$\boxed{\sum a_i = \sum b_j}$

$C = \left[\quad \right]_{m\times n}$ cost matrix

16 Lecture 16

Transportation problems

[16.1] **Definition** `(What is a balanced transportation problem?)`

a) Suppose that a_i amount of goods is available at the source S_i, $i = 1, \ldots, m$ and that b_j amount is demanded at the sink T_j, $j = 1, \ldots, n$.

b) Assume that $\sum a_i = \sum b_j$.

c) Let c_{ij} be the cost of transportation of 1 unit of goods from S_i to T_j.

d) We want to transport the goods from the sources to the sinks with the minimum transportation cost. This is called a BALANCED TRANSPORTATION PROBLEM (BTP).

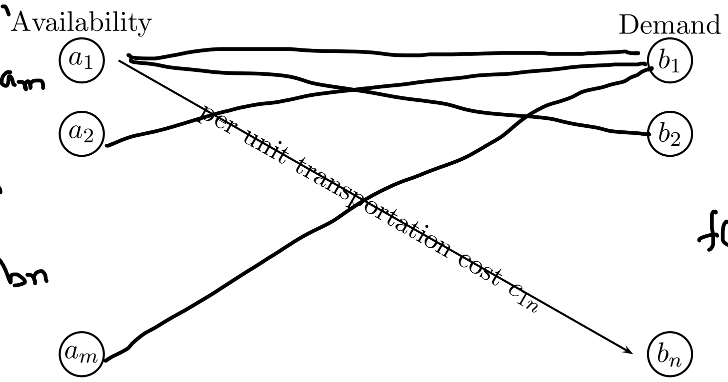
$x_{11}+x_{12}+ \cdots + x_{1n} = a_1$

$x_{m1}+ \cdots + x_{mn} = a_m$

$x_{11}+x_{21}+ \cdots + x_{m1} = b_1$

$\vdots$

$x_{1n}+ \cdots + x_{mn} = b_n$



x_{ij} = no of units take fr source i to sink j

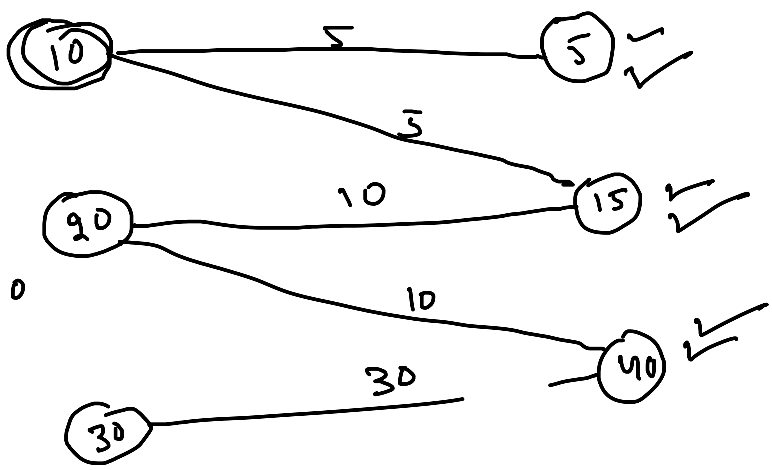
$f(x) = \sum_{i,j} c_{ij} x_{ij}$

e) `(Modeling a btp)` To model it, let x_{ij} be the amount of goods to be transported from S_i to T_j. Then our problem is the lpp

$$
\begin{array}{lll}
(P) & \min & \sum_{i,j} c_{ij}x_{ij} \\
& \text{s.t.} & \sum_j x_{ij} = a_i,\ \forall i; \quad \sum_i x_{ij} = b_j,\ \forall j; \quad x_{ij} \geq 0.
\end{array}
$$

$X = (x_{ij})$

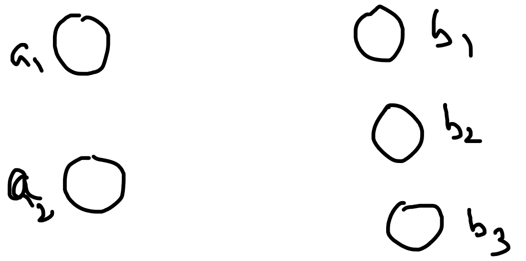
$\left(\quad \right)$



20 make 60 posts

[16.2] **Lemma** Consider the BTP in (P). Then the set of feasible solutions is always nonempty.

