✓

l) Can we conclude that $D_d f(a) \geq 0$ for each $d \in \mathcal{D}(a)$?

Answer. Yes. As $D(a) = \mathcal{D}(a)$.

m) So $Z(a) = \emptyset$ and $a = (0, 1, 1/3)$ is the only KT point.

`(Second (algebraic) way.)` Recall the problem

$$\begin{array}{ll} \min & x - y + z \\ \hline \text{s.t.} & \begin{array}{l} g_1 \equiv 1 - x \geq 0, g_2 \equiv 1 - y \geq 0, g_3 \equiv 1 - z \geq 0, g_4 \equiv x \geq 0, \\ g_5 \equiv y \geq 0, g_6 \equiv z \geq 0, h_1 \equiv x + 2y + 3z - 3 = 0 \end{array} \end{array}$$

a) Suppose that a feasible point a is a KT point. So there exist $\lambda_i \geq 0, w \in \mathbb{R}$ with $\lambda_i g_i(a) = 0$ such that

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

b) Must $\lambda_2 > 0$? What do we get from here? $\lambda_2 \geq 0$ $\quad \lambda_2 = 0$ $\quad$ 2nd coordinate

we have $w < 0$. so $\lambda_4 > 0$, $\lambda_6 > 0$

$\lambda_4 g_4(a) = 0$ $\quad \Rightarrow x = 0$, $z = 0$

$a = (0, 1.5, 0) \notin T$

Answer. Yes. Note that $\nabla f(a)$ has second coordinate -1. We must have $\boxed{\lambda_2 > 0}$.

[Why? If $\lambda_2 = 0$, then $w \leq -\frac{1}{2}$. And so $\lambda_4 > 0$ and $\lambda_6 > 0$. As $\lambda_4 g_4 = 0$ and $\lambda_6 g_6 = 0$, we see that the point $a = (0, *, 0)$. But as a lies on the plane, we must have $a = (0, 1.5, 0)$. But then $a \notin T$.]

As $\lambda_2 g_2(a) = 0$, we get that $\boxed{a(2) = 1}$. Also $\boxed{\lambda_5 = 0}$.

c) Can $\lambda_3 > 0$? What do get from here?

Answer. Suppose that $\lambda_3 > 0$. As $\lambda_3 g_3(a) = 0$, we get $a(3) = 1$. Hence $a = (\geq 0, 1, 1) \notin T$. We get $\boxed{\lambda_3 = 0}$.

$$\min \frac{x_1+x_2}{\begin{array}{c} x_1 \le 1,\ x_2 \le 1 \\ x_1 \ge 0,\ x_2 \ge 0 \end{array}}$$

33 Lecture 33

KT theory for lpp

[33.1] **Example** Consider minimizing $f(x) = x_1 + x_2$ on our favorite set $T = \{(x_1, x_2) \mid 0 \le x_1, x_2 \le 1\}$. What are the KT points?

Answer. a) Write the constraints properly first:

$$g_1 = 1-x_1 \quad , \quad g_2 = 1-x_2 \qquad g_3 = x_1 \qquad g_4 = x_2$$

Constraints: $g_1(x) \equiv 1 - x_1 \ge 0$, $g_2(x) \equiv 1 - x_2 \ge 0$, $g_3(x) = x_1 \ge 0$ and $g_4(x) = x_2 \ge 0$.

b) Write the expression $\bigtriangledown L(a, \lambda) = 0$ and try to identify the points a at which the equality holds, while satisfying the two other conditions of KT points. For that, start like the following and argue.

Let a be a kt pt. So $\exists\ \lambda \ge 0$ s.t $\bigtriangledown L(a,\lambda) = 0$, $a \in T$ and

vector

$\lambda_i g_i(a) = 0 \quad i = 1,2,3,4$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so we must have $\lambda_3 > 0$, $\lambda_4 > 0$. So $x_1 = 0$, $x_2 = 0$.

so $a = (0,0)$.

Take $\lambda = (0,0,1,1)$. Then $a = (0,0) \in T$, $\bigtriangledown L(a,\lambda) = 0$ and $\lambda_i g_i(a) = 0\ \forall i$. so $a = (0,0)$ is the only KT point.

Let a be a KT point. Then $a \in T$ and $\exists \lambda \ge 0$

$$\bigtriangledown f(a) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \lambda_i g_i(a) = 0,\ i = 1, 2, 3, 4.$$

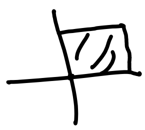
As entries of $\bigtriangledown f(a)$ are positive, we see that λ_3 and λ_4 must be positive. This forces, the point to be $a = (0, 0)$.

