9 Lecture 9

[9.1] **Theorem** (`Bfs means a vertex`) A point w is a bfs of $Ax = b$ iff w is a vertex of the polyhedron $T = \{x \mid Ax = b, x \geq 0\}$.

Vertex v of $T = \{x \mid \underbrace{\begin{bmatrix} A \\ -A \\ I \end{bmatrix}}_{\tilde{A}} x \geq \underbrace{\begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}}_{\tilde{b}} \}$

rank $\tilde{A}_v = n$. $\tilde{A}_v$ contain A, $-A$, some rows of I

Suppose $v_{k+1}, \dots, v_n = 0$.

Then

cols are lin ind.

$$\begin{array}{c} k \text{ cols} \\ \begin{bmatrix} A_1 & A_2 \\ 0 & I_{n-k} \end{bmatrix} \end{array} \longrightarrow \text{rank } n$$

cols of A corresp to +ve entries of v are lin ind.
$\Rightarrow v$ is a bfs.

Proof. Let w be a bfs of $Ax = b$ with a basis matrix, say, B. For simplicity, assume $A = [B|C]$. First write $T = \{x \mid A'x \leq b'\}$, where $A' = \begin{bmatrix} A \\ \hline -A \\ \hline -I \end{bmatrix} = \begin{bmatrix} B & C \\ \hline -B & -C \\ \hline -I & 0 \\ \hline 0 & -I \end{bmatrix}$ and $b' = \begin{bmatrix} b \\ \hline -b \\ \hline 0 \end{bmatrix}$. The submatrix $\begin{bmatrix} B & C \\ \hline 0 & -I \end{bmatrix}$ has rank n and rows in this matrix correspond equalities in $A'w \leq b'$. By [5.22], w is a vertex of T.

Conversely, let w be a vertex of T. This means $A'w \leq b'$ and A'_w has rank n. Notice that $A'w \leq b'$ itself implies that $Aw = b$. Hence A'_w must contain A, $-A$ and a few rows from $-I$. For simplicity, assume that A'_w contains the last $n - k$ rows of $-I$. (This means $w_{k+1} = \cdots = w_n = 0$.) As the rows in A'_w which are from $-A$ are negatives of those which are from A, we see that the matrix obtained from A'_w by deleting the rows of $-A$, also has rank n. But the matrix must look like

$$\left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ \vdots & & & & & \\ a_{m1} & \cdots & a_{mk} & a_{m,k+1} & \cdots & a_{mn} \\ \hline 0 & \cdots & 0 & -1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & 0 & 0 & \cdots & -1 \end{array}\right].$$

Further, as the rank of this matrix is n, we see the top left k columns are linearly independent. (These are columns of A corresponding to the nonzero entries in w.) So by [8.22], w is a bfs. ■

[9.2] **Corollary** (`Existence of a bfs`) If $\{x \mid Ax = b, x \geq 0\} \neq \emptyset$, then $Ax = b$ has a bfs.

Proof. The set $T = \{x \mid Ax = b, x \geq 0\}$ being a nonempty polyhedron which is bounded below, has a vertex. By [9.1], a vertex of T is a bfs of $Ax = b$. ■

> How to define a bfs when rank $A_{m\times n} = k < m$? If the system $Ax = b$ is inconsistent, then the set is empty and we do not have to do anything.
> If it is consistent, then one way is to remove some redundant equations to make it look like $A_{k\times n}x = b'$. We proceed from there.
> Another way is to take some linearly independent columns $i_1, i_2, \cdots, i_k$. Then a basic solution of $Ax = b$ is a solution with $x_j = 0$ for each $j \notin \{i_1, \cdots, i_k\}$. A nonnegative basic solution is a bfs.

Drawing basic solutions

Consider

$$\begin{array}{ll} \min & x_2 - 5x_1 \\ \text{s.t.} & x_1 \leq 2, x_2 \leq 2, x_1 + x_2 \leq 3, x_1 - x_2 \leq 3, x_i \geq 0 \end{array}.$$

