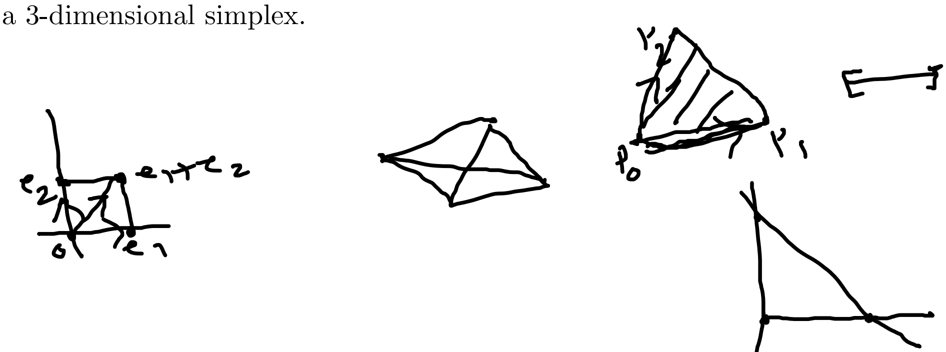
3 Lecture 3

[3.1] **Definitions** Fix $S = \{p_0, p_1, \dots, p_k\} \subseteq \mathbb{R}^n$. If $\{p_1 - p_0, \dots, p_k - p_0\}$ is linearly independent, then we say that S is AFFINE INDEPENDENT, otherwise it is called AFFINE DEPENDENT. If $S = \{p_0, p_1, \dots, p_k\}$ is affine independent, then we call $\mathsf{conv}(S)$ a SIMPLEX of dimension k.

[3.2] **Example** The sets $S_1 = \{0, e_1, e_2\} \subseteq \mathbb{R}^2$ and $S_2 = \{e_1, e_2, e_3\} \subseteq \mathbb{R}^3$ are affine independent. Whereas related to our favorite set, the set $\{0, e_1, e_2, e_1 + e_2\}$ is affine dependent.

[3.3] **Example** A point is regarded as a zero dimensional simplex. A closed line segment is a 1-dimensional simplex. A closed triangular plate is a 2-dimensional simplex (it not the n of $\mathbb{R}^n$) and a closed solid tetrahedron is a 3-dimensional simplex.



[3.4] **Fact** `(Similar to linear dependence)` The set $S = \{p_0, p_1, \dots, p_k\}$ is affine dependent iff some p_i is an affine combination of the remaining.‼

[3.5] **Fact** `(Similar to linear independence)` Let $S = \{p_0, p_1, \dots, p_k\} \subseteq \mathbb{R}^n$. Then S is affine independent iff each $x \in \mathsf{aff}(S)$ can be written as an affine combination of p_i's in a unique way.‼

The following is an immediate corollary. It will be used in the sequel.

[3.6] **Corollary** Let $S = \{p_0, p_1, \dots, p_k\}$ be affine independent. Then each $x \in \mathsf{conv}(S)$ can be written as a convex combination of p_i's in a unique way.‼

Some exercises

[3.7] **Exercise** Related to our favorite set in $\mathbb{R}^2$, the set $\{0, e_1, e_2, e_1 + e_2\}$ is affine dependent. So one element should be an affine combination of the rest. Verify this.

[3.8] **Exercise** `(Not every element will be an affine combination of the rest)` Take $S = \{0, e_1, e_2, 2e_1\}$. It is affine dependent. But we cannot express e_2 as an affine combination of the rest, as e_2 does not lie on the affine span of the other three. Does this contradict our previous results?

[3.9] **Exercise** Write our favorite set as a union of two simplices of dimension 2, in two different ways.

[3.10] **Exercise** Recall that our favorite set is $\mathsf{aff}(S)$, where $S = \{0, e_1, e_2, e_1 + e_2\}$ is affine dependent. So, there should be a point $x \in \mathsf{aff}(S)$ which can be written as an affine combination of points of S in two different ways. Find such a point.