Take $\lambda = (0, 0, 1, 1)$. Then a is a feasible point, $\bigtriangledown L(a, \lambda) = 0$ and $\lambda_i g_i(a) = 0$ for each i. So it is the only KT point. ┘

[33.2] **Example** Consider maximizing $f(x) = x_1 + x_2$ on our favorite set $T = \{(x_1, x_2) \mid 0 \le x_1, x_2 \le 1\}$. What are the KT points?

Answer. a) Constraints: $g_1(x) \equiv 1 - x_1 \ge 0$, $g_2(x) \equiv 1 - x_2 \ge 0$, $g_3(x) = x_1 \ge 0$ and $g_4(x) = x_2 \ge 0$.

b) First we change the problem to a minimization problem. So $h(x) = -x_1 - x_2$.



c) $\min\ -x_1 - x_2$

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad \lambda_1 > 0,\ \lambda_2 > 0 \quad \Rightarrow \quad a = (1,1)$

Take $\lambda = (1,1,0,0)$.

Then $a \in T$, $\bigtriangledown L(a,\lambda) = 0$, and $\lambda_i g_i(a) = 0\ \forall i$.

so $a = (1,1)$ is the only KT point.

Let a be a KT point. Then $a \in T$ and $\exists \lambda \geq 0$ such that

$$\bigtriangledown h(a) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \lambda_i g_i(a) = 0,\ i = 1, 2, 3, 4.$$

As entries of $\bigtriangledown h(a)$ are negative, we see that λ_1 and λ_2 must be positive. In fact they must be 1 each. This forces, the point to be $a = (1, 1)$.

Take $\lambda = (1, 1, 0, 0)$. Then a is a feasible point, $\bigtriangledown L(a, \lambda) = 0$ and $\lambda_i g_i(a) = 0$ for each i. So it is the only KT point. $\lrcorner$

[33.3] **Example** Consider minimizing $f(x) = x_1$ on our favorite set $T = \{(x_1, x_2) \mid 0 \leq x_1, x_2 \leq 1\}$. What are the KT points?

Answer. Constraints: $g_1(x) \equiv 1 - x_1 \geq 0$, $g_2(x) \equiv 1 - x_2 \geq 0$, $g_3(x) = x_1 \geq 0$ and $g_4(x) = x_2 \geq 0$.

b)

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \lambda_3 > 0 \quad \Rightarrow \quad x_1 = 0$. so $a = (0, t)$, $t \in [0, 1]$.

Take $\lambda = (0, 0, 1, 0)$. Then $a \in T$, $\nabla L(a, \lambda) = 0$ and $\lambda_i g_i(a) = 0$ $\forall i$.

These are KT points.

Let a be a KT point. Then $a \in T$ and and $\exists \lambda \geq 0$ such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \lambda_i g_i(a) = 0,\ i = 1, 2, 3, 4.$$

As first entry of $\bigtriangledown f(a)$ is positive, we see that $\lambda_3 = 1$. This forces $a = (0, t)$, $0 \leq t \leq 1$.

[Unnecessary: if we take $\lambda_4 > 0$, then we have to take $\lambda_2 > 0$. This is not possible.]

Take $\lambda = (0, 0, 1, 0)$. Then $a = (0, t)$, $0 \leq t \leq 1$ are feasible points with $\bigtriangledown L(a, \lambda) = 0$ and $\lambda_i g_i(a) = 0$ $\forall i$. So these are the only KT points. ■

In all the previous three cases, the KT points are the points of minimum. Is it true in general? Yes, see the next result.

[33.4] **Theorem** Let $A \in M_{m,n}(\mathbb{R})$. Consider minimizing $f(x) = c^t x$ over $T = \{x \mid Ax \geq b\}$. Let a be a KT point. Then a is a point of minimum.

Proof.

$$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ \\ b_m \end{bmatrix}$$

$g_1 \equiv a_{11}x_1 + \cdots + a_{1n}x_n - b_1 \geq 0$

$\vdots$

$g_m \equiv a_{m1}x_1 + \quad + a_{mn}x_n - b_m \geq 0$

For simplicity let $g_1, \cdots g_k$ be active at a. $\qquad g_i \equiv A(i,:)x - b_i \geq 0$

Given a is a KT pt. means $\exists \lambda \geq 0$ s.t $\nabla L(a, \lambda) = 0$, $\lambda_i g_i(a) = 0$ $\forall i$

$$c = \lambda_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} + \quad + \lambda_k \begin{bmatrix} a_{k1} \\ \vdots \\ a_{kn} \end{bmatrix} + \overset{0}{\lambda_{k+1}} \begin{bmatrix} & \end{bmatrix} \quad + \overset{0}{\lambda_m} \begin{bmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$c^T = \lambda_1 [a_{11} \cdots a_{1n}] + \cdots + \lambda_k [a_{k1} \cdots a_{kn}]$

Note that $A_a = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{k1} & & a_{kn} \end{bmatrix}$ so c^T is a nonnegative combination of rows of A_a. so $f(x) = c^T x$ is min at a.