Proof. Distribute the a_i amount available at S_i among the sinks in the ratio $b_1 : b_2 : \cdots : b_n$. Do this for each $i = 1, \ldots, n$. This gives a feasible solution of the BTP. ∎



$$\begin{aligned}
x_{11} + x_{12} + x_{13} &= a_1 \\
x_{21} + x_{22} + x_{23} &= a_2 \\
x_{11} + x_{21} &= b_1 \\
x_{12} + x_{22} &= b_2 \\
x_{13} + x_{23} &= b_3
\end{aligned}$$

[16.3] **Class workout** `(Just do not see, observe)` Take a 2×3 btp. Write the constraints. Can we write the slpp? Notice the column of $x_{2,3}$. It has two parts, roughly the a_i-part and the b_j-part. Can you see these two parts? Do you see the two parts for x_{11}?

$$\begin{array}{cc}
 & \begin{array}{cccccc} \checkmark & & & & & \checkmark \\ x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \end{array} \\
A = & \left[\begin{array}{cccccc}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right] \left[\ \ \right] = \left[\begin{array}{c} a_1 \\ a_2 \\ \hline b_1 \\ b_2 \\ b_3 \end{array}\right]
\end{array}
\qquad
\begin{array}{c} b \end{array}
\qquad
x_{ij}\ \left[\begin{array}{c} e_i \\ \hline e_j \end{array}\right]$$

e_1 e_2 e_1 e_3

$m+n$ rows

mn cols

[16.4] **Observe** (`Writing slpp`) We can write the problem (P) as an slpp, in the following way:

$$\begin{array}{ll} \min & \underline{c^t x} \\ \text{s.t.} & Ax = b,\ x \geq 0 \end{array}, \tag{12}$$

where $x = \begin{bmatrix} x_{11} & \cdots & x_{mn} \end{bmatrix}^t$, $c^t = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{mn} \end{bmatrix}$, $b^t = \begin{bmatrix} a_1 & \cdots & a_m & b_1 & \cdots & b_n \end{bmatrix}$ and

$$A_{(m+n)\times mn} = \left[\begin{array}{cccc|cccc|c|cccc}
1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\
\hline
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & 1
\end{array}\right]$$

[16.5] **Definition** We shall refer to the matrix A in (12) as the CONSTRAINT MATRIX. Notice that the size of A is $(m+n) \times mn$. Also notice that the column corresponding to the variable $x_{i,j}$ is $C_{i,j} := \left[\begin{array}{c} e_i \\ \hline e_j \end{array}\right]$.

We need the information about its rank, in order to use simplex method to solve it.

[16.6] **Theorem** (`Rank of the constraint matrix is` $m+n-1$.) Consider the matrix A in (12). Then $\mathsf{rank}\, A = m + n - 1$. Let X_i be the matrix obtained by deleting row i from A. Then $\mathsf{rank}\, X_i = \mathsf{rank}\, A$.

We shall give three proofs. One now. Second as exercise and the third (very important) one a little latter (so that it does not become overwhelming).

$R_1 \to$ row 1 $\qquad R_1 + R_2 + \cdots + R_m = R_{m+1} + \cdots + R_{m+n}$

$R_1 = -R_2 \cdots - R_m + R_{m+1} + \cdots + R_{m+n}$

$\underline{C_{1,1}, C_{1,2}, \cdots, C_{1,n}, C_{2,1}}, C_{3,1}, \quad C_{m,1} \longrightarrow m+n-1$

Proof. Let us use R_i for row i. Notice that, $R_1 + \cdots + R_m = R_{m+1} + \cdots + R_{m+n}$. So $R_1 = -R_2 - \cdots - R_m + R_{m+1} + \cdots + R_{m+n}$. Similarly, we can express any row as a linear combination of the remaining rows. It now follows that $\mathsf{rank}\, X_i = \mathsf{rank}\, A \leq m + n - 1$.

Notice that the columns $C_{m,1}, C_{m-1,1}, \ldots, C_{1,1}, C_{1,2}, \ldots, C_{1,n}$ are linearly independent. To see that let $\alpha_{m,1}C_{m,1} + \cdots + \alpha_{1,1}C_{1,1} + \alpha_{1,2}C_{1,2} + \cdots + \alpha_{1,n}C_{1,,n} = 0$. Then

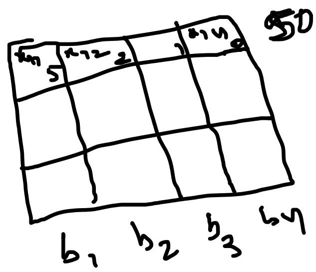
$$\left[\frac{(\alpha_{m,1}e_m + \cdots + \alpha_{1,1}e_1) + (\alpha_{1,2} + \cdots + \alpha_{1,n})e_1}{(\alpha_{m,1} + \cdots + \alpha_{1,1})e_1 + \alpha_{1,2}e_2 + \cdots + \alpha_{1,n}e_n}\right] = \left[\frac{0}{0}\right].$$

