Real Analysis
Course No. MA 224

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Contents

1 Measure on the Real Line

We wish to extend the notion of length of intervals to a much larger class of subsets of $\mathbb{R}$ containing the intervals. We would like that notion to be suitably additive and invariant under translations of sets. In the upcoming part when we say interval we mean intervals of the form $I = [a, b)$ where a and b are finite. When $a = b$, I is the empty set ϕ. We will write $l(I)$ for the length of I which is $b - a$.

Definition 1.1. *The Lebesgue outer measure of a subset A of $\mathbb{R}$ is given by $m^*(A) = \inf \sum l(I_n)$, where the infimum is taken over all countable collections of intervals $\{I_n\}$ such that $A \subset \cup I_n$.*

Theorem 1.2.

1. $m^*(A) \geq 0$.
2. $m^*(\phi) = 0$.
3. $m^*(A) \leq m^*(B)$ *if* $A \subset B$.
4. $m^*(\{x\}) = 0$ *for all* $x \in \mathbb{R}$.

Proof. (1), (2) and (3) are obvious. Since $\{x\} \subset [x, x + \frac{1}{n})$ for all n, $m^*(\{x\}) \leq 1/n$ for all n. Hence (4) follows. □

Problem: Show that for any set A, $m^*(A) = m^*(A+x)$ where $A+x = \{y+x : y \in A\}$. Thus outer measure is translation invariant.

Proof. For each $\epsilon > 0$ there exists a collection $\{I_n\}$ such that $A \subset \cup I_n$ and $m^*(A) \geq \sum l(I_n) - \epsilon$. But clearly $A + x \subset \cup(I_n + x)$. So, for any $\epsilon > 0$, $m^*(A + x) \leq \sum l(I_n + x) = \sum l(I_n) \leq m^*(A) + \epsilon$. So $m^*(A + x) \leq m^*(A)$. But $A = (A + x) - x$, so we have $m^*(A) \leq m^*(A + x)$. □

Theorem 1.3. *The outer measure of an interval equals its length. (Here by interval we mean all kinds of intervals)*

Proof. Case 1: Suppose $I = [a, b]$. Then for any $\epsilon > 0$, we have $I \subset [a, b + \epsilon)$. Thus $m^*(I) \leq b - a + \epsilon$. Hence $m^*(I) \leq b - a = l(I)$.
Now for the reverse inequality, for any $\epsilon > 0$ there exists a collection of intervals $\{I_n\}$ such that $I \subset \cup I_n$ and $m^*(I) \geq \sum l(I_n) - \epsilon$. If $I_n = [a_n, b_n)$, define $I'_n = (a_n - \frac{\epsilon}{2^n}, b_n)$. Then $I \subset \cup I'_n$. Now since I is compact, there exists a finite subcollection of I'_n, say $J_1, \ldots, J_N$ which covers I. Suppose $J_k = (c_k, d_k)$. Then as we may suppose

that no J_k is contained in any other, we have, supposing that $c_1 < c_2 < \ldots < c_N$, $d_N - c_1 \leq \sum_{k=1}^{N} l(J_k)$. Thus we have,

$$\begin{aligned} m^*(I) &\geq \sum l(I_n) - \epsilon \geq \sum l(I'_n) - 2\epsilon \geq \sum_{k=1}^{N} l(J_k) - 2\epsilon \\ &\geq d_N - c_1 - 2\epsilon > b - a - 2\epsilon > l(I) - 2\epsilon. \end{aligned}$$

Hence the proof.
Case 2: Suppose that $I = (a, b]$, $a > -\infty$. For $0 < \epsilon < b - a$, $m^*(I) \geq m^*([a + \epsilon, b]) = l(I) - \epsilon$. Again $I \subset [a, b + \epsilon)$. Thus $m^*(I) \leq l(I) + \epsilon$. Hence we are done.
Other cases can be dealt similarly and is left as an exercise. □

Theorem 1.4. *For any sequence of sets $\{E_i\}$, $m^*\left(\cup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$.*

Proof. For each i, and for any $\epsilon > 0$, there exists a sequence of intervals $\{I_{i,j}, j = 1, 2, \ldots\}$ such that $E_i \subset \cup_{j=1}^{\infty} I_{i,j}$ and $m^*(E_i) \geq \sum_{j=1}^{\infty} l(I_{i,j}) - \frac{\epsilon}{2^i}$. Then $\cup_{i=1}^{\infty} E_i \subset \cup_{i=1}^{\infty} \cup_{j=1}^{\infty} I_{i,j}$. Thus

$$m^*\left(\cup_{i=1}^{\infty} E_i\right) \leq \sum_{i,j=1}^{\infty} l(I_{i,j}) \leq \sum_{i=1}^{\infty} m^*(E_i) + \epsilon.$$

Since, ϵ was arbitrary, the result follows. □

Fact: Suppose that in the definition of outer measure instead of $I_n = [a_n, b_n)$, we take $(i) I_n$ is open, $(ii) I_n = (a_n, b_n]$, $(iii) I_n$ is closed or (v) mixtures are allowed, for different n, of the various types of intervals, we still get the same outer measure.

