10 Lecture 10

What to do if $\overline{c}_C$ has a negative entry?

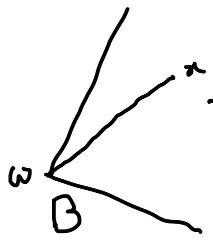
1. Suppose that we are minimizing an slpp we are at a bfs (vertex) w. If $\overline{c} \geq 0$, the we know that w is a point of minimum.
2. Otherwise, in view of [9.6], we are looking for a point x such that $\overline{c}^t x$ is minimum as the new objective function $\overline{c}^t x$ differs from the old objective function $c^t x$ by $c^t w$ which a constant independent of x.
3. So our problem

$$\begin{array}{ll} \min & c^t x \\ \hline \text{s.t.} & Ax = b, x \geq 0 \end{array} \equiv \begin{array}{ll} \min & \overline{c}^t x + const \\ \hline \text{s.t.} & Ax = b,\ x \geq 0 \end{array} \equiv \begin{array}{ll} \min & \overline{c}^t x \\ \hline \text{s.t.} & B^{-1}Ax = B^{-1}b,\ x \geq 0 \end{array} \equiv \begin{array}{ll} \min & \overline{c}^t x \\ \hline \text{s.t.} & \overline{A}x = \overline{b},\ x \geq 0 \end{array},$$

as the feasible sets over which the minimization is being done for all these problems are the same and for the last expression we have used the notations $\overline{\mathbf{A}} := B^{-1}A$, $\overline{\mathbf{b}} := B^{-1}b = w_B \geq 0$.

$$\begin{array}{l} \min\ c^T x \\ \hline Ax = b, x \geq 0 \end{array} \equiv \begin{array}{l} \min\ \overline{c}^T x \\ \hline Ax = b, x \geq 0 \end{array}$$

T →



$\overline{c}^T x = c^T x - \overline{c}^T w$ const

$$\equiv \begin{array}{l} \min\ \overline{c}^T x \\ \hline B^{-1}Ax = B^{-1}b, x \geq 0 \end{array} \longrightarrow \text{revised new problem}$$

$\overline{c}_B^T x_B + \overline{c}_C^T x_C$

$A = [B \quad C] \longrightarrow B^{-1}A = [I \quad B^{-1}C]$

4. For simplicity, let $(x_1, \ldots, x_m)$ be the basis for w. So $A = \begin{bmatrix} B & C \end{bmatrix}$. The slpp $\begin{array}{ll} \min & \overline{c}^t x \\ \hline \text{s.t.} & \overline{A}x = \overline{b}, x \geq 0 \end{array}$ is

$$\begin{array}{ll} \min & \begin{bmatrix} \overline{c}_B^t & \overline{c}_C^t \end{bmatrix} \begin{bmatrix} x_B \\ x_C \end{bmatrix} \\ \hline \text{s.t.} & \begin{bmatrix} I & B^{-1}C \end{bmatrix} \begin{bmatrix} x_B \\ x_C \end{bmatrix} = \overline{b}, x \geq 0 \end{array} = \begin{array}{ll} \min & \sum_{i=m+1}^{n} \overline{c}_i x_i \\ \hline & x_1 + \overline{a}_{1,m+1}x_{m+1} + \cdots + \overline{a}_{1,n}x_n = \overline{b}_1 \\ \text{s.t.} & \vdots \\ & x_m + \overline{a}_{m,m+1}x_{m+1} + \cdots + \overline{a}_{m,n}x_n = \overline{b}_m,\ x_i \geq 0. \end{array} \tag{5}$$

5. This slpp (5) can be represented by a table from which it is easy to gather the information. It is shown below. We call it the SIMPLEX TABLE at the basis $(x_1, \ldots, x_m)$. It also shows $\overline{c}^t$ at the bottom and the value $-f(w) = -c^t w$ at the bottom right corner.

bv	$x_1, \ldots, x_m, x_{m+1}, \ldots, x_n$	$\overline{b}$
x_1 ⋮ x_m	$B^{-1}A$	$B^{-1}b$ ⋮
$-f$	$\overline{c}^T$	$-f(w)$

bv	x_1	x_2	$\cdots$	x_m	x_{m+1}	$\cdots$	x_n	$\overline{b}$
x_1	1	0	$\cdots$	0	$\overline{a}_{1,m+1}$	$\cdots$	$\overline{a}_{1,n}$	$\overline{b}_1$
$\vdots$	$\vdots$		$B^{-1}A$					$\vdots$ $B^{-1}b$
x_m	0	0	$\cdots$	1	$\overline{a}_{m,m+1}$	$\cdots$	$\overline{a}_{m,n}$	$\overline{b}_m$
$-f$	0	0	$\cdots$	0	$\overline{c}_{m+1}$	$\cdots$	$\overline{c}_n$	$-f(w)$
			$c^t - c_B^t B^{-1}A$					