[3.11] **NoPen** a) T/F? The affine combinations of 5 points always form a subspace.

b) Take the points $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$. What do I get if I collect all the convex combinations? Affine combinations? Nonnegative combinations? Linear Combinations?

c) Is the convex hull of 5 different points in $\mathbb{R}^2$ always a pentagon?

d) Let $S \subseteq \mathbb{R}^4$ be a nonempty finite set. Is $\mathsf{conv}(S)$ necessarily a bounded set?

e) Let $\mathcal{A} = \{S \mid \mathsf{conv}(S) \neq S, S \subseteq \mathbb{R}^4 \text{ is convex}\}$. Can $\mathcal{A}$ have infinitely many elements?

f) Is there a bounded infinite subset of $\mathbb{R}^3$ such that $\mathsf{conv}(S)$ is open?

g) I have a set of 10 points in $\mathbb{R}^8$. Can it be affine independent?

h) Is a linearly independent set necessarily affine independent?

i) Can a set containing the zero vector be affine independent?

[3.12] **Exercise** `(Imagine cones)` Determine the cones $\mathsf{cone}(S)$ for the following sets S.

a) $S = \{(1,1),(1,0),(0,0)\}$, b) $S = \{(1,0),(-1,0)\}$, c) $S = \{e_1, e_2, e_3\} \subseteq \mathbb{R}^3$

[3.13] **Exercise** `(Pictures of unit balls)` Draw $B_1(0)$ in $\mathbb{R}^2$ with respect to the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$. Can you imagine the pictures of the unit ball in $\mathbb{R}^3$ with respect to the above norms?

[3.14] **Exercise** `(Norms and convex sets)` Fix $0 < p < 1$. On $\mathbb{R}^n$, $n \geq 2$, define $\|x\|_p = \left[\sum_{i=1}^n |x_i|^p\right]^{1/p}$. Show that the set $S = \{x \mid \|x\|_p \leq 1\}$ is not convex. Conclude that $\|x\|_p$ is not a norm on $\mathbb{R}^n$. Why was $n \geq 2$ required?

[3.15] **Exercise** `(How to find out?)` Consider

$$p_0 = \begin{bmatrix}1\\0\\2\end{bmatrix}, p_1 = \begin{bmatrix}2\\1\\0\end{bmatrix}, p_2 = \begin{bmatrix}2\\2\\2\end{bmatrix}, p_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}, p_4 = \begin{bmatrix}0\\1\\3\end{bmatrix}.$$

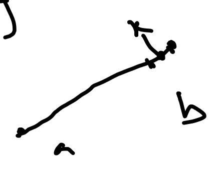
Which point is an affine combination of the rest? Give a general procedure that will work for many vectors in $\mathbb{R}^n$.

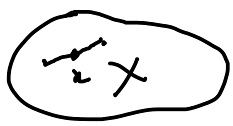
Vertices, hyperplanes

[3.16] **Definition** A point x of a convex set S is called a VERTEX if it is not on any open line segment in S. That is, $\nexists a, b \in S$, $a \neq b$ and $\lambda \in (0,1)$ such that $x = \lambda a + (1-\lambda)b$. Equivalently, it means $x \notin \mathsf{conv}(S \setminus \{x\})$. It means, you cannot write x as a convex combination of two points different from x.

[3.17] **Example** If we take $a \neq b$ in $\mathbb{R}^n$, then by definition, $(.1)a + (.9)b$ is not a vertex of $[a, b]$.

~~$x \in span(S)$~~





$x = \lambda_1 s_1 + \cdots + \lambda_k s_k$ $\quad$ conv(S)

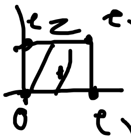
$\lambda_1 s_1 + (1-\lambda_1)\left(\dfrac{\lambda_2 s_2 + \cdots + \lambda_k s_k}{\lambda_2 + \cdots + \lambda_k}\right)$ $\in$ conv(S)

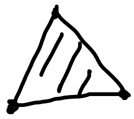
[3.18] **Fact** Let $\emptyset \neq S \subseteq \mathbb{R}^n$. Then the vertices of $\mathsf{conv}(S)$ must be elements of S. [2]

[3.19] **Example** So, our favorite set can have at most 4 vertices, namely, $0, e_1, e_2, e_1 + e_2$.

[3.20] **Theorem** Let $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{R}^n$ be affine independent. Then p_i are precisely the vertices of $\mathsf{conv}(S)$.[3]

$S = \{0, e_1, e_2, e_1+e_2, \cancel{\frac{e_1+e_2}{2}}\}$





[3.21] **Example** It follows that a closed triangular plate has 3 vertices. As our favorite set as $S = \{0, e_1, e_2, e_1 + e_2\}$ is affine dependent, the previous results only tell that the vertices of $\mathsf{conv}(S)$ are in S.

