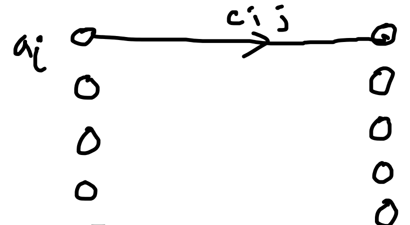
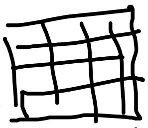


A ~~mxn~~ $m+n$, mn → const matrix

rank $m+n-1$

Transp array  m rows

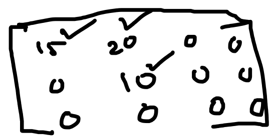
n cols

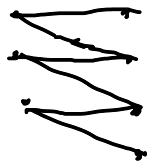
18 Lecture 18

A graph theoretic look at basis

[18.1] **Remember** When striking off the rows and columns in B, it is better you write them as a sequence using R1 for row one and C1 for column one. For example, for the B in the previous example, the strike of sequence is

$$R_6C_4, R_2C_3, R_3C_5, R_4C_1, R_1C_2.$$





Similarly, for the strike off in T you can write

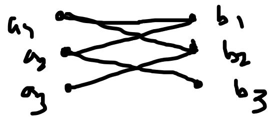
1R1 + 1R2 + 1Rn = 0 $\begin{bmatrix} & \end{bmatrix}$ $C_3, R_2, R_3, C_1, R_1.$

[18.2] **Discussion** `(A bipartite graph for a given set of variables)` a) Consider an $m \times n$ BTP. Put vertices $a_1, \ldots, a_m$ on the left and vertices $b_1, \ldots, b_n$ on the right, in a top down manner. The variable x_{ij} can be viewed to stand for the edge from a_i to b_j. In this way, we see a complete bipartite graph $K_{m,n}$.

b) Let S be a set of some highlighted variables. The graph $\mathbf{\Gamma}_S$ of S is the spanning subgraph of the previous graph, where we only show the edges for the variables in S.

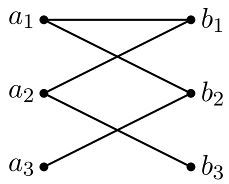
S → some set of variables Γ_S

c) For example



x_{11}	x_{12}	
x_{21}		x_{23}
	x_{32}	

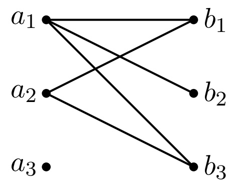
→



and

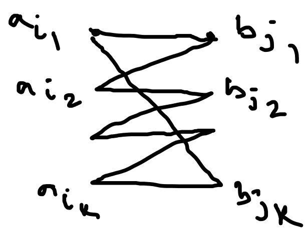
x_{11}	x_{12}	x_{13}
x_{21}		x_{23}

→



Recall that the column corresponding to x_{ij} of the transportation matrix A is $C_{ij} = \begin{bmatrix} e_i \\ e_j \end{bmatrix}$, where e_i has size m and e_j has size n.

[18.3] **Lemma** Let S be a set of variables. Suppose that Γ_S contains a cycle. Then S is linearly dependent.



$C = [a_{i_1}, b_{j_1}, a_{i_2}, b_{j_2}, \ldots, a_{i_k}, b_{j_k}, a_{i_1}]$

$x_{i_1j_1}$ $x_{i_2j_1}$ $x_{i_2j_2}$ $x_{i_1j_k}$

$$+\begin{bmatrix} e_{i_1} \\ \hline e_{j_1} \end{bmatrix} - \begin{bmatrix} e_{i_2} \\ \hline e_{j_1} \end{bmatrix} + \begin{bmatrix} e_{i_2} \\ \hline e_{j_2} \end{bmatrix} - \cdots - \begin{bmatrix} e_{i_1} \\ \hline e_{j_k} \end{bmatrix} = 0$$

Proof. Suppose that Γ_S contains a cycle $[a_{i_1}, b_{j_1}, a_{i_2}, b_{j_2}, \ldots, a_{i_k}, b_{j_k}, a_{i_1}]$. The corresponding edges are

$$x_{i_1,j_1}, x_{i_2,j_1}, x_{i_2,j_2}, x_{i_3,j_2}, \ldots, x_{i_k,j_k}, x_{i_1,j_k},$$

in that order. Then we have

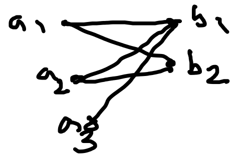
$$C_{i_1j_1} - C_{i_2j_1} + C_{i_2j_2} - C_{i_3j_2} + \cdots + C_{i_kj_k} - C_{i_1j_k} = \begin{bmatrix} e_{i_1} \\ e_{j_1} \end{bmatrix} - \begin{bmatrix} e_{i_2} \\ e_{j_1} \end{bmatrix} + \begin{bmatrix} e_{i_2} \\ e_{j_2} \end{bmatrix} - \cdots - \begin{bmatrix} e_{i_1} \\ e_{j_k} \end{bmatrix} = 0.$$