(6)

$$\begin{bmatrix}5&2&6\\7&8&6\\1&4&0\end{bmatrix}^{-1}\begin{bmatrix}5&2&6\\7&8&6\\1&4&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

bv	x_1	x_2	x_3	x_4	b
*	7	1	1	0	20
	3	-1	0	-1	12
PT → *	2	-3	0	0	*

[10.1] **Discuss in class** Suppose that we have applied some elementary row operations on A to get B.

$$\begin{bmatrix}5&2&6\\7&8&6\\1&4&0\end{bmatrix}^{-1} \qquad A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} * & 0 & * & * & 1 & 0 \\ * & 1 & * & * & 0 & 0 \\ * & 0 & * & * & 0 & 1 \end{bmatrix}$$

Recall that applying elementary row operations is as good as left-multiplying the matrix with an invertible matrix. Can you guess which matrix we have multiplied here from left to A in order to get B?

Writing a problem table from slpp. We can write a table to represent a given slpp. For example,

$$\begin{array}{ll} \text{opt} & 2x_1 - 3x_2 \\ \hline & 7x_1 + x_2 \leq 20 \\ \text{s.t.} & 3x_1 - x_2 \geq 12 \\ & x_i \geq 0 \end{array}$$

$\rightarrow$

bv	x_1	x_2	x_3	x_4	b
:	7	1	1	0	20
:	3	-1	0	-1	12
$*$	2	-3	0	0	$*$

Two ways of writing a simplex table at a given ordered basis. A simplex table can be formed with a given ordered basis $(x_{i_1}, \ldots, x_{i_m})$ in the following ways.

1. First way.
   a) Write the problem table.
   b) Write 0 below the 'constant' column and $-f$ below the 'bv' column.
   c) Write the basis variables under the bv column in order.
   d) Use elementary row operations to convert the column below x_{i_k} to e_k. This amounts to changing A into $\overline{A}$. $= B^{-1}A$
   e) Add suitable multiples of rows of $\overline{A}$ to c^t to make its entries below the basic variables 0.[a]
2. Second way.
   a) Make the design, that is, write the top labels and draw the lines.
   b) Write the basis elements in order in the bv column.
   c) Obtain the basis matrix B.
   d) Write $B^{-1}A$ in the matrix place and $B^{-1}b$ in the last column.
   e) Write $\overline{c}^t$ in the last row and $-f(w)$ at the bottom right corner.

[a]This amounts to taking a vector d such that $c^t - d^tB^{-1}A$ has B-part 0. That is, $c^t_B - d^tI$ or $c^t_B = d^t$. So, by doing this elementary operation, we are actually getting $\overline{c}^t$.

[10.2] **Example** Write the simplex table for $\begin{array}{ll} \text{opt} & \underline{x_1 + x_2} \\ \text{s.t.} & x_1 \le 2, x_2 \le 1, x \ge 0 \end{array}$ at the ordered basis (x_2, x_1).

Answer.

1st way

PT →

bv	x_1	x_2	x_3	x_4	b
*	1	0	1	0	2
–	0	1	0	1	1
*	$\boxed{1\ \ 1}$		0	0	*

A

$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = B^{-1}$

$w = (1,1)$

ST

1st bv → x_2

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	1	0	1	0	2
x_1	0	1	0	1	1
$-f$	1	1	0	0	0

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	0	1	0	1	1
x_1	1	0	1	0	2
$-f$	0	0	-1	-1	-3

$\bar{A}$, $\to \bar{b}$

2nd way

$w(2,1,0,0)$

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	0	1	0	1	1
x_1	1	0	1	0	2
$-f$	0	0	-1	-1	-3

$B^{-1}A$

$\bar{c}^T = 1\ 1\ 0\ 0 - [1\ 1]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$= 1\ 1\ 0\ 0 - [1\ 1]\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$0\ 0\ {-1}\ {-1}$

1. Problem table:

bv	x_1	x_2	x_3	x_4	b
	1	0	1	0	2
	0	1	0	1	1
$*$	1	1	0	0	$*$

2. Write the basis elements in order, write a $-f$ below them and a 0 below the constants:

bv	x_1	x_2	x_3	x_4	b
x_2	1	0	1	0	2
x_1	0	1	0	1	1
$-f$	1	1	0	0	0

3. Make the column below the first basis element (here it is x_2) e_1, by using elementary row operations; even the entry of the cost vector is converted to 0. Here we use a row interchange and then subtract the first row from the third row. Entries of the constant column are also changed accordingly.

