1 Stochastic Integration

1.1 Motivation

We are interested in defining the integral of the form

$$\int_0^T f(t)dW_t\,,$$

where f is a stochastic process satisfying suitable assumptions. Defining it as

$$\int_0^T f(t)W'(t)dt$$

is not possible since Brownian motion paths are nowhere differentiable almost surely. So let us try in the Riemann-Stieltjes sense for the process $f(t) = W(t)$, i.e., we are trying to define the integral

$$\int_0^T W(t)dW_t\,.$$

Take a partition $\Pi = \{t_0, t_1, \ldots, t_n\}$ of $[0, T]$. Then

$$L_n = \sum_{i=1}^{n} W(t_{i-1})(W(t_i) - W(t_{i-1}))\,,$$

and

$$R_n = \sum_{i=1}^{n} W(t_i)(W(t_i) - W(t_{i-1}))\,.$$

If the integral has to exist in the Riemann-Stieltjes sense then the difference $R_n - L_n$ should go to 0 as $||\Pi||$ goes to 0. But

$$\lim_{||\Pi||\to 0}(R_n - L_n) = \lim_{||\Pi||\to 0}\sum_{i=1}^{n}(W(t_i) - W(t_{i-1}))^2 = T\,.$$

Ideally we would like the stochastic integral to be a martingale. Now

$$\begin{aligned} R_n + L_n &= \sum_{i=1}^{n}(W(t_i) + W(t_{i-1}))(W(t_i) - W(t_{i-1})) \\ &= \sum_{i=1}^{n}(W^2(t_i) - W^2(t_{i-1})) \\ &= W^2(T) - W^2(0)\,. \end{aligned}$$

Thus

$$R_n = \frac{1}{2}(R_n + L_n + R_n - L_n), \quad L_n = \frac{1}{2}(R_n + L_n - R_n + L_n)\,.$$

So,

$$\lim_{||\Pi||\to 0} R_n = \frac{1}{2}(W^2(T) + T)\,,\ \lim_{||\Pi||\to 0} L_n = \frac{1}{2}(W^2(T) - T)\,.$$

In the second case it is indeed a martingale.

1.2 Definition and Properties

Definition 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $W(\cdot)$ be a Brownian motion defined on it. Let $\mathcal{F}_t$ be a Brownian filtration. For $T > 0$, define the space $L^2_{ad}([0,T] \times \Omega)$ to denote the space of all stochastic processes $f(t,\omega), 0 \le t \le T, \omega \in \Omega$ satisfying the following two properties:*

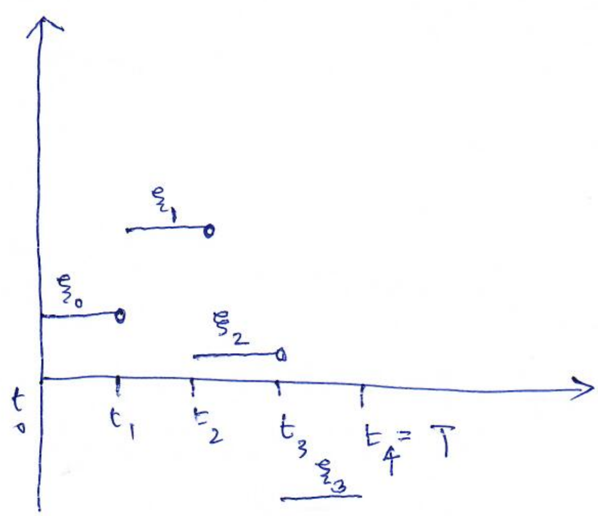
- *$f(t,\omega)$ is adapted to the filtration $\{\mathcal{F}_t\}$.*
- *$\int_0^T \mathbb{E}|f(t)|^2 dt < \infty$.*

We will construct the integral in several steps like we did for expectation.

Definition 1.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $W(\cdot)$ be a Brownian motion defined on it. Let $\mathcal{F}_t$ be a Brownian filtration. A stochastic process $f(t), t \ge 0$ is said to be a simple stochastic process if there is a finite sequence of numbers $0 = t_0, t_1, \dots, t_n = T$ and square integrable random variables $\xi_0, \xi_1, \dots, \xi_{n-1}$ $(\mathbb{E}[\xi_i^2] < \infty)$ such that*

$$f(t,\omega) = \sum_{i=1}^{n} \xi_{i-1} 1_{[t_{i-1}, t_i)}(t) ,$$

where $\{t_0, t_1, \dots, t_n\}$ is a partition of $[0,T]$ and ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$ measurable and $\mathbb{E}(\xi_{i-1}^2) < \infty$.



