13 Lecture 13

Finding an initial bfs

Recall that our simplex algorithm requires a bfs to start with. How do we find one? Some texts call this the first phase of simplex method and the original algorithm the second phase. The first phase algorithm is similar to the first one and it is based on the following result.

[13.1] **Theorem** `(Initial bfs)` Let $\mathsf{rank}\, A_{m\times n} = m$ and $\underline{b \ge 0}$ (needed). Consider the problem

$$\begin{aligned} &\min\ \mathbf{1}^t y \\ &\text{s.t.}\ \ [A \mid I]\begin{bmatrix} x \\ y\end{bmatrix} = b, x \ge 0, y \ge 0 \end{aligned} \tag{11}$$

and consider running the simplex method starting with the bfs $(y_1, \ldots, y_m)$. Then the simplex method always terminates. If the minimum value is positive, then the set $\{x \mid Ax = b, x \ge 0\}$ is empty. If the minimum value is 0 and the algorithm stops at a minimum bfs $\begin{bmatrix} x_0 \\ y_0\end{bmatrix}$, then x_0 is a bfs for the system $Ax = b$.

original $\dfrac{\min c^T x}{Ax = b, x \geq 0}$, $b \geq 0$ → T

related $\dfrac{\min \mathbb{1}^t y = y_1 + y_2 + \cdots + y_m}{[A_{m\times n}\ I_{m\times m}]\begin{bmatrix} x \\ y\end{bmatrix} = b, x \geq 0, y \geq 0}$ → T^*

$\begin{bmatrix} 0 \\ b\end{bmatrix} \in T^* \Rightarrow T^* \neq \emptyset$. obj fn $\mathbb{1}^t y \geq 0$

so by FTLP min exists.

→ bfs corresp to basis $(y_1, y_2, \ldots, y_m)$. Apply simplex method.

$\begin{bmatrix} x_0 \\ y_0\end{bmatrix}$ → minimal ~~vertex~~ bfs. value of obj ≥ 0 $\alpha = 0$.

if value $> 0 \Rightarrow T = \emptyset$. ($p \in T \Rightarrow \begin{bmatrix} p \\ 0\end{bmatrix} \in T^*$ of value 0)

value $= 0 \Rightarrow y_0 = 0$. $\Rightarrow x_0$ is a bfs of $Ax = b$.

Proof. The feasible set of (11) is bounded below and it is nonempty as it contains the point $\begin{bmatrix} 0 \\ b\end{bmatrix}$. (We could not have said this if we did not know that $b \ge 0$.) Observe that it is a bfs corresponding to the basis $(y_1, \ldots, y_m)$. The function $\mathbf{1}^t y$ is bounded below by 0. Hence the simplex method will never show the unboundedness situation. As there are only finitely many bfs, the method must terminate (using Bland's rule).

Suppose the minimum value is $\alpha > 0$. Then the set $\{x \mid Ax = b, x \ge 0\}$ must be empty. Suppose that it is not and let $Az = b$, $z \ge 0$. Then $\begin{bmatrix} z \\ 0\end{bmatrix}$ is a feasible solution at which the value of the objective function is 0. This is a contradiction.

Suppose that the minimum value is zero and the minimal bfs is $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ with a basis $(x_{i_1}, \ldots, x_{i_k}, y_{j_1}, \ldots, y_{j_p})$ in some order. This means the set consisting of the columns $A_{:i_1}, \ldots, A_{:i_k}$ and p more vectors, are linearly independent. In particular, the columns $A_{:i_1}, \ldots, A_{:i_k}$ are linearly independent. Since the minimum values is 0, it follows that $y_0 = 0$ and hence $Ax_0 = b$. Since the columns of A corresponding to the nonzero entries of x_0 must be a subset of the columns $\{A_{:i_1}, \ldots, A_{:i_k}\}$, it follows that they are linearly independent. Hence x_0 is a bfs for $Ax = b$. To supply a basis for x_0, we just need to add $m - k$ many more columns to these k columns. This is possible as the rank is m. ■

<u>To find an initial bfs</u> Let rank $A_{m \times n} = m$, $b \geq 0$ be given. Do the following to get an initial bfs of $Ax = b$.

a) Write the simplex table for
$$\min \ \mathbf{1}^t y \quad \text{s.t.} \quad [A \mid I] \begin{bmatrix} x \\ y \end{bmatrix} = b, x \geq 0, y \geq 0$$
at the basis $(y_1, \ldots, y_m)$. The variables $y_1, \ldots, y_m$ are called ARTIFICIAL VARIABLES.

b) Apply simplex method to find the minimum value α of $\mathbf{1}^t y$.[a]

c) If $\alpha > 0$, conclude that $T = \emptyset$.

d) If $\alpha = 0$ is attained at a bfs $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, then conclude that x_0 is a bfs of $Ax = b$.

