$\partial S = \bar{S} \cap \overline{S^c} = \emptyset$

$S \to$ closed

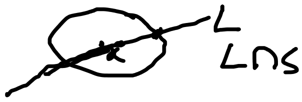
$S \cup S^c = \mathbb{R}^n \Rightarrow \bar{S} \cup \overline{S^c} = \mathbb{R}^n$

$S = (\bar{S})^c = (\overline{S^c})^c$

$S =$ open

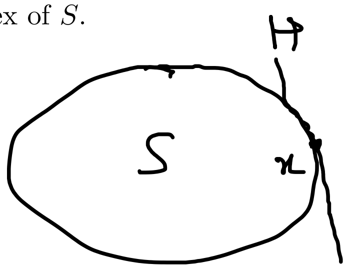
5 Lecture 5

[5.1] **Fact** Let $S \subseteq \mathbb{R}^n$ be a nonempty compact set. Then $\partial S \neq \emptyset$. (This follows from definition. You can give a more geometrical proof if S is given convex.)‼



[5.2] **Theorem** A nonempty compact convex set in $\mathbb{R}^n$ has a vertex.

Proof. Use induction on n. For $n = 1$, the statement is trivial. Assume the statement $\forall n < m$. Let S be a nonempty compact convex subset of $\mathbb{R}^m$. Then $\partial S \neq \emptyset$. Let $p \in \partial S$ and H a supporting hyperplane of S at p. Then $H \cap S$ may be seen as a nonempty compact convex set in $\mathbb{R}^{m-1}$. By induction hypothesis, $H \cap S$ has a vertex w. By [4.14], w is a vertex of S. ∎



$H \cap S \to$ nonempty cpt, conv

(induction $\to$ has a vertex.

That is also a vertex of S.

The following is the finite dimensional version of a famous theorem.

[5.3] **Krein-Milman-theorem.** (1940-Stud Math.) If E is the set of vertices of a nonempty, compact, convex set $S \subseteq \mathbb{R}^n$, then $S = \mathsf{conv}(E)$.

E = set of vertices of S. Then $S = \mathrm{conv}(E)$.

$E \subseteq S \Rightarrow \mathrm{conv}(E) \subseteq S$. Conversely $x \in S$.

first assume $x \in \partial S$. $H \cap S \to$ nonempty, cpt, convex.

$x \in \mathrm{conv}(E)$.

Proof. It is clear that $\mathsf{conv}(E) \subseteq S$. We shall now show that every point in S is a convex combination of some points in E. We will use induction (on the dimension n) to show that. For $n = 1$, the statement is trivial. Suppose the statement is true for all $n < m$. Let $\emptyset \neq S$ be a compact convex subset of $\mathbb{R}^m$ and $x \in S$. We have to show that $x \in \mathsf{conv}(E)$.

If $x \in \partial S$, then by [4.13], there is a supporting hyperplane H of S at x. Thus $x \in H \cap S$, which may be viewed as compact convex subset of $\mathbb{R}^{m-1}$. So by induction hypothesis x is in the convex hull of the vertices of $H \cap S$. Since vertices of $H \cap S$ are vertices of S, we see that $x \in \mathsf{conv}(E)$.

The only other possibility is that $x \in S^\circ$. Let L be any line that passes through x. Then $S \cap L$ is a closed line segment of positive length.‼ Let it be $[w, z]$. Notice that and $x \in [w, z]$ and $w, z \in \partial S$. Hence $w, z \in \mathsf{conv}(E)$. Since x is convex combination of w and z, it follows that $x \in \mathsf{conv}(E)$. ∎

Some exercises

[5.4] **Exercise(E)** (General separation of two nonempty disjoint convex sets.) Let S and T be nonempty disjoint convex sets in $\mathbb{R}^n$. Then there exists a hyperplane which separates S and T. (Consider separating $S - T$ and 0.)

[5.5] **Exercise(M)** `(Separating hyperplane.)` Can you strictly separate $S = \{(x, y) \mid y \geq 1/x, x > 0\}$ and $T = \{(x, 0) \mid x \in \mathbb{R}\}$ by a hyperplane? By two open disjoint subsets?

