[24.3] <u>**Class workout**</u> Solve the ap for the cost matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Answer.

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0^* & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0^* & 0' & 0 \\ 0' & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

| not part of Hung algo

'-*-sequence $0'_{21},\ 0^*_{11},\ 0'_{12}$

$$\longrightarrow \begin{bmatrix} 0 & 0^* & 0 \\ 0^* & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0^* & 0' \\ 0^* & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0^* & 0' \\ 0^* & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 2 & 0^* & 0' \\ 0^* & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0^* & 0' \\ 0^* & 0' & 1 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0^* & 0' \\ 0^* & 0' & 1 \\ 0' & 0 & 2 \end{bmatrix}$$

'-*-sequence $0'_{31}\ \ \underline{0^*_{21}}\ \ 0'_{22}\ \ \underline{0^*_{12}}\ \ 0'_{13}$

$$\begin{bmatrix} 2 & 0 & 0^* \\ 0 & 0^* & 1 \\ 0^* & 0 & 2 \end{bmatrix} \longrightarrow \text{zero cost diagonal}$$

so ~~the~~ a min cost solution is

$\{a_{13}, a_{22}, a_{31}\}$

cost = 10.

Reduced cost matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$.

Computer assigns zero-stars and covers them by columns: $\begin{bmatrix} 0^* & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$.

A zero-prime found. There is a zero-star in its row. Column changed to row: $\begin{bmatrix} 0^* & 0' & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$.

Another zero-prime found. There is no zero-star in its row: $\begin{bmatrix} 0^* & 0' & 0 \\ 0' & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$.

So a prime-start sequence found: $c'_{21}, c^*_{11}, c'_{12}$. Primes and stars in this sequence are interchanged. After that all lines and primes are removed (not the stars). $\begin{bmatrix} 0 & 0^* & 0 \\ 0^* & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$.

Cover all zero-stars by columns: $\begin{bmatrix} 0 & 0^* & 0 \\ 0^* & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$.

A zero-prime found. There is a zero-star in its row. Column changed to row: $\begin{bmatrix} 0 & 0^* & 0' \\ 0^* & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$.

All zeros are covered with minimum number of lines. Let m be the minimum of the uncovered elements. Subtract m from each uncovered element and add m to each intersection points.

$$\begin{bmatrix} 2 & 0^* & 0 \\ 0^* & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

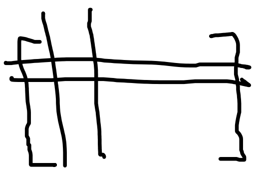
New zeros created. Find an uncovered zero and prime it. Continue.

$$\begin{bmatrix} 2 & 0^* & 0' \\ 0^* & 0' & 1 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0^* & 0' \\ 0^* & 0' & 1 \\ 0' & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0^* \\ 0 & 0^* & 1 \\ 0^* & 0 & 2 \end{bmatrix}.$$

So the optimum assignment is c_{13}, c_{22}, c_{31}.

[24.4] **Theorem** The Hungarian algorithm converges in finitely many steps for integer costs.

$C_{n\times n} \to$ reduced



$c + r < n$

$y =$ min uncovered elem.

$r\,c\,y$

$-(n-r)(n-c)y$

$= -\left[(n-r)(n-c)y - rcy\right] = -\left[n^2 y - n(r+c)y\right]$

$= -n(n-r-c)y \le -n$

$\Sigma_i c_{ij}$ $\qquad$ $\Sigma c'_{ij}$ $\qquad$ $c^1 \to$ next stage matrix

Proof. Assume that $n > 1$. Let C_1 be the matrix after Step b) and c_1, r_1 be the number of covered columns and rows, respectively. Suppose that $c_1 + r_1 < n$, so that we visit c). In c), we choose $y = \min\{$ uncovered elements $\} > 0$. The matrix C_2 is obtained by subtracting y from each uncovered element and adding y at intersections of the lines. So in this case

$$\sum_{i,j} C_1(i,j) - \sum_{i,j} C_2(i,j) = (n - c_1)(n - r_1)y - c_1 r_1 y = [n^2 - (c_1 + r_1)n]y \geq n(1)y \geq n.$$

So each time we visit c), the total cost decreases at least by n. Hence the algorithm will stop within at most $\frac{\sum c_{ij}}{n}$ iterations, where C is the reduced cost matrix with nonnegative integer entries. ∎

[24.5] **Exercise(M)** Suppose that the costs are nonnegative real numbers. I want to implement the Hungarian algorithm and get a minimum cost diagonal in finitely many steps. Can this be done?