[3.22] **Careful** A nonempty convex cone may not have a vertex, for example take $\mathbb{R}^n$ itself. But if it has one, then it is the point 0. This is because if $x \neq 0$, then it is the midpoint of $\frac{1}{2}x$ and $\frac{3}{2}x$.

[3.23] **Definitions** Take $0 \neq c \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then the set $H = \{x \in \mathbb{R}^n \mid c^t x = \alpha\}$ is called a HYPERPLANE. We often write $H : c^t x = \alpha$, to mean the hyperplane $H = \{x \in \mathbb{R}^n \mid c^t x = \alpha\}$. When $\alpha = 0$, we call H a LINEAR HYPERPLANE.

Give $c \neq 0, \alpha \in \mathbb{R}$, $\{x \mid c^T x = \alpha\} \to$ hyperplane

[3.24] **Example** ◦ In $\mathbb{R}^3$, a hyperplane is a plane. $\quad \{x \mid c^T x = 0\}$

◦ In $\mathbb{R}^2$, a hyperplane is a line.

◦ In $\mathbb{R}^1$, a point is a hyperplane. $\quad \{c\} \to$ extend to an orthonormal basis.

[3.25] **Fact** Let $n > 1$. Then any linear hyperplane in $\mathbb{R}^n$ is an $n-1$ dimensional subspace. So it is isomorphic to $S = \{y \mid y(n) = 0\}$ which can also be viewed as $\mathbb{R}^{n-1}$.

$c^\perp$ = span(remaining vectors in the extension)

[3.26] **Fact** Any hyperplane in $\mathbb{R}^n$ is a translated linear hyperplane. So it is an affine subspace, but in

[2] *Proof.* If $p \in \mathsf{conv}(S) \setminus S$, then $p \notin S$ and so $S \subseteq \mathsf{conv}(S) \setminus \{p\}$. So $p \in \mathsf{conv}(\mathsf{conv}(S) \setminus \{p\})$. So p is not a vertex. ■

[3] *Proof.* We know that vertices of $\mathsf{conv}(S)$ must be elements of S. Now, suppose p_0 is not a vertex. So there exist distinct points $x = \sum \lambda_i p_i$ and $y = \sum \mu_i p_i$ in $\mathsf{conv}(S)$ such that for some $\lambda \in (0,1)$, we have

$$p_0 = \lambda x + (1-\lambda)y = (\lambda\lambda_0 + (1-\lambda)\mu_0)p_0 + (\lambda\lambda_1 + (1-\lambda)\mu_1)p_1 + \cdots + (\lambda\lambda_k + (1-\lambda)\mu_k)p_k.$$

As $p_0 = (1)p_0 + (0)p_1 + \cdots + (0)p_k$, and S is affine independent, we have $1 = \lambda\lambda_0 + (1-\lambda)\mu_0$ and $0 = \lambda\lambda_1 + (1-\lambda)\mu_1 = \cdots = \lambda\lambda_k + (1-\lambda)\mu_k$. Observe that, if a convex combination of two nonnegative numbers is 0, then they both must be 0. Hence $\lambda_1 = \mu_1 = \cdots = \lambda_k = \mu_k = 0$. So $\lambda_0 = \mu_0 = 1$ and $x = y = p_0$, a contradiction to the fact that x and y are distinct. ■

$[1\ \ -1]\begin{bmatrix}x\\y\end{bmatrix}=1$ $\quad [2\ \ 1]\begin{bmatrix}x\\y\end{bmatrix}=8$ $\quad H: c^T x=\alpha$ hypplane

general, affine subspaces can have dimensions other than $n-1$.

[3.27] **Fact** It is easy to see that the distance from 0 to the hyperplane $H : c^t x = \alpha$ is $\frac{|\alpha|}{\|c\|}$. [4]

[3.28] **Definition** Hyperplanes $c_i^t x = \alpha_i$, $i = 1, \ldots, k$ are called LINEARLY INDEPENDENT if $\{c_1, \ldots, c_k\}$ is linearly independent.

$H_i : c_i^t x = \alpha_i \quad , i = 1, \ldots, k$

[3.29] **Example** The hyperplanes $x - y = 1$ and $2x + y = 8$ are linearly independent in $\mathbb{R}^2$, whereas the hyperplanes $x - y = 1$ and $x - y = 2$ are not linearly independent.