[5.6] **Exercise(E)** `(Illustration of [5.3].)` Take our favorite set T in $\mathbb{R}^2$. Take the point $x = (.5, .6)$. Take a line passing through x, say $x_1 + x_2 = 1.1$.

a) What are the points y, z for this line?

b) What are the supporting hyperplanes H_y and H_z at y and z, respectively?

c) What are $H_y \cap T$ and $H_z \cap T$?

d) Express y as a convex combination of vertices of $H_y \cap T$.

e) Express z as a convex combination of vertices of $H_z \cap T$.

f) Thus you obtain x as a convex combination of vertices of T, what is it?

[5.7] **Exercise(M)** `(A point in` $\partial\,\mathsf{cone}(S)$ `is a nonnegative multiple of a point in` ∂S `if` $0 \notin \overline{S}$`.)` Let $S \subseteq \mathbb{R}^n$ be nonempty, bounded and convex such that $0 \notin \overline{S}$. Take any $y \in \partial\,\mathsf{cone}(S)$. Show that there is a point $x \in \partial S$ such that $y = \lambda x$ for some $\lambda \geq 0$. Why did we need $0 \notin \overline{S}$?

[5.8] **Exercise(M)** Let $n > 1$ and $S \subseteq \mathbb{R}^n$ be a nonempty bounded convex set not containing the origin. Show that there is a supporting linear hyperplane of S. Can we lift any of the conditions bounded or convex or $n > 1$?

[5.9] **Exercise(E)** `(Writing as intersections)` Show that a closed convex set can be written as the intersection of a class of closed half spaces. Show that a closed convex cone is the intersection of a class of closed linear half spaces.

[5.10] **Exercise(E)** `(Subspaces of nonempty interior)` Let X be a subspace of $\mathbb{R}^n$ and $x \in X$. Assume that $B_\alpha(x) \subseteq X$ for some $\alpha > 0$. Show that $X = \mathbb{R}^n$.

[5.11] **Exercise(E)** `(Point of maximum distance)` Let $\emptyset \neq S \subseteq \mathbb{R}^n$ be compact and convex. Let $a \notin S$. Show that there is a point of maximum distance in S from a. Shall it be unique?

[5.12] **Exercise(E)** `(Illustration of points of shortest distance and strict separation)` Consider our favorite set T in $\mathbb{R}^2$ and take $x = (-1, \alpha)$. There is a unique $x_0 \in T$ with minimum distance from x. Which point is x_0? Write the corresponding hyperplane H talked in the strict separation theorem.

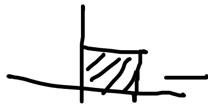
[5.13] **Exercise(E)** Let S and T be disjoint closed convex sets in $\mathbb{R}^n$ and T be bounded. Then show that there exists a hyperplane that strictly separates S and T.

[5.14] **Exercise(E)** `(Existence of a vertex)` Let $\emptyset \neq S \subseteq \mathbb{R}^n$ be compact and convex. Let $a \notin S$. Argue that the point $b \in S$ which is at a maximum distance from a is a vertex of S.

Polytopes and polyhedrons

Why do we need them? They will be used to prove many results about the feasible set, one of them being the fact that the feasible set $T = \{x \mid Ax = b, x \geq 0\}$ has only finitely many vertices.

[5.15] **Definition** Recall that a polytope is the convex hull of a nonempty finite set and a polyhedron is



$\to [0,1]^2$ $\qquad a \leq x \ \ \forall x \in S$

coordinatewise

the intersection of finitely many (at least one) closed half-spaces. We say a set $S \subseteq \mathbb{R}^n$ is BOUNDED BELOW, if there is a point $a \in \mathbb{R}^n$ such that $a(i) \leq x(i)$, $\forall i, \forall x \in S$.

[5.16] **Facts** Polytopes and polyhedrons are closed convex sets. A polytope is always bounded.‼

[5.17] **Recall** We already know that the vertices of a polytope $\mathsf{conv}(S)$ form a subset of S.

[5.18] **Example** Our favorite set is a polytope. It is also a polyhedron. In general, a solid hypercube (that is, a product of n many closed nontrivial closed intervals) in $\mathbb{R}^n$ is a polytope and a polyhedron.