[a] To save time in computation, we drop an artificial variable as soon as it becomes nonbasic. This means, we do not show the corresponding column in the next table. To justify, imagine we had not introduced this artificial variable and write the simplex table at the current basis. Then we would get the current table.

[13.2] **Example** Consider
$$\min \ f(x) = 2x_1 + 2x_2 + 4x_3 + 2x_4 \quad \text{s.t.} \quad 2x_1 - x_2 + 3x_3 - 2x_4 = 3,\ -x_1 + 3x_2 - 4x_3 + 4x_4 = 1,\ x_i \geq 0.$$
Find an initial bfs. Then proceed to solve the original problem.

PT

bv	x_1	x_2	x_3	x_4	b
	2	-1	3	-2	3
	-1	3	-4	4	1
$c \rightarrow$	2	2	4	2	

bv	x_1	x_2	x_3	x_4	y_1	y_2	$\bar{b}$
y_1	$\boxed{2}$	-1	3	-2	1	0	3
y_2	-1	3	-4	4	0	1	1
	0	0	0	0	1	1	0 ⟵X
	-1	-2	1	-2	0	0	-4
	↑						

bv	x_1	x_2	x_3	x_4	y_2	$\bar{b}$
x_1	1	-1/2	3/2	-1	0	3/2
y_2	0	$\boxed{5/2}$	$-\frac{5}{2}$	3	1	5/2
	0	$-\frac{5}{2}$	$\frac{5}{2}$	-3	0	$-\frac{5}{2}$

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_1	1	0	1	-2/5	2
x_2	0	1	-1	6/5	1
	0	0	0	0	0 → val
	~~2~~	~~2~~	~~4~~	~~2~~	

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_1	1	0	1	-2/5	2
x_2	0	1	-1	6/5	1
	2	2	4	2	0 X
	0	0	4	2/5	-6

min is 6 attained at $x_1 = 2$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$

Answer. To find an initial bfs consider the problem

$$\begin{array}{ll} \min & y_1+y_2 \\ \text{s.t.} & 2x_1-x_2+3x_3-2x_4+y_1=3,\ -x_1+3x_2-4x_3+4x_4+y_2=1,\ x_i,y_j\geq 0. \end{array}$$

We follow the simplex method starting with the initial basis (y_1, y_2).

bv	x_1	x_2	x_3	x_4	y_1	y_2	$\overline{b}$
y_1	2	−1	3	−2	1	0	3
y_2	−1	$\boxed{3}$	−4	4	0	1	1
$-f$	−1	−2	1	−2	0	0	−4

$\rightarrow$

bv	x_1	x_2	x_3	x_4	y_1	$\overline{b}$
y_1	$\boxed{\frac{5}{3}}$	0	$\frac{5}{3}$	$-\frac{2}{3}$	1	$\frac{10}{3}$
x_2	$-\frac{1}{3}$	1	$-\frac{4}{3}$	$\frac{4}{3}$	0	$\frac{1}{3}$
$-f$	$-\frac{5}{3}$	0	$-\frac{5}{3}$	$\frac{2}{3}$	0	$-\frac{10}{3}$

$\rightarrow$

bv	x_1	x_2	x_3	x_4	$\overline{b}$
x_1	1	0	1	$-\frac{2}{5}$	2
x_2	0	1	−1	$\frac{6}{5}$	1
$-f$	0	0	0	0	0

To continue solving the original problem, we have to make a simplex table at the basis (x_1, x_2). For this we use the previous table by recalculating $\overline{c}$

bv	x_1	x_2	x_3	x_4	$\overline{b}$
x_1	1	0	1	$-\frac{2}{5}$	2
x_2	0	1	−1	$\frac{6}{5}$	1
$-f$	0	0	4	$\frac{2}{5}$	−6

and continue with simplex method.