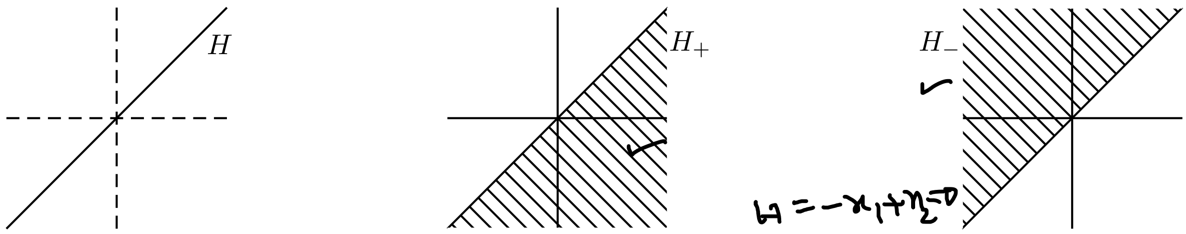
[3.30] **Definition** Let $H : c^t x = \alpha$ be a hyperplane. Then the set $H_+ = \{x \mid c^t x \geq \alpha\}$ is called the positive CLOSED HALFSPACE of H. The set $H_+^\circ = \{x \mid c^t x > \alpha\}$ is called the positive OPEN HALFSPACE of H. Negative closed and open halfspaces are defined similarly.

$H: c^t x = \alpha$

$H_+ = \{x \mid c^T x \geq \alpha\} \rightarrow$ positive closed half space

$H_- = \{x \mid c^T x \leq \alpha\} \rightarrow$ neg "

[3.31] **Example** Take $c = [1 \quad -1]^t$ and $\alpha = 0$. Then the linear hyperplane $H : c^t x = 0$ in $\mathbb{R}^2$ is the line $x_1 - x_2 = 0$. The positive and negative closed halfspaces are shown below.



$H = -x_1 + x_2 = 0$

$c^t = [-1, 1]$

But if we take $c = [-1 \quad 1]^t$, then H_+ and H_- change.

[3.32] **NoPen** T/F? Our favorite set is the intersection of 4 closed halfspaces namely, $x_1 \leq 1, x_1 \geq 0, x_2 \leq 1$ and $x_2 \geq 0$.

[3.33] **Fact** Hyperplanes, halfspaces are convex. Linear hyperplanes, linear closed halfspaces are convex cones.

Some exercises

[4] If $x \in H$, then $\|x\|^2 = \|x - \frac{\alpha c}{\|c\|^2} + \frac{\alpha c}{\|c\|^2}\|^2 = \|x - \frac{\alpha c}{\|c\|^2}\|^2 + \|\frac{\alpha c}{\|c\|^2}\|^2$, as $\langle x - \frac{\alpha c}{\|c\|^2}, c\rangle = 0$. So $\|x\|^2 \geq \|\frac{\alpha c}{\|c\|^2}\|^2$. Equality is attained at $x = \frac{\alpha c}{\|c\|^2}$. So the distance is $\|x\| = \frac{|\alpha|}{\|c\|}$.

[3.34] **Exercise+** `(Largest ball inside)` Let $S = \{\pm e_i \mid i = 1, \cdots, 9\} \subseteq \mathbb{R}^9$. Find the largest value of δ such that $B_\delta(0) \subseteq \mathsf{conv}(S)$.

[3.35] **Exercise** `(Minimal generating set)` Let $S \subseteq \mathbb{R}^n$ be nonempty and finite. Throw out, one by one, points from S which are convex combinations of the others and let $T = \{x_1, \ldots, x_k\}$ be the set which remains at the end. a) Is $\mathsf{conv}(T) = \mathsf{conv}(S)$? b) Notice that $x_1 = 1x_1 + 0x_2 + \cdots + 0x_k$. Argue that there cannot be another way to express x_1 as a convex combination of points in T. c) Hence prove that x_1 is a vertex of $\mathsf{conv}(S)$.