[24.6] **Class workout** Four jobs are available and three candidates are available. The cost matrix $C = (c_{ij})$, where c_{ij} is the cost of making the ith person do the jth job. Minimize the cost while assigning exactly one job to each person.

	J1	J2	J3	J4
P1	12	9	12	9
P2	15	Unsuitable	13	20
P3	4	8	10	6

Answer.

$$C = \begin{bmatrix} 12 & 9^* & 12 & 9 \\ 15^* & \infty & 13 & 20 \\ 4 & 8 & 10 & 6^* \end{bmatrix} \qquad C' = \begin{bmatrix} 19 & 9^* & 12 & 9 \\ 15^* & \infty & 13 & 20 \\ 4 & 8 & 10 & 6^* \\ 0 & 0 & 0^* & 0 \end{bmatrix}$$

Let C' be the cost matrix obtained from C by adding a new zero row for a virtual person who wants to do any one job for free. Then a solution for C naturally corresponds to a solution of C' of the same cost. Then

the value of a minimum solution for C = the value of the corresponding solution of $C' \geq$ the value of a minimum cost solution of C' = the value of the corresponding solution of $C \geq$ the value of a minimum cost solution for C.

Hence, it is enough to find a minimum cost solution for C'.

$$\begin{bmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \end{bmatrix} \to \begin{bmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 3 & 0 \\ 2 & \infty & 0 & 7 \\ 0 & 4 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0^* & 3 & 0 \\ 2 & \infty & 0^* & 7 \\ 0^* & 4 & 6 & 2 \\ 0 & 0 & 0 & 0^* \end{bmatrix}.$$

Some exercises

[24.7] **NoPen**

a) If the cost matrix c of an assignment problem has some negative entries what do we do?

b) In case we have to maximize profit, what do we do?

c) Consider an assignment problem with reduced cost matrix $c = \begin{bmatrix} 0 & 3 & 1 & 2 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 2 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Suppose that we apply the Hungarian algorithm. Is it true that we have to revise the matrix at least 6 times before we get an optimum?

[24.8] **Practice** Take a bipartite graph with parts of size 8 each. Consider the adjacency matrix. Interchange the ones with zeros. Check how star-prime algorithm gives you a maximum matching in the graph.

[24.9] **Exercise(E)** `(Wrong method)` Think of the following method to determine the smallest number of lines to cover all zeros: select a line which has the most uncovered zeros in it, cover it and repeat the process. Give an example to show that the method is not correct.

[24.10] **Exercise(M)** `(Wrong method)` Think of the following method to get a maximum independent set when the minimum number of lines required to cover the zeros in the matrix $A_{n\times n}$ is n: pick a row/column with minimum number of zeros. Box any arbitrary zero on it. Cross all zeros in the column/row of the boxed zero. Repeat the procedure. Give an example to show that the method is not correct.

[24.11] **Exercise(E)** `(Wrong method)` Think of the following method to get a maximum independent set when the minimum number of lines required to cover the zeros in the matrix $A_{n\times n}$ is n: pick a zero with minimum number of zeros in its column and row. Box it and cross all zeros in its column and row. Repeat the procedure. Give an example to show that the method is not correct.

[24.12] **Exercise(E)** Next semester, the department of mathematics has to offer five theoretical courses to the first year MSc students. There are four teachers available for this and they have given their top three choices to the course allocator. The "teaching value" of a particular course is 7, if the first choice is met; 4, if the second choice is met; 2, if the third choice is met; 0, else. The choices of the teachers (T1 for teacher1 and C1 for course1) are given in the following table.