Definition 1.5. *The set E is lebesgue measurable if for each set A we have,*

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Since m^* is sub-additive, in order to show that E is measurable we only need to show that for each A,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

Example: It follows from Theorem 1.2 that, if $m^*(E) = 0$ then E is measurable.

Definition 1.6. *A class of subsets of a non-empty set X is said to be a $\sigma-$algebra if X belongs to the class and the class is closed under the formation of countable unions and of complements.*

If in the above definition we replace countable unions by finite unions then the collection is called an algebra.
Exercises: Suppose $\mathcal{S}$ is a $\sigma-$algebra. If $A, B \in \mathcal{S}$, then show that $A - B$ is in $\mathcal{S}$.
Show that arbitrary intersection of $\sigma-$algebras is again a $\sigma-$algebra.
Show that union of two $\sigma-$algebras need not be a $\sigma-$algebra.
Let M denote the collection of all Lebesgue measurable sets.

Theorem 1.7. *M is a $\sigma-$algebra.*

Proof. Since for any A, $m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \phi) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c)$, we have $\mathbb{R} \in M$. Also since the definition of measurability is symmetric in E and E^c, if $E \in M$, then $E^c \in M$. Now suppose $\{E_i\} \subset M$. We need to show that $\cup_{i=1}^{\infty} E_i \in M$. Let A be any set. Then

$$
\begin{aligned}
m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c) \\
&= m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap E_1^c \cap E_2^c).
\end{aligned}
$$

Continuing in this way we obtain, for $n \geq 2$

$$
\begin{aligned}
m^*(A) &= m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap (\cap_{j<i} E_j^c)) + m^*(A \cap (\cap_{j=1}^{n} E_j^c)) \\
&= m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap (\cup_{j<i} E_j)^c) + m^*(A \cap (\cup_{j=1}^{n} E_j)^c) \\
&\geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap (\cup_{j<i} E_j)^c) + m^*(A \cap (\cup_{j=1}^{\infty} E_j)^c)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
m^*(A) &\geq m^*(A \cap E_1) + \sum_{i=2}^{\infty} m^*(A \cap E_i \cap (\cup_{j<i} E_j)^c) + m^*(A \cap (\cup_{j=1}^{\infty} E_j)^c) \\
&\geq m^*(A \cap (\cup_{j=1}^{\infty} E_j)) + m^*(A \cap (\cup_{j=1}^{\infty} E_j)^c) \geq m^*(A). \qquad (1.1)
\end{aligned}
$$

The second inequality follows by sub-additivity of m^* and by the fact that $A \cap (\cup_{j=1}^{\infty} E_j) = (A \cap E_1) \cup (\cup_{i=2}^{\infty} A \cap E_i \cap (\cup_{j<i} E_j)^c)$. Hence, the proof. $\square$

Theorem 1.8 (Countable Additivity)**.** *If $\{E_i\}$ is any sequence of disjoint measurable sets then*

$$
m^*(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i)\,.
$$

Proof. Take $A = \cup_{i=1}^{\infty} E_i$ in (1.1) and use the fact that E_is are disjoint. $\square$

Is m^* countably additive on $\mathcal{P}(\mathbb{R})$ (power set of $\mathbb{R}$)?
Example: If $x, y \in [0,1]$, let $x \sim y$ if $y - x \in \mathbb{Q}_1 = \mathbb{Q} \cap [-1,1]$. Then check that $\sim$ is an equivalence relation and thus partitions $[0,1]$ into equivalence classes, i.e., $[0,1] = \cup_\alpha E_\alpha$, where E_α are disjoint sets such that x and y are in the same E_α iff $x \sim y$. Using Axiom of Choice we form a subset V of $[0,1]$ containing exactly one element x_α from each E_α. Let r_i be an enumeration of $\mathbb{Q}_1$. Define $V_n = V + r_n$ for each n. Claim: $V_n \cap V_m = \phi$. Because if $y \in V_n \cap V_m$ then there exists x_α and x_β in V such that $y = x_\alpha + r_n$ and $y = x_\beta + r_m$. But then $x_\alpha - x_\beta \in \mathbb{Q}_1$, which contradicts

the construction of V. Now for any $x \in [0,1]$, $x \in E_\alpha$ for some α. Thus $x = x_\alpha + r_n$ for some n. Hence $x \in V_n$. So we get, $[0,1] \subset \cup_{n=1}^{\infty} V_n \subset [-1,2]$. But then since m^* is translation invariant and outer measure of an interval is equal to its length,

$$1 = m^*([0,1]) \leq m^*(\cup_{n=1}^{\infty} V_n) = \sum_{n=1}^{\infty} m^*(V_n) \leq m^*([-1,2]) = 3\,.$$

But the sum can only be 0 or ∞. Thus m^* is not countably additive on $\mathcal{P}(\mathbb{R})$. This also tells that M is a proper subset of $\mathcal{P}(\mathbb{R})$.