[3.36] **NoPen** a) Is it necessary that a nonempty convex set should have at least one vertex?

b) What are the vertices of a closed circular disc?

c) Is it necessary for a nonempty convex cone in $\mathbb{R}^3$ to have at least one vertex?

d) Is the number of bounded nonempty convex cones in $\mathbb{R}^4$ at least 4?

e) Is every subspace of the vector space $\mathbb{R}^n$ a linear hyperplane?

f) Is every linear hyperplane of the vector space $\mathbb{R}^n$ is a subspace?

g) Which unit vectors are perpendicular to the hyperplane $\{x \mid c^t x = 0\}$?

h) There are k hyperplanes in $\mathbb{R}^5$ which intersect only at a point. At the maximum, how many of them could form a linearly independent subset?

i) T/F? If S and T are disjoint nonempty finite sets, then $\mathsf{conv}(S)$ and $\mathsf{conv}(T)$ cannot be equal.

j) Fix $A_{m\times n}$ and $b \in \mathbb{R}^m$. Express $\{x \in \mathbb{R}^n \mid Ax \le b\}$ as intersection of some convex sets.

k) In $\mathbb{R}^3$ take $S = \mathsf{conv}(e_1, e_1 + e_2)$. Can we write S as an intersection of some hyperplanes?

l) In $\mathbb{R}^3$ take $S = \mathsf{conv}(e_1, e_1 + e_2)$. Can we write S as an intersection of closed halfspaces?

m) Let $S \subseteq \mathbb{R}^n$ be convex and A be an $m \times n$ matrix. Is the set $\{Ax \mid x \in S\}$ is convex?

n) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $S \subseteq \mathbb{R}^n$ be convex. Is $T(S)$ convex?

o) What is the distance of the hyperplane $\{x \mid a^t x = b\}$ from 0? What is the distance between the hyperplanes $\{x \mid a^t x = b\}$ and $\{x \mid a^t x = b'\}$?

[3.37] **Exercise** Given two nonzero vectors a and b in $\mathbb{R}^5$, write the conditions so that $\{x \in \mathbb{R}^5 \mid a^t x \le \alpha\} \subseteq \{x \in \mathbb{R}^5 \mid b^t x \le \beta\}$. When are they equal?

Sums of sets

[3.38] **Definition** Let C and D be two subsets of $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then we define $\boldsymbol{\alpha C}:= \{\alpha x \mid x \in C\}$ and we define $\boldsymbol{C + D}:= \{x + y \mid x \in C, y \in D\}$.

[3.39] **Example** `(Warning! C + C = 2C may not be correct)` Take the set $C = \{e_1, e_2\}$. Then $C + C = \{2e_1, 2e_2, e_1 + e_2\} \neq 2C$.

[3.40] **Fact** It is easy to show that, if C is convex, then $C + C = 2C$.

$\{2e_1, 2e_2\}$

[3.41] **Example** We have

$C + C = \{x + y \mid y, x \in C\} \neq \{e_1 + e_1, e_1 + e_2, e_2 + e_2\}$

$\{(0, y) \mid y \in [0,1]\}$ $\{(x, 0) \mid x \in [0,1]\}$ $\{(x, y) \mid x, y \in [0,1]\}$

C D

[3.42] **Fact** Let C, D be convex, $\lambda \in \mathbb{R}$. Then λC, $\overline{C}$ (closure), and $C + D$ are convex.‼

Some exercises

[3.43] **Exercise** In $\mathbb{R}^2$, let $S = \mathsf{conv}(e_1 + e_2, e_1 - e_2, -e_1 + e_2, -e_1 - e_2)$ and T be the closed disc of radius $\frac{1}{2}$ centered at 0. Can you draw $S + T$?

[3.44] **Exercise+** `(From our past knowledge)` Let $x, y \in \mathbb{R}^n$ be linearly independent. Show that there is a nonzero $w \in \mathsf{span}(x, y)$ such that the sum of the entries of w is zero. Do this in three ways: a) by a direct formula for w using x, y, b) by using dimensions, c) by using intermediate value theorem.

[3.45] **NoPen** a) Let C and D be convex sets. Is $C - D$ necessarily convex?

b) Can $C + D$ be convex when neither is convex?

c) Let $S \subseteq \mathbb{R}^2$ be the set $\mathsf{conv}((1,1),(3,1),(3,3),(1,3))$. What is $S + (-S)$?

d) Let C and D be convex sets. Is $C \times D$ necessarily convex?

e) In $\mathbb{R}^4$, what is $B_1(0) + B_1(0)$?

f) In $\mathbb{R}^4$, what is $B_1(0) + B_5(5e_1)$?