Teacher	first choice	second choice	third choice
T1	C2	C4	C5
T2	C2	C3	C4
T3	C3	C1	C2
T4	C5	C1	C4

Help the allocator to allocate these 5 course to these 4 teachers in such a way that the overall teaching value is maximized and each teacher gets at least one course to teach.

[24.13] **Exercise(M)** `(Different cases)` There are three men who can do any of the six jobs. The time (in days) taken by each man to do various jobs is given in the table. Solve the following problems minimizing the total time taken.

i) If each man has to do only one job, find which three jobs will be left undone in an optimal assignment.

ii) If all the jobs have to be completed, find an optimal assignment so that each person gets at least one job.

iii) Find an optimal assignment if men M_1, M_2, M_3 are asked to do at least 1 job, 2 jobs, 1 job respectively, and all the jobs have to be done.

iv) Find an optimal assignment if each man gets exactly two jobs.

	J_1	J_2	J_3	J_4	J_5	J_6
M_1	10	9	7	6	8	5
M_2	7	4	8	6	5	9
M_3	3	6	9	8	4	7

[24.14] **Practice** Apply star-prime method to determine the minimum number of lines required to cover all zeros in the following matrix starting with the given position $\begin{bmatrix} 0 & 3 & 2 & 3 \\ 0 & 2 & 2 & 0^* \\ 0 & 2 & 0^* & 2 \\ 0^* & 0 & 0 & 0 \end{bmatrix}$.

[24.15] **Exercise(M)** There are 4 jobs and two men are available for it. If each of them has to be assigned exactly 2 jobs, find an optimal assignment. The amount of payment needed to assign a particular job to a particular man is shown in

Men	Job J1	J2	J3	J4
M1	5	4	5	8
M2	6	9	4	5

[24.16] **Practice** Illustrate the effect of the star-prime algorithm on the bipartite graph for the following matrix, where zeros mean the edges. Use pencils.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Always read from top-left for searching anything.

1) First find an independent set of zero stars (a matching). In the graph draw the lines. Do not show all edges of the graph.

2) Apply the star-prime algorithm to increase the matching size by 1. Show the primes in the graph by dashed lines. Show the augmented path.

[24.17] **Practice** Solve the assignment problem with the cost matrix $\begin{bmatrix} 6 & 4 & 5 & 7 & 5 & 8 \\ 5 & 6 & 7 & 4 & 8 & 7 \\ 4 & 8 & 5 & 6 & 7 & 6 \\ 7 & 8 & 7 & 4 & 5 & 4 \\ 6 & 7 & 6 & 4 & 6 & 5 \\ 4 & 4 & 5 & 4 & 5 & 4 \end{bmatrix}$.

[24.18] **Exercise(H)** There are five candidates who can do any of the six jobs. The cost demanded by each to do various jobs is given in the table. Solve the problem minimizing the total cost giving each candidate at least one job while doing all the jobs.

	$J1$	$J2$	$J3$	$J4$	$J5$	$J6$
$C1$	6	4	5	7	5	8
$C2$	5	6	7	4	8	7
$C3$	4	8	5	6	7	6
$C4$	7	8	7	3	5	4
$C5$	6	7	6	5	5	5

[24.19] **Practice** Find the minimum cost assignment from the given matrix. The computer reads from top-left, goes to the right, then down.

$$A = \begin{bmatrix} 2 & 3 & 1 & 1 & 5 & 3 & 3 & 1 & 2 & 2 \\ 4 & 2 & 2 & 7 & 1 & 5 & 1 & 4 & 1 & 4 \\ 2 & 3 & 4 & 6 & 2 & 1 & 3 & 5 & 4 & 1 \\ 4 & 1 & 2 & 4 & 2 & 5 & 6 & 7 & 4 & 2 \\ 3 & 4 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 2 \\ 4 & 1 & 5 & 3 & 6 & 3 & 2 & 4 & 4 & 5 \\ 1 & 6 & 4 & 2 & 7 & 6 & 4 & 3 & 5 & 6 \\ 4 & 1 & 3 & 5 & 2 & 3 & 4 & 2 & 5 & 3 \\ 3 & 3 & 1 & 4 & 3 & 3 & 4 & 6 & 5 & 4 \\ 2 & 1 & 2 & 2 & 4 & 3 & 3 & 5 & 6 & 3 \end{bmatrix}$$

