (0, 1/4, 1/3, 1/2, 2/3, 3/4, 1)$ is a partition of $[0,1]$, $Q = (0, 1/4, 3/8, 1/2, 3/4, 7/8, 1)$ is another partition of of $[0,1]$.
Riemann sums:- Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. Let $P \in \mathcal{P}[a,b]$ (i.e, $P = (x_0, x_1, \cdots, x_n)$, where $a = x_0 < x_1 < \cdots < x_n = b$). Since f is bounded on $[a,b]$, f is bounded on $[x_{r-1}, x_r]$, for $r = 1, 2, \cdots, n$. Let $M = \sup_{x\in[a,b]} f(x)$, $m = \inf_{x\in[a,b]} f(x)$; $M_r = \sup_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf_{x\in[x_{r-1},x_r]} f(x)$; for $r = 1, 2, \cdots, n$. Then $m \leq m_r \leq M_r \leq M$, for $r = 1, 2, \cdots, n$. The sum $U(P,f) := \sum_{i=1}^{n} M_r(x_r - x_{r-1})$ is said to be the upper Riemann sum and the sum $L(P,f) := \sum_{i=1}^{n} m_r(x_r - x_{r-1})$ is said to be lower Riemann sum.

Here $U(P,f)$ is the blue shaded area (region) and $L(P,f)$ is the red shaded area (region) of Figure 1. Note that $m(x_r - x_{r-1}) \leq m_r(x_r - x_{r-1}) \leq M_r(x_r - x_{r-1}) \leq M(x_r - x_{r-1})$, for $r = 1, 2, \cdots, n$. Therefore,

$$m\sum_{r=1}^{n}(x_r - x_{r-1}) \leq \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \leq \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \leq M\sum_{r=1}^{n}(x_r - x_{r-1}),$$

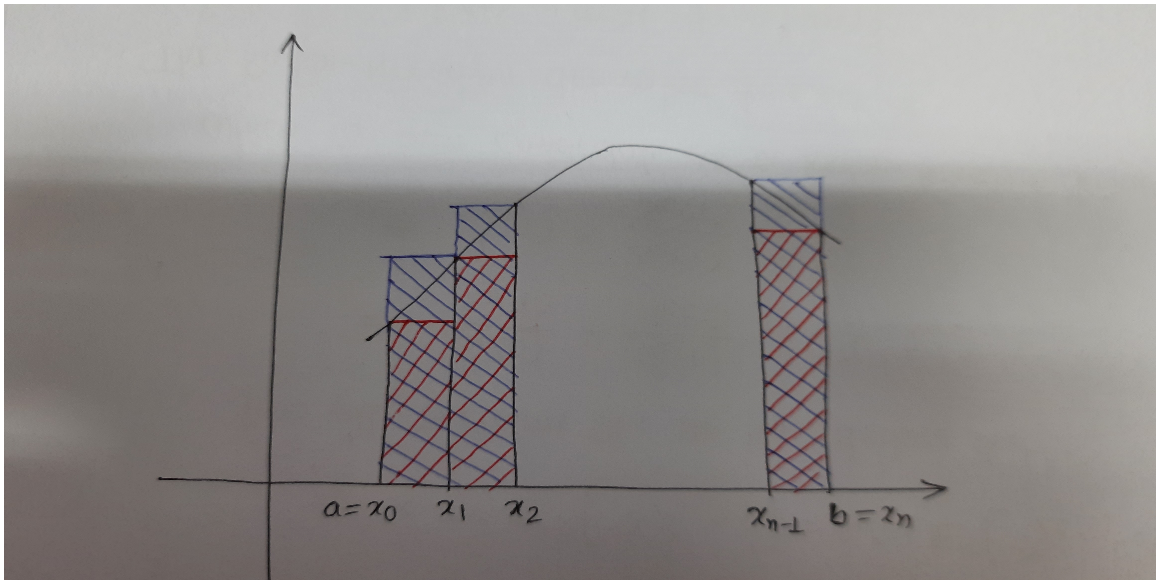

Figure 1:

or, $m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$. We have two sets of real numbers $\{U(P,f) : P \in \mathcal{P}[a,b]\}$ and $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ both sets are bounded. The supremum of the set $\{L(P,f) : P \in P \in \mathcal{P}[a,b]\}$ exists and it is called the lower integral of f on $[a,b]$ and is denoted by $\underline{\int_a^b} f(x)dx$. The infimum of the set $\{U(P,f); P \in P \in \mathcal{P}[a,b]\}$ exists and it is called the upper integral of f on $[a,b]$ and is denoted by $\overline{\int_a^b} f(x)dx$. f is said to be Riemann integral on $[a,b]$ if