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \quad 1 \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The alternating signs match perfectly as a cycle must have an even length in a bipartite graph. Thus, S is linearly dependent. ∎

[18.4] Workout Show that the columns of A corresponding to the set of variables $S = \{x_{11}, x_{12}, x_{21}, x_{22}, x_{31}\}$ is linearly dependent.

~~$[a_1, b_1]$~~

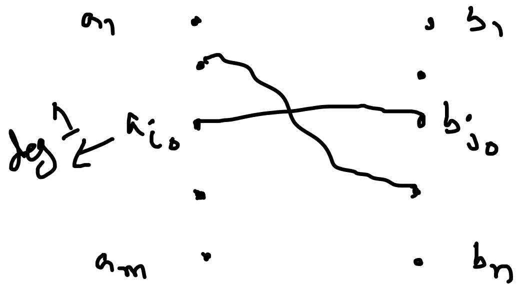


$[a_1, b_1, a_2, b_2, a_1]$

Answer. In Γ_S we have a cycle $[a_1, b_1, a_2, b_2, a_1]$. Hence we have $C_{11} - C_{21} + C_{2,2} - C_{1,2} = 0$. So it is linearly dependent.

[18.5] Lemma Let S be a set of variables and suppose that Γ_S does not have a cycle. Then the columns of A corresponding to the variables in S are linearly independent.

pendant vertex → deg 1 vertex



suppose that a_{i_0} is degree 1 in Γ_S.

$x_{i_0 j_0} \in S$ $\quad \left[\frac{e_{i_0}}{e_{j_0}}\right] \longrightarrow C_{i_0 j_0} \notin \text{span}\left(C_{ij} \mid x_{ij} \in S \setminus \{x_{i_0 j_0}\}\right)$

Take any other $x_{ij} \in S$ $\quad C_{ij} = \left[\frac{e_i}{}\right]$

$S' = S \setminus \{x_{i_0 j_0}\}$. $\Gamma_{S'}$ will not have a cycle. $\Rightarrow S'$ is lin ind.

Proof. If $S = \emptyset$, the statement is vacuously true. So, let $S \neq \emptyset$. Then Γ_S must contain a pendant vertex (degree one vertex).[13] Let that vertex be a_{i_0}. (The other case is also similar.) Suppose that b_{j_0} is adjacent to this vertex. Then x_{i_0,j_0} is the edge. Notice that the corresponding column is $C_{i_0 j_0} = \left[\frac{e_{i_0}}{e_{j_0}}\right]$. None of the remaining edges is incident on e_{i_0} and hence their columns will not have e_{i_0} in the upper half. Hence, $C_{i_0 j_0}$ is independent of the columns of the remaining edges. (Means $C_{i_0 j_0}$ is not in the span of those columns.)

Now consider the set $S' = S \setminus \{x_{i_0 j_0}\}$. Then $\Gamma_{S'}$ does not have a cycle as it is a subgraph of Γ_S. Hence the columns of A corresponding to the elements in S' are linearly independent. As $C_{i_0 j_0}$ is not in the span of these columns, we are done by induction. ∎

for the $\boxed{}$ $m \times n$ transp problem $\Rightarrow m+n-1$ rows

$m+n$ vertex

[18.6] Theorem A set S of variables forms a basis iff Γ_S is a tree.

Proof. Follows from the previous two results. ∎

$$A = \begin{array}{c} m \\ n \end{array}\left[\begin{array}{c} \\ \hline \\ \end{array}\right]$$

Another way to check if a set S of variables is a basis
Draw the bipartite graph Γ_S. If it is tree then S is a basis. Otherwise not.

[18.7] Exercise In a 5×6 BTP, is $\{x_{1,2}, x_{1,3}, x_{2,1}, x_{2,3}, x_{3,3}, x_{3,4}, x_{4,5}, x_{5,5}, x_{5,6}, x_{5,1}\}$ a basis?

[13] There is a positive degree vertex. Let it be u_1. As $d(u_1) > 0$, let it be adjacent to u_2. As $d(u_2) > 1$, there is a vertex u_3 adjacent to u_2. Continue. Repetition is bound to happen, as the graph is finite. Stop at the first repetition. You have got a cycle.