[13.3] **Remark** For a maximization problem the optimal condition is $\overline{c} \leq 0$ and in order to find a better bfs we look at a positive entry of $\overline{c}$.

[13.4] **Example** `(Self)` Consider $\begin{array}{ll} \min & x_1+x_2+3x_3-x_4 \\ \text{s.t.} & x_1+x_2=1, x_2+x_3+x_4=2, x_1+2x_2+x_3+x_4=3,\ x_i\geq 0. \end{array}$

Notice that the matrix A has rank 2. One way to approach it is to drop one equation which is a linear combination of the other two. But suppose that the computer did not notice it and proceeded to find an initial bfs by introducing artificial variables y_1, y_2, y_3.

PT:

bv	x_1	x_2	x_3	x_4	y_1	y_2	y_3	$\overline{b}$
*	1	1	0	0	1	0	0	1
*	0	1	1	1	0	1	0	2
*	1	2	1	1	0	0	1	3
$-f$	0	0	0	0	1	1	1	0

PhaseI:

bv	x_1	x_2	x_3	x_4	y_1	y_2	y_3	$\overline{b}$
y_1	1	1	0	0	1	0	0	1
y_2	0	1	1	$\boxed{1}$	0	1	0	2
y_3	1	2	1	1	0	0	1	3
$-f$	−2	−4	−2	−2	0	0	0	−6

$\rightarrow$

bv	x_1	x_2	x_3	x_4	y_1	y_3	$\overline{b}$
y_1	1	1	0	0	1	0	1
x_4	0	1	1	1	0	0	2
y_3	1	$\boxed{1}$	0	0	0	1	1
$-f$	0	−2	0	0	0	0	−2

$\rightarrow$

bv	x_1	x_2	x_3	x_4	y_1	$\overline{b}$
y_1	0	0	0	0	1	0
x_4	−1	0	1	1	0	1
x_2	1	1	0	0	0	1
$-f$	2	0	0	0	0	0

Notice that the first row of the simplex table is zero except the y_1 entry. This means the corresponding equation of $Ax = b$ was a linear combinations of the other two. So, we proceed for phaseII with basis (x_4, x_2) recalculating $\overline{c}$.

PhaseII:

bv	x_1	x_2	x_3	x_4	$\overline{b}$
x_4	−1	0	1	1	1
x_2	$\boxed{1}$	1	0	0	1
$-f$	−1	0	4	0	0

$\rightarrow$

bv	x_1	x_2	x_3	x_4	$\overline{b}$
x_4	0	1	1	1	2
x_1	1	1	0	0	1
$-f$	0	1	4	0	1

$A_{m\times n}x=b$

Some exercises

[13.5] **NoPen** Let $\mathsf{rank}\, A_{3,4} = 3$.

a) While finding an initial bfs our final simplex table is the following. What do we do now?

bv	x_1	x_2	x_3	x_4	y_1	$\overline{b}$
y_1	0	0	0	0	1	.1
x_4	-1	0	1	1	0	1
x_2	1	1	0	0	0	1
$-f$	2	0	0	0	0	$-.1$

b) While finding an initial bfs our final simplex table is the following. What do we do now?

bv	x_1	x_2	x_3	x_4	y_1	$\overline{b}$
y_1	1	0	0	0	1	.1
x_4	-1	0	1	1	0	1
x_2	1	1	0	0	0	1
$-f$	2	0	0	0	0	$-.1$

c) While finding an initial bfs our final simplex table is the following. What do we do now?

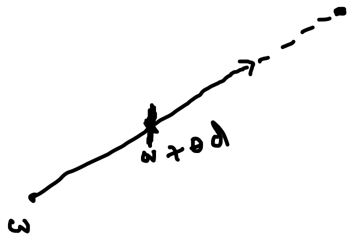
bv	x_1	x_2	x_3	x_4	y_1	$\overline{b}$
y_1	1	0	0	0	1	0
x_4	-1	0	1	1	0	1
x_2	1	1	0	0	0	1
$-f$	2	0	0	0	0	0

[13.6] **Application(M)** `(Nonempty sets)` Let $A_{m\times n}$ be a real matrix and $b \in \mathbb{R}^m$. We want to know whether the set $\{x \in \mathbb{R}^n \mid Ax = b\}$ is nonempty. We can use GJE to solve this problem. Write another method.

