MA 372 : Stochastic Calculus for Finance

July - November 2021

Department of Mathematics, Indian Institute of Technology Guwahati

Total Marks: 70 End-Semester Examination Duration: Two Hours

- Answer **all** questions.
- Justify all your answers. Answers without justification carry no marks.
- Throughout this exam $\{W(t), \quad 0 \leq t \leq T\}$ denotes a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{F}(t),\ 0 \leq t \leq T\}$ denotes the filtration generated by $W(t)$.

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1. Show that if $M(t)$ is a martingale with respect to the filtration $\mathcal{F}(t),\ t \geq 0$, then
$$\mathbb{E}[M(t)] = \mathbb{E}[M(0)]$$
for all $t \geq 0$. Give an example of a stochastic process $M(t)$ satisfying
$$\mathbb{E}[M(t)] = \mathbb{E}[M(0)],\ \forall t \geq 0$$
and which is not a martingale with respect to its own filtration, (i.e., $\mathcal{F}(t) := \sigma\{M(s) | s \leq t\}$). [5+5]

2. Suppose f, g are square integrable, deterministic functions and that there exist constants C, D such that
$$C + \int_0^T f(s)dW(s) = D + \int_0^T g(s)dW(s),\ a.e.w \in \Omega.$$
(i) What is the relationship between C and D?
(ii) What is the relationship between $f(t)$ and $g(t)$? [4+6]

3. Let
$$X(t) = \int_0^t W(s)\ ds$$
for $t \geq 0$. Find the mean, variance and distribution function of the random variable X(2). [10]

4. Suppose that the price of a stock $\{S(t); t \geq 0\}$ follows geometric Brownian motion with drift 0.1 and volatility 0.05 so that it satisfies the stochastic differential equation
$$dS(t) = 0.05\ S(t)\ dW(t) + 0.1\ S(t)\ dt.$$
If the price of the stock at time zero is 35, determine the probability that the price of the stock at time $t = 5$ is less than 48. (If Z is a normal random variable with mean 0 and variance 1, then $\mathbb{P}(Z \leq -1.5911) = 0.0558$) [10]

5. Let $(W_1(t), W_2(t), W_3(t))$ be a 3-dimensional Brownian motion and

$X(t) = \int_0^t \sin(W_3(s))dW_1(s) + \int_0^t \cos(W_3(s))dW_2(s)$

$Y(t) = \int_0^t \cos(W_3(s))dW_1(s) + \int_0^t \sin(W_3(s))dW_2(s)$

(i) Is $X(t)$ a Brownian motion?

(ii) Is $(X(t), Y(t))$ a two-dimensional Brownian motion? [4+6]

6. Let $(W_1(t), W_2(t)),\ \ 0 \leq t \leq T$ be a 2-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t),\ 0 \leq t \leq T$ be a filtration for this Brownian motion. Consider a financial market consisting of a risk-free asset $B(t)$ and two stocks (risky assets) $S_1(t)$ and $S_2(t)$, whose price at time $t,\ t > 0$ satisfy the following differentials:

$$\begin{aligned} dB(t) &= 5B(t)dt \\ dS_1(t) &= S_1(t)\Big(7\ dt + dW_1(t) + dW_2(t)\Big) \\ dS_2(t) &= S_2(t)\Big(\mu\ dt + dW_1(t) + \sigma\ dW_2(t)\Big) \end{aligned}$$

where μ, σ are positive constants.

(i) When the above market is arbitrage free?

(ii) When the above market is complete?

(iii) If $\mu = 8$ and $\sigma = 2$ then find the risk-neutral probability measure $\mathbb{Q}$ for the above market.

(iv) If $\mu = 8$ and $\sigma = 2$ then find $dS_1(t)$ in terms of $\tilde{W}(t) = (\tilde{W}_1(t), \tilde{W}_2(t))$, where $\tilde{W}(t) = (\tilde{W}_1(t), \tilde{W}_2(t))$ is a 2-dimensional Brownian motion under the risk-neutral probability measure $\mathbb{Q}$.

[4+4+6+6]