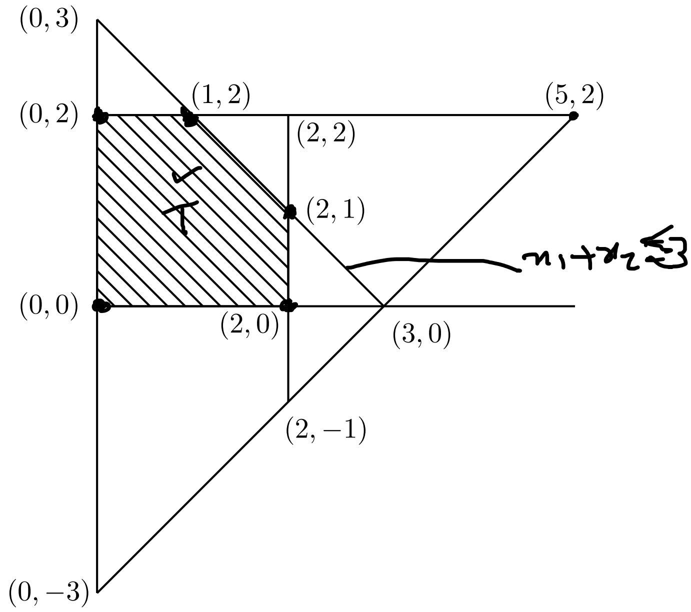
a) Write the set T in $\{x \mid \tilde{A}x \leq \tilde{b}\}$ form.

$$T = \{x \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 2 \\ 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}\}.$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ 3 \\ 3 \end{bmatrix}}_{b}$$

b) Draw it's feasible region T.

$p = (5,2)$ — $p' = (5, 2, -3, 0, -4, 0)$ bs of SLPP



c) Observe that the point $p = (5, 2)$ has been obtained due to n linearly independent hyperplanes passing through it. What are they? Here they are given by rows two and four of $\tilde{A}$.

d) Write the slpp.

Slpp: $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, x_i \geq 0. \end{array}$, where $c^t = \begin{bmatrix} -5 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 3 \end{bmatrix}$.

e) What is the basic solution corresponding to $p = (5, 2)$?

Take $x_1 = 5$ and $x_2 = 2$. Use them in the slpp to get $p' = (5, 2, -3, 0, -4, 0)$.

f) Could you tell before hand that p' will not be a bfs for the slpp?

Yes. There is a half space, namely $x_1 \leq 2$ which does not contain p. Hence for the slpp, the corresponding slack variable (namely x_3) must be negative.

g) Find out the different basic solutions for the slpp. What are the points T related to these basic solutions?

$^6C_4 = 15$

basis	basic solution	point in figure
x_1, x_2, x_3, x_4	$(3, 0, -1, 2, 0, 0)$	$(3, 0)$
x_1, x_2, x_3, x_5	$(5, 2, -3, 0, -4, 0)$	$(5, 2)$
x_1, x_2, x_3, x_6	$(1, 2, 1, 0, 0, 4)$	$(1, 2)$
x_1, x_2, x_4, x_5	$(2, -1, 0, 3, 2, 0)$	$(2, -1)$
x_1, x_2, x_4, x_6	$(2, 1, 0, 1, 0, 2)$	$(2, 1)$
x_1, x_2, x_5, x_6	$(2, 2, 0, 0, -1, 3)$	$(2, 2)$
x_1, x_3, x_4, x_5	$(3, 0, -1, 2, 0, 0)$ repeat	$(3, 0)$
x_1, x_3, x_4, x_6	$(3, 0, -1, 2, 0, 0)$ repeat	$(3, 0)$
x_1, x_3, x_5, x_6	not a basis	
x_1, x_4, x_5, x_6	$(2, 0, 0, 2, 1, 1)$	$(2, 0)$
x_2, x_3, x_4, x_5	$(0, -3, 2, 5, 6, 0)$	$(0, -3)$
x_2, x_3, x_4, x_6	$(0, 3, 2, -1, 0, 6)$	$(0, 3)$
x_2, x_3, x_5, x_6	$(0, 2, 2, 0, 1, 5)$	$(0, 2)$
x_2, x_4, x_5, x_6	not a basis	
x_3, x_4, x_5, x_6	$(0, 0, 2, 2, 3, 3)$	$(0, 0)$

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Some exercises

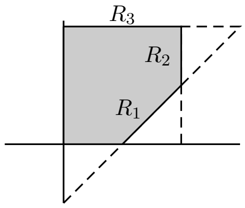
[9.3] **Exercise(E)** Let $\mathsf{rank}\, A_{m \times n} = r \leq m$ and $T = \{x \mid Ax = b, x \geq 0\} \neq \emptyset$. It is given that w is a vertex of T. How many nonzero entries can w have at the most?