[3.46] **Exercise** Let S and T be nonempty closed sets. Is $S + T$ necessarily closed? What if S is compact?

[3.47] **Exercise** What is $S + T$?

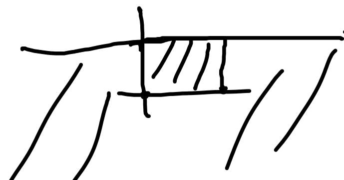
a) Take $S = \{x \in \mathbb{R}^2 \mid x(2) \leq 0\}$ and $T = \{x \in \mathbb{R}^2 \mid x(2) = 1/x(1), x(1) < 0\}$.

b) Take $S = \{x \in \mathbb{R}^2 \mid x(2) \leq 0\}$ and $T = \{x \in \mathbb{R}^2 \mid x(2) = 1/x(1), x(1) > 0\}$.

c) Take $S = \{x \in \mathbb{R}^2 \mid x(2) = 0\}$ and $T = \{x \in \mathbb{R}^2 \mid x(2) = 1/x(1), x(1) > 0\}$.

Caratheodory theorem and its applications

Why do we need Caratheodory theorem? Caratheodory theorem proves that a polytope is compact. That will be used to prove that a polytope is a bounded polyhedron. That result is used to prove Farka's lemma, which is very useful. For example, it is used to prove the existence of the generalized Lagrange multipliers in nonlinear optimization.



$\{x \mid x_1 \le 1\} \cap \{x \mid x_1 \ge 0\} \cap \{x \mid x_2 \le 1\} \cap \{x \mid x_2 \ge 0\}$

[3.48] **Definitions** A POLYTOPE means $\mathsf{conv}(S)$ where $S \subseteq \mathbb{R}^n$ is a nonempty finite set. A POLYHEDRON in $\mathbb{R}^n$ is the intersection of a nonempty finite class of closed half spaces.

[3.49] **Caratheodory theorem** Let $\emptyset \ne S \subseteq \mathbb{R}^n$. Then each $x \in \mathsf{conv}(S)$ is a convex combination of at most $n+1$ points of S.

$\exists\, x \in \mathrm{conv}(S)$ s.t. it cannot be written as a conv. comb of at most $n+1$ pts. Let $m > n+1$ be smallest s.t.

$x = \lambda_1 x_1 + \ \ + \lambda_m x_m$, where $\lambda_i \in (0,1)$, $\sum_{i=1}^{m} \lambda_i = 1$.

$-\alpha(0 = w_1 x_1 + \cdots + w_m x_m)$

$x = (\underbrace{\lambda_1 - \alpha w_1}_{\ge 0,\ 0}) x_1 + \cdots + (\underbrace{\lambda_m - \alpha w_m}_{\ge 0}) x_m$ $\quad X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}$. $\exists\, w \in \mathrm{null\,sp}(X)$, $w_1 + w_2 + \cdots + w_m = 0$

$w \in \mathrm{null\,sp}(X) \Rightarrow w_1 x_1 + \cdots + w_m x_m = 0$

Proof. Suppose that the statement is not true. So $\exists x \in \mathsf{conv}(S)$ which cannot be written as a convex combination of $n+1$ or less number of elements of S. But as $x \in \mathsf{conv}(S)$, there exists $\{x_1, \ldots, x_m\} \subseteq S$ of the smallest size such that

$$x = \lambda_1 x_1 + \cdots + \lambda_m x_m,\ 0 < \lambda_i < 1,\ \sum \lambda_i = 1.$$

Thus $m \ge n+2$. Form a matrix X with columns $x_1, \ldots, x_m$. Then $\mathsf{nullity}(X) \ge 2$. By [3.44], $\exists w \ne 0$ with

$$Xw = w(1)x_1 + \cdots + w(m)x_m = 0, \qquad w(1) + \cdots + w(m) = 0.$$

Take $t = \min\left\{\frac{\lambda_i}{w(i)} \mid w(i) > 0\right\} > 0$ and put $\gamma_i = \lambda_i - t w(i)$, $i = 1, \ldots, m$. So each $\gamma_i \ge 0$ and at least one $\gamma_i = 0$. As $\sum_{i=1}^{m} \gamma_i = 1$, we see that

$$x = \lambda_1 x_1 + \cdots + \lambda_m x_m - t(w(1)x_1 + \cdots + w(m)x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m$$

is a convex combination of at most $m-1$ vectors of S, a contradiction. ∎

[3.50] **Remark** `(Relation between conv(S) and cone(S))` If $x = \sum_{i=1}^{n} \lambda_i x_i \in \mathsf{cone}(S)$, $\lambda_i > 0$, then $\frac{x}{\sum \lambda_i} \in \mathsf{conv}(S)$. Thus, each nonzero point in $\mathsf{cone}(S)$ is a positive multiple of some point in $\mathsf{conv}(S)$.