Computing a basic solution for a given basis

Suppose that we have a basis S. How do I get the bs (basic solution) corresponding to S?

Answer. Suppose that $x_{i,j}$ is the first we strike off in the transportation array T with a line, say a column. Then, put $x_{i,j} = b_j$. (Since $x_{i,j}$ was the only highlighted variable in that column, we see that the value $x_{i,j} = b_j$ is correct.) Modify $a_i = a_i - b_j$. (Notice that for the struck off array T', the set $S' = S \setminus \{x_{i,j}\}$ is a basis. So its value can be determined correctly by induction.) In other words, we go for the next strike off and decide the value of another variable and so on.

[18.8] **Example** Compute the basic solution of the following BTP for the highlighted basis. Do we have a basic solution which is not feasible?

c_1 , c_2 , R_3 , c_3

x_{11} 150	x_{12} 200	x_{13} 250	x_{14} −400	200
			x_{24} 250	250
			x_{34} 300	300
150	200	250	150	

.

$x_{11} = 150$, $x_{12} = 200$

$x_{34} = 300$, $x_{13} = 250$

$x_{14} = -400$, $x_{24} = 250$

Answer.

x_{11}	x_{12}	x_{13}	x_{14}	200
			x_{24}	250
			300	300
150	200	250	150	

$\rightarrow$

x_{11}	x_{12}	x_{13}	x_{14}	200
			250	250
			300	300
150	200	250	150	

$\rightarrow$

150	x_{12}	x_{13}	x_{14}	200
			200	250
			300	300
150	200	250	150	

$\rightarrow \rightarrow$

150	200	250	−400	200
			250	250
			300	300
150	200	250	150	

.

Yes, it is a basic solution which is not feasible.

[18.9] **Exercise** `(Alternate way to compute a basic solution from a basis)` How would we compute the bs for a given basis using the tree structure Γ_S of the basis S?

[18.10] **Discussion** We already know that any arbitrary basis will not give us a bfs. So, we have the following algorithm to find an initial bfs.

General method to find an initial bfs.

1) Select any unstruck variable x_{ij} in the transportation table. Put $x_{ij} = \min\{a_i, b_j\}$.
2) If $a_i = \min\{a_i, b_j\}$, then strike off row i from the table and modify $b_j = b_j - a_i$ else strike off column j from the table and modify $a_i = a_i - b_j$.
3) If at any stage there is just a column (or row) left, then start striking off the rows (columns).
4) If there is any unstruck variable available, then go to 1).

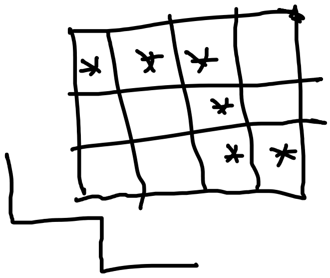
[18.11] **Example** Consider the following transportation array. Find an initial bfs.

40 50 10

60 60

30 90

x_{11}	x_{12} 10	x_{13}	x_{14}	x_{15} 90	~~100~~ 10
x_{21} 10	x_{22} 40	x_{23} 70	x_{24}	x_{25}	~~120~~ ~~50~~ ~~10~~
x_{31} 30	x_{32}	x_{33}	x_{34} 90	x_{35}	~~120~~ 30
40	~~50~~ 40	~~70~~	~~90~~	90	



	x_{12}	x_{13}	x_{14}	x_{15}	100
40	x_{22}	x_{23}	x_{24}	x_{25}	80
	x_{32}	x_{33}	x_{34}	x_{35}	120
0	50	70	90	90	

→

	x_{12}	70	x_{14}	x_{15}	30
40	x_{22}		x_{24}	x_{25}	80
	x_{32}		x_{34}	x_{35}	120
0	50	0	90	90	

→

	30	70			0
40	x_{22}		x_{24}	x_{25}	80
	x_{32}		x_{34}	x_{35}	120
0	20	0	90	90	

→

	30	70			0
40	20		x_{24}	x_{25}	60
			x_{34}	x_{35}	120
0	0	0	90	90	

→

	30	70			0
40	20		60		0
			x_{34}	x_{35}	120
0	0	0	30	90	

→

	30	70			0
40	20		60		0
			30	90	0
0	0	0	0	0	

The initial bfs is given by $x_{12} = 30$, $x_{13} = 70$, $x_{21} = 40$, $x_{22} = 20$, $x_{24} = 60$, $x_{34} = 30$, and $x_{35} = 90$.

