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1. (a) $x_{i+1} = (5x_i + 3) \bmod 16$, $x_0 = 7$

~~i=0~~ $x_1 = (5 \times 7 + 3) \bmod 16 = 6$

~~i=1~~ $x_2 = (5 \times 6 + 3) \bmod 16 = 1$

~~i=2~~ $x_3 = (5 \times 1 + 3) \bmod 16 = 8$

~~i=3~~ $x_4 = (5 \times 8 + 3) \bmod 16 = 11$

~~i=4~~ $x_5 = (5 \times 11 + 3) \bmod 16 = 10$

$x_6 = (5 \times 10 + 3) \bmod 16 = 5$

$x_7 = (5 \times 5 + 3) \bmod 16 = 12$

$x_8 = (5 \times 12 + 3) \bmod 16 = 15$

$x_9 = (5 \times 15 + 3) \bmod 16 = 14$

$x_{10} = (5 \times 14 + 3) \bmod 16 = 9$

$x_{11} = (5 \times 9 + 3) \bmod 16 = 0$

$x_{12} = (5 \times 0 + 3) \bmod 16 = 3$

$x_{13} = (5 \times 3 + 3) \bmod 16 = 2$

$x_{14} = (5 \times 2 + 3) \bmod 16 = 13$

$x_{15} = (13 \times 5 + 3) \bmod 16 = 4$

$x_{16} = (5 \times 4 + 3) \bmod 16 = 7 = x_0 \longrightarrow$ reptition starts.

Period = 15

(b) $x_{i+1} = (5x_i + 3) \bmod 16$ ② Done

$x_1 = (5 \times 5 + 3) \bmod 16 = 12$

$x_2 = (5 \times 12 + 3) \bmod 16 = 15$

$x_3 = (15 \times \cancel{2}5 + 3) \bmod 16 = 14$

$x_4 = 9$

$x_5 = 0$

$x_6 = 3$

$x_7 = 2$

$x_8 = 13$

$x_9 = 4$

$x_{10} = \cancel{8}\,7$

$x_{11} = \cancel{1}\,6$

$x_{12} = \cancel{8}\,1$

$x_{13} = \cancel{4}\,8$

$x_{14} = \cancel{10}\,11$

$x_{15} = \cancel{7}\,10 = \cancel{x_0}$

$x_{16} = 5 = x_0$

Period = 15

(c) $x_{i+1} = 5x_i \bmod 16$, $x_0 = 5$

$x_1 = 9$

$x_2 = 13$

$x_3 = 1$

$x_4 = 5 = x_0 \Rightarrow$ Repitition

~~$x_5 = 9$~~

Period = $\cancel{4}\,3$

③ Done

(2.) $f(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}$

$\frac{x^2}{2} \geq |x| - \frac{1}{2} \Rightarrow \frac{x^2}{2} \geq x - \frac{1}{2} \quad \because x \geq 0$

$-\frac{x^2}{2} \leq \frac{1}{2} - x$

$e^{-\frac{x^2}{2}} \leq e^{\frac{1}{2} - x}$

$\sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} \leq \sqrt{\frac{2}{\pi}} e^{\frac{1}{2} - x}$

$\sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} \leq \sqrt{\frac{2}{\pi}} e^{\frac{1}{2}} e^{-x}$

$\therefore$ Taking $g(x) = e^{-x}$, $c = \sqrt{\frac{2}{\pi}} e^{1/2}$

$\cancel{c = \sqrt{\frac{2}{\pi}} e^{\lambda/2}}$

~~out~~ for considering various $\lambda : (x - \lambda)^2 \geq 0$ will be used.