For $T > 0$, define the space $L^2_{step}([0,T] \times \Omega)$ to denote the space of all simple stochastic processes $f(t,\omega), 0 \le t \le T, \omega \in \Omega$. Note that

- *ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$ measurable $\Rightarrow f(t,\omega)$ is adapted to the filtration $\{\mathcal{F}_t\}$.*
- *$\mathbb{E}(\xi_{i-1}^2) < \infty \Rightarrow \int_0^T \mathbb{E}|f(t)|^2 dt < \infty$.*

Step 1: f is a step stochastic process in $L^2_{\text{step}}([0,T] \times \Omega)$, i.e.,

$$f(t,\omega) = \sum_{i=1}^{n} \xi_{i-1} 1_{[t_{i-1}, t_i)}(t) ,$$

where $\{t_0, t_1, \ldots, t_n\}$ is a partition of $[0, T]$ and ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$ measurable and $\mathbb{E}(\xi_{i-1}^2) < \infty$.

In this case we define

$$I(f) = \int_0^T f(t)dW_t = \sum_{i=1}^n \xi_{i-1}(W(t_i) - W(t_{i-1}))\,. \tag{1}$$

1.3 Stochastic integral as a gain process

- Let $W(t)$ be the price per share of an asset at time t (Since Brownian motion can take negative as well as positive values, it is not a good model of the price of a asset such as a stock. For the sake of this illustration, we ignore that issue)
- Think of $t_0, t_1, \ldots, t_{n-1}$ as the trading dates
- $f(t_0), f(t_1), \ldots, f(t_{n-1})$ as the position (number of shares) taken in the asset at each trading date and held to the next trading date

The gain $I_t(f)$ from trading at each time t is given by

$$\begin{aligned}
I_t(f) &= f(t_0)(W(t) - W(t_0)) = f(0)W(t) \;\text{ if }\; t_0 < t \le t_1 \\
I_t(f) &= f(0)W(t) + f(t_1)(W(t) - W(t_1)) \;\text{ if }\; t_1 < t \le t_2 \\
I_t(f) &= f(0)W(t) + f(t_1)(W(t_2) - W(t_1)) + f(t_2)(W(t) - W(t_2)) \;\text{ if }\; t_2 < t \le t_3
\end{aligned}$$

In general, if $t_k < t \le t_{k+1}$ then

$$I_t(f) = \sum_{i=0}^{k-1} f(t_i)(W(t_{i+1}) - W(t_i)) + f(t_k)(W(t) - W(t_k)) := \int_0^t f(s)dW(s)$$

Exercise: If f and g are two step processes, then show that $I(af + bg) = aI(f) + bI(g)$ for any $a, b \in \mathbb{R}$.

Lemma 1.3. *(Ito's Isometry) Let $I(f)$ be as in* (1)*, then $\mathbb{E}(I(f)) = 0$ and $\mathbb{E}(I(f))^2 = \int_0^T \mathbb{E}|f(t)|^2 dt$.*

Proof: For each $1 \le i \le n$ in (1),

$$\begin{aligned}
&\mathbb{E}(\xi_{i-1}(W(t_i) - W(t_{i-1})) \\
&= \mathbb{E}(\mathbb{E}[\xi_{i-1}(W(t_i) - W(t_{i-1})|\mathcal{F}_{t_{i-1}}]) \\
&= \mathbb{E}(\xi_{i-1}\mathbb{E}((W(t_i) - W(t_{i-1}))) = 0\,.
\end{aligned}$$

Thus $\mathbb{E}(I(f)) = 0$. Now

$$(I(f))^2 = \sum_{i,j=1}^n \xi_{i-1}\xi_{j-1}(W(t_i) - W(t_{i-1}))(W(t_j) - W(t_{j-1}))\,.$$

For $i \ne j$, suppose $i < j$, then

$$\begin{aligned}
&\mathbb{E}\left(\xi_{i-1}\xi_{j-1}(W(t_i) - W(t_{i-1}))(W(t_j) - W(t_{j-1}))\right) \\
&= \mathbb{E}\left(\mathbb{E}[\xi_{i-1}\xi_{j-1}(W(t_i) - W(t_{i-1}))(W(t_j) - W(t_{j-1}))|\mathcal{F}_{t_{j-1}}]\right) \\
&= \mathbb{E}(\xi_{i-1}\xi_{j-1}(W(t_i) - W(t_{i-1}))\mathbb{E}(W(t_j) - W(t_{j-1}))) = 0\,.
\end{aligned}$$