Equating the second line, as $e_1, \ldots, e_n$ are linearly independent, we see that $\alpha_{1,2} = \cdots = \alpha_{1,n} = 0$. Use this in the first line, as $e_1, \ldots, e_m$ are linearly independent, we see that $\alpha_{m,1} = \cdots = \alpha_{1,1} = 0$. Thus $\mathsf{rank}\, A = m + n - 1$. ■

[16.7] **<u>Exercise</u>** `(Second proof of the preceding theorem)` Follw this argument to write an alternate proof of the previous theorem. Consider X_{m+n}. Suppose that a linear combination of the rows $v = \sum_{i=1}^{m+n-1} \alpha_i A_{i:} = 0$. Since $v(n) = 0$, we see that $\alpha_1 = 0$. As $v(1) = 0$ and $\alpha_1 = 0$, we have that $\alpha_{m+1} = \alpha_{m+2} = \cdots = \alpha_{m+n-1} = 0$. Continue to show that these rows are linearly independent.

[16.8] **<u>Definition</u>** `(Transportation array)` Now that we know the rank of A in (12) is $m + n - 1$, one should now ask 'how to test whether set of $m + n - 1$ variables will form a basis?' We will see later that the transportation algorithm is nothing but an implementation of simplex algorithm in a smaller sized table (than the simplex table). This table is called the TRANSPORTATION ARRAY. In fact, the transportation algorithm uses the structure of the matrix A and provides a more effective implementation of the simplex algorithm. The transportation array is shown below.

x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}		x_{1n} c_{1n}	a_1
x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}		x_{2n} c_{2n}	a_2
			$\ddots$		
x_{21} c_{m1}	x_{m2} c_{m2}	x_{m3} c_{m3}		x_{mn} c_{mn}	a_m
b_1	b_2	b_3		b_n	



Notice that a row in A of (12), corresponds to a row or a column in the transportation array.

[16.9] **<u>Example</u>** A company has plants in three cities Guwahati, Shillong, Jorhat with capacities of production (in tons) of 200, 250, and 300, respectively. The company has signed contracts to supply goods of weight (in tons) 150, 200, 250, and 150 to mills (1), (2), (3), and (4), respectively, every month. The cost (in thousand) of transportation of the product per ton is

Plant	Mill (1)	(2)	(3)	(4)
G	1.5	2.5	2	1.1
S	2.0	3.0	1.5	1.2
J	1.8	3.2	1.8	1.0

Write the transportation array. Observe how easily it describes the problem.

x_{11} 1.5	x_{12} 2.5	2	1.1	200
2.0	3	1.5	1.2	250
1.8	3.2	1.8	1.0	300
150	200	250	150	X

$x_{12}+x_{22}+x_{32}=b_2$

Answer. The transportation array is

x_{11} 1.5	x_{12} 2.5	x_{13} 2	x_{14} 1.1	200
x_{21} 2	x_{22} 3	x_{23} 1.5	x_{24} 1.2	250
x_{31} 1.8	x_{32} 3.2	x_{33} 1.8	x_{34} 1	300
150	200	250	150	

.

$x_{31}+x_{32}+x_{33}+x_{34}=a_3$

[16.10] <u>**Class workout**</u> In a 3×4 transportation problem, the matrix A has size $\underline{\ 7\ } \times \underline{\ 12\ }$.

Which row/column of the transportation array correspond to the rows of A?

Row/column of the transportation array	Row of A
3rd row	3rd
2nd column	5th

[16.11] <u>**Class workout**</u> Let A be a 4×6 matrix with $a_{3,5} = \alpha$. Give two permutation matrices P, Q such that α appears in the $(1,1)$th position of PAQ. Give your answer using E_{ij} which means a matrix all of whose entries are zero except that the (i,j)th entry is 1. (Can this be an unique answer? Of course not.)

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{P,\ 4\times 4} \begin{bmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & a_{3,5} & \square \\ \square & \square & \square & \square & \square & \square \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{6\times 6}$$

$PP^T = I$

$I - E_{11} - E_{55} + E_{51} + E_{15}$

Answer. Take $P = I - E_{33} - E_{11} + E_{31} + E_{13}$ and $Q = I - E_{11} - E_{55} + E_{15} + E_{51}$. In fact, any permutation matrix $P_{7\times 7}$ with $p_{1,3} = 1$ and any permutation matrix $Q_{9\times 9}$ with $q_{5,1} = 1$ will do.

