DEPARTMENT OF MATHEMATICS
Indian Institute of Technology, Guwahati

Midsem	MA321 Optimization	23-09-2021
Maximum Score : 20 of 33		Time : 14:00–15:59
Instructor : Sukanta Pati		Submit before : 15:59

Write appropriate and precise justifications. Draw neatly. Use pencils for convenience. Submit in the portal. If that does not work, only then send it to my email pati@iitg.ac.in before 16:05.

1. Consider the problem table

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
$*$	1	1	1	0	1	0	0	0	1
$*$	1	1	0	1	0	1	0	0	1
$*$	1	0	1	1	0	0	1	0	1
$*$	0	1	1	1	0	0	0	1	1
$*$	-1	-1	-1	-1	0	0	0	0	$*$

Write the simplex table for the basis (x_2, x_1, x_5, x_6). $\boxed{3}$

Answer.

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{b}$
x_2	0	1	1	1	0	0	0	1	1
x_1	1	0	1	1	0	0	1	0	1
x_5	0	0	-1	-2	1	0	-1	-1	-1
x_6	0	0	-2	-1	0	1	-1	-1	-1
$-f$	0	0	1	1	0	0	1	1	2

2. Continue from the previous table. Taking x_5 as the outgoing variable, use dual simplex method to reach the next simplex table. $\boxed{3}$

Answer. $\min_{a_{3i}<0} \frac{\bar{c}_i}{|a_{3i}|} = \frac{1}{2}$ occurs for x_4. So x_4 comes in. The next table is

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\bar{b}$
x_2	0	1	1/2	0	1/2	0	−1/2	1/2	1/2
x_1	1	0	1/2	0	1/2	0	1/2	−1/2	1/2
x_4	0	0	1/2	1	−1/2	0	1/2	1/2	1/2
x_6	0	0	−3/2	0	−1/2	1	−1/2	−1/2	−1/2
$-f$	0	0	1/2	0	1/2	0	1/2	1/2	3/2

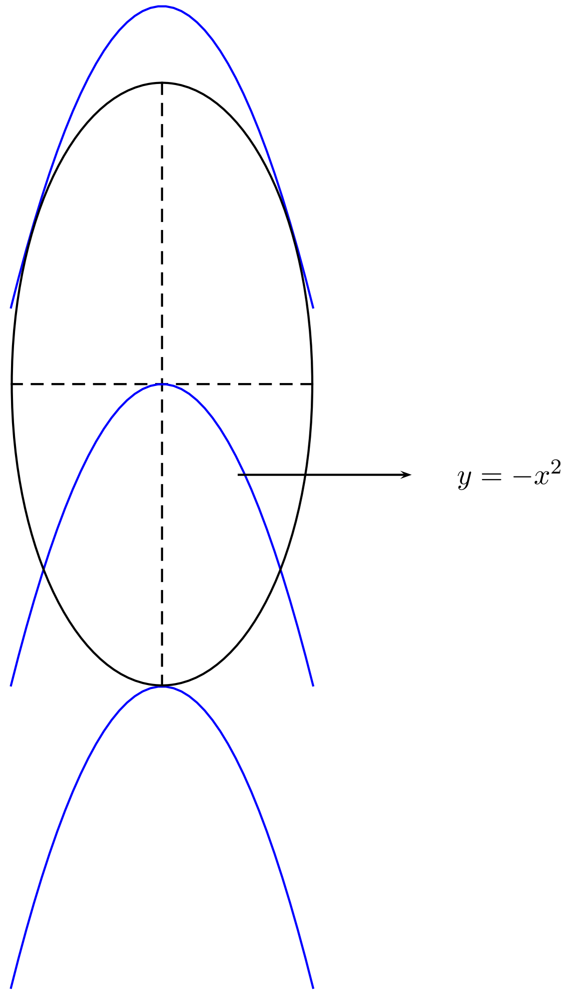
3. Consider the problem table given above. Write the simplex table at the ordered basis (x_4, x_5, x_6, x_7). What is the current vertex? What is the direction given by the nonbasic variable x_1? What is the next ordered basis to consider using Bland's rule? $\boxed{2+1+2+1}$

Answer.

bv	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
x_4	0	1	1	1	0	0	0	1	1
x_5	1	1	1	0	1	0	0	0	1
x_6	1	0	−1	0	0	1	0	−1	0
x_7	1	−1	0	0	0	0	1	−1	0
$-f$	−1	0	0	0	0	0	0	1	1

$$w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \qquad (x_4, x_5, x_1, x_7)$$

4. Consider optimizing $x^2 + y$ over the set $P = \{(x, y) \mid \frac{x^2}{4} + \frac{y^2}{16} = 1\}$. Use graphical method to solve it. $\boxed{4}$



Answer.

It is a matter of drawing $y = -x^2$ which you have learned in calculus. As y varies from -4 to 4 over the ellipse, the minimum -4 is attained at $(0, 4)$.

The maximum occur at points with $x \neq 0$. At those points the curves will have common tangents. Hence $y' = -2x = -4x/y$ and so $y = 2$. Hence $x = \pm\sqrt{3}$. Thus the maximum value of the function is $3 + 2 = 5$ attained at two point $(\pm\sqrt{3}, 2)$.

