20 Lecture 20

Unbalanced Transportation Problem (ubtp)

We now consider some unbalanced problems, that is, $\sum a_i \neq \sum b_j$ meaning the availability and the demand are unequal. In each of the cases, it will be mentioned whether meeting the demands and other conditions are requirements or not. In each of the cases, the strategy is to first convert it to a balanced problem and solve that. Then we retrieve the solution of the ubtp from the solution of the btp.

Surplus

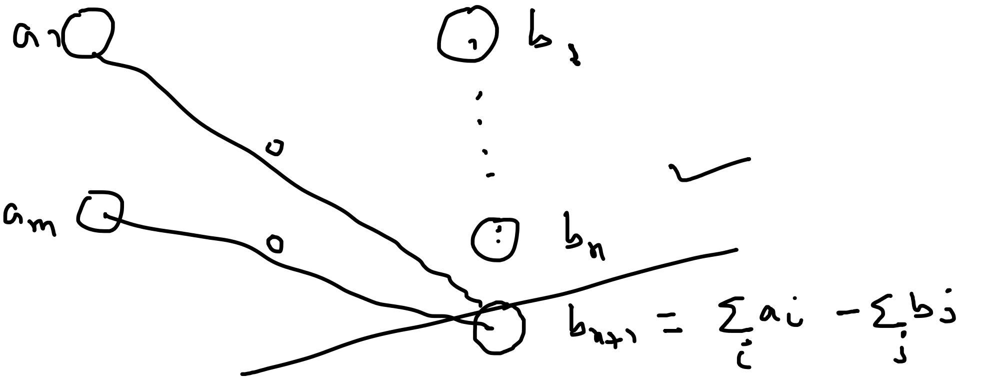
a) Here $\sum a_i > \sum b_j$. The objective is to meet the exact demand (some product will remain at the sources) and minimize the cost.

b)`(Lpp for surplus)` So the lpp for the ubtp is

$$\frac{\min \ \sum c_{ij} x_{ij}}{\sum_j x_{ij} \leq a_i \ \forall i, \quad \sum_i x_{ij} = b_j, \quad x_{ij} \geq 0}$$

$$\begin{array}{ll} \min & \sum_{i,j} c_{ij} x_{ij} \\ \hline \text{s.t.} & \sum_j x_{ij} \leq a_i, \ \forall i, \qquad \sum_i x_{ij} = b_j, \ \forall j, \qquad x_{ij} \geq 0. \end{array}$$

c)`(Converting to btp)`



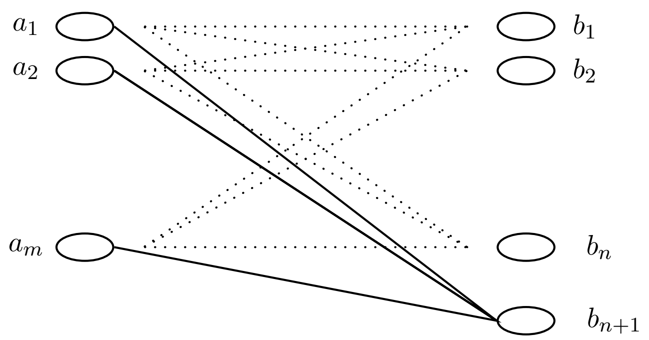
soln of btp

$$X = \begin{bmatrix} x_{11} & x_{1n} & x_{1,n+1} \\ x_{m1} & x_{mn} & x_{m,n+1} \end{bmatrix} \xleftrightarrow{\text{bij}} X = \begin{bmatrix} x_{11} & x_{1n} \\ x_{m1} & x_{mn} \end{bmatrix}$$

soln of ubtp

cost = cost

This problem can be converted to a btp, by introducing a virtual sink T_{n+1} with demand $b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} \sum b_j$ and putting the cost of transportation of goods from each source to the new sink T_{n+1} as 0.



d) `(Retrieving solution)` Observe that there is a natural bijection between the solutions of the btp and the solutions of the ubtp.