For $i = j$, we have

$$\begin{aligned}
&\mathbb{E}\left(\xi_{i-1}^2(W(t_i) - W(t_{i-1}))^2\right) \\
&= \mathbb{E}\left(\mathbb{E}[\xi_{i-1}^2\xi_{j-1}(W(t_i) - W(t_{i-1}))^2|\mathcal{F}_{t_{i-1}}]\right) \\
&= \mathbb{E}(\xi_{i-1}^2\mathbb{E}(W(t_i) - W(t_{i-1}))^2) \\
&= \mathbb{E}(\xi_{i-1}^2(t_i - t_{i-1}))\,.
\end{aligned}$$

Thus $\mathbb{E}(I(f))^2 = \sum_{i=1}^n \mathbb{E}(\xi_{i-1}^2)(t_i - t_{i-1}) = \int_0^T \mathbb{E}|f(t)|^2 dt$. □

Theorem 1.4. *Define the process $I_t(f) = \int_0^t f(s)dW(s)$ for $0 \le t \le T$. Then $I_t(f)$ is a martingale.*

Proof: Let $0 \le s \le t \le T$. We assume that s and t are in different sub-intervals, i.e., there are t_l and t_k with $t_l < t_k$ and $s \in [t_l, t_{l+1})$, $t \in [t_k, t_{k+1})$. If they are in the same interval the following proof simplifies and is left as an exercise. Now

$$\begin{aligned}
I_t(f) = &\sum_{i=0}^{l-1} \xi_i(W(t_{i+1}) - W(t_i)) + \xi_l(W(t_{l+1}) - W(t_l)) + \sum_{i=l+1}^{k-1} \xi_i(W(t_{i+1}) - W(t_i)) \\
&+ \xi_k(W(t) - W(t_k))\,.
\end{aligned}$$

$$\mathbb{E}[\sum_{i=0}^{l-1} \xi_i(W(t_{i+1}) - W(t_i))|\mathcal{F}_s] = \sum_{i=0}^{l-1} \xi_i(W(t_{i+1}) - W(t_i))\,.$$

$$\begin{aligned}
\mathbb{E}[\xi_l(W(t_{l+1}) - W(t_l))|\mathcal{F}_s] &= \xi_l\mathbb{E}[W(t_{l+1}) - W(t_l))|\mathcal{F}_s] \\
&= \xi_l\mathbb{E}[W(t_{l+1}) - W(s) + W(s) - W(t_l))|\mathcal{F}_s] \\
&= \xi_l(W(s) - W(t_l))\,.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\sum_{i=l+1}^{k-1} \xi_i(W(t_{i+1}) - W(t_i))|\mathcal{F}_s] &= \sum_{i=l+1}^{k-1} \mathbb{E}[\xi_i(W(t_{i+1}) - W(t_i))|\mathcal{F}_s] \\
&= \sum_{i=l+1}^{k-1} \mathbb{E}[\mathbb{E}[\xi_i(W(t_{i+1}) - W(t_i))|\mathcal{F}_{t_i}]|\mathcal{F}_s] \\
&= \sum_{i=l+1}^{k-1} \mathbb{E}[\xi_i\mathbb{E}(W(t_{i+1}) - W(t_i))|\mathcal{F}_s] = 0\,.
\end{aligned}$$

Similarly,

$$\mathbb{E}[\xi_k(W(t) - W(t_k))|\mathcal{F}_s] = 0\,.$$

Thus adding all together we get,

$$\mathbb{E}[I_t(f)|\mathcal{F}_s] = \sum_{i=0}^{l-1} \xi_i(W(t_{i+1} - W(t_i)) + \xi_l(W(s) - W(t_l)) = I_s(f)\,.$$

□

Theorem 1.5. *The quadratic variation of the Ito integral on $[0, T]$ is $\int_0^T f^2(u)du$.*

Proof: We compute the quadratic variation of the Ito integral on one of the sub-intervals $[t_j, t_{j+1}]$ on which f is constant. For that choose partition points $t_j = s_0 < s_1 < \ldots < s_m = t_{j+1}$ and consider

$$\begin{aligned}
&\sum_{i=0}^{m-1}[I(s_{i+1}) - I(s_i)]^2 \\
&= \sum_{i=0}^{m-1}[\xi_{t_j}(W(s_{i+1}) - W(s_i))]^2 \\
&= \xi_{t_j}^2 \sum_{i=0}^{m-1}(W(s_{i+1}) - W(s_i))^2 \,.
\end{aligned}$$

Now as $m \to \infty$ and the norm of the partition goes to 0,

$$\sum_{i=0}^{m-1}(W(s_{i+1}) - W(s_i))^2 \to t_{j+1} - t_j$$

in L^2. Hence

$$\lim_{||\Pi||\to 0} \sum_{i=0}^{m-1}[I(s_{i+1}) - I(s_i)]^2 = \xi_{t_j}^2(t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \xi_{t_j}^2 \, dt \,.$$

The result now follows by adding up the sub-intervals. □

Symbolically we have, $dI(t) = f(t)dW(t)$ and $dI(t)dI(t) = f^2(t)dt$.