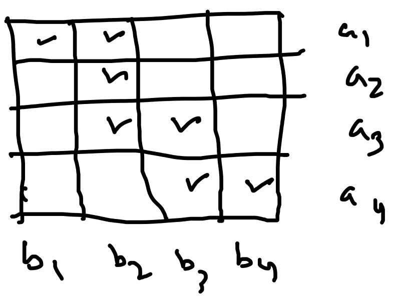
[18.12] **Theorem** The previous algorithm gives us a bfs correctly.

Proof. The fact that we are striking off a line before selecting the next variable ensures that we are going to get a basis. The fact that the value of a new variable $x_{l,k}$ is decided by the minimum of the modified availabilities a_l and b_k, ensures that the value of the variable is nonnegative. The nonbasic variables are zero, anyway. So it is a bfs. ■

> Some traditional common rules to find an initial bfs To find an initial bfs the following two methods are commonly used, though none of them are beneficial, in general.
>
> a) North-West rule Always Select the variable in the north-west (top-left) corner of the unstruck part of the table.
>
> b) Minimum cost rule Always select the variable with the minimum cost from the unstruck part of the table.

[18.13] **Proposition** `(Values of the variables in a bfs)` Values of the basic variables in a bfs can be written as a sum of some $\pm a_i$'s and $\pm b_j$'s, allowing repetition.

Assume the line is a row. Assume some of the already struck off variables were there in that row.



$C_4, R_4, C_3, R_3, R_2, C_2, C_1$

$x_{44} \to b_4$

$x_{i_1 j_1}, x_{i_2 j_2}, \cdots, x_{i_k j_k}$

$x_{i_{k+1} j_{k+1}}$ a line was struck off

$x_{i_{k+1} j_{k+1}} = \underline{a_{i_{k+1}} - \sum \pm a_i - \sum \pm b_j}$

Proof. Suppose, $x_{i_1 j_1}, \cdots, x_{i_s j_s}$ are the basic variables in a bfs, written in the strike off order. We employ induction on the stage number k.

In the first stage of the strike off, if a row was struck off, then $x_{i_1 j_1} = a_{i_1}$, otherwise $x_{i_1 j_1} = b_{j_1}$. So the statement holds for $k = 1$.

Assume that the statements hold up to stage $k-1$. At a stage k, if a row is struck off, then

$$x_{i_k j_k} = a_{i_k} - \sum \text{already found basic variables in that row.}$$

By induction hypothesis, the values of the already found variables in that row are in the required form. So the value of $x_{i_k j_k}$ has the required form. Argument is similar, if a column is struck off at stage k. ■

[18.14] **Corollary** (`Nonnegative integer bfs`) Suppose that in a BTP, all a_i's and b_j's are nonnegative integers. Then in any bfs, the values of the variables are nonnegative integers. In particular, if all a_i and b_j are 1, then the value any variable in that bfs is 0 or 1.

Tomorrow
at 5.05
{
5.10

Proof. As the values of the variables are sums of the $\pm a_i$'s and $\pm b_j$'s, they must be integers. As it is a bfs, they must be nonnegative. For the next question, the value any variable in that bfs is 0 or 1, as the value of a variable (in a bfs) cannot be more than the a_i's and b_j's. ■

The minimality test

[18.15] **Discussion** Consider a BTP with a bfs w. Let $\tilde{A}$ be the matrix obtained from A by dropping the last row and $\tilde{B}$ be a basis matrix for w (this is a submatrix of $\tilde{A}$). For minimality, we need to check the nonnegativity of the vector $\overline{c}^t = c^t - c^t_{\tilde{B}}\tilde{B}^{-1}\tilde{A}$.

a) Suppose that we have a vector $\alpha^t = \begin{bmatrix} p_1 & \cdots & p_m & q_1 & \cdots & q_{n-1} \end{bmatrix}$ such that $c^t - \alpha^t\tilde{A} \geq 0$ with basic entries 0. In view of our discussions on lpp, we know that $\alpha^t = c^t_{\tilde{B}}\tilde{B}^{-1}$ and $\overline{c}^t = c^t - \alpha^t\tilde{A}$.

b) Suppose that we have a vector $\beta^t = \begin{bmatrix} u_1 & \cdots & u_m & v_1 & \cdots & v_{n-1} & 0 \end{bmatrix}$ such that $c^t - \beta^t A \geq 0$ with basic entries 0. As the last entry of β is 0, we see that

$$\beta^t = \begin{bmatrix} \alpha^t & 0 \end{bmatrix} \quad \text{and} \quad \overline{c}^t = c^t - \alpha^t\tilde{A} = c^t - \beta^t A.$$