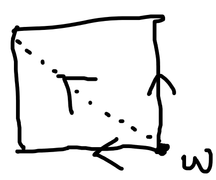
[13.7] **Application(E)** `(Nonempty sets)` We want to know whether the set $\{x \mid Ax \le b\}$ is nonempty. Write a method to do this.

Traveling via edges (optional)

Let $\mathsf{rank}\, A_{m\times n} = m$ and consider the slpp $\begin{array}{ll} \min & c^t y \\ \text{s.t.} & Ax = b, x \ge 0. \end{array}$ Let w be a bfs for the slpp with basis $(x_1, \cdots, x_m)$. Then the columns $\overline{A}_{:m+1}, \cdots, \overline{A}_{:n}$ provide us with some directions in which we can possibly move out of w. The extent could be infinite if that column is nonpositive. Otherwise the extent is decided by the calculation of δ (this could be zero). Consider the direction d provided by $v := \overline{A}_{:m+1}$ and suppose that it is possible to move by a positive extent, say θ, in that direction. Then the new point is



$$p = \begin{bmatrix} \overline{b}_1 \\ \vdots \\ \overline{b}_m \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \theta \begin{bmatrix} -v_1 \\ \vdots \\ -v_m \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = w + \theta d \quad \text{(say).}$$



$\exists x \quad w+\theta d \ne \Sigma x \in T$

$A\, w = b$

$A\, d = 0$

$A\, x = 0$

$$A = \begin{array}{c} x_1 \; x_2 \quad x_m \; x_{m+1} \\ \left[\begin{array}{c|c} \tilde{A} & \tilde{\tilde{A}} \end{array}\right] \end{array} \begin{bmatrix} x_1 \\ x_{m+1} \\ \hline 0 \\ 0 \end{bmatrix} = \begin{array}{l} \tilde{A}\tilde{x} = 0 \\ \tilde{A}\tilde{d} = 0 \end{array} \quad \Big| \quad \tilde{A}(\tilde{d} - \tilde{x}) = 0$$

Can there exist a vector $y \neq 0$ of small magnitude, not proportional to d, such that $p \pm x$ are both feasible?

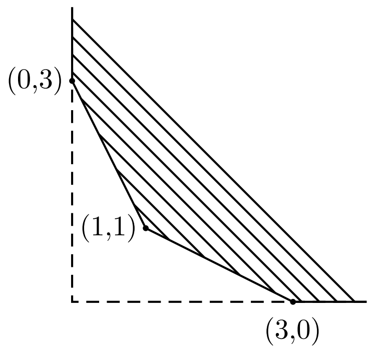
If it is so, then as we have $Aw = b$, $A(w + \theta d) = b$, and $A(w + \theta d + x) = b$, we must have $Ad = 0 = Ax$. Notice that as $p = w + \theta d$ has entries 0 at positions $m+2, \cdots, n$, it follows that, in order that $p \pm x$ have nonnegative coordinates, we must have $x_{m+2} = \cdots = x_n = 0$. Hence

$$\sum_{i=1}^{m+1} d_i A_{:i} = 0 = \sum_{i=1}^{m+1} x_i A_{:i}.$$

That is d and x are in the null space of the matrix $A_{:,\{1,2,\ldots,m+1\}}$ which has rank m. Hence, we must have x a scalar multiple of d, a contradiction. So the line segment or ray that is obtained by starting from w and moving in the direction d is an edge of the feasible set (polyhedron). Thus, for an slpp, the simplex algorithm moves from a vertex to another vertex along an edge.

[13.8] Example Consider $\min \; f(x) = x_2$ s.t. $x_1 + 2x_2 \geq 3,\ 2x_1 + x_2 \geq 3,\ x_i \geq 0.$ Draw the feasible region. Solve it by simplex method starting with the bfs corresponding to the vertex $(0,3)$. Which vertex is considered next by simplex method?

Answer. We first draw the feasible region of the lpp to see that there are three vertices $(0,3)$, $(1,1)$ and $(3,0)$.