8×10 matrix

Lower triangular basis

[16.12] <u>**Definition**</u> Let $A \in M_{m,n}$. Recall that A is called LOWER TRIANGULAR if each $a_{i,j}$ with $j > i$ is 0.

[16.13] <u>**Lemma**</u> Let $k \geq 2$ and $Y \in M_{k,k-1}$. Assume that

a) each column of Y has only two nonzero entries, one 1 and one -1,

b) each row of Y is a linear combination of the remaining rows and

c) $\operatorname{rank} Y = k - 1$.

Then there exist permutation matrices P and Q such that PYQ is lower triangular.

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \quad 6\times 6$$

Proof.

$$\begin{bmatrix} -1 & & & \\ & -1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

The lemma is true for $k = 2$. Let $k > 2$.

Since each row of Y is a combination of the remaining rows, the deletion of any row from Y keeps the rank unchanged. Let Y_r be the matrix obtained from Y by deleting the rth row. As $\mathsf{rank}\, Y = k - 1$, we see that Y_r is nonsingular.

Is it possible that each column of Y_r still contains two nonzero entries? No, otherwise, it would be singular, as the column sum is zero. Hence Y_r can have at most $2(k-1) - 1$ nonzero entries.

As Y_r is nonsingular, it cannot have a zero row and as it has $k - 1$ many rows, it must have a row with exactly one nonzero entry.

Hence, the original matrix Y has a row containing exactly one nonzero entry. So, we can find permutation matrices P_1 and Q_1 such that $P_1 Y Q_1$ has the form $\left[\begin{array}{c|c} 1 & 0 \\ \hline * & X \end{array}\right]$.

Observe that X is $k - 1 \times k - 2$ matrix satisfying the hypothesis of the lemma.[10] Hence by induction, we can find permutation matrices P_x and Q_x such that $T_x := P_x X Q_x$ is lower triangular. Thus

$$\left(\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_x \end{array}\right] P_1\right) Y \left(Q_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q_x \end{array}\right]\right) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_x \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline * & X \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q_x \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline * & P_x X Q_x \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline * & T_x \end{array}\right]$$

is lower triangular. As the product of permutation matrices is a permutation matrix[11], the proof is complete.■

[10] Since the row above X is a zero row and in Y each row was a linear combination of the remaining rows, it follows that each row of X is also a linear combination of the remaining rows. It is easy to see that the other two conditions also hold.

[11] Try multiplying the all ones vector from left and right and use that they are invertible.

[16.14] **Theorem** `(Any basis is a lower triangular basis.)` Let $(x_{i_1,j_1}, \cdots, x_{i_{m+n-1},j_{m+n-1}})$ be a basis for (12). Form B by taking the corresponding columns of A. Then there exist permutation matrices P and Q such that PBQ is lower triangular.

Proof. Note that $m+n \geq 2$ and $\operatorname{rank} B = m+n-1$. Since each row of A is a combination of the remaining rows, each row of B will also be a combination of the remaining rows.

Let D be obtained from B by multiplying the last n rows with -1. Then $\operatorname{rank} D = \operatorname{rank} B$.[12] Notice that each column of D contains exactly one 1, exactly one -1 and the rest are 0. Also each row of D is a linear combination of the remaining rows. So, the theorem follows by the previous lemma. ∎

[16.15] **Why do we need this lower triangular results?** It gives us a very useful STRIKE-OFF PROCEDURE for the transportation algorithm. First let us understand the previous results a little more.

a) From the previous result, we see that the matrix B contains a row that has exactly one nonzero entry.

b) If we delete that row and the column corresponding to the nonzero entry (in B), then the new matrix contains a row that has exactly one nonzero entry.

c) Again, if we delete that row and the column corresponding to the nonzero entry, then the new matrix contains a row that has exactly one nonzero entry and so on.

d) It continues till the resulting matrix is empty.

This gives us a 'test' for a basis. Before we mention the general process, let us look at one observation.

[16.16] **Class workout** Consider a BTP with $m = 3, n = 3$. Consider $\{x_{11}, x_{12}, x_{22}, x_{32}\}$. Take the corresponding submatrix B of A and draw the table highlighting these variables.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$T =$

x_{11}	x_{12}	
	x_{22}	
	x_{32}	

a) Row 6 is a zero row in B. It corresponds to the _____ column in T. A row/column of T is called a LINE.
b) Row 6 in B does not have a 1. What does that means for T?
c) T/F? Number of 1's in Row 5 equals number of highlighted variables in the corresponding line of T?

[12] As D is obtained from B by multiplying a nonsingular matrix.