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

The common value is called the Riemann integral of f on $[a,b]$ and it is denoted by $\int_a^b f(x)dx$.

Exercise:-

(1) Let $f(x) = c$, $x \in [a,b]$. Prove that f is Riemann integral on $[a,b]$.

(2) A function f is defined on $[0,1]$ by

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational.} \end{cases}$$

Show that f is not Riemann integral on $[0,1]$.

(3) Prove that the function f is defined on $[a,b]$ by $f(x) = x$, $x \in [a,b]$ is Riemann integral on $[a,b]$. Evaluate $\int_a^b f(x)dx$.

(4) $f(x) = x^2$.

(5) $f(x) = e^x$.

Refinement of a partition:- Let $P = (x_0, x_1, x_2, \cdots, x_n)$ be a partition of $[a,b]$. A partition Q of $[a,b]$ is said to be a refinement of P if P is a proper subset of Q. That is Q is obtained by adjoining a finite number of additional points to P.

For example, let $P = (0, 1/4, 1/2, 3/4, 1)$ be a parttion of $[0, 1]$ and $Q = (0, 1/8, 1/4, 1/2, 3/4, 7/8, 1)$, then Q is a refinement of P. If $R = (0, 1/8, 1/4, 3/8, 1/2, 3/4, 1)$, then R is a refinement of P but not a refinement of Q.

Lemma 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and P be a partition of $[a, b]$. If Q is a refinement of P, then $U(P, f) \geq U(Q, f)$ and $L(P, f) \leq L(Q, f)$.*

Norm of partition:- Let $P = (x_0, x_1, \cdots, x_n)$ be a partition of $[a, b]$. Then norm of a partition denoted by $\|P\|$, is defined by

$$\|P\| = \max_{r \in \{1,2,\cdots,n\}} |x_r - x_{r-1}|.$$

If Q is a refinement of P, then $\|Q\| \leq \|P\|$.

Lemma 1.2. *Let $f : [a, b] \to \mathbb{R}$ be bounded. If $\{P_n\}$ is a sequence of partition of $[a, b]$ such that $\|P_n\| \to 0$, then*

(i) $\lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f$

(ii) $\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f.$

Condition for integrability:- Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there exists a partition of P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ iff for each $\varepsilon > 0$ there exists a positive δ such that

$$U(P, f) - L(P, f) < \varepsilon$$

for every partition P of $[a, b]$ satifying $\|P\| \leq \delta$.

Properties:-

(1) Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be both Riemann integrable on $[a, b]$. Then $f + g$ is Riemann integrable on $[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

(2) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and $c \in \mathbb{R}$. Then cf is integrable on $[a, b]$ and $\int_a^b cf = c \int_a^b f$.

(3) $|f|$, f^2, $f \cdot g$ are Riemann integrable. If $g \geq k > 0$ then $1/g$ is also Riemann integrable.

Ex. A function f is defined by $f(x) = x^2$, $x \in [a, b]$, where $a > 0$. Find $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$. Deduce that f is integrable on $[a, b]$.