From now on, if $E \in M$, then we will write $m(E)$ in place of $m^*(E)$. $m(E)$ is called the Lebesgue measure of E. Thus m is set function defined on the $\sigma-$algebra M such that $m(\phi) = 0$ and m is countably additive.

Proposition 1.9. *Every interval is measurable.*

Proof. Since M is a $\sigma-$algebra, it is enough to prove it for interval of the form $[a, \infty)$. For any set A we need to show,

$$m^*(A) \geq m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty)).$$

Let $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap [a, \infty)$. Now for any $\epsilon > 0$ there exist intervals $\{I_n\}$ such that $A \subset \cup I_n$ and $m^*(A) \geq \sum l(I_n) - \epsilon$. Let $I_n' = I_n \cap (-\infty, a)$ and $I_n'' = I_n \cap [a, \infty)$. Thus $l(I_n) = l(I_n') + l(I_n'')$.Also, $A_1 \subset \cup I_n'$ and $A_2 \subset \cup I_n''$.Thus

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum l(I_n') + \sum l(I_n'') \\ &\leq \sum l(I_n) \leq m^*(A) + \epsilon. \end{aligned}$$

Hence the result. □

Since every open set in $\mathbb{R}$ can be written as a countable union of open intervals, every open set is measurable, which then implies that every closed set is measurable.

Definition 1.10. *For any sequence of sets* $\{E_i\}$ *define*

$$\limsup E_i = \cap_{n=1}^{\infty} \cup_{i \geq n} E_i, \qquad \liminf E_i = \cup_{n=1}^{\infty} \cap_{i \geq n} E_i.$$

Thus, $\limsup E_i$ is the set of points which belongs to infinitely many of the sets E_i and $\liminf E_i$ is the set of points which belongs to all but finitely many of the sets E_i. It is easy to see that $\liminf E_i \subset \limsup E_i$. When the sets are equal we denote the common set by $\lim E_i$. If $E_1 \subset E_2 \subset \ldots$, then $\lim E_i = \cup_{i=1}^{\infty} E_i$ and if $E_1 \supset E_2 \supset \ldots$, then $\lim E_i = \cap_{i=1}^{\infty} E_i$.

Theorem 1.11 (Continuity Properties of Measure)**.** *Let* $\{E_i\}$ *be a sequence of measurable sets. Then*

1. *if* $E_1 \subset E_2 \subset \ldots$*, we have* $m(\lim E_i) = \lim m(E_i)$*,*

2. *if* $E_1 \supset E_2 \supset \ldots$, *and* $m(E_1) < \infty$, *then* $m(\lim E_i) = \lim m(E_i)$.

Proof. (1) Write $F_1 = E_1$, $F_i = E_i - E_{i-1}$ for $i > 1$. Then $\cup E_i = \cup F_i$ and F_i are disjoint. Thus,

$$m(\lim E_i) = m(\cup E_i) = m(\cup F_i) = \sum m(F_i) = \lim \sum_{i=1}^{n} m(F_i) = \lim m(\cup_{i=1}^{n} F_i) = \lim m(E_n).$$

(2) We have $E_1 - E_1 \subset E_1 - E_2 \subset E_1 - E_3 \ldots$, so by (1),

$$m(\lim(E_1 - E_i)) = \lim m(E_1 - E_i) = m(E_1) - \lim m(E_i).$$

But $\lim(E_1 - E_i) = \cup(E_1 - E_i) = E_1 - \cap E_i = E_1 - \lim E_i$.Thus

$$m(E_1) - m(\lim E_i) = m(\lim(E_1 - E_i)) = m(E_1) - \lim m(E_i).$$

Since $m(E_1) < \infty$, we are done. $\square$

Theorem 1.12. *The following are equivalent.*

1. E *is measurable.*
2. *For any* $\epsilon > 0$, *there exists an open set* O, *such that* $E \subset O$ *and* $m^*(O - E) \leq \epsilon$.
3. *For any* $\epsilon > 0$, *there exists a closed set* F, *such that* $E \supset F$ *and* $m^*(E - F) \leq \epsilon$.