(Not part of proof: we have to write down the constraints first.) Let $g_i(x) \equiv A(i,:)x - b_i$.

(Not part of proof: next we start like 'let a be a KT point...' and proceed to find out something about the KT points which will help to prove that such points are minimums.)

Let a be a KT point. Then $a \in T$ and $\exists \lambda \geq 0$ such that $\triangledown L(a, \lambda) = 0$ and $\lambda_i g_i(a) = 0 \ \forall i$.

The expression $\triangledown L(a, \lambda) = 0$ converts to

$$c = \lambda_1 \triangledown g_1 + \cdots + \lambda_m \triangledown g_m = A^t \lambda \quad \text{implying that} \quad c^t = \lambda^t A. \tag{13}$$

For simplicity, let g_i, $i = 1, \ldots, k$ be the active constraints at a. That is, these are the (positively) supporting hyperplanes at a. Also, as the other g_i are inactive at a, we have $\lambda_{k+1} = \cdots = \lambda_m = 0$.

It now follows from the previous equation that $c^t = \begin{bmatrix} \lambda_1 & \cdots & \lambda_k \end{bmatrix} A_a$, that is, c^t is a nonnegative combination of the rows of A_a.

It now follows from [7.5] that $f(x) = c^t x$ is minimized at a. ∎

[33.5] **Remark** The converse for the above theorem is also true. That is, for an lpp, a point of minimum is a KT point. This is so, as a point of minimum always satisfies $D_d f(a) \geq 0$, along any feasible direction $d \in \overline{D}(a)$, by FONC. But we already know that $D(a) = \mathcal{D}(a)$. So $Z(a) = \emptyset$ and so a is a KT point.

min $f(x) = c^T x$ / $Ax \geq b$

a is min $\overset{?}{\Rightarrow}$ a KT pt

For each $d \in \overline{D}(a)$, we have $\langle \nabla f, d \rangle \geq 0$

$\overline{D}(a) = \mathcal{D}(a) \Rightarrow Z(a) = \emptyset$

Question Did you just see another way to solve an lpp? Yes. Just find the KT points.

Checking whether a given point is a KT point using simplex method

Checking whether a given point is a KT point can be done using simplex method, as we have to find some $\lambda_i \geq 0$ and w_i which satisfy certain equalities or inequalities.

[33.6] **Example** Check using simplex method, whether $a = (1, .5)$ is a KT point of

1) $a \in T$
2) $\nabla L(a, \lambda, w) = 0$
3) $\lambda_i g_i(a) = 0$

$$\begin{array}{ll} \min & x_1^3 - 5x_2 \\ \hline \text{s.t.} & g_1 \equiv x_1^2 - 2x_2 \geq 0,\ g_2 \equiv x_1 + x_2 \geq 0,\ h \equiv x_1 - 2x_2 = 0. \end{array}$$

Answer. a) First find out the active constraints.

only g_1 is active here.

Here only g_1 is active.

Thrown away the inactive g_i

$$\begin{bmatrix}3\\-5\end{bmatrix} = \lambda_1 \begin{bmatrix}2\\-2\end{bmatrix} + w \begin{bmatrix}1\\-2\end{bmatrix} \qquad \lambda_1 \geq 0, \; w \in \mathbb{R}$$

b) So if a is a KT point, then $\lambda_2 = 0$. (This reduces some of our work as opposed to directly starting with $\bigtriangledown L(a, \lambda, w) = 0$ and ultimately concluding that $\lambda_2 = 0$ if a is a KT point.)

c) So a is a KT point iff we can find $\lambda_1 \geq 0$, $w \in \mathbb{R}$ such that

$$\begin{bmatrix}3\\-5\end{bmatrix} = \lambda_1 \begin{bmatrix}2\\-2\end{bmatrix} + w \begin{bmatrix}1\\-2\end{bmatrix}.$$

$2\lambda_1 + w = 3$

$\cancel{-2\lambda_1 - 2w = -5}$

$2\lambda_1 + 2w = 5$

d) That is, we have to see whether the set

$$T = \left\{ (\lambda_1, w_1, w_2) \mid \begin{bmatrix} 2\lambda_1 + w_1 - w_2 = 3 \\ 2\lambda_1 + 2w_1 - 2w_2 = 5 \end{bmatrix}, \lambda_1, w_i \geq 0 \right\} \neq \emptyset.$$

