Definition 0.1. *If X is an integrable random variable then the expectation of X is defined to be*

$$\mathbb{E}(X) = \int_\Omega X d\mathbb{P} = \mathbb{E}(X^+) - \mathbb{E}(X^-)\,.$$

Definition 0.2. *Let X be an integrable random variable or a non-negative random variable and $A \in \mathcal{F}$. Then*

$$\int_A X d\mathbb{P} = \int_\Omega X 1_A d\mathbb{P}\,.$$

Proposition 0.3. *Suppose $X \geq 0$ and $\mathbb{E}(X) = 0$. Then $\mathbb{P}(X = 0) = 1$.*

Proof: Let $E = \{\omega \in \Omega : X(\omega) > 0\}$ and let $E_n = \{\omega \in \Omega : X(\omega) \geq 1/n\}$. Then by definition $0 = \mathbb{E}(X) \geq \int \frac{1}{n} 1_{E_n} d\mathbb{P} = \frac{1}{n}\mathbb{P}(E_n)$. Thus $\mathbb{P}(E_n) = 0$. Hence

$$\mathbb{P}(E) = \mathbb{P}(\cup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(E_n) = 0\,.$$

Thus the conclusion. □

Exercise: Suppose $X > 0$ on A and $\int_A X d\mathbb{P} = 0$, then show that $\mathbb{P}(A) = 0$.

Theorem 0.4. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let g be a Borel-measurable function on $\mathbb{R}$. Then*

$$\mathbb{E}[|g(X)|] = \int_{\mathbb{R}} |g(x)| d\mu_x(x)$$

and if this quantity is finite, then

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) d\mu_x(x).$$

Definition:- Let $f(x)$ be a real-valued function defined on $\mathbb{R}$. The function $f(x)$ is said to be Borel measurable if for every Borel subset B of $\mathbb{R}$, the set $\{x : f(x) \in B\}$ is also a Borel subset of $\mathbb{R}$.

Theorem 0.5. *Let X be a random variable with density f. Then for any Borel-measurable function g on $\mathbb{R}$, we have,*

$$\mathbb{E}[|g(X)|] = \int_{-\infty}^{\infty} |g(x)| f(x) dx.$$

If this quantity is finite, then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

Proof. **Step 1:-** If $g(x) = \mathbb{I}_B(x)$. Then L.H.S= $\mathbb{E}[\mathbb{I}_B(X)] = 1 \cdot \mathbb{P}(X \in B) = \mu_x(B)$. Since X has density so, $\mu_x(B) = \int_B f(x)dx = \int_{-\infty}^{+\infty} g(x) f(x) dx$=R.H.S.

Step 2:-If $g(x) = \sum_{k=1}^{n} \alpha_k \mathbb{I}_{B_k}$, then

$$\begin{aligned}
\text{L.H.S.} &= \mathbb{E}(g(x)) = \mathbb{E}(\sum_{k=1}^{n} \alpha_k \mathbb{I}_{B_k}) = \sum_{k=1}^{n} \alpha_k \mathbb{E}(\mathbb{I}_{B_k}(x)) \\
&= \sum_{k=1}^{n} \alpha_k \int_{-\infty}^{+\infty} \mathbb{I}_{B_k}(x) f(x) dx = \int_{-\infty}^{+\infty} \sum_{k=1}^{n} \alpha_k \mathbb{I}_{B_k}(x) f(x) dx \\
&= \int_{-\infty}^{+\infty} g(x) f(x) dx.
\end{aligned}$$

Step 3:- Let $g(x)$ be a given non-negative Borel-measurable function. Then $\exists$ a sequence of simple functions $0 \leq g_1 \leq g_2 \leq \cdots$ such that

$$\lim_{n\to\infty} g_n(x) = g(x).$$

Now by Monotone Convergence Theorem and previous step we have

$$\begin{aligned}\mathbb{E}(g(X)) &= \lim_{n\to\infty} \mathbb{E}(g_n(X)) \\ &= \lim_{n\to\infty} \int_{-\infty}^{+\infty} g_n(x)f(x)dx = \int_{-\infty}^{+\infty} g(x)f(x)dx.\end{aligned}$$