[5.19] **Example** Let $A \in M_{m,n}$ and $b \in \mathbb{R}^n$. Then the nonnegative solution space $\{x \mid Ax = b, x \geq 0\}$ is a polyhedron, as it is the intersection of $2m + n$ halfspaces.

$2x+3y+5z=1 \implies 2x+3y+5z \leq 1$, ≥ 1

[5.20] **Example** The set $\{x \in \mathbb{R}^3 \mid x(i) \geq -i^2\}$ is bounded below by $(-1,-4,-9)$. Also by $(-10,-10,-10)$. But the set $\{x \in \mathbb{R}^3 \mid x(1) = x(2) = 5\}$ is not bounded below.

Suppose that someone gives us a set of inequations in $\mathbb{R}^9$ and gives a point in the polyhedron described by it. How do we know whether it is a vertex? The following result gives an answer. Before that let discuss a notation.

$x_1 - x_3 \leq 1$, $2x_1 + 3x_2 + x_3 \leq 6$ $\qquad A_{i:} \to$ ith row, $A_{:i} \to$ ith col

[5.21] **Notation** Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 6 \\ 1 \\ 6 \end{bmatrix}$. Then $z = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of the system $Ax \leq b$.

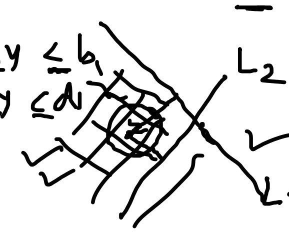
Then by $\boldsymbol{A_z}$ we denote the submatrix of A, formed by the rows of A (in that order) satisfying $A_{i:}z = b_i$ (that is, the rows corresponding to the equalities in $Az \leq b$). Here $A_z = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$. We use $\boldsymbol{b_z}$ to denote the part of b corresponding to A_z. We use $\overline{\boldsymbol{A}}_{\boldsymbol{z}}$ and $\overline{\boldsymbol{b}}_{\boldsymbol{z}}$ to denote the remaining part of A and b, respectively.

[5.22] **Lemma** (Test for vertex of a polyhedron) Let $z \in P = \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$. Then z is a vertex of P iff $\mathsf{rank}\, A_z = n$. Geometrically, z is a vertex iff there are n linearly independent supporting hyperplanes at z.

$z \in P = \{x \mid Ax \leq b\}$ is a vertex iff A_z has rank n.

If $A_z = \emptyset$, then $z \in P^o$. Then z is not a vertex.

$a_1 x + a_2 y \leq b_1$, $c_1 x + c_2 y \leq d$



$A_z = \begin{bmatrix} \text{—} \\ \text{—} \\ \text{—} \end{bmatrix}$. rank $A_z < n \implies \exists\, w \neq 0$ s.t. $A_z w = 0$

$A_z(z + \alpha w) = b_z$

$\overline{A}_z z < \overline{b}_z$. So by continuity, $\exists\, \delta > 0$ s.t. $x \in B_\delta(z) \implies \overline{A}_z x < \overline{b}_z$

$\overline{A}_z\left(z + \frac{\delta}{2}\frac{w}{\|w\|}\right), z - \frac{\delta}{2}\frac{w}{\|w\|} < \overline{b}_z$

$A_z\left(z + \frac{\delta}{2}\frac{w}{\|w\|}\right) = b_z$

Converse rank $A_z = n$. Assume z is not a vertex.

$\exists\, u, v \in P$, $(u \neq v)$, $\lambda \in (0,1)$ s.t. $z = \lambda u + (1-\lambda)v$.

$(b_z = A_z z = A_z \lambda u + A_z (1-\lambda) v \leq \lambda b_z + (1-\lambda) b_z = b_z) \Rightarrow A_z u = b_z = A_z v$

$A_z(u-v) = 0 \Rightarrow\Leftarrow$

Proof. Let z be a vertex of P. If A_z is empty, then $Az < b$. Since $f(x) = Ax$ is a continuous function of x, $\exists\, \epsilon > 0$ such that $y \in B_\epsilon(z) \Rightarrow Ay < b$. So $B_\epsilon(z) \subseteq P$, implying that z is not a vertex.