Proof. ($1 \implies 2$). Suppose first that $m(E) < \infty$. Now given $\epsilon > 0$ there exists a sequence of intervals $\{I_n\}$ such that $E \subset \cup I_n$ and $m(E) \geq \sum l(I_n) - \epsilon/2$. Now if $I_n = [a_n, b_n)$, let $I'_n = (a_n - \frac{\epsilon}{2^{n+1}}, b_n)$. Then $E \subset O = \cup I'_n$, O is an open set and

$$m(O) \leq \sum l(I'_n) = \sum l(I_n) + \frac{\epsilon}{2} \leq m(E) + \epsilon.$$

Hence $m(O - E) = m(O) - m(E) \leq \epsilon$. Now if $m(E) = \infty$, then write $\mathbb{R} = \cup I_n$, a union of disjoint finite intervals. Then if $E_n = E \cap I_n$, we have $m(E_n) < \infty$. So there exists open sets O_n such that $E_n \subset O_n$ and $m(O_n - E_n) \leq \epsilon/2^n$. Let $O = \cup O_n$. Then

$$O - E = \cup O_n - \cup E_n \subset \cup(O_n - E_n).$$

So $m(O - E) \leq \sum m(O_n - E_n) \leq \epsilon$.
($2 \implies 1$). For each n, let O_n be an open set such that $E \subset O_n$ and $m^*(O_n - E) < 1/n$. Let $G = \cap O_n$. Then, $E \subset G$ and $m^*(G - E) \leq m^*(O_n - E) < 1/n$. Thus $m^*(G - E) = 0$, which is therefore measurable. Also G is measurable. Hence $E = G - (G - E)$ is measurable.
($1 \implies 3$). Since E^c is measurable, there exists open set O, such that $E^c \subset O$ and $m(O - E^c) \leq \epsilon$. But $O - E^c = E - O^c$. Thus taking $F = O^c$, we are done.
($3 \implies 1$). Exercise. $\square$

Application: If E is measurable, then for each y, $E+y$ is also measurable and there measures are same.

Proof. Given $\epsilon > 0$, there exists an open set $O \supset E$ such that $m(O-E) \leq \epsilon$. For any y, $y+O$ is open, $y+E \subset y+O$. Now, $(y+O)-(y+E) = y+(O-E)$. Also we have already shown that outer measure is translation invariant. Thus, $m^*((O+y)-(E+y)) = m^*(O-E) \leq \epsilon$. Hence by the above Theorem, $y+E$ is measurable and their measures are same. $\square$

1.1 Measurable Functions

We will use the following conventions:

- $a + \infty = \infty (a \text{ real, or } a = \infty)$,
- $a \cdot \infty = \infty (a > 0)$,
- $a \cdot \infty = -\infty (a < 0)$, $\infty \cdot \infty = \infty$,
- $0 \cdot \infty = 0$, $0 \cdot -\infty = 0$.

Definition 1.13. *Let f be an extended real valued function defined on a measurable set E. Then f is said to be a Lebesgue measurable function if, for each $\alpha \in \mathbb{R}$, the set $\{x : f(x) > \alpha\}$ is measurable.*

Theorem 1.14. *The following are equivalent.*

1. *f is measurable.*
2. *For all α, $\{x : f(x) \geq \alpha\}$ is measurable.*
3. *For all α, $\{x : f(x) < \alpha\}$ is measurable.*
4. *For all α, $\{x : f(x) \leq \alpha\}$ is measurable.*

Proof. ($1 \implies 2$). Let f be measurable. Then

$$\{x : f(x) \geq \alpha\} = \cap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$$

is measurable.
($2 \implies 3$). Let $\{x : f(x) \geq \alpha\}$ be measurable, then $\{x : f(x) < \alpha\} = \{x : f(x) \geq \alpha\}^c$ is measurable.
($3 \implies 4$). Let $\{x : f(x) < \alpha\}$ be measurable. Then

$$\{x : f(x) \leq \alpha\} = \cap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$$

is measurable.
($4 \implies 1$). If $\{x : f(x) \leq \alpha\}$ is measurable, so is its complement $\{x : f(x) > \alpha\}$. $\square$

Exercises: Show that a constant function is measurable.
Show that the indicator function 1_A of the set A is measurable if and only if A is measurable.
Example: Continuous functions are measurable.