Theorem 1.6. *(Properties of Ito Integral) Let T be a positive constant. Let $f, g \in L^2_{step}([0,T]\times\Omega)$. The the following are true:*

1. *$I_t(af + bg) = aI_t(f) + bI_t(g)$ for all real constants a, b and for all $t \in [0,T]$.*
2. *$I_t(f), 0 \le t \le T$ has continuous sample paths almost surely.*
3. *For each t, $I_t(f)$ is $\mathcal{F}_t$ measurable, where $\mathcal{F}_t$ is a filtration for Brownian motion.*
4. *$I_t(f), 0 \le t \le T$ is a martingale.*
5. *$\mathbb{E}(I_t(f)) = 0$ for all $t \in [0,T]$.*
6. *(Ito's Isometry) $\mathbb{E}|I_t(f)|^2 = \int_0^t \mathbb{E}|f(u)|^2 du$ for all $t \in [0,T]$.*
7. *The quadratic variation is given by $[I(f), I(f)](t) = \int_0^t f^2(u)du$.*

Now what about general integrands.

Lemma 1.7. *Suppose $f \in L^2_{ad}([0,T]\times\Omega)$. Then there exists a sequence of step stochastic processes $\{f_n\}$ such that*

$$\lim_{n\to\infty} \int_0^T \mathbb{E}|f_n(t) - f(t)|^2 dt = 0 \,.$$

Fact: Let L^2 be the space of all random variables X such that $\mathbb{E}(X^2) < \infty$. Then L^2 is complete, i.e., every Cauchy sequence converges. So if $\{X_n\}$ is a sequence of random variables such that $\mathbb{E}|X_n - X_m|^2$ goes to 0 as $m, n \to \infty$,

then there exists a random variable X in L^2 such that $\mathbb{E}|X_n - X|^2$ goes to 0 as $n \to \infty$.
Let $\{f_n\}$ and f be as in Lemma 1.7. Then we have $I(f_n) - I(f_m) = I(f_n - f_m)$. So by Ito's isometry,

$$\mathbb{E}(I(f_n) - I(f_m))^2 = \int_0^T \mathbb{E}(f_n(t) - f_m(t))^2 dt \to 0\,,$$

as $m, n \to \infty$. Thus $\{I(f_n)\}$ is Cauchy in L^2 and hence has a limit in L^2. So define $I(f)$ to be

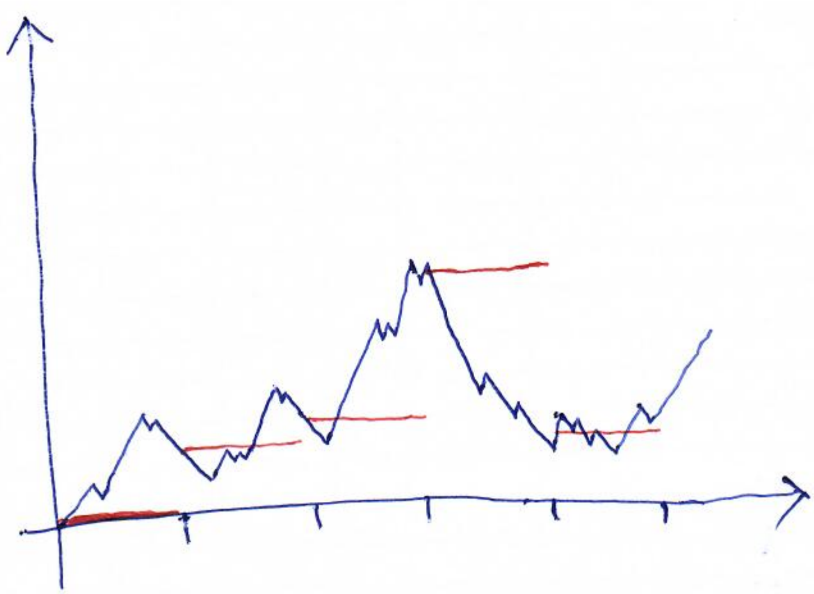
$$I(f) = \int_0^T f(t)dW_t = \lim_{n\to\infty} I(f_n)\,.$$

Theorem 1.8. *(Properties of Ito Integral) Let T be a positive constant. Let $f, g \in L^2_{ad}([0,T]\times\Omega)$. The the following are true:*