bv	x_1	x_2	x_3	x_4	$\bar{b}$
x_2	0	1	0	1	1
x_1	1	0	1	0	2
$-f$	1	0	0	-1	-3

4. Make the column below the second basis element look like e_2. Here it is already there, except that we need to make the entry of the cost vector 0. We get

bv	x_1	x_2	x_3	x_4	$\overline{b}$
x_2	0	1	0	1	1
x_1	1	0	1	0	1
$-f$	0	0	-1	-1	-2

5. This is the simplex table. Notice that the basis (x_2, x_1) corresponds to the bfs $(1, 1, 0, 0)$ with a cost 2.
6. Try the second way too.

Some exercises

[10.3] Exercise(E) Consider the slpp for our favorite lpp. Compute the relative cost for the bfs corresponding to $(0, 1)$. Form the simplex table at this bfs.

[10.4] Exercise(E) Consider the unit cube T in $\mathbb{R}^3$ with sides e_1, e_2, e_3.

a) View it as the intersection of following 7 halfspaces: 6 faces and the halfspace $x_1 + x_2 + x_3 \le 3$. Write it as $\{x \mid A_{4\times 3}x \le b,\ x \ge 0\}$.

b) Add slack variables to write the corresponding region T' in $\mathbb{R}^7$ in the form $A'x = b'$, $b' \ge 0, x \ge 0$. (Use x_7 in $x_1 + x_2 + x_3 + x_7 = 3$.)

c) Which vertex v' of T' corresponds to the vertex $v = (1, 1, 1) \in T$?

d) Supply all the bases corresponding to v', written in increasing subscripts.

e) Consider $f(x) = x_1 + x_2 + x_3$. Write the simplex table for this problem at this bfs with the basis (x_1, x_2, x_3, x_7).

Information obtained from the simplex table

The simplex table not only shows the revised problem in simple way but also gives us many geometric information which helps us to get a better bfs, in case we are not an optimal bfs. We discuss this with an example. A general discussion is similar.

[10.5] Example

a) Consider the unit cube T in $\mathbb{R}^3$ with sides e_1, e_2 and e_3. Imagine minimizing $f(x) := x_1 + x_2 + x_3$. We know that it will be at $(0, 0, 0)$.

b) Write the slpp:

$x_1 \le 1, x_2 \le 1, x_3 \le 1$
$x \ge 0$

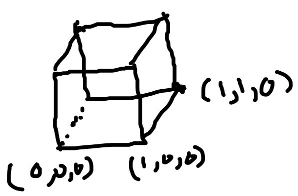
$$\begin{array}{ll} \min & x_1 + x_2 + x_3 + 0x_4 + 0x_5 + 0x_6 \\ \hline \text{s.t.} & x_1 + x_4 = 1, x_2 + x_5 = 1, x_3 + x_6 = 1, x_i \ge 0. \end{array} \tag{7}$$

Let T^* be the feasible set of the slpp. This is in $\mathbb{R}^6$.

c) Write the problem table:

	bv	x_1	x_2	x_3	x_4	x_5	x_6	$\overline{b}$
		1	0	0	1	0	0	1
$A \rightarrow$		0	1	0	0	1	0	1
		0	0	1	0	0	1	1
	$*$	1	1	1	0	0	0	$*$ 0

bv	x_1	x_2	x_3	x_4	x_5	x_6	$\overline{b}$
$\rightarrow x_2$	0	1	0	0	1	0	1
x_1	1	0	0	1	0	0	1
x_3	0	0	1	0	0	1	1
	0	0	1	-1	-1	0	-2



✓ $\quad x_1 \; x_2 \qquad\qquad x_6$

$\omega^* = (1, 1, 0, 0, 0, 1)$ bfs

d) Consider the vertex $w = (1, 1, 0) \in T$.

1. What is the corresponding point w^* of T^*?

   Answer. Using (7), we see that $w^* = (1, 1, 0, 0, 0, 1)$.

2. Is w^* a bfs of $Ax = b$?

   Answer. Yes, as it is nonnegative and the columns corresponding to the nonzero entries are linearly independent. (Otherwise, you can also say, in view of [9.4] that since w was a vertex of T, the point w^* is a vertex of T^*.)

