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1.) $\Omega = \{a, b, c, d\}$

$\sigma(X) = \{\phi, \Omega, \{a, b\}, \{c, d\}\} = \{\phi, \{a, b, c, d\}, \{a, b\}, \{c, d\}\}$

~~(ii) $E[XY \mid X] = \begin{cases} X=2, [& X = 2 \\ (\frac{1}{8} + \frac{1}{4}) & \\ X=4 & X = 4 \end{cases}$~~

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ii) $E[XY|X] = X\,E[Y|X]$

$=$

$$= \frac{P(Y=1, X=2) + 2 \times P(Y=2, X=2)}{P(X=2)} I_{\{X=2\}}$$

$$+ \frac{2P(Y=2, X=4) + P(Y=1, X=4)}{P(X=4)} I_{\{X=4\}}$$

$$= \frac{P(\{a\}) + P(\{d\})}{P(\{a,d\})} I_{\{X=2\}} + \frac{2P\{b\} + P\{c\}}{P\{b,c\}} I_{\{X=4\}}$$

$$= \frac{\left(1 \times \frac{1}{8} + 2 \times \frac{1}{4}\right)}{3/8} I_{\{X=2\}} + \frac{\left(2 \times \frac{1}{8} + 1 \times \frac{1}{2}\right)}{5/8} I_{\{X=4\}}$$

$$= \frac{5}{3} I_{\{X=2\}} + \frac{6}{5} I_{\{X=4\}}$$

$$= \frac{5}{3} I_{\{a,d\}} + \frac{6}{5} I_{\{b,c\}}$$

$$\therefore\ E[XY|X] = \frac{10}{3} I_{\{a,d\}} \quad 5$$

$$E[X|Y] = \begin{cases} \dfrac{X(\{a\})P(\{a\}) + X(\{c\})P(\{c\})}{P(\{a\}) + P(\{c\})} & Y = 1 \\ & \cancel{Y=(2)} \\ \dfrac{X(\{b\})P(\{b\}) + X(\{d\})P(\{d\})}{P(\{b\}) + P(\{d\})} & Y = 2 \end{cases}$$

$$= \begin{cases} \dfrac{2 \times \frac{1}{8} + 4 \times \frac{1}{2}}{\frac{1}{8} + \frac{1}{2}} & \omega \in \{a, c\} \\ \dfrac{4 \times \frac{1}{8} + 2 \times \frac{1}{4}}{\frac{1}{8} + \frac{1}{4}} & \omega \in \{b, d\} \end{cases}$$

$$= \begin{cases} \dfrac{\frac{1}{4} + 2}{5/8} & \omega \in \{a, c\} \\ \dfrac{1}{3/8} & \omega \in \{b, d\} \end{cases}$$

$$= \begin{cases} \dfrac{9 \times 8}{4 \times 5} & \omega \in \{a, c\} \\ \dfrac{8}{3} & \omega \in \{b, d\} \end{cases}$$

$$= \begin{cases} \dfrac{18}{5} & \omega \in \{a, c\} \\ \dfrac{8}{3} & \omega \in \{b, d\} \end{cases}$$

$$E[X|Y] = \frac{18}{5} I_{\{a,c\}} + \frac{8}{3} I_{\{b,d\}}$$

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2.) $X(t) = W(t+4) - W(4)$

$X(0) = W(4) - W(4) = 0$

as $W(t)$ is continuous, $X(t)$ is continuous.

for $0 = t_0 < t_1 < t_2 < t_3 \cdots < t_m$

$\cancel{X(t_1 - t_0)}$

$X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$ --- $X(t_m) - X(t_{m-1})$

$$X(t_{p+1}) - X(t_p)$$
$$= W(t_{p+1}+4) - W(4) - [W(t_p+4) - W(4)]$$
$$= W(t_{p+1}+4) - W(t_p+4) \quad \text{--- (i)}$$

$$= W(t'_{p+1}) - W(t'_p)$$

Let $t_i + 4 = t'_i$

They are $N[0, t'_{p+1} - t'_p]$

$N[0, t_{p+1} + 4 - t_p - 4]$

$N[0, t_{p+1} - t_p]$

$\therefore$ These are $N(0, t_{p+1} - t_p)$

by (i) because $W(t_{p+1}+4) - W(t_p+4)$ are independent $\forall$ valid p.