Proof. If f is continuous, $\{x : f(x) > \alpha\}$ is open and therefore measurable. $\square$

Theorem 1.15. *Let c be any real number and let f and g be real-valued measurable functions defined on the same measurable set E. Then $f + c$, cf, $f + g$, $f - g$ and fg are also measurable.*

Proof. For each α, $\{x : f(x)+c > \alpha\} = \{x : f(x) > \alpha - c\}$ is measurable. Thus $f+c$ is measurable. For $c > 0$, $\{x : cf(x) > \alpha\} = \{x : f(x) > c^{-1}\alpha\}$ is measurable. Similarly, for $c < 0$, $\{x : cf(x) > \alpha\} = \{x : f(x) < c^{-1}\alpha\}$ is measurable. Thus cf is measurable. Now, to show that $f + g$ is measurable, observe that $x \in A = \{x : f(x) + g(x) > \alpha\}$ only if $f(x) > \alpha - g(x)$, that is, only if there exists a rational r_i such that $f(x) > r_i > \alpha - g(x)$, where $\{r_i, i = 1, 2, \ldots\}$ is an enumeration of $\mathbb{Q}$. Thus $x \in \{x : f(x) > r_i\} \cap \{x : g(x) > \alpha - r_i\}$. Hence $A \subset B = \cup_{i=1}^{\infty}\{x : f(x) > r_i\} \cap \{x : g(x) > \alpha - r_i\}$. It is trivial to see that $B \subset A$. Thus, $A = B$ and B is measurable since f and g are measurable functions.
Since $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$, it is sufficient to show that f^2 is measurable whenever f is. If $\alpha < 0$, then $\{x : f^2(x) > \alpha\} = \mathbb{R}$ is measurable. If $\alpha \geq 0$, $\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$ is a measurable set. $\square$

Theorem 1.16. *Let $\{f_n\}$ be a sequence of measurable functions defined on the same measurable set. Then*

1. $\sup_{1\leq i\leq n} f_i$ *is measurable for each n.*
2. $\inf_{1\leq i\leq n} f_i$ *is measurable for each n.*
3. $\sup f_n$ *is measurable.*
4. $\inf f_n$ *is measurable.*
5. $\limsup f_n$ *is measurable.*
6. $\liminf f_n$ *is measurable.*

Proof. (1) Since $\{x : \sup_{1\leq i\leq n} f_i > \alpha\} = \cup_{i=1}^{n}\{x : f_i(x) > \alpha\}$, we have the result.

(2) $\inf_{1\leq i\leq n} f_i = -\sup_{1\leq i\leq n}(-f_i)$.

(3) $\{x : \sup f_n > \alpha\} = \cup_{n=1}^{\infty}\{x : f_n(x) > \alpha\}$.
(4) $\inf f_n = -\sup(-f_n)$.

(5) $\limsup f_n = \inf(\sup_{i \geq n} f_i)$.

(6) $\liminf f_n = -\limsup(-f_n)$. □

Definition 1.17. *If a property holds except on a set of measure zero, we say that it holds almost everywhere, and abbreviate as a.e.*

Theorem 1.18. *Let f be a measurable function and let $f = g$ a.e. Then g is also measurable.*

Proof. First we show that if F is measurable and $m^*(F\Delta G) = m^*((F-G)\cup(G-F)) = 0$, then G is also measurable. Now by given condition, $F-G$ and $G-F$ are measurable. Thus $F \cap G = F-(F-G)$ is measurable. So, $G = (F\cap G)\cup(G-F)$ is measurable. Now $\{x : f(x) > \alpha\}\Delta\{x : g(x) > \alpha\} \subset \{x : f(x) \neq g(x)\}$. Now the result follows. □

Exercises: Let $\{f_n\}$ be a sequence of measurable functions converging point-wise to another function f. The show that f is measurable.
Show that, if f is measurable, so are $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

1.2 Lebesgue Integration:

A non-negative finite-valued function φ, taking only a finite number of different values, is called a simple function. If $a_1, a_2, \ldots, a_n$ are the distinct values taken by φ and $A_i = \{x : \varphi(x) = a_i\}$, then clearly

$$\varphi(x) = \sum_{i=1}^{n} a_i 1_{A_i}(x). \tag{1.2}$$

The sets A_i are measurable if φ is measurable.

Definition 1.19. *Let φ be a measurable simple function. Then*

$$\int \varphi(x)dx = \sum_{i=1}^{n} a_i m(A_i),$$

where a_i and A_i are as in (1.2)*, is called the integral of φ.*

Definition 1.20. *For any non-negative measurable function f, the integral of f, $\int f dx$, is given by $\int f dx = \sup \int \varphi dx$, where the supremum is taken over all measurable simple functions φ, $\varphi \leq f$.*

Definition 1.21. *For any measurable set E, and any non-negative measurable function f, $\int_E f dx = \int f 1_E dx$ is the integral of f over E.*

Theorem 1.22. *If φ is a measurable simple function, then in the notation of* (1.2)

1. *$\int_E \varphi dx = \sum_{i=1}^{n} a_i m(A_i \cap E)$ for any measurable set E.*

2. $\int_{A \cup B} \varphi dx = \int_A \varphi dx + \int_B \varphi dx$ *for any disjoint measurable sets* A *and* B.

3. $\int a\varphi dx = a \int \varphi dx$ *if* $a > 0$.

4. *If* ψ *is another measurable simple function, then* $\int (\varphi + \psi) dx = \int \varphi dx + \int \psi dx$.