This bijection f is given by

$a_1 - (x_{11} + \cdots + x_{1n})$

$$f: \quad \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} & x_{1,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m,1} & \cdots & x_{m,n} & x_{m,n+1} \end{bmatrix} \leftarrow \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \vdots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix}.$$

e) The plan is to take a minimum solution X of the btp and claim that $f^{\cancel{-1}}(X)$ is a minimum solution for the ubtp.

f) `(Argument for e))`

cost of min soln of btp = cost of the corresp soln of the ubtp

$\rightarrow$ = $\geq$ cost of the min soln of the ubtp

= cost of the corresp soln of the btp

$\geq$ cost of the min soln of the btp.

Notice that cost of a minimum solution of the ubtp

$$
\begin{aligned}
&= \text{cost of corresponding solution of the btp}\\
&\geq \text{cost of an minimum solution of the btp}\\
&= \text{cost of the corresponding solution of the ubtp}\\
&\geq \text{cost of a minimum solution of ubtp.}
\end{aligned}
$$

Hence the minimum solution of the ubtp naturally corresponds to the minimum solution of the btp.

[20.1] **Example** Solve the following ubtp.

Factories	Product	Stores	Demands
A	60	A	70
B	90	B	70

, Cost $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Answer.

Trans array for btP

1	2	0	60
2	1	0	90
70	70	10	

X =

60 1	2	0	−1
10 2	70 1	10 0	0
2	1	0	

$\bar{C} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So it is a minimum soln of the btP.

So min soln for the ubtp is $X' = \begin{bmatrix} 60 & 0 \\ 10 & 70 \end{bmatrix}$.

cost is 150.

Btp array:

60 1	2	0	60 −1
10 2	70 1	10 0	90 0∗
70 2	70 1	10 0	

.

Ibfs (nw rule), simplex multipliers, and $\overline{c}$ are shown below.

60 / 1	/ 2	/ 0	60 / −1
10 / 2	70 / 1	10 / 0	90 / 0∗
70 / 2	70 / 1	10 / 0	

$$\overline{c} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Minimum soluntion for the btp: $X = \begin{bmatrix} 60 & 0 & 0 \\ 10 & 70 & 10 \end{bmatrix}$.

Minimum solution for the ubtp: $X = \begin{bmatrix} 60 & 0 \\ 10 & 70 \end{bmatrix}$. Minimum cost is 150.

[20.2] <u>**NoPen**</u> Will it also work, if we take the cost of transportation of goods from each source to the new sink a constant, say 5?

Shortage

a) Here $\sum a_i < \sum b_j$. The objective is to transport all product to the sinks and minimize the cost.

b) This can be treated in a way similar to the surplus case.

Surplus with storage cost and with possibility of sending more

a) Here $\sum a_i \geq \sum b_j$. The problem is to minimize the cost under the following conditions.

'Sinks may get more than they demanded. Part of the product may remain at the source with a per unit storage cost θ_i at source S_i.'

b) The lpp for the ubtp is

$$\min \quad \sum c_{ij} x_{ij} + \sum_i \theta_i (a_i - x_{i1} - \cdots - x_{in})$$

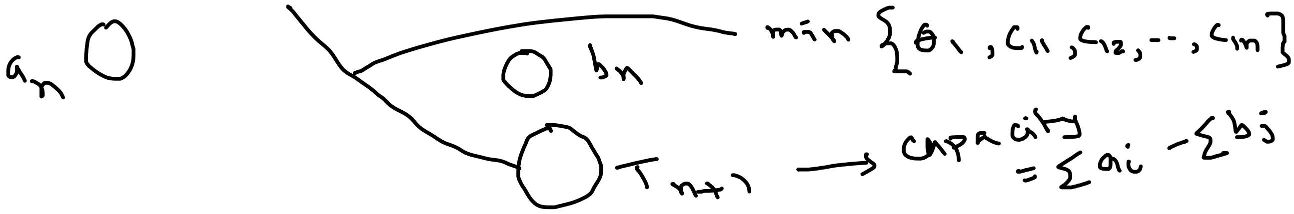
$$x_{i1} + \cdots + x_{in} \le a_i \quad \forall i$$

