1. Compound Poisson models are commonly used for rainfall. Here, we will look at stratifying such a model. In our model setting, the number of rainfall events (storms) in the coming month is $S \sim Poi(\lambda)$ with $\lambda = 2.9$. The depth of rainfall (in centimeters) in storm i is $D_i \sim Weib(k, \sigma)$ with shape $k = 0.8$ and scale $\sigma = 3$ (centimeters) and the storms are independent. The PDF of $Weib(k, \sigma)$ distribution is given by

$$f(x) = k\lambda x^{k-1} e^{-\lambda x^k} \quad \text{for } x > 0.$$

If the total rainfall is below 5 centimeters then an emergency water allocation will be imposed. Goal is to approximate the probability of imposing emergency water allocation in the coming month. Use simple Monte Carlo and stratification method to approximate the probability based on $n = 10^2$ and 10^4. Also provide the 99% confidence interval for the probability using both the methods. [Hint: Note that total rain fall is $\sum_{i=1}^{S} D_i$. For stratification, you may take the strata base one S, *viz.*, $S = 0$, $S = 1$, $S = 2$, $S = 3$, $S = 4$, $S = 5$, and $S \geq 6$. Can you think of the justification for clubbing $S \geq 6$ into one strata?]

2. Let $\boldsymbol{X} = (X_1, \ldots, X_{38})^T$ be a random variable having a Dirichlet distribution with parameters $\alpha_1, \ldots, \alpha_{38}$. Here $\alpha_j > 0$ for all $j = 1, 2, \ldots, 38$. The corresponding probability density function is given by

$$f(x_1, \ldots, x_{38}) = \begin{cases} \frac{\Gamma\left(\sum_{j=1}^{38} \alpha_j\right)}{\prod_{j=1}^{38} \Gamma(\alpha_j)} \prod_{j=1}^{38} x_j^{\alpha_j - 1} & \text{if } x_j > 0 \text{ and } \sum_{i=1}^{38} x_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the $\mu = P(X_{19} = \max_i X_i)$ using conditional Monte Carlo technique, where the values of the parameters are given in the following table.

j	1	2	3	4	5	6	7	8	9	10	11	12	13
α_j	2082	1999	2008	2047	2199	2153	1999	2136	2053	2121	1974	2110	2110
j	14	15	16	17	18	19	20	21	22	23	24	25	26
α_j	2168	2035	2019	2044	2191	2284	1912	2196	2099	2041	2192	2188	1984
j	27	28	29	30	31	32	33	34	35	36	37	38	
α_j	2158	2019	2032	2051	2192	2133	2142	2113	2150	2221	2046	2127	

Note: Generation from a Dirichlet distribution can be performed using the following Lemma. Prove it by yourself. Do not need to submit the proof.

Lemma: Let Y_i, $i = 1, 2, \ldots, k$, be n i.i.d. gamma random variables with shape parameter $\alpha_1, \ldots, \alpha_k$, respectively. Define $X_j = \frac{Y_j}{\sum_{i=1}^{k} Y_i}$. Then $\boldsymbol{X} = (X_1, \ldots, X_k)^T$ has a Dirichlet distribution with parameters $\alpha_1, \ldots, \alpha_k$.

Clearly, X_{19} is the largest X_j if and only if Y_{19} is the largest Y_j. A direct Monte Carlo estimate of μ can be found by repeatedly sampling $\boldsymbol{X} \in [0, \infty)^{38}$. This procedure needs $38n$ generation from different gamma distribution. Here you may condition on Y_{19}. Given that $Y_{19} = y_{19}$, the probability that Y_{19} is largest is

$$h(y_{19}) = \prod_{j=1, j \neq 19}^{38} G_{\alpha_j}(y_{19}),$$

where $G_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x e^{-t} t^{\alpha-1} dt$ (Why?).

3. Compute $\mu = E(f(\boldsymbol{X}))$, where $f(\boldsymbol{x}) = \max\left\{0, \frac{1}{5}\sum_{i=1}^{5} x_i\right\}$ and X_i are independent log-normal random variables with parameters (μ_i, σ_i^2), $i = 1, 2 \ldots, 5$, using covariate technique. Choose (μ_i, σ_i^2), $i = 1, 2 \ldots, 5$ and explicitly mention the choices in the report.

   Hint: You may take $h(x) = \max\left\{0, \frac{1}{5}\prod_{i=1}^{5} x_i\right\}$

Submission Deadline: October 19, 2021, 11:50 PM