$x^2 - 2\lambda x + \lambda^2 \geq 0$

$\frac{x^2}{2} \geq \lambda x - \frac{\lambda^2}{2}$

$-\frac{x^2}{2} \leq \frac{\lambda^2}{2} - \lambda x$

$e^{-\frac{x^2}{2}} \leq e^{\frac{\lambda^2}{2} - \lambda x}$

$\sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} \leq \sqrt{\frac{2}{\pi}} e^{\frac{\lambda^2}{2}} e^{-\lambda x}$

$\therefore c = \sqrt{\frac{2}{\pi}} e^{\frac{\lambda^2}{2}}$

$\therefore \lambda = 1$ gives smallest c.

$G(x) = 1 - e^{-x}$

$G^{-1}(x) = -\log(1-x)$

$G^{-1}(X) = -\log(1-X)$

$\therefore$ but X and $1-X$ has same distribution

$G^{-1}(X) = -\log(X)$

~~repeat~~

~~generate~~

Algorithm

repeat

generate U_1 from Uniform(0,1)

$X = G^{-1}(U_1) = -\log(U_1)$

generate U_2 from Uniform(0, 1)

until $U_2 \le f(X)/cg(X)$

return X

(3) $f(x) = cp(1-p)^{x-1}$, $x = 1, 2 \ldots n$

$0 < p < 1$, $n \geq 1$ and integer, c = normali. const.

$$cp(1-p)^0 + cp(1-p)^1 + cp(1-p)^2 \ldots cp(1-p)^{n-1} = 1$$

$$cp(1 + (1-p) + (1-p)^2 \ldots (1-p)^{n-1}) = 1$$

$$cp \frac{1 \cdot (1-p)^n - 1}{1-p-1} = 1$$

$$cp \frac{(1-p)^n - 1}{-p} = 1$$

$$c = \frac{-1}{(1-p)^n - 1} = \frac{1}{1-(1-p)^n}$$

$cp \quad cp(1-p) \quad cp(1-p)^2 \ldots$

first generate $U(0,1)$ and set

$$X = \begin{cases} 1 & \text{if } U < cp \\ 2 & \text{if } cp \leq U < cp + cp(1-p) \\ 3 & \text{if } cp + cp(1-p) < U < cp + cp(1-p) + cp(1-p)^2 \\ \vdots & \\ j & \text{if } cp + cp(1-p) \ldots cp(1-p)^{j-2} < U < cp + cp(1-p) \ldots cp(1-p)^{j-1} \\ \vdots & \end{cases}$$

$j \leq n$

$\therefore X = j$ if $cp(1 + (1-p) \ldots (1-p)^{j-2}) < U < cp(1 + (1-p) \ldots (1-p)^{j-1})$

$X = j$ if $cp\left[\frac{(1-p)^{j-1} - 1}{1-p-1}\right] < U < cp\left[\frac{(1-p)^j - 1}{1-p-1}\right]$

$X = j$ if $cp\left[\frac{1-(1-p)^{j-1}}{p}\right] < U < cp\left[\frac{1-(1-p)^j}{p}\right]$

$$X = j \quad \text{if} \quad c\left(1-(1-p)^{j-1}\right) < U < c\left(1-(1-p)^{j}\right)$$

$$1-(1-p)^{j-1} < \frac{U}{c} < \left(1-(1-p)^{j}\right)$$

$$X = j \quad \text{if} \quad (1-p)^{j} < 1-\frac{U}{c} < (1-p)^{j-1}$$

As $1-U$ has the same distribution as U, we can thus define X by

$$X = \min\left\{ j : (1-p)^{j} < \frac{U}{c} \right\} = \min\left\{ j : j > \frac{\log(U/c)}{\log(1-p)} \right\}$$

$$\therefore \quad X = 1 + \left\lfloor \frac{\log(U/c)}{\log(1-p)} \right\rfloor$$

$\therefore$ <u>Algorithm</u>:

Generate uniform $U \sim U(0,1)$

~~Ein~~ return $1 + \left\lfloor \frac{\log(U/c)}{\log(1-p)} \right\rfloor$, where $c = \frac{1}{1-(1-p)^{n}}$