Problem: If f is a non-negative measurable function, then $f = 0$ a.e. if, and only if, $\int f dx = 0$.

Proof. If $f = 0$ a.e. and φ is a measurable simple function, $\varphi \leq f$, then clearly $\int \varphi dx = 0$. So by definition $\int f dx = 0$. Conversely, if $\int f dx = 0$ and $E_n = \{x : f(x) \geq 1/n\}$, then $\int f dx \geq \int n^{-1} 1_{E_n} = n^{-1} m(E_n)$. So $m(E_n) = 0$. But $\{x : f(x) > 0\} = \cup_{n=1}^{\infty} E_n$, so $f = 0$ a.e. $\square$

Theorem 1.23. *Let* f *and* g *be non-negative measurable functions.*

1. *If* $f \leq g$*, then* $\int f dx \leq \int g dx$.

2. *If* A *is a measurable set and* $f \leq g$ *on* A*, then* $\int_A f dx \leq \int_A g dx$.

3. *If* $a \geq 0$*, then* $\int a f dx = a \int f dx$.

4. *If* A *and* B *are measurable sets and* $A \subset B$*, then* $\int_A f dx \leq \int_B f dx$.

Theorem 1.24 (Fatou's Lemma)**.** *Let* $\{f_n\}$ *be a sequence of non-negative measurable functions. Then*

$$\int \liminf f_n dx \leq \liminf \int f_n dx.$$

Theorem 1.25 (Monotone Convergence Theorem)**.** *Let* $\{f_n\}$ *be a sequence of non-negative measurable functions such that* $\{f_n(x)\}$ *is monotone increasing for each* x. *Let* $f = \lim f_n$. *Then* $\int f dx = \lim \int f_n dx$.

Proof. Clearly $\int f_n dx \leq \int f dx$ for all n. Thus $\limsup \int f_n dx \leq \int f dx$.

But, by Fatou's lemma, $\int f dx = \int \liminf f_n dx \leq \liminf \int f_n dx$. Hence the result follows. $\square$

Example: We give an example where strict inequality occurs in Fatou's Lemma. Let $f_{2n-1} = 1_{[0,1]}$ and $f_{2n} = 1_{(1,2)}$ for $n \geq 1$. Then $\liminf f_n(x) = 0$ for all x, but $\int f_n dx = 1$ for all n.

Theorem 1.26. *Let* f *be a non-negative measurable function. Then there exists a sequence* $\{\varphi_n\}$ *of measurable simple functions such that, for each* x, $\varphi_n(x) \uparrow f(x)$.

Proof. For each n, let $E_{nk} = \{x : (k-1)/2^n < f(x) \le k/2^n\}$, $k = 1, 2, \ldots, n2^n$ and $F_n = \{x : f(x) > n\}$. Define φ_n by

$$\varphi_n = \sum_{k=1}^{2^n} \frac{k-1}{2^n} 1_{E_{nk}} + n 1_{F_n}.$$

It is easy to see that φ_n are measurable simple functions and $\varphi_n \le \varphi_{n+1}$. Now if $f(x)$ is finite, then $x \in F_n^c$ for all n large enough, and thus $|f(x) - \varphi_n(x)| \le 2^{-n}$. So $\varphi_n(x) \uparrow f(x)$. If $f(x) = \infty$, then $x \in \cap_{n=1}^{\infty} F_n$, so $\varphi_n(x) = n$ for all n, and again $\varphi_n(x) \uparrow f(x)$. □

Corollary 1.27. *If f and φ_n are as in Theorem above, then* $\lim \int \varphi_n(x)dx = \int f dx$.

Corollary 1.28. *Let f, g be non-negative measurable functions. Then*

$$\int (f+g)dx = \int f dx + \int g dx.$$

Proof. Exercise. □

Now we try to extend the definition of integral to a wider class of functions. For any function f

$$f^+(x) = \max(f(x), 0) \quad f^-(x) = \max(-f(x), 0),$$

are said to be the positive and negative parts of f, respectively. It is easy to see that $f = f^+ - f^-$, $|f| = f^+ + f^-$ and f is measurable iff f^+ and f^- are both measurable.

Definition 1.29. *If f is a measurable function and $\int f^+ dx < \infty$, $\int f^- dx < \infty$, we say that f is integrable and its integral is given by*

$$\int f dx = \int f^+ dx - \int f^- dx.$$

Definition 1.30. *If E is a measurable set, f is a measurable function and $f1_E$ is integrable, we say that f is integrable over E, and its integral is given by $\int_E f dx = \int f 1_E dx$.*

Theorem 1.31. *Let f and g be integrable functions.*

1. *af is integrable and $\int af dx = a \int f dx$.*
2. *$f + g$ is integrable and $\int (f+g)dx = \int f dx + \int g dx$.*
3. *If $f = 0$ a.e., then $\int f dx = 0$.*
4. *If $f \le g$ a.e., then $\int f dx \le \int g dx$.*

5. If A and B are disjoint measurable sets, then

$$\int_A f dx + \int_B f dx = \int_{A\cup B} f dx.$$

Proof. Exercise. □

Problem: Show that if f is an integrable function, then $|\int f dx| \leq \int |f| dx$.