Slpp : $\min \; f(x) = x_2$ s.t. $x_1 + 2x_2 - x_3 = 3,\ 2x_1 + x_2 - x_4 = 3,\ x_i \geq 0$.

Let T denote the feasible set for the lpp and T^* denote the feasible set for the slpp. Points of P^* corresponding to the vertices $(0,3)$, $(1,1)$, and $(3,0)$ are $(0,3,3,0)$, $(1,1,0,0)$, and $(3,0,0,3)$, respectively.

Now we apply simplex method to the slpp starting with the bfs $(0,3,3,0)$ with basis (x_2, x_3).

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	2	1	0	-1	3
x_3	$\boxed{3}$	0	1	-2	3
$-f$	-2	0	0	1	-3

$\rightarrow$

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	0	1	$-2/3$	$\boxed{1/3}$	1
x_1	1	0	$1/3$	$-2/3$	1
$-f$	0	0	$2/3$	$-1/3$	-1

Now we are at the bfs $(1,1,0,0) \simeq (1,1)$, where $f = 1$ and we have not reached the optimum.

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_4	0	3	-2	1	3
x_1	1	2	-1	0	3
$-f$	0	1	0	0	0

Now we are at the bfs $(3,0,0,3) \simeq (3,0)$, where $f = 0$ and we have reached the optimum.

Duality

The following result is a recast of [8.6]. We supply an alternate proof via simplex method. Recall that

$$\text{Primal } \min\ c^t x \quad \text{s.t. } x \in T_p := \{x \mid Ax \geq b,\ x \geq 0\} \qquad\qquad \text{Dual } \max\ y^t b \quad \text{s.t. } y \in T_d := \{y \mid y^t A \leq c^t,\ y \geq 0\}.$$

[13.9] **Theorem** (Primal-Dual theorem) For each $x \in T_p$ and for each $y \in T_d$ the relation $\boxed{c^t x \geq b^t y}$ holds. Furthermore, if the primal has a minimum, then there exist $w \in T_p$ and $z \in T_d$ such that $c^t w = z^t b$. That is, w is minimum for the primal, z is maximum for the dual and both the optimal values are the same.

~~Primal min x_0~~

$$\min\ c^T x \quad \text{s.t. } Ax \geq b,\ x \geq 0$$

$$\min\ \tilde{c}^T \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{s.t. } \begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = b,\ x, y \geq 0$$

$$\begin{bmatrix} c^T & 0^T \end{bmatrix}$$

Let (by simplex method) the min bfs $\omega = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $B \to$ basis matrix

$$\tilde{c}^T - \tilde{c}_B^T B^{-1} \tilde{A} = \begin{bmatrix} c^T & 0^T \end{bmatrix} - \tilde{c}_B^T B^{-1} \begin{bmatrix} A & -I \end{bmatrix} \geq 0$$

$$c^T - \tilde{c}_B^T B^{-1} A \geq 0 \qquad \tilde{c}_B^T B^{-1} \geq 0 \to z_0^T \qquad \to \qquad z_0^T A \leq c^T$$

$$z_0 \in T_d$$

$$\begin{bmatrix} c^T & 0^T \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = z_0^T b$$

$$\tilde{c}^T \omega = \tilde{c}_B^T \omega_B = \tilde{c}_B^T B^{-1} b = z_0^T b$$

Proof. The first statement follows from the fact that $c^t x \geq y^t A x \geq y^t b$. To prove the second one, assume that the given lpp has a minimal solution. Thus the lpp

$$\min\ \begin{bmatrix} c^t & 0 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix} \quad \text{s.t. } \begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix} = b,\ x, x' \geq 0$$

also has a minimal solution. By simplex method we get a minimal bfs $\tilde{w} = \begin{bmatrix} w \\ w' \end{bmatrix}$ at which the relative cost vector is nonnegative. Taking B as the basis matrix and $z^t = \begin{bmatrix} c^t & 0 \end{bmatrix}_B B^{-1}$, we have

$$\begin{bmatrix} c^t & 0 \end{bmatrix} - \begin{bmatrix} c^t & 0 \end{bmatrix}_B B^{-1} \begin{bmatrix} A & -I \end{bmatrix} = \begin{bmatrix} c^t - z^t A & z^t \end{bmatrix} \geq 0.$$