$2\lambda_1 + w_1 - w_2 = 3$

$2\lambda_1 + 2w_1 - 2w_2 = 5$

$\lambda_1, w_1, w_2 \geq 0$

e) We have already dealt with such problems. As T is bounded below, if it is nonempty, it will have a vertex. That will lead to a bfs. So we can use the same simplex method we used to find an initial bfs.

f) This can be achieved by the solving lpp

$$\begin{array}{ll} \min & y_1 + y_2 \\ \text{s.t.} & 2\lambda_1 + w_1 - w_2 + y_1 = 3 \\ & 2\lambda_1 + 2w_1 - 2w_2 + y_2 = 5, \; \lambda_i, w_i, y_i \geq 0. \end{array}$$

bv	λ_1	w_1	w_2	y_1	y_2	$\bar{b}$
y_1	$\boxed{2}$	1	-1	1	0	3
y_2	2	2	-2	0	1	5
$-f$	-4	-3	3	0	0	-8

bv	λ_1	w_1	w_2	y_2	$\bar{b}$
λ_1	1	.5	$-.5$	0	1.5
y_2	0	$\boxed{1}$	-1	1	2
$-f$	0	-1	1	0	-2

$\rightarrow$

bv	λ_1	w_1	w_2	$\bar{b}$
λ_1	1	0	0	.5
w_1	0	1	-1	2
$-f$	0	0	0	0

$\lambda_1 = \frac{1}{2}$

$w_1 = 2.$

$w_2 = 0$

e) We see that $\lambda_1 = .5$, $w = w_1 - w_2 = 2$ satisfies our requirement. So $(1, .5)$ is a KT point.

$\lambda = 1/2, \; w = 2$

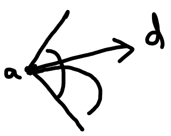
Regularity conditions

We will only talk about the FIRST ORDER KUHN-TUCKER CONSTRAINT QUALIFICATION(~~KTCQ1~~ ktcq1) here.

[33.7] **Definition** First order Kuhn-Tucker constraint qualification (ktcq1) is said to hold at $a \in T$, if each nonzero direction d in the linearizing cone $\mathcal{D}(a)$ is the tangent to some $\mathcal{C}^1$ curve in T at a. That is, $\exists \alpha$ such that

i) $\alpha(0) = a$, ii) $\exists \epsilon > 0$ such that $\alpha(t) \in T$, $\forall t \in [0, \epsilon]$, and iii) $d = \lim_{t \to 0+} \dfrac{\alpha(t) - \alpha(0)}{t}$.

Notice that, this definition is about the feasible set and it is independent of the objective function. ┘



~~for each~~ Take a $d \in \mathcal{D}(a)$. Suppose that we can find a C^1 curve $\alpha(t)$ s.t.

1) $\alpha(0) = a$ 2) $\exists \, \varepsilon > 0$ s.t. $\alpha(t) \in T$ $\forall \, t \in [0, \varepsilon]$

for each $d \in \mathcal{D}(a)$ $\{$ 3) $d = \lim_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t}$

[33.8] **Example** `(ktcq1 at interior points.)` Consider (P2). Show that ktcq1 holds at each $a \in T^{\circ}$.

Answer. a) Let $a \in T^{\circ}$. What is $\mathcal{D}(a)$?

$D(a) = \mathbb{R}^n$

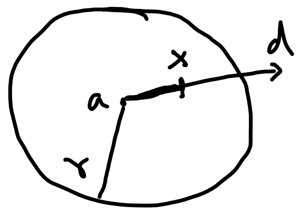
$\mathcal{D}(a) = \mathbb{R}^n$

As $a \in T^{\circ}$, suppose that $B_{\delta}(a) \subseteq T$. We know that $D(a) = \mathbb{R}^n$. So $\mathcal{D}(a)$ has to be $\mathbb{R}^n$.

b) To show ktcq1 holds here, let $d \in \mathcal{D}(a)$ be nonzero.

$B_{\gamma}(a) \subseteq T$

c) Now, we need to define a suitable curve.



$\alpha(t) = a + t d$

$t \in \left[0, \frac{\gamma}{2\|d\|}\right]$

$\alpha(t) \in B_{\gamma}(a) \subseteq T$

We take the straight line segment that starts at a, goes in the direction of d up to a distance $\delta/2$. (You can take other, but this is one of the simplest.)