$X(t_{p+1}) - X(t_p)$ are also independent.

$\therefore$ $X(t)$ is a standard brownian motion.

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3.) Supermartingle ⇒ $E[X(t) | \mathcal{F}_s] \leq X(s)$

To Prove : $E[X(t) | \mathcal{F}_s] = X(s)$

EE $= \cancel{E[X(s)]}$

∴ $E[X(t)]$ 6 is independent of time

∴ $E[X(t) | \mathcal{F}_s] = E[X(t)] = 6 \leq X(S)$

4. $X \to$ standard normal. 190123066

$Z \to$ independent random variable.

$$P\{Z=0\} = P\{Z=1\} = 1/2$$

$$Y = ZX$$

(i) $\operatorname{cov}(X^2, Y^2) = E[(X^2 - E[X^2])[Y^2 - E[Y^2]]]$

$$= E[(X^2 - E(X^2))(Z^2X^2 - E(Z^2X^2))]$$

$$= E[Z^2X^4 - Z^2X^2E[X^2] - E[X^2Z^2]X^2 + E[X^2] \cdot E[Z^2X^2]]$$

~~(ii) X and Y are independent if~~

$$= E[Z^2]E[X^4] - E[Z^2](E[X^2])^2 - E[X^2]^2E[Z^2] + E[X^2]^2E[Z^2]$$

$$= E[Z^2](E[X^4] - (E[X^2])^2)$$

$$= E[Z^2]\operatorname{var}(X^2)$$

$$= \left(0 \times \frac{1}{2} + 1 \times \frac{1}{2}\right)(\operatorname{var}(X^2))$$

$$= \frac{1}{2}\operatorname{var}(X^2)$$

~~$E[X^2]\operatorname{var}(X) = \sigma^2$~~ $\operatorname{var}(X^2) = 2\sigma^4$

$\because X$ is a standard normal

~~$\operatorname{var}[X^2] =$~~ $\sigma^2 = 1$

$$\operatorname{var}(X^2) = 2$$

$$\therefore \operatorname{cov}(X^2Y^2) = \frac{2}{2} = 1$$

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(ii) $\operatorname{cov}(X,Y) = E[(X-E(X))(Y-E(Y))]$

$= E[(X-E(X))(XZ - E[X]E[Z])]$

$= \cancel{E[(X^2Z -}$

$= E[X^2]E[Z] - (E[X])^2 E[Z]$

$- \cancel{(E[X])^2 E[Z]} + \cancel{E[X]^2 E[Z]}$

$= (E[X^2] - (E[X])^2) E[Z]$

$= \operatorname{var}(X) E[Z]$

$= 1 \times (0 \times \frac{1}{2} + 1 \times \frac{1}{2})$

$= \frac{1}{2}$

$\therefore \operatorname{cov}(X,Y) \neq 0$ X, Y are not independent

(iii) $\cancel{E[X^2 - E[X}$

$E((X^2 - E[X^2])(Y_1 - E[Y_1]))$

$E((X^2 - E[X^2])(Y^2 - E[Y^2 | X^2]$

(5)

$T = \min\{t \geq 0 \mid W(t) = -a \text{ or } W(t) = b\}$

$\frac{2}{\sqrt{2\pi}} \int$ $\cancel{\frac{a}{a+b}}$

Probability of hitting b first than -a is $\frac{a}{a+b}$

and Prob of hitting -a first than b is $\frac{b}{a+b}$