DEPARTMENT OF MATHEMATICS
Indian Institute of Technology, Guwahati

Quiz1 MA321

26-08-2021 Instructor : Sukanta Pati

Time : 11.10am–12.10pm Maximum Score : 20 of 30

Write appropriate and precise justifications. Use pencils for convenience. Submit by 12.20pm. Try to submit using the teams app. If that does not work, only then send it to my email pati@iitg.ac.in

1. `(The story of four techniques)` Let $T = \mathsf{conv}(0, e_1, 3e_2, 2e_3) \subseteq \mathbb{R}^3$.

a) What are the vertices of T? $\boxed{1}$

Answer. Let $S = \{0, e_1, 3e_2, 2e_3\}$. Then S is affine independent, as $e_1, 3e_2, 2e_3$ are linearly independent. Hence the vertices of $\mathsf{conv}\, S$ are precisely points of S.

b) Assume that initially each point in T was given a value 0. On the first day the value of each point was increased by the amount of its first coordinate. On the second day the value of each point was decreased by twice the amount of its second coordinate and on the third day the value of each point was increased by thrice the amount of its third coordinate. On the final day we are supposed to find the maximum of the value at each point on T.

c) Formulate the problem as an LPP. $\boxed{1}$

Answer. lpp:
$$\begin{array}{ll} \text{max} & x_1 - 2x_2 + 3x_3 \\ \hline \text{s.t.} & x_1 + \frac{x_2}{3} + \frac{x_3}{2} \le 1,\ x \ge 0. \end{array}$$

d) Write its slpp in matrix form. $\boxed{1}$

Answer. slpp:
$$\begin{array}{ll} \text{max} & \begin{bmatrix} 1 & -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ \hline \text{s.t.} & \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 1,\ x \ge 0. \end{array}$$

e) Can we use FTLP to tell the maximum solution for c)? $\boxed{2}$

Answer. Yes. As T is a nonempty compact convex set, and $f(x) = x_1 - 2x_2 + 3x_3$ is a linear function, the maximum will be attained at a vertex. We now evaluate the function at the four vertices. We see that $2e_3$ is the maximum solution and the value is 6.

f) Write the dual lpp of the lpp in c). $\boxed{2}$

Answer. Primal: $\max\ \underline{x_1 - 2x_2 + 3x_3}$ $\equiv$ $\min\ \underline{-x_1 + 2x_2 - 3x_3}$

s.t. $x_1 + \frac{x_2}{3} + \frac{x_3}{2} \le 1,\ x \ge 0$ s.t. $-x_1 - \frac{x_2}{3} - \frac{x_3}{2} \ge -1,\ x \ge 0$

Dual: $\max\ \underline{-y}$ $\equiv$ $\min\ \underline{y}$

s.t. $y\begin{bmatrix}-1 & -\frac{1}{3} & -\frac{1}{2}\end{bmatrix} \le \begin{bmatrix}-1 & 2 & -3\end{bmatrix},\ y \ge 0$ s.t. $y \ge 1,\ \frac{y}{3} \ge -2,\ \frac{y}{2} \ge 3,\ y \ge 0$

g) Verify the primal-dual theorem for c) and f) and hence show that one can find the maximum value of the function in c) in this way too. $\boxed{1}$

Answer. The primal-dual theorem says that the primal has a minimum iff the dual has a maximum. Here the dual in the second form has the minimum 6. Hence the dual in its first form has maximum value -6. By primal-dual theorem, the primal in its second form has minimum value -6. Hence, the primal in its first form has maximum value 6.

h) You can justify your answer to c) in a third way. Write the form of all linear functions that are strictly maximized at your vertex and continue. $\boxed{3}$

Answer. The set is $T = \{x \mid \begin{bmatrix}-1 & -\frac{1}{3} & -\frac{1}{2}\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} x \ge \begin{bmatrix}-1\\ 0\\ 0\\ 0\end{bmatrix}\}$

For $w = 2e_3$, we have $A_w = \begin{bmatrix}-1 & -\frac{1}{3} & -\frac{1}{2}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

All linear functions that are uniquely maximized at w are negative combinations of rows of A_w. Indeed, if we take $-6R_1 - 5R_2 - 4R_3$ we get $\begin{bmatrix}1 & -2 & 3\end{bmatrix}$. So $x_1 - 2x_2 + 3x_3$ is uniquely maximized at w.

i) You can answer your question in a fourth way. Compute the corresponding bfs and compute the relative cost vector. $\boxed{3}$

Answer. The bfs corresponding to $w = (0,0,2)$ is $w^* = (0,0,2,0)$. Basis is (x_3) and $B = [\frac{1}{2}]$. So

$$\overline{c}^t = c^t - c_B^t B^{-1} A = \begin{bmatrix}1 & -2 & 3 & 0\end{bmatrix} - 3 \times 2\begin{bmatrix}1 & \frac{1}{3} & \frac{1}{2} & 1\end{bmatrix} = \begin{bmatrix}-5 & -4 & 0 & -6\end{bmatrix},$$

which is entrywise nonpositive. Further, as $\overline{c}_C^t < 0$, the bfs w^* is a strict maximum. Hence w is a strict maximum and the value is 6.

