

MA 372 : Stochastic Calculus for Finance

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Exercises 6

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1. Suppose that the price of a stock $\{S(t); t \geq 0\}$ follows geometric Brownian motion with drift 0.06 and volatility 0.2 so that it satisfies the stochastic differential equation

$$dS(t) = 0.06 S(t) dW(t) + 0.2 S(t) dt.$$

If the price of the stock at time zero is 40, determine the probability that the price of the stock at time $t = 5$ is less than 50.

2. Let $r(t)$ and $\sigma(t)$ be non-random functions. Suppose $S(t)$ satisfies the following:

$$S(t) = S(0) \exp \left\{ \int_0^t \sigma(s) d\tilde{W}(s) - \frac{1}{2} \int_0^t (r(s) - \frac{1}{2} \sigma^2(s)) ds \right\}$$

where $\tilde{W}(t)$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. The price of an European call at time t , given by the risk-neutral valuation formula is

$$c(0, S(0)) = \tilde{\mathbb{E}} \left[\exp \left\{ - \int_0^T r(s) ds \right\} (S(T) - K)^+ \right]$$

Let

$$\begin{aligned} BSM(T, x, K, R, b) &= xN \left(\frac{1}{b\sqrt{T}} \left[\log \left(\frac{x}{K} \right) + \left(R + \frac{b^2}{2} \right) T \right] \right) \\ &\quad - \exp \{ -RT \} K N \left(\frac{1}{b\sqrt{T}} \left[\log \left(\frac{x}{K} \right) + \left(R - \frac{b^2}{2} \right) T \right] \right) \end{aligned}$$

Show that

$$c(0, S(0)) = BSM(T, S(0), K, \frac{1}{T} \int_0^T r(t) dt, \sqrt{\frac{1}{T} \int_0^T \sigma^2(t) dt})$$

3. The BSM price of an European call at time zero, if the stock price is x is given by

$$\begin{aligned} c(0, x) &= \tilde{\mathbb{E}} \left[\exp \left\{ -rT \right\} (S(T) - K)^+ \right] \\ &= \tilde{\mathbb{E}} \left[\exp \left\{ -rT \right\} \left(x \exp \left\{ \sigma \tilde{W}(T) + \left(r - \frac{1}{2} \sigma^2 \right) T \right\} - K \right)^+ \right] \end{aligned}$$

Now if $h(s) = (s - K)^+$, then

$$\begin{aligned} \frac{dh}{ds} &= 0, \text{ if } s < K \\ &= 1, \text{ if } s > K \end{aligned}$$

Using this and the fact that $\mathbb{P}(S(T) = K) = 0$. Show that $c_x(0, x) = \hat{\mathbb{P}}(S(T) > K)$ where $\hat{\mathbb{P}}$ is a probability measure equivalent to \mathbb{P} . Show that $\hat{W}(t) = \tilde{W}(t) - \sigma t$ is a Brownian motion under $\hat{\mathbb{P}}$. Rewrite $S(T)$ in terms of $\hat{W}(T)$. Finally conclude that $\hat{\mathbb{P}}(S(T) > K) = N(d_+(T, x))$ where $N(x)$ is the normal distribution function and

$$d_+(T, x) = \frac{1}{\sigma\sqrt{T}} \left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T \right].$$

4. Let $W(t)$, $0 \leq t \leq T$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \leq t \leq T$ be a filtration for this Brownian motion. Consider a stock price process whose differential is

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad \mu, \sigma \in \mathbb{R}, \sigma > 0.$$

- (i) Write down the probability measure \mathbb{Q} under which the discounted stock price $Y(t) = e^{-rt}S(t)$ is a martingale with respect to $\mathcal{F}(t)$.
 - (ii) Determine $d(e^{-rt}S(t))$ under the risk-neutral probability measure \mathbb{Q} .
5. Suppose the market has an arbitrage. So there is a portfolio value process satisfying $X_1(0) = 0$ and $\mathbb{P}(X_1(T) \geq 0) = 1$, $\mathbb{P}(X_1(T) > 0) > 0$, for some positive time T .
- a) Show that if $X_2(0)$ is positive, then there exists a portfolio value process $X_2(t)$ satisfying at $X_2(0)$ and satisfying

$$\mathbb{P}\left(X_2(T) \geq \frac{X_2(0)}{D(T)}\right) = 1 \text{ and } \mathbb{P}\left(X_2(T) > \frac{X_2(0)}{D(T)}\right) > 0$$

- b) Suppose that the market has a portfolio process $X_2(t)$ such that $X_2(0)$ is positive and the above holds. Then show that the model has a portfolio value process $X_1(t)$ which is an arbitrage.
6. Consider a financial market consisting of a risk-free asset $B(t)$ and a stock $S(t)$, whose price at time t , $t > 0$ satisfy the following differentials:

$$\begin{aligned} dB(t) &= 2 B(t)dt \\ dS(t) &= S(t) \left(7 dt + 2 dW(t) \right) \end{aligned}$$

- (i) Is the above market arbitrage free?
- (ii) Is the above market complete?
- (iii) Find the risk-neutral probability measure \mathbb{Q} for the above market.
- (iv) Find $dS(t)$ in terms of $\tilde{W}(t)$, where $\tilde{W}(t)$ is a Brownian motion under \mathbb{Q} .
- (v) Find the price of a European call whose payoff at time T is represented by $(S(T) - K)^+$ by using risk-neutral pricing formula.