25 Lecture 25

Integer linear problems

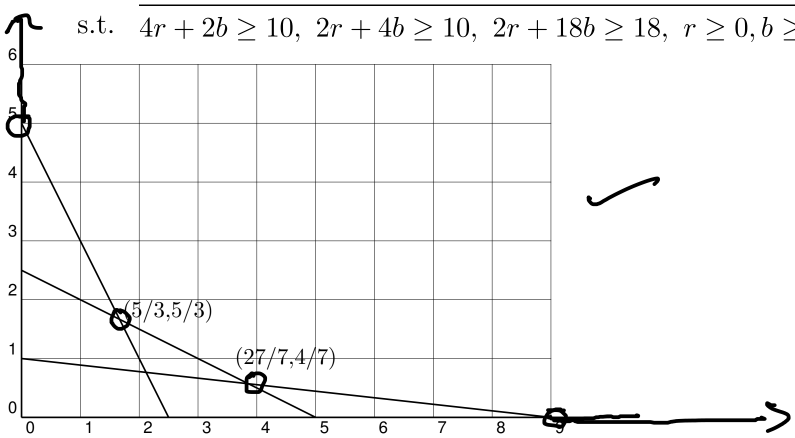
[25.1] **Discussion**

a) `(Remember a similar problem?)` Doctors advice me to take a minimum of 10mg of vitamin A, 10mg of vitamin B and 18mg of vitamin C, daily. There are two type of pills available in the market.

pill	A content	B content	C content	cost
red	4	2	2	6
blue	2	4	18	5

What amount (real number) of pills do I take to meet the requirements while minimizing the cost?

b) We can model the problem to $\begin{array}{ll} \min & 6r + 5b \\ \hline \text{s.t.} & 4r + 2b \geq 10,\ 2r + 4b \geq 10,\ 2r + 18b \geq 18,\ r \geq 0, b \geq 0. \end{array}$



As there are only four vertices, we can see that the minimum solution is at $(5/3, 5/3)$ and has value $55/3 = 18.33$. But what if, we wanted integer solutions?

c) Such problems are called INTEGER LINEAR PROGRAMMING PROBLEMS (ilp). The coefficients in the objective functions and constraints are integers and and additional $x \in \mathbb{Z}^n$ condition is there.

d) Such problems are harder. You can reduce the minimum vertex cover problem to an ilp. Given a graph $G = (V, E)$, consider the ilp

$$\left.\begin{array}{ll} \min & \sum_{v \in V} f(v) \\ \hline \text{s.t.} & f(u) + f(v) \geq 1 \quad \forall [u, v] \in E, \\ & f(u) \in \mathbb{Z}_+ \cancel{u \in V} \quad \forall u \in V \end{array}\right\}$$

A solution to this gives us a minimum vertex cover of G. The minimum vertex cover problem is known to be NP-complete, as proved by Karp (1972).