Proof. $|f| - f \geq 0$, so $\int (|f| - f) dx \geq 0$, or $\int |f| dx \geq \int f dx$. Again $|f| + f \geq 0$, so $\int |f| dx \geq -\int f dx$. Hence we get the result. □

Problem: Show that if f is integrable, then f is finite a.e.

Proof. If $|f| = \infty$ on a set E with $m(E) > 0$, then $\int |f| dx \geq \int_E |f| dx \geq n\, m(E)$ for all n, giving a contradiction. □

Problem: Let f be an integrable function such that $\int_E f dx = 0$ for every measurable set E, then $f = 0$ a.e.

Proof. Let $A = \{x : f(x) > 0\}$. Suppose, $m(A) > 0$. Let $A_n = \{x : f(x) \geq 1/n\}$. Then $A = \cup A_n$. So there exists N such that $m(A_N) > 0$. Then

$$\int_A f dx \geq \int_{A_N} f dx \geq \frac{1}{N} m(A_N) > 0,$$

which is a contradiction. Thus $m(A) = 0$. Similarly, it can be shown that $m(B) = 0$ where $B = \{x : f(x) < 0\}$. Thus $f = 0$ a.e. □

Definition 1.32. *Let $\{f_n\}$ be a sequence of measurable functions. We say that* $\lim f_n = f$ *a.e., if* $m\{x : f_n(x) \to f(x)\}^c = 0$.

Theorem 1.33 (Dominated Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$, where g is integrable. Let* $\lim f_n = f$ *a.e. Then f is integrable and*

$$\lim \int f_n dx = \int f dx.$$

Proof. Since for each n, $|f_n| \leq g$, we have $|f| \leq g$ a.e. Thus f_n and f are integrable. Now $-g \leq f_n \leq g$. Thus $g + f_n$ is a sequence of non-negative measurable functions, so by Fatou's Lemma

$$\liminf \int (g + f_n) dx \geq \int \liminf (g + f_n) dx.$$

So $\int g dx + \liminf \int f_n dx \geq \int g dx + \int f dx$. Since $\int g dx$ is finite, we get

$$\liminf \int f_n dx \geq \int f dx.$$

Again, $\{g - f_n\}$ is also a sequence of measurable functions, so

$$\liminf \int (g - f_n)dx \geq \int \liminf (g - f_n)dx.$$

So, $\int gdx - \limsup \int f_n dx \geq \int gdx - \int fdx$. So,

$$\limsup \int f_n dx \leq \int fdx.$$

Hence the result follows. $\square$

Corollary 1.34. *With the same hypotheses as the above Theorem, we have*

$$\lim \int |f_n - f|dx = 0.$$

Proof. Exercise. $\square$

Theorem 1.35. *Let $\{f_n\}$ be a sequence of measurable functions such that*

$$\sum_{n=1}^{\infty} \int |f_n|dx < \infty.$$

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. Further, if f denotes its sum, then f is integrable and

$$\int fdx = \sum_{n=1}^{\infty} \int f_n dx.$$

Proof. Let $\phi(x) = \sum_{n=1}^{\infty} |f_n(x)|$. Then by given hypothesis,

$$\int \phi dx = \int \sum_{n=1}^{\infty} |f_n(x)|dx = \sum_{n=1}^{\infty} \int |f_n|dx < \infty. \qquad \text{(By MCT)}$$

Thus ϕ is integrable and so ϕ is finite a.e. It follows that $\sum_{n=1}^{\infty} f_n$ is absolutely convergent a.e., its sum $f(x)$ is defined a.e. and $|f| \leq \phi$. Thus f is also integrable. Now if we write $g_n(x) = \sum_{i=1}^{n} f_i(x)$. Then $|g_n(x)| \leq \phi(x)$ for all n and $\lim g_n(x) = f(x)$ a.e. Now we are done by applying DCT. $\square$

Theorem 1.36. *If f is bounded and Riemann integrable over the finite interval $[a, b]$, then f is integrable on $[a, b]$ and $R\int_a^b fdx = \int_a^b fdx$.*

Example: Now we see an example of a function which is not Riemann integrable but Lebesgue integrable. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 0$ if x is irrational and $f(x) = 1$ if x is rational. Then it is known that f is not Riemann integrable. Now since every singleton set has measure zero, by countable additivity every countable subset of $\mathbb{R}$ has also measure zero. Thus $f = 0$ a.e and hence $\int_0^1 fdx = 0$.