Step 4:- For any Borel measurable function g, we have $\mathbb{E}[g^+(X)] = \int_{-\infty}^{+\infty} g^+(x)f(x)dx$ and $\mathbb{E}[g^-(X)] = \int_{-\infty}^{+\infty} g^-(x)f(x)dx$. Thus $\mathbb{E}|g(X)| = \int_{-\infty}^{+\infty} |g(x)|f(x)dx$ and $\mathbb{E}(g(X)) = \int_{-\infty}^{+\infty} g(x)f(x)dx$ provided $\mathbb{E}[|g(X)|] < +\infty$. $\square$

1 Change of Measure:-

Theorem 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely non-negative random variable with $\mathbb{E}[Z] = 1$, for $A \in \mathcal{F}$, define*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega).$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is non-negative random variable, then Then

$$\tilde{\mathbb{E}}[X] = \int_\Omega X(\omega)d\tilde{\mathbb{P}} = \mathbb{E}[XZ]. \tag{1}$$

If Z is almost surely strictly positive, we have

$$\mathbb{E}[Y] = \tilde{\mathbb{E}}\left[\frac{Y}{Z}\right], \tag{2}$$

for every non-negative random variable Y.

Proof. $\tilde{\mathbb{P}}(\Omega) = \int_\Omega Z(\omega)d\mathbb{P}(\omega) = \mathbb{E}[Z] = 1$. Let $A_1, A_2, \cdots$ be a sequence of disjoint sets in $\mathcal{F}$, define $B_n = \cup_{k=1}^n A_k$ and $B_\infty = \cup_{k=1}^\infty A_k$, then $\mathbb{I}_{B_n}(w) = \sum_{k=1}^n 1_{A_k}(w)$ and $\mathbb{I}_{B_\infty}(w) = \sum_{k=1}^\infty \mathbb{I}_{A_k}(w)$ and $\mathbb{I}_{B_n}(w) \uparrow \mathbb{I}_{B_\infty}(w)$. By MCT

$$\begin{aligned}\tilde{\mathbb{P}}(B_\infty) &= \int_\Omega \mathbb{I}_{B_\infty}(\omega)Z(\omega)d\mathbb{P} \\ &= \lim_{n\to\infty}\int_\Omega \mathbb{I}_{B_n}(\omega)Z(\omega)d\mathbb{P} \\ &= \lim_{n\to\infty}\int_\Omega \sum_{k=1}^n \mathbb{I}_{A_k}(\omega)Z(\omega)d\mathbb{P}(\omega) \\ &= \lim_{n\to\infty}\sum_{k=1}^n \tilde{\mathbb{P}}(A_k) = \sum_{k=1}^\infty \mathbb{P}(A_k).\end{aligned}$$

Therefore $\tilde{\mathbb{P}}$ is a probability measure. Now suppose X is a non-negative random variable. If $X = \mathbb{I}_A$, then $\tilde{\mathbb{E}}[X] = \tilde{\mathbb{P}}(A) = \int_\Omega \mathbb{I}_A(\omega)Z(\omega)d\mathbb{P}(\omega) = \mathbb{E}[\mathbb{I}_A Z] = \mathbb{E}[XZ]$. Now one can complete the proof using standard machine develoved in previous theorem. When $Z > 0$ a.s., $\frac{Y}{Z}$ is defined and we may replace X in (1) by $\frac{Y}{Z}$ to obtain (2). $\square$

Definition:- Let Ω be a non-empty set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are said to be equivalent if they agree on which sets in $\mathcal{F}$ have probability zero.

Under the assumptions of the above theorem and $Z > 0$ a.s., $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent. Let $A \in \mathcal{F}$ and $\mathbb{P}(A) = 0$. Then the random variable $1_A Z$ is $\mathbb{P}$ a.s. zero $\Rightarrow \tilde{\mathbb{P}}(A) = \int_\Omega 1_A(\omega) Z(\omega) d\mathbb{P}(\omega) = 0$. On the other hand, suppose $B \in \mathcal{F}$ satisfies $\tilde{\mathbb{P}}(B) = 0$. Then $\frac{1}{Z} 1_B = 0$ almost surely under $\tilde{\mathbb{P}}$, so $\tilde{\mathbb{E}}\left[\frac{1}{Z} 1_B\right] = 0 = \mathbb{P}(B)$. Hence $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent.