3. Which ordered basis corresponds to w^*?

   Answer. We can take $B = (x_1, x_2, x_6)$ or $B = (x_2, x_1, x_6)$ or any of 3! ways of writing x_1, x_2, x_6.

e) Form the simplex table at the ordered basis $B = (x_2, x_1, x_6)$. ✓

bv	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	0	1	0	0	1	0	1
x_1	1	0	0	1	0	0	1
x_6	0	0	1	0	0	1	1
$-f$	0	0	1	-1	-1	0	-2

(8)

From this table, we see that $-f(w^*) = -2$. Indeed, we had $f(w) = 2$.

f) Select a nonbasic variable, say x_4. We are at $w^* = (1, 1, 0, 0, 0, 1)$. The value of x_4 is now 0.

1. Suppose that we want to increase the value of x_4 by 1, keeping the other <u>nonbasic</u> variables unchanged.
2. That is, we are looking at a point $x := w^* + d$, where $d_4 = 1, d_3 = 0, d_5 = 0$.

$$\overset{\omega^*}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} + \alpha \begin{bmatrix} d_1 \\ d_2 \\ 0 \\ 1 \\ 0 \\ d_6 \end{bmatrix}$$

$\bar{A}(\omega^* + \alpha d) = \bar{b} \quad , \because \; \bar{A}\omega^* = \bar{b}$

$\bar{A} d = 0$

$\bar{A}_{:,1} d_1 + \bar{A}_{:,2} d_2 + \bar{A}_{:,3} d_3 + \bar{A}_{:,4} d_4 + \bar{A}_{:,5} d_5 + \bar{A}_{:,6} d_6 = 0$

$$d = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} d_2 \\ d_1 \\ d_6 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3. We want the new point x to satisfy $\overline{A}x = \overline{b}$ and $x \geq 0$, so that we stay in the feasible region.

4. Suppose that $\overline{A}x = \overline{b}$. As $\overline{A}w^*$ is already $\overline{b}$, we must have $\overline{A}d = 0$. Thus we must have

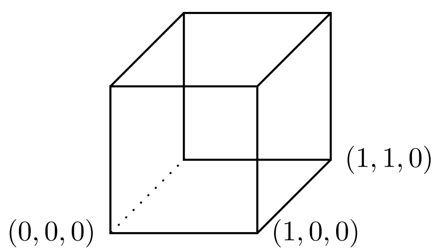
$$d_1\overline{A}_{:1} + d_2\overline{A}_{:2} + d_6\overline{A}_{:6} + d_4\overline{A}_{:4} = 0 \quad \text{(as other } d_i\text{s are 0)} \quad \text{that is, } \begin{bmatrix} d_2 \\ d_1 \\ d_6 \end{bmatrix} = -\overline{A}_{:4} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

5. So a point in the direction of d is

$$w^*_\alpha = w^* + \alpha d = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ 1 \\ 0 \\ \alpha \\ 0 \\ 1 \end{bmatrix}. \tag{9}$$

6. As long as $w^*_\alpha \geq 0$, it is inside T^*. In this example, what is the maximum value α_{max} of α so that $w^*_\alpha \in T^*$? We have $\alpha_{max} = 1$.

7. The corresponding physical point $w_\alpha \in T$ is $\begin{bmatrix} 1-\alpha \\ 1 \\ 0 \end{bmatrix}$. That is, we are heading from $w = (1,1,0)$ towards $u = (0,1,0)$ on the edge (imagine increasing α slowly).



g) The cost difference is $c^t w^*_\alpha - c^t w^* = \overline{c}^t w^*_\alpha$ (see [9.6]).

But notice that except fourth entry of w^*_α which is α, all other nonzero entries are basic places for w^* (hence in $\overline{c}$, those places will be 0). So

$$c^t w^*_\alpha - c^t w^* = \overline{c}^t w^*_\alpha = \sum \overline{c}_i (w^*_\alpha)_i = \overline{c}_4 \alpha. \tag{10}$$

h) This means, for moving a step in the direction provided by the the nonbasic variable x_4, the cost will change by $\overline{c}_4$.

I)(Draw and explain below) Points in the direction given by x_5 are $w^*_\alpha = \begin{bmatrix}1 & 1-\alpha & 0 & 0 & \alpha & 1\end{bmatrix}^t$. Here $\alpha_{max} = 1$.

[10.6] **<u>Theorem</u>** Let $A \in M_{m,n}$ have rank m. Consider opt $\dfrac{f(x) = c^t x}{\text{s.t. } Ax = b,\ x \geq 0.}$ Consider the simplex table at a bfs w^* of $Ax = b$ with the basis matrix B. Let x_r be a nonbasic variable.

a) Then the direction given by x_r is the vector

$$d \text{ with } d_B = -\overline{A}_{:r},\ d_r = 1, \text{ and other entries zero.}$$

b) Also $f(w^* + \alpha d) = f(w^*) + \alpha \overline{c}_r$.