Definition 1.37. *If, for each a and b, f is bounded and Riemann integrable on $[a,b]$ and*

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f dx$$

exists, then f is said to be Riemann integrable on $(-\infty,\infty)$, and the integral is written as $R\int_{-\infty}^{\infty} f dx$.

Theorem 1.38. *Let f be bounded and let f and $|f|$ be Riemann integrable on $(-\infty,\infty)$. Then f is integrable and*

$$\int_{-\infty}^{\infty} f dx = R\int_{-\infty}^{\infty} f dx.$$

1.3 L^p Spaces

For $1 \le p < \infty$, we define $L^p(\mathbb{R}, m)$ to be the set of all measurable functions on $\mathbb{R}$ such that $\int |f|^p dx < \infty$. For any measurable set E, we say that $f \in L^p(E, m)$ if $f1_E \in L^p(\mathbb{R}, m)$. All results that we state below for $L^p(\mathbb{R}, m)$ is also true for $L^p(E, m)$. We follow the convention that any two function are equal in L^p if they are equal almost everywhere.

Theorem 1.39. *Let $f, g \in L^p$ and let a, b be constants. Then $af + bg \in L^p$.*

Proof. It is easy to see that af and bg is in L^p. Thus we need to show that $f + g \in L^p$. But that follows because

$$|f+g|^p \le 2^p(|f|^p + |g|^p).$$

□

Theorem 1.40. *(Holder's Inequality) Let $1 < p < \infty$, $1 < q < \infty$, $1/p + 1/q = 1$ and let $f \in L^p, g \in L^q$. Then $fg \in L^1$ and*

$$\int |fg| dx \le \left(\int |f|^p dx\right)^{1/p} \left(\int |g|^q dx\right)^{1/q}.$$

Proof. If either $\int |f|^p dx = 0$ or $\int |g|^q dx = 0$, then $fg = 0$ a.e. and the inequality follows trivially. So suppose that $\int |f|^p dx > 0$ and $\int |g|^q dx > 0$. Let

$$a = \frac{|f|^p}{\int |f|^p dx} \qquad \text{and} \qquad b = \frac{|g|^q}{\int |g|^q dx}.$$

Then by Young's inequality ($a^{1/p} b^{1/q} \le a/p + b/q$) we get,

$$\frac{|fg|}{\left(\int |f|^p dx\right)^{1/p} \left(\int |g|^q dx\right)^{1/q}} \le \frac{1}{p}\frac{|f|^p}{\int |f|^p dx} + \frac{1}{q}\frac{|g|^q}{\int |g|^q dx}.$$

The right hand side is integrable and so $fg \in L^1$. Now we get the result by integrating on both sides. □

Theorem 1.41. *(Minkowski's Inequality) Let $1 \leq p < \infty$ and let $f, g \in L^p$. Then*

$$\left(\int |f+g|^p dx\right)^{1/p} \leq \left(\int |f|^p dx\right)^{1/p} + \left(\int |g|^p dx\right)^{1/p}.$$

Proof. The case $p = 1$ is trivial. So suppose that $p > 1$ and that p, q are as in Holder's inequality. Then

$$\begin{aligned}
\int |f+g|^p dx &\leq \int |f||f+g|^{p-1} dx + \int |g||f+g|^{p-1} dx \\
&\leq \left(\int |f|^p dx\right)^{1/p} \left(\int |f+g|^{(p-1)q} dx\right)^{1/q} + \left(\int |g|^p dx\right)^{1/p} \left(\int |f+g|^{(p-1)q} dx\right)^{1/q} \\
&= \left[\left(\int |f|^p dx\right)^{1/p} + \left(\int |g|^p dx\right)^{1/p}\right] \left[\left(\int |f+g|^p dx\right)^{1/q}\right],
\end{aligned}$$

since $(p-1)q = p$. Now the result follows. $\square$

Thus $||f||_p = \left(\int |f|^p dx\right)^{1/p}$ is a norm on L^p.

Definition 1.42. *Let f be a measurable function, then* $\inf\{\alpha : f \leq \alpha$ *a.e.*$\}$ *is called the essential supremum of f, denoted by ess sup f. Similarly,* $\sup\{\alpha : f \geq \alpha$ *a.e.*$\}$ *is called the essential infimum of f, denoted by ess inf f. If f is a measurable function such that ess sup $|f| < \infty$, then f is said to be essentially bounded.*

The set of all essentially bounded functions will be denoted by L^∞. Actually L^∞ is a NLS with the norm $||f||_\infty =$ess sup $|f|$. (check that)

Theorem 1.43. *For $1 \leq p \leq \infty$, L^p is a complete metric space.*