j) Suppose that we had to maximize the square of the value at each point on the final day. Can that be dealt with given our limited knowledge of lpp? Show that it can be. Do the points of maximum form a convex set here? $\boxed{3}$

2. `(Proper writing)` Prove/disprove. Let S be a nonempty compact convex subset of $\mathbb{R}^5$ and $a \in \mathbb{R}^5$ be an arbitrary fixed point. Then a point of S which is at the maximum distance from a must be a vertex of S. $\boxed{3}$

   Answer. Let b be such a point and suppose that it is not a vertex. Let $b = \lambda u + (1-\lambda)v$, $\lambda \in (0,1)$, $u, v \in S$, $u \ne v$. So $\|a-b\| = \|a - \lambda u - (1-\lambda)v\| = \|\lambda(a-u) + (1-\lambda)(a-v)\| \le \lambda\|a-u\| + (1-\lambda)\|a-v\| \le \|a-b\|$.

Hence, we must have equality in the previous inequalities. Thus, we must have that $a-u=\alpha(a-v)$, $\alpha>0$ and $\|a-v\|=\|a-b\|=\|a-u\|$. So $\alpha=1$ and so $u=v$. This is a contradiction.

3. (Proper writing) Let S and T be nonempty disjoint convex sets in $\mathbb{R}^n$. Then show that there exists a hyperplane which separates S and T. $\boxed{3}$

   Answer. Note that $S-T$ is convex. As $0\notin S-T$, either $0\in\partial(S-T)$, or $0\notin\overline{S-T}$. Suppose first that $0\in\partial(S-T)$. Then by supporting hyperplane theorem, $\exists\, c\neq 0$, such that $c^t(s-t)\geq 0$, $\forall s\in S, t\in T$. That is, for each $s\in S$ and $t\in T$, we have $c^ts\geq c^tt$. It follows that $\{c^ts\mid s\in S\}$ is nonempty and bounded below, $\{c^tt\mid t\in S\}$ is nonempty and bounded above in $\mathbb{R}$ and that $\inf_{s\in S}c^ts\geq\alpha\geq\sup_{t\in T}c^tt$. Take any α such that $\inf_{s\in S}c^ts\geq\alpha\geq\sup_{t\in T}c^tt$. Then the hyperplane $H=\{z\mid c^tz=\alpha\}$ separates S and T. If $0\notin\overline{S-T}$, then use the strict separation theorem and proceed similarly.

4. (Proper writing) In $\mathbb{R}^5$, consider the sets $A=\mathsf{conv}(0,e_1,3e_2,2e_3)$ and $B=\mathsf{conv}(0,e_4,e_5,e_4+e_5)$. Is $A+B$ a polytope? $\boxed{7}$

   Answer. Claim: Let S and T be two nonempty affine independent subsets of $\mathbb{R}^n$ such that $S\perp T$ (that is, they are orthogonal). Then the vertices of $\mathsf{conv}(S)+\mathsf{conv}(T)$ are precisely $S+T$.

   Proof: We known that elements of S are vertices of $\mathsf{conv}(S)$ and the elements of T are vertices of $\mathsf{conv}(T)$. Let $s\in S$ and $t\in T$. We argue that that $s+t$ is a vertex of $S+T$. On the contrary, suppose that for some $s_1+t_1\neq s_2+t_2$ and $\lambda(0,1)$, we have

$$s+t=\lambda(s_1+t_1)+(1-\lambda)(s_2+t_2)=\lambda s_1+(1-\lambda)s_2+\lambda t_1+(1-\lambda)t_2.$$

   It follows that $s-\lambda s_1-(1-\lambda)s_2=\lambda t_1+(1-\lambda)t_2-t$. But as the lhs and the rhs are orthogonal vectors, we must have that they both are 0. Hence, $s=\lambda s_1+(1-\lambda)s_2$ and $t=\lambda t_1+(1-\lambda)t_2$. But as s is known to be a vertex and $\lambda\in(0,1)$, it follows that $s_1=s_2$. Similarly, $t_1=t_2$. But then $s_1+t_1=s_2+t_2$, a contradiction. The proof of the claim is complete.

   As $S=\{0,e_1,3e_2,2e_3\}$ is affine independent, the vertices of A are precisely the elements of S.

   Let $B_1=\mathsf{conv}(0,e_4)$ and $B_2=\mathsf{conv}(0,e_5)$. Then $B=B_1+B_2$ and by applying the above Claim, we see that the vertices of B are precisely the elements of $T=\{0,e_4,e_5,e_4+e_5\}$.

   Again by applying the Claim, the vertices of $A+B$ are precisely $S+T$. So $A+B$ is a compact convex set with finitely many vertices. By Krein-Milman theorm, it is a polytope.