If possible, let $\mathsf{rank}\, A_z < n$. Take $v \in \mathsf{nullsp}\, A_z$, $\|v\| = 1$. Then $A_z(z + \alpha v) = b_z$ for each $\alpha \in \mathbb{R}$.

Also we have $\overline{A}_z z < \overline{b}_z$. Thus $\exists\ \epsilon > 0$ such that $y \in B_\epsilon(z) \Rightarrow \overline{A}_z y < \overline{b}_z$.

From the previous two observations, we see that $A_z(z \pm \frac{\epsilon}{2}v) = b_z$, and $\overline{A}_z(z \pm \frac{\epsilon}{2}v) < \overline{b}_z$. Thus $z \pm \frac{\epsilon}{2}v \in P$, implying that z is not a vertex. So we must have $\mathsf{rank}\, A_z = n$.

Conversely, suppose that $\mathsf{rank}\, A_z = n$ and z is not a vertex of P. So $z = \lambda u + (1-\lambda)v$, where $\lambda \in (0,1)$, $u \neq v$, $u, v \in P$. Notice that

$$b_z = A_z z = \lambda A_z u + (1-\lambda) A_z v \leq \lambda b_z + (1-\lambda) b_z = b_z.$$

Hence we must have $A_z u = A_z v = b_z$. So $A_z(u - v) = 0$, where $u - v \neq 0$. This means $\mathsf{rank}\, A_z < n$, a contradiction. ∎

[5.23] **Remark** By [5.22], for two distinct vertices z and w, we cannot have that A_z as a submatrix of A_w, otherwise, $A_z z = b_z = A_z w \Rightarrow A_z(z - w) = 0 \Rightarrow \mathsf{rank}\, A_z < n$, a contradiction. Thus any rank n submatrix of A can correspond to at most one vertex of the polyhedron.

$A_z \to n \times n$

$z \quad \neq \quad w$

$A_z \subseteq A_w$

$A_z z = b_z \qquad A_z w = b_z$

$\Rightarrow A_z(z - w) = 0 \Rightarrow\Leftarrow$

To find vertices of a polyhedron. To find the vertices of $\{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$, do the following.

1. Find a $n \times n$ submatrix B of A of rank n and let b_B be the corresponding subvector of b.
2. Find $z = B^{-1} b_B$. If z satisfies $Az \leq b$, then z is a vertex, otherwise it is not.
3. Repeat this for each submatrix of A of rank n.

∘ A list of vertices of the polyhedron $\{(x, y, z) \mid x + y \leq 2, y + z \leq 4, x + z \leq 3, -2x - y \leq 3, -y - 2z \leq 3, -2x - z \leq 2\}$ is

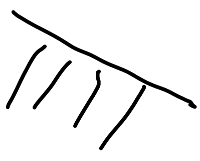
$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{matrix} 2 \\ 4 \\ 3 \end{matrix}$$

$$\begin{bmatrix} 1/2 & 3/2 & 5/2 \\ -4/3 & 10/3 & 2/3 \\ 11/3 & -5/3 & -2/3 \\ 1/5 & 9/5 & -12/5 \\ -4/3 & -1/3 & 13/3 \\ -9/4 & 3/2 & 5/2 \\ 3/2 & -6 & 3/2 \\ -2/3 & -5/3 & -2/3 \end{bmatrix}.$$

$A_{6,3}$

[5.24] **Corollary** A nonempty polyhedron has finitely many vertices. (It may have no vertices at all.)

Proof. By [5.22] and [5.23], the number of vertices of the polyhedron $\{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$ is at most the number of square submatrices of A with rank n, which is finite. ∎



Some exercises

[5.25] <u>**Exercise(E)**</u> `(Illustration of [5.22].)` Let T be the solid unit cube in $\mathbb{R}^3_+$ with one corner at $(0,0,0)$.

1. It can be written as intersection of six halfspaces. Add one more halfspace: $x_1 + x_2 + x_3 \le 3$. Express the set in the form $A_{7\times 3}\, x \le b$.
2. Take $z = (1,1,1)$. Write A_z. What is $\mathsf{rank}\, A_z$? Is z a vertex by the test?
3. Take $z = (.5,1,1)$. Write A_z. What is $\mathsf{rank}\, A_z$? Apply the test to conclude that z is not a vertex. Give a small line segment around z in T. Is the difference of the endpoints a null vector for A_z?