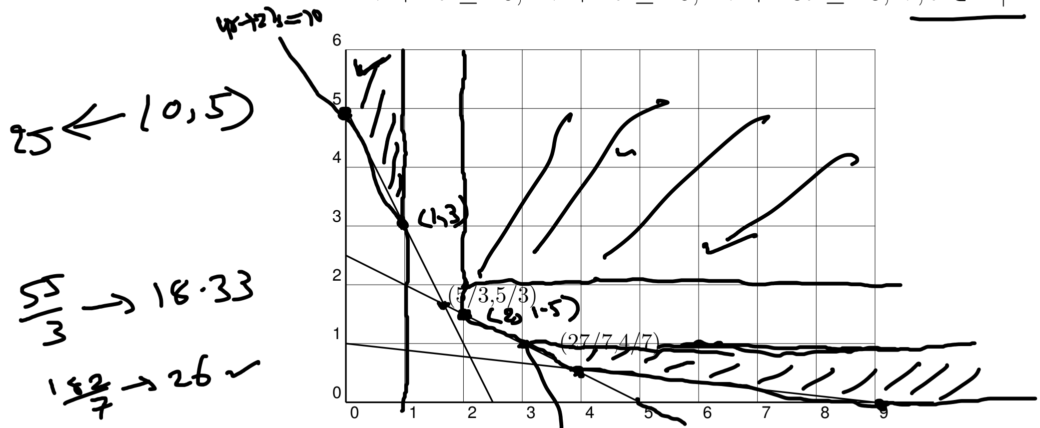
[25.2] **One method to solve an ilp is by complete enumeration** Consider the problem

$$\begin{array}{ll} \min & 6r + 5b \\ \hline \text{s.t.} & 4r + 2b \geq 10,\ 2r + 4b \geq 10,\ 2r + 18b \geq 18,\ r, b \in \mathbb{Z}_+, r \leq 9, b \leq 6. \end{array}$$

So you evaluate (make the computer evaluate) the function at every lattice point in the feasible set. Time taking. If the feasible set has only a few points then fine.

[25.3] **The branch and bound method**

a) Consider the problem $\begin{array}{ll} \min & 6r + 5b \\ \hline \text{s.t.} & 4r + 2b \geq 10,\ 2r + 4b \geq 10,\ 2r + 18b \geq 18,\ r, b \in \mathbb{Z}_+. \end{array}$



b) First consider the lpp and find a minimum. 2r+4b=10

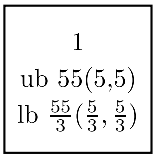
93 (3,1)

lpp1: $\begin{array}{ll} \min & \overset{6}{\cancel{2}}r + 5b \\ \hline \text{s.t.} & 4r + 2b \geq 10,\ 2r + 4b \geq 10,\ 2r + 18b \geq 18,\ r, b \geq 0. \end{array}$ 21 (1,3) | 19.5 (2,1.5)

Minimum is at $(5/3, 5/3)$ of value 18.33. ✓

c) So the solution to ilp, if exists must have value at least 19. This is a lower bound on the value. At this stage, if you know an integer feasible point and the value at that point, then that can serve as an upper bound. If we do not have that, we can take it to be ∞. Here let us take $(5, 5)$. So upper bound is 55.

We store this information as node 1.



(r, b)

d) The feasible set can be divided (this is branching) into two parts $r \leq 1$ and $r \geq 2$. Accordingly, we will have two lpp.

lpp2: min $\overset{6}{\cancel{2}}r + 5b$
s.t. $4r + 2b \geq 10,\ 2r + 4b \geq 10,\ 2r + 18b \geq 18,\ \underline{r \leq 1},\ r, b \geq 0.$

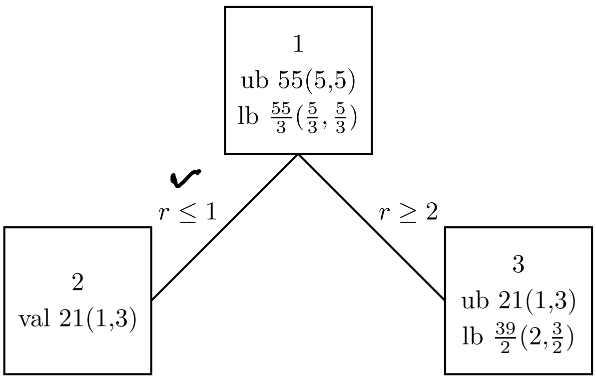
lpp3: min $\overset{6}{\cancel{2}}r + 5b$
s.t. $4r + 2b \geq 10,\ 2r + 4b \geq 10,\ 2r + 18b \geq 18,\ \underline{r \geq 2},\ r, b \geq 0.$