[9.4] **Exercise(H)** `(One-one correspondence of corner points of lpp and slpp.)`

$\{x \mid Ax = b,\ x \geq 0\}$

Let $\mathsf{rank}\, A_{m \times n} = m$. Take the lpp $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \leq b, x \geq 0 \end{array}$. The slpp $\begin{array}{ll} \min & \begin{bmatrix} c^t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{s.t.} & \begin{bmatrix} A & I_m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = b, x \geq 0, y \geq 0 \end{array}$ is in a higher dimensional space. Let T and T' be the feasible sets for the lpp and the slpp, respectively. Assume that both are nonempty. Then show that v is a corner point (through which n linearly independent hyperplanes pass) of T iff $\begin{bmatrix} v \\ b - Av \end{bmatrix}$ is a bs of $\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = b$ (equivalently iff $\begin{bmatrix} v \\ b - Av \end{bmatrix}$ is a corner point of T').

[9.5] **Exercise(E+)** Consider the region $P = \mathsf{conv}((0,0),(1,0),(2,1),(2,2),(0,2)) \subseteq \mathbb{R}^2$.



a) Write this region as $\{x \mid \hat{A}_{3\times 2}x \le \hat{b}, x \ge 0\}$, where the ith row of $\hat{A}$ should correspond to the line R_i in the figure.

b) Consider minimizing $x_1 + x_2$ on this region. Write the lpp using the matrix $\hat{A}$ used in the previous item. Write the corresponding slpp in the form $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, x \ge 0 \end{array}$.

c) List down the basic solutions of $Ax = b$ and the points of $\mathbb{R}^2$ which correspond to these basic solutions. Fill the table. Write 'NA' for 'not applicable'.

choice	basis?	basic sol	corresponding pt of $\mathbb{R}^2$
x_1, x_2, x_3	yes	$(2,2,1,0,0)$	$(2,2)$
x_1, x_2, x_4			
x_1, x_2, x_5			
x_1, x_3, x_4	no	NA	NA
x_1, x_3, x_5			
x_1, x_4, x_5			
x_2, x_3, x_4			
x_2, x_3, x_5			
x_2, x_4, x_5			
x_3, x_4, x_5			

Cost of basic feasible solutions

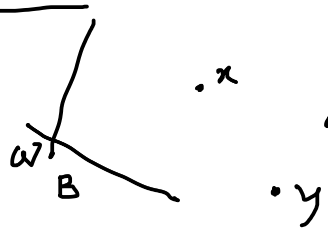
So, we have understood how to identify the vertices of the slpp. These are the bfs's. Now, suppose that I am at a bfs. How do I know whether it is cheaper than the remaining? The following result gives a way to answer that.

[9.6] **Lemma** (`Cost difference of any point w.r.t a given vertex`) Let $\mathsf{rank}(A_{m\times n}) = m$ and w be a bfs of $Ax = b$ with basis matrix B. Let p be any solution of $Ax = b$. Then their cost difference $c^t p - c^t w = \overline{c}^t p$, where $\overline{c}^t = c^t - c_B^t B^{-1} A$.

$\bar{c}x - \bar{c}w = c^T x - c_B^T w_B = c^T x - c_B^T B^{-1} b$

$= c^T x - c_B^T B^{-1} A x$

$= [c^T - c_B^T B^{-1} A] x$



$w = \begin{cases} w_B = B^{-1} b \\ w_C = 0 \end{cases}$

Proof. The cost difference is

$$c^t p - c^t w = c^t p - c_B^t w_B = c^t p - c_B B^{-1} b = c^t p - c_B^t B^{-1} A p = [c^t - c_B^t B^{-1} A] p,$$

as $w_B = B^{-1} b$ and $w_C = 0$. ∎

[9.7] **Definition** Let $\mathsf{rank}(A_{m\times n}) = m$ and w be a bfs of $Ax = b$ with basis matrix B. Then the vector $\overline{c}^t = c^t - c_B^t B^{-1} A$ used in [9.6] is called the RELATIVE COST vector at the vertex w.

relative cost — $\bar{c}^T = c^T - c_B^T B^{-1} A$

$$\bar{c}_B^T = c_B^T - c_B^T B^{-1} A_B = c_B^T - c_B^T B^{-1} B = \underline{0000}$$
$$\bar{c}_C^T = c_C^T - c_B^T B^{-1} C$$

[9.8] **Facts** (About $\bar{c}$) Let $\text{rank}(A_{m\times n}) = m$ and w be a bfs of $Ax = b$ with basis matrix B and $\bar{c}$ be the relative cost vector at w.

$\underline{c^T - c_B^T B^{-1} A}$

a) Then $\bar{c}_B^t = [c^t - c_B^t B^{-1} A]_B = [c_B^t - c_B B^{-1} B] = 0.$

b) Then $\bar{c}_C^t = [c^t - c_B^t B^{-1} A]_C = [c_C^t - c_B B^{-1} C].$ ≥ 0

c) The relative cost vector depends only on the basis matrix B of w.