That is, $z \in T_d$. (Mark it! This is how we get the dual maximal solution.) As $\begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} w \\ w' \end{bmatrix} = b$, we have $Aw \geq b$. As $w \geq 0$, we have $w \in T_p$. Furthermore, for these w and z, we have

$$c^t w = \begin{bmatrix} c^t & 0 \end{bmatrix} \begin{bmatrix} w \\ w' \end{bmatrix} \overset{\text{say}}{=} \begin{bmatrix} c^t & 0 \end{bmatrix} \tilde{w} = \begin{bmatrix} c^t & 0 \end{bmatrix} \begin{bmatrix} \tilde{w}_B \\ \tilde{w}_C \end{bmatrix} = \begin{bmatrix} c^t & 0 \end{bmatrix}_B B^{-1} b = z^t b.$$

The proof is complete. ■

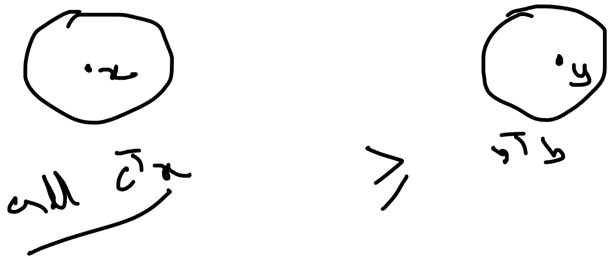
[13.10] Corollary If the dual has a maximum of value α, then the primal has a minimum of value α.

[13.11] Lemma `(Facts about duality)` The following statements are true.

a) If the primal objective function is unbounded below, then the dual feasible set is empty.

a1) If the dual objective function is unbounded above, then the primal feasible set is empty.

b) The primal has a minimum iff the dual has a maximum iff both T_p and T_d are nonempty.



Proof. a) Suppose that $T_d \neq \emptyset$. Let $y_0 \in T_d$. As $c^t x \geq b^t y_0$ for each $x \in T_p$, we have $f(x) = c^t x$ bounded below by $b^t y_0$ (a fixed value). A contradiction.

The proof of item a1) is similar.

To proof item b), note that we already know that 'primal lpp has a minimal solution $\Leftrightarrow$ dual lpp has a maximal solution'. Also we know that 'Primal has a minimal solution $\Rightarrow \{T_p \neq \emptyset, T_d \neq \emptyset\}$'. Thus, we only need to prove '$\{T_p \neq \emptyset, T_d \neq \emptyset\} \Rightarrow$ primal lpp has a minimal solution'.

Towards that assume $T_p \neq \emptyset$ and $T_d \neq \emptyset$. Thus the primal objective function $f(x) = c^t x$ is bounded below. By FTLP II, the primal lpp has a minimal solution. ■

> Attention.
> a) If you have an lpp $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax \geq b, x \geq 0 \end{array}$ where A has two rows, then the dual has only two variables.
> You may use graphical solution for the dual problem.
> b) Dual of an lpp given in other forms can be written by converting the lpp into the form $Ax \geq b, x \geq 0$ form first.

Some exercises

[13.12] **Exercise(E)** Consider minimizing $x_1 - x_2$ on $C(0, e_1, e_2, e_1 + e_2)$ in $\mathbb{R}^2$. We know it is minimized at the vertex e_2. What is the dual problem. Find the dual optimal solution in two ways. (See [8.6] and [13.9].)

[13.13] **NoPen** If the value of the primal objective function at a primal feasible solution and the value of the dual objective function at a dual feasible solution are the same, then is the common value necessarily optimal?

[13.14] **Exercise(E)** Give an example of a pair of primal and dual lpp such that neither has a feasible solution.

[13.15] **Exercise(E)** A company wants to produce a powder. The powder will be composed of three basic elements A,B,and C. Each kilo of the powder must contain at least 2200 calories of energy and 1200 units of vitamins. The vitamin and calorie content of each element is given below. Find the minimum cost to produce one kilo of the powder by: formulating the problem, writing the dual problem and solving the dual problem.

Element	Calorie amount	vitamin amount	cost per kilo
A	1800	1200	28
B	2500	1400	36
C	3000	1000	40