[16.17] <u>**Strike-off Theorem**</u> `(Test for a basis)` Let $S = \{x_{i_1,j_1}, \cdots, x_{i_{m+n-1},j_{m+n-1}}\}$ be a set of variables for (12). Highlight these variables in the transportation table T. Then the following are equivalent.

a) The set S is a basis.

b) The table T can be completely struck off by repeated application of the rule:
'find a line in T containing only one unstruck highlighted variable and strike that line off'.

Proof. a)⇒b) Suppose that S is a basis. Let B be the submatrix of A corresponding to S.

By [16.14], B has a row with exactly one nonzero entry 1 which corresponds to a variable, say, x_{lk}. Correspondingly, the table T has a line containing only the highlighted variable x_{lk}. In [16.14], we deleted the row and column of B corresponding to this entry. Call the new matrix B'. Correspondingly, <u>strike off</u> the line of T containing only x_{lk} to obtain T'.

By [16.14], B' has a row with only one nonzero entry 1 corresponding to the variable, say, $x_{l'k'}$. Correspondingly, T' has a line containing the <u>unstruck variable</u> $x_{l'k'}$. Strike that line off from T'.

Repeating our argument, we shall be able to strike off all the highlighted variables from the resulting table, one by one.

b)⇒a) Let B be the submatrix of A corresponding to S. Suppose that $x_{i,j}$ is the first highlighted variable we struck off from T. This means the corresponding row of B has only one nonzero entry 1 (below $x_{i,j}$).

Since the future strike offs in T will not involve $x_{i,j}$, we can strike off that row of B and the column corresponding to $x_{i,j}$.

Note that the rank of the old B is one more than that of the new B. If we are able to strike off another line from the new T, then it means that there is another row of the new B which contains only one nonzero entry and so on.

It follows that, if we are able to strike off all the $m + n - 1$ variables in T, then the new B would be empty as, at each step we strike off one column. Thus, the original B had rank $m + n - 1$. So, they form a basis. ∎

[16.18] **Example** Consider a BTP with $m = 3, n = 3$. Is $(x_{11}, x_{12}, x_{21}, x_{23}, x_{32})$ a basis?

a) First write the matrix B and make the corresponding transportation array. Highlight these variables.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

b) If this is a basis, then there should be a row in B with one nonzero entry. (The respective variable is never going to be used in the matrix X mentioned in the earlier proof.) This means, we can strike off that row and column from B to get X. Correspondingly, strike off that row/column from the transportation table.

$$B = \begin{bmatrix} 1 & 1 & 0 & \cancel{0} & 0 \\ 0 & 0 & 1 & \cancel{1} & 0 \\ 0 & 0 & 0 & \cancel{0} & 1 \\ 1 & 0 & 1 & \cancel{0} & 0 \\ 0 & 1 & 0 & \cancel{0} & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

c) Repeat this process with the new matrices and tables.

$$B = \begin{bmatrix} 1 & 1 & \cancel{0} & \cancel{0} & 0 \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1} & \cancel{0} \\ 0 & 0 & \cancel{0} & \cancel{0} & 1 \\ 1 & 0 & \cancel{1} & \cancel{0} & 0 \\ 0 & 1 & \cancel{0} & \cancel{0} & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

$$B = \begin{bmatrix} 1 & 1 & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \\ 1 & 0 & \cancel{1} & \cancel{0} & \cancel{0} \\ 0 & 1 & \cancel{0} & \cancel{0} & \cancel{1} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

$$B = \begin{bmatrix} \cancel{1} & 1 & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \\ \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{0} & 1 & \cancel{0} & \cancel{0} & \cancel{1} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \end{bmatrix}$$

x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} x_{11} & x_{12} & \\ x_{21} & & x_{23} \\ & x_{32} & \end{bmatrix}$$

Yes it is a basis.

> <u>Test for a set of $m + n - 1$ variables to be a basis.</u>
>
> <u>First way.</u> Form the matrix B by taking the columns $C_{i,j} = \begin{bmatrix} e_i \\ e_j \end{bmatrix}$ for a variable $x_{i,j}$. Find a row with only one nonzero entry. Delete that row and the column corresponding to the nonzero entry. Repeat. If you can finish the whole matrix in this way, then the given set is a basis. Otherwise not.
>
> <u>Second way.</u> Form the transportation table and highlight those variables. Find a line with only one unstruck variable and strike it. Repeat. If you can finish the whole table in this way, then the given set is a basis. Otherwise not.

[16.19] <u>**Example**</u> Consider a BTP with $m = 3, n = 3$. Is $(x_{11}, x_{12}, x_{21}, x_{22}, x_{32})$ a basis?