[10.7] **<u>Lemma</u>** `(Moving out of a vertex : unboundedness of objective function)` Let $\mathsf{rank}\, A_{m\times n} = m$. Consider the simplex table at a bfs w^* of $Ax = b$. Suppose that $\overline{c}_r < 0$. (It means, x_r is a nonbasic variable, otherwise $\overline{c}_r$ should have been 0. Also it means if we move in the direction d given by x_r, then $f(x)$ decreases.) Suppose that $\overline{A}_{:r} \leq 0$. (It means $d \geq 0$ and so, if we move in the direction d, then we always stay inside T in view of (9).) Then $c^t x$ is not bounded below on $\{x \mid Ax = b, x \geq 0\}$.‼

[10.8] **Lemma** `(Moving out of a vertex : better bfs)` Let $\mathsf{rank}\, A_{m\times n} = m$. Consider a simplex table at a bfs w^* of $Ax = b$ and let d be the direction given by a nonbasic variable x_r. Suppose that $\overline{c}_r < 0$ and $\overline{A}_{:r} \not\leq 0$. (It means, we can go at most a finite amount in the direction d while staying inside the feasible region.) Then $w^*_\delta = w^* + \delta d$ is a better bfs, where $\delta = \min\limits_{\overline{a}_{i,r}>0} \dfrac{\overline{b}_i}{\overline{a}_{i,r}}$. If $\delta > 0$, then $f(w^*_\delta) < f(w^*)$.

Proof. From [10.6] we know that $f(w^*_\delta) \leq f(w^*)$. We only need to show that w^*_δ is also a bfs. For simplicity, assume that $(x_1, \ldots, x_m)$ is the basis and $\delta = \min\limits_{\overline{a}_{i,r}>0} \frac{\overline{b}_i}{\overline{a}_{i,r}}$ is attained at $i = 1$. Consider the columns of $\overline{A}$ corresponding to $x_r, x_2, \cdots, x_m$. Since $\overline{a}_{1,r} > 0$, these columns are linearly independent. So w^*_δ is a bfs. ∎

Some exercises

[10.9] **Exercise(E)** Suppose that a minimum for $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, x \geq 0 \end{array}$ exists. Suppose also that we are at a nondegenerate bfs w and that $\overline{c}_r < 0$. Must we have $\delta > 0$? Can the new bfs be degenerate?

[10.10] **Exercise(M)** `(Converse of test of optimality.)` Let $\mathsf{rank}\, A_{m\times n} = m$. Consider the simplex table for the slpp $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, x \geq 0 \end{array}$ at a bfs w. The optimality test says that 'if $\overline{c} \geq 0$, then w is a point of minimum'.

a) Give an example to show that the converse is not true in general.

b) Argue that if w is a minimum nondegenerate bfs, then $\overline{c}$ must be nonnegative.

c) Conclude that, if $\overline{c}$ has a negative entry for a minimal bfs w, then w must be degenerate.

[10.11] **Exercise(E)** `(Forming the simplex table.)` Consider $\begin{array}{ll} \min & c^t x \\ \text{s.t.} & A_{m\times n} x = b,\ x \geq 0 \end{array}$, $\mathsf{rank}\, A = m$. Let w be a bfs of $Ax = b$ for some basis. Students X and Y are forming the simplex table. Student X first converts A to $\overline{A}$ so that he has identity matrix under the basic variables. Then he takes c^t; finds entries of c^t corresponding to the basic variables; makes them zero by subtracting appropriate multiples of rows of $\overline{A}$. He claims that he has got the simplex table at the given basis. Student Y first takes c^t; finds entries of c^t corresponding to the basic variables; makes them zero by subtracting appropriate multiples of rows of A. Then he converts A to $\overline{A}$. He claims that he has got the simplex table at the given basis. Who is correct?

[10.12] **Application(M)** `(When does` $\{x \mid A_{m,n}x \leq 0\}$ `have a nontrivial point?)` We wish to find a nontrivial point in the set $P = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$. Convert this to an lpp.

[10.13] **Application(H)** `(Using lpp to separate two finite sets.)` Consider two sets $V = \{v_1, \ldots, v_k\}$ and $W = \{w_1, \ldots, w_m\}$ in $\mathbb{R}^n$. We wish to find a hyperplane $H : a^t x = b$ which separates them. Convert this to a lpp.