That is, take $\epsilon = \delta/2$ and define

$$\alpha(t) = a + td, \qquad t \in [0, \epsilon].$$

$\lim_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t} = d$

The curve is a straight line segment here. Note that the curve $\alpha(t)$, $t \in [0, \epsilon]$ is inside $B_{\delta}(a) \subseteq T$.

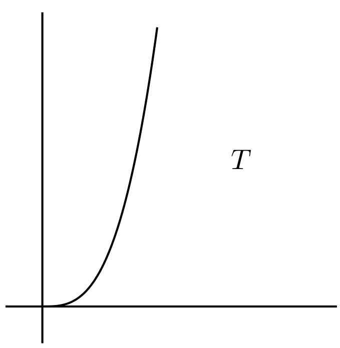
d) Is $\lim_{t \to 0+} \frac{\alpha(t) - a}{t} = d$?

Yes, $\lim_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t} = \lim_{t \to 0+} \frac{a + td - a}{t} \lim_{t \to 0+} \frac{td}{t} = d.$

e) So ktcq1 holds at a.

[33.9] **Example** Consider $T = \{x \in \mathbb{R}^2 \mid g_1(x) = x_1^3 - x_2 \geq 0,\ g_2(x) = x_1 \geq 0,\ g_3(x) = x_2 \geq 0\}$. Show that ktcq1 holds at all points in T.

Answer. The region is shown here.



a) There are four types of points here: interior points, point only on x-axis, point only on the curve, and $(0,0)$ which is common.

b) Do we know that ktcq1 holds at each interior point?

Yes.

c) Let $a = (0,0)$. Find $\mathcal{D}(a)$ and check whether all those directions are tangents of some $\mathcal{C}^1$-curves.

Here $A(a) = \{1,2,3\}$. So

$$\mathcal{D}(a) = \{d \mid \bigtriangledown g_i^t(a)d \geq 0, i \in A(a)\} = \left\{ d \mid \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} d \geq 0 \right\} = \{d \mid d_1 \geq 0, d_2 = 0\}.$$

Let $d = \begin{bmatrix} d_1 \\ 0 \end{bmatrix} \in \mathcal{D}(a)$. We take $\alpha(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$, $t \in [0,1]$. Then the curve is in T and

$$\lim_{t \to 0+} \frac{\alpha(t) - \alpha(0)}{t} = \lim_{t \to 0+} \frac{t \begin{bmatrix} d_1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}}{t} = \begin{bmatrix} d_1 \\ 0 \end{bmatrix} = d.$$

So we see that kctq1 holds at a.

d) Let $a = (a_1, 0)$, $a_1 > 0$. Find $\mathcal{D}(a)$ and check whether all those directions are tangents of some $\mathcal{C}^1$-curves.

Here $A(a) = \{3\}$ and $\mathcal{D}(a) = \{d \mid d_2 \geq 0\}$. Draw picture to understand.

Let $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathcal{D}(a)$. If $d = 0$, we have nothing to prove. We take

$$\alpha(t) = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \qquad t \in [0, \frac{a_1}{2\|d\|}].$$

We see (verify) that ktcq1 holds.

e) Let $a = (x > 0, x^3)$. Find $\mathcal{D}(a)$ and check whether all those directions are tangents of some $\mathcal{C}^1$-curves.

Here $A(a) = \{1\}$ and

$$\mathcal{D}(a) = \left\{ d \mid \begin{bmatrix} 3a_1^2 \\ -1 \end{bmatrix} d \geq 0 \right\} = \{d \mid 3a_1^2 d_1 \geq d_2\}.$$

Taking $\alpha(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

Is $\alpha(t)$ in T? (Thought: this is on the right side of tangent line at a. So $\alpha(t) \in T$ for small t.)

As both coordinates of a are positive, for t small, both the coordinates of $\alpha(t)$ will also be positive. To see whether $\alpha(t)$ satisfies g_1, observe that

$$\begin{aligned} g_1(\alpha(t)) &= (a_1 + td_1)^3 - (a_2 + td_2) \\ &= a_1^3 - a_2 + t(3a_1^2 d_1 - d_2) + 3a_1 t^2 d_1^2 + t^3 d_1^3 \\ &= 0 \qquad + t(\geq 0) \qquad + (3a_1 + td_1)t^2 d_1^2 \geq 0, \end{aligned}$$

if $3a_1 + td_1 \geq 0$, which holds for all small $t \geq 0$, as $a_1 > 0$. So $\alpha(t) \in T$ for all small t.

As $\alpha(t) = a + td$, we have $\lim_{t \to 0+} \frac{\alpha(t) - a}{t} = d$, as required. ┘