1. *$I_t(af + bg) = aI_t(f) + bI_t(g)$ for all real constants a, b and for all $t \in [0,T]$.*
2. *$I_t(f), 0 \le t \le T$ has continuous sample paths almost surely.*
3. *For each t, $I_t(f)$ is $\mathcal{F}_t$ measurable, where $\mathcal{F}_t$ is a filtration for Brownian motion.*
4. *$I_t(f), 0 \le t \le T$ is a martingale.*
5. *$\mathbb{E}(I_t(f)) = 0$ for all $t \in [0,T]$.*
6. *(Ito's Isometry) $\mathbb{E}|I_t(f)|^2 = \int_0^t \mathbb{E}|f(u)|^2 du$ for all $t \in [0,T]$.*
7. *The quadratic variation is given by $[I(f), I(f)](t) = \int_0^t f^2(u)du$.*

Example: Find $\int_0^T W^2(t)dW(t)$. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,T]$. Define the step stochastic process $f_n(t,\omega) = W^2(t_{i-1},\omega)$ for $t \in [t_{i-1}, t_i)$, $i = 1, 2, \ldots, n$. Then

$$\lim_{n\to\infty} \int_0^T \mathbb{E}|f_n(t) - W^2(t)|^2 dt = 0\,.$$



So by definition

$$\int_0^T W^2(t)dW(t) = \lim_{n\to\infty}\int_0^T f_n(t)dW(t) = \lim_{n\to\infty}\sum_{i=1}^n W^2(t_{i-1})(W(t_i)-W(t_{i-1})\,,$$

where the limit is in L^2 sense. It is easy to check that

$$3\sum_{i=1}^n W^2(t_{i-1})(W(t_i)-W(t_{i-1}) =$$
$$W^3(T)-W^3(0)-\sum_{i=1}^n(W(t_i)-W(t_{i-1}))^3 - 3\sum_{i=1}^n W(t_{i-1})(W(t_i)-W(t_{i-1}))^2\,.$$

Now

$$\mathbb{E}(\sum_{i=1}^n(W(t_i)-W(t_{i-1}))^3)^2 = \mathbb{E}\sum_{i=1}^n(W(t_i)-W(t_{i-1}))^6 = 15\sum_{i=1}^n(t_i-t_{i-1})^3 \le 15||\Pi||^2 T \to 0\,.$$

Thus $\sum_{i=1}^n(W(t_i)-W(t_{i-1}))^3$ converges to 0 in L^2. For the second summation term,

$$\mathbb{E}\left|\sum_{i=1}^n W(t_{i-1})(W(t_i)-W(t_{i-1})^2 - \sum_{i=1}^n W(t_{i-1})(t_i-t_{i-1})\right|^2$$
$$= \sum_{i=1}^n 2t_{i-1}(t_i-t_{i-1})^2 \le 2T(T)||\Pi|| \to 0\,.$$

From the above we can say that $\sum_{i=1}^n W(t_{i-1})(W(t_i)-W(t_{i-1}))^2$ converges to $\int_0^T W(t)dt$ in L^2. Thus we have

$$\int_0^T W^2(t)dW(t) = \frac{1}{3}W^3(T) - \int_0^T W(t)dt\,.$$

1.4 Ito's Formula

Chain rule in ordinary calculus tells us that if f and g are differentiable functions then, $\frac{d}{dt}f(g(t)) = f'(g(t))g'(t)$. Thus by Fundamental Theorem of Calculus,

$$f(g(T)) - f(g(0)) = \int_0^T f'(g(t))g'(t)dt\,.$$

If $g(t) = W(t)$, then $W'(t)$ does not make sense but if we write $dW(t)$ in place of $W'(t)dt$ then we get,

$$f(W(T)) - f(W(0)) = \int_0^T f'(W(t))dW(t)\,.$$

So in Ito Calculus is this formula correct? Let us take $f(x) = x^2$. By Taylor's Theorem,

$$f(y)-f(x) = f'(x)(y-x) + \frac{1}{2}f''(x)(y-x)^2\,.$$

Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,T]$. Then,

$$\begin{aligned} f(W(T)) - f(W(0)) &= \sum_{j=0}^{n-1} f(W(t_{j+1})) - f(W(t_j)) \\ &= \sum_{j=0}^{n-1} f'(W(t_j))(W(t_{j+1})-W(t_j)) + \frac{1}{2}\sum_{j=0}^{n-1} f''(W(t_j))(W(t_{j+1})-W(t_j))^2 \\ &= \sum_{j=0}^{n-1} 2(W(t_j))(W(t_{j+1})-W(t_j)) + \sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2 \end{aligned}$$

Thus

$$f(W(T)) - f(W(0)) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} 2(W(t_j))(W(t_{j+1}) - W(t_j)) + \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2$$
$$= \int_0^T 2W(t)dW(t) + T = \int_0^T f'(W(t))dW(t) + \frac{1}{2}\int_0^T f''(W(t))dt\,.$$

So the blue term is the extra term, it is sometimes referred to as Ito's correction term.