[5.26] <u>**Exercise(E)**</u> Consider $\left\{x \mid \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \\ 2 & -1 & -3 \\ 4 & 3 & 2 \\ 3 & -1 & 2 \\ 3 & 4 & 2 \end{bmatrix} x \le \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}\right\}$. Is $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ a vertex of this polyhedron?

[5.27] <u>**Exercise(E-)**</u> `(Program to find the vertices of a polyhedron.)` Write a Matlab/Octave program to find the vertices of a polyhedron $\{x \mid Ax \le b\}$, with the inputs A and b. The output should be a list of the vertices. That is, the command line "list=vertices(A,b)" should list the vertices of the polyhedron.

[5.28] <u>**NoPen**</u> a) Is it necessary for a nonempty polytope to have a vertex?

b) Is it necessary for a nonempty polyhedron to have a vertex?

c) T/F? The intersection of 10 half-spaces in $\mathbb{R}^7$ has at most $C(10,7)$ vertices.

d) How many vertices does the polyhedron $\{(x,y,z) \mid x \ge 0, y \ge 0, x+y \le 1\}$ have? What if we put one more half space $z \ge 0$?

e)`(Vertices with rational entries)` Consider a polyhedron $P = \{x \mid Ax \le b\} \subseteq \mathbb{R}^n$, where A and b have integer entries. Is it necessary that vertices of P have rational coordinates?

f) Imagine the closed convex region T in $\mathbb{R}^3$ bounded by the planes $x_1 = 1$, $x_1 = -1$, $x_2 = 2$, $x_2 = -2$, $x_3 = 3$, $x_3 = -3$. Write its vertices.

g) Imagine the closed region T in $\mathbb{R}^3$ bounded by the planes $x_1 = 1$, $x_1 = -1$, $x_2 = 2$, $x_2 = -2$, $x_3 = 3$, $x_3 = -3$. Write it as intersection of 7 closed halfspaces.

h) Imagine the closed region T in $\mathbb{R}^3$ bounded by the planes $x_1 = 1$, $x_1 = -1$, $x_2 = 2$, $x_2 = -2$, $x_3 = 3$, $x_3 = -3$. Write 3 supporting hyperplanes at $(1,2,2)$.

i) T/F? Imagine the closed region T in $\mathbb{R}^3$ bounded by the planes $x = 1$, $x = -1$, $y = 2$, $y = -2$, $z = 3$, $z = -3$. Then at each point on ∂T we have more than one supporting hyperplanes.

j) T/F? Let $S, T \subseteq \mathbb{R}^3$ be nonempty convex sets and H be a hyperplane separating them strictly. Then there is another hyperplane which also separates them strictly.

k) In $\mathbb{R}^3$, write two vectors c such that $c^t x = 1$ strictly separates $B_1(0)$ and $(1,1,1)$.

[5.29] <u>**Exercise(E)**</u> Determine the vertices of the following polyhedrons.

a) $\{(x,y) \mid x \ge 0, y \ge 0, y - x \le 2, x + y \le 8, x + 2y \le 10, x \le 4\}$.

b) $\{(x, y, z) \mid x + y \leq 2, y + z \leq 4, x + z \leq 3, -2x - y \leq 3, -y - 2z \leq 3, -2x - z \leq 2\}$.

c) $\{(x, y, z) \mid x + y \geq 1, x + z \geq 1, y - z \geq 0, x \geq 0, y \geq 0\}$.

[5.30] **Exercise(M)** Determine which of the following sets are polyhedrons. In case they are, express them in $\{x \mid Ax \leq b\}$ form and determine their vertices.

a) $S = \{\lambda a + \alpha b + \beta c \mid \lambda, \alpha, \beta \in [-1, 1]\}$, where $a, b, c \in \mathbb{R}^3$ are fixed vectors not coplanar.

b) $S = \{x \mid x \geq 0, x^t y \leq 1 \ \forall y \text{ with } \|y\| = 1\}$.

c) $S = \{x \mid x \geq 0, x^t y \leq 1 \ \forall y \text{ with } \|y\|_1 = 1\}$. (Here $\|y\|_1$ means $\sum |y(i)|$.)