$$x_{1j} + x_{2j} + \cdots + x_{mj} \ge b_j$$

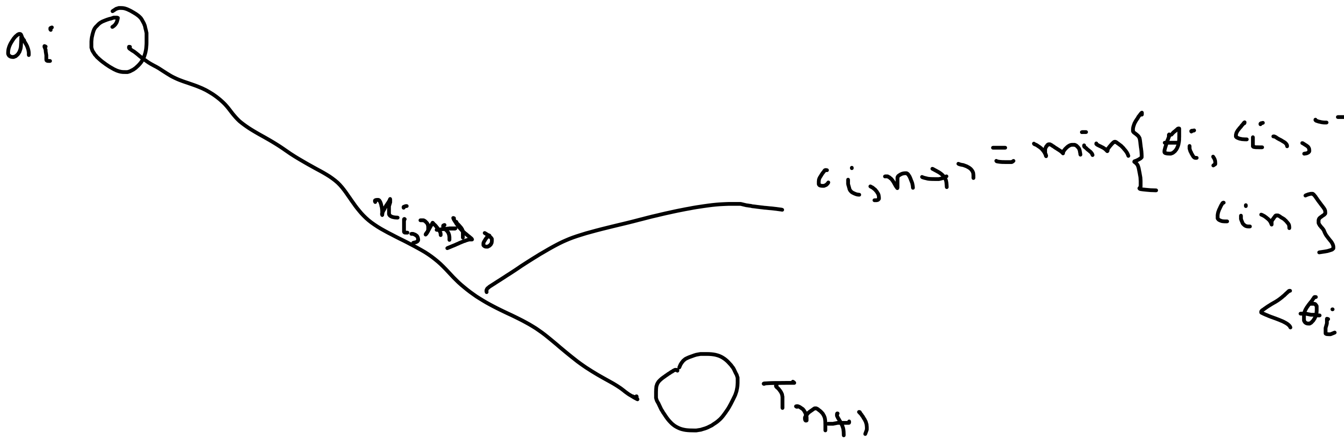
$$x_{ij} \ge 0$$

$$\begin{array}{ll} \min & \sum_{i,j} c_{ij} x_{ij} + \sum_i \theta_i \Big(a_i - \sum_j x_{ij} \Big) \\ \hline \text{s.t.} & \sum_j x_{ij} \le a_i, \ \forall i, \quad \sum_i x_{ij} \ge b_j, \ \forall j, \quad x_{ij} \ge 0. \end{array}$$

c) `(Converting to btp)` Imagine a solution where the demands have been met and there are some product



left with the source S_1. Now it is your wish whether to send it to some sink or to keep it at the source. This is as good as sending it to a new sink. So introduce a virtual sink T_{n+1} with demand $(\sum a_i - \sum b_j)$. Put $c_{i,n+1} = \min\{\theta_i, c_{i1}, \cdots, c_{in}\}$. Note that $c_{i,n+1} = \theta_i$ means that any surplus product at S_i goes to the storage and $c_{i,n+1} = c_{i1}$ means that the surplus product at S_i goes to T_1. This gives us a btp. Draw picture.



d)`(Retrieving the solution)` Suppose that X is a minimum solution of the btp with cost α.

soln Y for ubtp.