Theorem 1.9. *(Ito-Doeblin Formula for Brownian Motion) Let $f(t,x)$ be a function such that the partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ all exist and are continuous. Let $W(\cdot)$ be a Brownian motion. Then for every $T \geq 0$,*

$$f(T,W(T)) - f(0,W(0)) = \int_0^T f_t(t,W(t))dt + \int_0^T f_x(t,W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t,W(t))dt\,.$$

In differential form,

$$df(t,W(t)) = f_t(t,W(t))dt + f_x(t,W(t))dW(t) + \frac{1}{2} f_{xx}(t,W(t))dt\,.$$

Example: Use Ito's formula to show that $W^3(t) - 3\int_0^t W(t)dt$ is a martingale.
Using Ito's formula to the function $f(x) = x^3$, we get

$$W^3(t) - W^3(0) = \int_0^t 3W^2(t)dW(t) + 3\int_0^t W(t)dt\,.$$

Now using the fact that stochastic integral is a martingale we are done.

SKETCH OF PROOF (Ito-Doeblin Formula):
By Taylor's Theorem we have

$$\begin{aligned}
& f(t_{j+1},x_{j+1}) - f(t_j,x_j) \\
= \;& f_t(t_j,x_j)(t_{j+1}-t_j) + f_x(t_j,x_j)(x_{j+1}-x_j) + \frac{1}{2} f_{xx}(t_j,x_j)(x_{j+1}-x_j)^2 \\
+ \;& f_{tx}(t_j,x_j)(x_{j+1}-x_j)(t_{j+1}-t_j) + \frac{1}{2} f_{tt}(t_j,x_j)(t_{j+1}-t_j)^2 + \text{higher order terms}
\end{aligned}$$

We replace x_j by $W(t_j)$, replace x_{j+1} by $W(t_{j+1})$, and sum:

$$\begin{aligned}
& f(T,W(T)) - f(0,W(0)) \\
&= \sum_{j=0}^{n-1} f(t_{j+1},W(t_{j+1})) - f(t_j,W(t_j)) \\
&= \sum_{j=0}^{n-1} f_t(t_j,W(t_j))(t_{j+1}-t_j) + \sum_{j=0}^{n-1} f_x(t_j,W(t_j))(W(t_{j+1}) - W(t_j)) \\
&+ \frac{1}{2}\sum_{j=0}^{n-1} f_{xx}(t_j,W(t_j))(W(t_{j+1}) - W(t_j))^2 + \sum_{j=0}^{n-1} f_{tx}(t_j,W(t_j))(W(t_{j+1}) - W(t_j))(t_{j+1}-t_j) \\
&+ \frac{1}{2}\sum_{j=0}^{n-1} f_{tt}(t_j,W(t_j))(t_{j+1}-t_j)^2 + \text{higher order terms.}
\end{aligned}$$

Let

- $A(\Pi) = \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j)$
- $B(\Pi) = \sum_{j=0}^{n-1} f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j))$
- $C(\Pi) = \sum_{j=0}^{n-1} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2$
- $D(\Pi) = \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))(t_{j+1} - t_j)$
- $E(\Pi) = \sum_{j=0}^{n-1} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2$

If we take the limit as $||\Pi|| \to 0$, then we have

$$A(\Pi) \to \int_0^T f_t(t, W(t))dt$$

$$B(\Pi) \to \int_0^T f_x(t, W(t))dW(t)$$

If f_{xx} is bounded by some constant $M > 0$, then

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_{xx}(t_j, W(t_j))\Big[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)\Big] = 0 \text{ in } L^2.$$

Thus,

$$C(\Pi) \to \int_0^T f_{xx}(t, W(t))dt$$

Note that

$$\begin{aligned}
& \lim_{||\Pi|| \to 0} \Big| \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))(t_{j+1} - t_j)\Big| \\
\leq \quad & \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \Big| f_{tx}(t_j, W(t_j))\Big|\Big|(W(t_{j+1}) - W(t_j))\Big|\Big|(t_{j+1} - t_j)\Big| \\
\leq \quad & \lim_{||\Pi|| \to 0} \max_{0 \leq k \leq n-1} \Big|(W(t_{j+1}) - W(t_j))\Big| \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \Big| f_{tx}(t_j, W(t_j))\Big|\Big|(t_{j+1} - t_j)\Big| \\
\leq \quad & \lim_{||\Pi|| \to 0} \max_{0 \leq k \leq n-1} \Big|(W(t_{j+1}) - W(t_j))\Big| \int_0^T \Big| f_{tx}(t, W(t))\Big| dt = 0
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{||\Pi|| \to 0} \Big| \sum_{j=0}^{n-1} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2\Big| \\
\leq \quad & \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \Big| f_{tt}(t_j, W(t_j))\Big|(t_{j+1} - t_j)^2 \\
\leq \quad & \lim_{||\Pi|| \to 0} \max_{0 \leq k \leq n-1} (t_{j+1} - t_j) \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \Big| f_{tt}(t_j, W(t_j))\Big|(t_{j+1} - t_j) \\
\leq \quad & \lim_{||\Pi|| \to 0} \max_{0 \leq k \leq n-1} (t_{j+1} - t_j) \int_0^T \Big| f_{tt}(t, W(t))\Big| dt = 0
\end{aligned}$$