[5.31] **Exercise(M)** Let

$$S = \{(\cos\theta, \sin\theta) \mid \theta = \frac{2k\pi}{6},\ k = 1, 2, \ldots, 6\} \quad \text{and} \quad T = \{(\cos\theta, \sin\theta) \mid \theta = \frac{2k\pi}{12},\ k = 1, 2, \ldots, 12\}.$$

Can you find a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\mathsf{conv}(S)) = \mathsf{conv}(T)$?

[5.32] **Exercise(H)** Let $C = [-1, 1]^3 \subseteq \mathbb{R}^3$. We know there are linear transformations $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(C) = C$. How many are there in all?

Bounded polyhedrons and polytopes

Is a nonempty bounded polyhedron a polytope? Is a polytope a bounded polyhedron? In this section we will show these. We need to establish a few lemmas before that.

[5.33] **Lemma** Let $S \subseteq \mathbb{R}^n$ be nonempty, closed, convex and bounded below. Then S has a vertex. In particular, if $T = \{x \mid Ax = b, x \geq 0\} \neq \emptyset$, then T has a vertex.

Proof. Take a point $y \in S$. Pick any direction d (means a nonzero vector) with a negative entry and try to move out from y in that direction. You cannot move indefinitely, as S is bounded below. Let y_1 be the point where we stop. It is a boundary point of S.

Take a supporting hyperplane H_1 at y_1 and consider the new set $S_1 := H_1 \cap S$. Take a new direction d to move inside H_1 (with a negative entry).[9] You cannot move indefinitely, as S_1 (being a subset of S) is also bounded below. The point y_2 where you stop is a boundary point of S_1 in H_1.

Take a supporting hyperplane H_2 to S_1 at y_2 in H_1 and consider the new set $S_2 := H_2 \cap S_1$.

Repeat the steps a few more times to get $S_n = H_n \cap S_{n-1}$, where H_n is a supporting hyperplane to S_{n-1} at y_n in H_{n-1}.

As $S_n = \{y_n\}$, we see that y_n is vertex of S_n. Hence it is a vertex of S_{n-1} and so on. Finally, it is a vertex of S. ∎

[5.34] **Example** Consider the set $S = \mathsf{conv}(e_1, e_2, e_3)$ in $\mathbb{R}^3$ bounded below by $(0, 0, 0)$.

Pick $y = (.1, .2, .7)$. Pick a direction $d = (-1, -1, -1)$.

Then $y_1 = (.1, .2, .7)$. The hyperplane is $H_1 : x + y + z = 1$ and the set $S_1 = \mathsf{conv}(e_1, e_2, e_3)$. (But we are now looking at it inside H_1.)

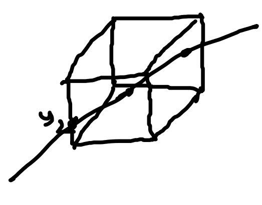
[9] To do that, pick any point $x \in H_1$ other than y_1 and take either $d = x - y_1$ or $y_1 - x$, whichever is suitable.

$y_2 \in \partial S$ $\quad H \cap S \quad x \in H,\ x \neq y_2$

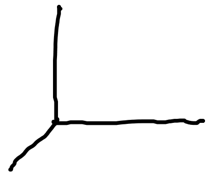
$y_1 + d\begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\quad a$

$d_2 \quad x - y_2,\ y_2 - x$

$y_2 + \alpha d_2 \to y_3 \to \cdots y_n$



$y_1 + \alpha d_1$



Pick any point in H_1 other than y_1: $x = (.3, .1, .6)$. Then $d_1 = x - y_1 = (.2, -.1, -.1)$ to move inside H_1. Then $y_2 = (.5, 0, .5)$. A supporting hyperplane to S_1 at y_2 in H_1 is nothing but the line given by e_1 and e_3. Next pick $x = (.3, 0, .7)$ and $d_2 = x - y_2 = (-.2, 0, .2)$. Then $y_3 = (0, 0, 1)$ and that is a vertex.

$y_2 + \alpha d_2$