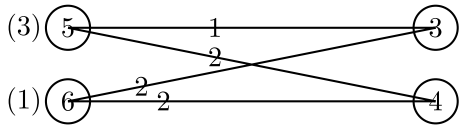
$$y_{ij} = \begin{cases} x_{ij} + x_{i,n+1} & \longrightarrow \text{if } x_{i,n+1} > 0 \text{ and } c_{i,n+1} < \theta_i \text{ and } j \text{ is the first one s.t. } c_{ij} = c_{i,n+1} \\ x_{ij} & \text{if } x_{i,n+1} > 0,\ c_{i,n+1} < \theta_i \text{ and } j \text{ is not the first one} \end{cases}$$

We can define a corresponding solution Y of the ubtp of the same cost in the following way. For $j = 1, \ldots, n$, define

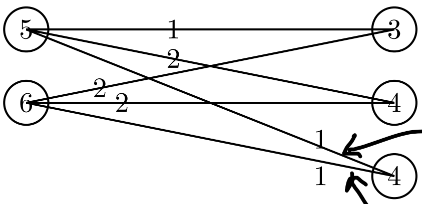
$$y_{ij} = \begin{cases} x_{ij} + x_{i,n+1} & \text{if } c_{i,j} = c_{i,n+1} < \theta_i,\ j \text{ smallest} \\ \cancel{x_{ij}} & \cancel{\text{if } c_{i,n+1} = \theta_i.} \end{cases}$$

or the prev case and j not smallest

[20.3] **Example** Consider the following ubtp.



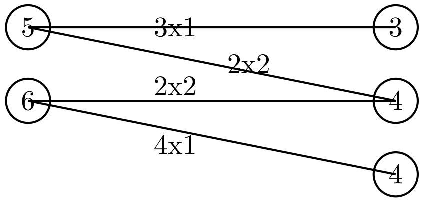
The corresponding btp:



$\min\{\overset{3}{\theta_1}, \overset{1}{c_{11}}, \overset{2}{c_{12}}\}$

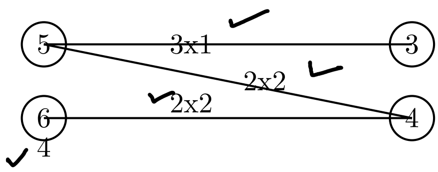
$\min\{\overset{1}{\theta_2}, \overset{2}{c_{21}}, \overset{2}{c_{22}}\}$

A minimum solution for btp:



$$x = \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 2 & 4 \end{array}\right]$$

A minimum solution for ubtp:



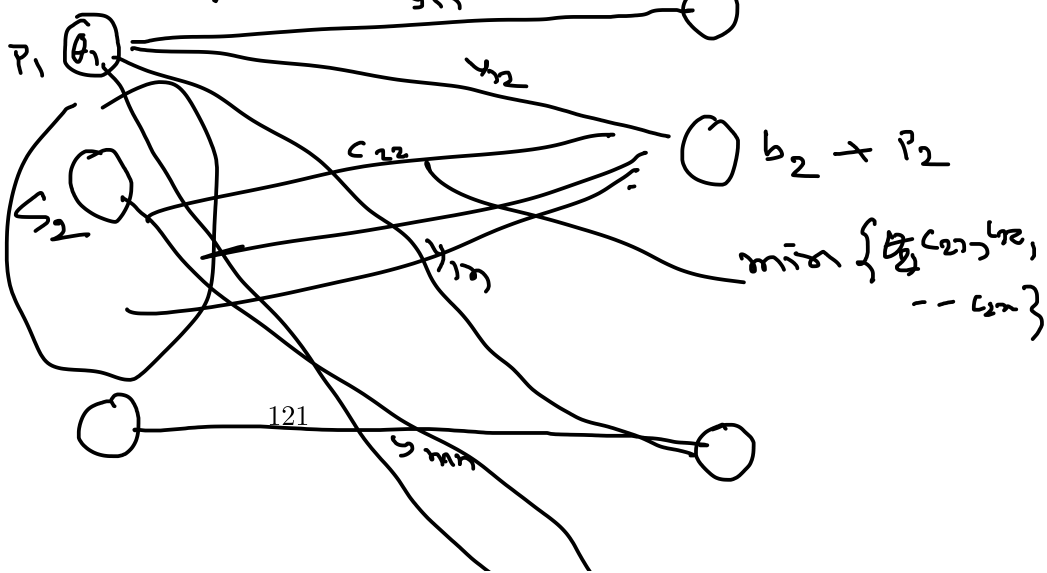
e) `(Arguing that it is indeed the correct solution)` In view of Item d), we see that the cost of the minimum cost solution of the btp is $\geq$ the cost of the minimum cost solution of the ubtp.