[9.9] **Theorem** (Minimality of a given bfs) Let $\text{rank}\, A_{m,n} = m$ and consider $\begin{array}{ll} \text{opt} & c^t x \\ \text{s.t.} & Ax = b, x \geq 0 \end{array}$.

Assume that w is a bfs of $Ax = b$. Then the following are true.

a) If $\bar{c}_C^t \geq 0$ (equivalently, if $\bar{c}^t \geq 0$), then w is a point of local minimum.

b) If $\bar{c}_C^t > 0$, then w is a strict minimum.

c) If $\bar{c}_C^t \leq 0$, then w is point of maximum and if $\bar{c}_C^t < 0$, then w is a point of strict maximum.

a) $c^T x - c^T w = \bar{c}^T x = \bar{c}_C^T x_C \geq 0$

b) $\bar{c}_C^T > 0$, then w is a strict min. $y \neq w$. $y_C = 0$?

$\bar{c} y - \bar{c} w = \bar{c}^T y = \bar{c}_C y_C > 0$ $\quad y_C = w_C$

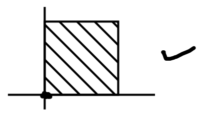
$y_B = B^{-1} b = w_B$

Proof. a) For any $x \in T$, we have $c^t x - c^t w = \bar{c}^t x \geq 0$, as $x \geq 0$.

b) Note that, if $x \neq w$ then x_C cannot be 0, because, in that case $x_B = B^{-1} b = w_B$ and so $w = x$, not possible. Now, as $\bar{c}_C > 0$ and $x \geq 0$ with x_C having at least one positive entry, we get $\bar{c}^t x > 0$.

c) Similar. ∎

[9.10] **Example** Consider $\begin{array}{ll} \text{opt} & x_1 + x_2 \\ \text{s.t.} & x_1 \leq 1, x_2 \leq 1, x_i \geq 0 \end{array}$.



a) Write the slpp.

$x_1 + x_3 = 1$

$x_2 + x_4 = 1$

$(0,0) \longrightarrow (0,0,1,1)$ slpp

w $\quad (x_3, x_4)$

$B = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Answer. It is $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, x \geq 0 \end{array}$, where $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c^t = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$.

b) The point $(0,0)$ is a vertex in our picture (of lpp). What is the corresponding vertex w of the slpp?

$\bar{c}^T = [1\ 1\ 0\ 0] - [0\ 0] \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $\quad C \quad B$

Answer. It is $w = (0,0,1,1)$.

$= [1\ 1\ \boxed{0\ 0}]$ $\quad \bar{c}_C^T \Rightarrow w$ is st min

c) By our earlier result, it must be a bfs of $Ax = b$ for some basis. Write one such basis?

Answer. One basis (x_3, x_4).

d) Compute the relative cost at w.

Answer. At w the relative cost is

$$\bar{c}^t = c^t - c_B^t B^{-1} A = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}.$$

e) Will the test conclude that w is a point of strict minimum?

Answer. Yes, as $\bar{c}_C$ has all entries positive.

f)(Self) The point $(1, 1)$ is a vertex in our picture. The corresponding vertex of the slpp is $w = (1, 1, 0, 0)$. It is a bfs of $Ax = b$ with basis (x_1, x_2). At w the relative cost is

$$\bar{c}^t = c^t - c_B^t B^{-1} A = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \end{bmatrix}.$$

So w is a point of strict maximum.

g)(Self) Compute and observe $\bar{c}$ at the two other vertices and see whether the test supplies any conclusions.

shift by 10 min

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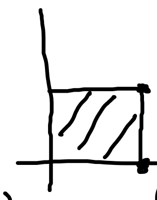
$x_1 \le 1, \; x_2 \le 1$

$x_1 + x_3 = 1$

$x_2 + x_4 = 1$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{\to A} \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\to b}$$



$(1,1) \to (1,1,0,0)$

$w \to$ max

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$f(x) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$f(x) = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ c^T

$$\bar{c}^T = c^T - c_B^T B^{-1} A = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = [\underbrace{0 \;\; 0}_{\bar{c}_B^T} \;\; \underbrace{-1 \;\; -1}_{\bar{c}_C^T}]$$

$\bar{c}_C^T < 0$

$(1,0) \to (1,0,0,1)$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} B^{-1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$$

$\bar{c} \ge 0$

$\bar{c} \le 0$

no conclusion