Therefore

$$|D(\Pi)| + |E(\Pi)| \to 0$$

The higher-order terms likewise contribute zero. Hence we have the result.

Definition 1.10. *Let $W(t), t \geq 0$ be a Brownian motion and let $\mathcal{F}_t, t \geq 0$ be a filtration for the Brownian motion. An Ito process is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du\,,$$

where $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes such that $\int_0^t \mathbb{E}\Delta^2(u)du < \infty$ and $\int_0^t |\Theta(u)|du < \infty$ for all $t \geq 0$.

Lemma 1.11. *The quadratic variation of the above Ito process is*

$$[X, X](t) = \int_0^t \Delta^2(u)du\,.$$

Remark: Symbolically, $dX(t) = \Delta(t)dW(t) + \Theta(t)dt$, so

$$dX_t dX_t = \Delta^2(t)dW_t dW_t + \Theta^2(t)dtdt + 2\Delta(t)\Theta(t)dW_t dt = \Delta^2(t)dt\,.$$

Proof: Suppose $I(t) = \int_0^t \Delta(u)dW(u)$ and $R(t) = \int_0^t \Theta(u)du$. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,t]$. Then,

$$\begin{aligned}\sum_{j=0}^{n-1}[X(t_{j+1}) - X(t_j)]^2 = \sum_{j=0}^{n-1}[I(t_{j+1}) - I(t_j)]^2 + \sum_{j=0}^{n-1}[R(t_{j+1}) - R(t_j)]^2 \\ + 2\sum_{j=0}^{n-1}[I(t_{j+1}) - I(t_j)][R(t_{j+1}) - R(t_j)]\,.\end{aligned}$$

The first term converges to $\int_0^t \Delta^2(u)du$ as $||\Pi||$ goes to 0. For the second term,

$$\begin{aligned}\sum_{j=0}^{n-1}[R(t_{j+1}) - R(t_j)]^2 &\leq \max_{0\leq k\leq n-1} |R(t_{j+1}) - R(t_j)| \sum_{j=0}^{n-1} |R(t_{j+1}) - R(t_j)| \\ &= \max_{0\leq k\leq n-1} |R(t_{j+1}) - R(t_j)| \sum_{j=0}^{n-1} |\int_{t_j}^{t_{j+1}} \Theta(u)du| \\ &\leq \max_{0\leq k\leq n-1} |R(t_{j+1}) - R(t_j)| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |\Theta(u)|du \\ &= \max_{0\leq k\leq n-1} |R(t_{j+1}) - R(t_j)| \int_0^t |\Theta(u)|du \to 0\end{aligned}$$

as $||\Pi|| \to 0$, since $R(\cdot)$ is continuous. The third term is bounded by

$$\begin{aligned}&2\max_{0\leq k\leq n-1} |I(t_{j+1}) - I(t_j)| \sum_{j=0}^{n-1} |R(t_{j+1}) - R(t_j)| \\ &= 2\max_{0\leq k\leq n-1} |I(t_{j+1}) - I(t_j)| \sum_{j=0}^{n-1} |\int_{t_j}^{t_{j+1}} \Theta(u)du| \\ &\leq 2\max_{0\leq k\leq n-1} |I(t_{j+1}) - I(t_j)| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |\Theta(u)|du \\ &= 2\max_{0\leq k\leq n-1} |I(t_{j+1}) - I(t_j)| \int_0^t |\Theta(u)|du \to 0\end{aligned}$$

as $||\Pi|| \to 0$, since $I(\cdot)$ is continuous. Hence we have the result. $\square$

Definition 1.12. *Let $X(t), t \geq 0$ be an Ito process. Let $\Gamma(t), t \geq 0$ be an adapted process. Further suppose that $\int_0^t \mathbb{E}(\Gamma^2(u)\Delta^2(u))du$ and $\int_0^t |\Gamma(u)\Theta(u)|du$ are finite for all $t \geq 0$. We define the integral with respect to an Ito process by,*

$$\int_0^t \Gamma(u)dX(u) = \int_0^t \Gamma(u)\Delta(u)dW(u) + \int_0^t \Gamma(u)\Theta(u)du\,.$$