Now to see the converse, let y be a minimum solution for the ubtp of cost β.

ubtp $\longrightarrow$ btp $\rightarrow$ got a min soln $X \rightarrow$ defined a soln y for the ubtp $\rightarrow$ min ?

$X \longrightarrow Y$

Given a soln Y for ubtp



get a soln
x for the btp. $T_{n+1} \to$

STEP1: First of all, identify the sources at which some product is left due to y. Send them to the $n+1$th sink with the same cost as of the storage cost.

Suppose $A_1 = a_1 - y_{11} - \cdots - y_{1n} > 0$. Then we must have $\theta_1 = \min\{\theta_1, c_{11}, \cdots, c_{1n}\}$, otherwise, we could send that extra amount at S_1 to another place decreasing the overall cost.

Thus, in this case that extra amount at S_1 may be viewed as being taken to T_{n+1} with a transportation cost $c_{1,n+1}$. We perform this operation at all sources S_i that have some extra product A_i left at it.

STEP2: Find all sinks that have received more. Return a proportionate amount they are receiving from the sources. Those extra amount can be sent to the $n+1$th sink with cost equal to the cost along the path where it was returned back.

Suppose that the sink T_1 has received $B_1 > b_1$. Assume that only $y_{11}, \ldots, y_{k1} > 0$. So $y_{11} + \ldots + y_{k1} > b_1$. Notice that we must have $c_{11} = \min\{\theta_1, c_{11}, \ldots, c_{1n}\}$. [If $c_{11} > \theta_1$, we could have kept a little amount at S_1, instead of sending it to T_1 and decrease the cost. If $c_{11} > c_{12}$, we could have sent a little more amount to T_2 instead of sending it to T_1.]

Similarly, we must have $c_{k1} = \min\{\theta_k, c_{k1}, \ldots, c_{kn}\}$.

The above two mean we can reduce the transportation from sources $S_1, \ldots, S_k$ to T_1 proportionately and divert the residual to T_{n+1}. In fact, we can do a similar scaling at all other sources too, as those transportation are zero.

That is,

$$y'_{11} = \frac{b_1}{B_1} y_{11}, \ \cdots, \ y'_{k1} = \frac{b_1}{B_1} y_{k1}, \ \cdots, y'_{k+1,1} = \cdots = y'_{n,1} = 0$$

Thus the final solution of the btp is given by

$$y'_{i,j} = \begin{cases} \frac{b_j}{B_j} y_{i,j} & \text{if } B_j > b_j, \ j = 1, \ldots, n \\ y_{i,j} & \text{if } B_j = b_j, \ j = 1, \ldots, n \end{cases}$$

and

$$y'_{i,n+1} = A_i + \sum_{j=1}^{n} [y_{i,j} - y'_{i,j}].$$

With this, we obtain a solution of the btp of the same cost. Hence the cost of the minimum cost btp is $\leq$ the cost of the minimum cost ubtp.

—— X —— End.

Below demand supply with storage cost and fine for unmet demands

a) Given a_i's, b_j's and the cost matrix C. Conditions: part of the product may remain at the source with a per unit storage cost θ_i at source S_i, sinks may get less while the fine amount for each unit of unmet demand of T_j is f_j. Problem: minimize the cost.

b) The lpp of the ubtp is

$$\begin{array}{ll} \min & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} \theta_i \Big(a_i - \sum_{j=1}^{m} x_{ij} \Big) + \sum_{j=1}^{n} f_j \Big(b_j - \sum_{i=1}^{m} x_{ij} \Big) \\ \text{s.t.} & \sum_{j=1}^{n} x_{ij} \leq a_i, \forall i, \quad \sum_{i=1}^{m} x_{ij} \leq b_j, \forall j, \quad x_{ij} \geq 0. \end{array}$$