[3.51] **Corollary** `(Application of Caratheodory theorem)` If $\emptyset \ne S \subseteq \mathbb{R}^n$ is compact, then $\mathsf{conv}(S)$ is compact. In particular, a polytope is compact.[5]

$\mathrm{conv}(S) = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i \;\middle|\; \lambda_i \in [0,1],\ \sum \lambda_i = 1,\ x_i \in S \right\}$ $\quad \underbrace{S \times S \times S \cdots \times S}_{n+1}$

$f(s_1, s_2, \ldots s_{n+1}, \lambda_1, \ldots \lambda_{n+1})$

$f: (S^{n+1} \times P_{n+1})$

Some exercises

[5] *Proof.* By Caratheodory theorem, $\mathsf{conv}(S) = \left\{\sum_{i=1}^{n+1} \lambda_i x_i \mid \lambda_i \ge 0, \sum_{i=1}^{n+1} \lambda_i = 1, x_i \in S\right\}$. Consider the set of probability vectors $P_{n+1} := \{\lambda = (\lambda_1, \ldots, \lambda_{n+1}) \mid \sum \lambda_i = 1, \lambda_i \ge 0\}$. This set being closed and bounded, is compact. Similarly, the set S^{n+1}, the $n+1$ fold Cartesian product of S is compact. So the set $P_{n+1} \times S^{n+1}$ is compact. Define $f : P_{n+1} \times S^{n+1} \to \mathbb{R}^n$ by $f(\lambda, x) = \sum_{i=1}^{n+1} \lambda_i x_i$. As f is a polynomial in λ_i, x_i, it is continuous. As $P_{n+1} \times S^{n+1}$ is compact, $\mathsf{conv}(S) = f(P_{n+1} \times S^{n+1})$ is compact. ∎

[3.52] **Exercise** One may ask a similar question for cones. For that, we have the following. a) Even if the set $S \subseteq \mathbb{R}^n$ is compact, the set $\mathsf{cone}(S)$ need not be closed. Give an example. b) Show that, further if $0 \notin \mathsf{conv}(S)$, then $\mathsf{cone}(S)$ is closed. (If you are unable to show this, at least remember the statement.)

[3.53] **Exercise** `(Projection of a convex set)` Let $\emptyset \neq S \subseteq \mathbb{R}^n$ be convex and $\hat{S}$ be obtained from S by making the nth entry 0 for each $x \in S$. Show that $\hat{S}$ is convex. How does $\mathsf{cone}(S)$ relate to $\mathsf{cone}(\hat{S})$?

[3.54] **Exercise** `(What if 0 is an interior point of cone(S)?)` Let $\emptyset \neq S \subseteq \mathbb{R}^n$.

1. Suppose that $0 \in \mathsf{cone}(S)^{\circ}$. Show that $0 \in \mathsf{conv}(S)^{\circ}$.
2. Suppose that $0 \in \partial\,\mathsf{conv}(S)$. Must $0 \in \partial\,\mathsf{cone}(S)$? (Here $\partial A := \overline{A} \cap \overline{A^c}$ means the boundary of A.)
3. Suppose that $0 \in \partial\,\mathsf{cone}(S)$. Can $\mathsf{cone}(S) = \mathbb{R}^n$?

[3.55] **Exercise** a) Give an example of a set S in $\mathbb{R}^2$ where $\mathsf{conv}(S)$ has a point which can only be written as a convex combination 3 points of S. b) Give an example of a set $S \subseteq \mathbb{R}^3$ and a point $x \in \mathsf{conv}(S)$ which is not a convex combination of less than 4 points of S.

[3.56] **NoPen** If $S \subseteq \mathbb{R}^n$ is a finite set, then is it necessary that $\mathsf{conv}(S)$ is closed?