Theorem 1.13. *(Ito-Doeblin Formula for Ito process) Let $X(t), t \geq 0$ be an Ito process. Let $f(t,x)$ be a function such that the partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ all exist and are continuous. Then for every $T \geq 0$,*

$$\begin{aligned} f(T, X(T)) - f(0, X(0)) &= \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))dX(t) \\ &+ \frac{1}{2}\int_0^T f_{xx}(t, X(t))dX(t)dX(t) \\ &= \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))\Delta(t)dW(t) + \int_0^T f_x(t, X(t))\Theta(t)dt \\ &+ \frac{1}{2}\int_0^T f_{xx}(t, X(t))\Delta^2(t)dt\,. \end{aligned}$$

Generalized geometric Brownian motion

Let $W(t), t \geq 0$ be a Brownian motion and let $\mathcal{F}_t, t \geq 0$ be a filtration for the Brownian motion and let $\alpha(u)$ and $\sigma(u)$ be adapted stochastic processes. Define the Ito process

$$X(t) = X(0) + \int_0^t \sigma(u)dW(u) + \int_0^t (\alpha(u) - \frac{1}{2}\sigma^2(u))du\,.$$

Then

$$dX(t) = \sigma(t)dW(t) + (\alpha(t) - \frac{1}{2}\sigma^2(t))dt\,,$$

and

$$dX(t)dX(t) = \sigma^2(t)dW(t)dW(t) = \sigma^2(t)dt\,.$$

Consider an asset price process given by

$$S(t) = S(0)\exp\{X(t)\} = S(0)\exp\left\{\int_0^t \sigma(u)dW(u) + \int_0^t (\alpha(u) - \frac{1}{2}\sigma^2(u))du\right\},$$

where S(0) is nonrandom and positive. We may write $S(t) = f(X(t))$, where $f(x) = S(0)e^x$, $f^{'}(x) = S(0)e^x$, and $f^{''}(x) = S(0)e^x$. According to the Ito Doeblin formula

$$\begin{aligned} dS(t) &= df(X(t)) \\ &= f^{'}(X(t))dX(t) + \frac{1}{2}f^{''}(X(t))dX(t)dX(t) \\ &= S(0)\exp\{X(t)\}dX(t) + \frac{1}{2}S(0)\exp\{X(t)\}dX(t)dX(t) \\ &= S(t)dX(t) + \frac{1}{2}S(t)dX(t)dX(t) \\ &= \alpha(t)S(t)dt + \sigma(t)S(t)dW(t). \end{aligned}$$

If $\alpha(t) = 0$, then

$$dS(t) = \sigma(t)S(t)dW(t).$$

Integration of both sides yields

$$S(t) = S(0) + \int_0^t \sigma(u)S(u)dW(u).$$

The right-hand side is the nonrandom constant $S(0)$ plus an Ito integral, which is a martingale, and hence (in the case $\alpha = 0$)

$$S(t) = S(0)\exp\Big\{\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\Big\}$$

is a martingale.

Problem: Let $W(\cdot)$ be a Brownian motion and let $\sigma(t)$ be a non random function. Define $I(t) = \int_0^t \sigma(u)dW(u)$. Then show that $I(t)$ is a normal random variable with mean 0 and variance $\int_0^t \sigma^2(u)du$.

Solution: Since $I(t)$ is a martingale and $I(0) = 0$, we must have $\mathbb{E}(I(t)) = I(0) = 0$. Ito's isometry implies that

$$var(I(t)) = \mathbb{E}(I^2(t)) = \int_0^t \sigma^2(u)du.$$

The challenge is to show that $I(t)$ is normally distributed. We shall do this by establishing that $I(t)$ has the moment-generating function of a normal random variable with mean zero and variance $\int_0^t \sigma^2(u)du$, which is

$$\begin{aligned}
& \mathbb{E}(e^{sI(t)}) = \exp\Big\{s^2\frac{1}{2}\int_0^t \sigma^2(u)du\Big\} \text{ for all } s \in \mathbb{R} \\
\Leftrightarrow \quad & \mathbb{E}\Big[\exp\Big\{sI(t) - s^2\frac{1}{2}\int_0^t \sigma^2(u)du\Big\}\Big] = 1 \\
\Leftrightarrow \quad & \mathbb{E}\Big[\exp\Big\{\int_0^t s\sigma(u)dW(u) - \frac{1}{2}\int_0^t (s\sigma(u))^2du\Big\}\Big] = 1
\end{aligned}$$

But the process

$$\exp\Big\{\int_0^t s\sigma(u)dW(u) - \frac{1}{2}\int_0^t (s\sigma(u))^2du\Big\}$$

is a martingale. Hence we have the result.