We solve both these lpp in our usual method. (Remember, adding a constraint?) We note the minimum solutions, using simplex method or otherwise (if you can use graphical method use it).

e) For lpp2, we have only two vertices $(0, 5)$ and $(3, 1)$. Evaluating the function we see that the minimum will be obtained at $(1, 3)$ at which the value is 21. We are done for this part, as we have got an integer solution. We note that for the other nodes, we can set our upper bound to 21.

f) For lpp3, we have only three vertices $(2, 3/2)$, $(27/7, 4/7)$ and $(9, 0)$. Evaluating the function we see that the minimum will be obtained at $(2, 1.5)$ at which the value is 19.5. Any further sub-branching will not gives a solution cheaper that 19.5. So we can set our upper bound to 21 and lower bound to 19.5.

g) We store these information as nodes 2 and 3.



h) From node 3, we can have the following branching. As $b = 1.5$, we can have $b \leq 1$ or $b \geq 2$. Accordingly, we have the following lpp.

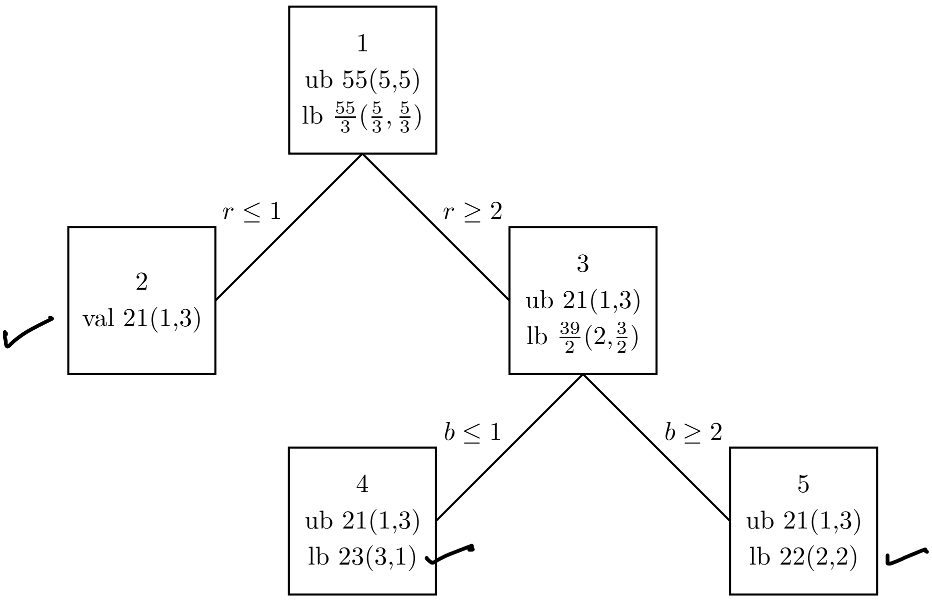
lpp4: min $2r + 5b$
s.t. $4r + 2b \geq 10,\ 2r + 4b \geq 10,\ 2r + 18b \geq 18,\ r \geq 2,\ b \leq 1,\ r, b \geq 0.$

Vertices $(3, 1)$, $(27/7, 4/7)$ and $(9, 0)$. The minimum value 23 is attained at $(3, 1)$.

lpp5: min $\underline{2r+5b}$
s.t. $4r+2b \geq 10,\ 2r+4b \geq 10,\ 2r+18b \geq 18,\ r \geq 2,\ b \geq 2,\ r,b \geq 0.$

The only vertex is $(2,2)$. The value is 22.

i) We store these information as nodes 4 and 5.



j) It is now clear, that we do not have any cheaper solution than $(1,3)$. So that is our minimum solution.

k) Sometimes, when the feasible set is empty, you can write ‘infeasible’ in the box.

l) The lpp’s that we considered are sometimes called ‘lp relaxations’.

[25.4] <u>**Exercise**</u> Solve max $\underline{10x+15y}$
s.t. $8x+4y \leq 40,\ 3x+6y \leq 40,\ x,y \in \mathbb{Z}_+.$