Example:- Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$ $\mathbb{P} = \mathcal{L}$, Lebesgue measure, and let $0 \leq a \leq b \leq 1$

$$\begin{aligned}\tilde{\mathbb{P}}[a,b] &= \int_a^b 2\omega d\omega = b^2 - a^2 \\ &= \int_a^b 2\omega d\mathbb{P}(\omega) \text{ [using the fact that } d\mathbb{P}(\omega) = d\omega].\end{aligned}$$

So, $\tilde{\mathbb{P}}(B) = \int_B 2\omega d\mathbb{P}(\omega)$ for every Borel set $B \in \mathcal{B}[0,1]$. Set $Z(\omega) = 2\omega > 0$ a.s. in $\mathbb{P}$ and $\mathbb{E}[Z] = \int_0^1 2\omega d\omega = 1$. By (1), for every non-negative random variable X, we have

$$\int_0^1 X(\omega) d\tilde{\mathbb{P}}(\omega) = \int_0^1 X(\omega) 2\omega d\mathbb{P}(\omega).$$

This suggests the notation

$$d\tilde{\mathbb{P}}(\omega) = 2\omega d\omega = 2\omega d\mathbb{P}(\omega).$$

Let X be a standard normal random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Define $Y = X + \theta, \theta > 0$. Then $\mathbb{E}[Y] = \theta$ and $var(Y) = 1$. We want to change to a new probability measure $\tilde{\mathbb{P}}$ on Ω under which Y is a standard normal random variable i.e., $\tilde{\mathbb{E}}[Y] = 0$ and $\tilde{Var}(Y) = \tilde{\mathbb{E}}(Y - \tilde{\mathbb{E}}(Y))^2 = 1$. Define the random variable

$$Z(\omega) = \exp\{-\theta X(\omega) - \frac{1}{2}\theta^2\} \; \forall \omega \in \Omega.$$

We see

$$Z(\omega) > 0 \;\text{ and }\; \mathbb{E}[Z] = 1.$$

$$\begin{aligned}\mathbb{E}[Z] &= \int_{-\infty}^{\infty} \exp\{-\theta x - \frac{1}{2}\theta^2\}\phi(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-\theta x - \frac{1}{2}\theta^2 - \frac{1}{2}x^2\} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{\frac{1}{2}(\theta + x)^2\} dx \;\text{ (put } \theta + x = y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{\frac{-y^2}{2}\} dy = 1.\end{aligned}$$

Define $\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \; \forall A \in \mathcal{F}$

$$\begin{aligned}\tilde{\mathbb{P}}(Y \leq b) &= \int_\Omega 1_{\{Y(\omega) \leq b\}} Z(\omega) d\mathbb{P}(\omega) = \int_\Omega 1_{\{X(\omega) \leq b - \theta\}} \exp\{-\theta X(\omega) - \frac{1}{2}\theta^2\} d\mathbb{P}(\omega) \\ &= \int_{-\infty}^{\infty} 1_{\{x \leq b - \theta\}} \exp\{-\theta x - \frac{1}{2}\theta^2\} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b-\theta} \exp\{-\theta x - \frac{1}{2}\theta^2 - \frac{1}{2}x^2\} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{y^2}{2}} dy \text{ (put } y = \theta + x).\end{aligned}$$

So, $\tilde{\mathbb{P}}(Y \leq b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^b e^{-\frac{y^2}{2}} dy$ which shows that Y is a standard normal random variable under the probability measure $\tilde{\mathbb{P}}$.

Theorem 1.2. ***(Radon-Nikodym Theorem):-*** *Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$. Then there exists an almost surely positive random variable Z such that $\mathbb{E}[Z] = 1$ and*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega),$$

for every $A \in \mathcal{F}$.