5. Consider a 4×5 btp with the cost matrix $C = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 3 & 2 & 1 & 2 & 2 \\ 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 3 & 3 \end{bmatrix}$, where the availabilities at the sources S_1, S_2, S_3, S_4 are $50, 60, 60, 70$, respectively and the demands at the sinks $T_1, \ldots, T_5$ are $40, 60, 30, 30, 80$, respectively.

   a) Write the corresponding transportation array. $\boxed{1}$

Answer.

x_{11} 1	x_{12} 2	x_{13} 1	x_{14} 3	x_{15} 1	50
x_{21} 3	x_{22} 2	x_{23} 1	x_{24} 2	x_{25} 2	60
x_{31} 2	x_{32} 1	x_{33} 1	x_{34} 3	x_{35} 1	60
x_{41} 1	x_{42} 1	x_{43} 3	x_{44} 3	x_{45} 3	70
40	60	30	30	80	

.

b) Is $\{x_{11}, x_{12}, x_{21}, x_{23}, x_{24}, x_{25}, x_{31}, x_{45}\}$ a basis? Argue using any of the three methods. $\boxed{2}$

Answer. Yes. As we can strike off the entire table by striking off lines containing only one unstruck variable from this set. The lines are $R_4, R_3, C_5, C_4, C_3, C_2, R_2, R_1$.

c) What is the corresponding basic solution to the above basis? Is it feasible? $\boxed{1}$

Answer.

-10 1	60 2	x_{13} 1	x_{14} 3	x_{15} 1	50
-10 3	x_{22} 2	30 1	30 2	10 2	60
60 2	x_{32} 1	x_{33} 1	x_{34} 3	x_{35} 1	60
x_{41} 1	x_{42} 1	x_{43} 3	x_{44} 3	70 3	70
40	60	30	30	80	

.

It is not feasible.

d) Select the initial bfs using nw-corner rule. $\boxed{1}$

Answer.

40 1	10 2	x_{13} 1	x_{14} 3	x_{15} 1	50
x_{21} 3	50 2	10 1	x_{24} 2	x_{25} 2	60
x_{31} 2	x_{32} 1	20 1	30 3	10 1	60
x_{41} 1	x_{42} 1	x_{43} 3	x_{44} 3	70 3	70
40	60	30	30	80	

.

e) Verify whether the bfs in d) is a minimal bfs. $\boxed{3}$

Answer.

40 1	10 2	x_{13} 1	x_{14} 3	x_{15} 1	50 0
x_{21} 3	50 2	10 1	x_{24} 2	x_{25} 2	60 0
x_{31} 2	x_{32} 1	20 1	30 3	10 1	60 $0*$
x_{41} 1	x_{42} 1	x_{43} 3	x_{44} 3	70 3	70 2
40 1	60 2	30 1	30 3	80 1	

,

$$\overline{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ -2 & -3 & 0 & -2 & 0 \end{bmatrix}$$

f) Find a better bfs. $\boxed{2}$

Answer. Entering: x_{42}. Cycle and the new bfs are shown below.

40 1	10 2	x_{13} 1	x_{14} 3	x_{15} 1	50
x_{21} 3	$50-\theta$ 2	$10+\theta$ 1	x_{24} 2	x_{25} 2	60
x_{31} 2	x_{32} 1	$20-\theta$ 1	30 3	$10+\theta$ 1	60
x_{41} 1	θ 1	x_{43} 3	x_{44} 3	$70-\theta$ 3	70
40	60	30	30	80	

, $\theta = 20$,

40 1	10 2	x_{13} 1	x_{14} 3	x_{15} 1	50
x_{21} 3	30 2	30 1	x_{24} 2	x_{25} 2	60
x_{31} 2	x_{32} 1	x_{33} 1	30 3	30 1	60
x_{41} 1	20 1	x_{43} 3	x_{44} 3	50 3	70
40	60	30	30	80	

,

6. `(Write properly)` Consider the set

$$P = \{x \in \mathbb{R}^4 \mid x \geq 0,\ x_1 + x_2 + x_3 \leq 1,\ x_1 + x_2 + x_4 \leq 1,\ x_1 + x_3 + x_4 \leq 1,\ x_2 + x_3 + x_4 \leq 1\}.$$

Which of the following statements are correct? $\boxed{7}$

a) It has a vertex with no coordinates 0.

b) It has a vertex with exactly one coordinate 0.

c) It has a vertex with exactly two coordinates 0.

d) It has a vertex with exactly three coordinates 0.

e) It has a vertex with all coordinates 0.

Answer. Note that $P = \{x \mid Ax \le b\}$, where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

a) is true. The point $w = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a vertex as $w \in P$ and A_w has rank 4.

b) is False. Suppose it has a vertex w with exactly one coordinate 0. Due to symmetry, let $w_1 = 0$. If row 5 is in A_w, then we have $w_2 + w_3 = 1$ and hence $w_4 = 0$, a contradiction. If row 5 is not in A_w, then row 6 must be in A_w and hence $w_3 = 0$, a contradiction.

c) is false. Suppose it has a vertex w with exactly two coordinates 0. Due to symmetry, let $w_1 = w_2 = 0$. If row 5 is in A_w, then we have $w_3 = 1$ and hence $w_4 = 0$, a contradiction. If row 6 is in A_w, then we have $w_4 = 1$ and hence $w_3 = 0$, a contradiction. If rows 5 and 6 are not in A_w, then A_w has rank less than 4, a contradiction.

d) is true. The point $w = (0, 0, 0, 1)$ is a vertex as $w \in P$ and A_w has rank 4.

e) is true. The point $w = (0, 0, 0, 0)$ is a vertex as $w \in P$ and A_w has rank 4.