Ans:- f is bounded on $[a, b]$. Let $P_n = (a, a + h, a + 2h, \cdots, a + nh)$ where $h = \frac{b-a}{n}$. Then P_n is partition of $[a, b]$ with $\|P_n\| = \frac{b-a}{n}$. Since f is increasing function on $[a, b]$,
$M_r = (a + rh)^2$, $m_r = [a + (r - 1)h]^2$ for $r = 1, 2, \cdots, n$.

$$\begin{aligned}
U(P_n, f) &= h\Big[(a + h)^2 + (a + 2h)^2 + \cdots + (a + nh)^2\Big] \\
&= h\Big[(a^2 + a^2 + \cdots + a^2) + 2ah(1 + 2 + 3 + \cdots + n) + h^2(1^2 + 2^2 + 3^3 + \cdots + n^2)\Big] \\
&= h\left[na^2 + 2ah\frac{n(n + 1)}{2} + h^2\frac{n(n + 1)(2n + 1)}{6}\right] \\
&= nha^2 + anh(nh + h) + \frac{nh(nh + h)(2nh + h)}{6} \\
&= (b - a)a^2 + a(b - a)^2(1 + \frac{1}{n}) + \frac{1}{6}(b - a)^3(1 + \frac{1}{n})(2 + \frac{1}{n})
\end{aligned}$$

and

$$\begin{aligned} L(P_n, f) &= h\left[a^2 + (a+h)^2 + (a+2h)^2 + \cdots + (a+(n-1)h)^2\right] \\ &= h\left[na^2 + 2ah\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6}\right] \\ &= nha^2 + anh(nh-h) + \frac{nh(nh-h)(2nh-h)}{6} \\ &= (b-a)a^2 + a(b-a)^2(1-\frac{1}{n}) + \frac{1}{6}(b-a)^3(1-\frac{1}{n})(2-\frac{1}{n}). \end{aligned}$$

Consider the sequence of partitions $\{P_n\}$ of $[a,b]$ with $\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \frac{b-a}{n} = 0$. Then $\overline{\int_a^b} f(x)dx = \lim_{n\to\infty} U(P_n, f) = (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3-a^3}{3}$ and

$$\begin{aligned} \underline{\int_a^b} f(x)dx &= \lim_{n\to\infty} L(P_n, f) \\ &= (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3-a^3}{3}. \end{aligned}$$

As $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, f is integrable on $[a,b]$ and $\int_a^b f(x)dx = \frac{b^3-a^3}{3}$.
Ex. A function f is defined on $[0,1]$ by

$$f(x) = \begin{cases} x \text{ if } x \in [0,1] \cap \mathbb{Q} \\ 0 \text{ if } x \in [0,1] \backslash \mathbb{Q}.. \end{cases}$$

Find $\underline{\int_0^1} f(x)dx$ and $\overline{\int_0^1} f(x)dx$. Deduce that f is not integrable on $[0,1]$.
Ans:- f is bounded on $[0,1]$. Let us take the partition P_n of $[0,1]$ defined by $P_n = (0, 1/n, 2/n, \cdots, n/n)$. Let $M_r = \sup_{x\in[\frac{r-1}{n}, \frac{r}{n}]} f(x)$, $m_r = \inf_{x\in[\frac{r-1}{n}, \frac{r}{n}]} f(x)$, for $r = 1, 2, \cdots, n$. Then $M_r = r/n$ and $m_r = 0$ for $r = 1, 2, \cdots n$.

$$\begin{aligned} U(P_n, f) &= M_1(\frac{1}{n} - 0) + M_2(\frac{2}{n} - \frac{1}{n}) + \cdots M_n(\frac{n}{n} - \frac{n-1}{n}) \\ &= \frac{1}{n}[\frac{1}{n} + \frac{2}{n} + \cdots \frac{n}{n}] \\ &= \frac{n(n+1)}{2n^2} = \frac{n+1}{2n} \end{aligned}$$

and

$$L(P_n, f) = m_1(\frac{1}{n} - 0) + m_2(\frac{2}{n} - \frac{1}{n}) + \cdots m_n(\frac{n}{n} - \frac{n-1}{n}) = 0.$$

Let us consider the sequence of partitions $\{P_n\}$ of $[0,1]$ with $\|P_n\| = \frac{1}{n}$ and $\lim_{n\to\infty} \|P_n\| = 0$. Then $\lim_{n\to\infty} U(P_n, f) = \overline{\int_0^1} f(x) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$ and $\lim_{n\to\infty} L(P_n, f) = \underline{\int_0^1} f(x)dx = 0$. Since $\overline{\int_0^1} f(x)dx \neq \underline{\int_0^1} f(x)dx$, f is not Reimann integrable on $[0,1]$.